(b) Repeat part (a) for $N = 2$ and $N = 10$ with random initial particle displacements between -0.5 and $+0.5$ and zero initial velocities. Can you detect all the normal modes in the power spectrum? Repeat for a different set of random initial displacements.

(c) Repeat part (a) for initial displacements corresponding to the equal sum of two normal modes. Does the power spectrum show two peaks? Are these peaks of equal height?

(d) Recompute the power spectrum for $N = 10$ with $T = 6.4$. Is this time long enough? How can you tell? ■

Problem 9.21 Quasiperiodic power spectra

(a) Write a program to compute the power spectrum of the circle map (6.62). Begin by exploring the power spectrum for $K = 0$. Plot $\ln P(\omega)$ versus ω, where $P(\omega)$ is proportional to the modulus squared of the Fourier transform of x_n. Begin with 256 iterations. How do the power spectra differ for rational and irrational values of the parameter Ω? How are the locations of the peaks in the power spectra related to the value of Ω?

(b) Set $K = 1/2$ and compute the power spectra for $0 < \Omega < 1$. Do the power spectra differ from the spectra found in part (a)?

(c) Set $K = 1$ and compute the power spectra for $0 < \Omega < 1$. How do the power spectra compare to those found in parts (a) and (b)? ■

In Problem 9.20 we found that the peaks in the power spectrum yield information about the normal mode frequencies. In Problems 9.22 and 9.23 we compute the power spectra for a system of coupled oscillators with disorder. Disorder can be generated by having random masses or random spring constants (or both). We will see that one effect of disorder is that the normal modes are no longer simple sinusoidal functions. Instead, some of the modes are localized, meaning that only some of the particles move significantly while the others remain essentially at rest. This effect is known as *Anderson localization*. Typically, we find that modes above a certain frequency are *localized*, and those below this threshold frequency are *extended*. The threshold frequency is well defined for large systems. All states are localized in the limit of an infinite chain with any amount of disorder. The dependence of localization on disorder in systems of coupled oscillators in higher dimensions is more complicated.

Problem 9.22 Localization with a single defect

(a) Modify your program developed in Problem 9.2 so that the mass of one oscillator is equal to one fourth that of the others. Set $N = 20$ and use fixed boundary conditions. Compute the power spectrum over a time $T = 51.2$ using random initial displacements between -0.5 and $+0.5$ and zero initial velocities. Sample the data at intervals of $\Delta = 0.1$. The normal mode frequencies correspond to the well-defined peaks in $P(\omega)$. Consider at least three different sets of random initial displacements to insure that you find all the normal mode frequencies.

(b) Apply an external force $F_e = 0.3 \sin \omega t$ to each particle. (The steady-state behavior occurs sooner if we apply an external force to each particle instead of just one particle.) Because the external force pumps energy into the system, it is necessary to add a damping force to prevent the oscillator displacements from becoming too large. Add a damping force equal to $-\gamma v_i$ to all the oscillators with $\gamma = 0.1$. Choose random initial displacements and zero initial velocities and use the frequencies found in part (a) as the driving frequencies ω. Describe the motion of the particles. Is the system driven to a normal mode? Take a "snapshot" of the particle displacements after the system has run for a sufficiently long time, so that the patterns repeat themselves. Are the particle displacements simple sinusoidal functions? Sketch the approximate normal mode patterns for each normal mode frequency. Which of the modes appear localized and which modes appear to be extended? What is the approximate cutoff frequency that separates the localized from the extended modes? ■

Problem 9.23 Localization in a disordered chain of oscillators

(a) Modify your program so that the spring constants can be varied by the user. Set $N = 10$ and use fixed boundary conditions. Consider the following set of 11 spring constants: 0.704, 0.388, 0.707, 0.525, 0.754, 0.721, 0.006, 0.479, 0.470, 0.574, and 0.904. To help you determine all the normal modes, we provide two of the normal mode frequencies: $\omega \approx 0.28$ and 1.15. Find the power spectrum using the procedure outlined in Problem 9.22a.

(b) Apply an external force $F_e = 0.3 \sin \omega t$ to each particle and find the normal modes as outlined in Problem 9.22b.

(c) Repeat parts (a) and (b) for another set of random spring constants for $N = 40$. Discuss the nature of the localized modes in terms of the specific values of the spring constants. For example, is the edge of a localized mode at a spring that has a relatively large or small spring constant?

(d) Repeat parts (a) and (b) for uniform spring constants but random masses between 0.5 and 1.5. Is there a qualitative difference between the two types of disorder? ■

In 1955 Fermi, Pasta, and Ulam used the Maniac I computer at Los Alamos to study a chain of oscillators. Their surprising discovery might have been the first time a qualitatively new result, instead of a more precise number, was found from a simulation. To understand their results, we need to discuss an idea from statistical mechanics that was discussed in Project 8.23.

A fundamental assumption of statistical mechanics is that an isolated system of particles is quasi-ergodic; that is, the system will evolve through all configurations consistent with the conservation of energy. A system of linearly coupled oscillators is not quasi-ergodic, because if the system is initially in a normal mode, it stays in that normal mode forever. Before 1955 it was believed that if the interaction between the particles is weakly nonlinear (and the number of particles is sufficiently large), the system would be quasi-ergodic and evolve through the different normal modes of the linear system. In Problem 9.24 we will find, as did Fermi, Pasta, and Ulam, that the behavior of the system is much more complicated. The question of ergodicity in this system is known as the FPU problem.