(d) Verify the following properties of the fixed points of $f^{(2)}(x)$. As r is increased, the fixed points of $f^{(2)}(x)$ move apart, and the slope of $f^{(2)}(x)$ at its fixed points decreases. What is the value of $r = s_2$ at which one of the two fixed points of $f^{(2)}$ equals 1/2? What is the value of the other fixed point? What is the slope of $f^{(2)}(x)$ at $x = 1/2$? What is the slope at the other fixed point? As r is further increased, the slopes at the fixed points become negative. Finally at $r = b_2 \approx 0.8623$, the slopes at the two fixed points of $f^{(2)}(x)$ equal -1, and the two fixed points of $f^{(2)}$ become unstable. (The exact value of b_2 is $b_2 = (1 + \sqrt{6})/4$.)

(e) Show that for r slightly greater than b_2, for example $r = 0.87$, there are four stable fixed points of $f^{(4)}(x)$. What is the value of $r = s_3$ when one of the fixed points equals 1/2? What are the values of the three other fixed points at $r = s_3$?

(f) Determine the value of $r = b_3$ at which the four fixed points of $f^{(4)}$ become unstable.

(g) Choose $r = s_3$ and determine the number of iterations that are necessary for the trajectory to converge to period 4 behavior. How does this number of iterations change when neighboring values of r are considered? Choose several values of x_0 so that your results do not depend on the initial conditions. ■

Problem 6.5 Periodic windows in the chaotic regime

(a) If you look closely at the bifurcation diagram in Figure 6.2, you will see that the range of chaotic behavior for $r > r_\infty$ is interrupted by intervals of periodic behavior. Magnify your bifurcation diagram so that you can look at the interval $0.957107 \leq r \leq 0.960375$, where a periodic trajectory of period 3 occurs. (Period 3 behavior starts at $r = (1 + \sqrt{8})/4$.) What happens to the trajectory for slightly larger r, for example, $r = 0.9604$?

(b) Plot $f^{(3)}(x)$ versus x at $r = 0.96$, a value of r in the period 3 window. Draw the line $y = x$ and determine the intersections with $f^{(3)}(x)$. The stable fixed points satisfy the condition $x^* = f^{(3)}(x^*)$. Because $f^{(3)}(x)$ is an eighth-order polynomial, there are eight solutions (including $x = 0$). Find the intersections of $f^{(3)}(x)$ with $y = x$ and identify the three stable fixed points. What are the slopes of $f^{(3)}(x)$ at these points? Then decrease r to $r = 0.957107$, the (approximate) value of r below which the system is chaotic. Draw the line $y = x$ and determine the number of intersections with $f^{(3)}(x)$. Note that at this value of r, the curve $y = f^{(3)}(x)$ is tangent to the diagonal line at the three stable fixed points. For this reason, this type of transition is called a *tangent bifurcation*. Note that there is also an unstable point at $x \approx 0.76$.

(c) Plot $x_{n+1} = f^{(3)}(x_n)$ versus n for $r = 0.9571$, a value of r just below the onset of period 3 behavior. How would you describe the behavior of the trajectory? This type of chaotic motion is an example of *intermittency*, that is, nearly periodic behavior interrupted by occasional irregular bursts.

(d) To understand the mechanism for the intermittent behavior, we need to "zoom in" on the values of x near the stable fixed points that you found in part (c). To do so change the arguments of the `setPreferredMinMax` method. You will see a narrow channel between the diagonal line $y = x$ and the plot of $f^{(3)}(x)$ near each fixed point. The trajectory can require many iterations to squeeze through the channel, and we see apparent period 3 behavior during this time. Eventually, the trajectory escapes from the channel and bounces around until it is again enters a channel at some unpredictable later time. ■

Table 6.1 Values of the control parameter $r = b_k$ for the onset of the kth bifurcation. Six decimal places are shown.

k	b_k
1	0.750 000
2	0.862 372
3	0.886 023
4	0.891 102
5	0.892 190
6	0.892 423
7	0.892 473
8	0.892 484

6.4 ■ UNIVERSAL PROPERTIES AND SELF-SIMILARITY

In Sections 6.2 and 6.3 we found that the trajectory of the logistic map has remarkable properties as a function of the control parameter r. In particular, we found a sequence of period doublings accumulating in a chaotic trajectory of infinite period at $r = r_\infty$. For most values of $r > r_\infty$, the trajectory is very sensitive to the initial conditions. We also found "windows" of period 3, 6, 12, ... embedded in the range of chaotic behavior. How typical is this type of behavior? In the following, we will find further numerical evidence that the general behavior of the logistic map is independent of the details of the form (6.5) of $f(x)$.

You might have noticed that the range of r between successive bifurcations becomes smaller as the period increases (see Table 6.1). For example, $b_2 - b_1 = 0.112398$, $b_3 - b_2 = 0.023624$, and $b_4 - b_3 = 0.00508$. A good guess is that the decrease in $b_k - b_{k-1}$ is geometric; that is, the ratio $(b_k - b_{k-1})/(b_{k+1} - b_k)$ is a constant. You can check that this ratio is not exactly constant, but converges to a constant with increasing k. This behavior suggests that the sequence of values of b_k has a limit and follows a geometrical progression:

$$b_k \approx r_\infty - C\delta^{-k}, \tag{6.8}$$

where δ is known as the *Feigenbaum number* and C is a constant. From (6.8) it is easy to show that δ is given by the ratio

$$\delta = \lim_{k\to\infty} \frac{b_k - b_{k-1}}{b_{k+1} - b_k}. \tag{6.9}$$

Problem 6.6 Estimation of the Feigenbaum constant

(a) Derive the relation (6.9) given (6.8). Plot $\delta_k = (b_k - b_{k-1})/(b_{k+1} - b_k)$ versus k using the values of b_k in Table 6.1 and determine the value of δ. Is the number of decimal places given in Table 6.1 for b_k sufficient for all the values of k shown? The best numerical determination of δ is

$$\delta = 4.669\,201\,609\,102\,991\ldots. \tag{6.10}$$