Table 15.2 The energy and magnetization of the 2^4 states of the zero-field Ising model on the 2×2 square lattice. The quantity $g(E, M)$ is the number of microstates with the same energy.

# Spins Up	$g(E, M)$	Energy	Magnetization
4	1	−8	4
3	4	0	2
2	4	0	0
2	2	8	0
1	4	0	−2
0	1	−8	−4

periodic boundary conditions. In Table 15.2 we group the sixteen states according to their total energy and magnetization.

We can compute all the quantities of interest using Table 15.2. The partition function is given by

$$Z = 2e^{8\beta J} + 12 + 2e^{-8\beta J}. \tag{15.113}$$

If we use (15.104) and (15.113), we find

$$\langle E\rangle = -\frac{\partial}{\partial\beta}\ln Z = -\frac{1}{Z}[2(8)e^{8\beta J} + 2(-8)e^{-8\beta J}]. \tag{15.114}$$

Because the other quantities of interest can be found in a similar manner, we only give the results:

$$\langle E^2\rangle = \frac{1}{Z}[(2\times 64)e^{8\beta J} + (2\times 64)e^{-8\beta J}] \tag{15.115}$$

$$\langle M\rangle = \frac{1}{Z}(0) = 0 \tag{15.116}$$

$$\langle |M|\rangle = \frac{1}{Z}[(2\times 4)e^{8\beta J} + 8\times 2] \tag{15.117}$$

$$\langle M^2\rangle = \frac{1}{Z}[(2\times 16)e^{8\beta J} + 8\times 4]. \tag{15.118}$$

The dependence of C and χ on βJ can be found by using (15.114) and (15.115) and (15.116) and (15.118), respectively.

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