• Introduction

Root-finding: to find the value of x such that it satisfies equation $f(x) = 0$.

How to find the one-thirds power of 12, $x = 12^{1/3}$, by four operations $(+, -, \times, /)$ calculator?

It is equivalent to find the root of $f(x) = x^3 - 12 = 0$.

Minimization of total energy:

$$\frac{\partial E(x)}{\partial x} = 0 \ .$$

Maximization of entropy:

$$\frac{\partial S(x)}{\partial x} = 0 \ .$$

Traveling sales person problem, etc.

- **Polynomial**

$$
\begin{aligned}
P_n(x) &= \sum_{k=0}^{n} a_k x^k \\
&= a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n \ . \qquad (1)
\end{aligned}
$$

1. *Quadratic Equation*

$$a_0 + a_1 x + a_2 x^2 = 0 \qquad (2)$$

Solution (**two** roots):

$$x_\pm = \frac{1}{2a_2}\left(a_1 \pm \sqrt{a_1^2 - 4a_0 a_2}\right) \qquad (3)$$

Question: how do you get this solution?

2. *Cubic Equation*

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 = 0 \qquad (4)$$

Solution: (first scale a_3 to 1).

$$\text{step 1}: x = y - \frac{a_2}{3} \quad \Rightarrow \quad y^3 + py + q = 0 \qquad (5)$$

$$p = -\frac{a_2^2}{3} + a_1, \qquad q = \frac{2a_2^3}{27} - \frac{a_2 a_1}{3} + a_0 \ . \qquad (6)$$

step 2: solve Eq. (5) (**three** roots)

$$y_1 = u + v; y_2 = \omega_1 u + \omega_2 v; y_3 = \omega_2 u + \omega_1 v \qquad (7)$$

where

$$\omega_1 = \frac{-1 + \sqrt{3}i}{2}, \qquad \omega_2 = \frac{-1 - \sqrt{3}i}{2} = \omega_1^2 \ , \qquad (8)$$

$$u^3 = -\frac{q}{2} + \sqrt{(\frac{q}{2})^2 + (\frac{p}{2})^3} \qquad (9)$$

$$v^3 = -\frac{q}{2} - \sqrt{(\frac{q}{2})^2 + (\frac{p}{2})^3} \qquad (10)$$

3. *Quartic Equation*

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 = 0 \qquad (11)$$

Solution (**four** roots): see reference book.

For any order n, there *always* exits n roots such that

$$P_n(x) = a_n(x - x_1)(x - x_2) \cdots (x - x_{n-1})(x - x_n)\ . \qquad (12)$$

x_i are complex numbers in general.

- **Taylor Series**

Assume function $f(x)$ has a continuous nth derivative in the interval of $a \leq x \leq b$. Then

$$\int_a^{x_1} dx_0 f^{[n]}(x_0) = f^{[n-1]}(x_1) - f^{[n-1]}(a) \ . \tag{13}$$

$$\begin{aligned} & \int_a^{x_2}\int_a^{x_1} dx_0 dx_1 f^{[n]}(x_0) \\ = & \int_a^{x_2} dx_1 \left(f^{[n-1]}(x_1) - f^{[n-1]}(a)\right) \\ = & f^{[n-2]}(x_2) - f^{[n-2]}(a) - (x_2 - a) f^{[n-1]}(a) \ . \end{aligned} \tag{14}$$

After n integrations, we have

$$\begin{aligned} & \int_a^{x_n} \cdots \int_a^{x_1} dx_0 \cdots dx_{n-1} f^{[n]}(x) \\ = & f(x_n) - f(a) - (x_n - a) f^{'}(a) - \frac{(x_n - a)^2}{2!} f^{''}(a) \\ - & \frac{(x_n - a)^3}{3!} f^{'''}(a) \cdots - \frac{(x_n - a)^{n-1}}{(n-1)!} f^{[n-1]}(a) \ . \end{aligned} \tag{15}$$

