is heard at a time. We define $Q = \omega_0/\Delta\omega$, where the width $\Delta\omega$ is the frequency interval between points on the resonance curve $I_{\max}(\omega)$ that are $\sqrt{2}/2$ of $I_{\max}$ at its maximum. Compute Q for the values of R, L, and C given in part (a). Change the value of R by 10% and compute the corresponding percentage change in Q. What is the corresponding change in Q if L or C is changed by 10%?

(c) Compute the time dependence of the voltage drops across each circuit element for approximately fifteen frequencies ranging from 1/10 to 10 times the resonant frequency. Plot the time dependence of the voltage drops.

(d) The ratio of the amplitude of the sinusoidal source voltage to the amplitude of the current is called the *impedance* Z of the circuit, that is, $Z = V_{\max}/I_{\max}$. This definition of Z is a generalization of the resistance that is defined by the relation $V = IR$ for direct current circuits. Use the plots of part (d) to determine $I_{\max}$ and $V_{\max}$ for different frequencies and verify that the impedance is given by

$$Z(\omega) = \sqrt{R^2 + (\omega L - 1/\omega C)^2}. \tag{4.25}$$

For what value of ω is Z a minimum? Note that the relation $V = IZ$ holds only for the maximum values of I and V and not for I and V at any time.

(e) Compute the phase difference ϕ_R between the voltage drop across the resistor and the voltage source. Consider $\omega \ll \omega_0$, $\omega = \omega_0$, and $\omega \gg \omega_0$. Does the current lead or lag the voltage in each case; that is, does the current reach a maxima before or after the voltage? Also compute the phase differences ϕ_L and ϕ_C and describe their dependence on ω. Do the relative phase differences between V_C, V_R, and V_L depend on ω?

(f) Compute the amplitude of the voltage drops across the inductor and the capacitor at the resonant frequency. How do these voltage drops compare to the voltage drop across the resistor and to the source voltage? Also compare the relative phases of V_C and V_L at resonance. Explain how an RLC circuit can be used to amplify the input voltage. ■

4.6 ■ ACCURACY AND STABILITY

Now that we have learned how to use numerical methods to find numerical solutions to simple first-order differential equations, we need to develop some practical guidelines to help us estimate the accuracy of the various methods. Because we have replaced a differential equation by a difference equation, our numerical solution is not identically equal to the true solution of the original differential equation, except for special cases. The discrepancy between the two solutions has two causes. One cause is that computers do not store numbers with infinite precision, but rather to a maximum number of digits that is hardware and software dependent. As we have seen, Java allows the programmer to distinguish between *floating point* numbers, that is, numbers with decimal points, and *integer* numbers. Arithmetic with numbers represented by integers is exact, but we cannot solve a differential equation using integer arithmetic. Arithmetic operations involving floating point numbers, such as addition and multiplication, introduce *roundoff error*. For example, if a computer only stored floating point numbers to two significant figures, the product 2.1 × 3.2 would be

stored as 6.7 rather than 6.72. The significance of roundoff errors is that they accumulate as the number of mathematical operations increases. Ideally, we should choose algorithms that do not significantly magnify the roundoff error; for example, we should avoid subtracting numbers that are nearly the same in magnitude.

The other source of the discrepancy between the true answer and the computed answer is the error associated with the choice of algorithm. This error is called the *truncation error*. A truncation error would exist even on an idealized computer that stored floating point numbers with infinite precision and hence had no roundoff error. Because the truncation error depends on the choice of algorithm and can be controlled by the programmer, you should be motivated to learn more about numerical analysis and the estimation of truncation errors. However, there is no general prescription for the best algorithm for obtaining numerical solutions of differential equations. We will find in later chapters that the various algorithms have advantages and disadvantages, and the appropriate selection depends on the nature of the solution, which you might not know in advance, and on your objectives. How accurate must the answer be? Over how large an interval do you need the solution? What kind of computer(s) are you using? How much computer time and personal time do you have?

In practice, we usually can determine the accuracy of a numerical solution by reducing the value of Δt until the numerical solution is unchanged at the desired level of accuracy. Of course, we have to be careful not to make Δt too small, because too many steps would be required and the computation time and roundoff error would increase.

In addition to accuracy, another important consideration is the *stability* of an algorithm. As discussed in Appendix 3A, it might happen that the numerical results are very good for short times, but diverge from the true solution for longer times. This divergence might occur if small errors in the algorithm are multiplied many times, causing the error to grow geometrically. Such an algorithm is said to be *unstable* for the particular problem. We consider the accuracy and the stability of the Euler algorithm in Problems 4.15 and 4.16.

Problem 4.15 Accuracy of the Euler algorithm

(a) Use the Euler algorithm to compute the numerical solution of $dy/dx = 2x$ with $y = 0$ at $x = 0$ and $\Delta x = 0.1, 0.05, 0.025, 0.01$, and 0.005. Make a table showing the difference between the exact solution and the numerical solution. Is the difference between these solutions a decreasing function of Δx? That is, if Δx is decreased by a factor of two, how does the difference change? Plot the difference as a function of Δx. If your points fall approximately on a straight line, then the difference is proportional to Δx (for $\Delta x \ll 1$). The numerical method is called nth order if the difference between the analytical solution and the numerical solution is proportional to $(\Delta x)^n$ for a fixed value of x. What is the order of the Euler algorithm?

(b) One way to determine the accuracy of a numerical solution is to repeat the calculation with a smaller step size and compare the results. If the two calculations agree to p decimal places, we can reasonably assume that the results are correct to p decimal places. What value of Δx is necessary for 0.1% accuracy at $x = 2$? What value of Δx is necessary for 0.1% accuracy at $x = 4$? ■

Problem 4.16 Stability of the Euler algorithm

(a) Consider the differential equation (4.23) with $Q = 0$ at $t = 0$. This equation represents the charging of a capacitor in an RC circuit with a constant applied voltage