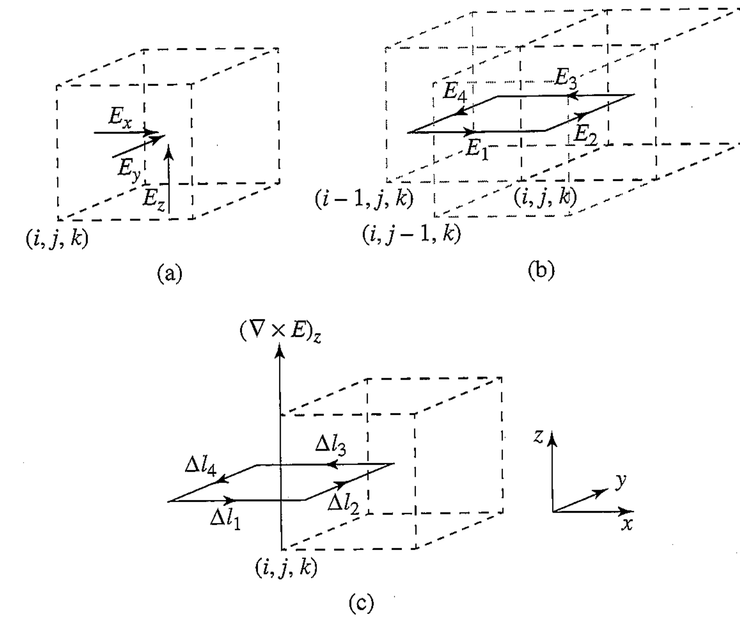

Figure 10.7 Calculation of the curl of the vector **E** defined on the faces of a cube. (a) The face vector **E** associated with the cube (i, j, k). The components associated with the left, front, and bottom faces are $E_x(i, j, k)$, $E_y(i, j, k)$, $E_z(i, j, k)$, respectively. (b) The components E_i on the faces that share the front left edge of the cube (i, j, k). $E_1 = E_x(i, j-1, k)$, $E_2 = E_y(i, j, k)$, $E_3 = -E_x(i, j, k)$, and $E_4 = -E_y(i-1, j, k)$. The cubes associated with E_1 and E_4 are also shown. (c) The vector components Δl_i on the faces that share the left front edge of the cube. (The z-component of the curl of **E** defined on the left edge points in the positive z direction.)

is similar to (10.52) with B_i replaced by E_i, where E_i and l_i are shown in Figures 10.7b and 10.7c, respectively. The z-component of $\nabla \times \mathbf{E}$ is along the left edge of the front face.

A coordinate free definition of the divergence of the vector field **W** is

$$\nabla \cdot \mathbf{W} = \lim_{V \to 0} \frac{1}{V} \oint_S \mathbf{W} \cdot d\mathbf{S}, \tag{10.53}$$

where V is the volume enclosed by the closed surface **S**. The divergence measures the average flow of the vector through a closed surface. An example of the discrete version of (10.53) is given in (10.54).

We now discuss where to define the quantities ρ, **j**, **E**, and **B** on the grid. It is natural to define the charge density ρ at the center of a cube. From the continuity equation (10.48), we see that this definition leads us to define **j** at the faces of the cube. Hence, each face of a cube has a number associated with it corresponding to the current density flowing parallel to the outward normal to that face. Given the definition of **j** on the grid, we see from (10.47) that the electric field **E** and **j** should be defined at the same places, and hence we define the electric field on the faces of the cubes. Because **E** is defined on the faces, it is natural to define the magnetic field **B** on the edges of the cubes. Our definitions of the vectors **j**, **E**, and **B** on the grid are now complete.

We label the faces of cube c by the symbol f_c. If we use the simplest finite difference method with a discrete time step Δt and discrete spatial interval $\Delta x = \Delta y = \Delta z \equiv \Delta l$,

we can write the continuity equation as

$$[\rho(c, t + \Delta t/2) - \rho(c, t - \Delta t/2)] = -\frac{\Delta t}{\Delta l} \sum_{f_c=1}^{6} j(f_c, t). \tag{10.54}$$

The factor of $1/\Delta l$ comes from the area of a face $(\Delta l)^2$ used in the surface integral in (10.53) divided by the volume $(\Delta l)^3$ of a cube. In the same spirit, the discretization of (10.47) can be written as:

$$E(f, t + \Delta t/2) - E(f, t - \Delta t/2) = \Delta t[\nabla \times \mathbf{B} - 4\pi j(f, t)]. \tag{10.55}$$

Note that **E** in (10.55) and ρ in (10.54) are defined at different times than **j**. As usual, we choose units such that $c = 1$.

We next need to define a square around which we can discretize the curl. If **E** is defined on the faces, it is natural to use the square that is the border of the faces. As we have discussed, this choice implies that we should define the magnetic field on the edges of the cubes. We write (10.55) as

$$E(f, t + \Delta t/2) - E(f, t - \Delta t/2) = \Delta t \left[\frac{1}{\Delta l} \sum_{e_f=1}^{4} B(e_f, t) - 4\pi j(f, t) \right], \tag{10.56}$$

where the sum is over e_f, the four edges of the face f (see Figure 10.7b). Note that B is defined at the same time as j. In a similar way we can write the discrete form of (10.46) as

$$B(e, t + \Delta t) - B(e, t) = -\frac{\Delta t}{\Delta l} \sum_{f_e=1}^{4} E(f_e, t + \Delta t/2), \tag{10.57}$$

where the sum is over f_e, the four faces that share the same edge e (see Figure 10.7b).

We now have a well-defined algorithm for computing the spatial dependence of the electric and magnetic field, the charge density, and the current density as a function of time. This algorithm was developed in 1966 by Yee, an electrical engineer, and independently in 1988 by Visscher, a physicist, who also showed that all of the integral relations and other theorems that are satisfied by the continuum fields are also satisfied for the discrete fields.

Usually, the most difficult part of this algorithm is specifying the initial conditions because we cannot simply place a charge somewhere. The reason is that the initial fields appropriate for this charge would not be present. Indeed, our rules for updating the fields and the charge densities reflect the fact that the electric and magnetic fields do not appear instantaneously at all positions in space when a charge appears, but instead evolve from the initial appearance of a charge. Of course, charges do not appear out of nowhere, but appear by disassociating from neutral objects. Conceptually, the simplest initial condition corresponds to two charges of opposite sign moving oppositely to each other. This condition corresponds to an initial current on one face. From this current a charge density and thus an electric field appears using (10.54) and (10.56), respectively, and a magnetic field appears using (10.57).

Because we cannot compute the fields for an infinite lattice, we need to specify the boundary conditions. The easiest method is to use fixed boundary conditions such that the fields vanish at the edges of the lattice. If the lattice is sufficiently large, fixed boundary