Problem 7.12 Synchronized random walks

(a) Randomly place two walkers on a one-dimensional lattice of L sites, so that both walkers are not at the same site. At each time step randomly choose whether the walkers move to the left or to the right. Both walkers move in the same direction. If a walker cannot move in the chosen direction because it is at a boundary, then this walker remains at the same site for this time step. A trial ends when both walkers are at the same site. Write a program to determine the mean time and the mean square fluctuations of the time for two walkers to reach the same site. This model is relevant to a method of doing cryptography using neural networks (see Rutter et al.).

(b) Change your program so that you use biased random walkers for which $p \neq q$. How does this change affect your results? ∎

Problem 7.13 Random walk on a continuum

One of the first continuum models of a random walk was proposed by Rayleigh in 1919. In this model the length a of each step is a random variable and the direction of each step is uniformly random. In this case the variable of interest is R, the distance of the walker from the origin after N steps. The model is known as the freely jointed chain in polymer physics (see Section 7.7) in which case R is the end-to-end distance of the polymer. For simplicity, we first consider a walker in two dimensions with steps of equal (unit) length at a random angle.

(a) Write a Monte Carlo program to compute $\langle R\rangle$ and determine its dependence on N.

(b) Because R is a continuous variable, we need to compute $p_N(R)\Delta R$, the probability that R is between R and $R+\Delta R$ after N steps. The quantity $p_N(R)$ is the *probability density*. Because the area of the ring between R and $R+\Delta R$ is $\pi(R+\Delta R)^2 - \pi R^2 = 2\pi R\Delta R + \pi(\Delta R)^2 \approx 2\pi R\Delta R$, we see that $p_N(R)\Delta R$ is proportional to $R\Delta R$. Verify that for sufficiently large N, $p_N(R)\Delta R$ has the form

$$p_N(R)\Delta R \propto 2\pi R\Delta R\, e^{-(R-\langle R\rangle)^2/2\Delta R^2}, \tag{7.16}$$

where $\Delta R^2 = \langle R^2\rangle - \langle R\rangle^2$. ∎

Problem 7.14 Random walks with steps of variable length

(a) Consider a random walk in one dimension with jumps of all lengths. The probability that the length of a single step is between a and $a+\Delta a$ is $f(a)\Delta a$, where $f(a)$ is the probability density. If the form of $f(a)$ is given by $f(a) = Ce^{-a}$ for $a > 0$ with the normalization condition $\int_0^\infty p(a)\,da = 1$, the code needed to generate step lengths according to this probability density is given by (see Section 11.5)

```
stepLength = -Math.log(1 - Math.random());
```

Modify `Walker` and `WalkerApp` to simulate walks of variable length with this probability density. Note that the bin width Δa is one of the input parameters. Consider $N \geq 100$ and visualize the motion of the walker. Generate many walks of N steps and determine $p(x)\Delta x$, the probability that the displacement is between x and $x+\Delta x$ after N steps. Plot $p(x)$ versus x and confirm that the form of $p(x)$ is consistent with a Gaussian distribution.

(b) Assume that the probability density $f(a)$ is given by $f(a) = C/a^2$ for $a \geq 1$. Determine the normalization constant C using the condition $C\int_1^\infty a^{-2}\,da = 1$. In this case we will learn in Section 11.5 that the statement

```
stepLength = 1.0/(1.0 - Math.random());
```

generates values of a according to this form of $f(a)$. Do a Monte Carlo simulation as in part (a) and determine $p(x)\Delta x$. Is the form of $p(x)$ a Gaussian? This type of random walk, for which $f(a)$ decreases as a power law $a^{-1-\alpha}$, is known as a *Levy flight* for $\alpha \leq 2$. ∎

Problem 7.15 Exploring the central limit theorem

Consider a continuous random variable x with probability density $f(x)$. That is, $f(x)\Delta x$ is the probability that x has a value between x and $x+\Delta x$. The mth *moment* of $f(x)$ is defined as

$$\langle x^m\rangle = \int x^m f(x)\,dx. \tag{7.17}$$

The mean value $\langle x\rangle$ is given by (7.17) with $m = 1$. The *variance* σ_x^2 of $f(x)$ is defined as

$$\sigma_x^2 = \langle x^2\rangle - \langle x\rangle^2. \tag{7.18}$$

Consider the sum y_n corresponding to the average of n values of x:

$$y = y_n = \frac{1}{n}(x_1 + x_2 + \cdots + x_n). \tag{7.19}$$

Suppose that we make many measurements of y. We know that the values of y will not be identical but will be distributed according to a probability density $p(y)$, where $p(y)\Delta y$ is the probability that the measured value of y is in the range y to $y+\Delta y$. The main quantities of interest are $\langle y\rangle$, $p(y)$, and an estimate of the probable variability of y in a series of measurements.

(a) Suppose that $f(x)$ is uniform in the interval $[-1, 1]$. Calculate $\langle x\rangle$, $\langle x^2\rangle$, and σ_x analytically.

(b) Write a program to make a sufficient number of measurements of y and determine $\langle y\rangle$ and $p(y)\Delta y$. Use the `HistogramFrame` class to determine and plot $p(y)\Delta y$. Choose at least 10^4 measurements of y for $n = 4, 16, 32$, and 64. What is the qualitative form of $p(y)$? Does the qualitative form of $p(y)$ change as the number of measurements of y is increased for a given value of n? Does the qualitative form of $p(y)$ change as n is increased?

(c) Each value of y can be considered to be a measurement. How much does the value of y vary (on the average) from one measurement to another? Make a rough estimate of this variability by comparing several measurements of y for a given value of n. Increase n by a factor of four and estimate the variability of y again. Does the variability from one measurement to another decrease (on the average) as n is increased?