$x \rightarrow x_n$, we obtain **Taylor series**:

$$\begin{aligned} f(x) = & f(a) + (x-a) f^{'}(a) + \frac{(x-a)^2}{2!} f^{''}(a) \\ + & \frac{(x-a)^3}{3!} f^{'''}(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!} f^{[n-1]}(a) \\ + & R_n(x) \ . \end{aligned} \tag{16}$$

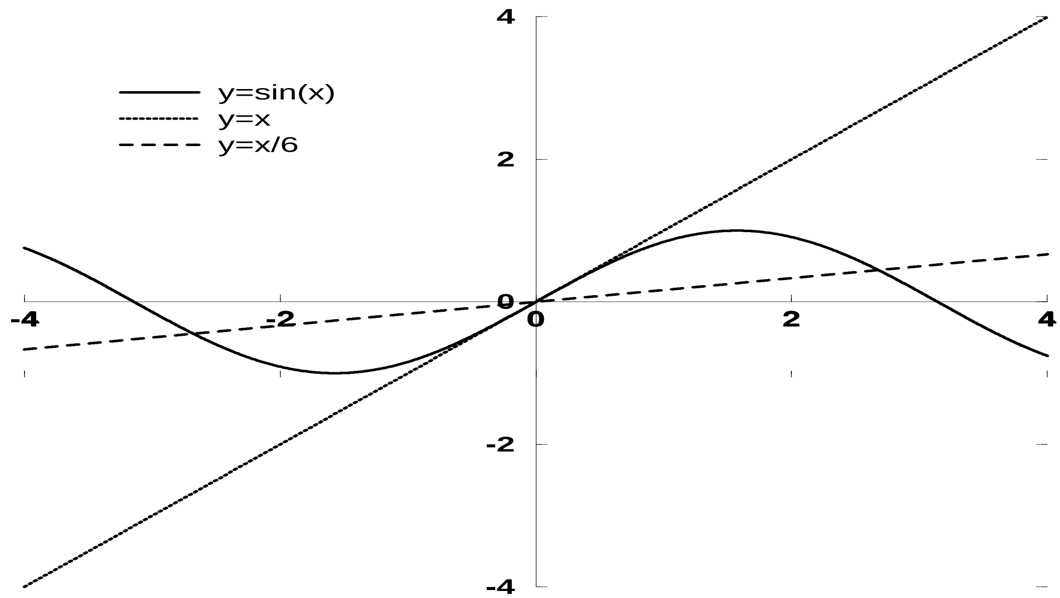
Remainder $R_n(x)$ can be written as

$$R_n(x) = \frac{(x-a)^n}{n!} f^{[n]}(\xi), \qquad a \leq \xi \leq x \ . \tag{17}$$

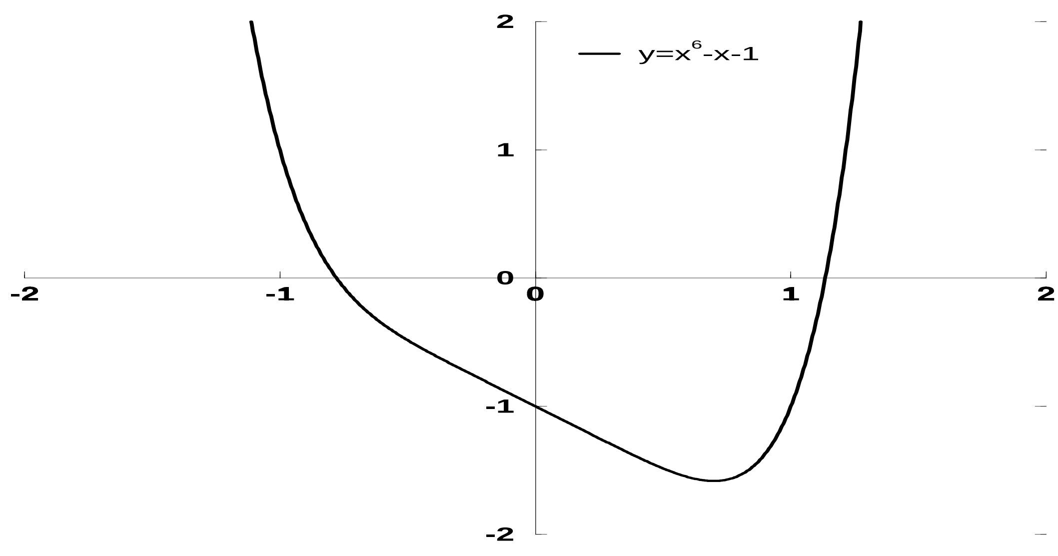
- **Graphic Solution**

The simplest way, unaccurate, but very powerful.

