Poisson's equation can also be solved using the random walk method. In this case the potential is given by

$$V(x,y) = \frac{1}{n}\sum_{\alpha} V(\alpha) + \frac{\pi\,\Delta x\,\Delta y}{n}\sum_{i,\alpha}\rho(x_{i,\alpha}, y_{i,\alpha}), \tag{10.27}$$

where α labels the walker and i labels the site visited by the walker. That is, each time a walker is at site i, we add the charge density at that site to the second sum in (10.27).

*10.7 ■ FIELDS DUE TO MOVING CHARGES

The fact that accelerating charges radiate electromagnetic waves is one of the more important results in the history of physics. In this section we discuss a numerical algorithm for computing the electric and magnetic fields due to the motion of charged particles. The algorithm is very general, but requires some care in its application.

To understand the algorithm, we need a few results that can be conveniently found in Feynman's lectures. We begin with the fact that the scalar potential at the observation point $\mathbf{R}$ due to a stationary particle of charge q is

$$V(\mathbf{R}) = \frac{q}{|\mathbf{R}-\mathbf{r}|}, \tag{10.28}$$

where $\mathbf{r}$ is the position of the charged particle. The electric field is given by

$$\mathbf{E}(\mathbf{R}) = -\frac{\partial V(\mathbf{R})}{\partial \mathbf{R}}, \tag{10.29}$$

where $\partial V(\mathbf{R})/\partial\mathbf{R}$ is the gradient with respect to the coordinates of the observation point. (Note that our notation for the observation point differs from that used in other sections of this chapter.) How do the relations (10.28) and (10.29) change when the particle is moving? We might guess that because it takes a finite time for the disturbance due to a charge to reach the point of observation, we should modify (10.28) by writing

$$V(\mathbf{R}) \stackrel{?}{=} \frac{q}{r_{\rm ret}}, \tag{10.30}$$

where

$$r_{\rm ret} = |\mathbf{R} - \mathbf{r}(t_{\rm ret})|. \tag{10.31}$$

The quantity $r_{\rm ret}$ is the separation of the charged particle from the observation point $\mathbf{R}$ at the retarded time $t_{\rm ret}$. The latter is the time at which the particle was at $\mathbf{r}(t_{\rm ret})$ such that a disturbance starting at $\mathbf{r}(t_{\rm ret})$ and traveling at the speed of light would reach $\mathbf{R}$ at time t; $t_{\rm ret}$ is given by the implicit equation

$$t_{\rm ret} = t - \frac{r_{\rm ret}(t_{\rm ret})}{c}, \tag{10.32}$$

where t is the observation time and c is the speed of light.

Although the above reasoning is plausible, the relation (10.30) is not quite correct (cf. Feynman et al. for a derivation of the correct result). We need to take into account that the potential due to the charge is a maximum if the particle is moving toward the observation point and a minimum if it is moving away. The correct result can be written as

$$V(\mathbf{R},t) = \frac{q}{r_{\rm ret}(1 - \hat{\mathbf{r}}_{\rm ret}\cdot\mathbf{v}_{\rm ret}/c)}, \tag{10.33}$$

where

$$\mathbf{v}_{\rm ret} = \frac{d\mathbf{r}(t)}{dt}\Big|_{t=t_{\rm ret}}, \tag{10.34}$$

and $\hat{\mathbf{r}} = \mathbf{r}/r$.

To find the electric field of a moving charge, we recall that the electric field is related to the time rate of change of the magnetic flux. Hence, we expect that the total electric field at the observation point $\mathbf{R}$ has a contribution due to the magnetic field created by the motion of the charge. We know that the magnetic field due to a moving charge is given by

$$\mathbf{B} = \frac{1}{c}\frac{q\mathbf{v}\times\mathbf{r}}{r^3}. \tag{10.35}$$

If we define the vector potential $\mathbf{A}$ as

$$\mathbf{A} = \frac{q}{r}\frac{\mathbf{v}}{c}, \tag{10.36}$$

we can express $\mathbf{B}$ in terms of $\mathbf{A}$ as

$$\mathbf{B} = \nabla\times\mathbf{A}. \tag{10.37}$$

As we did for the scalar potential V, we argue that the correct formula for $\mathbf{A}$ is

$$\mathbf{A}(\mathbf{R},t) = q\frac{\mathbf{v}_{\rm ret}/c}{r_{\rm ret}(1-\hat{\mathbf{r}}_{\rm ret}\cdot\mathbf{v}_{\rm ret}/c)}. \tag{10.38}$$

Equations (10.33) and (10.38) are known as the Liénard–Wiechert form of the potentials.

The contribution to the electric field $\mathbf{E}$ from V and $\mathbf{A}$ is given by

$$\mathbf{E} = -\nabla V - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}. \tag{10.39}$$

The derivatives in (10.39) are with respect to the observation coordinates. The difficulty associated with calculating these derivatives is that the potentials depend on $t_{\rm ret}$, which in turn depends on $\mathbf{R}$, $\mathbf{r}$, and t. The result can be expressed as

$$\mathbf{E}(\mathbf{R},t) = \frac{q r_{\rm ret}}{(\mathbf{r}_{\rm ret}\cdot\mathbf{u}_{\rm ret})^3}[\mathbf{u}_{\rm ret}(c^2 - v_{\rm ret}^2) + \mathbf{r}_{\rm ret}\times(\mathbf{u}_{\rm ret}\times\mathbf{a}_{\rm ret})], \tag{10.40}$$

where

$$\mathbf{u}_{\rm ret} \equiv c\hat{\mathbf{r}}_{\rm ret} - \mathbf{v}_{\rm ret}. \tag{10.41}$$

The acceleration of the particle is given by $\mathbf{a}_{\rm ret} = d\mathbf{v}(t)/dt|_{t=t_{\rm ret}}$. We can also show using (10.37) that the magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = \hat{\mathbf{r}}_{\rm ret}\times\mathbf{E}. \tag{10.42}$$