(g) What is the shape of the phase space trajectory for the initial condition $x(t=0) = 1$, $v(t=0) = 0$? Do you find a different phase space trajectory for other initial conditions?

(h) Why is $A(\omega = 0) < A(\omega)$ for small ω? Why does $A(\omega) \to 0$ for $\omega \gg \omega_0$?

(i) Does the mean kinetic energy resonate at the same frequency as does the amplitude? Compute the mean kinetic energy over one cycle once steady state conditions have been reached. Choose $\omega_0 = 3$ and $\gamma = 0.5$. ■

In Problem 4.8 we found that the response of the damped harmonic oscillator to an external driving force is linear. For example, if the magnitude of the external force is doubled, then the magnitude of the steady state motion is also doubled. This behavior is a consequence of the linear nature of the equation of motion. When a particle is subject to nonlinear forces, the response can be much more complicated (see Section 6.8).

For many problems, the sinusoidal driving force in (4.18) is not realistic. Another example of an external force can be found by observing someone pushing a child on a swing. Because the force is nonzero for only short intervals of time, this type of force is impulsive. In the following problem, we consider the response of a damped linear oscillator to an impulsive force.

***Problem 4.9 Response of a damped linear oscillator to nonsinusoidal external forces**

(a) Assume a swing can be modeled by a dampled linear oscillator. The effect of an impulse is to change the velocity. For simplicity, let the duration of the push equal the time step Δt. Introduce an integer variable for the number of time steps and use the `%` operator to ensure that the impulse is nonzero only at the time step associated with the period of the external impulse. Determine the steady state amplitude $A(\omega)$ for $\omega = 1.0, 1.3, 1.4, 1.5, 1.6, 2.5, 3.0$, and 3.5. The corresponding period of the impulse is given by $T = 2\pi/\omega$. Choose $\omega_0 = 3$ and $\gamma = 0.5$. Are your results consistent with your experience of pushing a swing and with the comparable results of Problem 4.8?

(b) Consider the response to a half-wave external force consisting of the positive part of a cosine function (see Figure 4.3). Compute $A(\omega)$ for $\omega_0 = 3$ and $\gamma = 0.5$. At what values of ω does $A(\omega)$ have a relative maxima? Is the half-wave cosine driving force equivalent to a sum of cosine functions of different frequencies? For example, does $A(\omega)$ have more than one resonance?

(c) Compute the steady state response $x(t)$ to the external force

$$\frac{1}{m}F(t) = \frac{1}{\pi} + \frac{1}{2}\cos t + \frac{2}{3\pi}\cos 2t - \frac{2}{15\pi}\cos 4t. \tag{4.20}$$

How does a plot of $F(t)$ versus t compare to the half-wave cosine function? Use your results to conjecture a principle of superposition for the solutions to linear equations. ■

In many of the problems in this chapter, we have asked you to draw a phase space plot for a single oscillator. This plot provides a convenient representation of both the position and velocity. When we study chaotic phenomena, such plots will become almost indispensable (see Chapter 6). Here we will consider an important feature of phase space trajectories for conservative systems.

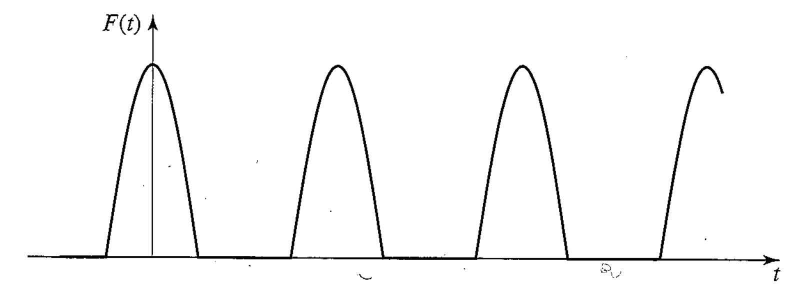

Figure 4.3 A half-wave driving force corresponding to the positive part of a cosine function.

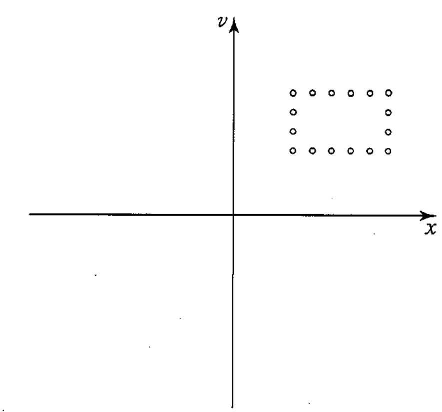

Figure 4.4 What happens to a given area in phase space for conservative systems?

If there are no external forces, the undamped simple harmonic oscillator and undamped pendulum are examples of conservative systems, that is, systems for which the total energy is a constant. In Problems 4.10 and 4.11, we will study two general properties of conservative systems, the nonintersecting nature of their trajectories in phase space and the preservation of area in phase space. These concepts will become more important when we study the properties of conservative systems with more than one degree of freedom.

Problem 4.10 Trajectory of a simple harmonic oscillator in phase space

(a) We explore the phase space behavior of a single harmonic oscillator by simulating N initial conditions simultaneously. Write a program to simulate N identical simple harmonic oscillators each of which is represented by a small circle centered at its position and velocity in phase space as shown in Figure 4.4. One way to do so is to adapt the `BouncingBallApp` class introduced in Section 2.6. Choose $N = 16$ and consider random initial positions and velocities. Do the phase space trajectories for different initial conditions ever cross? Explain your answer in terms of the uniqueness of trajectories in a deterministic system.

(b) Choose a set of initial conditions that form a rectangle (see Figure 4.4). Does the shape of this area change with time? What happens to the total area in comparison to the original area? ■