total solar eclipse by Eddington showed that the sum of the interior angles is not 180°. It is an experimental fact that the universe is non-Euclidian.

The Eddington experiment was remarkable because it confirmed Einstein's general theory of relativity and showed that space and time are not separate entities. We cannot measure space, only distances between events in space using rulers, light beams, and clocks. Furthermore, the separation between events is not the same for different observers unless they incorporate both spatial and temporal separations into their definition of distance. In the absence of gravitational fields, observers moving at constant relative velocity can reconcile (18.2) and (18.3) and obtain the same "distance" only if they agree that the distance between events $\Delta\sigma$ includes time and is measured as

$$(\Delta\sigma)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2(\Delta t)^2, \tag{18.9}$$

where $c$ is the speed of light, $\Delta t$ is the temporal separation, and $\Delta x$, $\Delta y$, and $\Delta z$ are the spatial coordinate separations. Equation (18.9) is based on Einstein's special theory of relativity and is known as the Minkowski metric. It follows from Einstein's assumption that Maxwell's equations must be the same for all observers in uniform relative motion and leads naturally to the equivalence of mass and energy embodied in the famous equation $E = mc^2$.

Einstein's great insight that acceleration and gravity are indistinguishable enabled him to incorporate gravity into the spacetime fabric by generalizing (18.9). Imagine an elevator compartment resting on the surface of Earth in which the occupants perform experiments, such as dropping an object or observing a swinging pendulum, that reveal the presence of Earth's gravity. Then the occupants are placed in a compartment far from any gravitational object, and the compartment accelerates at 9.8 m/s$^2$. According to Einstein, the experimental results must be identical. Furthermore, if the near-Earth elevator cable is cut to produce a freely falling reference frame, then the occupants will be unable to detect the gravitational field. The implication is that we can do away with gravity and regard it as a consequence of an accelerated reference frame in four-dimensional spacetime. It took Einstein ten years to incorporate this equivalence of gravitational forces and accelerated motion to the special theory of relativity to produce the general theory.

Einstein's general theory of relativity produces ten simultaneous coupled nonlinear partial differential equations. Calculations using this theory are truly daunting and require sophisticated mathematical techniques such as tensor analysis and Riemannian geometry. All forms of energy gravitate (attract), and nonlinearities arise because a body's gravitational field is itself a form of energy and therefore gravitates. Few analytic results are known. Two of the most important are the Schwarzschild and Kerr metrics in the vicinity of a spherically symmetric mass. Except for very special cases or very weak fields, the dynamical equations derived from these metrics and the principle that the path taken by an objects is a maximum as measured by a watch carried with the object (maximum aging) must be solved numerically to predict how particles move and how they appear when seen by an observer.

## 18.3 ■ DYNAMICS IN POLAR COORDINATES

General relativistic trajectories of particles and light in the vicinity of spherically symmetric gravitational fields are conveniently described using polar coordinates. We therefore reformulate the classical two-body problem (see Chapter 5) using polar coordinates. If the motion is confined to a plane, rectangular coordinates $(x, y)$ and polar coordinates $(r, \phi)$ are related by



$$x = r\cos\phi, \quad y = r\sin\phi, \tag{18.10}$$

and

$$r = \sqrt{x^2 + y^2}, \quad \phi = \arctan\frac{y}{x}. \tag{18.11}$$

The radial velocity is given by

$$\dot{r} = \frac{dr}{dt} = \frac{\mathbf{r}\cdot\mathbf{v}}{r}, \tag{18.12}$$

and the angular velocity is given by

$$\dot{\phi} = \frac{d\phi}{dt} = \frac{L}{mr^2}, \tag{18.13}$$

where $L$ is the magnitude of the conserved angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$.

To construct the appropriate differential equations, the radial and angular accelerations can be obtained by differentiating (18.12) and (18.13) with respect to time. Another approach is to use the Lagrangian

$$\mathcal{L} = \frac{1}{2}[\dot{r}^2 + r^2\dot{\phi}^2] + \frac{GM}{r}, \tag{18.14}$$

and apply Lagrange's equations of motion:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = 0, \quad \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} - \frac{\partial\mathcal{L}}{\partial\phi} = 0. \tag{18.15}$$

If we do the differentiation, we obtain the following rate equations for the polar state vector $(r, \dot{r}, \phi, \dot{\phi}, t)$:

$$\frac{dr}{dt} = \dot{r} \tag{18.16a}$$

$$\frac{d\dot{r}}{dt} = r\dot{\phi}^2 - \frac{GM}{r^2} \tag{18.16b}$$

$$\frac{d\phi}{dt} = \dot{\phi} \tag{18.16c}$$

$$\frac{d\dot{\phi}}{dt} = -\frac{2}{r}\dot{\phi}\dot{r} \tag{18.16d}$$

$$\frac{dt}{dt} = 1. \tag{18.16e}$$

**Exercise 18.5 Angular momentum**

Show that (18.16) leads to the polar coordinate expression for conservation of angular momentum

$$r\ddot{\phi} + 2\dot{\phi}\dot{r} = 0. \tag{18.17}$$

■