CHAPTER 5

Few-Body Problems: The Motion of the Planets

We apply Newton's laws of motion to planetary motion and other systems of a few particles and explore some of the counterintuitive consequences of Newton's laws.

5.1 ■ PLANETARY MOTION

Planetary motion is of special significance because it played an important role in the conceptual history of the mechanical view of the universe. Few theories have affected Western civilization as much as Newton's laws of motion and the law of gravitation, which together relate the motion of the heavens to the motion of terrestrial bodies.

Much of our knowledge of planetary motion is summarized by Kepler's three laws, which can be stated as

1. Each planet moves in an elliptical orbit with the Sun located at one of the foci of the ellipse.
2. The speed of a planet increases as its distance from the Sun decreases such that the line from the Sun to the planet sweeps out equal areas in equal times.
3. The ratio T^2/a^3 is the same for all planets that orbit the Sun, where T is the period of the planet and a is the semimajor axis of the ellipse.

Kepler obtained these laws by a careful analysis of the observational data collected over many years by Tycho Brahe.

Kepler's first and third laws describe the shape of the orbit rather than the time dependence of the position and velocity of a planet. Because it is not possible to obtain this time dependence in terms of elementary functions, we will obtain the numerical solution of the equations of motion of planets and satellites in orbit. In addition, we will consider the effects of perturbing forces on the orbit and problems that challenge our intuitive understanding of Newton's laws of motion.

5.2 ■ THE EQUATIONS OF MOTION

The motion of the Sun and Earth is an example of a *two-body problem.* We can reduce this problem to a one-body problem in one of two ways. The easiest way is to use the fact that the mass of the Sun is much greater than the mass of the Earth. Hence, we can assume that, to a good approximation, the Sun is stationary and is a convenient choice of the origin of our coordinate system.

If you are familiar with the concept of a *reduced mass,* you know that the reduction to a one-body problem is more general. That is, the motion of two objects of mass m and M, whose total potential energy is a function of only their relative separation, can be reduced to an equivalent one-body problem for the motion of an object of reduced mass μ given by

$$\mu = \frac{Mm}{m+M}. \tag{5.1}$$

Because the mass of the Earth, $m = 5.99 \times 10^{24}$ kg, is so much smaller than the mass of the Sun, $M = 1.99 \times 10^{30}$ kg, we find that for most practical purposes, the reduced mass of the Sun and the Earth is that of the Earth alone. In the following, we consider the problem of a single particle of mass m moving about a fixed center of force, which we take as the origin of the coordinate system.

Newton's universal law of gravitation states that a particle of mass M attracts another particle of mass m with a force given by

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}, \tag{5.2}$$

where the vector $\mathbf{r}$ is directed from M to m (see Figure 5.1). The negative sign in (5.2) implies that the gravitational force is attractive; that is, it tends to decrease the separation r. The gravitational constant G is determined experimentally to be

$$G = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg} \cdot \text{s}^2}. \tag{5.3}$$

The force law (5.2) applies to the motion of the center of mass for objects of negligible spatial extent. Newton delayed publication of his law of gravitation for twenty years while he invented integral calculus and showed that (5.2) also applies to any uniform sphere or spherical shell of matter if the distance r is measured from the center of each mass.

The gravitational force has two general properties: its magnitude depends only on the separation of the particles, and its direction is along the line joining the particles. Such a force is called a central force. The assumption of a central force implies that the orbit of the Earth is restricted to a plane (x-y), and the angular momentum $\mathbf{L}$ is conserved and lies in the third (z) direction. We write L_z in the form

$$L_z = (\mathbf{r} \times m\mathbf{v})_z = m(xv_y - yv_x), \tag{5.4}$$

where we have used the cross-product definition $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ and $\mathbf{p} = m\mathbf{v}$. An additional constraint on the motion is that the total energy E is conserved and is given by

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}. \tag{5.5}$$

If we fix the coordinate system at the mass M, the equation of motion of the particle of mass m is

$$m\frac{d^2\mathbf{r}}{dt^2} = -\frac{GMm}{r^3}\mathbf{r}. \tag{5.6}$$