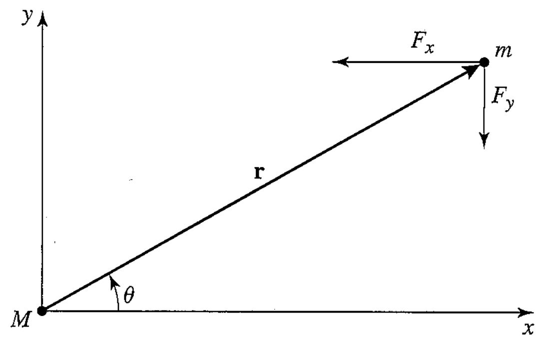

Figure 5.1 An object of mass m moves under the influence of a central force F. Note that $\cos\theta = x/r$ and $\sin\theta = y/r$, which provide useful relations for writing the equations of motion in component form suitable for numerical solutions.

It is convenient to write the force in Cartesian coordinates (see Figure 5.1):

$$F_x = -\frac{GMm}{r^2}\cos\theta = -\frac{GMm}{r^3}x \tag{5.7a}$$

$$F_y = -\frac{GMm}{r^2}\sin\theta = -\frac{GMm}{r^3}y. \tag{5.7b}$$

Hence, the equations of motion in Cartesian coordinates are

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^3}x \tag{5.8a}$$

$$\frac{d^2y}{dt^2} = -\frac{GM}{r^3}y, \tag{5.8b}$$

where $r^2 = x^2 + y^2$. Equations (5.8a) and (5.8b) are examples of coupled differential equations because each equation contains both x and y.

5.3 ■ CIRCULAR AND ELLIPTICAL ORBITS

Because many planetary orbits are nearly circular, it is useful to obtain the condition for a circular orbit. The magnitude of the acceleration a is related to the radius r of the circular orbit by

$$a = \frac{v^2}{r}, \tag{5.9}$$

where v is the speed of the object. The acceleration is always directed toward the center and is due to the gravitational force. Hence, we have

$$\frac{mv^2}{r} = \frac{GMm}{r^2}, \tag{5.10}$$

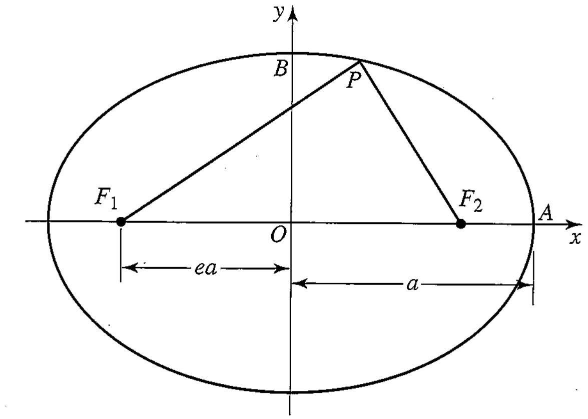

Figure 5.2 The characterization of an ellipse in terms of the semimajor axis a and the eccentricity e. The semiminor axis b is the distance OB. The origin O in Cartesian coordinates is at the center of the ellipse.

and

$$v = \left(\frac{GM}{r}\right)^{1/2}. \tag{5.11}$$

The relation (5.11) between the radius and the speed is the general condition for a circular orbit.

We can also find the dependence of the period T on the radius of a circular orbit using the relation

$$T = \frac{2\pi r}{v}, \tag{5.12}$$

in combination with (5.11) to obtain

$$T^2 = \frac{4\pi^2}{GM}r^3. \tag{5.13}$$

The relation (5.13) is a special case of Kepler's third law with the radius r corresponding to the semimajor axis of an ellipse.

A simple geometrical characterization of an elliptical orbit is shown in Figure 5.2. The two *foci* of an ellipse, F_1 and F_2, have the property that for any point P, the distance $F_1P + F_2P$ is a constant. In general, an ellipse has two perpendicular axes of unequal length. The longer axis is the major axis; half of this axis is the semimajor axis a. The shorter axis is the minor axis; the semiminor axis b is half of this distance. It is common to specify an elliptical orbit by a and by the eccentricity e, where e is the ratio of the distance between the foci to the length of the major axis. Because $F_1P + F_2P = 2a$, it is easy to show that

$$e = \sqrt{1 - \frac{b^2}{a^2}}, \tag{5.14}$$