• Interpolation

Experimental Measurements $\rightarrow$ *Data* $\rightarrow$ *Theoretical Analysis (curves, formula, etc.)* $\rightarrow$ *Theory* $\rightarrow$ *Experiments.*

Given two distinct points (x_0, y_0) and (x_1, y_1), draw a straight line through them gives a linear polynomial

$$\begin{aligned} P_1(x) &= \frac{(x_1 - x)y_0 + (x - x_0)y_1}{x_1 - x_0} \qquad (P_1(x_i) = y_i) \\ &= y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \ . \end{aligned} \tag{1}$$

With three data points $(x_i, f(x_i), i = 0, 1, 2)$,

$$\begin{aligned} P_2(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) \rightarrow L_{0,2}(x) f(x_0) \qquad (2) \\ &+ \frac{(x - x_2)(x - x_0)}{(x_1 - x_2)(x_1 - x_0)} f(x_1) \rightarrow L_{1,2}(x) f(x_1) \\ &+ \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \rightarrow L_{2,2}(x) f(x_2) \end{aligned}$$

Figs. 4.1, 4.2, 4.3
Examples 1 & 2

Polynomial Interpolation (Lagrange):

nth order polynomials require $n+1$ data points:

$$P_n(x) = \sum_{j=0}^{n} L_{j,n}(x) f(x_j) \ , \qquad j = 0, 1, \cdots, n, \qquad (3)$$

$$L_{j,n}(x) = \frac{(x - x_0)\cdots(x - x_{j-1})(x - x_{j+1})\cdots(x - x_n)}{(x_j - x_0)\cdots(x_j - x_{j-1})(x_j - x_{j+1})\cdots(x_j - x_n)}.$$

$$L_{j,n}(x_i) = \delta_{ij} \quad = 1 \ if \ i = j, \quad = 0 \ if \ i \neq j. \qquad (4)$$

Formula (3) is called *Lagrange's formula* for the degree n interpolating polynomial.

The interpolating polynomial is *unique*. (proof?)

For $n+1$ data points, the interpolating polynomial $P_n(x)$ may have degree less than n, eg, three points lie on a straight line.

Error in Interpolation: Let $n \geq 0$, $f(x)$ has $n+1$ continuous derivatives on $[a, b]$, $x_0, x_1, \cdots, x_n$ are distinct node points in $[a, b]$. Then

$$f(x) - P_n(x) = \frac{(x - x_0)(x - x_1)\cdots(x - x_n)}{(n+1)!} f^{(n+1)}(c_x) \quad (5)$$

for $a \leq x \leq b$, where

$$Min\{x_0, x_1, \cdots, x_n, x\} \leq c_x \leq Max\{x_0, x_1, \cdots, x_n, x\} .$$

Behavior of the Error: The polynomial

$$\Psi_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$$

is the most important quantity. Assume the node points $x_0, x_1, \cdots, x_n$ are evenly spaced, then for larger values of $n (\geq 5)$, the values of $\Psi_n(x)$ change greatly through the interval $x_0 \leq x \leq x_n$. The values in $[x_0, x_1]$ and $[x_{n-1}, x_n]$ become much larger than the values in the middle.

Examples.

Figs. 4.5, 4.6
Examples 1, 2, 3

Hermite Interpolation

Use the information about the function and its derivatives. Example, determine a_i for

$$P_3(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \ , \quad \text{with}$$

$$P_3(x_0) = f(x_0), \qquad P_3(x_1) = f(x_1); \qquad j = 0, 1$$

$$P_3'(x_0) = f'(x_0), \qquad P_3'(x_1) = f'(x_1) \ .$$

We obtain: (check Eq. (7))

$$\begin{aligned} P_3(x) \ &= \ \left[1 - 2\frac{x - x_0}{x_0 - x_1}\right] \frac{(x - x_1)^2}{(x_0 - x_1)^2} f(x_0) \\ &+ \ \left[1 - 2\frac{x - x_1}{x_1 - x_0}\right] \frac{(x - x_0)^2}{(x_1 - x_0)^2} f(x_1) \\ &+ \ (x - x_0)\frac{(x - x_1)^2}{(x_0 - x_1)^2} f'(x_0) \\ &+ \ (x - x_1)\frac{(x - x_0)^2}{(x_1 - x_0)^2} f'(x_1) \ . \end{aligned} \tag{6}$$

The general Hermite interpolation polynomial is

$$P_{2n+1}(x) = \sum_{j=0}^{n} h_{j,n}(x) f(x_j) + \sum_{j=0}^{n} \bar{h}_{j,n}(x) f'(x_j) \ , \tag{7}$$

$$h_{j,n}(x) = [1 - 2(x - x_j) L_{j,n}'(x_j)] L_{j,n}^2(x),$$

$$\bar{h}_{j,n}(x) = (x - x_j) L_{j,n}^2(x) \ .$$

- **Cubic Splines and Interpolation**

  Problems with Polynomial Interpolation:

  Information on the function derivatives may not be available, so cannot do Hermite interpolation.