Example 1: $\sin(x) = ax$.



Example 2: $f(x) = x^6 - x - 1$.



- **The Bisection Algorithm**

  Probably the simplest method, but very powerful.

  Task: find the root x such that $f(x) = 0$.

  1. Choose two values, x_{left} and x_{right}, with $x_{left} < x_{right}$, such that $f(x_{left})f(x_{right}) < 0$. There must be a value of x such that $f(x) = 0$ in the interval $[x_{left}, x_{right}]$.

  2. Choose the midpoint, $x_{mid} = x_{left} + \frac{1}{2}(x_{right} - x_{left})$ $= \frac{1}{2}(x_{right} + x_{left})$, as the guess for x.

  3. If $f(x_{mid})$ has the same sign as $f(x_{left})$, then replace x_{left} by x_{mid}; otherwise, replace x_{right} by x_{mid}. Thus, we halved the interval for the location of the root.

  4. Repeat steps 2 and 3 until the desired level of precision is achieved.

  <u>Error Bounds:</u> let $a = x_{left}$ and $b = x_{right}$ at beginning with error tolerance ϵ and number of iterations n, then (see **Error Analysis**)

  $$\frac{1}{2^n}(b - a) \leq \epsilon \ .$$

- **Error Analysis**

$$\begin{aligned}&\textit{Absolute error}\\ &= \ |\ \textit{true value} \ - \ \textit{approximate value}\ |\ .\end{aligned}$$

$$\begin{aligned}&\textit{Relative error}\\ &= \left|\ \frac{\textit{true value} \ - \ \textit{approximate value}}{\textit{true value}}\ \right|\ .\end{aligned}$$

$$\begin{aligned}&\textit{Approximate relative error}\\ &= \left|\ \frac{\textit{best approximation} \ - \ \textit{approximate value}}{\textit{best approximation}}\ \right|\ .\end{aligned}$$

Example: the bisection method

$$\begin{aligned}|x - x_{mid}| \ &\le \ x_{mid} - x_{left} = x_{right} - x_{mid}\\ &= \ \frac{1}{2}(x_{right} - x_{left}) = \cdots = \frac{1}{2^n}(b-a)\ .\end{aligned}$$

$$|x - x^n_{mid}| \ \le \ \frac{1}{2^n}(b-a) \le \epsilon\ .$$

$$n \ \ge \ \frac{\log((b-a)/\epsilon)}{log2}\ .$$

- **The Newton-Raphson Method**

  Assume that we have a good "guess" of solution x^* so that $(x-x^*)$ is a small number. By Taylor expansion, we have

$$0 = f(x^*) \approx f(x) + (x^* - x)f^{'}(x) \ .$$

  Thus

$$x^* = x - \frac{f(x)}{f'(x)} \quad \Rightarrow \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ . \qquad (18)$$

Example 1: $f(x) = x^6 - x - 1, f^{'}(x) = 6x^5 - 1$.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	1.5	8.89E+1	
1	1.30049088	2.54E+1	-2.00E-1
2	1.18148042	5.38E-1	-1.19E-1
3	1.13945559	4.92E-2	-4.20E-2
4	1.13477763	5.50E-4	-4.68E-3
5	1.13472415	7.11E-8	-5.35E-5
6	1.13472414	1.55E-15	-6.91E-9

Example 2: $f(x) = x^2 + 1, f^{'}(x) = 2x$.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
0	0.57735027	1.3333	
1	-0.57735027	1.3333	-1.1547
2	0.57735027	1.3333	1.1547
3	-0.57735027	1.3333	-1.1547
4	0.57735027	1.3333	1.1547
5	-0.57735027	1.3333	-1.1547
6	0.57735027	1.3333	1.1547

