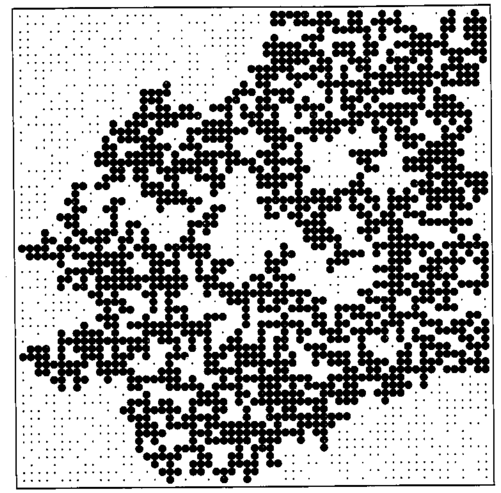

Figure 13.1 Example of a spanning percolation cluster generated at $p = 0.5927$ on a $L = 124$ square lattice. The other occupied sites are not shown.

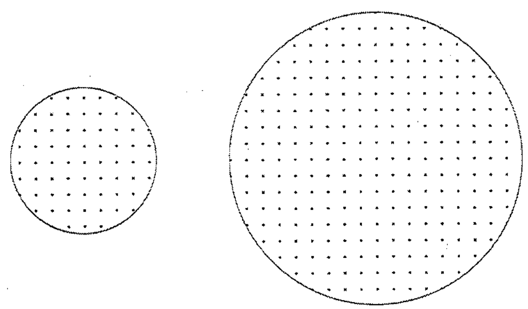

Figure 13.2 The number of dots per unit area in each circle is uniform. How does the total number of dots (mass) vary with the radius of the circle?

Because $D < d$, we see that a fractal object becomes less dense at larger length scales. The scale dependence of the density is a quantitative measure of the ramified or stringy nature of fractal objects. In addition, another characteristic of fractal objects is that they have holes of all sizes. This property follows from (13.3) because if we replace R by Rb, where b is some constant, we obtain the same power law dependence for $\rho(R)$. Thus, it does not matter what scale of length is used, and thus all hole sizes must be present.

Another important characteristic of fractal objects is that they look the same over a range of length scales. This property of *self-similarity* or *scale invariance* means that if we take part of a fractal object and magnify it by the same magnification factor in all directions, the magnified picture is similar to the original. This property follows from the scaling argument given for $\rho(R)$.

The percolation cluster shown in Figure 13.1 is an example of a *random* or statistical fractal because the mass-length relation (13.1) is satisfied only on the average, that is, only if the quantity $M(R)$ is averaged over many different origins in a given cluster and over many clusters.

In physical systems, the relation (13.1) does not extend over all length scales but is bounded by both upper and lower cut-off lengths. For example, a lower cut-off length is provided by the lattice spacing or the mean distance between the constituents of the object. In computer simulations, the maximum length is usually the finite system size. The presence of these cut-offs complicates the determination of the fractal dimension.

In Problem 13.1 we compute the fractal dimension of percolation clusters using straightforward Monte Carlo methods. Remember that data extending over several decades is required to obtain convincing evidence for a power law relationship between M and R and to determine accurate estimates for the fractal dimension. Hence, conclusions based on the limited simulations posed in the problems need to be interpreted with caution.

Problem 13.1 The fractal dimension of percolation clusters

(a) Generate a site percolation configuration on a square lattice with $L \geq 61$ at $p = p_c \approx 0.5927$. Why might it be necessary to generate several configurations before a spanning cluster is obtained? Does the spanning cluster have many dangling ends?

(b) Choose a point on the spanning cluster and count the number of points in the spanning cluster $M(b)$ within a square of area b^2 centered about that point. Then double b and count the number of points within the larger box. Can you repeat this procedure indefinitely? Repeat this procedure until you can estimate the b-dependence of the number of points. Use the b-dependence of $M(b)$ to estimate D according to the definition $M(b) \sim b^D$, that is, estimate D from a log-log plot of $M(b)$ versus b. Choose another point in the cluster and repeat this procedure. Are your results similar? A better estimate for D can be found by averaging $M(b)$ over several origins in each spanning cluster and averaging over many spanning clusters.

(c) If you have not already done Problem 12.8a, compute D by determining the mean size (mass) M of the spanning cluster at $p = p_c$ as a function of the linear dimension L of the lattice. Consider $L = 11, 21, 41$, and 61 and estimate D from a log-log plot of M versus L. ■

***Problem 13.2 Renormalization group calculation of the fractal dimension**

Compute $\langle M^2 \rangle$, the average of the square of the number of occupied sites in the spanning cluster at $p = p_c$, and the quantity $\langle M'^2 \rangle$, the average of the square of the number of occupied sites in the spanning cluster on the renormalized lattice of linear dimension $L' = L/b$. Because $\langle M^2 \rangle \sim L^{2D}$ and $\langle M'^2 \rangle \sim (L/b)^{2D}$, we can obtain D from the relation $b^{2D} = \langle M^2 \rangle / \langle M'^2 \rangle$. Choose the length rescaling factor to be $b = 2$ and adopt the same blocking procedure as was used in Section 12.5. An average over ten spanning clusters for $L = 16$ and $p = 0.5927$ is sufficient for qualitative results. ■

In Problems 13.1 and 13.2, we were interested only in the properties of the spanning clusters. For this reason, our algorithm for generating percolation configurations by randomly occupying each site is inefficient because it generates many clusters. A more efficient way of generating single percolation clusters is due independently to Hammersley, Leath, and Alexandrowicz. This algorithm, commonly known as the Leath or the single cluster growth algorithm, is equivalent to the following steps (see Figure 13.3):

1. Occupy a single seed site on the lattice. The nearest neighbors (four on the square lattice) of the seed represent the *perimeter* sites.