Housekeeping methods such as `reset` and `initialize` are similar to methods in other simulations and are not shown.

The class `Lorenz` draws the attractor in the three-dimensional (x, y, z) space defined by (6.33). The state of the system is shown as a red ball in this 3D space, and the state's trajectory is shown as a trail. An easy way to show the time evolution is to extend the 3D `Group` class and create the ball and the trail inside the group. When points are added to the group, the trail is extended and the position of the ball is set. The `Lorenz` class imports `org.opensourcephysics.display3d.simple3d.*`. The ball and trail are then instantiated and added to the group as follows:

```
public class Lorenz extends Group implements ODE {
   ElementEllipsoid ball = new ElementEllipsoid();
   ElementTrail trail = new ElementTrail();
   addElement(trail);      // adds trace to Lorenz group
   addElement(ball);       // adds ball to Lorenz group
   ...
}
```

The properties of the `ball` and `trail` objects are set by

```
// sets size of ball in world coordinates
ball.setSizeXYZ(1, 1, 1);
ball.getStyle().setFillColor(java.awt.Color.RED);
```

To plot each part of the trajectory through state space, we use `trail.addPoint(x,y,z)` to add to the trail and `ball.setXYZ(x,y,z)` to show the current state. The user can project onto two dimensions using the frame's menu or rotate the three-dimensional plot using the mouse because these capabilities are built into the frame. The `getRate` and `getState` methods model (6.33) by implementing the `ODE` interface.

Problem 6.16 The Lorenz model

(a) Use a Runge–Kutta algorithm such as RK4 or RK45 (see Appendix 3A) to obtain a numerical solution of the Lorenz equations (6.33). Generate three-dimensional plots using `Display3DFrame`. Explore the basin of the attractor with $\sigma = 10$, $b = 8/3$, and $r = 28$.

(b) Determine qualitatively the sensitivity to initial conditions. Start two points very close to each other and watch their trajectories for approximately 10^4 time steps.

(c) Let z_m denote the value of z where z is a relative maximum for the mth time. You can determine the value of z_m by finding the average of the two values of z when the right-hand side of (6.33) changes sign. Plot z_{m+1} versus z_m and describe what you find. This procedure is one way that a continuous system can be mapped onto a discrete map. What is the slope of the z_{m+1} versus z_m curve? Is its magnitude always greater than unity? If so, then this behavior is an indication of chaos. Why? ■

The application of the Lorenz equations to weather prediction has led to a popular metaphor known as the *butterfly effect*. This metaphor is made even more meaningful by inspection of Figure 6.10. The "butterfly effect" is often ascribed to Lorenz (see Hilborn). In a 1963 paper he remarked that:

> One meteorologist remarked that if the theory were correct, one flap of a seagull's wings would be enough to alter the course of the weather forever.

By 1972 the seagull had evolved into the more poetic butterfly and the title of his talk was "Predictability: Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"

6.8 ■ FORCED DAMPED PENDULUM

We now consider the dynamics of nonlinear systems described by classical mechanics. The general problem in classical mechanics is the determination of the positions and velocities of a system of particles subjected to certain forces. For example, we considered in Chapter 5 the celestial two-body problem and were able to predict the motion at any time. We will find that we cannot make long-time predictions for the trajectories of nonlinear classical systems when these systems exhibit chaos.

A familiar example of a nonlinear mechanical system is the simple pendulum (see Chapter 3). To make its dynamics more interesting, we assume that there is a linear damping term present and that the pivot is forced to move vertically up and down. Newton's second law for this system is (cf. McLaughlin or Percival and Richards)

$$\frac{d^2\theta}{dt^2} = -\gamma\frac{d\theta}{dt} - [\omega_0{}^2 + 2A\cos\omega t]\sin\theta, \tag{6.34}$$

where θ is the angle the pendulum makes with the vertical axis, γ is the damping coefficient, $\omega_0{}^2 = g/L$ is the natural frequency of the pendulum, and ω and A are the frequency and amplitude of the external force. The effect of the vertical acceleration of the pivot is equivalent to a time-dependent gravitational field, because we can write the total vertical force due to gravity $-mg$ plus the pivot motion $f(t)$ as $-mg(t)$ where $g(t) \equiv g - f(t)/m$.

How do we expect the driven, damped simple pendulum to behave? Because there is damping present, we expect that if there is no external force, the pendulum would come to rest. That is, $(x = 0, v = 0)$ is a stable attractor. As A is increased from zero, this attractor remains stable for sufficiently small A. At a value of A equal to A_c, this attractor becomes unstable. How does the driven nonlinear oscillator behave as we increase A?

It is difficult to determine whether the pendulum has some kind of underlying periodic behavior by plotting only its position or even plotting its trajectory in phase space. We expect that if it does, the period will be related to the period of the external time-dependent force. Thus, we analyze the motion by plotting a point in phase space after every cycle of the external force. Such a phase space plot is called a *Poincaré map*. Hence, we will plot $d\theta/dt$ versus θ for values of t equal to nT for n equal to $1, 2, 3, \ldots$. If the system has a period T, then the Poincaré map consists of a single point. If the period of the system is nT, there will be n points.

`PoincareApp` uses the fourth-order Runge–Kutta algorithm to compute $\theta(t)$ and the angular velocity $d\theta(t)/dt$ for the pendulum described by (6.34). This equation is modeled in the `DampedDrivenPendulum` class, but is not shown here because it is similar to other ODE implementations. A phase diagram for $d\theta(t)/dt$ versus $\theta(t)$ is shown in one frame. In the other frame, the Poincaré map is represented by drawing a small box at the point $(\theta, d\theta/dt)$ at time $t = nT$. If the system has period 1, that is, if the same values of $(\theta, d\theta/dt)$ are drawn at $t = nT$, we would see only one box; otherwise, we would see several boxes. Because the first few values of $(\theta, d\theta/dt)$ show the transient behavior, it is desirable to clear the display and draw a new Poincaré map without changing A, θ, or $d\theta/dt$.