Error in Newton-Raphson

Let $\epsilon_n = x^* - x_n$ and from Eq. (18), we have

$$\epsilon_{n+1} = \epsilon_n + \frac{f(x_n)}{f'(x_n)} \; .$$

Do Taylor expansion to second order, we have

$$\begin{aligned} f(x^*) &= f(x_n) + (x^* - x_n) f^{'}(x_n) \\ &+ \frac{(x^* - x_n)^2}{2!} f^{''}(x_n) + \cdots \; . \end{aligned} \tag{19}$$

Since $0 = f(x^*)$ so we have

$$f(x_n) = -\epsilon_n f^{'}(x_n) - \frac{\epsilon_n^2}{2} f^{''}(x_n) \; .$$

$$\frac{f(x_n)}{f'(x_n)} = -\epsilon_n - \frac{\epsilon_n^2 f^{''}(x_n)}{2 f'(x_n)} \; .$$

$$\epsilon_{n+1} = -\frac{\epsilon_n^2 f^{''}(x_n)}{2 f'(x_n)} \; . \tag{20}$$

The error in x_{n+1} is proportional to the *square* of the error in $x_n (p = 2)$. Convergence could be very rapid!

Error Estimation

$$x^* - x_n \approx x_{n+1} - x_n \qquad f^{'}(x^*) \neq 0 \; . \tag{21}$$

This method fails if $f^{'}(x^*) = 0$! Now what?

- **Rates of Convergence**

  One must concern the speed of convergence of an iteration method. We say that a sequence $x_n; n \geq 0$ converges to x^* with an *order of convergence* $p > 0$ if

$$|x^* - x_{n+1}| \leq c|x^* - x_n|^p, \qquad n \geq 0 \tag{22}$$

  for some constant $c \geq 0$.

  1. *Linear convergence*: $p = 1$, $c <?$

     e.g., Bisection method.

  2. *Quadratic convergence*: $p = 2$.

     e.g., Newton-Raphson method.

  3. *Cubic convergence*: $p = 3$.

- **The Secant Method**

It could be difficult to calculate $f^{'}(x)$.

By Taylor expansion

$$f(x_n) \approx f(x_{n-1}) + (x_n - x_{n-1})f^{'}(x_n) \ .$$

$$f^{'}(x_n) \approx \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \ ,$$

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &\approx x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \end{aligned} \tag{23}$$

Error Analysis

One can show that $|x^* - x_{n+1}| \leq c|x^* - x_n|^p$, with $p = (\sqrt{5}+1)/2 = 1.62$.

Example 1: $f(x) = x^6 - x - 1, x_0 = 2.0, x_1 = 1.0$.

n	x_n	$f(x_n)$	$x_n - x_{n-1}$
2	1.01612903	-9.15E-1	1.61E-2
3	1.19057777	6.57E-2	1.74E-1
4	1.11765583	-1.68E-1	-7.29E-2
5	1.13253155	-2.24E-2	1.49E-2
6	1.13481681	9.54E-4	2.29E-3
7	1.13472365	-5.07E-6	-9.32E-5
8	1.13472414	-1.13E-8	4.92E-7

- **The False Position Method**

  Approximate

  $$f(x) \approx \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b) \ ,$$

  then

  $$f^{'}(x) = \frac{f(b)-f(a)}{b-a} \ ,$$

  and (Newton-Raphson)

  $$\overline{x} = x - \frac{f(x)}{f'(x)} = \frac{af(b)-bf(a)}{f(b)-f(a)} \ .$$

  If $f(a)f(\overline{x}) < 0$, replace b by $\overline{x}$;

  If $f(b)f(\overline{x}) < 0$, replace a by $\overline{x}$.

  Continue $\cdots$.

- **The Hybrid Bisection/Newton-Raphson Method**

  Bisection – *always* gives solution but *slowly* converges.

  Newton-Raphson – converges *fast* but could *fail.*

  Naturally, hybridize both of them.

