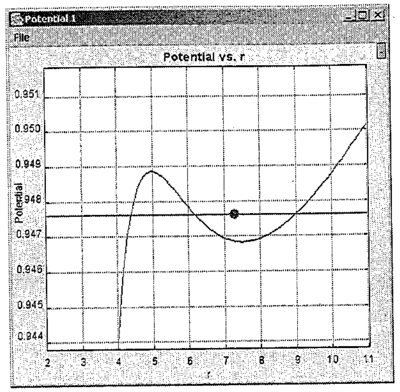

Figure 18.3 The effective potential $V(r)$ of a particle in the vicinity of a black hole.

In analogy with the classical effective potential function for a particle in a gravitational field, we use (18.35) to define a *relativistic effective potential* (see Figure 18.3):

$$\left(\frac{V(r)}{m}\right)^2 = \left(1 - \frac{2M}{r}\right)\left[1 + \left(\frac{\ell}{r}\right)^2\right]. \tag{18.36}$$

Exercise 18.15 Energy and angular momentum

Show that the energy and angular momentum are conserved for the orbits you observed in Exercise 18.11. ■

Exercise 18.16 Effective potential

Add a plot of the effective potential $V(r)$ to your program for Exercise 18.11. Add a horizontal line showing the energy per unit mass and place a red marker on this line showing the particle's radial position. Describe the effective potential and the motion of the marker when the orbit is circular, when the orbit precesses, and when the orbit plunges toward the event horizon. ■

*18.8 ■ THE KERR METRIC

Because almost all astronomical objects rotate, most black holes likely have angular momentum. The metric for a spinning black hole was derived by Kerr in 1964. For simplicity, we show the metric for particle motion in the equatorial plane. Note that this metric contains a new angular momentum parameter a:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma^2}{r}\,dt\,d\phi - \left(1 - \frac{2M}{r} + \frac{a^2}{r^2}\right)^{-1} dr^2 - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right) r^2\,d\phi^2. \tag{18.37}$$

Because there are two values at which the coefficient of dr^2 increases without limit, $r_h = M \pm \sqrt{M^2 - a^2}$, there are two horizons. We also see that the largest real value of a consistent with real values of r_h is $a = M$. This maximum value of a limits the angular momentum of a black hole. Because we are interested in maximizing the effect of rotation, we simplify (18.37) by letting the angular momentum parameter take on its maximum value. The metric for this extreme Kerr black hole is

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 + \frac{4Ma}{r}\,dt\,d\phi - \left(1 - \frac{M}{r}\right)^{-2} dr^2 - R^2\,d\phi^2, \tag{18.38}$$

where

$$R^2 \equiv r^2 + M^2 + \frac{2M^3}{r}. \tag{18.39}$$

We recast this metric as a Lagrangian and follow the derivation by Hanc and Tuleja and obtain the rate

$$\frac{dr}{dt} = \dot{r} \tag{18.40a}$$

$$\frac{d\dot{r}}{dt} = -\frac{(M-r)^2(M - 2M^2\dot{\phi} + M^3\dot{\phi}^2 - r^3\dot{\phi}^2)}{r^4} + \frac{2M^3 - 2M^4\dot{\phi} + 3Mr^2 - M^2r(1 + 6r\dot{\phi})}{r^2(M-r)^2}\dot{r}^2 \tag{18.40b}$$

$$\frac{d\phi}{dt} = \dot{\phi} \tag{18.40c}$$

$$\frac{d\dot{\phi}}{dt} = \frac{4M^3\dot{\phi} - 2M^4\dot{\phi}^2 + 6Mr^2\dot{\phi} - 2r^3\dot{\phi} - 2M^2(1 + 3r^2\dot{\phi}^2)}{r^2(M-r)^2}\dot{r} \tag{18.40d}$$

$$\frac{dt}{dt} = 1. \tag{18.40e}$$

Problem 18.17 Falling into a spinning black hole

(a) Write a program that plots the general relativistic trajectory of a particle near an extreme black hole using (18.40).
(b) Follow the trajectory of a particle that starts from rest far from the center of the extreme black hole. Describe the trajectory.
(c) A particle is thrown with an angular momentum opposite to the hole's spin starting at $r = 3M$. Write a program to simulate this situation and describe the particle's motion. ■

A space ship near a black hole must fire its rockets radially to keep from falling into a black hole. It has an angular momentum appropriate for that radius, so that the remote stars do not move overhead, and therefore it does not fire its rockets tangentially. However, if the space ship moves inward, it must fire its rockets tangentially or it will be swept sideways with respect to the remote stars. (The ship must only fire its rockets while moving inward.) This