- **Electrical Circuit Oscillations**

  Physics deals with laws of nature. Many seeming different phenomena can be described by the same set of physics laws (equations).

  Example: RLC circuit and Linear Oscillator.

element	**voltage drop**	**units**
resistor	$V_R = IR$	resistance R, ohms(Ω)
capacitor	$V_C = Q/C$	capacitance C, farads (F)
inductor	$V_L = LdI/dt$	inductance L, henries (H)

$$V_L + V_R + V_C = V_s(t)\ (emf)\ .$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_s(t)\ ,$$

$$I = \frac{dQ}{dt}\ .$$

Electric circuit	**Mechanical system**
charge Q	displacement x
current $I = dQ/dt$	velocity $v = dx/dt$
voltage drop	force
inductance L	mass m
inverse capacitance $1/C$	spring constant k
resistance R	damping γ

- **Linear Oscillator**

Consider a driven damped linear oscillator, the equation of motion is

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma\frac{dx}{dt} + \frac{1}{m}F(t) \ ,$$

where ω_0 is its natural frequency, γ is the *damping coefficient* measuring the magnitude of dissipative force, and F is an external force (perturbation), to which response of the system reveals the nature of the system.

Force $F(t)$ could have arbitrary forms and usually we do our analysis with

$$F(t) = A_0 cos\omega t \ ,$$

where ω is the angular frequency of the driving force.

This is because for a *linear system* we can apply the *superposition principle*, e.g., $\omega_n = n\omega$,

$$F(t) = a_0 + \sum_{n=1}^{N} a_n \cos\omega_n t + b_n \sin\omega_n t \ . \tag{1}$$

This is a form of *Fourier Series*.

- **The Fourier Series**

Eq. (1) is an example of *Fourier series*: any arbitrary periodic function $f(x)$ of period $2L$ can be expressed as a Fourier series of sines and cosines:

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}) \ , \tag{2}$$

The quantity $\frac{\pi}{L}$ (sometime defined as ω_0) is the fundamental frequency. Terms in Eq. (2) for $k = 2, 3, \cdots$ represent higher order harmonics. a_k and b_k are called the *Fourier coefficients*, given by

$$\begin{aligned} a_k &= \frac{1}{L}\int_{-L}^{L} dx f(x) \cos\frac{k\pi x}{L} \qquad k = 0, 1, 2, \cdots \\ b_k &= \frac{1}{L}\int_{-L}^{L} dx f(x) \sin\frac{k\pi x}{L} \qquad k = 1, 2, 3, \cdots \end{aligned} \tag{3}$$

Functions:

$$1, \sin\frac{\pi x}{L}, \cos\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \cos\frac{2\pi x}{L}, \cdots$$

form *a complete orthogonal set* in $(-L, L)$:

$$\begin{aligned} \frac{1}{L}\int_{-L}^{L} dx \ \cos\frac{k\pi x}{L} \ \cos\frac{m\pi x}{L} &= \delta_{k,m} \\ \frac{1}{L}\int_{-L}^{L} dx \ \sin\frac{k\pi x}{L} \ \sin\frac{m\pi x}{L} &= \delta_{k,m} \\ \frac{1}{L}\int_{-L}^{L} dx \ \cos\frac{k\pi x}{L} \ \sin\frac{m\pi x}{L} &= 0 \ . \end{aligned} \tag{4}$$

Dirichlet Theorem: Suppose that

1. $f(x)$ is defined and single-valued except possibly at a finite number of points in $[-L, L]$
2. $f(x)$ is periodic outside $[-L, L]$ with period $2L$
3. $f(x)$ and $f'(x)$ are piecewise continuous in $[-L, L]$.

Then the series (2) with coefficients (3) converges to

(a) $f(x)$ if x is a point of continuity.

(b) $(f(x+0)+f(x-0))/2$ if x is a point of discontinuity.

Conditions (1), (2), and (3) are *sufficient* but not necessary.

Parseval's Identity:

$$\frac{1}{L}\int_{-L}^{L} dx\ f^2(x) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\ . \tag{5}$$

This identity can be used for summation of a series.

Proof: $\cdots$

Examples 1 and 2

Approximation: In general, an infinite number of terms is needed to represent an arbitrary periodic function exactly. But in practice, we usually only use a few terms to make approximation. A Fourier series can approximate a function at *all* points.