  Derivative of the Lagrange interpolating polynomial is *not continuous*.

  Examples: see the attached.

  Table 4.3
  Figs. 4.7, 4.8, 4.9

Spline Interpolation:

Assume we have n data points $(x_i, y_i), i = 1, \cdots, n$ such that

$$a = x_1 < x_2 < \cdots < x_n = b \ .$$

We seek a function $s(x)$ defined on $[a, b]$ that interpolates the data

$$s(x_i) = y_i, \qquad i = 1, \cdots, n \ .$$

For smoothness of $s(x)$, we require that $s^{'}(x)$ and $s^{''}(x)$ be continuous. We also want the curve to follow the general shape given by the piecewise linear function connecting the data points. We want $s^{'}(x)$ not change too rapidly between node points, i.e., requiring the second derivatives $s^{''}(x)$ to be as small as possible.

There is a unique solution $s(x)$ to this problem, and it satisfies the following:

1. $s(x)$ is a polynomial of degree ≤ 3 on *each* subinterval $[x_{j-1}, x_j]$, for $j = 2, 3, \cdots, n$;
2. $s(x)$, $s^{'}(x)$, and $s^{''}(x)$ are continuous for $a \leq x \leq b$;
3. $s^{''}(x_1) = s^{''}(x_n) = 0$.

The function $s(x)$ is called the *natural cubic spline function* that interpolates the data $\{(x_i, y_i)\}$.

Construction of $s(x)$:

Define:

$$h_j \equiv x_{j+1} - x_j, \qquad M_j \equiv s^{''}(x_j)$$

Step 1: $s(x)$ is cubic on $[x_j, x_{j+1}]$,

$$s(x) = a_j(x-x_j)^3 + b_j(x-x_j)^2 + c_j(x-x_j) + d_j, \tag{8}$$

$$y_j = d_j, \qquad y_{j+1} = a_j h_j^3 + b_j h_j^2 + c_j h_j + d_j. \tag{9}$$

Step 2: take derivatives twice,

$$s^{''}(x) = 6a_j(x - x_j) + 2b_j\ . \tag{10}$$

$$M_j = s^{''}(x_j) = 2b_j, \quad \Rightarrow \quad b_j = M_j/2\ . \tag{11}$$

$$M_{j+1} = 6a_j h_j + 2b_j, \quad \Rightarrow \quad a_j = \frac{M_{j+1} - M_j}{6h_j}\ . \tag{12}$$

Step 3: from Eq. (9),

$$c_j = \frac{y_{j+1} - y_j}{h_j} - \frac{h_j M_{j+1} + 2h_j M_j}{6} \ . \tag{13}$$

Step 4: M_js are unknown and they are determined by continuity condition of $s'(x)$ on subintervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$, $j = 2, \cdots, n-1$,

$$s'(x) = 3a_j(x - x_j)^2 + 2b_j(x - x_j) + c_j \ , \tag{14}$$

$$s'(x_j + 0) = s'(x_j - 0) \ . \tag{15}$$

$$h_{j-1}M_{j-1} + (2h_{j-1} + 2h_j)M_j + h_j M_{j+1}$$

$$= \ 6\left(\frac{y_{j+1} - y_j}{h_j} - \frac{y_j - y_{j-1}}{h_{j-1}}\right) \tag{16}$$

Step 5: $M_1 = M_n = 0$, as assumed.

So we need to solve a *tridiagonal system* (16) to find M_js. Finally,

$$\begin{aligned} s(x) \ &= \ \frac{(x_j - x)^3 M_{j-1} + (x - x_{j-1})^3 M_j}{6(x_j - x_{j-1})} \\ &+ \ \frac{(x_j - x)y_{j-1} + (x - x_{j-1})y_j}{x_j - x_{j-1}} \\ &- \ \frac{1}{6}(x_j - x_{j-1})\left[(x_j - x)M_{j-1} + (x - x_{j-1})M_j\right] \ . \end{aligned} \tag{17}$$

Fig. 4.10

- **Derivatives and Integration**

  Given data points $\{x_i, f(x_i)\}$, how to calculate derivatives and integrations?

