Chapter 7 Wave propagation and Fourier optics

Fourier optics describes propagation of light in optical systems using Fourier transform techniques. These techniques are useful since many operations are linear and spatially shift-invariant. They form the basis for analyzing and designing optical imaging and computation systems.

7.1 Propagation of light in the paraxial approximation

Although the classical wave description of light is as a transverse electromagnetic wave, many effects can be studied using a scalar rather than the full vector wave equation. In free space, we have

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \tag{7.1}$$

In this equation ψ represents a component of the electric or magnetic field. For monochromatic, coherent light, we can write

$$\psi(x,y,z,t)=\psi(x,y,z,0)\mathrm{e}^{-\mathrm{j}\omega t}. \tag{7.2}$$

Substituting this into the wave equation, we obtain Helmholtz's equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}=-k^2\psi, \tag{7.3}$$

where $\omega/k=c$.

Consider propagation which is nearly parallel to the z axis, so that

$$\psi(x,y,z,0)=f_z(x,y)\mathrm{e}^{\mathrm{j}kz}, \tag{7.4}$$

where $f_z(x,y)$ varies slowly with z. (Note, for example that for a plane wave travelling parallel to the z axis, $f_z(x,y)$ is constant).

Substituting (7.4) into Helmholtz's equation (7.3) yields

$$\mathrm{e}^{\mathrm{j}kz}\left[\frac{\partial^2 f_z}{\partial x^2}+\frac{\partial^2 f_z}{\partial y^2}+\frac{\partial^2 f_z}{\partial z^2}+2\mathrm{j}k\frac{\partial f_z}{\partial z}-k^2 f_z\right]=-k^2 f_z\mathrm{e}^{\mathrm{j}kz}. \tag{7.5}$$

The *paraxial approximation* neglects $\partial^2 f_z/\partial z^2$ since f_z is assumed to vary slowly with z. This yields the paraxial wave equation

$$\boxed{\frac{\partial^2 f_z}{\partial x^2}+\frac{\partial^2 f_z}{\partial y^2}+2\mathrm{j}k\frac{\partial f_z}{\partial z}=0} \tag{7.6}$$

If we are given $f_{z_1}(x,y)$, the solution of this partial differential equation will give the amplitude distribution at $z=z_2$.

7.2 Solving the paraxial wave equation

The paraxial wave equation is most easily solved by computing its two-dimensional Fourier transform. The two-dimensional Fourier transform of a function $g(x, y)$ is defined as

$$G(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) \exp[-\mathrm{j}2\pi(ux + vy)] \, \mathrm{d}x \, \mathrm{d}y \tag{7.7}$$

The corresponding inverse Fourier transform is

$$g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u, v) \exp[+\mathrm{j}2\pi(ux + vy)] \, \mathrm{d}u \, \mathrm{d}v \tag{7.8}$$

Exercise: Prove that the above transforms are inverses, using the properties of the usual one-dimensional Fourier transform.

We now consider the paraxial wave-equation (7.6). Let $F_z(u, v)$ denote the two-dimensional Fourier transform of $f_z(x, y)$ so that the Fourier transformed paraxial wave equation is

$$(\mathrm{j}2\pi u)^2 F_z + (\mathrm{j}2\pi v)^2 F_z + 2\mathrm{j}k\frac{\partial F_z}{\partial z} = 0 \tag{7.9}$$

or

$$\frac{\partial F_z}{\partial z} = \left(\frac{2\pi^2}{\mathrm{j}k}\right)(u^2 + v^2) F_z(u, v) \tag{7.10}$$

This may be integrated directly, yielding

$$\boxed{F_z\left(u, v\right) = F_0\left(u, v\right) \exp\left[-\frac{\mathrm{j}2\pi^2}{k}\left(u^2 + v^2\right) z\right]} \tag{7.11}$$

Since the (two-dimensional) inverse Fourier transform of $\exp[-\mathrm{j}2\pi^2(u^2 + v^2)z/k]$ is

$$h(x, y) = \frac{1}{\mathrm{j}\lambda z} \exp\left[\frac{\mathrm{j}k}{2z}(x^2 + y^2)\right] \tag{7.12}$$

where $\lambda = 2\pi/k$ (check this!), we can take the inverse Fourier transform of the product in (7.11) to obtain the convolutional relationship

$$f_z(x, y) = (f_0 * h)(x, y) \tag{7.13}$$

or, writing out the convolution in full

$$\boxed{f_z\left(x, y\right) = \frac{1}{\mathrm{j}\lambda z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_0\left(x_0, y_0\right) \exp\left[\frac{\mathrm{j}k}{2z}\left((x - x_0)^2 + (y - y_0)^2\right)\right] \mathrm{d}x_0 \mathrm{d}y_0.} \tag{7.14}$$

,This is the *paraxial diffraction integral.* It states that the field amplitudes at the plane at z are related to those at the plane at $z = 0$ by a linear, shift-invariant filtering operation with "impulse response" $h(x, y)$.

