where $p_1 \equiv p_x, q_1 \equiv x$, etc. This notation emphasizes that the p_i and the q_i are generalized coordinates. For example, in some systems p can represent the angular momentum and q can represent an angle. For a system of N particles in three dimensions, the sum in (6.36) runs from 1 to $3N$, where $3N$ is the number of degrees of freedom.

The methods for constructing the generalized momenta and the Hamiltonian are described in standard classical mechanics texts. The time dependence of the generalized momenta and coordinates is given by

$$\dot{p}_i \equiv \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \tag{6.37a}$$

$$\dot{q}_i \equiv \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}. \tag{6.37b}$$

Check that (6.37) leads to the usual form of Newton's second law by considering the simple example of a single particle in a potential $U(x)$, where $q = x$ and $p = m\dot{x}$.

As we found in Chapter 4, an important property of conservative systems is preservation of areas in phase space. Consider a set of initial conditions of a dynamical system that form a closed surface in phase space. For example, if phase space is two-dimensional, this surface would be a one-dimensional loop. As time evolves, this surface in phase space will typically change its shape. For Hamiltonian systems, the volume (area for a two-dimensional phase space) enclosed by this surface remains constant in time. For dissipative systems, this volume will decrease, and hence dissipative systems are not described by a Hamiltonian. One consequence of the constant phase space volume is that Hamiltonian systems do not have phase space attractors.

In general, the motion of Hamiltonian systems is very complex. In some systems the motion is regular, and there is a constant of the motion (a quantity that does not change with time) for each degree of freedom. Such a system is said to be *integrable*. For time-independent systems, an obvious constant of the motion is the total energy. The total momentum and angular momentum are other examples. There may be others as well. If there are more degrees of freedom than constants of the motion, then the system can be chaotic. When the number of degrees of freedom becomes large, the possibility of chaotic behavior becomes more likely. An important example that we will consider in Chapter 8 is a system of interacting particles. Their chaotic motion is essential for the system to be described by the methods of statistical mechanics.

For regular motion the change in shape of a closed surface in phase space is uninteresting. For chaotic motion, nearby trajectories must exponentially diverge from each other, but are confined to a finite region of phase space. Hence, there will be local stretching of the surface accompanied by repeated folding to ensure confinement. There is another class of systems whose behavior is in between; that is, the system behaves regularly for some initial conditions and chaotically for others. We will study these *mixed* systems in this section.

Consider the Hamiltonian for a system of N particles. If the system is integrable, there are $3N$ constants of the motion. It is natural to identify the generalized momenta with these constants. The coordinates that are associated with each of these constants will vary linearly with time. If the system is confined in phase space, then the coordinates must be periodic. If we have just one coordinate, we can think of the motion as a point moving on a circle in phase space. In two dimensions the motion is a point moving in two circles at once, that is, a point moving on the surface of a torus. In three dimensions we can imagine a generalized

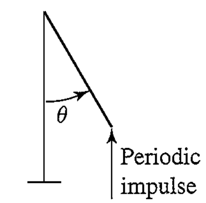

Figure 6.11 Model of a kicked rotor consisting of a rigid rod with moment of inertia I. Gravity and friction at the pivot is ignored. The motion of the rotor is given by the standard map in (6.39).

torus with three circles, and so on. If the period of motion along each circle is a rational fraction of the period of all the other circles, then the torus is called a resonant torus, and the motion in phase space is periodic. If the periods are not rational fractions of each other, then the torus is called nonresonant.

If we take an integrable Hamiltonian and change it slightly, what happens to these tori? A partial answer is given by a theorem due to Kolmogorov, Arnold, and Moser (KAM), which states that, under certain circumstances, the tori will remain. When the perturbation of the Hamiltonian becomes sufficiently large, these KAM tori are destroyed.

To understand the basic ideas associated with mixed systems, we consider a simple model of a rotor known as the *standard* map (see Figure 6.11). The rod has a moment of inertia I and length L and is fastened at one end to a frictionless pivot. The other end is subjected to a vertical periodic impulsive force of strength k/L applied at time $t = 0, \tau, 2\tau, \ldots$. Gravity is ignored. The motion of the rotor can be described by the angle θ and the corresponding angular momentum p_θ. The Hamiltonian for this system can be written as

$$H(\theta, p_\theta, t) = \frac{{p_\theta}^2}{2I} + k\cos\theta \sum_n \delta(t - n\tau). \tag{6.38}$$

The term $\delta(t - n\tau)$ is zero everywhere except at $t = n\tau$; its integral over time is unity if $t = n\tau$ is within the limits of integration. If we use (6.37) and (6.38), it is easy to show that the corresponding equations of motion are given by

$$\frac{dp_\theta}{dt} = k\sin\theta \sum_n \delta(t - n\tau) \tag{6.39a}$$

$$\frac{d\theta}{dt} = \frac{p_\theta}{I}. \tag{6.39b}$$

From (6.39) we see that p_θ is constant between kicks (remember that gravity is assumed to be absent), but changes discontinuously at each kick. The angle θ varies linearly with t between kicks and is continuous at each kick.

It is convenient to know the values of θ and p_θ at times just after the kick. We let θ_n and p_n be the values of $\theta(t)$ and $p_\theta(t)$ at times $t = n\tau + 0^+$, where 0^+ is a infinitesimally small positive number. If we integrate (6.39a) from $t = (n+1)\tau - 0^+$ to $t = (n+1)\tau + 0^+$, we