(d) Perturb the circular orbit at $r = 9$ by giving the particle an initial tangential velocity of $v = 0.345c$. At what rate does the perihelion of the orbit advance? ■

The equations for light can be obtained by adding a constraint to (18.25) using a Lagrange multiplier. The constraint is the condition that the proper time along a light worldline is zero:

$$0 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\phi^2. \tag{18.29}$$

If we add the Lagrange multiplier, do the differentiation, and simplify terms, we obtain rate equations that can be solved using standard numerical techniques. (The use of a computer algebra program would be helpful.)

$$\frac{dr}{dt} = \dot{r} \tag{18.30a}$$

$$\frac{d\dot{r}}{dt} = \frac{-4M^2 + 2Mr + (r - 5M)r^3\dot{\phi}^2}{r^3} \tag{18.30b}$$

$$\frac{d\phi}{dt} = \dot{\phi} \tag{18.30c}$$

$$\frac{d\dot{\phi}}{dt} = \frac{2(-3M + r)\dot{r}\dot{\phi}}{(2M - r)r} \tag{18.30d}$$

$$\frac{dt}{dt} = 1. \tag{18.30e}$$

Exercise 18.12 Light trajectories

(a) Write a program that plots the general relativistic trajectory of light using Schwarzschild coordinates. Demonstrate the deflection of starlight passing near a gravitational mass by plotting the trajectory of a light ray.

(b) Verify that light orbits a black hole at $r = 3$ and $M = 1$.

(c) Show that a gravitational mass can act as a lens by plotting the trajectory of two light rays that leave a point source at different angles but later cross. The two light rays should pass on opposite sides of the mass. Do the two light beams always arrive at the crossing point at the same time? ■

18.6 ■ SEEING

Because of the nonlinearity of the Schwarzschild metric, simulation plays an essential role. A calculation of a view of the stars in the vicinity of a black hole, for example, would require the solution of the light-ray trajectory for angles within the eye's field of view.

The angles drawn on a Schwarzschild map are not the same as the angles seen by an observer because distances on the map are distorted. A stationary observer at a constant r-value is known as a *shell observer* because he is on a stationary shell at fixed (r, ϕ) coordinates. Launch angles measured by such a shell observer can easily be converted to angles on the Schwarzschild map by taking into account the contraction by $\sqrt{1 - 2M/r}$ in the radial direction:

$$\tan\theta_{\text{shell}} = (1 - 2M/r)^{1/2} \tan\theta_{\text{schw}}. \tag{18.31}$$

Use this transformation in Exercise 18.13 and Project 18.20.

Exercise 18.13 Knife-edge trajectory

Many important properties of light rays can be expressed in terms of an impact parameter b defined as

$$b = r(1 - 2M/r)^{-1/2} \sin\theta_{\text{shell}}. \tag{18.32}$$

For example, light that is launched with $b = \sqrt{27}M$ enters an unstable orbit that teeters between an escape to infinity and a plunge into the black hole. This trajectory is known as a *knife-edge* trajectory because the result is very sensitive to the initial conditions and numerical roundoff error and cannot be predicted. What will a shell observer see if he looks into space at an angle that has this impact parameter? ■

Problem 18.14 Seeing near a black hole

Imagine a grid of light beacons located far away from a black hole in the $\phi = \pi$ direction. A shell observer at $\phi = 0$ with an arbitrary value of r attempts to view the grid by looking toward the black hole. What will the observer see? One way to answer this question is to assume a reasonable field of view (for example, 180°) for the eye and calculate light rays leaving the eye at equal angular intervals. Compute the light paths and tabulate where the ray crosses the beacon grid as a function of viewing angle. Because it is unlikely that the light rays will intersect a beacon location, use interpolation to determine the angles at which beacons appear. Plot these locations to show the observer's view. ■

18.7 ■ GENERAL RELATIVISTIC DYNAMICS

In general relativity, the magnitude of the angular momentum L per unit mass m of a particle is

$$\ell \equiv \frac{L}{m} = r^2 \frac{d\phi}{d\tau}, \tag{18.33}$$

and the energy E per unit mass is

$$e \equiv \frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}. \tag{18.34}$$

We can solve (18.33) for $d\phi$ and (18.34) for dt and substitute the result into the time-like form of the metric and obtain a relation for $dr/d\tau$:

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \left(\frac{\ell}{r}\right)^2\right]. \tag{18.35}$$