D. Stauffer, "Percolation clusters as teaching aid for Monte Carlo simulation and critical exponents," Am. J. Phys. **45**, 1001–1002 (1977); D. Stauffer, "Scaling theory of percolation clusters," Physics Reports **54**, 1–74 (1979).

Dietrich Stauffer and Amnon Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, 1994). A delightful book by two of the leading workers in the field. An efficient Fortran implementation of the Hoshen–Kopelman algorithm is given in Appendix A.3.

B. P. Watson and P. L. Leath, "Conductivity in the two-dimensional site percolation problem," Phys. Rev. B **9**, 4893–4896 (1974). A research paper on the conductivity of chicken wire.

John C. Wierman and Dora Passen Naor, "Criteria for evaluation of universal formulas for percolation thresholds," Phys. Rev. E **71**, 036143-1–7 (2005). Wierman and Naor evaluate several universal formulas that predict approximate values for p_c for various lattices. Percolation was first conceived by chemistry Nobel laureate P. J. Flory as a model for polymer gelation, for example, the sol-gel transition of jello (P. J. Flory, "Molecular size distribution in three-dimensional polymers," J. Am. Chem. Soc. **63**, 3083–3100 (1941)). Further important work in this context was done by another chemist, Walter H. Stockmayer. The term "percolation" was coined by mathematicians Broadbent and Hammersley in 1957 who considered percolation on a lattice for the first time (S. R. Broadbent and J. M. Hammersley, "Percolation processes I. Crystals and mazes," Proceedings Cambridge Philosophical Society **53**, 629–641 (1957)).

Kenneth G. Wilson, "Problems in physics with many scales of length," Sci. Am. **241** (8), 158–179 (1979). An accessible article on the renormalization group method and its applications in particle and condensed matter physics. See also K. G. Wilson, "The renormalization group and critical phenomena," Rev. Mod. Phys. **55**, 583–600 (1983). The latter article is the text of Wilson's lecture on the occasion of the presentation of the 1982 Nobel Prize in Physics. In this lecture he claims that he "... found it very helpful to demand that a correctly formulated field theory be soluble by computer, the same way an ordinary differential equation can be solved on a computer ..."

W. Xia and M. F. Thorpe, "Percolation properties of random ellipses," Phys. Rev. A **38**, 2650–2656 (1988). The authors consider continuum percolation and show that the area fraction remaining after punching out holes at random is given by $\phi = e^{-A\rho}$, where A is the area of a hole, and ρ is the number density of the holes. This relation does not depend on the shape of the holes.

Richard Zallen, *The Physics of Amorphous Solids* (Wiley–Interscience, 1983). Chapter 4 discusses many of the applications of percolation concepts to realistic systems.

R. M. Ziff and M. E. J. Newman, "Convergence of threshold estimates for two-dimensional percolation," Phys. Rev. E **66**, 016129-1–10 (2002).

CHAPTER 13 Fractals and Kinetic Growth Models

We introduce the concept of fractal dimension and discuss several processes that generate fractal objects.

13.1 ■ THE FRACTAL DIMENSION

One of the more interesting geometrical properties of objects is their shape. As an example, we show in Figure 13.1 a spanning cluster generated at the percolation threshold. Although the visual description of such a cluster is subjective, such a cluster can be described as ramified, airy, tenuous, and stringy, rather than compact or space-filling.

In the 1970s a new *fractal* geometry was developed by Mandelbrot and others to describe the characteristics of ramified objects. One quantitative measure of the structure of these objects is their *fractal dimension* D. To define D, we first review some simple ideas of dimension in ordinary Euclidean geometry. Consider a circular or spherical object of mass M and radius R. If the radius of the object is increased from R to $2R$, the mass of the object is increased by a factor of 2^2 if the object is circular or by 2^3 if the object is spherical. We can express this relation between mass and the radius or a characteristic length as

$$M(R) \sim R^D \quad \text{(mass dimension)}, \tag{13.1}$$

where D is the dimension of the object. Equation (13.1) implies that if the linear dimensions of an object are increased by a factor of b while preserving its shape, then the mass of the object is increased by b^D. This mass-length scaling relation is closely related to our intuitive understanding of spatial dimension.

If the dimension of the object D and the dimension of the Euclidean space in which the object is embedded d are identical, then the mass density $\rho = M/R^d$ scales as

$$\rho(R) \propto M(R)/R^d \sim R^0; \tag{13.2}$$

that is, its density is constant. An example of such a two-dimensional object is shown in Figure 13.2. An object whose mass-length relation satisfies (13.1) with $D = d$ is said to be *compact*.

Equation (13.1) can be generalized to define the fractal dimension. We denote objects as fractals if they satisfy (13.1) with a value of D different from the spatial dimension d. If an object satisfies (13.1) with $D < d$, its density is not the same for all R but scales as

$$\rho(R) \propto M/R^d \sim R^{D-d}. \tag{13.3}$$