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CHAPTER 6

The Chaotic Motion of Dynamical Systems

We study simple nonlinear deterministic models that exhibit chaotic behavior. We will find that the use of the computer to do numerical experiments will help us gain insight into the nature of chaos.

6.1 ■ INTRODUCTION

Most natural phenomena are intrinsically nonlinear. Weather patterns and the turbulent motion of fluids are everyday examples. Although we have explored some of the properties of nonlinear systems in Chapter 4, it is easier to introduce some of the important concepts in the context of ecology. Our first goal will be to motivate and analyze the one-dimensional difference equation

$$x_{n+1} = 4rx_n(1 - x_n), \tag{6.1}$$

where x_n is the ratio of the population in the nth generation to a reference population. We shall see that the dynamical properties of (6.1) are surprisingly intricate and have important implications for the development of a more general description of nonlinear phenomena. The significance of the behavior of (6.1) is indicated by the following quote from the ecologist Robert May:

> Its study does not involve as much conceptual sophistication as does elementary calculus. Such study would greatly enrich the student's intuition about nonlinear systems. Not only in research but also in the everyday world of politics and economics we would all be better off if more people realized that simple nonlinear systems do not necessarily possess simple dynamical properties.

The study of chaos is of much current interest, but the phenomena is not new and has been of interest, particularly to astronomers and mathematicians, for over one hundred years. Much of the current interest is due to the use of the computer as a tool for making empirical observations. We will use the computer in this spirit.

6.2 ■ A SIMPLE ONE-DIMENSIONAL MAP

Imagine an island with an insect population that breeds in the summer and leaves eggs that hatch the following spring. Because the population growth occurs at discrete times, it is appropriate to model the population growth by a difference equation rather than by a