  Assume the root is bounded between $x = a$ and $x = b$, and the best current guess is $x = r$, then we want to know if the *next Newton-Raphson guess* is within the bounds, i.e.,

$$
\begin{aligned}
a \; &\leq \; \tilde{r} = r - \frac{f(r)}{f'(r)} \leq b \\
\Rightarrow 0 \; &\leq \; (r-a)f^{'}(r) - f(r) = A(r) \\
0 \; &\geq \; (r-b)f^{'}(r) - f(r) = B(r) \; . \qquad (24)
\end{aligned}
$$

  Thus, if $A(r) \times B(r) \leq 0$, Newton-Raphson.

  if $A(r) \times B(r) > 0$, Bisection.

  Alternatively, just calculate $(\tilde{r} - a) \times (\tilde{r} - b)$.

  The Hybrid Bisection/Secant Method

  Same idea: if $a < x_{n+1} < b$ (Secant), no (Bisection).

  Or make a simple replacement in the Hybrid Bisection/Newton-Raphson routine.

- **Iteration in General**

Consider solving the equation $f(x) = 0$. One can rewrite it in various form such that

$$x = G(x) \quad \Rightarrow \quad x_{n+1} = G(x_n) \ ,$$

and iterative x_n with given error tolerance ϵ, say, $|x_{n+1} - x_n| \leq \epsilon$.

The solution is also called *fixed point*: $x^* = G(x^*)$.

Convergence

1. Let $G(x)$ be a continuous function for an interval $[a, b]$, and suppose G satisfies the property

$$a \leq x \leq b \Rightarrow a \leq G(x) \leq b \ . \tag{25}$$

Then the equation $x = G(x)$ has at least one solution x^* in the interval $[a, b]$.

Proof: Define the function $f(x) = x - G(x)$. It is continuous for $a \leq x \leq b$. Moreover, $f(a) \leq 0$ and $f(b) \geq 0$. By the intermediate value theorem, there exists a point x^* in $[a, b]$ at which $f(x^*) = 0$.

2. Assume $G(x)$ and $G^{'}(x)$ are continuous for $[a, b]$, and assume G satisfies Eq. (25). Further assume that $\lambda \equiv \text{Maximum}_{a \le x \le b} |G^{'}(x)| < 1$. Then

(a) There is a unique solution x^* of $x = G(x)$ in the interval $[a, b]$.

(b) For any initial estimate x_0 in $[a, b]$, the iterates x_n will converge to x^*.

(c) $$|x^* - x_n| \le \frac{\lambda^n}{1-\lambda} |x_0 - x_1|, \quad n \ge 0 \ . \qquad (26)$$

(d) $$Lim_{n \to \infty} \frac{x^* - x_{n+1}}{x^* - x_n} = G^{'}(x^*) \ . \qquad (27)$$

Proof: By using the mean value theorem, we have that for any two points w and z in $[a, b]$,

$$G(w) - G(z) = G^{'}(c)(w - z)$$

for some c between w and z. Then

$$|G(w) - G(z)| = |G^{'}(c)||(w - z)| \le \lambda |(w - z)|.$$

(a) Suppose that there are two solutions, α and β, then

$$\alpha - \beta = G(\alpha) - G(\beta) \ .$$

$$|\alpha - \beta| \le \lambda |\alpha - \beta|$$

$$(1-\lambda)|\alpha-\beta| \leq 0$$

Since $\lambda < 1$, we must have $\alpha = \beta$.

(b) For any initial guess x_0, the iterates x_n will all remain in $[a, b]$. Then

$$x^* - x_{n+1} = G(x^*) - G(x_n) = G^{'}(c)(x^* - x_n)$$

$$|x^* - x_{n+1}| \leq \lambda |x^* - x_n| \leq \lambda^n |x^* - x_0|$$

Since $\lambda < 1$, $x_n \to x^*$ as $n \to \infty$.