Let us now consider the physical interpretation of this result. The impulse response gives the amplitude of the light on a plane at distance z away from a point source of light (e.g. a pinhole in an opaque screen) located at the origin. If we put back the full dependence on z and t we find that in the paraxial approximation, a point source gives

$$\psi(x,y,z,t) = \frac{1}{\mathrm{j}\lambda z}\exp\left[\frac{\mathrm{j}k}{2z}\left(x^2+y^2\right)\right]\exp(\mathrm{j}kz)\exp(-\mathrm{j}\omega t) \tag{7.15}$$

The spatial dependence of the phase term is

$$\exp\left[\mathrm{j}kz\left(1+\frac{x^2+y^2}{2z^2}\right)\right] \tag{7.16}$$

On the other hand, by Huygen's construction we expect it to be

$$\exp\left[\mathrm{j}k\sqrt{x^2+y^2+z^2}\right] \tag{7.17}$$

However if $z \gg x, y$ (which is true in the paraxial approximation),

$$\mathrm{j}k(x^2+y^2+z^2)^{\frac{1}{2}} = \mathrm{j}kz\left(1+\frac{x^2+y^2}{z^2}\right)^{\frac{1}{2}} \tag{7.18}$$

$$\approx \mathrm{j}kz\left[1+\frac{1}{2}\left(\frac{x^2+y^2}{z^2}\right)-\frac{1}{8}\left(\frac{x^2+y^2}{z^2}\right)^2+\ldots\right] \tag{7.19}$$

and so the phasefronts calculated by the paraxial approximation are very close to the true spherical wavefronts.

Note that

- The $1/z$ factor in the amplitude represents spherical spreading (inverse square law for intensity).
- The additional phase term $-\mathrm{j}$ is actually present, although it is not predicted by a simple application of Huygen's construction.

Exercises:

1. Show that if $f_0(x,y) = 1$, then the diffraction formula predicts that $f_z(x,y) = 1$ for all z. This means that a plane-wave propagating along the z axis is a solution.
2. Show that if $f_0(x,y) = \exp(\mathrm{j}ky\sin\theta)$, the diffraction formula predicts that

   $$f_z(x,y) = \exp(\mathrm{j}ky\sin\theta)\exp\left(-\mathrm{j}\frac{kz}{2}\sin^2\theta\right).$$

   Interpret this result physically, and compare it with the exact result (i.e., without the paraxial approximation).

7.3 Fresnel and Fraunhofer Diffraction

Whenever the paraxial diffraction integral is valid, the observer is said to be in the region of *Fresnel diffraction*. The quadratic terms in the exponential of the diffraction integral can be rewritten as

$$\exp\left[\frac{\mathrm{j}k}{2z}\left((x-x_0)^2+(y-y_0)^2\right)\right] = \exp\left[\frac{\mathrm{j}k}{2z}\left(x^2+y^2\right)\right]\exp\left[-\frac{\mathrm{j}k}{z}\left(xx_0+yy_0\right)\right]\exp\left[\frac{\mathrm{j}k}{2z}\left(x_0^2+y_0^2\right)\right] \tag{7.20}$$

Hence we may write

$$f_z(x,y) = \frac{1}{\mathrm{j}\lambda z} P_z(x,y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{f_0(x_0,y_0)\, P_z(x_0,y_0)\} \exp\left[-\mathrm{j}2\pi\left(\frac{xx_0+yy_0}{z\lambda}\right)\right] \mathrm{d}x_0\mathrm{d}y_0 \quad (7.21)$$

where

$$P_z(x,y) = \exp\left[\frac{\mathrm{j}k}{2z}\left(x^2+y^2\right)\right] \quad (7.22)$$

is a phase factor with unit modulus. Thus the paraxial diffraction integral for propagating a field through a distance z in free space may be interpreted as a sequence of three operations

- Multiplication of $f_0(x_0, y_0)$ by the phase factor $P_z(x_0, y_0)$,
- Calculation of a two-dimensional Fourier transform with spatial-frequency variables $x/(z\lambda)$ and $y/(z\lambda)$,
- Multiplication of the result by a further phase factor $P_z(x, y)$.

