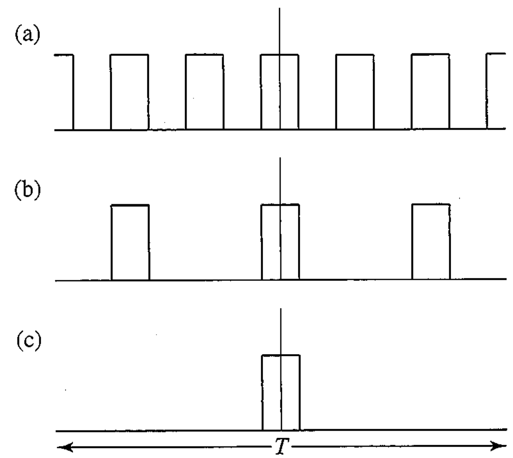

Figure 9.2 A series of pulses with increasing periods.

9.6 ■ POWER SPECTRUM

The power output of a periodic electrical signal $f(t)$ is proportional to the integral of the signal squared:

$$P = \frac{1}{T}\int_0^T |f(t)|^2\, dt. \tag{9.46}$$

Another way to look at the power is to calculate the power P associated with the Fourier components ω_k of a signal that has been sampled at regular intervals. If we substitute (9.31) into (9.46), rearrange terms, and apply the orthogonality condition (9.14), we obtain

$$P = \sum_{j=-N/2}^{N/2} \sum_{k=-N/2}^{N/2} c_j^* c_k \frac{1}{T}\int_0^T e^{i(\omega_k-\omega_j)t}\, dt = \sum_{k=-N/2}^{N/2} |c_k|^2. \tag{9.47}$$

From (9.47) we conclude that the average power of $|f(t)|^2$ is the sum of the power in each frequency component $P(\omega_k) = |c_k|^2$. This result is one form of Parseval's theorem.

In most measurements, the function $f(t)$ corresponds to an amplitude, and the power or intensity is proportional to the square of this amplitude, or for complex functions, the modulus squared. The *power spectrum* $P(\omega_k)$ is proportional to the power associated with a particular frequency component embedded in the quantity of interest. Well-defined peaks in $P(\omega_k)$ often correspond to normal mode frequencies.

What happens to the power associated with frequencies greater than the Nyquist frequency? To answer this question, consider two choices of the Nyquist frequency ω_{Q_a} and $\omega_{Q_b} > \omega_{Q_a}$ and the corresponding sampling times $\Delta_b < \Delta_a$. The calculation with $\Delta = \Delta_b$ is more accurate because the sampling time is smaller. Suppose that this calculation of the spectrum yields the result that $P(\omega > \omega_a) > 0$. What happens if we compute the power spectrum using $\Delta = \Delta_a$? The power associated with $\omega > \omega_a$ must be "folded" back into the $\omega < \omega_a$ frequency components. For example, the frequency component at $\omega + \omega_a$ is added to the true value at $\omega - \omega_a$ to produce an incorrect value at $\omega - \omega_a$ in the computed

power spectrum. This phenomenon is called *aliasing* and leads to spurious results. Aliasing occurs in calculations of $P(\omega)$ if the latter does not vanish above the Nyquist frequency. To avoid aliasing, it is necessary to sample more frequently or to remove the high frequency components from the signal before sampling the data.

Although the power spectrum can be computed by a simple modification of `AnalyzeApp`, it is a good idea to use the FFT for many of the following problems.

Problem 9.18 Aliasing

Sample the sinusoidal function, $\sin 2\pi t$, and display the resulting power spectrum using sampling frequencies above and below the Nyquist frequency. Start with a sampling time of $\Delta = 0.1$ and increase the time until $\Delta = 10.0$.

(a) Is the power spectrum sharp? That is, is all the power located in a single frequency? Does your answer depend on the ratio of the period to the sampling time?

(b) Explain the appearance of the power spectrum for the values $\Delta = 1.25$, $\Delta = 1.75$, and $\Delta = 2.5$.

(c) What is the power spectrum if you sample at the Nyquist frequency or twice the Nyquist frequency? ■

Problem 9.19 Examples of power spectra

(a) Create a data set with N points corresponding to $f(t) = 0.3\cos(2\pi t/T) + r$, where r is a uniform random number between 0 and 1 and $T = 4$. Plot $f(t)$ versus t in time intervals of $\Delta = 4T/N$ for $N = 128$. Can you visually detect the periodicity? Compute the power spectrum using the same sampling interval $\Delta = 4T/N$. Does the frequency dependence of the power spectrum indicate that there are any special frequencies? Repeat with $T = 16$. Are high or low frequency signals easier to pick out from the random background?

(b) Simulate a one-dimensional random walk and compute the time series $x^2(t)$, where $x(t)$ is the distance from the origin of the walk after t steps. Average $x^2(t)$ over several trials. Compute the power spectrum for a walk of $t \le 256$. In this case $\Delta = 1$, the time between steps. Do you observe any special frequencies?

(c) Let f_n be the nth member of a random number sequence. As in part (b), $\Delta = 1$. Compute the power spectrum of the random number generator. Do you detect any periodicities? If so, is the random number generator acceptable? ■

Problem 9.20 Power spectrum of coupled oscillators

(a) Modify your program developed in Problem 9.2 so that the power spectrum of the position of one of the N particles is computed at the end of the simulation. Set $\Delta = 0.1$ so that the Nyquist frequency is $\omega_Q = \pi/\Delta \approx 31.4$. Choose the time of the simulation equal to $T = 25.6$ and let $k/m = 1$. Plot the power spectrum $P(\omega)$ at frequency intervals equal to $\Delta\omega = \omega_0 = 2\pi/T$. First choose $N = 2$ and choose the initial conditions so that the system is in a normal mode. What do you expect the power spectrum to look like? What do you find? Then choose $N = 10$ and choose initial conditions corresponding to various normal modes. Is the power spectrum the same for all particles?