(c) Consider the effects of air resistance on the range and optimum angle of a steel ball. For a ball of mass 7 kg and cross-sectional area 0.01 m^2, the parameter $C_2 \approx 0.1$. What are the units of C_2? It is convenient to exaggerate the effects of air resistance so that you can more easily determine the qualitative nature of the effects. Hence, compute the optimum angle for $h = 2$ m, $v_0 = 30$ m/s, and $C_2/m = 0.1$ and compare your answer to the value found in part (b). Is R more or less sensitive to changes in θ_0 from θ_{max} than in part (b)? Determine the optimum launch angle and the corresponding range for the more realistic value of $C_2 = 0.1$. A detailed discussion of the maximum range of the ball has been given by Lichtenberg and Wills. ■

Problem 3.11 Comparing the motion of two objects

Consider the motion of two identical objects that both start from a height h. One object is dropped vertically from rest and the other is thrown with a horizontal velocity v_0. Which object reaches the ground first?

(a) Give reasons for your answer assuming that air resistance can be neglected.

(b) Assume that air resistance cannot be neglected and that the drag force is proportional to v^2. Give reasons for your anticipated answer for this case. Then perform numerical simulations using, for example, $C_2/m = 0.1$, $h = 10$ m, and $v_0 = 30$ m/s. Are your qualitative results consistent with your anticipated answer? If they are not, the source of the discrepancy might be an error in your program. Or the discrepancy might be due to your failure to anticipate the effects of the coupling between the vertical and horizontal motion.

(c) Suppose that the drag force is proportional to v rather than to v^2. Is your anticipated answer similar to that in part (b)? Do a numerical simulation to test your intuition. ■

3.9 ■ DECAY PROCESSES

The power of mathematics when applied to physics comes in part from the fact that seemingly unrelated problems frequently have the same mathematical formulation. Hence, if we can solve one problem, we can solve other problems that might appear to be unrelated. For example, the growth of bacteria, the cooling of a cup of hot water, the charging of a capacitor in a RC circuit, and nuclear decay can all be formulated in terms of equivalent differential equations.

Consider a large number of radioactive nuclei. Although the number of nuclei is discrete, we may often treat this number as a continuous variable because the number of nuclei is very large. In this case the law of radioactive decay is that the rate of decay is proportional to the number of nuclei. Thus we can write

$$\frac{dN}{dt} = -\lambda N, \tag{3.19}$$

where N is the number of nuclei and λ is the decay constant. Of course, we do not need to use a computer to solve this decay equation, and the analytical solution is

$$N(t) = N_0 e^{-\lambda t}, \tag{3.20}$$

where N_0 is the initial number of particles. The quantity λ in (3.19) or (3.20) has dimensions of inverse time.

Problem 3.12 Single nuclear species decay

(a) Write a class that solves and plots the nuclear decay problem. Input the decay constant λ from the control window. For $\lambda = 1$ and $\Delta t = 0.01$, compute the difference between the analytical result and the result of the Euler algorithm for $N(t)/N(0)$ at $t = 1$ and $t = 2$. Assume that time is measured in seconds.

(b) A common time unit for radioactive decay is the half-life $T_{1/2}$, the time it takes for one-half of the original nuclei to decay. Another natural time scale is the time τ it takes for $1/e$ of the original nuclei to decay. Use your modified program to verify that $T_{1/2} = \ln 2/\lambda$. How long does it take for $1/e$ of the original nuclei to decay? How is $T_{1/2}$ related to τ?

(c) Because it is awkward to treat very large or very small numbers on a computer, it is convenient to choose units so that the computed values of the variables are not too far from unity. Determine the decay constant λ in units of s^{-1} for $^{238}U \to {}^{234}Th$ if the half-life is 4.5×10^9 years. What units and time step would be appropriate for the numerical solution of (3.19)? How would these values change if the particle being modeled was a muon with a half-life of 2.2×10^{-6} s?

(d) Modify your program so that the time t is expressed in terms of the half-life. That is, at $t = 1$, one half of the particles would have decayed and at $t = 2$, one quarter of the particles would have decayed. Use your program to determine the time for 1000 atoms of ^{238}U to decay to 20% of their original number. What would be the corresponding time for muons? ■

Multiple nuclear decays produce systems of first-order differential equations. Problem 3.13 asks you to model such a system using the techniques similar to those that we have already used.

Problem 3.13 Multiple nuclear decays

(a) ^{76}Kr decays to ^{76}Br via electron capture with a half-life of 14.8 h, and ^{76}Br decays to ^{76}Se via electron capture and positron emission with a half-life of 16.1 h. In this case there are two half-lives, and it is convenient to measure time in units of the smallest half-life. Write a program to compute the time dependence of the amount of ^{76}Kr and ^{76}Se over an interval of one week. Assume that the sample initially contains 1 gm of pure ^{76}Kr.

(b) ^{28}Mn decays via beta emission to ^{28}Al with a half-life of 21 h, and ^{28}Al decays by positron emission to ^{28}Si with a half-life of 2.31 min. If we were to use minutes as the unit of time, our program would have to do many iterations before we would see a significant decay of the ^{28}Mn. What simplifying assumption can you make to speed up the computation?

(c) ^{211}Rn decays via two branches as shown in Figure 3.4. Make any necessary approximations and compute the amount of each isotope as a function of time, assuming that the sample initially consists of 1 μg of ^{211}Rn. ■