Fourier series with complex coefficients:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt} , \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} dt f(t) e^{-ikt} .$$

We have these relations (see Eq. (3)):

$$c_0 = \frac{1}{2} a_0, \qquad c_k = \frac{1}{2}(a_k - ib_k), \qquad c_{-k} = (c_k)^*.$$

Half Range Fourier Sine or Cosine Series:

1. If $f(x)$ is an *odd* function on $[-L, L]$, then

$$a_n = 0, \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

2. If $f(x)$ is an *even* function on $[-L, L]$, then

$$b_n = 0, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.$$

One can *extend* $f(x)$, defined on $[0, L]$ to either odd or even periodic function with period $2L$.

Solving Differential Equation

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma\frac{dx}{dt} + \frac{1}{m}F(t)\ .$$

Let

$$x(t) \ = \ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega x(\omega)e^{i\omega t} \tag{16}$$

$$x(\omega) \ = \ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt x(t)e^{-i\omega t} \tag{17}$$

Then

$$\frac{dx}{dt} \ = \ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\ (i\omega)\ x(\omega)e^{i\omega t} \tag{18}$$

$$\frac{d^2x}{dt^2} \ = \ \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\ (i\omega)^2\ x(\omega)e^{i\omega t} \tag{19}$$

Differential equation becomes

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{i\omega t}\left[(\omega^2 - \omega_0^2 - i\omega\gamma)x(\omega) + \frac{f(\omega)}{m}\right] = 0$$

This leads to an *algebric equation:*

$$(\omega^2 - \omega_0^2 - i\omega\gamma)x(\omega) = -\frac{f(\omega)}{m} \tag{20}$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega x(\omega)e^{i\omega t}\frac{f(\omega)}{m}\frac{-1}{\omega^2 - \omega_0^2 - i\omega\gamma} \tag{21}$$

- **The Fourier Transform**

Let $t \to \pi t/T$, then

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{in\pi t/T} = \sum_{-\infty}^{\infty} c_n e^{in\Delta\omega t} ,$$

$$c_n = \frac{1}{2T} \int_{-T}^{T} dt f(t) e^{-in\pi t/T} = \frac{\Delta\omega}{2\pi} \int_{-T}^{T} dt f(t) e^{-in\Delta\omega t} ,$$

where

$$\omega = \frac{n\pi}{T}, \qquad \Delta\omega = \frac{\pi}{T} .$$

Define

$$c_n = \frac{\Delta\omega}{\sqrt{2\pi}} g(n\Delta\omega)$$

so that

$$g(n\Delta\omega) = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} dt f(t) e^{-in\Delta\omega t}$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{-\infty}^{\infty} \Delta\omega g(n\Delta\omega) e^{in\Delta\omega t}$$

$T \to \infty$, we get the *Fourier transform* and its *inverse*.

$$\mathcal{F}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt f(t) e^{-i\omega t} \tag{6}$$

$$\mathcal{F}^{-1}[g(\omega)] = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega g(\omega) e^{i\omega t} \tag{7}$$

Properties of the Fourier Transform

1. The Fourier Transform is linear:

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t}[f_1(t) + f_2(t)] = g_1(\omega) + g_2(\omega)$$

2. Scaling relation:

$$\mathcal{F}[f(\alpha t)] = \begin{cases} \frac{1}{\alpha} g(\frac{\omega}{\alpha}) & \alpha > 0 \\ -\frac{1}{\alpha} g(\frac{\omega}{\alpha}) & \alpha < 0 \end{cases} = \frac{1}{|\alpha|} g(\frac{\omega}{\alpha}).$$

   Same for inverse transform.

   The more *localized* in time, the more *delocalized* in frequency. Heisenberg Uncertainly principle.

3. Shifting relation:

$$\mathcal{F}[f(t - t_0)] = e^{-i\omega t_0} g(\omega), \mathcal{F}^{-1}[g(\omega - \omega_0)] = e^{i\omega t_0} f(t).$$

4. Real/Imaginary/Odd/Even relations: page 290.

5. Derivatives:

$$\mathcal{F}[f'(t)] = i\omega g(\omega).$$

6. Parsevel's identity:

$$I = \int_{-\infty}^{\infty} dt f_1^*(t) f_2(t) = \int_{-\infty}^{\infty} d\omega g_1^*(\omega) g_2(\omega).$$

Examples 4-6

- **Convolution**

Consider two functions, $p(t)$ and $q(t)$. Mathematically, the convolution of the two functions is defined as

$$p \otimes q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau p(\tau) q(t-\tau) \ . \tag{8}$$

