from its maximum value. Is the value of τ constant throughout the motion? Compute τ for the values of γ considered in part (b) and discuss the qualitative dependence of τ on γ.

(d) Plot the total energy as a function of time for the values of γ considered in part (b). If the decrease in energy is not monotonic, explain.

(e) Compute the time dependence of $x(t)$ and $v(t)$ for $\gamma = 4$, 5, 6, 7, and 8. Is the motion oscillatory for all γ? How can you characterize the decay? For fixed ω_0, the oscillator is said to be *critically damped* at the smallest value of γ for which the decay to equilibrium is monotonic. For what value of γ does critical damping occur for $\omega_0 = 4$ and $\omega_0 = 2$? For each value of ω_0, compute the value of γ for which the system approaches equilibrium most quickly.

(f) Compute the phase space diagram for $\omega_0 = 3$ and $\gamma = 0.5$, 2, 4, 6, and 8. Why does the phase space trajectory converge to the origin, $x = 0$, $v = 0$? This point is called an *attractor*. Are these qualitative features of the phase space plot independent of γ? ■

Problem 4.7 Damped nonlinear pendulum

Consider a damped pendulum with $\omega_0 = \sqrt{g/L} = 3$ and a damping term equal to $-\gamma\, d\theta/dt$. Choose $\gamma = 1$ and the initial condition $\theta(t = 0) = 0.2$, $d\theta(t = 0)/dt = 0$. In what ways is the motion of the damped nonlinear pendulum similar to the damped linear oscillator? In what ways is it different? What is the shape of the phase space trajectory for the initial condition $\theta(t = 0) = 1$, $\omega(t = 0) = 0$? Do you find a different phase space trajectory for other initial conditions? Remember that θ is restricted to be between $-\pi$ and $+\pi$. ■

4.4 ■ RESPONSE TO EXTERNAL FORCES

How can we determine the period of a pendulum that is not already in motion? The obvious way is to disturb the system, for example, to displace the bob and observe its motion. We will find that the nature of the response of the system to a small perturbation tells us something about the nature of the system in the absence of the perturbation.

Consider the driven damped linear oscillator with an external force $F(t)$ in addition to the linear restoring force and linear damping force. The equation of motion can be written as

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma v + \frac{1}{m}F(t). \tag{4.17}$$

It is customary to interpret the response of the system in terms of the displacement x rather than the velocity v.

The time dependence of $F(t)$ in (4.17) is arbitrary. Because many forces in nature are periodic, we first consider the form

$$\frac{1}{m}F(t) = A_0 \cos \omega t, \tag{4.18}$$

where ω is the angular frequency of the driving force.

Problem 4.8 Response of a driven damped linear oscillator

(a) Modify your simple harmonic oscillator program so that an external force of the form (4.18) is included. Add this force to the class that encapsulates the equations of motion without changing the target class. The angular frequency of the driving force should be added as an input parameter.

(b) Choose $\omega_0 = 3$, $\gamma = 0.5$, $\omega = 2$ and the amplitude of the external force $A_0 = 1$ for all runs unless otherwise stated. For these values of ω_0 and γ, the dynamical behavior in the absence of an external force corresponds to an underdamped oscillator. Plot $x(t)$ versus t in the presence of the external force with the initial condition $x(t = 0) = 1$, $v(t = 0) = 0$. How does the qualitative behavior of $x(t)$ differ from the non-perturbed case? What is the period and angular frequency of $x(t)$ after several oscillations? Repeat the same observations for $x(t)$ with $x(t = 0) = 0$, $v(t = 0) = 1$. Identify a transient part of $x(t)$ that depends on the initial conditions and decays in time and a steady state part that dominates at longer times and is independent of the initial conditions.

(c) Compute $x(t)$ for several combinations of ω_0 and ω. What is the period and angular frequency of the steady state motion in each case? What parameters determine the frequency of the steady state behavior?

(d) A measure of the long-term behavior of the driven harmonic oscillator is the amplitude of the steady state displacement $A(\omega)$, which can be be computed for a given value of ω if the simulation is run until a steady state has been achieved. One way to determine A is to check the position after every time step to see if a new maximum has been reached as is done by the following code:

```
if (x > Math.abs(amplitude)) {
   amplitude = Math.abs(x);
   control.println("new amplitude = " + amplitude);
}
```

(e) Measure the amplitude and phase shift to verify that the steady state behavior of $x(t)$ is given by

$$x(t) = A(\omega)\cos(\omega t + \delta). \tag{4.19}$$

The quantity δ is the phase difference between the applied force and the steady state motion. Compute $A(\omega)$ and $\delta(\omega)$ for $\omega_0 = 3$, $\gamma = 0.5$, and $\omega = 0$, 1.0, 2.0, 2.2, 2.4, 2.6, 2.8, 3.0, 3.2, and 3.4. Choose the initial condition $x(t = 0) = 0$, $v(t = 0) = 0$. Repeat the simulation for $\gamma = 3.0$, and plot $A(\omega)$ and $\delta(\omega)$ versus ω for the two values of γ. Discuss the qualitative behavior of $A(\omega)$ and $\delta(\omega)$ for the two values of γ. If $A(\omega)$ has a maximum, determine the angular frequency ω_{max} at which the maximum of A occurs. Is the value of ω_{max} close to the natural angular frequency ω_0? Compare ω_{max} to ω_0 and to the frequency of the damped linear oscillator in the absence of an external force.

(f) Compute $x(t)$ and $A(\omega)$ for a damped linear oscillator with the amplitude of the external force $A_0 = 4$. How do the steady state results for $x(t)$ and $A(\omega)$ compare to the case $A_0 = 1$? Does the transient behavior of $x(t)$ satisfy the same relation as the steady state behavior?