(e) Choose an arbitrary initial position for the particle in a stadium with $L = 1$ and a small hole as in part (d). Choose at least 5000 values of the initial value p_{x0} uniformly distributed between 0 and 1. Choose p_{y0} so that $|\mathbf{p}| = 1$. Plot the escape time versus p_{x0} and describe the visual pattern of the trajectories. Then choose 5000 values of p_{x0} in a smaller interval centered about the value of p_{x0} for which the escape time was greatest. Plot these values of the escape time versus p_{x0}. Do you see any evidence of self-similarity?

(f) Repeat steps (a)–(e) for the Sinai billiard geometry. ■

Project 6.27 The circle map and mode locking

The driven damped pendulum can be approximated by a one-dimensional difference equation for a range of amplitudes and frequencies of the driving force. This difference equation is known as the *circle map* and is given by

$$\theta_{n+1} = \left(\theta_n + \Omega - \frac{K}{2\pi}\sin 2\pi\theta_n\right) \quad \text{(modulo 1)}. \tag{6.62}$$

The variable θ represents an angle, and Ω represents a frequency ratio, the ratio of the natural frequency of the pendulum to the frequency of the periodic driving force. The parameter K is a measure of the strength of the nonlinear coupling of the pendulum to the external force. An important quantity is the winding number which is defined as

$$W = \lim_{m\to\infty} \frac{1}{m}\sum_{n=0}^{m-1} \Delta\theta_n, \tag{6.63}$$

where $\Delta\theta_n = \Omega - (K/2\pi)\sin 2\pi\theta_n$.

(a) Consider the linear case $K = 0$. Choose $\Omega = 0.4$ and $\theta_0 = 0.2$ and determine W. Verify that if Ω is a ratio of two integers, then $W = \Omega$ and the trajectory is periodic. What is the value of W if $\Omega = \sqrt{2}/2$, an irrational number? Verify that $W = \Omega$ and that the trajectory comes arbitrarily close to any particular value of θ. Does θ_n ever return exactly to its initial value? This type of behavior of the trajectory is termed *quasiperiodic*.

(b) For $K > 0$, we will find that $W \neq \Omega$ and "locks" into rational frequency ratios for a range of values of K and Ω. This type of behavior is called *mode locking*. For $K < 1$, the trajectory is either periodic or quasiperiodic. Determine the value of W for $K = 1/2$ and values of Ω in the range $0 < \Omega \leq 1$. The widths in Ω of the various mode-locked regions where W is fixed increase with K. Consider other values of K, and draw a diagram in the $K\Omega$-plane ($0 \leq K, \Omega \leq 1$) so that those areas corresponding to frequency locking are shaded. These shaded regions are called *Arnold tongues*.

(c) For $K = 1$, all trajectories are frequency-locked periodic trajectories. Fix K at $K = 1$ and determine the dependence of W on Ω. The plot of W versus Ω for $K = 1$ is called the *Devil's staircase*. ■

Project 6.28 Chaotic scattering

In Chapter 5 we discussed the classical scattering of particles off a fixed target, and found that the differential cross section for a variety of interactions is a smoothly varying function

of the scattering angle. That is, a small change in the impact parameter b leads to a small change in the scattering angle θ. Here we consider examples where small changes in b lead to large changes in θ. Such a phenomenon is called *chaotic scattering* because of the sensitivity to initial conditions that is characteristic of chaos. The study of chaotic scattering is relevant to the design of electronic nanostructures, because many experimental structures exhibit this type of scattering.

A typical scattering model consists of a target composed of a group of fixed hard disks and a scatterer consisting of a point particle. The goal is to compute the path of the scatterer as it bounces off the disks and measure θ and the time of flight as a function of the impact parameter b. If a particle bounces inside the target region before leaving, the time of flight can be very long. There are even some trajectories for which the particle never leaves the target region.

Because it is difficult to monitor a trajectory that bounces back and forth between the hard disks, we consider instead a two-dimensional map that contains the key features of chaotic scattering (see Yalcinkaya and Lai for further discussion). The map is given by

$$x_{n+1} = a\left[x_n - \frac{1}{4}(x_n + y_n)^2\right] \tag{6.64a}$$

$$y_{n+1} = \frac{1}{a}\left[y_n + \frac{1}{4}(x_n + y_n)^2\right], \tag{6.64b}$$

where a is a parameter. The target region is centered at the origin. In an actual scattering experiment, the relation between (x_{n+1}, y_{n+1}) and (x_n, y_n) would be much more complicated, but the map (6.64) captures most of the important features of realistic chaotic scattering experiments. The iteration number n is analogous to the number of collisions of the scattered particle off the disks. When x_n or y_n is significantly different from zero, the scatterer has left the target region.

(a) Write a program to iterate the map (6.64). Let $a = 8.0$ and $y_0 = -0.3$. Choose 10^4 initial values of x_0 uniformly distributed in the interval $0 < x_0 < 0.1$. Determine the time $T(x_0)$, the number of iterations for which $x_n \leq -5.0$. After this time, x_n rapidly moves to $-\infty$. Plot $T(x_0)$ versus x_0. Then choose 10^4 initial values in a smaller interval centered about a value of x_0 for which $T(x_0) > 7$. Plot these values of $T(x_0)$ versus x_0. Do you see any evidence of self-similarity?

(b) A trajectory is said to be *uncertain* if a small change ϵ in x_0 leads to a change in $T(x_0)$. We expect that the number of uncertain trajectories N will depend on a power of ϵ, that is, $N \sim \epsilon^{\alpha}$. Determine $N(\epsilon)$ for $\epsilon = 10^{-p}$ with $p = 2$ to 7 using the values of x_0 in part (a). Then determine the uncertainty dimension $1 - \alpha$ from a log-log plot of N versus ϵ. Repeat these measurements for other values of a. Does α depend on a?

(c) Choose 4×10^4 initial conditions in the same interval as in part (a) and determine the number of trajectories $S(n)$ that have not yet reached $x_n = -5$ as a function of the number of iterations n. Plot $\ln S(n)$ versus n and determine if the decay is exponential. It is possible to obtain algebraic decay for values of a less than approximately 6.5.

(d) Let $a = 4.1$ and choose 100 initial conditions uniformly distributed in the region $1.0 < x_0 < 1.05$ and $0.60 < y_0 < 0.65$. Are there any trajectories that are periodic and hence have infinite escape times? Due to the accumulation of roundoff error, it