In a diffraction experiment, we usually consider the field at the plane $z = 0$ to be non-zero only over a relatively small region, specified by the aperture or mask placed in that plane. If we suppose that z is chosen to be so large that the phase factor $P_z(x_0, y_0) \approx 1$ over the entire region of the (x_0, y_0) plane in which $f_0(x_0, y_0)$ is non-zero. The equation (7.21) may then be written as

$$f_z(x,y) = \frac{1}{\mathrm{j}\lambda z} P_z(x,y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x_0,y_0) \exp\left[-\mathrm{j}2\pi\left(\frac{xx_0+yy_0}{z\lambda}\right)\right] \mathrm{d}x_0\mathrm{d}y_0 \quad (7.23)$$

In this regime, $f_z(x, y)$ is just the two-dimensional Fourier transform of $f_0(x_0, y_0)$ except for a multiplicative phase factor which does not affect the intensity of the diffracted light. This is called the *Fraunhofer approximation*. It is valid provided that $z \gg k(x_0^2 + y_0^2)_{\max}/2$.

7.4 The diffraction grating

For simplicity, specialize to one dimension so that

$$f_z(x,y) = \frac{1}{\sqrt{\mathrm{j}\lambda z}} \exp\left(\mathrm{j}\frac{kx^2}{2z}\right) \int_{-\infty}^{\infty} f_0(x_0) \exp\left(-\mathrm{j}\frac{2\pi x x_0}{z\lambda}\right) \mathrm{d}x_0. \quad (7.24)$$

Consider a screen placed at $z = 0$ illuminated by a plane wave travelling along the z axis as shown in Figure 7.1

If the transmission function of the screen is $t(x_0)$, this is also the amplitude of the field immediately after the screen for an incident plane wave of unit amplitude. For a a diffraction grating with $2N+1$ rectangular slits of width w separated by distance d,

$$f_0(x_0) = t(x_0) = \left(\sum_{k=-N}^{N} \delta\left(x_0 - kd\right)\right) * \Pi(x_0/w) \quad (7.25)$$

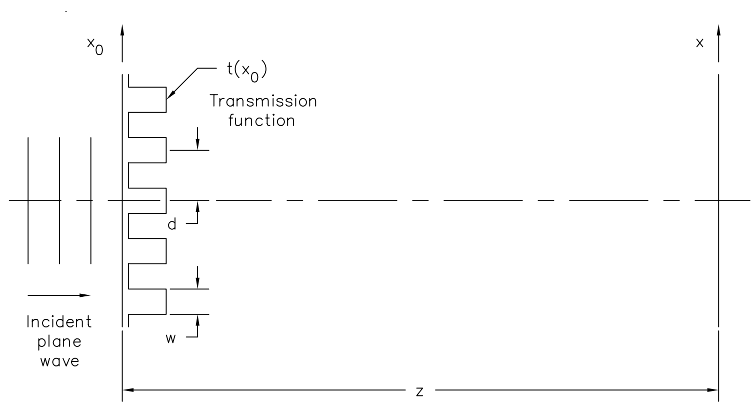


Figure 7.1 Diffraction configuration

where the asterisk denotes convolution and $\Pi(x)$ denotes the "top hat" function which is one for $|x| \leq \frac{1}{2}$ and zero otherwise. The Fourier transform of this is

$$F_0(u) = \left\{ \frac{1}{d} \sum_{k=-\infty}^{\infty} \delta(u - k/d) * (2N+1)\, d \operatorname{sinc}[(2N+1)\, du] \right\} w \operatorname{sinc}(wu) \tag{7.26}$$

$$= (2N+1)\, w \operatorname{sinc}(wu) \sum_{k=-\infty}^{\infty} \operatorname{sinc}[(2N+1)(du-k)]. \tag{7.27}$$

The intensity of the Fourier diffraction pattern is

$$|f_z(x)|^2 = \frac{1}{\lambda z} \left| F_0\left(\frac{x}{\lambda z}\right) \right|^2. \tag{7.28}$$

Figure 7.2 is a plot of $F_0(u)$. The diffraction pattern consists of bright lines positioned at $x_n = n\lambda z/d$. Each line is a sinc function whose first zero is at $\lambda z/[(2N+1)d]$ away from the peak. Hence if N is large, these lines are very sharp. If the incoming light consists of many wavelengths, these components are separated spatially by the grating.

