that follows. We use the chain rule to write

$$\dot{r}_i = \sum_j J_{i,j}\dot{q}_j, \tag{19.6}$$

and substitute (19.6) into (19.5) and obtain

$$\sum_{i,j} F_i^{(c)} J_{i,j}\dot{q}_j = 0, \tag{19.7}$$

which is valid for all $\dot{q}$. Because of this independence, the prefactor for each term must be identically zero:

$$\sum_i F_i^{(c)} J_{i,j} = 0. \tag{19.8}$$

Equation (19.8) gives us a way to eliminate the constraint force. That is, we multiply (19.4) by J_{ij} and sum over i. If we use the condition (19.8), the $F^{(c)}$ term will disappear, and we are left with

$$\sum_i J_{i,j}\mu_i q_i \ddot{r}_i = \sum_i J_{i,j} F_i^{(a)}. \tag{19.9}$$

Note that we have eliminated the constraint force from the equation of motion.

Because we want to describe the system by the generalized coordinates, we need to know $\ddot{q}$. Fortunately, this information is embedded in $\ddot{r}$. We differentiate (19.6) with respect to time and switch the summation index from j to k and write

$$\ddot{r}_i = \sum_k (\dot{J}_{i,k}\dot{q}_k + J_{i,k}\ddot{q}_k). \tag{19.10}$$

We still need to determine $\dot{J}_{i,k}(q, \dot{q})$. We again apply the chain rule and write

$$\dot{J}_{i,k} = \sum_l (\partial J_{i,k}/\partial q_l)\dot{q}_l. \tag{19.11}$$

We substitute (19.11) into (19.10), rearrange terms, and obtain our desired result:

$$\sum_{i,k} J_{i,j}\mu_i J_{i,k}\ddot{q}_k = \sum_{i,k} (J_{i,j}F_i^{(a)} - J_{i,j}\mu_i \dot{J}_{i,k}\dot{q}_k), \tag{19.12}$$

or in matrix notation,

$$(J^T MJ)\ddot{q} = J^T F^{(a)} - J^T M\dot{J}\dot{q}, \tag{19.13}$$

where $M_{i,j} = \delta_{i,j}\mu_i$, and J^T is the transpose of J. The unknown in (19.13) is $\ddot{q}$, which can be obtained at each time step by inverting the matrix $J^T MJ$.

To make the above matrix manipulations more concrete, we consider the motion of two particles of mass m_1 and mass m_2 connected by a spring and constrained to move on the curve given by $y(x) = x^4 - 2x^2$. The equilibrium length of the spring is L_0. We choose q_1

and q_2 to be x_1 and x_2, respectively, and calculate the various matrices in (19.13). In this case the matrix J can be written as

$$J = \begin{pmatrix} \dfrac{\partial r_1}{\partial q_1} & \dfrac{\partial r_1}{\partial q_2} \\ \vdots & \vdots \\ \dfrac{\partial r_4}{\partial q_1} & \dfrac{\partial r_4}{\partial q_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ y_1' & 0 \\ 0 & 1 \\ 0 & y_2' \end{pmatrix}, \tag{19.14}$$

where $y_1' = 4x_1^3 - 4x_1$ and $y_2' = 4x_2^3 - 4x_2$. We also have

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_2 \end{pmatrix} \tag{19.15}$$

$$MJ = \begin{pmatrix} m_1 & 0 \\ m_2 y_1' & 0 \\ 0 & m_2 \\ 0 & m_2 y_2' \end{pmatrix} \tag{19.16}$$

$$J^T MJ = \begin{pmatrix} m_1 + m_1 y_1'^2 & 0 \\ 0' & m_2 + m_2 y_2'^2 \end{pmatrix}. \tag{19.17}$$

Because $J^T MJ$ is a diagonal matrix, it is straightforward to calculate its inverse.

The force due to gravity can be written as

$$F^g = \begin{pmatrix} 0 \\ -m_1 g \\ 0 \\ -m_2 g \end{pmatrix}. \tag{19.18}$$

We write the force due to the spring connecting the two masses as $|F^s| = k(L - L_0)$, where L is the length of the spring and is given by $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. It can be shown that F^s can be written as

$$F^s = \begin{pmatrix} (x_2 - x_1)k_{\text{eff}} \\ (y_2 - y_1)k_{\text{eff}} \\ -(x_2 - x_1)k_{\text{eff}} \\ -(y_2 - y_1)k_{\text{eff}} \end{pmatrix}, \tag{19.19}$$

where $k_{\text{eff}} = k(1 - L_0/L)$.

The class `ConstraintApp` solves the equation of motion (19.13) using the Open Source Physics ODE solver in the usual way. To see how straightforward it is to implement the constrained dynamics in this case, we show the `getRate` method in Listing 19.1.