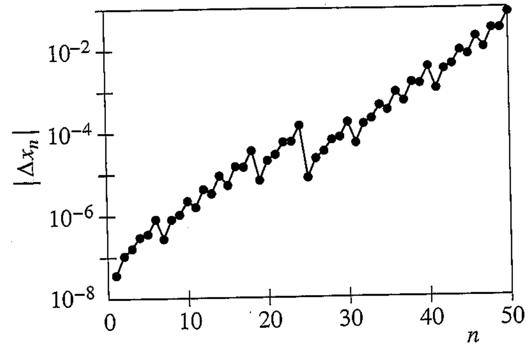

Figure 6.8 The evolution of the difference Δx_n between the trajectories of the logistic map at $r = 0.91$ for $x_0 = 0.5$ and $x_0 = 0.5001$. The separation between the two trajectories increases with n, the number of iterations, if n is not too large. (Note that $|\Delta x_1| \sim 10^{-8}$ and that the trend is not monotonic.)

by the relation

$$|\Delta x_n| = |\Delta x_0|\, e^{\lambda n}, \tag{6.14}$$

where Δx_n is the difference between the trajectories at time n. If the Lyapunov exponent λ is positive, then nearby trajectories diverge exponentially. Chaotic behavior is characterized by the *exponential divergence of nearby trajectories*.

A naive way of measuring the Lyapunov exponent λ is to run the same dynamical system twice with slightly different initial conditions and measure the difference of the trajectories as a function of n. We used this method to generate Figure 6.8. Because the rate of separation of the trajectories might depend on the choice of x_0, a better method would be to compute the rate of separation for many values of x_0. This method would be tedious because we would have to fit the separation to (6.14) for each value of x_0 and then determine an average value of λ.

A more important limitation of the naive method is that because the trajectory is restricted to the unit interval, the separation $|\Delta x_n|$ ceases to increase when n becomes sufficiently large. Fortunately, there is a better way of determining λ. We take the natural logarithm of both sides of (6.14), and write λ as

$$\lambda = \frac{1}{n} \ln \left| \frac{\Delta x_n}{\Delta x_0} \right|. \tag{6.15}$$

Because we want to use the data from the entire trajectory after the transient behavior has ended, we use the fact that

$$\frac{\Delta x_n}{\Delta x_0} = \frac{\Delta x_1}{\Delta x_0} \frac{\Delta x_2}{\Delta x_1} \cdots \frac{\Delta x_n}{\Delta x_{n-1}}. \tag{6.16}$$

Hence, we can express λ as

$$\lambda = \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| \frac{\Delta x_{i+1}}{\Delta x_i} \right|. \tag{6.17}$$

The form (6.17) implies that we can interpret x_i for any i as the initial condition.

We see from (6.17) that the problem of computing λ has been reduced to finding the ratio $\Delta x_{i+1}/\Delta x_i$. Because we want to make the initial difference between the two trajectories as small as possible, we are interested in the limit $\Delta x_i \to 0$.

The idea of the more sophisticated procedure is to compute dx_{i+1}/dx_i from the equation of motion at the same time that the equation of motion is being iterated. We use the logistic map as an example. From (6.5) we have

$$\frac{dx_{i+1}}{dx_i} = f'(x_i) = 4r(1 - 2x_i). \tag{6.18}$$

We can consider x_i for any i as the initial condition and the ratio dx_{i+1}/dx_i as a measure of the rate of change of x_i. Hence, we can iterate the logistic map as before and use the values of x_i and the relation (6.18) to compute $f'(x_i) = dx_{i+1}/dx_i$ at each iteration. The Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right|, \tag{6.19}$$

where we begin the sum in (6.19) after the transient behavior is finished. We have explicitly included the limit $n \to \infty$ in (6.19) to remind ourselves to choose n sufficiently large. Note that this procedure weights the points on the attractor correctly; that is, if a particular region of the attractor is not visited often by the trajectory, it does not contribute much to the sum in (6.19).

Problem 6.9 Lyapunov exponent for the logistic map

(a) Modify `IterateMapApp` to compute the Lyapunov exponent λ for the logistic map using the naive approach. Choose $r = 0.91$, $x_0 = 0.5$, and $\Delta x_0 = 10^{-6}$, and plot $\ln |\Delta x_n/\Delta x_0|$ versus n. What happens to $\ln |\Delta x_n/\Delta x_0|$ for large n? Determine λ for $r = 0.91$, $r = 0.97$, and $r = 1.0$. Does your result for λ for each value of r depend significantly on your choice of x_0 or Δx_0?

(b) Modify `BifurcateApp` to compute λ using the algorithm discussed in the text for $r = 0.76$ to $r = 1.0$ in steps of $\Delta r = 0.01$. What is the sign of λ if the system is not chaotic? Plot λ versus r and explain your results in terms of behavior of the bifurcation diagram shown in Figure 6.2. Compare your results for λ with those shown in Figure 6.9. How does the sign of λ correlate with the behavior of the system as seen in the bifurcation diagram? For what value of r is λ a maximum?

(c) In Problem 6.3b we saw that roundoff errors in the chaotic regime make the computation of individual trajectories meaningless. That is, if the system's behavior is chaotic, then small roundoff errors are amplified exponentially in time, and the actual numbers we compute for the trajectory starting from a given initial value are not "real." Repeat your calculation of λ for $r = 1$ by changing the roundoff error as you did in Problem 6.3b. Does your computed value of λ change? How meaningful is your computation of the Lyapunov exponent? We will encounter a similar question in Chapter 8 where we compute the trajectories of chaotic systems of many particles. We will find that although the "true" trajectories cannot be computed for long times, averages over the trajectories yield meaningful results. ■