The number of decimal places in (6.10) is shown to indicate that δ is known precisely. Use (6.8) and (6.10) and the values of b_k to determine the value of r_∞.

(b) In Problem 6.4 we found that one of the four fixed points of $f^{(4)}(x)$ is at $x^* = 1/2$ for $r = s_3 \approx 0.87464$. We also found that the convergence to the fixed points of $f^{(4)}(x)$ for this value of r is more rapid than at nearby values of r. In Appendix 6A we show that these *superstable* trajectories occur whenever one of the fixed points is at $x^* = 1/2$. The values of $r = s_m$ that give superstable trajectories of period 2^{m-1} are much better defined than the points of bifurcation, $r = b_k$. The rapid convergence to the final trajectories also gives better numerical results, and we always know one member of the trajectory, namely $x = 1/2$. Assume that δ can be defined as in (6.9) with b_k replaced by s_m. Use $s_1 = 0.5$, $s_2 \approx 0.809017$, and $s_3 = 0.874640$ to determine δ. The numerical values of s_m are found in Project 6.22 by solving the equation $f^{(m)}(x = 1/2) = 1/2$ numerically; the first eight values of s_m are listed in Table 6.2 in Section 6.11. ■

We can associate another number with the series of "pitchfork" bifurcations. From Figures 6.3 and 6.5, we see that each pitchfork bifurcation gives birth to "twins" with the new generation more densely packed than the previous generation. One measure of this density is the maximum distance M_k between the values of x describing the bifurcation (see Figure 6.5). The disadvantage of using M_k is that the transient behavior of the trajectory is very long at the boundary between two different periodic behaviors. A more convenient measure of the distance is the quantity $d_k = x_k^* - 1/2$, where x_k^* is the value of the fixed point nearest to the fixed point $x^* = 1/2$. The first two values of d_k are shown in Figure 6.6 with $d_1 \approx 0.3090$ and $d_2 \approx -0.1164$. The next value is $d_3 \approx 0.0460$. Note that the fixed point nearest to $x = 1/2$ alternates from one side of $x = 1/2$ to the other. We define the quantity α by the ratio

$$\alpha = \lim_{k\to\infty} -\left(\frac{d_k}{d_{k+1}}\right). \tag{6.11}$$

The ratios $\alpha = (0.3090/0.1164) = 2.65$ for $k = 1$ and $\alpha = (0.1164/0.0460) = 2.53$ for $k = 2$ are consistent with the asymptotic value $\alpha = 2.5029078750958928485\ldots$.

We now give qualitative arguments that suggest that the general behavior of the logistic map in the period doubling regime is independent of the detailed form of $f(x)$. As we have seen, period doubling is characterized by self-similarities; for example, the period doublings look similar except for a change of scale. We can demonstrate these similarities by comparing $f(x)$ for $r = s_1 = 0.5$ for the superstable trajectory with period 1 to the function $f^{(2)}(x)$ for $r = s_2 \approx 0.809017$ for the superstable trajectory of period 2 (see Figure 6.7). The function $f(x, r = s_1)$ has unstable fixed points at $x = 0$ and $x = 1$ and a stable fixed point at $x = 1/2$. Similarly, the function $f^{(2)}(x, r = s_2)$ has a stable fixed point at $x = 1/2$ and an unstable fixed point at $x \approx 0.69098$. Note the similar shape but different scale of the curves in the square boxes in part (a) and part (b) of Figure 6.7. This similarity is an example of scaling. That is, if we scale $f^{(2)}$ and change *(renormalize)* the value of r, we can compare $f^{(2)}$ to f. (See Chapter 12 for a discussion of scaling and renormalization in another context.)

This graphical comparison is meant only to be suggestive. A precise approach shows that if we continue the comparison of the higher-order iterates, for example, $f^{(4)}(x)$ to $f^{(2)}(x)$,

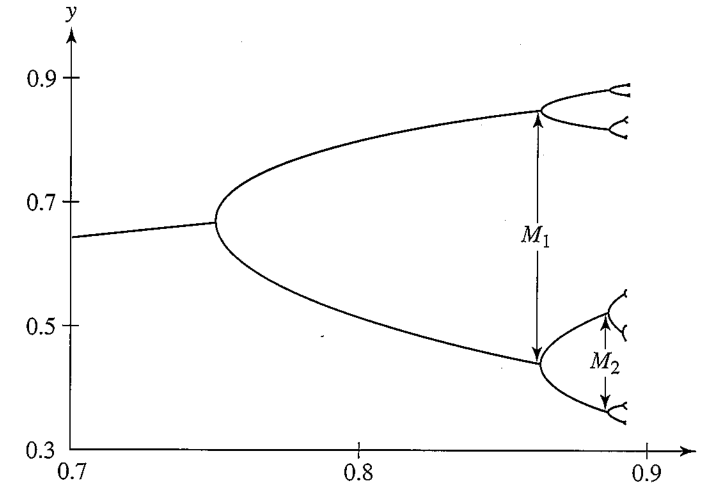

Figure 6.5 The first few bifurcations of the logistic equation showing the scaling of the maximum distance M_k between the asymptotic values of x describing the bifurcation.

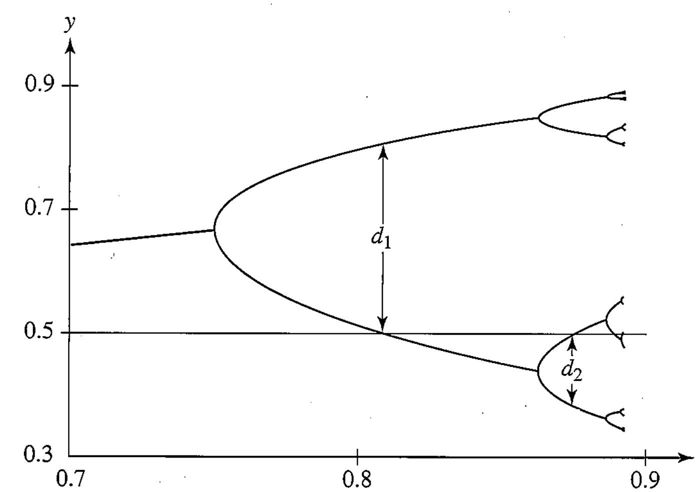

Figure 6.6 The quantity d_k is the distance from $x^* = 1/2$ to the nearest element of the attractor of period 2^k. It is convenient to use this quantity to determine the exponent α.

etc., the superposition of functions converges to a universal function that is independent of the form of the original function $f(x)$.

Problem 6.7 Further determinations of the exponents α and δ

(a) Determine the appropriate scaling factor and superimpose f and the rescaled form of $f^{(2)}$ found in Figure 6.7.

(b) Use arguments similar to those discussed in the text and in Figure 6.7 and compare the behavior of $f^{(4)}(x, r = s_3)$ in the square about $x = 1/2$ with $f^{(2)}(x, r = s_2)$ in