(c) Let $n = 1$, we have

$$\begin{aligned} |x^* - x_0| &\leq |x^* - x_1| + |x_1 - x_0| \\ | &\leq \lambda |x^* - x_0| + |x_1 - x_0| \end{aligned}$$

$$(1-\lambda)|x^* - x_0| \leq |x_1 - x_0|$$

$$|x^* - x_0| \leq \frac{1}{1-\lambda}|x_1 - x_0|$$

(d) Use previous results to write

$$Lim_{n\to\infty}\frac{x^* - x_{n+1}}{x^* - x_n} = Lim_{n\to\infty}G^{'}(c) = G^{'}(x^*)$$

3. If $|G^{'}(x^*)| > 1$, then the iteration $x_{n+1} = G(x_n)$ will not converge to x^*.

Atiken's error estimate

From

$$x^* - x_n = G(x^*) - G(x_{n-1}) \approx G'(x^*)(x^* - x_{n-1})$$

we have

$$x^* - x_n \approx \lambda(x^* - x_{n-1})$$

then

$$x^* = x_n + \frac{\lambda}{1-\lambda}(x_n - x_{n-1})$$

Let's estimate λ,

$$\lambda_n = \frac{x_n - x_{n-1}}{x_{n-1} - x_{n-2}}, \qquad n \geq 2 \ .$$

Then we get

$$x^* - x_n \approx \frac{\lambda_n}{1-\lambda_n}(x_n - x_{n-1}) \ .$$

Example: $f(x) = x^2 - 5 = 0$, $x = \pm\sqrt{5} = \pm 2.2361$.

We can write

$$
\begin{aligned}
&(I1) \quad x_{n+1} = 5 + x_n - x_n^2 \\
&(I2) \quad x_{n+1} = 5/x_n \\
&(I3) \quad x_{n+1} = 1 + x_n - \frac{1}{5}x_n^2 \\
&(I4) \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right) .
\end{aligned}
$$

Start from $x_0 = 2.5$,

n	$x_n(I1)$	$x_n(I2)$	$x_n(I3)$	$x_n(I4)$
0	2.5	2.5	2.5	2.5
1	1.25	2.0	2.25	2.25
2	4.6875	2.5	2.2375	2.2361
3	-12.2852	2.0	2.2362	2.2361
$G'(\sqrt{5})$	$1 - 2\sqrt{5}$	-1.0	$1 - \frac{2}{5}\sqrt{5}$	0

Start from $x_0 = -2.5$,

n	$x_n(I1)$	$x_n(I2)$	$x_n(I3)$	$x_n(I4)$
0	-2.5	-2.5	-2.5	-2.5
1	-3.75	-2.0	-2.75	-2.25
2	-12.8125	-2.5	-3.2625	-2.2361
3	-171.9726	-2.0	-4.3913	-2.2361
$G'(\sqrt{5})$	$1 + 2\sqrt{5}$	-1.0	$1 + \frac{2}{5}\sqrt{5}$	0

What have we learned here?

• **Continue ...**

1. **Accelerating the rate of convergence**

   In the Hybrid Bisection/Secant method, a *linear approximation* of the function is used to obtain the next approximation to the root, and then the end-points of the interval are *adjusted* to keep the root *bounded.* How about higher order approximation?

   Consider three points x_0, x_1, and x_2, and function evaluated at these points. Then approximate $f(x)$ by a *quadratic* function (2nd order)

   $$f(x) \approx p(x) = a_2(x - x_2)^2 + a_1(x - x_2) + a_0 \ .$$

   Let $f(x_3) \approx p(x_3) = 0$, we get

   $$x_3 - x_2 = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2}$$

   We expect x_3 approaches x_2, thus

   $$x_3 = x_2 - \frac{2a_0}{a_1 + \sqrt{a_1^2 - 4a_2a_0}}, \qquad a_1 \geq 0 \ .$$

   $$x_3 = x_2 - \frac{2a_0}{a_1 - \sqrt{a_1^2 - 4a_2a_0}}, \qquad a_1 \leq 0 \ .$$

   It is *robust, virtually failsafe, and no derivatives.*

2. **Multiple Root**

$$f(x) = (x - a)^m h(x)$$

When the Newton and Secant methods are applied to the calculation of a multiple root, the convergence of $x^* - x_n$ to zero is much slower than it would be for a simple root. In addition, there is a large *interval of uncertainty* as to where the root actually lies, because of the *noise* in evaluating $f(x)$.

