High efficiency means on the average more players are in the minority. We might think that the efficiency increases as the number of past outcomes increases, because then the players have more information to choose their strategy. However, you might be surprised!

(a) Write a program to simulate the minority game. For simplicity, give each player only two strategies chosen at random. Run your program for a memory m varying from 2 to about 12. A reasonable choice for N is 101, but for testing purposes choose $N = 11$. Each game should be run for at least 1000 iterations, and your results should be averaged over at least 10 independent runs for the same m, with different strategies for the players. Plot the average of σ versus m and describe the behavior for different values of N. Explain why there is a minimum in these plots.

(b) The results of the minority game scale unambiguously. Plot the average of σ^2/N versus $2^m/N$ for different values of N. You should find that your data fall on the same curve. What does 2^m represent? Discuss this scaling behavior and describe the behavior of the efficiency on either side of the minimum. Can you describe your results as a phase transition? Where is the ordered phase and where is the disordered phase?

(c) Plot the spread in the values of σ versus m. The spread can be taken to be the standard deviation of each game's value of σ over many games. Discuss the significance of your results. ■

Project 14.25 A cellular automaton for Burger's equation

In Section 14.6 we mentioned that the partial differential equation describing the flow of incompressible fluids, the Navier–Stokes equation, is very difficult to solve numerically. A one-dimensional approximation of the Navier–Stokes equation was given by Burgers, and is given by

$$\frac{\partial n}{\partial t} + c\frac{\partial}{\partial x}\left(n - \frac{n^2}{2}\right) = D\frac{\partial^2 n}{\partial x^2}, \tag{14.10}$$

where $n(x, t)$ corresponds to the velocity field at position x at time t, c is the linear advection (drift) coefficient, and D is a diffusion coefficient. Equation (14.10) is of general interest because it can be solved analytically and its solutions exhibit discontinuities (shock waves) depending on the values of the parameters and the initial conditions.

Boghosian and Levermore have proposed a cellular automaton that is equivalent to (14.10). The study of this cellular automaton raises many of the same issues as the lattice gas models of the incompressible Navier–Stokes equation considered in Section 14.6. Its study also illustrates the idea that many partial differential equations can be formulated as cellular automata.

We know that if all particles on the lattice move one lattice site to either the right or the left in one time step, then the density of the particles obeys the diffusion equation (see Appendix 7A),

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}, \tag{14.11}$$

where $D = (\Delta x)^2/2\Delta t$, Δx is the lattice spacing, and Δt is the time between successive steps of the random walk. If you add a bias so that the probability of a step to the right is

Table 14.2 Rules for the collision substep.

$b_1(i,t)$	$b_0(i,t)$	$\tilde{b}_1(i,t)$	$\tilde{b}_0(i,t)$
0	0	0	0
0	1	$(1-\alpha(i,t))/2$	$(1+\alpha(i,t))/2$
1	0	$(1-\alpha(i,t))/2$	$(1+\alpha(i,t))/2$
1	1	1	1

$(1 + \alpha)/2$ and the probability of a step to the left is $(1 - \alpha)/2$, the density of the walkers satisfies

$$\frac{\partial n}{\partial t} + c\frac{\partial n}{\partial x} = D\frac{\partial^2 n}{\partial x^2}, \tag{14.12}$$

where $c = \alpha\Delta x/\Delta t$. To incorporate the quadratic term, we add the rule that no two particles occupying the same site may be moving in the same direction. In this way the state of each site is specified by two bits. The right bit is 1 if a particle moving to the right is present and is 0 otherwise. Similarly, the left bit stores information about the presence of a particle moving to the left. Thus each site has four possible states labeled by the binary numbers 00, 01, 10, and 11.

In the first part of the step, the collision substep, the particles change their direction at random at their present lattice sites subject to the exclusion rule. In the second substep, the particles move to the neighboring lattice site in their new direction. We follow Boghosian and Levermore and denote the right (left) bit at lattice site i and time step t by $b_0(x, t)$ ($b_1(x, t)$). After the collision substep, the new states are $\tilde{b}_{0,1}(x, t)$ and are given in Table 14.2, where $\alpha(x, t) = \pm 1$ with mean α. The rules in Table 14.2 may be written in the form

$$\tilde{b}_0(x,t) = \frac{1+\alpha(x,t)}{2}b_0(x,t)|b_1(x,t) + \frac{1-\alpha(x,t)}{2}b_0(x,t)\,\&\,b_1(x,t) \tag{14.13a}$$

$$\tilde{b}_1(x,t) = \frac{1-\alpha(x,t)}{2}b_0(x,t)|b_1(x,t) + \frac{1+\alpha(x,t)}{2}b_0(x,t)\,\&\,b_1(x,t), \tag{14.13b}$$

where | is the inclusive *or* operator and & denotes the *and* operation on a pair of bits. In the advection substep, the particles move to the neighboring lattice site in their new direction. The rules for these moves are

$$\tilde{b}_0(x+1,t+1) = \tilde{b}_0(x,t) \tag{14.14a}$$

$$\tilde{b}_1(x-1,t+1) = \tilde{b}_0(x,t). \tag{14.14b}$$

We can combine these two substeps to arrive at the rule for one full time step of the cellular automaton:

$$b_0(x+1,t+1) = \frac{1+\alpha(x,t)}{2}b_0(x,t)|b_1(x,t) + \frac{1-\alpha(x,t)}{2}b_0(x,t)\,\&\,b_1(x,t) \tag{14.15a}$$

$$b_1(x-1,t+1) = \frac{1-\alpha(x,t)}{2}b_0(x,t)|b_1(x,t) + \frac{1+\alpha(x,t)}{2}b_0(x,t)\,\&\,b_1(x,t). \tag{14.15b}$$