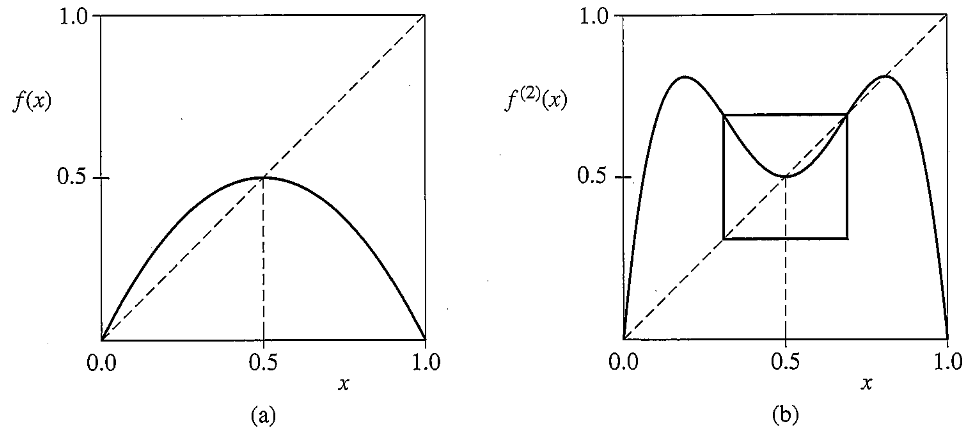

Figure 6.7 Comparison of $f(x, r)$ for $r = s_1$ with the second iterate $f^{(2)}(x)$ for $r = s_2$. (a) The function $f(x, r = s_1)$ has unstable fixed points at $x = 0$ and $x = 1$ and a stable fixed point at $x = 1/2$. (b) The function $f^{(2)}(x, r = s_1)$ has a stable fixed point at $x = 1/2$. The unstable fixed point of $f^{(2)}(x)$ nearest to $x = 1/2$ occurs at $x \approx 0.69098$, where the curve $f^{(2)}(x)$ intersects the line $y = x$. The upper right-hand corner of the square box in (b) is located at this point, and the center of the box is at $(1/2, 1/2)$. Note that if we reflect this square about the point $(1/2, 1/2)$, the shape of the reflected graph in the square box is nearly the same as it is in part (a) but on a smaller scale.

its square about $x = 1/2$. The size of the squares are determined by the unstable fixed point nearest to $x = 1/2$. Find the appropriate scaling factor and superimpose $f^{(2)}$ and the rescaled form of $f^{(4)}$. ■

***Problem 6.8 Other one-dimensional maps**

It is easy to modify your programs to consider other one-dimensional maps. Determine the qualitative properties of the one-dimensional maps

$$f(x) = xe^{r(1-x)} \tag{6.12}$$

$$f(x) = r \sin \pi x. \tag{6.13}$$

Do they also exhibit the period doubling route to chaos? The map in (6.12) has been used by ecologists (cf. May) to study a population that is limited at high densities by the effect of epidemics. Although it is more complicated than (6.5), its advantage is that the population remains positive no matter what (positive) value is taken for the initial population. There are no restrictions on the maximum value of r, but if r becomes sufficiently large, x eventually becomes effectively zero. What is the behavior of the time series of (6.12) for $r = 1.5$, 2, and 2.7? Describe the qualitative behavior of $f(x)$. Does it have a maximum?

The sine map (6.13) with $0 < r \leq 1$ and $0 \leq x \leq 1$ has no special significance, except that it is nonlinear. If time permits, determine the approximate value of δ for both maps. What limits the accuracy of your determination of δ? ■

The above qualitative arguments and numerical results suggest that the quantities α and δ are *universal*, that is, independent of the detailed form of $f(x)$. In contrast, the values of the accumulation point r_∞ and the constant C in (6.8) depend on the detailed form of $f(x)$. Feigenbaum has shown that the period doubling route to chaos and the values of δ and α are universal properties of maps that have a quadratic maximum, that is, $f'(x)|_{x=x_m} = 0$ and $f''(x)|_{x=x_m} < 0$.

Why is the universality of period doubling and the numbers δ and α more than a curiosity? The reason is that because this behavior is independent of the details, there might exist realistic systems whose underlying dynamics yield the same behavior as the logistic map. Of course, most physical systems are described by differential rather than difference equations. Can these systems exhibit period doubling behavior? Several workers (cf. Testa et al.) have constructed nonlinear RLC circuits driven by an oscillatory source voltage. The output voltage shows bifurcations, and the measured values of the exponents δ and α are consistent with the predictions of the logistic map.

Of more general interest is the nature of turbulence in fluid systems. Consider a stream of water flowing past several obstacles. We know that at low flow speeds, the water flows past obstacles in a regular and time-independent fashion called *laminar* flow. As the flow speed is increased (as measured by a dimensionless parameter called the Reynolds number), some swirls develop, but the motion is still time independent. As the flow speed is increased still further, the swirls break away and start moving downstream. The flow pattern as viewed from the bank becomes time dependent. For still larger flow speeds, the flow pattern becomes very complex and looks random. We say that the flow pattern has made a transition from laminar flow to *turbulent* flow.

This qualitative description of the transition to chaos in fluid systems is superficially similar to the description of the logistic map. Can fluid systems be analyzed in terms of the simple models of the type we have discussed here? In a few instances such as turbulent convection in a heated saucepan, period doubling and other types of transitions to turbulence have been observed. The type of theory and analysis we have discussed has suggested new concepts and approaches, and the study of turbulent flow is a subject of much current interest.

6.5 ■ MEASURING CHAOS

How do we know if a system is chaotic? The most important characteristic of chaos is *sensitivity to initial conditions*. In Problem 6.3, for example, we found that the trajectories starting from $x_0 = 0.5$ and $x_0 = 0.5001$ for $r = 0.91$ become very different after a small number of iterations. Because computers only store floating numbers to a certain number of digits, the implication of this result is that our numerical predictions of the trajectories of chaotic systems are restricted to small time intervals. That is, sensitivity to initial conditions implies that even though the logistic map is deterministic, our ability to make numerical predictions of its trajectory is limited.

How can we quantify this lack of predictably? In general, if we start two identical dynamical systems from slightly different initial conditions, we expect that the difference between the trajectories will increase as a function of n. In Figure 6.8 we show a plot of the difference $|\Delta x_n|$ versus n for the same conditions as in Problem 6.3a. We see that, roughly speaking, ln $|\Delta x_n|$ is a linearly increasing function of n. This result indicates that the separation between the trajectories grows exponentially if the system is chaotic. This divergence of the trajectories can be described by the *Lyapunov* exponent λ, which is defined