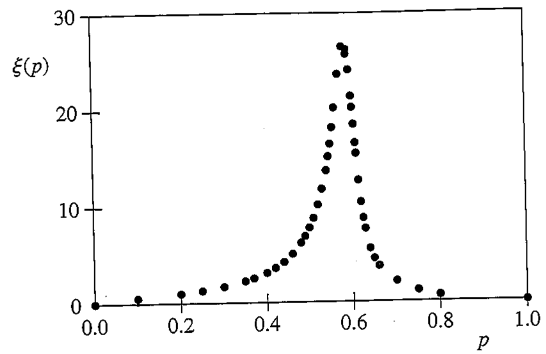

Figure 12.10 The qualitative p-dependence of the connectedness length $\xi(p)$ for a square lattice with $L = 128$. The results were averaged over approximately 2000–6000 configurations for each value of p. Note that ξ is finite for a finite lattice.

We have found that the essential physics near the percolation threshold is associated with the existence of large clusters. For example, for $p \neq p_c$, we found in Problem 12.7 that n_s decays rapidly with s. However for $p = p_c$, the s-dependence of n_s is qualitatively different, and n_s decreases much more slowly. This different behavior of n_s at $p = p_c$ is due to the presence of clusters of all length scales, for example, the "infinite" spanning cluster and the finite clusters of all sizes. In Figure 12.10 we show the mean connectedness length $\xi(p)$ for a lattice with $L = 128$. We see that ξ is finite, and an increasing function of p for $p < p_c$, and a decreasing function of p for $p > p_c$. Moreover, we know that $\xi(p = p_c)$ is approximately equal to L and hence diverges as $L \to \infty$. These qualitative considerations lead us to conjecture that in the limit $L \to \infty$, $\xi(p)$ grows rapidly in the *critical region*, $|p - p_c| \ll 1$.

We can describe the quantitative behavior of $\xi(p)$ for p near p_c by introducing the *critical exponent* ν defined by the relation

$$\xi(p) \sim |p - p_c|^{-\nu}. \tag{12.10}$$

Of course, there is no *a priori* reason why the divergence of $\xi(p)$ can be characterized by a simple power law. Note that the exponent ν is assumed to be the same above and below p_c.

How do the other quantities that we have considered behave in the critical region in the limit $L \to \infty$? According to the definition (12.2) of P_∞, $P_\infty = 0$ for $p < p_c$ and is an increasing function of p for $p > p_c$. We conjecture that in the critical region, the increase of P_∞ with increasing p is characterized by the exponent β defined by the relation

$$P_\infty(p) \sim (p - p_c)^{\beta}. \tag{12.11}$$

Note that P_∞ is assumed to approach zero continuously as p approaches p_c from above; that is, the percolation transition is an example of a *continuous* phase transition. In the language of critical phenomena, P_∞ is an example of an *order parameter*; that is, P_∞ is nonzero in the ordered phase $p > p_c$ and zero in the disordered phase $p < p_c$. We will see that at $p = p_c$, the spanning cluster is a *fractal* and approaches zero density as the size of the system becomes larger.

Table 12.1 Several of the critical exponents for the percolation and magnetism phase transitions in $d = 2$ and $d = 3$ dimensions. Ratios of integers correspond to known exact results. The critical exponents for the Ising model are discussed in Chapter 15.

Quantity	Functional Form	Exponent	$d = 2$	$d = 3$
Percolation				
order parameter	$P_\infty \sim (p - p_c)^{\beta}$	β	5/36	0.41
mean size of finite clusters	$S(p) \sim \lvert p - p_c\rvert^{-\gamma}$	γ	43/18	1.80
connectedness length	$\xi(p) \sim \lvert p - p_c\rvert^{-\nu}$	ν	4/3	0.88
cluster numbers	$n_s \sim s^{-\tau}\ (p = p_c)$	τ	187/91	2.19
Ising model				
order parameter	$M(T) \sim (T_c - T)^{\beta}$	β	1/8	0.32
susceptibility	$\chi(T) \sim \lvert T - T_c\rvert^{-\gamma}$	γ	7/4	1.24
correlation length	$\xi(T) \sim \lvert T - T_c\rvert^{-\nu}$	ν	1	0.63

The mean number of sites in the finite clusters $S(p)$ also diverges in the critical region. Its critical behavior is written as

$$S(p) \sim |p - p_c|^{-\gamma}, \tag{12.12}$$

which defines the critical exponent γ. The common critical exponents for percolation are summarized in Table 12.1. The analogous critical exponents of a magnetic critical point are also shown.

Because we can simulate only finite lattices, a direct fit of the measured quantities ξ, P_∞, and $S(p)$ to their assumed critical behavior for an infinite lattice would not yield good estimates for the corresponding exponents ν, β, and γ (see Problem 12.8). The problem is that if p is close to p_c, the connectedness length of the largest cluster becomes comparable to L, and the nature of the clusters is affected by the finite size of the system. In contrast, for p far from p_c, $\xi(p)$ is small in comparison to L, and the measured values of ξ, and hence the values of other physical quantities, are not appreciably affected by the finite size of the lattice. Hence, for $p \ll p_c$ and $p \gg p_c$, the properties of the system are indistinguishable from the corresponding properties of a truly macroscopic system ($L \to \infty$). However, if p is close to p_c, $\xi(p)$ is comparable to L and the nature of the system differs from that of an infinite system. In particular, a finite lattice cannot exhibit a true phase transition characterized by divergent physical quantities. Instead, ξ reaches a finite maximum at $p = p_c(L)$.

The effects of the finite system size can be made more quantitative by the following argument. Consider, for example, the critical behavior (12.11) of P_∞. If $\xi \gg 1$ but is much less than L, the power law behavior given by (12.11) is expected to hold. However, if ξ is comparable to L, ξ cannot change appreciably and (12.11) is no longer applicable. This qualitative change in the behavior of P_∞ and other physical quantities occurs for

$$\xi(p) \sim L \sim |p - p_c|^{-\nu}. \tag{12.13}$$

We invert (12.13) and write

$$|p - p_c| \sim L^{-1/\nu}. \tag{12.14}$$