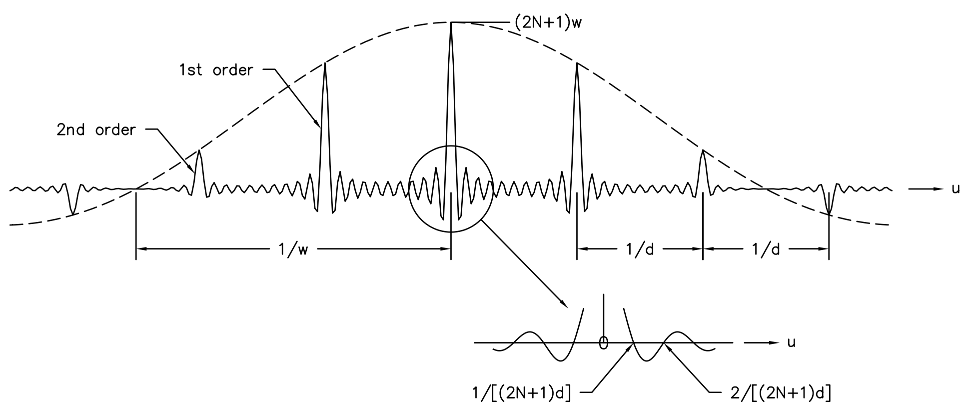


Figure 7.2 Fraunhofer diffraction amplitude

The overall envelope of the diffraction pattern which determines the amount of light diffracted into each order of the spectrum is determined by the width of each slit. Gratings can be designed to diffract most of the light into a particular order of the spectrum or to suppress unwanted orders.

Exercises:

1. Show that for red light ($\lambda \approx 600\,\mathrm{nm}$) and an aperture width of 1 cm, we require $z \gg 120\,\mathrm{m}$ to satisfy the Fraunhofer approximation.

2. Consider a sinusoidal amplitude grating illuminated by a plane wave travelling along the z axis for which the transmission function of the grating is

$$t(x,y) = \frac{1}{2}\left[1 + m\cos\left(2\pi f_0 x\right)\right]\Pi(x/l)\Pi(y/l),$$

   where $\Pi(x)$ is the top-hat function which is unity if $|x| < \frac{1}{2}$. Show that the intensity of the Fraunhofer diffraction pattern at z is given by

$$I(x,y) = \left[\frac{l^2}{2\lambda z}\right]^2 \mathrm{sinc}^2\left(\frac{ly_0}{\lambda z}\right) \times$$
$$\left\{\mathrm{sinc}\left(\frac{lx_0}{\lambda z}\right) + \frac{m}{2}\,\mathrm{sinc}\left(\frac{l\left(x_0 + f_0\lambda z\right)}{\lambda z}\right) + \frac{m}{2}\,\mathrm{sinc}\left(\frac{l\left(x_0 - f_0\lambda z\right)}{\lambda z}\right)\right\}^2$$

   Sketch the form of $I(x,y)$.

3. Compute the Fraunhofer diffraction pattern of a circular aperture of diameter d illuminated by a plane wave along the z axis. Find where the first zero of the diffraction pattern occurs.

   You may find the following identities useful

$$J_0(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z\cos\theta)\,\mathrm{d}\theta$$

$$\int_0^z tJ_0(t)\,\mathrm{d}t = zJ_1(z).$$

7.5 Fresnel diffraction – numerical calculation

Numerical techniques for calculating Fresnel diffraction patterns are often the only feasible methods for practical problems. The two analytically equivalent methods turn out to be useful in different regimes.

