interval $[x_i, x_{i+1}]$ is of order $f''''(x_i)(\Delta x)^5$, and that the total error in the interval $[a, b]$ associated with Simpson's rule is $O(n^{-4})$.

The error estimates can be extended to two dimensions in a similar manner. The two-dimensional integral of $f(x, y)$ is the volume under the surface determined by $f(x, y)$. In the "rectangular" approximation, the integral is written as a sum of the volumes of parallelograms with cross-sectional area $\Delta x \Delta y$ and a height determined by $f(x, y)$ at one corner. To determine the error, we expand $f(x, y)$ in a Taylor series:

$$f(x, y) = f(x_i, y_i) + \frac{\partial f(x_i, y_i)}{\partial x}(x - x_i) + \frac{\partial f(x_i, y_i)}{\partial y}(y - y_i) + \cdots, \tag{11.68}$$

and write the error as

$$\Delta_i = \left[\int\!\!\int f(x, y)\, dx\, dy\right] - f(x_i, y_i)\Delta x \Delta y. \tag{11.69}$$

If we substitute (11.68) into (11.69) and integrate each term, we find that the term proportional to f cancels and the integral of $(x - x_i)\, dx$ yields $\frac{1}{2}(\Delta x)^2$. The integral of this term with respect to dy gives another factor of Δy. The integral of the term proportional to $(y - y_i)$ yields a similar contribution. Because Δy is also order Δx, the error associated with the intervals $[x_i, x_{i+1}]$ and $[y_i, y_{i+1}]$ is to leading order in Δx:

$$\Delta_i \approx \frac{1}{2}[f_x'(x_i, y_i) + f_y'(x_i, y_i)](\Delta x)^3. \tag{11.70}$$

We see that the error associated with one parallelogram is order $(\Delta x)^3$. Because there are n parallelograms, the total error is order $n(\Delta x)^3$. However, in two dimensions, $n = A/(\Delta x)^2$, and hence the total error is order $n^{-1/2}$. In contrast, the total error in one dimension is order n^{-1}, as we saw earlier.

The corresponding error estimates for the two-dimensional generalizations of the trapezoidal approximation and Simpson's rule are order n^{-1} and n^{-2}, respectively. In general, if the error goes as order n^{-a} in one dimension, then the error in d dimensions goes as $n^{-a/d}$. In contrast, Monte Carlo errors vary as $n^{-1/2}$ independent of d. Hence, for large enough d, Monte Carlo integration methods will lead to smaller errors for the same choice of n.

APPENDIX 11B: THE STANDARD DEVIATION OF THE MEAN

In Section 11.4 we found empirically that the probable error associated with a single measurement consisting of n samples is $\sigma/\sqrt{n}$, where σ is the standard deviation associated with n data points. We derive this relation in the following. The quantity of experimental interest is denoted as x. Consider m sets of measurements each with n samples for a total of mn samples. For simplicity, we will assume that $n \gg 1$. We use the index α to denote a particular measurement and the index i to designate the ith sample within a measurement. We denote $x_{\alpha,i}$ as sample i in the measurement α. The value of a measurement is given by

$$M_\alpha = \frac{1}{n}\sum_{i=1}^{n} x_{\alpha,i}. \tag{11.71}$$

The mean $\overline{M}$ of the *total* mn individual samples is given by

$$\overline{M} = \frac{1}{m}\sum_{\alpha=1}^{m} M_\alpha = \frac{1}{mn}\sum_{\alpha=1}^{m}\sum_{i=1}^{n} x_{\alpha,i}. \tag{11.72}$$

The difference between measurement α and the mean of all the measurements is given by

$$e_\alpha = M_\alpha - \overline{M}. \tag{11.73}$$

We can write the variance of the means as

$$\sigma_m^2 = \frac{1}{m}\sum_{\alpha=1}^{m} e_\alpha^2. \tag{11.74}$$

We now wish to relate σ_m to the variance of the individual measurements. The discrepancy $d_{\alpha,i}$ between an individual sample $x_{\alpha,i}$ and the mean is given by

$$d_{\alpha,i} = x_{\alpha,i} - \overline{M}. \tag{11.75}$$

Hence, the variance σ^2 of the mn individual samples is

$$\sigma^2 = \frac{1}{mn}\sum_{\alpha=1}^{m}\sum_{i=1}^{n} d_{\alpha,i}^2. \tag{11.76}$$

We write

$$e_\alpha = M_\alpha - \overline{M} = \frac{1}{n}\sum_{i=1}^{n}(x_{\alpha,i} - \overline{M}) \tag{11.77a}$$

$$= \frac{1}{n}\sum_{i=1}^{n} d_{\alpha,i}. \tag{11.77b}$$

If we substitute (11.77b) into (11.74), we find

$$\sigma_m^2 = \frac{1}{m}\sum_{\alpha=1}^{n}\left(\frac{1}{n}\sum_{i=1}^{n} d_{\alpha,i}\right)\left(\frac{1}{n}\sum_{j=1}^{n} d_{\alpha,j}\right). \tag{11.78}$$

The sum in (11.78) over samples i and j in set α contains two kinds of terms—those with $i = j$ and those with $i \neq j$. We expect that $d_{\alpha,i}$ and $d_{\alpha,j}$ are independent and equally positive or negative on the average. Hence, in the limit of a large number of measurements, we expect that only the terms with $i = j$ in (11.78) survive, and we write

$$\sigma_m^2 = \frac{1}{mn^2}\sum_{\alpha=1}^{n}\sum_{i=1}^{n} d_{\alpha,i}^2. \tag{11.79}$$

If we combine (11.79) with (11.76), we arrive at the result

$$\sigma_m^2 = \frac{\sigma^2}{n}. \tag{11.80}$$