(b) Our first task is to determine the optimum value of the parameter b. Let $\Delta x = 1$ and $N \geq 100$ and try the following combinations of c and Δt: $c = 1$, $\Delta t = 0.1$; $c = 1$, $\Delta t = 0.5$; $c = 1$, $\Delta t = 1$; $c = 1$, $\Delta t = 1.5$; $c = 2$, $\Delta t = 0.5$; and $c = 2$, $\Delta t = 1$. Verify that the value $b = (c\Delta t)^2 = 1$ leads to the best results; that is, for this value of b, the initial form of the wave is preserved..

(c) It is possible to show that the discrete form of the wave equation with $b = 1$ is exact up to numerical roundoff error (cf. DeVries). Hence, we can replace (9.56) by the simpler algorithm

$$u(x, t + \Delta t) = u(x + \Delta x, t) + u(x - \Delta x, t) - u(x, t - \Delta t). \tag{9.57}$$

That is, the solutions of (9.57) are equivalent to the solutions of the original partial differential equation (9.50). Try several different initial waveforms and show that if the displacements have the form $f(x \pm ct)$, the waveform maintains its shape with time. For the remaining problems, we will use (9.57) corresponding to $b = 1$. Unless otherwise specified, choose $c = 1$, $\Delta x = \Delta t = 1$, and $N \geq 100$ in the following problems. ■

Problem 9.26 Velocity of waves

(a) Use the waveform given in Problem 9.25a and verify that the speed of the wave is unity by determining the distance traveled in a given amount of time. Because we have set $\Delta x = \Delta t = 1$ and $b = 1$, the speed $c = 1$. (A way of incorporating different values of c is discussed in Problem 9.27c.)

(b) Replace the waveform considered in part (a) by a sinusoidal wave that fits exactly; that is, choose $u(x, t) = \sin(qx - \omega t)$ such that $\sin q(N + 1) = 0$. Measure the period T of the wave by measuring the time it takes for successive maxima to pass a given point. What is the wavelength λ of the wave? Does it depends on the value of q? The frequency of the wave is given by $f = 1/T$. Verify that $\lambda f = c$. ■

Problem 9.27 Reflection of waves

(a) Consider a wave of the form $u(x, t) = e^{-(x-10-ct)^2}$. Use fixed boundary conditions so that $u_0 = u_{N+1} = 0$. What happens to the reflected wave?

(b) Modify your program so that free boundary conditions are incorporated: $u_0 = u_1$ and $u_N = u_{N+1}$. Compare the phase of the reflected wave to your result for the phase from part (a).

(c) What happens to a pulse at the boundary between two media? Set $c = 1$ and $\Delta t = 1$ on the left side of your grid and $c = 2$ and $\Delta t = 0.5$ on the right side. These choices of c and Δt imply that $b = 1$ on both sides, but that the right side is updated twice as often as the left side. What happens to a pulse that begins on the left side and moves to the right? Is there both a reflected and transmitted wave at the boundary between the two media? What is their relative phase? Find a relation between the amplitude of the incident pulse and the amplitudes of the reflected and transmitted pulses. Repeat for a pulse starting from the right side. ■

Problem 9.28 Superposition of waves

(a) Consider the propagation of the wave determined by $u(x, t = 0) = \sin(4\pi x/N)$. What must $u(x, -\Delta t)$ be so that the wave moves in the positive x direction? Test your answer by doing a simulation. Use periodic boundary conditions. Repeat for a wave moving in the negative x direction.

(b) Simulate two waves moving in opposite directions each with the same spatial dependence given by $u(x, 0) = \sin(4\pi x/N)$. Describe the resultant wave pattern. Repeat the simulation for $u(x, 0) = \sin(8\pi x/N)$.

(c) Assume that $u(x, 0) = \sin q_1 x + \sin q_2 x$ with $q_1 = 10\pi/N$ and $q_2 = 12\pi/N$. Describe the qualitative form of $u(x, t)$ for fixed t. What is the distance between modulations of the amplitude? Estimate the wavelength associated with the fine ripples of the amplitude. Estimate the wavelength of the envelope of the wave. Find a simple relation for these two wavelengths in terms of the wavelengths of the two sinusoidal terms. This phenomena is known as *beats*.

(d) Consider the motion of the two Gaussian pulses moving in opposite directions, $u_1(x, 0) = e^{-(x-10)^2}$ and $u_2(x, 0) = e^{-(x-90)^2}$. Choose the array at $t = -\Delta t$ as in Problem 9.25. What happens to the two pulses when they overlap or partially overlap? Do they maintain their shape? While they are going through each other, is the displacement $u(x, t)$ given by the sum of the displacements of the individual pulses? ■

Problem 9.29 Standing waves

(a) In Problem 9.28c we considered a *standing wave*, the continuum analog of a normal mode of a system of coupled oscillators. As is the case for normal modes, each point of the wave has the same time dependence. For fixed boundary conditions, the displacement is given by $u(x, t) = \sin qx \cos \omega t$, where $\omega = cq$, and the wavenumber q is chosen so that $\sin qN = 0$. Choose an initial condition corresponding to a standing wave for $N = 100$. Describe the motion of the particles and compare it with your observations of standing waves on a rope.

(b) Establish a standing wave by displacing one end of a system periodically. The other end is fixed. Let $u(x, 0) = u(x, -\Delta t) = 0$, and $u(x = 0, t) = A \sin \omega t$ with $A = 0.1$. How long must the simulation run before you observe standing waves? How large is the standing wave amplitude? ■

We have seen that the wave equation can support pulses that propagate indefinitely without distortion. In addition, because the wave equation is linear, the sum of any two solutions is also a solution, and the principle of superposition is satisfied. As a consequence, we know that two pulses can pass through each other unchanged. We have also seen that similar phenomena exist in the discrete system of linearly coupled oscillators. What happens if we create a pulse in a system of nonlinear oscillators? As an introduction to nonlinear wave phenomena, we consider a system of N coupled oscillators with the potential energy