Example:

First Course in Computational Physics, Fig. 6.4

$$\frac{V_{in} - V_{out}}{R} = C\frac{d(V_{out} - V_c)}{dt} = C\frac{dV_{out}}{dt} \ ,$$

$$V_{out}(t) = e^{-t/RC}\left[\int_{-\infty}^{t} dt e^{\tau/RC} + C_1\right] \ ,$$

C_1 depends on the initial initial condition.

$$V_{in}(t) = \delta(t) \quad \rightarrow \quad r(t) \equiv V_{out}(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{RC} e^{-t/RC}, & t \geq 0. \end{cases}$$

A continuous function $f(t)$ could be treated as an integral over delta functions,

$$V_{in}(t) = \int_{-\infty}^{\infty} d\tau V_{in}(\tau)\delta(t-\tau) \ ,$$

$$V_{out}(t) = \int_{-\infty}^{\infty} d\tau V_{in}(\tau) r(t-\tau) = V_{in}(t) \otimes r(t) \ .$$

<u>The Fourier Transform of the Convolution</u>

$$\begin{aligned}
\mathcal{F}[p \otimes q] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau p(\tau) q(t-\tau) \right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau p(\tau) \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} q(t-\tau) \right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} p(\tau) Q(\omega) = P(\omega) Q(\omega)
\end{aligned}$$

The Fourier Convolution Theorem:

$$\mathcal{F}[p \otimes q] = \mathcal{F}[p]\mathcal{F}[q] \ . \tag{9}$$

Example of *Deconvolution*:

$$V_{out} = V_{in} \otimes r \ \rightarrow \ \mathcal{F}[V_{out}] = \mathcal{F}[V_{in}]\mathcal{F}[r] \ \rightarrow$$

$$\mathcal{F}[V_{in}] = \mathcal{F}[V_{out}]/\mathcal{F}[r] \ \rightarrow \ V_{in} = \mathcal{F}^{-1}[\mathcal{F}[V_{out}]/\mathcal{F}[r]] \ .$$

- **Correlation**

$$p \odot q = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\tau p^*(\tau) q(t+\tau). \tag{10}$$

It measures how much one function is similar to the other.

Example:

First Course in Computational Physics, page 299

Autocorrelation: $q = p$.

Average Correlation Function

$$[p \odot q]_{average} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} d\tau p^*(\tau) q(t+\tau). \qquad (11)$$

For periodic function with period T_0,

$$[p \odot q]_{average} = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} d\tau p^*(\tau) q(t+\tau).$$

Fluctuation

Let $p(\tau) =< p > +\delta_p(\tau)$ and $q(\tau) =< q > +\delta_q(\tau)$. where $<>$ defines mean value and δ the derivation. Then

$$p \odot q =< p >< q > + \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} d\tau \delta_p(\tau) \delta_q(t+\tau).$$

If uncorrelated,

$$p \odot q =< p >< q > . \qquad (12)$$

Useful Information

1. Trigonometry:

$$2\ \cos A \cos B \ = \ \cos(A-B)+\cos(A+B)$$

$$2\ \sin A \sin B \ = \ \cos(A-B)-\cos(A+B)$$

$$2\ \sin A \cos B \ = \ \sin(A-B)+\sin(A+B)$$

$$2\ \sin^2 A \ = \ 1-\cos 2A$$

$$2\ \cos^2 A \ = \ 1+\cos 2A$$

2. $$\int dx\ \cos(nx)=\frac{\sin(nx)}{n}$$

$$\int dx\ \sin(nx)=-\frac{\cos(nx)}{n}$$

3. $$\int dx\ e^{\alpha x}\cos(mx)=\int dx\ Re\left(e^{\alpha x+imx}\right)=Re\left(\frac{e^{\alpha x+imx}}{\alpha+im}\right)$$

$$\int dx\ e^{\alpha x}\sin(mx)=\int dx\ Im\left(e^{\alpha x+imx}\right)=Im\left(\frac{e^{\alpha x+imx}}{\alpha+im}\right)$$

4. Recursion relations:

$$\int dx\ x^k\cos(nx)=\frac{\sin(nx)}{n}x^k-\frac{k}{n}\int dx\ x^{k-1}\sin(nx)$$

$$\int dx\ x^k\sin(nx)=-\frac{\cos(nx)}{n}x^k+\frac{k}{n}\int dx\ x^{k-1}\cos(nx)$$