To calculate the critical exponent ν, we recall that all lengths are reduced on the renormalized lattice by a factor of b in comparison to the lengths in the original system. Hence, the connectedness length transforms as

$$\xi' = \xi / b. \tag{12.20}$$

Because $\xi(p) = A|p - p_c|^{-\nu}$ for p near p_c, where A is a constant, we have

$$|p' - p^*|^{-\nu} = b^{-1}|p - p^*|^{-\nu}, \tag{12.21}$$

where we have identified p_c with p^*. To find the relation between p' and p near p_c, we expand the renormalization transformation (12.17) in a Taylor series about p^* and obtain to first order in $(p - p^*)$

$$p' - p^* = R(p) - R(p^*) \approx \lambda\,(p - p^*), \tag{12.22}$$

where

$$\lambda = \frac{dR}{dp}\Big|_{p=p^*}. \tag{12.23}$$

We need to do a little algebra to obtain an explicit expression for ν. We first raise both sides of (12.22) to the νth power and write

$$|p' - p^*|^{\nu} = \lambda^{\nu}(p - p^*)^{\nu}. \tag{12.24}$$

We then compare (12.24) and (12.21) and obtain

$$b = \lambda^{\nu}. \tag{12.25}$$

Finally, we take the logarithm of both sides of (12.25) and obtain the desired relation for the critical exponent ν:

$$\nu = \frac{\log b}{\log \lambda}. \tag{12.26}$$

As an example, let us calculate λ for a square lattice with $b = 2$. We write (12.18) in the form $R(p) = -p^4 + 2p^2$. The derivative of $R(p)$ with respect to p yields $\lambda = 4p(1 - p^2) = 1.5279$ at $p = p^* = 0.61804$. We then use the relation (12.26) to obtain

$$\nu = \frac{\log 2}{\log 1.5279} \approx 1.635. \tag{12.27}$$

A comparison of (12.27) with the exact result $\nu = 4/3$ (see Table 12.1) for two dimensions shows reasonable agreement for such a simple calculation. (What would we be able to conclude if we were to measure $\xi(p)$ directly on a 2×2 lattice?)

Our calculation of ν does not give us an estimate of the error. What is the nature of our approximations? Our major assumption has been that the occupancy of each cell is independent of all other cells. This assumption is correct for the original sites, but after one renormalization, we lose some of the original connecting paths and gain connecting paths that are not present in the original lattice. An example of this interface problem is shown in

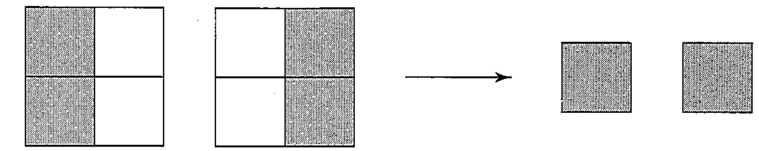

Figure 12.15 Example of the interface problem between cells. Two cells that are not connected at the original site level but that are connected at the cell level.

Figure 12.15. Because this surface effect becomes less important with increasing cell size, one way to improve the renormalization group calculation is to consider larger cells. In Project 12.12 we consider a cell-to-cell method that does not require large cells and yields comparable accuracy.

Problem 12.10 Renormalization group method for small cells

(a) Enumerate the spanning configurations for a $b = 2$ cell, assuming that a cell is occupied if a spanning path exists in either the vertical or the horizontal directions. Obtain the recursion relation and solve for the fixed point p^*. Use either a root finding algorithm or simple trial and error to find the value of $p = p^*$ such that $R(p) - p$ is zero. How do p^* and ν compare to their values using the vertical spanning criterion?

(b) Repeat the simple renormalization group calculation in part (a) using the criterion that a cell is occupied only if a spanning path exists in both directions.

*(c) The association of p_c with p^* is not the only possible one. Two alternatives involve the derivative $R'(p) = dR/dp$. For example, we could let $p_c = \int_0^1 p R'(p)\,dp$. Alternatively, we could choose $p_c = p_{\max}$, where $p_{\max}$ is the value of p at which $R'(p)$ has its maximum value. Compute p_c using these two alternative definitions and the various spanning criteria. In the limit of large cells, all three definitions should lead to the same values of p_c.

*(d) Enumerate the possible spanning configurations of a $b = 3$ cell, assuming that a cell is occupied if a cluster spans the cell vertically. Determine the probability of each configuration and verify the renormalization transformation $R(p) = p^9 + 9p^8q + 36p^7q^2 + 67p^6q^3 + 59p^5q^4 + 22p^4q^5 + 3p^3q^6$, where $q = 1 - p$ is the probability of an empty site. Solve the recursion relation (12.19) for p^*. Use this value of p^* to find the slope λ and the exponent ν. Then assume a cell is occupied if a cluster spans the cell both vertically and horizontally and obtain $R(p)$. Determine $p^*(b = 3)$ and $\nu(b = 3)$ for the two spanning criteria. Are your results for p^* and ν closer to their known values than for $b = 2$ for the same spanning criteria? ■

Problem 12.11 Renormalization group method for the triangular lattice

(a) For the triangular lattice, a cell can be formed by grouping three sites that form a triangle into one renormalized site. The only reasonable spanning criterion is that the cell spans if any two sites are occupied. Verify that $R(p) = p^3 + 3p^2(1 - p)$ and find $p_c = p^*$. How does p^* compare to the exact result $p_c = 1/2$?