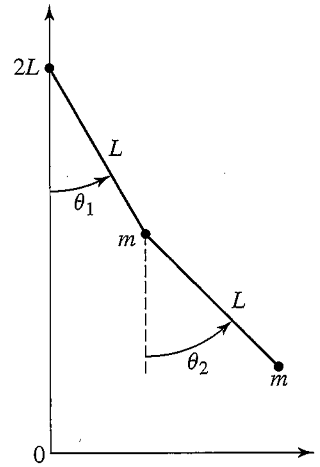

Figure 6.12 The double pendulum.

The constant a for Saturn's rings is approximately $2 \times 10^{12}\,\text{km}^3$. We can show, using a similar technique as before, that the volume in (r, θ) space is preserved, and hence (6.47) is a Hamiltonian map.

The purpose of the above discussion was only to motivate and not to derive the form of the map (6.47). In Problem 6.20 we investigate how the map (6.47) yields the qualitative structure of Saturn's rings. In particular, what happens to the values of r_n if the period of a moon is related to the period of Mimas by the ratio of two integers?

Problem 6.20 A simple model of the rings of Saturn

(a) Write a program to implement the map (6.47). Be sure to save the last two values of r so that the values of r_n are updated correctly. The radius of Saturn is 60.4×10^3 km. Express all lengths in units of 10^3 km. In these units $a = 2000$. Plot the points $(r_n \cos\theta_n, r_n \sin\theta_n)$. Choose initial values for r between the radius of Saturn and σ, the distance of Mimas from Saturn, and find the bands of r_n values where stable trajectories are found.

(b) What is the effect of changing the value of a? Try $a = 200$ and $a = 20{,}000$ and compare your results with part (a).

(c) Vary the force function. Replace $\cos\theta$ by other trigonometric functions. How do your results change? If the changes are small, does that give you some confidence that the model has something to do with Saturn's rings? ■

A more realistic dynamical system is the double pendulum, a system that can be demonstrated in the laboratory. This system consists of two equal point masses m, with one suspended from a fixed support by a rigid weightless rod of length L and the other suspended from the first by a similar rod (see Figure 6.12). Because there is no friction, this system is an example of a Hamiltonian system. The four rectangular coordinates x_1, y_1, x_2, and y_2 of

the two masses can be expressed in terms of two generalized coordinates θ_1, θ_2:

$$x_1 = L\sin\theta_1 \tag{6.48a}$$

$$y_1 = 2L - L\cos\theta_1 \tag{6.48b}$$

$$x_2 = L\sin\theta_1 + L\sin\theta_2 \tag{6.48c}$$

$$y_2 = 2L - L\cos\theta_1 - L\cos\theta_2. \tag{6.48d}$$

The kinetic energy is given by

$$K = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2) = \frac{1}{2}mL^2[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)], \tag{6.49}$$

and the potential energy is given by

this choice makes $E_{min} = 0$ ($V = 0 \Rightarrow y = 0$)

$$U = mgL[3 - 2\cos\theta_1 - \cos\theta_2]. \tag{6.50}$$

For convenience, U has been defined so that its minimum value is zero.

To use Hamilton's equations of motion (6.37), we need to express the sum of the kinetic energy and potential energy in terms of the generalized momenta and coordinates. In rectangular coordinates, the momenta are equal to $p_i = \partial K/\partial \dot{q}_i$, where, for example, $q_i = x_1$ and p_i is the x-component of $m\mathbf{v}_1$. This relation works for generalized momenta as well, and the generalized momentum corresponding to θ_1 is given by $p_1 = \partial K/\partial\dot{\theta}_1$. If we calculate the appropriate derivatives, we can show that the generalized momenta can be written as

$$p_1 = mL^2[2\dot{\theta}_1 + \dot{\theta}_2\cos(\theta_1 - \theta_2)] \tag{6.51a}$$

$$p_2 = mL^2[\dot{\theta}_2 + \dot{\theta}_1\cos(\theta_1 - \theta_2)]. \tag{6.51b}$$

The Hamiltonian or total energy becomes

$$H = \frac{1}{2mL^2}\frac{p_1^2 + 2p_2^2 - 2p_1p_2\cos(q_1 - q_2)}{1 + \sin^2(q_1 - q_2)} + mgL(3 - 2\cos q_1 - \cos q_2), \tag{6.52}$$

where $q_1 = \theta_1$ and $q_2 = \theta_2$. The equations of motion can be found by using (6.52) and (6.37).

Figure 6.13 shows a Poincaré map for the double pendulum. The coordinate p_1 is plotted versus q_1 for the same total energy $E = 15$, but for two different initial conditions. The map includes the points in the trajectory for which $q_2 = 0$ and $p_2 > 0$. Note the resemblance between Figure 6.13 and plots for the standard map above the critical value of k; that is, there is a regular trajectory and a chaotic trajectory for the same parameters but different initial conditions.

Problem 6.21 Double pendulum

(a) Use either the fourth-order Runge–Kutta algorithm (with $\Delta t = 0.003$) or the second-order Euler–Richardson algorithm (with $\Delta t = 0.001$) to simulate the double pendulum. Choose $m = 1$, $L = 1$, and $g = 9.8$. The input parameter is the total energy