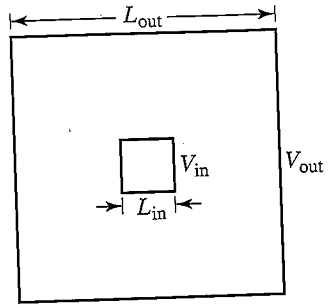

Figure 10.2 The geometry of the two concentric squares considered in Problem 10.12.

where $V_{\text{ave}}(x, y)$ is the average of the potential of the four neighbors of (x, y). The overrelaxation parameter w is in the range $1 < w < 2$. The effect of w is to cause the potential to change by a greater amount than in the simple relaxation procedure. Explore the dependence of the rate of convergence on w. A relaxation method that increases the rate of convergence is explored in Project 10.26. ■

Problem 10.12 The capacitance of concentric squares

(a) Use a relaxation method to compute the potential distribution between the two concentric square cylinders shown in Figure 10.2. The potential of the outer square conductor is $V_{\text{out}} = 10$, and the potential of the inner square conductor is $V_{\text{in}} = 5$. The linear dimensions of the exterior and interior squares are $L_{\text{out}} = 25$ and $L_{\text{in}} = 5$, respectively. Modify your program so that the potential of the interior square is fixed. Sketch the equipotential surfaces.

(b) A system of two conductors with charge Q and $-Q$, respectively has a capacitance C that is defined as the ratio of Q to the potential difference ΔV between the two conductors. Determine the capacitance per unit length of the concentric cylinders considered in part (a). In this case $\Delta V = 5$. The charge Q can be determined from the fact that near a conducting surface, the surface charge density σ is given by $\sigma = E_n/4\pi$, where E_n is the magnitude of the electric field normal to the surface. E_n can be approximated by the relation $-\delta V/\delta r$, where δV is the potential difference between a boundary site and an adjacent interior site a distance δr away. Use the result of part (a) to compute δV for each site adjacent to the two square surfaces. Use this information to determine E_n for the two surfaces and the charge per unit length on each conductor. Are the charges equal and opposite in sign? Compare your numerical result to the capacitance per unit length, $1/2 \ln r_{\text{out}}/r_{\text{in}}$, of a system of two concentric circular cylinders of radii r_{out} and r_{in}. Assume that the circumference of each cylinder equals the perimeter of the corresponding square, that is, $2\pi r_{\text{out}} = 4L_{\text{out}}$ and $2\pi r_{\text{in}} = 4L_{\text{in}}$.

(c) Move the inner square 1 cm off center and repeat the calculations of parts (a) and (b). How do the potential surfaces change? Is there any qualitative difference if we set the inner conductor potential equal to -5? ■

Laplace's equation holds only in charge-free regions. If there is a charge density $\rho(x, y, z)$ in the region, we need to use *Poisson's* equation which can be written as

$$\nabla^2 V(\mathbf{r}) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -4\pi\rho(\mathbf{r}), \tag{10.16}$$

where $\rho(\mathbf{r})$ is the charge density. The difference form of Poisson's equation is given in two dimensions by

$$\begin{aligned} V(x, y) \approx \frac{1}{4}[&V(x + \Delta x, y) + V(x - \Delta x, y) + V(x, y + \Delta y) + V(x, y - \Delta y)] \\ &+ \frac{1}{4}\Delta x \Delta y\, 4\pi\rho(x, y). \end{aligned} \tag{10.17}$$

Note that the product $\rho(x, y)\Delta x \Delta y$ is the total charge in a $\Delta x \times \Delta y$ region centered at (x, y).

Problem 10.13 Surface charge

(a) Poisson's equation can be used to find the surface charge on a conductor after Laplace's equation has been solved. The potential is fixed at the boundary sites. If we assume the boundary is a conductor with some thickness, then we can assume that the potential for the next layer of sites outside the boundary has the same potential as the boundary. If we use this assumption, then after we have solved numerically for the potential of the interior sites, we will find that the average value of the neighbors of a boundary site will not equal the imposed potential. From (10.17) the difference will equal $\Delta x \Delta y \pi\rho(x, y)$. Modify `LaplaceApp` to calculate and display the surface charge density, assuming $\Delta x = \Delta y = 1$. Notice that because we are in two dimensions the "surface" charge density, $\Delta x \Delta y \rho(x, y)$, is a linear density of charge per unit length.

(b) Consider the same system as in Problem 10.10c and find the surface charge density on the boundary sites. Make a reasonable choice for assigning the potential at the corner sites.

(c) Model a system with the boundary at a potential $V = 0$ and a centered interior rectangle of 6×12 at a potential of $V = 10$. Where is the charge density the highest?

(d) Repeat if the interior rectangle is placed close to an edge. ■

Problem 10.14 Numerical solution of Poisson's equation

(a) Consider a square of linear dimension $L = 25$ whose boundary is fixed at a potential equal to $V = 10$. Assume that the interior region has a uniform charge density ρ such that the total charge is $Q = 1$. Modify `LaplaceApp` to compute the potential distribution for this case. Compare the equipotential surfaces obtained for this case to that found in Problem 10.12.

(b) Find the potential distribution if the charge distribution of part (a) is restricted to a 5×5 square at the center.

(c) Find the potential distribution if the charge distribution of part (a) is restricted to a 1×1 square at the center. How does the potential compare to that of a point charge without the boundary? ■