where n is the number of darts in a given cell, and $\langle n \rangle$ is the mean number, $\langle n \rangle = \sum_{n=0}^{N} nP(n)$. Because $N \gg 1$, we can take the upper limit of this sum to be ∞ when it is convenient.

Problem 7.19 Darts and the Poisson distribution

(a) Write a program to compute $\sum_{n=0} P(n)$, $\sum_{n=0} nP(n)$, and $\sum_{n=0} n^2 P(n)$ using the form (7.31) for $P(n)$ and reasonable values of p and N. Verify that $P(n)$ in (7.31) is normalized. What is the value of $\sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2$ for the Poisson distribution?

(b) Modify the program that you developed in Problem 7.18 to compute $\langle n \rangle$ as well as $P(n)$. Choose $N = 50$ and $M = 1000$. How do your computed values of $P(n)$ compare to the Poisson distribution in (7.31) using your measured value of $\langle n \rangle$ as input? If time permits, use larger values of N and M.

(c) Choose $N = 50$ and $M = 100$ and redo part (b). Are your results consistent with a Poisson distribution? What happens if $M = N = 50$? ■

Now that we are more familiar with the Poisson distribution, we consider the decay of radioactive nuclei. We know that a collection of radioactive nuclei will decay; however, we cannot know a priori which nucleus will decay next. If all nuclei of a particular type are identical, why do they not all decay at the same time? The answer is based on the uncertainty inherent in the quantum description of matter at the microscopic level. In the following, we will see that a simple model of the decay process leads to exponential decay. This approach complements the continuum approach discussed in Section 3.9.

Because each nucleus is identical, we assume that during any time interval Δt, each nucleus has the same probability per unit time p of decaying. The basic algorithm is simple—choose an unstable nucleus and generate a random number r uniformly distributed in the unit interval $0 \leq r < 1$. If $r \leq p$, the unstable nucleus decays; otherwise, it does not. Each unstable nucleus is tested once during each time interval. Note that for a system of unstable nuclei, there are many events that can happen during each time interval; for example, $0, 1, 2, \ldots, n$ nuclei can decay. Once a nucleus decays, it is no longer in the group of unstable nuclei that is tested at each time interval. Class `Nuclei` in Listing 7.5 implements the nuclear decay algorithm.

Listing 7.5 The `Nuclei` class.

```
package org.opensourcephysics.sip.ch07;
public class Nuclei {
   // accumulated data on number of unstable nuclei, index is time
   int n[];
   int tmax; // maximum time to record data
   int n0;   // initial number of unstable nuclei
   double p; // decay probability

   public void initialize() {
      n = new int[tmax+1];
   }

   public void step() {
      n[0] += n0;
      int nUnstable = n0;
      for(int t = 0;t<tmax;t++) {
         for(int i = 0;i<nUnstable;i++) {
            if(Math.random()<p) {
               nUnstable--;
            }
         }
         n[t+1] += nUnstable;
      }
   }
}
```


Problem 7.20 Monte Carlo simulation of nuclear decay

(a) Write a target class that extends `AbstractSimulation`, does many trials, and plots the average number of unstable nuclei as a function of time. Assume that the time interval Δt is one second. Choose the initial number of unstable nuclei `n0` $= 10{,}000$, $p = 0.01$, and `tmax` $= 100$ and average over 100 trials. Is your result for $n(t)$, the mean number of unstable nuclei at time t, consistent with the expected behavior $n(t) = n(0)\, e^{-\lambda t}$ found in Section 3.9? What is the value of λ for this value of p?

(b) There are a very large number of unstable nuclei in a typical radioactive source. We also know that over any reasonable time interval, only a relatively small number decay. Because $N \gg 1$ and $p \ll 1$, we expect that $P(n)$, the probability that n nuclei decay during a specified time interval, is a Poisson distribution. Modify your target class so that it outputs the probability that n unstable nuclei decay during the first time interval. Choose `n0` $= 1000$, $p = 0.001$, and `tmax` $= 1$ and average over 1000 trials. What is the mean number $\langle n \rangle$ of nuclei that decay during this interval? What is the associated variance? Plot $P(n)$ versus n and compare your results to the Poisson distribution (7.31) with your measured value of $\langle n \rangle$ as input. Then consider $p = 0.02$ and determine if $P(n)$ is a Poisson distribution.

(c) Modify your target class so that it outputs the probability that n unstable nuclei decay during the first two time intervals. Choose `n0` $= 10000$, $p = 0.001$, and `tmax` $= 2$. Average over 1000 trials. Compare the probability you obtain with your results from part (b). How do your results change as the time interval becomes larger?

(d) Increase p for fixed `n0` $= 10{,}000$ and determine $P(n)$ for a fixed time interval. Estimate the values of p and n for which the Poisson distribution is no longer applicable.

(e) Modify your program so that it flashes a small circle on the screen or makes a sound (like that of a Geiger counter) when a nucleus decays. You can have the computer make a beep by using the method `Toolkit.getDefaultToolkit().beep()` in `java.awt`. Choose the location of the small circle at random. Do a single run and describe the qualitative differences between the visual or audio patterns for the cases in parts (a)–(d)? Choose `n0` ≥ 5000. Such a visualization might be somewhat misleading on a serial computer because only one nuclei can be considered at a time. In contrast, for a real system, the nuclei can decay simultaneously. ■

7.5 ■ PROBLEMS IN PROBABILITY

Why have we bothered to simulate many random processes that can be solved by analytical methods? The main reason is that it is simpler to introduce new methods in a familiar context. Another reason is that if we change the nature of many random processes slightly,