The proper distance $d\sigma$ is the distance measured by a surveyor placing meter sticks in space between two locations. This distance is clearly greater than the result predicted by (18.22) when the events occur at different r-coordinates. In fact, the rate of change of the proper length with respect to the r-coordinate becomes infinite as we approach what is known as the event horizon $r = 2M$. The distance around a gravitational mass has no such singularity, which is why this distance is used to define the r-coordinate. (The singularity at the event horizon is an artifact of the Schwarzschild coordinate system. An object falling into a black hole passes through the event horizon without incident and is crushed only at $r = 0$.)

Exercise 18.8 Measuring distance

Although the rate of change of distance with respect to r becomes infinite, the distance from a point outside the event horizon to the event horizon is finite. Write a test program that integrates $d\sigma$ from a point $r = a$ outside the event horizon to a point arbitrarily close to the event horizon. What is the distance from $r = 4$ to $r = 2$ when $M = 1$? ■

Exercise 18.9 Event horizon

(a) Although general relativity predicts the shape of orbits around any spherically symmetric gravitational object, not all objects have an event horizon. (The objects that do are black holes.) The event horizon assumes that the mass of the entire object is within the horizon, which implies very high mass densities. Calculate the r-coordinate of the event horizon for an object having the mass of the Earth and compare it to the radius of the Earth. Repeat the calculation for the Sun.

(b) The event horizon for the black hole believed to exist at the center of our galaxy has an event horizon of $r = 7.6 \times 10^9$ m. What is its mass in units of our own Sun (solar mass)? ■

The variable t in the Schwarzschild metric is the time as measured by a faraway observer. Time as measured by a local observer is known as the proper time τ. Observers experience time as measured by their wristwatches and (18.19) shows that the wristwatch time interval $d\tau$ depends on location. A faraway observer who measures the time between two light flashes records a value of dt, while an observer standing next to these flashes measures a time interval $d\tau$ given by

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2. \tag{18.24}$$

Proper time intervals near a gravitational mass are clearly smaller than faraway time intervals. This result gives rise to the gravitational red shift when applied to light.

Exercise 18.10 Local and faraway time

Estimate the difference due to gravitational effects between local time and faraway time during one hour for an observer on Earth. Does special relativity play a role in an actual measurement? ■

18.5 ■ PARTICLE AND LIGHT TRAJECTORIES

The physics describing the trajectory of a particle in the vicinity of a gravitational mass can be formulated using the principle of stationary aging (see Hanc). This principle states that a particle takes a path through spacetime such that the elapsed time $\delta\tau$ recorded by the wristwatch attached to the particle is a maximum. (In general, $\delta\tau$ is an extrenum so it could also be a minimum or a saddle point.) Because Lagrangian dynamics is based on the principle that the integral of the Lagrangian over time (called the action) is also stationary, we construct a Lagrangian using the Schwarzschild metric:

$$\mathcal{L}(r, \dot{r}, \phi, \dot{\phi}) = \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2\right]^{1/2}, \tag{18.25}$$

such that

$$\tau = \int_{\text{initial event}}^{\text{final event}} d\tau = \int_{\text{initial event}}^{\text{final event}} \mathcal{L}(r, \dot{r}, \phi, \dot{\phi})\, dt. \tag{18.26}$$

Because the Lagrangian in (18.25) satisfies (18.15), we can take the required derivatives and simplify terms to obtain the following system of first-order differential equations:

$$\frac{dr}{dt} = \dot{r} \tag{18.27a}$$

$$\frac{d\dot{r}}{dt} = \frac{4M^3 - 4M^2r - 4M^2r^3\dot{\phi}^2 + 4Mr^4\dot{\phi}^2 - r^5\dot{\phi}^2 + r^2(M - 3M\dot{r}^2)}{(2M - r)r^3} \tag{18.27b}$$

$$\frac{d\phi}{dt} = \dot{\phi} \tag{18.27c}$$

$$\frac{d\dot{\phi}}{dt} = \frac{2(-3M + r)\dot{r}\dot{\phi}}{(2M - r)r} \tag{18.27d}$$

$$\frac{dt}{dt} = 1. \tag{18.27e}$$

Note that the independent variable in (18.27) is faraway time. The metric provides an additional differential equation if we wish to track the particle's proper time τ:

$$\frac{d\tau}{dt} = \left[\left(1 - \frac{2M}{r}\right) - \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2\right]^{1/2}. \tag{18.28}$$

Exercise 18.11 General relativistic trajectories

(a) Write a program that plots the general relativistic trajectory of a particle using Schwarzschild coordinates. Verify that circular orbits are obtained for $v = \sqrt{M/r}$ for $r \geq 6M$.

(b) Show that there are no stable circular orbits for $r < 6M$.

(c) Add the differential equation for proper time. What is the proper time for one complete orbit at $r = 6$? This interval is the orbital period as measured by an observer traveling with the particle. Compare this wristwatch orbital period to the faraway orbital period and to the time interval predicted by (18.24). Explain any discrepancies in your numerical values.