```
      // the tools menu, following statement does so explicitly
      frame.showDataTable(true);
   }

   public void reset() {
      control.setValue("f(t)", "sin(pi*t/10)");
      control.setValue("delta", 0.1);
      control.setValue("N", 200);
      control.setValue("number of coefficients", 10);
      calculate();
   }

   public static void main(String[] args) {
      CalculationControl.createApp(new AnalyzeApp());
   }
}
```

In Problem 9.11 we compute the Fourier coefficients for several functions. We will see that if $f(t)$ is a sum of sinusoidal functions with different periods, it is essential that the period $T = N\Delta$ in the Fourier analysis program be an integer multiple of the periods of all the functions in the sum. If T does not satisfy this condition, then the results for some of the Fourier coefficients will be spurious. In practice, the solution to this problem is to vary the sampling interval Δ and the total time over which the signal $f(t)$ is sampled. Fortunately, the results for the power spectrum (see Section 9.6) are less ambiguous than the values for the Fourier coefficients themselves.

Problem 9.11 Fourier analysis

(a) Use the `AnalyzeApp` class with $f(t) = \sin \pi t/10$. Determine the first three nonzero Fourier coefficients by doing the integrals in (9.26) analytically before running the program. Choose the number of data points to be $N = 200$ and the sampling time $\Delta = 0.1$. Which Fourier components are nonzero? Repeat your analysis for $N = 400$, $\Delta = 0.1$; $N = 200$, $\Delta = 0.05$; $N = 205$, $\Delta = 0.1$; and $N = 500$, $\Delta = 0.1$, and other combinations of N and Δ. Explain your results by comparing the period of $f(t)$ with $N\Delta$, the assumed period. If the combination of N and Δ are not chosen properly, do you find any spurious results for the coefficients?

(b) Consider functions $f_1(t) = \sin \pi t/10 + \sin \pi t/5$, $f_2(t) = \sin \pi t/10 + \cos \pi t/5$, and $f_3(t) = \sin \pi t/10 + \frac{1}{2}\cos \pi t/5$, and answer the same questions as in part (a) for each function. What combinations of N and Δ give reasonable results for each function?

(c) Consider a function that is not periodic, but goes to zero for $|t|$ large. For example, try $f(t) = t^4 e^{-t^2}$ and $f(t) = t^3 e^{-t^2}$. Interpret the difference between the Fourier coefficients of these two functions. ■

As shown in Appendix 9A, sine and cosine functions in a Fourier series can be combined into exponential functions with complex coefficients and complex exponents. We express $f(t)$ as

$$f(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{i\omega_k t}, \tag{9.31}$$

where

$$\omega_k = k\omega_0 \quad \text{and} \quad \omega_0 = \frac{2\pi}{T}, \tag{9.32}$$

and use (9.24) to express the complex coefficients c_k in terms of a_k and b_k:

$$c_k = \frac{1}{2}(a_k - ib_k) \tag{9.33a}$$

$$c_0 = \frac{1}{2}a_0 \tag{9.33b}$$

$$c_{-k} = \frac{1}{2}(a_k + ib_k). \tag{9.33c}$$

The coefficients c_k can be expressed in terms of $f(t)$ by using (9.33) and (9.26) and the fact that $e^{\pm i\omega_k t} = \cos \omega_k t \pm i \sin \omega_k t$. The result is

$$c_k = \frac{1}{T}\int_{T/2}^{T/2} f(t)\, e^{-i\omega_k t}\, dt. \tag{9.34}$$

As in (9.30), we can approximate the integral in (9.34) using the rectangular approximation. We write

$$g(\omega_k) \equiv c_k \frac{T}{\Delta} \approx \sum_{j=N/2}^{N/2} f(j\Delta)\, e^{-i\omega_k j\Delta} = \sum_{j=N/2}^{N/2} f(j\Delta)\, e^{-i2\pi kj/N}. \tag{9.35}$$

If we multiply (9.35) by $e^{i2\pi kj'/N}$, sum over k, and use the orthogonality condition

$$\sum_{k=N/2}^{N/2} e^{i2\pi kj/N}\, e^{-i2\pi kj'/N} = N\delta_{j,j'}, \tag{9.36}$$

we obtain the inverse Fourier transform

$$f(j\Delta) = \frac{1}{N}\sum_{k=N/2}^{N/2} g(\omega_k)\, e^{i2\pi kj/N} = \frac{1}{N}\sum_{k=N/2}^{N/2} g(\omega_k)\, e^{i\omega_k t_j}. \tag{9.37}$$

The frequencies ω_k for $k > N/2$ are greater than the Nyquist frequency ω_Q. We can interpret the frequencies for $k > N/2$ as negative frequencies equal to $(k - N)\omega_0$ (see Problem 9.13). The occurrence of negative frequency components is a consequence of the use of the exponential functions rather than sines and cosines. Note that $f(t)$ is real if $g(-\omega_k) = g(\omega_k)$ because the $\sin \omega_k$ terms in (9.37) cancel due to symmetry.

The calculation of a single Fourier coefficient using (9.30) requires approximately $\mathcal{O}(N)$ multiplications. Because the complete Fourier transform contains N complex coefficients, the calculation requires $\mathcal{O}(N^2)$ multiplications and may require hours to complete if the sample contains just a few megabytes of data. Because many of the calculations are redundant, it is possible to organize the calculation so that the computational time is order $N \log N$. Such an algorithm is called a *fast Fourier transform* (FFT) and is discussed in