7.5.1 The convolutional approach

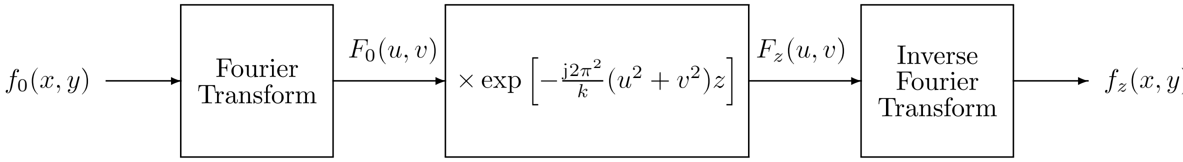
The paraxial diffraction integral is essentially a convolutional relationship

$$f_z(x,y) = \frac{1}{\mathrm{j}\lambda z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_0(x_0,y_0)\exp\left[\frac{\mathrm{j}k}{2z}\left((x-x_0)^2 + (y-y_0)^2\right)\right]\mathrm{d}x_0\,\mathrm{d}y_0. \qquad (7.29)$$

This convolution can be calculated by multiplying together the Fourier transforms

$$F_z(u,v) = F_0(u,v)\exp\left[-\frac{\mathrm{j}2\pi^2}{k}(u^2+v^2)z\right] \qquad (7.30)$$

The following block diagram shows the steps involved in the computation. This is useful for small z since the variation in the phase term would cause aliasing if the change in $2\pi^2(u^2+v^2)z/k$ is too great between adjacent sample points. As $z \to 0$, we see that $f_z(x,y) \to f_0(x,y)$ which corresponds to the "geometrical shadow" of the diffracting obstacle.



7.5.2 The Fourier transform approach

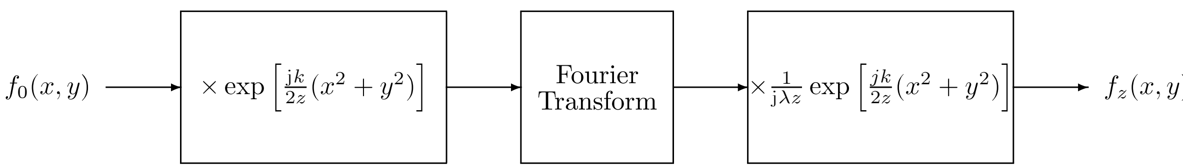
By expanding the quadratic phase term, we saw that the paraxial diffraction integral can be written as

$$f_z(x,y) = \frac{1}{\mathrm{j}\lambda z} P_z(x,y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{f_0(x_0,y_0)\, P_z(x_0,y_0)\} \exp\left[-\mathrm{j}2\pi\left(\frac{xx_0+yy_0}{z\lambda}\right)\right] \mathrm{d}x_0\, \mathrm{d}y_0 \quad (7.31)$$

where

$$P_z(x,y) = \exp\left[\frac{\mathrm{j}k}{2z}\left(x^2+y^2\right)\right] \quad (7.32)$$

The block diagram shows the steps involved in the computation. This is useful for large z since the phase terms change slowly when z is large. As $z \to \infty$, we see that $f_z(x,y)$ tends to the Fourier transform of $f_0(x,y)$ which is the usual result for Fraunhofer diffraction.



7.6 Matlab code for computing Fresnel diffraction patterns

The following MATLAB function computes a one-dimensional Fresnel diffraction pattern using the algorithms discussed above. A fast Fourier transform is used to compute a discrete approximation to the Fourier transform.

```
function [f1,dx1,x1]=fresnel(f0,dx0,z,lambda)
%
% [fz,dx1,xbase1]=fresnel(f0,dx,z,lambda)
%  computes the Fresnel diffraction pattern at distance
%  "z" for light of wavelength "lambda" and an input
%  vector "f0" (must have a power of two points)
%  in the plane z=0, with intersample distance "dx0".
%  Returns the diffraction pattern in "f1", with
%  intersample separation in "dx1", and a baseline
%  against which f1 may be plotted as "x1".
%
N = length(f0); k = 2*pi/lambda;
%
% Compute the critical distance which selects between
```

```
%  the two methods
%
zcrit = N * dx0^2/lambda;
%
if z < zcrit
   %
   % Carry out the convolution with the Fresnel
   %  kernel by multiplication in the Fourier domain
   %
   du = 1./(N*dx0);
   u = [0:N/2-1 -N/2:-1]*du;        % Note order of points for FFT
   H = exp(-i*2*pi^2*u.^2*z/k);     % Fourier transform of kernel
   f1 = ifft( fft(f0) .* H );       % Convolution
   dx1 = dx0;
   x1 = [-N/2:N/2-1]*dx1;           % Baseline for output
else
   %
   % Multiply by a phase factor, compute the Fourier
   %  transform, and multiply by another phase factor
   %
   x0 = [-N/2:N/2-1] * dx0;         % Input f0 is in natural order
   g = f0 .* exp(i*0.5*k*x0.^2/z);  % First phase factor
   G = fftshift(fft(fftshift(g)));  % Fourier transform
   du = 1./(N*dx0); dx1 = lambda*z*du;
   x1 = [-N/2:N/2-1]*dx1;           % Baseline for output
   f1 = G .* exp(i*0.5*k*x1.^2/z);  % Second phase factor
   f1 = f1 .* dx0 ./ sqrt(i*lambda*z);
end
```

