CHAPTER

12 Percolation

We introduce several geometrical concepts associated with percolation, including the percolation threshold, clusters, and cluster finding algorithms. We also introduce the ideas of critical phenomena in the context of the percolation transition, including critical exponents, scaling relations, and the renormalization group.

12.1 ■ INTRODUCTION

If a container is filled with small glass beads, and a battery is connected to the ends of the container, no current would pass and the system would be an insulator. Suppose that we choose a glass bead at random and replace it by a small steel ball. Clearly, the system would still be an insulator. If we continue randomly replacing glass beads with steel balls, eventually a current would pass. What percentage of steel balls is needed for the container to become a conductor? The change from the insulating to the conducting state that occurs as the percentage of steel balls is increased is an example of a *percolation phase transition*.

Another example of percolation is from the kitchen. Imagine a large metal sheet on which we randomly place drops of cookie dough. Assume that each drop of cookie dough spreads while the cookies are baking in an oven. If two cookies touch, they coalesce to form one cookie. If we are not careful, we might find a very large cookie that spans from one edge of the sheet to the opposite edge (see Figure 12.1). If such a spanning cookie exists, we say that there has been a percolation transition. As we will discuss in more detail, percolation has to do with *connectivity*.

Our discussion of percolation will require little background in physics, for example, no classical or quantum mechanics and little statistical physics. All that is required is some understanding of geometry and probability. Much of the appeal of percolation is its game-like aspects and intuitive simplicity. If you have a background in physics, this chapter will be more meaningful and can serve as an introduction to phase transitions and to important ideas such as scaling relations, critical exponents, and the renormalization group.

We first discuss a simple model of the cookie example to make the concept of percolation more explicit. We represent the cookie sheet by a lattice where each site can be in one of two states, occupied or empty. Each site is occupied independently of its neighbors with probability p. This model of percolation is called *site* percolation. The occupied sites form *clusters*, which are groups of occupied nearest neighbor lattice sites (see Figure 12.2).

An easy way to study site percolation is to generate a uniform random number r in the unit interval $0 < r \leq 1$ for each site in the lattice. A site is occupied if its random number satisfies the condition $r \leq p$. If p is small, we expect that only small isolated clusters will be present (see Figure 12.3a). If p is near unity, we expect that most of the lattice will

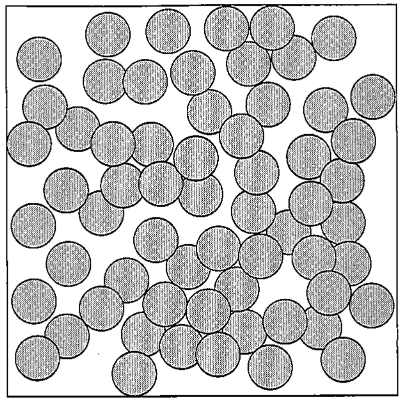
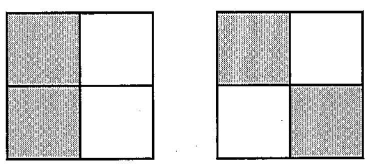
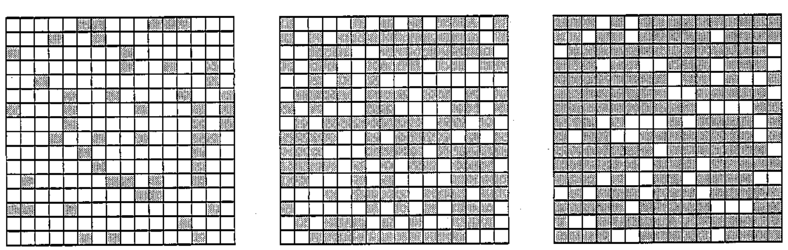


Figure 12.1 Cookies (circles) placed at random on a large sheet. Note that in this case there is a path of overlapping circles that connects the bottom and top edges of the cookie sheet. If such a path exists, we say that the cookies percolate, and there is a spanning path. See Problem 12.4e for a discussion of the algorithm used to generate this configuration.



(a) (b)

Figure 12.2 Example of a site percolation cluster on a square lattice of linear dimension $L = 2$. The two nearest neighbor occupied sites (shaded) in (a) are part of a cluster of size two. The two occupied sites in (b) are not nearest neighbor sites and do not belong to the same cluster; each occupied site is a cluster of size one.



$p = 0.2$ (a) $p = 0.59$ (b) $p = 0.8$ (c)

Figure 12.3 Examples of site percolation clusters on a square lattice of linear dimension $L = 16$ for $p = 0.2$, 0.59, and 0.8. On average, the fraction of occupied sites (shaded squares) is equal to p. Note that in this example, there exists a cluster that spans the lattice horizontally and vertically for $p = 0.59$.