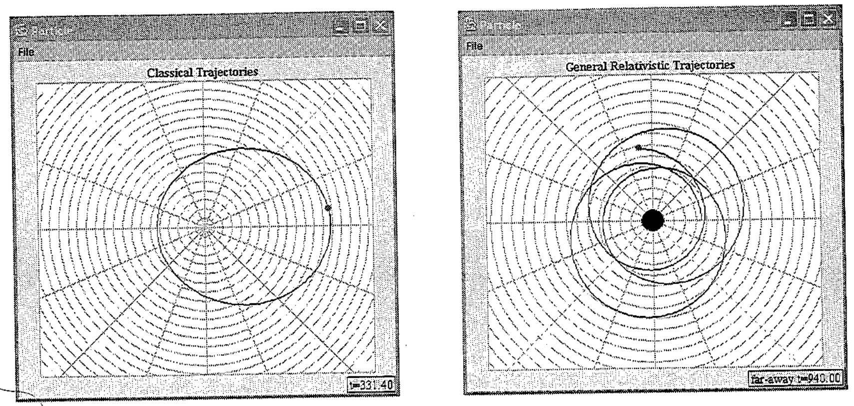

(a) Classical. (b) General relativistic.

Figure 18.2 Comparison of classical and general relativistic particle trajectories in the vicinity of a spherically symmetric mass.

Exercise 18.6 Classical trajectories

Modify the `PlanetApp` program introduced in Chapter 5 so that the classical trajectory of a particle is calculated using polar variables rather than Cartesian variables. Use (18.10)–(18.13) to determine the initial state and compare your results with those of the `PlanetApp` program. ■

The Open Source Physics plotting panel contains an axis object that displays a Cartesian coordinate grid by default. This grid can be replaced by a polar grid by using the method `setPolar` (see Figure 18.2):

```
plotFrame.setPolar("Trajectory",1.0);
```

The first parameter is the plot's title and the second parameter is the radial grid separation. The new plotting panel also displays the polar coordinates in the bottom left when the mouse is clicked or dragged.

Exercise 18.7 Polar coordinates

Modify Exercise 18.6 so that polar coordinate values are displayed when the mouse is click-dragged within the display. ■

18.4 ■ BLACK HOLES AND SCHWARZSCHILD COORDINATES

Our goal is to compute the worldlines (trajectories in spacetime) of particles and light in the vicinity of spherically symmetric gravitational objects. Readers should consult the classic text *Exploring Black Holes* by Edwin Taylor and John Wheeler for a more complete discussion of the physics near objects that have undergone gravitational collapse. Such an object is known as a *black hole* because light cannot escape from its vicinity. We will calculate the general relativistic trajectories of particles and light near a spherically symmetric gravitational mass. Because physical space is non-Euclidian, a two-dimensional plot of these trajectories will be distorted. Unlike classical orbits, the general relativistic orbits appear very different when seen by a viewer in the real world. We must calculate the trajectories of multiple light rays to construct the view as seen by a single observer.

Because time is incorporated as a fourth dimension and because space is curved, a general relativistic coordinate system centered on a spherically symmetric mass is more complicated than a three-dimensional Euclidean coordinate system. The azimuthal angle ϕ can still be defined as the ratio of the arc length to the circumference on an imaginary circle because the spherically symmetric gravitational mass M is located at the origin. However, the radial coordinate is not defined as the physical distance from the center. Rather, it is calculated using a path that circumnavigates the central mass:

$$r = \text{circumference}/(2\pi). \tag{18.18}$$

The time coordinate is defined using a wristwatch located far from the center of attraction. Note the *nonlocal* character of the (r, ϕ, t) spacetime coordinates. The wristwatch worn by the surveyor circumnavigating the mass used to measure r is not the time used to record events at that value of r.

This (r, ϕ, t) spacetime coordinate system is known as *Schwarzschild coordinates* and is a universal bookkeeping device that enables us to translate observations from one reference frame to another. The Schwarzschild coordinates give rise to a metric, known as the Schwarzschild metric, that enables us to calculate the four-dimensional distance between adjacent spacetime events. This metric is given by

$$d\sigma^2 = -d\tau^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\phi^2, \tag{18.19}$$

where t, r, and ϕ refer to the faraway time, the radial coordinate, and the azimuthal coordinate, respectively. Because we associate distance with a positive number, it is common to use $d\sigma^2$ when the right-hand side of (18.19) is positive and to use $d\tau^2$ when the right-hand side of (18.19) is negative. As in special relativity, these two forms are referred to as the space-like form and the time-like form of the metric, respectively. Note that both time and distance have units of length in (18.19). The speed of light c is the conversion factor

$$t_{\text{meters}} = c t_{\text{seconds}}. \tag{18.20}$$

Mass also has units of meters, and the conversion factor to kilograms is given in terms of the speed of light and Newton's gravitational constant G:

$$M = \frac{G}{c^2} M_{\text{kg}}. \tag{18.21}$$

If we freeze time so that $dt = 0$, then the Schwarzschild metric predicts that two simultaneous events far from the central mass are separated by the Euclidian metric in polar coordinates,

$$d\sigma^2 = dr^2 + r^2\, d\phi^2. \tag{18.22}$$

Strange things happen if two events are near the gravitational mass. The separation (known as the proper distance) between two adjacent events becomes

$$d\sigma^2 = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\phi^2. \tag{18.23}$$