Figure 7.3 shows the results of applying this program to the calculation of the diffraction pattern of a single slit of width 1 mm illuminated with red light of wavelength 600 nm. A grid of 1024 points is used in which the slit occupies the central 50 points. The Fresnel diffraction pattern is calculated at distances of 0.01 m, 0.05 m, 0.2 m, 1 m and 2 m.

7.7 Fourier transforming and imaging properties of lenses

Consider a thin convex lens of focal length f. Plane waves incident on the lens parallel to the axis are converted into spherical wavefronts which collapse onto the focus. The action of the lens is to introduce a position-dependent phase shift. Consider this phase shift as a function of x and y. The portion of the wavefront at a distance r from the centre must travel an additional distance

$$d \approx \frac{r^2}{2f} \tag{7.33}$$

relative to the centre. This corresponds to a phase shift at (x, y) of

$$\exp\left[-\mathrm{j}\frac{k\left(x^2+y^2\right)}{2f}\right] \tag{7.34}$$

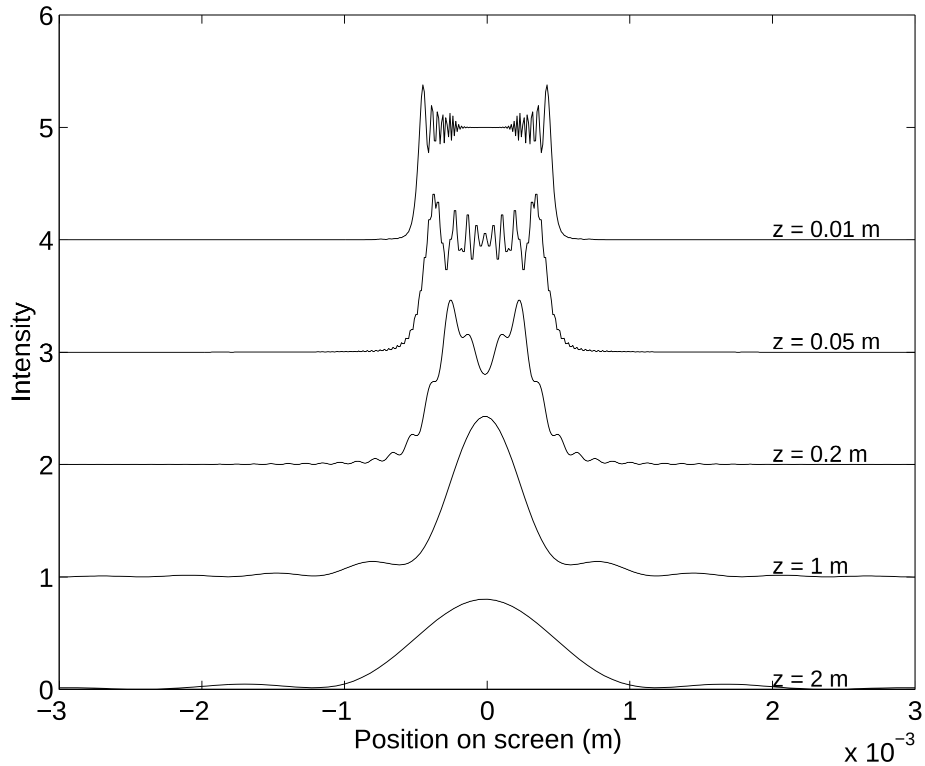


Figure 7.3 One-dimensional Fresnel diffraction from a slit

relative to the centre. Thus the effect of a thin lens is to multiply $f(x,y)$ by $\exp(-\mathrm{j}k(x^2+y^2)/(2f))$. (*Note:* In fact the phase change introduced is $\phi_0 - k(x^2+y^2)/(2f)$ since we have to retard the centre rather than advance the edges.)

