Problem 9.40 Optical resolution

Consider a mask containing two circular openings of radius 500λ separated by 100λ. Do a simulation to determine how far the screen can be placed from the aperture mask and still observe two distinct shadows. ■

APPENDIX 9A: COMPLEX FOURIER SERIES

A function $f(t)$ with period T can be expressed in terms of a trigonometric Fourier series:

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos\omega_k t + b_k \sin\omega_k t), \tag{9.70}$$

where $\omega_k = k\omega_0$ and $\omega_0 = 2\partial/T$. To derive the exponential form of this series, we express the sine and cosine functions as

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{9.71a}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}. \tag{9.71b}$$

We substitute (9.71) into (9.70) and obtain

$$f(t) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left[e^{ik\omega_0 t}\frac{a_k - ib_k}{2} + e^{-ik\omega_0 t}\frac{a_k + ib_k}{2}\right]. \tag{9.72}$$

We use $1/i = -i$ and define new Fourier coefficients as follows:

$$c_0 \equiv \frac{1}{2}a_0 \tag{9.73a}$$

$$c_k \equiv \frac{a_k - ib_k}{2} \tag{9.73b}$$

$$c_{-k} \equiv \frac{a_{-k} - ib_{-k}}{2} = \frac{a_k + ib_k}{2}, \tag{9.73c}$$

where the right-hand side of (9.73c) follows from $a_k = a_{-k}$ and $b_k = -b_{-k}$. We substitute these coefficients into (9.72) and find

$$f(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{ik\omega_0 t} + \sum_{k=1}^{\infty} c_{-k} e^{-ik\omega_0 t}, \tag{9.74}$$

or

$$f(t) = \sum_{k=0}^{\infty} c_k e^{ik\omega_0 t} + \sum_{k=1}^{\infty} c_{-k} e^{-ik\omega_0 t}. \tag{9.75}$$

Finally, we re-index the second sum from -1 to $-\infty$

$$f(t) = \sum_{k=0}^{\infty} c_k e^{ik\omega_0 t} + \sum_{k=-1}^{-\infty} c_k e^{ik\omega_0 t}, \tag{9.76}$$

and combine the summations to obtain the exponential form:

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t}. \tag{9.77}$$

APPENDIX 9B: FAST FOURIER TRANSFORM

The fast Fourier transform (FFT) was discovered independently in a variety of contexts by many workers. There are several variations of the algorithm, and we describe a version due to Danielson and Lanczos. The goal is to compute the Fourier transform $g(\omega_k)$ given the data set $f(j\Delta) \equiv f_j$ of (9.35). For convenience we rewrite the relation

$$g_k \equiv g(\omega_k) = \sum_{j=0}^{N-1} f(j\Delta)\, e^{-i2\pi kj/N}, \tag{9.78}$$

and introduce the complex number W given by

$$W = e^{-i2\pi/N}. \tag{9.79}$$

The following algorithm works with any complex data set if N is a power of two. Real data sets can be transformed by setting the array elements corresponding to the imaginary part equal to 0.

To understand the FFT algorithm, we consider the case $N = 8$ and rewrite (9.78) as

$$g_k = \sum_{j=0,2,4,6} f(j\Delta)\, e^{-i2\pi kj/N} + \sum_{j=1,3,5,7} f(j\Delta)\, e^{-i2\pi kj/N} \tag{9.80a}$$

$$= \sum_{j=0,1,2,3} f(2j\Delta)e^{-i2\pi k2j/N} + \sum_{j=0,1,2,3} f((2j+1)\Delta)e^{-i2\pi k(2j+1)/N} \tag{9.80b}$$

$$= \sum_{j=0,1,2,3} f(2j\Delta)e^{-i2\pi kj/(N/2)} + W^k \sum_{j=0,1,2,3} f((2j+1)\Delta)\, e^{-i2\pi kj/(N/2)} \tag{9.80c}$$

$$= g_k^{\rm e} + W^k g_k^{\rm o}, \tag{9.80d}$$

where $W^k = e^{-i2\pi k/N}$. The quantity $g^{\rm e}$ is the Fourier transform of length $N/2$ formed from the even components of the original $f(j\Delta)$; $g^{\rm o}$ is the Fourier transform of length $N/2$ formed from the odd components.

We can continue this decomposition if N is a power of two. That is, we can decompose $g^{\rm e}$ into its $N/4$ even and $N/4$ odd components, $g^{\rm ee}$ and $g^{\rm eo}$, and decompose $g^{\rm o}$ into its $N/4$ even and $N/4$ odd components, $g^{\rm oe}$ and $g^{\rm oo}$. We find

$$g_k = g_k^{\rm ee} + W^{2k} g_k^{\rm eo} + W^k g_k^{\rm oe} + W^{3k} g_k^{\rm oo}. \tag{9.81}$$

One more decomposition leads to

$$\begin{aligned} g_k = {} & g_k^{\rm eee} + W^{4k} g_k^{\rm eeo} + W^{2k} g_k^{\rm eoe} + W^{6k} g_k^{\rm eoo} \\ & + W^k g_k^{\rm oee} + W^{5k} g_k^{\rm oeo} + W^{3k} g_k^{\rm ooe} + W^{7k} g_k^{\rm ooo}. \end{aligned} \tag{9.82}$$