One can show that when we use Newton's method to calculate a root of multiplicity m, the ratio of the error in successful iteration

$$\frac{x^* - x_n}{x^* - x_{n-1}} \to \lambda = \frac{m-1}{m} \ ,$$

so the error decreases at about the constant rate.

$f(x) = (x - 1.1)^3(x - 2.1)$

n	x_n	$f(x_n)$	$x^* - x_n$	Ratio
0	0.800000	0.03510	0.300000	
1	0.892857	0.01073	0.207143	0.690
2	0.958176	0.00325	0.141824	0.685
3	1.00344	0.00099	0.09656	0.681
4	1.03486	0.00029	0.06514	0.675
5	1.05581	0.00009	0.04419	0.678
6	1.07028	0.00003	0.02972	0.673
7	1.08092	0.00000	0.01908	0.642

Thus, with any root of multiplicity $m \geq 2$, the Bisection method is always better!

The only way to obtain accurate values for multiple roots is to analytically remove the multiplicity, obtaining a new function for which x^* is a simple root. How to proceed?

Determine m experimentally, then work on

$$F(x) = f^{(m-1)}(x) \ .$$

3. **Stability of Roots**

Ill-conditioned or *unstable* problems.

Very small errors in evaluating $f(x)$ will lead to very large changes in the roots of the function.

Example

$$
\begin{aligned}
f(x) &= (x-1)(x-2)(x-3)(x-4)(x-5)(x-6)(x-7) \\
&= x^7 - 28x^6 + \cdots \\
F(x) &= x^7 - 28.002x^6 + \cdots
\end{aligned}
$$

Root of f(x)	Root of F(x)
1	1.0000028
2	1.9989382
3	3.0331253
4	3.8195692
5	5.4586758 + 0.54012578 i
6	5.4586758 - 0.54012578 i
7	7.2330128

There is not much that can be done except to go to higher precision arithmetic. Or find other ways to formulate the problem.

4. **Exhaustive Searching**

Find all roots?

5. **Function of n Variables**

e.g., spin glass $E(\{S_i\}) = \Sigma_{ij} J_{ij} S_i S_j$, Hartree-Fock (mean-field) solution of many-body problems.

The task is to find out the roots of the row vector function

$$F(X) = [f_1(X), f_2(X), \cdots, f_n(X)]$$

with $X = [x_1, x_2, \cdots, x_n]$.

Using the Taylor expansion,

$$F(X^*) = F(X_n) - \delta X D(X_n) + O(\delta X^2) \ ,$$

where D is called the Jacobian matrix,

$$D_{ij}(X) = \frac{\partial f_j(X)}{\partial x_i} \ .$$

Thus Newton's method (matrix form)

$$X_{n+1} = X_n - \delta X = X_n - F(X_n) D^{-1}(X_n) \ .$$

One can apply other iteration schemes.

6. *There are many other root-finding methods!*

e.g., Chapter 9 of *Numerical Recipes.*

$\vdots$

Mean Value Theorems

- **Intermediate Value Theorem**

  Let $f(x)$ be a continuous function on the interval $a \leq x \leq b$. Let

$$M = max\{f(x)\}, \qquad m = min\{f(x)\}$$

  Then for every value v satisfying $m \leq v \leq M$, there is at least one point c in $[a, b]$ for which $f(c) = v$.

  It is rather easy to understand this *intuitive result* graphically.

- **Mean Value Theorem**

  Let $f(x)$ be continuous on the interval $a \leq x \leq b$, and also let it be differentiable for $a < x < b$. Then there is at least one point c in (a, b) for which

$$f(b) - f(a) = f^{'}(c)(b - a)$$

- **Integral Mean Theorem**

  Let $\omega(x)$ be a nonnegative integrable function on $[a, b]$, and let $f(x)$ be continuous on $[a, b]$. Then there is at least one point c in $[a, b]$ for which

  $$\int_a^b dx \omega(x) f(x) = f(c) \int_a^b dx \omega(x)$$

  In particular, if we take $\omega(x) = 1$, then

  $$\int_a^b dx f(x) = f(c)(b - a)$$

  for some c in $[a, b]$.