We thus see that:

- Propagation through space by a distance z leads us to *convolve* the amplitude $f_z(x,y)$ with the propagation Green's function

$$\frac{1}{\mathrm{j}\lambda z}\exp\left[\mathrm{j}\frac{k\left(x^2+y^2\right)}{2z}\right]. \tag{7.35}$$

- Passing through a thin lens of focal length f leads us to *multiply* the amplitude $f_z(x,y)$ by the transmission function of the lens.

7.7.1 Transparency placed against lens

Consider a transparency with amplitude transmission function $t(x,y)$ placed against a thin convex lens of focal length f. The transparency is illuminated by plane waves travelling along the z axis and we wish to calculate the image $g(x,y)$ on a screen at distance f from the lens. The transformations undergone by the light may be represented by the block diagram:

$$t(x,y) \longrightarrow \boxed{\times \exp\left[-\frac{\mathrm{j}k(x^2+y^2)}{2f}\right]} \longrightarrow \boxed{*\frac{1}{\mathrm{j}\lambda f}\exp\left[\frac{\mathrm{j}k(x^2+y^2)}{2f}\right]} \longrightarrow g(x,y)$$

From the block diagram we immediately write down the expression for $g(x,y)$. This is

$$\begin{aligned} g\left(x_1,y_1\right) &= \frac{1}{\mathrm{j}\lambda f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\left(x_0,y_0\right)\exp\left[-\mathrm{j}\frac{k\left(x_0^2+y_0^2\right)}{2f}\right]\exp\left[\mathrm{j}\frac{k\left(\left(x_1-x_0\right)^2+\left(y_1-y_0\right)^2\right)}{2f}\right]\mathrm{d}x_0\,\mathrm{d}y_0 \\ &= \frac{1}{\mathrm{j}\lambda f}\exp\left[\mathrm{j}\frac{k\left(x_1^2+y_1^2\right)}{2f}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\left(x_0,y_0\right)\exp\left[-\mathrm{j}2\pi\left(\frac{x_1x_0+y_1y_0}{f\lambda}\right)\right]\mathrm{d}x_0\,\mathrm{d}y_0 \\ &= \frac{1}{\mathrm{j}\lambda f}\exp\left[\mathrm{j}\frac{k\left(x_1^2+y_1^2\right)}{2f}\right]T\left(\frac{x_1}{f\lambda},\frac{y_1}{f\lambda}\right) \end{aligned}$$

where $t \longleftrightarrow T$ form a two-dimensional Fourier transform pair. This configuration allows us to achieve Fraunhofer diffraction conditions with relatively small object-to-screen distances.

7.7.2 Transparency placed in front of lens

Now consider the transparency with amplitude transmission function $t(x,y)$ at a distance d_1 in front of a thin convex lens of focal length f. If we place the screen at distance d_2 behind the lens, the block diagram for this configuration is

$$t(x,y) \longrightarrow \boxed{*\frac{1}{j\lambda d_1}\exp\left[\frac{\mathrm{j}k(x^2+y^2)}{2d_1}\right]} \longrightarrow \boxed{\times \exp\left[-\frac{\mathrm{j}k(x^2+y^2)}{2f}\right]} \longrightarrow \boxed{*\frac{1}{j\lambda d_2}\exp\left[\frac{\mathrm{j}k(x^2+y^2)}{2d_2}\right]} \longrightarrow g(x,y)$$

From which we find

$$\begin{aligned} g\left(x_2,y_2\right) &= -\frac{1}{\lambda^2 d_1 d_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\mathrm{j}\frac{k\left(\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2\right)}{2d_2}\right]\exp\left[\mathrm{j}\frac{k\left(x_1^2+y_1^2\right)}{2f}\right]\times \\ &\quad \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\left(x_0,y_0\right)\exp\left[\frac{\mathrm{j}k}{2d_1}\left(\left(x_1-x_0\right)^2+\left(y_1-y_0\right)^2\right)\right]\mathrm{d}x_0\,\mathrm{d}y_0\right)\mathrm{d}x_1\,\mathrm{d}y_1 \\ &= -\frac{1}{\lambda^2 d_1 d_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\left(x_0,y_0\right)\exp\left[\frac{\mathrm{j}k}{2}\left\{\left(\frac{x_0^2+y_0^2}{d_1}\right)+\left(\frac{x_2^2+y_2^2}{d_2}\right)+\right.\right. \\ &\quad \left.\left.\left(\frac{1}{d_2}+\frac{1}{d_1}-\frac{1}{f}\right)\left(x_1^2+y_1^2\right)-2\left(\frac{x_1x_2+y_1y_2}{d_2}\right)-2\left(\frac{x_1x_0+y_1y_0}{d_1}\right)\right\}\right]\mathrm{d}x_0\,\mathrm{d}y_0\mathrm{d}x_1\,\mathrm{d}y_1 \end{aligned}$$

Two interesting cases arise

1. $d_1 = d_2 = f$.

