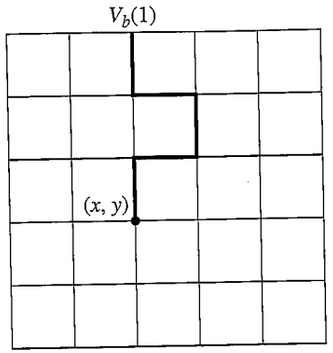

Figure 10.4 A random walk on a 6×6 grid starting at the point $(x, y) = (3, 3)$ and ending at the boundary site $V_b(3, 6)$ where the potential is recorded.

closed surface with fixed potential. The random walk algorithm for computing the solution to Laplace's equation can be stated as:

1. Begin at a point (x, y) where the value of the potential is desired and take a step in a random direction.
2. Continue taking steps until the walker reaches the surface. Record $V_b(i)$, the potential at the boundary site i. A typical walk is shown in Figure 10.4.
3. Repeat steps (1) and (2) n times and sum the potential found at the surface each time.
4. The value of the potential at the point (x, y) is estimated by

$$V(x, y) = \frac{1}{n} \sum_{i=1}^{n} V_b(i), \tag{10.25}$$

where n is the total number of random walkers.

Problem 10.17 Random walk solution of Laplace's equation

(a) Consider the square region shown in Figure 10.1 and compare the results of the random walk method with the results of the relaxation method (see Problem 10.10c). Try $n = 100$ and $n = 1000$ walkers and choose a point near the center of the square.

(b) Repeat part (a) for other points within the square. Do you need more or less walkers when the potential near the surface is desired? How quickly do your answers converge as a function of n? ■

The disadvantage of the random walk method is that it requires many walkers to obtain a good estimate of the potential at each site. However, if the potential is needed at only a small number of sites, then the random walk method might be more appropriate than the relaxation method, which requires the potential to be computed at all points within the region. Another case where the random walk method is appropriate is when the geometry of the boundary is fixed, but the potential in the interior for a variety of different boundary potentials is needed. In this case the quantity of interest is $G(x, y, x_b, y_b)$, the number of

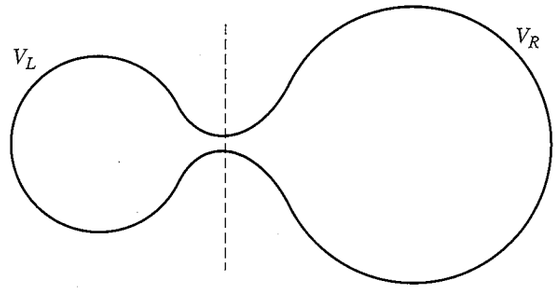

Figure 10.5 Two regions of space connected by a narrow neck. The boundary of the left region has a potential V_L, and the boundary of the right region has a potential V_R.

times that a walker from the point (x, y) lands at the boundary (x_b, y_b). The random walk algorithm is equivalent to the relation

$$V(x, y) = \frac{1}{n} \sum_{x_b, y_b} G(x, y, x_b, y_b) V(x_b, y_b), \tag{10.26}$$

where the sum is over all sites on the boundary. We can use the same function G for different distributions of the potential on a given boundary. G is an example of a Green's function, a function that you will encounter in advanced treatments of electrodynamics and quantum mechanics (cf. Section 16.9). Of course, if we change the geometry of the boundary, we have to recompute the function G.

Problem 10.18 Green's function solution of Laplace's equation

(a) Compute the Green's function $G(x, y, x_b, y_b)$ for the same geometry considered in Problem 10.17. Use at least 200 walkers at each interior site to estimate G. Because of the symmetry of the geometry, you can determine some of the values of G from other values without doing an additional calculation. Store your results for G in a file.

(b) Use your results for G found in part (a) to determine the potential at each interior site when the boundary potential is the same as in part (a), except for five boundary sites which are held at $V = 20$. Find the locations of the five boundary sites that maximize the potential at the interior site located at (3, 5). Repeat the calculation to maximize the potential at (5, 3). Use trial and error guided by your physical intuition. ■

The random walk algorithm can help us gain additional insight into the nature of the solutions of Laplace's equation. Suppose that you have a boundary similar to the one shown in Figure 10.5. The potentials on the left and right boundaries are V_L and V_R, respectively. If the neck between the two sides is narrow, it is clear that a random walker starting on the left side has a low probability of reaching the other side. Hence, we can conclude that the potential in the interior of the left side is approximately V_L, except very near the neck.