Each time a particle is deposited, the time t is increased by unity. Our main interest is how the width of the surface changes with t. We define the width of the surface by

$$w^2 = \frac{1}{N_s}\sum_{i=1}^{N_s}(h_i - \overline{h})^2. \tag{13.12}$$

In general, the width w, which is a measure of the surface roughness, depends on L and t. For short times we expect that

$$w(L,t) \sim t^{\beta}. \tag{13.13}$$

The exponent β describes the growth of the correlations with time along the vertical direction.

Figure 13.12 illustrates the evolution of the surface generated according to the Eden model. After a characteristic time, the length over which the fluctuations are correlated becomes comparable to L, and the width reaches a steady state value that depends only on L. We write

$$w(L, t \gg 1) \sim L^{\alpha}, \tag{13.14}$$

where α is known as the roughness exponent.

From (13.14) we see that in the steady state, the width of the surface in the direction perpendicular to the substrate grows as L^{α}. This scaling behavior of the width is characteristic of a *self-affine fractal*. Such a fractal is invariant (on the average) under anisotropic scale changes; that is, different scaling relations exist along different directions. For example, if we rescale the surface by a factor b in the horizontal direction, then the surface must be rescaled by a factor of b^{α} in the direction perpendicular to the surface to preserve the similarity along the original and rescaled surfaces.

Note that on short length scales, that is, lengths shorter than the width of the interface, the surface is rough and its roughness can be characterized by the exponent α. (Imagine an ant walking on the surface.) For length scales much larger than the width of the surface, the surface appears to be flat and, in our example, it is a one-dimensional object. The properties of the surface generated by several growth models are explored in Problem 13.11.

Problem 13.11 Growing surfaces

(a) In the Eden model a perimeter site is chosen at random and occupied. The growth rule is the same as the usual Eden model, but the growth is started from a line of length L rather than a single site. Hence, there can be "overhangs" as shown in Figure 13.12. Use periodic boundary conditions in the horizontal direction to determine the perimeter sites. The height h_i corresponds to the height of column i. Consider $L = 64$. Describe the visual appearance of the surface as the surface grows. Is the surface well defined visually? Where are most of the perimeter sites?

(b) To estimate the exponents α and β, plot the width $w(t)$ as a function of t for $L = 32$, 64, and 128 on the same graph. What type of plot is most appropriate? Does the width initially grow as a power law? If so, estimate the exponent β. Is there an L-dependent crossover time after which the width of the surface approaches its steady-state value?

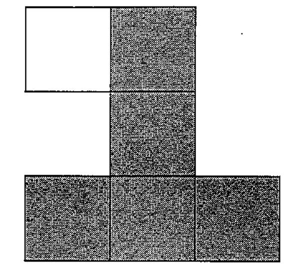

Figure 13.13 Example of the growth of a surface according to the ballistic deposition model. Note that if column one is chosen, the next site that would be occupied (not shaded) would leave an unoccupied site below it.

How can you estimate the exponent α? The best numerical estimates for β and α are consistent with the exact values $\beta = 1/3$ and $\alpha = 1/2$.

*(c) The dependence of $w(L,t)$ on t and L can be combined into the scaling form

$$w(L,t) \approx L^{\alpha} f(t/L^{\alpha/\beta}), \tag{13.15}$$

where

$$f(x) = \begin{cases} Ax^{\beta} & x \ll 1 \\ \text{constant} & x \gg 1, \end{cases} \tag{13.16}$$

where A is a constant. Verify the existence of the scaling form (13.15) by plotting the ratio $w(L,t)/L^{\alpha}$ versus $t/L^{\alpha/\beta}$ for the different values of L considered in part (b). If the scaling form holds, the results for w for the different values of L should fall on a universal curve. Use either the estimated values of α and β that you found in part (b) or the exact values.

(d) The Eden model is not really a surface growth model, because any perimeter site can become part of the cluster. In the simplest *random deposition* model, a column is chosen at random and a particle is deposited at the top of the column of already deposited particles. There is no horizontal correlation between neighboring columns. Do a simulation of this growth model and visually inspect the surface of the interface. Show that the heights of the columns follow a Poisson distribution (see (7.31)) and that $\overline{h} \sim t$ and $w \sim t^{1/2}$. This structure does not depend on L and hence $\alpha = 0$.

(e) In the *ballistic deposition* model, a column is chosen at random and a particle is assumed to fall vertically until it reaches the first perimeter site that is a nearest neighbor of a site that already is part of the surface. See Figure 13.13. This condition allows for growth parallel to the substrate. Only one particle falls at a time. How do the rules for this growth model differ from those of the Eden model? How does the surface compare to that of the Eden model? Suppose that instead of the particle falling vertically, we let it do a random walk as in DLA. Would the resultant surface be the same? ■