   In this situation we can carry out the integrals over x_1 and y_1 to obtain

$$g(x_2,y_2) = \frac{1}{\mathrm{j}\lambda f}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} t\left(x_0,y_0\right)\exp\left[-\mathrm{j}2\pi\left(\frac{x_0x_2+y_0y_2}{f\lambda}\right)\right]\mathrm{d}x_0\,\mathrm{d}y_0 \tag{7.36}$$

   This is an exact two-dimensional Fourier transform.

2. $1/d_1 + 1/d_2 = 1/f$.

   Again we can carry out the integrations to obtain

$$g\left(x_2, y_2\right) = \frac{1}{\mu} t\left(\frac{x_2}{\mu}, \frac{y_2}{\mu}\right) \exp\left[\frac{\mathrm{j}k}{2d_2}\left(x_2^2 + y_2^2\right)\left(\frac{\mu - 1}{\mu}\right)\right] \tag{7.37}$$

   where $\mu = -d_2/d_1$ is the magnification. Hence

$$|g(x_2, y_2)|^2 = \frac{1}{\mu^2}\left|t\left(\frac{x_2}{\mu}, \frac{y_2}{\mu}\right)\right|^2 \tag{7.38}$$

The distribution of intensity on the screen is a magnified version of that of the transparency function. Since μ is negative (if $d_1 > 0$ and $d_2 > 0$), the image is inverted and the sense of the coordinate system is reversed. The leading term $1/\mu^2$ indicates that the intensity of the image is inversely proportional to the square of the magnification.

Exercise: Carry out the manipulations which lead to the above results from the general expression for $g(x_2, y_2)$.

7.7.3 Vignetting

Since a real lens has finite size, instead of multiplying the amplitude by $\exp[-jk(x^2 + y^2)/(2f)]$, it multiplies by $p(x, y)\exp[-jk(x^2+y^2)/(2f)]$ where $p(x, y)$ is called the *pupil function*. This function is unity within the lens and zero outside.

Exercise: In the imaging configuration $(1/d_1 + 1/d_2 = 1/f)$ show that if the pupil function is not too small,

$$g(x_2, y_2) \approx -\frac{1}{\lambda^2 d_1 d_2}\left(t * h\right)\left(\frac{x_2}{\mu}, \frac{y_2}{\mu}\right) \exp\left[\mathrm{j}\frac{k}{2d_2}\left(x_2^2 + y_2^2\right)\left(\frac{\mu - 1}{\mu}\right)\right] \tag{7.39}$$

where

$$h(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x_1, y_1) \exp\left[\mathrm{j}2\pi\left(\frac{xx_1 + yy_1}{d_1\lambda}\right)\right] \mathrm{d}x_1\,\mathrm{d}y_1 \tag{7.40}$$

Thus the lens smears each pixel in the original object into a region (called the *point-spread function*) whose shape depends on the Fourier transform of the pupil function. For a large pupil, $h(x, y) = d_1^2\lambda^2\delta(x)\delta(y)$ and so this simplifies to the previous result.

7.8 Gaussian Beams

Fresnel diffraction integrals are often difficult to solve analytically. However an important special case is when $f_z(x, y)$ is Gaussian in x and y – these are called Gaussian beams.

Suppose that at $z = 0$, we have a Gaussian distribution of amplitude

$$f_0(x, y) = A_0 \exp[-B_0(x^2 + y^2)] \tag{7.41}$$

To find the amplitude at $z = z_1$, we convolve the amplitude with $h(x, y)$, the Green's function for free propagation, i.e.,

$$f_1(x, y) = (f_0 * h)(x, y) = f_0(x, y) * \frac{1}{\mathrm{j}\lambda z_1} \exp\left[\mathrm{j}k\left(\frac{x^2 + y^2}{2z_1}\right)\right] \tag{7.42}$$