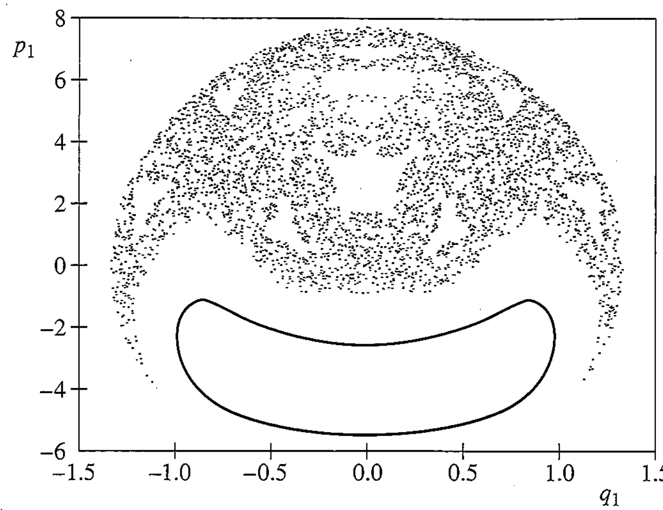

Figure 6.13 Poincaré plot for the double pendulum with p_1 plotted versus q_1 for $q_2 = 0$ and $p_2 > 0$. Two sets of initial conditions, $(q_1, q_2, p_1) = (0, 0, 0)$ and $(1.1, 0, 0)$, respectively, were used to create the plot. The initial value of the coordinate p_2 is found from (6.52) by requiring that $E = 15$.

E. The initial values of q_1 and q_2 can be chosen either randomly within the interval $|q_i| < \pi$ or by the user. Then set the initial $p_1 = 0$ and solve for p_2 using (6.52) with $H = E$. First explore the pendulum's behavior by plotting the generalized coordinates and momenta as a function of time in four windows. Consider the energies $E = 1, 5, 10, 15$, and 40. Try a few initial conditions for each value of E. Visually determine whether the steady state behavior is regular or appears to be chaotic. Are there some values of E for which all the trajectories appear regular? Are there values of E for which all trajectories appear chaotic? Are there values of E for which both types of trajectories occur?

(b) Repeat part (a) but plot the phase space diagrams p_1 versus q_1 and p_2 versus q_2. Are these plots more useful for determining the nature of the trajectories than those drawn in part (a)?

(c) Draw the Poincaré plot with p_1 plotted versus q_1 only when $q_2 = 0$ and $p_2 > 0$. Overlay trajectories from different initial conditions but with the same total energy on the same plot. Duplicate the plot shown in Figure 6.13. Then produce Poincaré plots for the values of E given in part (a) with at least five different initial conditions for each energy. Describe the different types of behavior.

(d) Is there a critical value of the total energy at which some chaotic trajectories first occur?

(e) Animate the double pendulum, showing the two masses moving back and forth. Describe how the motion of the pendulum is related to the behavior of the Poincaré plot. ■

Hamiltonian chaos has important applications in physical systems such as the solar system, the motion of the galaxies, and plasmas. It also has helped us understand the foundation for statistical mechanics. One of the most fascinating applications has been to quantum mechanics, which has its roots in the Hamiltonian formulation of classical mechanics. A current area of interest is the quantum analogue of classical Hamiltonian chaos. The meaning of this analogue is not obvious because well-defined trajectories do not exist in quantum mechanics. Moreover, Schrödinger's equation is linear and can be shown to have only periodic and quasiperiodic solutions.

6.10 ■ PERSPECTIVE

As the many books and review articles on chaos can attest, it is impossible to discuss all aspects of chaos in a single chapter. We will revisit chaotic systems in Chapter 13 where we introduce the concept of fractals. We will find that one of the characteristics of chaotic dynamics is that the resulting attractors often have an intricate geometrical structure.

The most general ideas that we have discussed in this chapter are that *simple systems can exhibit complex behavior* and that chaotic systems exhibit *extreme sensitivity to initial conditions*. We have also learned that computers allow us to explore the behavior of dynamical systems and visualize the numerical output. However, the simulation of a system does not automatically lead to understanding. If you are interested in learning more about the phenomena of chaos and the associated theory, the suggested readings at the end of the chapter are a good place to start. We also invite you to explore chaotic phenomenon in more detail in the following projects.

6.11 ■ PROJECTS

The first several projects are on various aspects of the logistic map. These projects do not exhaust the possible investigations of the properties of the logistic map.

Project 6.22 A more accurate determination of δ and α

We have seen that it is difficult to determine δ accurately by finding the sequence of values of b_k at which the trajectory bifurcates for the kth time. A better way to determine δ is to compute it from the sequence s_m of superstable trajectories of period 2^{m-1}. We already have found that $s_1 = 1/2$, $s_2 \approx 0.80902$, and $s_3 \approx 0.87464$. The parameters $s_1, s_2, \ldots$ can be computed directly from the equation

$$f^{(2^{m-1})}\left(x = \frac{1}{2}\right) = \frac{1}{2}. \tag{6.53}$$

For example, s_2 satisfies the relation $f^{(2)}(x = 1/2) = 1/2$. This relation, together with the analytical form for $f^{(2)}(x)$ given in (6.7), yields

$$8r^2(1 - r) - 1 = 0. \tag{6.54}$$

If we wish to solve (6.54) numerically for $r = s_2$, we need to be careful not to find the irrelevant solutions corresponding to a lower period. In this case we can factor out the solution $r = 1/2$ and solve the resultant quadratic equation analytically to find $s_2 = (1 + \sqrt{5})/4$. Clearly $r = s_1 = 1/2$ solves (6.54) with period 1, because from (6.53), $f^{(1)}(x = 1/2) = 4r\frac{1}{2}(1 - \frac{1}{2}) = r = 1/2$ only for $r = 1/2$.