(c) Use the value of g that you found in part (b) and compute the distribution of the number of sites s_f on fire. If the distribution is critical, determine the exponent α that characterizes this distribution. Also compute the distribution for the number of trees s_t. Is there any relation between these two distributions?

*(d) To obtain reliable results, it is frequently necessary to average over many initial configurations. However, the behavior of many systems is independent of the initial configuration, and averaging over many initial configurations is unnecessary. This latter possibility is called *self-averaging*. Repeat parts (b) and (c), but average your results over ten initial configurations. Is this forest fire model self-averaging? ■

Problem 14.11 Another forest fire model

Consider a simple variation of the model discussed in Problem 14.10. At $t = 0$ each site is occupied by a tree with probability p; otherwise, it is empty. The system is updated in successive iterations as follows:

(i) Randomly grow new trees at time t with a small probability g from sites that are empty at time $t - 1$.

(ii) A tree that is not on fire at $t - 1$ catches fire due to lightning with probability f.

(iii) Trees on fire ignite neighboring trees, which in turn ignite their neighboring trees, etc. The spreading of the fire occurs instantaneously.

(iv) Trees on fire at time $t - 1$ die (become empty sites) and are removed at time t (after they have set their neighbors on fire).

As in Problem 14.10, the changes in each site occur synchronously.

(a) Determine $N(s)$, the number of clusters of trees of size s that catch fire in each iteration. Two trees are in the same cluster if they are nearest neighbors. Is the behavior of $N(s)$ consistent with $N(s) \sim s^{-\alpha}$? If so, estimate the exponent α for several values of g and f.

*(b) The balance between the mean rate of birth and burning of trees in the steady state suggests a value for the ratio f/g at which this model is likely to be scale invariant. If the average steady-state density of trees is ρ, then at each iteration the mean number of new trees appearing is $gN(1 - \rho)$, where $N = L^2$ is the total number of sites. In the same spirit, we can say that for small f, the mean number of trees destroyed by lightning is $f\rho N\langle s\rangle$, where $\langle s\rangle$ is the mean number of trees in a cluster. Is this reasoning consistent with the results of your simulation? If we equate these two rates, we find that $\langle s\rangle \sim [(1 - \rho)/\rho](g/f)$. Because $0 < \rho < 1$, it follows that $\langle s\rangle \to \infty$ in the limit $f/g \to 0$. Given the relation $\langle s\rangle = \sum_{s=1}^{\infty} sN(s)/\sum_s N(s)$ and the divergent behavior of $\langle s\rangle$, why does it follow that $N(s)$ must decay more slowly than exponentially with s? This reasoning suggests that $N(s) \sim s^{-\alpha}$ with $\alpha < 2$. Is this expectation consistent with the results that you obtained in part (a)?

In this model there are three well-separated time scales, that is, the time for lightning to strike ($\propto f^{-1}$), the time for trees to grow ($\propto g^{-1}$), and the instantaneous spreading of fire through a connected cluster. This separation of time scales seems to be an essential ingredient for self-organized criticality (see Grinstein and Jayaprakash). ■

Problem 14.12 Model of punctuated equilibrium

(a) The idea of *punctuated equilibrium* is that biological evolution occurs episodically rather than as a steady, gradual process. That is, most of the major changes in life forms occur in relatively short periods of time. Bak and Sneppen have proposed a simple model that exhibits some of the behavior of punctuated equilibrium. The model consists of a one-dimensional cellular automaton of linear dimension L, where cell i represents the biological fitness of species i. Initially, all cells receive a random fitness f_i between 0 and 1. Then the cell with the lowest fitness and its two nearest neighbors are randomly given new fitness values. This update rule is repeated indefinitely. Write a program to simulate the behavior of this model. Use periodic boundary conditions and display the fitness of each cell as a column of height f_i. Begin with $L = 64$ and describe what happens to the distribution of fitness values after a long time.

(b) We can crudely think of the update process as replacing a species and its neighbors by three new species. In this sense the fitness represents a barrier to creating a new species. If the barrier is low, it is easier to create a new species. Do the low fitness species die out? What is the average value of fitness of the species after the model is run for a long time (10^4 or more iterations)? Compute the distribution of fitness values $N(f)$ averaged over all cells and over many iterations. Allow the system to come to a fluctuating steady state before computing $N(f)$. Plot $N(f)$ versus f. Is there a critical value f_c below which $N(f)$ is much less than the values above f_c? Is the update rule reasonable from a evolutionary point of view?

(c) Modify your program to compute the distance x between successive fitness changes and the distribution of these distances $P(x)$. Make a log-log plot of $P(x)$ versus x. Is there any evidence of self-organized criticality (power law scaling)?

(d) Another way to visualize the results is to make a plot of the time at which a cell is changed versus the position of the cell. Is the distribution of the plotted points approximately uniform? We might expect that the survival time of a species depends exponentially on its fitness, and hence each update corresponds to an elapsed time of e^{-cf_i}, where the constant c sets the time scale, and f_i is the fitness of the cell that has been changed. Choose $c = 100$ and make a similar plot with the time axis replaced by the logarithm of the time, that is, the quantity $100f_i$. Is this plot more meaningful?

(e) Another way of visualizing punctuated equilibrium is to plot the number of times groups of cells change as a function of time. Divide the time into units of 100 updates and compute the number of fitness changes for cells $i = 1$ to 10 as a function of time. Do you see any evidence of punctuated equilibrium? ■

14.3 ■ THE HOPFIELD MODEL AND NEURAL NETWORKS

Neural network models have been motivated in part by how neurons in the brain collectively store and recall memories. Usually, a neuron is in one of two states, a resting potential (not firing) or firing at the maximum rate. A neuron "fires" once it receives electrical inputs from other neurons whose strength reaches a certain threshold. An important characteristic of a