charge and write a program to compute $V(z)$, and hence $E_z = -\partial V(z)/\partial z$ for points along the z-axis and perpendicular to the sheet. Take $L = 1$, $Q = 1$, and $p = 10$ for your initial calculations. Increase p until your results for $V(z)$ do not change significantly. Plot $V(z)$ and E_z as a function of z and compare their z-dependence to their infinite sheet counterparts. ■

***Problem 10.8 Electrostatic shielding**

We know that the (static) electric field is zero inside a conductor, all excess charges reside on the surface of the conductor, and the surface charge density is greatest at the points of greatest curvature. Although these properties are plausible, it is instructive to do a simulation to see how these properties follow from Coulomb's law. For simplicity, consider the conductor to be two-dimensional so that the potential energy is proportional to $\ln r$ rather than $1/r$ (see Problem 10.4). It is also convenient to choose the surface of the conductor to be an ellipse.

(a) If we are interested only in the final distribution of the charges and not in the dynamics of the system, we can use a Monte Carlo method. Our goal is to find the minimum energy configuration beginning with the N charges randomly placed within a conducting ellipse. One method is to choose a charge i at random and make a trial change in the position of the charge. The trial position should be no more than δ from the old position and still be within the ellipse. Choose $\delta \approx b/10$, where b is the semiminor axis of the ellipse. Compute the change in the total potential energy given by (in arbitrary units)

$$\Delta U = -\sum_j [\ln r_{ij}^{(\text{new})} - \ln r_{ij}^{(\text{old})}]. \tag{10.10}$$

The sum is over all charges in the system not including i. If $\Delta U > 0$, then reject the trial move; otherwise accept it. Repeat this procedure many times until very few trial moves are accepted. Write a program to implement this Monte Carlo algorithm. Run the simulation for $N \geq 20$ charges inside a circle and then repeat the simulation for an ellipse. How are the charges distributed in the (approximately) minimum energy distribution? Which parts of the ellipse have a higher charge density?

(b) Repeat part (a) for a two-dimensional conductor, but assume that the potential energy $U \sim 1/r$. Do the charges move to the surface?

(c) Is it sufficient that the interaction be repulsive for the results of parts (a) and (b) to hold?

(d) Repeat part (a) with the added condition that there is a fixed positive charge of magnitude $N/2$ located outside the ellipse. How does this fixed charge affect the charge distribution? Are the excess free charges still at the surface? Try different positions for the fixed charge.

(e) Repeat parts (a) and (b) for $N = 50$ charges located within an ellipsoid in three dimensions. ■

10.5 ■ NUMERICAL SOLUTIONS OF BOUNDARY VALUE PROBLEMS

In Section 10.1 we found the electric fields and potentials due to a fixed distribution of charges. Suppose that we do not know the positions of the charges, but instead know only

the potential on a set of boundaries surrounding a charge-free region. This information is sufficient to determine the potential $V(\mathbf{r})$ at any point within the charge-free region.

The direct method of solving for $V(x, y, z)$ is based on Laplace's equation which can be expressed in Cartesian coordinates as

$$\nabla^2 V(x, y, z) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{10.11}$$

The problem is to find the function $V(x, y, z)$ that satisfies (10.11) and the specified boundary conditions. This type of problem is an example of a *boundary value* problem. Because analytic methods for regions of arbitrary shape do not exist, the only general approach is to use numerical methods. Laplace's equation is not a new law of physics, but can be derived directly from (10.7) and the relation $\nabla \cdot \mathbf{E} = 0$ or indirectly from Coulomb's law in regions of space where there is no charge.

For simplicity, we consider only two-dimensional boundary value problems for $V(x, y)$. We use a finite difference method and divide space into a discrete grid of sites located at the coordinates (x, y). In Problem 10.9b, we show that in the absence of a charge at (x, y), the discrete form of Laplace's equation satisfies the relation

$$V(x, y) \approx \frac{1}{4}[V(x + \Delta x, y) + V(x - \Delta x, y) + V(x, y + \Delta y) + V(x, y - \Delta y)] \quad \text{(two dimensions)}, \tag{10.12}$$

where $V(x, y)$ is the value of the potential at the site (x, y). Equation (10.12) says that $V(x, y)$ is the average of the potential of its four nearest neighbor sites. This remarkable property of $V(x, y)$ can be derived by approximating the partial derivatives in (10.11) by finite differences (see Problem 10.9b).

In Problem 10.9a we verify (10.12) by calculating the potential due to a point charge at a point in space that we select and at the four nearest neighbors. As the form of (10.12) implies, the average of the potential at the four neighboring sites should equal the potential at the center site. We assume the form (10.9) for the potential $V(r)$ due to a point charge, a form that satisfies Laplace's equation for $r \neq 0$.

Problem 10.9 Verification of the difference equation for the potential

(a) Modify `PotentialFieldApp` to compare the computed potential at a point to the average of the potential at its four nearest neighbor sites. Choose reasonable values for the spacings Δx and Δy and consider a point that is not too close to the source charge. Do similar measurements for other points. Does the relative agreement with (10.12) depend on the distance of the point to the source charge? Choose smaller values of Δx and Δy and determine if your results are in better agreement with (10.12). Does it matter whether Δx and Δy have the same value?

(b) Derive the finite difference equation (10.12) for $V(x, y)$ using the second-order Taylor expansion:

$$V(x + \Delta x, y) = V(x, y) + \Delta x \frac{\partial V(x, y)}{\partial x} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 V(x, y)}{\partial x^2} + \cdots \tag{10.13}$$

$$V(x, y + \Delta y) = V(x, y) + \Delta y \frac{\partial V(x, y)}{\partial y} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 V(x, y)}{\partial y^2} + \cdots. \tag{10.14}$$