energy at time $t_n = t_0 + n\Delta t$. (It is necessary to consider only the energy per unit mass.) Plot the difference ΔE_n as a function of t_n for several cycles for a given value of Δt. Choose $x(t=0)=1$, $v(t=0)=0$, and $\omega_0^2 = k/m = 9$ and start with $\Delta t = 0.05$. Is the difference ΔE_n uniformly small throughout the cycle? Does ΔE_n drift, that is, become bigger with time? What is the optimum choice of Δt?

(b) Implement the Euler–Cromer algorithm by writing an Euler–Cromer `ODESolver` and answer the same questions as in part (a).

(c) Modify your program so that the Euler–Richardson or Verlet algorithms are used and answer the same questions as in part (a). (The Verlet algorithm is discussed in Appendix 3A.)

(d) Describe the qualitative differences between the time dependence of ΔE_n using the various algorithms. Which algorithm is most consistent with the requirement of conservation of energy? For fixed Δt, which algorithm yields better results for the position in comparison to the analytical solution (4.4)? Is the requirement of conservation of energy consistent with the relative accuracy of the computed positions?

(e) Choose the best algorithm based on your criteria, and determine the values of Δt that are needed to conserve the total energy to within 0.1% over one cycle for $\omega_0 = 3$ and for $\omega_0 = 12$. Can you use the same value of Δt for both values of ω_0? If not, how do the values of Δt correspond to the relative values of the period in the two cases? ■

Problem 4.2 Analysis of simple harmonic motion

(a) Use your results from Problem 4.1 to select an appropriate numerical algorithm and value of Δt for the simple harmonic oscillator, and modify your program so that the time dependence of the potential and kinetic energies is plotted. Where in the cycle is the kinetic energy (potential energy) a maximum?

(b) Compute the average value of the kinetic energy and the potential energy during a complete cycle. What is the relation between the two averages?

(c) Compute $x(t)$ for different values of A and show that the shape of $x(t)$ is independent of A; that is, show that $x(t)/A$ is a *universal* function of t for a fixed value of ω_0. In what units should the time be measured so that the ratio $x(t)/A$ is independent of ω_0?

(d) The dynamical behavior of the one-dimensional oscillator is completely specified by $x(t)$ and $p(t)$, where p is the momentum of the oscillator. These quantities may be interpreted as the coordinates of a point in a two-dimensional space known as *phase space*. As the time increases, the point $(x(t), p(t))$ moves along a trajectory in phase space. Modify your program so that the momentum p is plotted as a function of x; that is, choose p and x as the vertical and horizontal axes, respectively. Choose $\omega_0 = 3$ and compute the phase space trajectory for the initial condition $x(t=0) = 1$, $v(t=0)=0$. What is the shape of this trajectory? What is the shape for the initial conditions $x(t=0)=0$, $v(t=0)=1$ and $x(t=0)=4$, $v(t=0)=0$? Do you find a different phase space trajectory for each initial condition? What physical quantity distinguishes the phase space trajectories? Is the motion of a representative point (x, p) always in the clockwise or counterclockwise direction? ■

Problem 4.3 Lissajous figures

A computer display can be used to simulate the output seen on an oscilloscope. Imagine that the vertical and horizontal inputs to an oscilloscope are sinusoidal in time, that is, $x = A_x \sin(\omega_x t + \phi_x)$ and $y = A_y \sin(\omega_y t + \phi_y)$. If the curve that is drawn repeats itself, such a curve is called a *Lissajous figure*. Write a program to plot y versus x as t advances from $t=0$. First choose $A_x = A_y = 1$, $\omega_x = 2$, $\omega_y = 3$, $\phi_x = \pi/6$, and $\phi_y = \pi/4$. For what values of the angular frequencies ω_x and ω_y do you obtain a Lissajous figure? How do the phase factors ϕ_x and ϕ_y and the amplitudes A_x and A_y affect the curves? ■

Waves are ubiquitous in nature and give rise to important phenomena such as beats and standing waves. We investigate their behavior in Problem 4.4. We will study the behavior of waves more systematically in Chapter 9.

Problem 4.4 Superposition of waves

(a) Write a program to plot $A\sin(kx + \omega t)$ from $x = x_{\min}$ to $x = x_{\max}$ as a function of t. (Implement an `AbstractSimulation` rather than an `AbstractCalculation`.) For simplicity, take $A = 1$, $\omega = 2\pi$, and $k = 2\pi/\lambda$, with $\lambda = 2$.

(b) Modify your program so that it plots the sum of $y_1 = \sin(kx - \omega t)$ and $y_2 = \sin(kx + \omega t)$. The quantity $y_1 + y_2$ corresponds to the superposition of two waves. Choose $\lambda = 2$ and $\omega = 2\pi$. What kind of a wave do you obtain?

(c) Use your program to demonstrate beats by plotting $y_1 + y_2$ as a function of time in the range $x_{\min} = -10$ and $x_{\max} = 10$. Determine the beat frequency for each of the following superpositions:

$$y_1(x,t) = \sin[8.4(x - 1.1t)],\ y_2(x,t) = \sin[8.0(x - 1.1t)]$$
$$y_1(x,t) = \sin[8.4(x - 1.2t)],\ y_2(x,t) = \sin[8.0(x - 1.0t)]$$
$$y_1(x,t) = \sin[8.4(x - 1.0t)],\ y_2(x,t) = \sin[8.0(x - 1.2t)]$$

What differences do you observe between these superpositions? ■

4.2 ■ THE MOTION OF A PENDULUM

A common example of a mechanical system that exhibits oscillatory motion is the simple pendulum (see Figure 4.2). A simple pendulum is an idealized system consisting of a particle or bob of mass m attached to the lower end of a rigid rod of length L and negligible mass; the upper end of the rod pivots without friction. If the bob is pulled to one side from its equilibrium position and released, the pendulum swings in a vertical plane.

Because the bob is constrained to move along the arc of a circle of radius L about the center O, the bob's position is specified by its arc length or by the angle θ (see Figure 4.2). The linear velocity and acceleration of the bob as measured along the arc are given by

$$v = L\frac{d\theta}{dt} \tag{4.8}$$

$$a = L\frac{d^2\theta}{dt^2}. \tag{4.9}$$