measurements of a single quantity over time. The dimension $D_c(d)$ computed by increasing the dimension of the space d using the delayed coordinate τ eventually saturates when d is approximately equal to the number of variables that actually determine the dynamics. Hence, if we have extensive data for a single variable, for example, the atmospheric pressure or a stock market index, we can use this method to determine the number of independent variables that determine the dynamics of the variable. This information can then be used to help create models of the dynamics.

13.5 ■ MANY DIMENSIONS

So far we have discussed three ways of defining the fractal dimension: the mass dimension (13.1), the box dimension (13.17), and the correlation dimension (13.19). These methods do not always give the same results for the fractal dimension. Indeed, there are many other dimensions that we could compute. For example, instead of just counting the boxes that contain a part of an object, we can count the number of points of the object in each box n_i and compute $p_i = n_i/N$, where N is the total number of points. A generalized dimension D_q can be defined as

$$D_q = \frac{1}{q-1} \lim_{\ell \to 0} \frac{\ln \sum_{i=1}^{N(\ell)} p_i^q}{\ln \ell}. \tag{13.20}$$

The sum in (13.20) is over all the boxes and involves the probabilities raised to the qth power. For $q = 0$, we have

$$D_0 = -\lim_{\ell \to 0} \frac{\ln N(\ell)}{\ln \ell}. \tag{13.21}$$

If we compare the form of (13.21) with (13.17), we can identify D_0 with the box dimension. For $q = 1$, we need to take the limit of (13.20) as $q \to 1$. Let

$$u(q) = \ln \sum_i p_i{}^q, \tag{13.22}$$

and do a Taylor-series expansion of $u(q)$ about $q = 1$. We have

$$u(q) = u(1) + (q-1)\frac{du}{dq} + \cdots. \tag{13.23}$$

The quantity $u(1) = 0$ because $\sum_i p_i = 1$. The first derivative of $u(q)$ is given by

$$\frac{du}{dq} = \frac{\sum_i p_i{}^q \ln p_i}{\sum_i p_i{}^q} = \sum_i p_i \ln p_i, \tag{13.24}$$

where the last equality follows by setting $q = 1$. If we use the above relations, we find that D_1 is given by

$$D_1 = \lim_{\ell \to 0} \frac{\sum_i p_i \ln p_i}{\ln \ell} \quad \text{(information dimension)}. \tag{13.25}$$

D_1 is called the *information dimension* because of the similarity of the $p \ln p$ term in the numerator of (13.24) to the information form of the entropy.

It is possible to show that D_2 as defined by (13.20) is the same as the mass dimension defined in (13.1) and the correlation dimension D_c. That is, box counting gives D_0 and correlation functions give D_2 (cf. Sander et al.).

There are many objects in nature that differ in appearance but have similar fractal dimension. An example is the different visual appearance in three dimensions of diffusion limited aggregation clusters and the percolation clusters at the percolation threshold. (Both objects have a fractal dimension of approximately 2.5.) In some cases this difference can be accounted for by the *multifractal* properties of an object. For *multifractals* the various D_q are different, in contrast to *monofractals* for which the different measures are the same. Percolation clusters are an example of a monofractal because $p_i \sim \ell^{D_0}$, the number of boxes $N(\ell) \sim \ell^{-D_0}$, and from (13.20) $D_q = D_0$ for all q. Multifractals occur when the growth quantities are not the same throughout the object, as frequently happens for the strange attractors produced by chaotic dynamics. Diffusion limited aggregation is an example of a multifractal.

13.6 ■ PROJECTS

Although the kinetic growth models we have considered yield beautiful pictures, there is much we do not understand. For example, the fractal dimension of DLA clusters can be calculated only by approximate theories whose accuracy is unknown. Why do the fractal dimensions have the values that we estimated by various simulations? Can we trust our numerical estimates of the various exponents, or is it necessary to consider much larger systems to obtain their true asymptotic values? Can we find unifying features for the many kinetic growth models that presently exist? What is the relation of the various kinetic growth models to physical systems? What are the essential quantities needed to characterize the geometry of an object?

One of the reasons that kinetic growth models are difficult to understand is that the final cluster typically depends on the history of the growth. We say that these models are examples of "nonequilibrium behavior." The combination of simplicity, beauty, complexity, and relevance to many experimental systems suggests that the study of fractal objects will continue to involve a wide range of workers in many disciplines.

Project 13.15 The percolation cluster size distribution

Use the Leath algorithm to determine the critical exponent τ of the cluster size distribution n_s for percolation clusters at $p = p_c$:

$$n_s \sim s^{-\tau} \quad (s \gg 1). \tag{13.26}$$

Modify class `SingleCluster` so that many clusters are generated and n_s is computed for a given probability p. Remember that the number of clusters of size s that are grown from a seed is the product sn_s, rather than n_s itself (see Problem 13.3a). Grow at least 100 clusters on a square lattice with $L \geq 61$. If time permits, use bigger lattices and average over more clusters and also estimate the accuracy of your estimate of τ. See Grassberger for a discussion of an extension of this approach to estimating the value of p_c in higher dimensions. ■