```
      control.addButton("addMemory", "Remember");
      control.addButton("randomizeState", "Randomize");
   }
}
```

Problem 14.13 Memory recall in the Hopfield model

(a) Use the `HopfieldApp` class to explore the ability of the Hopfield neural network to store and recall memories. Begin with $N = 10$ neurons and click on the cells to choose a pattern to remember. Then click on the Randomize button to randomize the spins. Does the neural network find a pattern similar to the one you saved? Consider other values of N and various patterns to obtain a feel for how the algorithm works.

(b) Store two memories of 20 bits,

$$11111\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}11111 \quad \text{and} \quad 11\bar{1}\bar{1}11\bar{1}\bar{1}11\bar{1}\bar{1}11\bar{1}\bar{1}11\bar{1}\bar{1},$$

where we have written -1 as $\bar{1}$. Recall a particular memory using the input string $1111111\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}1111111$. This input is similar to the first memory. Record the Hamming distance between the final state and the closest memory, where the Hamming distance is the number of bits that differ between two strings. Repeat this procedure for several different values of the number of neurons and the memory length.

(c) Estimate how many memories can be stored for a given number of neurons before recall becomes severely reduced. Make estimates for $N = 10$, 20, and 40. What is your criteria for the recall to be considered correct?

(d) In the Hopfield model, every neuron is linked to every other neuron. (The value of the links w_{ij} is determined by the stored memories.) Is the spatial dimension of the system relevant? Describe how the `HopfieldApp` class can store two-dimensional patterns. ■

Neural networks can also be used for difficult optimization problems. In Problem 14.14 we consider the problem of finding the minimum energy of a model *spin glass*. The latter is a magnetic analog of an ordinary glass in which the positions of the molecules are not ordered as in a crystal. In a spin glass the local magnetic moment is disordered because random magnetic interactions are "frozen in" and do not change. The simplest model of a spin glass is based on the simplest model of magnetism, the Ising model (see Section 15.5). In the Ising model the magnetic moment is represented by a spin s_i which can take on two values, ± 1. The spins are located on the sites of a lattice. Each spin is assumed to interact with all other spins, and the total energy of the system is given by

$$E = -\sum_{i,j \neq i} J_{ij} V_i V_j, \tag{14.6}$$

where the sum is over all pairs of spins. We have let $w \to J$ so that the notation is the same as the Ising model. If $J_{ij} > 0$, the spins i and j lower their energy by lining up in the same direction. If $J_{ij} < 0$, the spins lower their energy by lining up in opposite directions (see Figure 15.1).

We are interested in finding the ground state when the coupling constant J_{ij} randomly takes on the values $\pm J_0/N$, where N is the number of spins and J_0 is an arbitrary constant.

To find the ground state, we need to find the configurations of spins that give the lowest value of the energy. Finding the ground state of a spin glass is particularly difficult because there are many configurations that correspond to local minima of the energy. In fact the problem of finding the exact ground state is an example of a computationally difficult problem called *NP-complete*. (Another example of such a problem is considered in Problem 15.31.) In Problem 14.14 we explore if the Hopfield algorithm can find a good approximation to the global minimum.

Problem 14.14 Minimum energy of an Ising spin glass

(a) Choose $J_0 = 4$ in (14.6) and modify the `HopfieldApp` class so that it applies to a model spin glass. Display the output string and the energy after every N attempts to change a spin. Begin with $N = 20$.

(b) What happens to the energy after a long time? For different initial states but the same set of the J_{ij}, is the value of the energy the same after the system has evolved for a long time? Explain your results in terms of the number of local energy minima.

(c) What is the behavior of the system? Do you find periodic behavior, random behavior, or does the system evolve to a state that does not change? ■

14.4 ■ GROWING NETWORKS

A network is a collection of points called nodes that are connected by lines called links. Mathematicians refer to networks as graphs, and graph theory has been an active field of mathematics for many years. A mathematical network can represent an actual network by defining what a node represents and the kind of relationship represented by a link. For example, in an airline network, the nodes represent airports and the links represent flights between airports. In an acquaintance network, the nodes represent individuals, and the links represent the state of two people knowing each other. In a biochemical network, the nodes represent various molecular types, and the links represent a reaction between molecules.

One reason for the recent interest in networks is that data on existing networks is now more readily available due to the widespread use of computers. Indeed, one of the networks of current interest is the network of websites. Another reason for the interest in networks is that some new models of networks have been developed.

We first discuss one of the original network models, the Erdös–Rényi model. In this model we start with N nodes and then form n links between pairs of nodes such that each pair has either one link or no links. The probability of a link between any pair of nodes is $p = n/(N(N-1)/2)$. One quantity of interest is the degree distribution $D(\ell)$, which is the fraction of nodes that have ℓ links. An example of the determination of $D(\ell)$ is shown in Figure 14.3. In the Erdös–Rényi model, this distribution is a Poisson distribution for large N. Thus, there is a peak in $D(\ell)$, and for large ℓ, $D(\ell)$ decreases exponentially.

In some network models there is a path between any pair of nodes. In other models, such as the Erdös–Rényi model, there are some nodes that cannot be reached from other nodes (see Figure 14.3). In these networks there are other quantities of interest that are analogous to those in percolation theory. The main difference is that in network models the position of the nodes is irrelevant, and only their connectivity is relevant. In particular, there is no