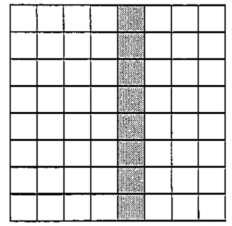

Figure 12.4 An example of a spanning cluster with a probability proportional to p^L on a $L = 8$ lattice. The probability of a spanning cluster with more sites will be proportional to a higher power of p.

Problem 12.1 Site percolation on the square lattice

(a) Use `PercolationApp` to generate random site percolation configurations on a square lattice. Estimate $p_c(L)$ by finding the mean value of p at which a spanning cluster first occurs. For a given seed, the `calculate` method assigns a random number to each site and determines the occupancy of each site by comparing the sites's random number to p. Choose one of the spanning rules and begin with a value of p for which a spanning cluster is unlikely to be present. Then systematically increase p until you find a spanning cluster. Then choose a new seed and, hence, a new set of random numbers. Repeat this procedure for at least ten configurations and find the average value of $p_c(L)$. (Each configuration corresponds to a different set of random numbers.)

(b) Repeat part (a) for larger values of L. Is $p_c(L)$ better defined for larger L; that is, are the values of $p_c(L)$ spread over a smaller range of values? How quickly can you visually determine the existence of a spanning cluster? Describe your visual algorithm for determining if a spanning cluster exists.

(c) Choose $L \geq 1024$ and generate a configuration of sites at $p = p_c$. For this value of L, you won't be able to distinguish the individual sites. Click on the lattice until you generate some large clusters. Describe their visual appearance. For example, are they compact or ramified? ■

The value of p_c depends on the symmetry of the lattice and on its dimension. In addition to the square lattice, the most common two-dimensional lattice is the triangular lattice. As discussed in Chapter 8, the essential difference between the square and triangular lattices is the number of nearest neighbors.

***Problem 12.2 Site percolation on the triangular lattice**

Modify `PercolationApp` to simulate random site percolation on a triangular lattice. Assume that a connected path connects the top and bottom sides of the lattice (see Figure 12.5). Do you expect p_c for the triangular lattice to be smaller or larger than the value of p_c for the square lattice? Estimate $p_c(L)$ for increasing values of L. Are your results for p_c consistent with your expectations? As we will discuss in the following, the exact value of p_c for the triangular lattice is $p_c = 1/2$. ■

In *bond* percolation each lattice site is occupied, but only a fraction of the sites have connections or occupied bonds between them and their nearest neighbor sites (see Figure 12.6). Each bond is either occupied with probability p or not occupied with probability

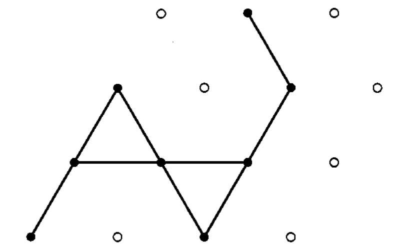

Figure 12.5 Example of a spanning site cluster on a $L = 4$ triangular lattice. The filled circles represent the occupied sites.

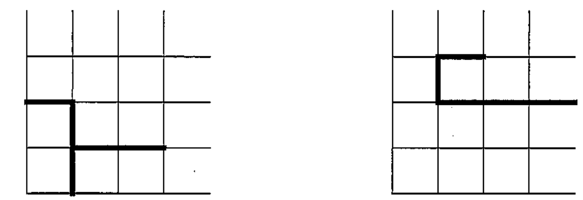

Figure 12.6 Two examples of bond clusters. The occupied bonds are shown as bold lines.

$1 - p$. A cluster is a group of sites connected by occupied bonds. The wire mesh described in Section 12.1 is an example of bond percolation if we imagine cutting the bonds between the nodes rather than removing the nodes themselves. An application of bond percolation to the description of gelation is discussed in Problem 12.3.

For bond percolation on the square lattice, the exact value of p_c can be obtained by introducing the *dual* lattice. The nodes of the dual lattice are the centers of the squares between the nodes in the original lattice (see Figure 12.7). The occupied bonds of the dual lattice are those that do not cross an occupied bond of the original lattice. Because every occupied bond on the dual lattice crosses exactly one unoccupied bond of the original lattice, the probability $\tilde{p}$ of an occupied bond on the dual lattice is $1 - p$, where p is the probability of an occupied bond on the original lattice. If we assume that the dual lattice percolates if and only if the original lattice does not, and vice versa, then $p_c = 1 - p_c$ or $p_c = 1/2$. This assumption holds for bond percolation on a square lattice because if a cluster in the original lattice spans in both directions, then because the occupied dual lattice bonds can only cross unoccupied bonds of the original lattice, the dual lattice clusters are blocked from spanning. An example is shown in Figure 12.7. This argument does not work for cubic lattices in three dimensions, but it can be used for site percolation on a triangular lattice to yield $p_c = 1/2$.

***Problem 12.3 Bond percolation on a square lattice**

Suppose that all the lattice sites of a square lattice are occupied by monomers, each with functionality four; that is, each monomer can form a maximum of four bonds. This model is equivalent to bond percolation on a square lattice. Assume that the presence or absence of a bond between a given pair of monomers is random and is characterized by the probability p. For small p, the system consists of only finite polymers (groups of monomers) and the system is in the *sol* phase. For some threshold value p_c, there will be a single polymer that spans the lattice. We say that for $p \geq p_c$, the system is in the *gel* phase. How does a bowl of jello, an example of a gel, differ from a bowl of broth? Write a program to simulate