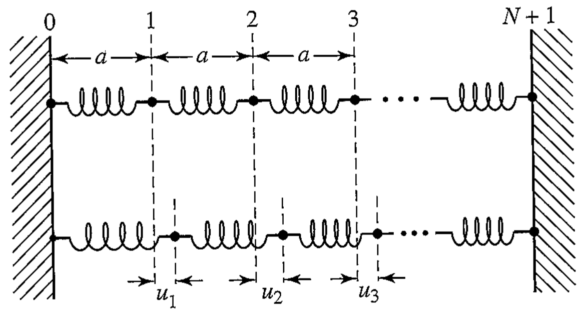


**Figure 9.1** A one-dimensional chain of $N$ particles of mass $m$ coupled by massless springs with force constant $k$. The first and last particles (0 and $N+1$) are attached to fixed walls. The top chain shows the oscillators in equilibrium. The bottom chain shows the oscillators displaced from equilibrium.

$$u_j(t) = u_j \cos \omega t, \tag{9.2}$$

where $u_j$ is the amplitude of the displacement of the $j$th particle. If we substitute the form (9.2) into (9.1), we obtain

$$-\omega^2 u_j = -\frac{k}{m}[2u_j - u_{j+1} - u_{j-1}]. \tag{9.3}$$

We next assume that the amplitude $u_j$ depends sinusoidally on the distance $ja$:

$$u_j = C \sin qja, \tag{9.4}$$

where the constants $q$ and $C$ will be determined. If we substitute (9.4) into (9.3), we find the following condition for $\omega$:

$$-\omega^2 \sin qja = -\frac{k}{m}[2\sin qja - \sin q(j-1)a - \sin q(j+1)a]. \tag{9.5}$$

We write $\sin q(j \pm 1)a = \sin qja \cos qa \pm \cos qja \sin qa$ and find that (9.4) is a solution if

$$\omega^2 = 2\frac{k}{m}(1 - \cos qa). \tag{9.6}$$

We need to find the values of the wavenumber $q$ that satisfy the boundary conditions $u_0 = 0$ and $u_{N+1} = 0$. The former condition is automatically satisfied by assuming a sine instead of a cosine solution in (9.4). The latter boundary condition implies that

$$q = q_n = \frac{\pi n}{a(N+1)} \quad \text{(fixed boundary conditions)}, \tag{9.7}$$

where $n = 1, \ldots, N$. The corresponding possible values of the wavelength $\lambda$ are related to $q$ by $q = 2\pi/\lambda$, and the corresponding values of the angular frequencies are given by

$$\omega_n{}^2 = 2\frac{k}{m}[1 - \cos q_n a] = 4\frac{k}{m}\sin^2\frac{q_n a}{2}, \tag{9.8}$$



or

$$\omega_n = 2\sqrt{\frac{k}{m}}\sin\frac{q_n a}{2}. \tag{9.9}$$

The relation (9.9) between $\omega_n$ and $q_n$ is known as a *dispersion relation.*

A particular value of the integer $n$ corresponds to the $n$th normal mode. We write the (time-independent) normal mode solutions as

$$u_{j,n} = C \sin q_n ja. \tag{9.10}$$

The linear nature of the equation of motion (9.1) implies that the time dependence of the displacement of the $j$th particle can be written as a superposition of normal modes:

$$u_j(t) = C\sum_{n=1}^{N}(A_n \cos \omega_n t + B_n \sin \omega_n t)\sin q_n ja. \tag{9.11}$$

The coefficients $A_n$ and $B_n$ are determined by the initial conditions:

$$u_j(t=0) = C\sum_{n=1}^{N} A_n \sin q_n ja \tag{9.12a}$$

$$v_j(t=0) = C\sum_{n=1}^{N} \omega_n B_n \sin q_n ja. \tag{9.12b}$$

To solve (9.12) for $A_n$ and $B_n$, we note that the normal mode solutions $u_{j,n}$ are orthogonal; that is, they satisfy the condition

$$\sum_{j=1}^{N} u_{j,n} u_{j,m} \propto \delta_{n,m}. \tag{9.13}$$

The Kronecker $\delta$ symbol $\delta_{n,m} = 1$ if $n = m$ and is zero otherwise. It is convenient to normalize the $u_{j,n}$ so that they are orthonormal, that is,

$$\sum_{j=1}^{N} u_{j,n} u_{j,m} = \delta_{n,m}. \tag{9.14}$$

It is easy to show that the choice, $C = 1/\sqrt{(N+1)/2}$, in (9.4) and (9.10) insures that (9.14) is satisfied.

We now use the orthonormality condition (9.14) to determine the $A_n$ and $B_n$ coefficients. If we multiply both sides of (9.12) by $C \sin q_m ja$, sum over $j$, and use the orthogonality condition (9.14), we obtain

$$A_n = C\sum_{j=1}^{N} u_j(0)\sin q_n ja \tag{9.15a}$$

$$B_n = C\sum_{j=1}^{N} (v_j(0)/\omega_n)\sin q_n ja. \tag{9.15b}$$