  Naive way: use data directly,

$$f^{'}(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \approx \cdots .$$

$$\int_a^b dx f(x) \approx \sum_i f(x_i)(x_i - x_{i-1}) \approx \cdots .$$

  Better way: have approximation for function $f(x)$.

  Example: linear polynomial approximation $[x_i, x_{i+1}]$,

$$P_1(x) = \frac{(x_{i+1} - x)f(x_i) + (x - x_i)f(x_{i+1})}{x_{i+1} - x_i} ,$$

$$\begin{aligned} \int_{x_i}^{x_{i+1}} dx f(x) &\approx \int_{x_i}^{x_{i+1}} dx P_1(x) \\ &= \frac{x_{i+1} - x_i}{2}(f(x_i) + f(x_{i+1})) , \end{aligned}$$

  leads to Trapezoidal rule.

  Approximating $f(x)$ by $P_2(x)$ leads to Simpson's rule.

  <u>Differentiation</u>

  Let $P_2(x) \approx f(x)$, $x_0 = x_1 - h, x_2 = x_1 + h$, then

$$P_2(x_1) = \frac{f(x_2) - f(x_1)}{2h} = \frac{f(x_1 + h) - f(x_1 - h)}{2h} ,$$

  the central formula.

- **The Best Approximation Problem**

Consider $f(x) = e^x$ on $-1 \leq x \leq 1$.

Fig. 4.12

1. linear approximation, $t_1(x) = 1 + x$,

   $$max_{-1 \leq x \leq 1} |e^x - t_1(x)| = 0.718 \ .$$

   Take $m_1(x) = 1.2643 + 1.1752x$,

   $$max_{-1 \leq x \leq 1} |e^x - m_1(x)| = 0.279 \ .$$

2. cubic approximation, $t_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$,

   $$max_{-1 \leq x \leq 1} |e^x - t_3(x)| = 5.16E - 2 \ .$$

   $m_3(x) = 0.994579 + 0.995668x + 0.542973x^2 + 0.179533x^3$,

   $$max_{-1 \leq x \leq 1} |e^x - m_3(x)| = 5.53E - 3 \ .$$

We note that:

1. $m_n(x)$ is usually a significant improvement on the Taylor polynomial $t_n(x)$.
2. The error $f(x) - m_n(x)$ has its error dispersed over the entire interval $[a, b]$.
3. The error $f(x) - m_n(x)$ is oscillatory on $[a, b]$. It changes sign at least $n + 1$ times.

Examples: Taylor and Minimax Errors for e^x on $[-1, 1]$.

Table 4.5
Figs. 4.13, 4.14
Table 4.6

Example(Interpolation): Values of e^x.

x	e^x	x	e^x
0.80	2.225541	0.85	2.339647
0.81	2.247908	0.86	2.363161
0.82	2.270500	0.87	2.386911
0.83	2.293319	0.88	2.410900
0.84	2.316367	0.89	2.435130

1. Estimate $e^{0.826}$ by $P_1(x)$.

Take $x_0 = 0.82, x_1 = 0.83$,

$$\begin{aligned} P_1(0.826) &= \frac{(0.83 - 0.826)e^{0.82} + (0.826 - 0.82)e^{0.83}}{0.01} \\ &= 2.2841914, 2.2841638 \end{aligned}$$

(Exact value)

Error:

$$e^x - P_1(x) = \frac{(x - x_0)(x - x_1)e^{c_x}}{2}$$

$$|e^x - P_1(x)| \le \frac{(x_1 - x_0)^2 e}{8}$$

2. By $P_2(x)$. Take $x_0 = 0.82, x_1 = 0.83, x_2 = 0.84$,

$$P_2(0.826) = 2.2841639$$

Fig. 4.4

$$\begin{aligned} |e^x - P_2(x)| &\le \frac{|(x - x_0)(x - x_1)(x - x_2)|}{6} e^1 \\ &\le \frac{h^3 e}{9\sqrt{3}}, \qquad h = x_1 - x_0 = x_2 - x_1 \end{aligned}$$

Example: Table 4.2
Interpolation to cos(x)

Example(Error behavior):

Fig. 4.5

Let $f(x) = cos(x), h = 0.2, n = 8$, interpolate at $x = 0.9$.

1. $x_0 = 0.8, x_8 = 2.4$, so $x = 0.9$ is in $[x_0, x_1]$.

$$cos(0.9) - P_8(0.9) = -5.51 \times 10^{-9}$$

2. $x_0 = 0.2, x_8 = 1.8$, so $x = 0.9$ is in $[x_3, x_4]$.

$$cos(0.9) - P_8(0.9) = 2.26 \times 10^{-10}$$