find the value of m for which $\sigma_s/\sqrt{s}$ becomes approximately independent of m. This ratio is our estimate of the error of the mean.

We see that we can make the probable error as small as we wish by either increasing n, the number of data points, or by reducing the variance σ^2. Several *reduction of variance* methods are introduced in Sections 11.6 and 11.7.

Problem 11.10 Estimate of the Monte Carlo error

(a) Estimate the integral of $f(x) = e^{-x}$ in the interval $0 \leq x \leq 1$ using the sample mean Monte Carlo method with $n = 10^4$, $n = 10^6$, and $n = 10^8$. Determine the exact integral analytically and estimate the n-dependence of the actual error. How does your estimated error compare with the error estimate obtained from the relation (11.23)?

(b) Generate 19 additional measurements of the integral each with $n = 10^6$ samples. Compute the standard deviation of the 20 measurements. Is the magnitude of this standard deviation of the means consistent with your estimates of the error obtained in part (a)? Compute the histogram of the additional measurements and confirm that the distribution of the measurements is consistent with a Gaussian distribution.

(c) Divide your first measurement of $n = 10^6$ samples into $s = 10$ subsets of 10^5 samples each. Is the value of $\sigma_s/\sqrt{s}$ consistent with your previous error estimates?

(d) Estimate the integral

$$\int_0^1 e^{-x^2}\, dx, \tag{11.24}$$

to two decimal places using $\sigma/\sqrt{n}$ as an estimate of the probable error.

(e) Estimate the integral $\int_0^{2\pi} \cos^2\theta\, d\theta$ using $n = 10^6$, where $\theta_i = \theta_{i-1} + (2r-1)\delta$, r is uniformly distributed between 0 and 1, and $\delta = 0.1$. Note that because $\cos\theta = \cos\theta + 2k\pi$ for any integer k, we do not have to restrict the range of θ_i. Estimate the error using (11.23). Is this error estimate accurate? Also, estimate the error by grouping the data into $m = 10, 10^2, 10^3, 10^4$, and 10^5 data points and compute $\sigma_s/\sqrt{s}$ for $s = 10^5$, 10^4, 10^3, 10^2, and 10, respectively. How large must m be so that the error estimates for different values of m are approximately the same? Discuss the relation between this result and the correlation of the data points. ■

***Problem 11.11 Importance of randomness**

We learned in Chapter 7 that the random number generator included with many programming languages is based on the linear congruential method. In this method each term in the sequence can be found from the preceding one by the relation

$$x_{n+1} = (ax_n + c) \bmod m, \tag{11.25}$$

where x_0 is the seed, and a, c, and m are nonnegative integers. The random numbers r in the unit interval $0 \leq r < 1$ are given by $r_n = x_n/m$. To examine the effect of a poor random number generator, we choose values of x_0, m, a, and c such that (11.25) has poor statistical properties, for example, a short period. What is the period for $x_0 = 1$, $a = 5$, $c = 0$, and $m = 32$? Estimate the integral in Problem 11.10 by making a single measurement of $n = 10^4$

samples using the linear congruential method (11.25) with these values of x_0, a, c, and m. Analyze your measurement by computing $\sigma_s/s^{1/2}$ for $s = 20$ subsets. Then divide your data into $s = 10$ subsets. Is the value of $\sigma_s/s^{1/2}$ consistent with what you obtained for $s = 20$? ■

***Problem 11.12 Error estimating by bootstrapping**

Suppose that we have made a series of measurements but do not know the underlying probability distribution of the data. How can we estimate the errors of the quantities of interest in an unbiased way? One way is to use a method known as *bootstrapping*, a method that uses random sampling to estimate the errors.

Consider a set of n measurements, such as n values of the pairs (x_i, y_i), and suppose we want to fit this data to the best straight line. If we label the original set of measurements $M = \{m_1, m_2, \ldots, m_n\}$, then the kth *resampled* data set M_k consists of n measurements that are randomly chosen from the original set. This procedure means that some of the m_i may not appear in M_k and some may appear more than once. We then compute the quantity G_k from the resampled data set. For example, G_k could be the slope found from a least squares calculation. If we do this resampling n_r times, a measure of the error in the quantity G is given by σ_G^2, where

$$\sigma_G = \frac{1}{n_r - 1}\sum_{k=1}^{n_r}[G_k - \langle G_k\rangle]^2, \tag{11.26}$$

with

$$\langle G_k\rangle = \frac{1}{n_r}\sum_{k=1}^{n_r} G_k. \tag{11.27}$$

(a) To see how this procedure works, consider $n = 15$ pairs of points x_i randomly distributed between 0 and 1 with the corresponding values of y given by $y_i = 2x_i + 3 + s_i$, where s_i is a uniform random number between -1 and $+1$. First compute the slope m and the intercept b using the least squares method and their corresponding errors using (7.42).

(b) Resample the same set of data 200 times, computing the slope and intercept each time using the least squares method. From your results estimate the probable error for the slope and intercept using (11.26). How well do the estimates from bootstrapping compare with the direct error estimates found in part (a)? Does the average of the bootstrap values for the slope and intercept equal m and b, respectively, from the least squares fits. If not, why not? Do your conclusions change if you resample 800 times? ■

11.5 ■ NONUNIFORM PROBABILITY DISTRIBUTIONS

In Sections 11.2 and 11.4, we learned how uniformly distributed random numbers can be used to estimate definite integrals. We will find that it is more efficient to sample the integrand $f(x)$ more often in regions of x where the magnitude of $f(x)$ is large or rapidly varying. Because *importance sampling* methods require nonuniform probability distributions, we first