Table 6.2 Values of the control parameter s_m for the superstable trajectories of period 2^{m-1}. Nine decimal places are shown.

m	Period	s_m
1	1	0.500 000 000
2	2	0.809 016 994
3	4	0.874 640 425
4	8	0.888 660 970
5	16	0.891 666 899
6	32	0.892 310 883
7	64	0.892 448 823
8	128	0.892 478 091

(a) It is straightforward to adapt the bisection method discussed in Section 6.6. Adapt the class `RecursiveFixedPointApp` to find the numerical solutions of (6.53). Good starting values for the left-most and right-most values of r are easy to obtain. The left-most value is $r = r_\infty \approx 0.8925$. If we already know the sequence $s_1, s_2, \ldots, s_m$, then we can determine δ by

$$\delta_m = \frac{s_{m-1} - s_{m-2}}{s_m - s_{m-1}}. \tag{6.55}$$

We use this determination for δ_m to find the right-most value of r:

$$r_{\text{right}}^{(m+1)} = \frac{s_m - s_{m-1}}{\delta_m}. \tag{6.56}$$

We choose the desired precision to be 10^{-16}. A summary of our results is given in Table 6.2. Verify these results and determine δ.

(b) Use your values of s_m to obtain a more accurate determination of α and δ. ■

Project 6.23 From chaos to order

The bifurcation diagram of the logistic map (see Figure 6.2) has many interesting features that we have not explored. For example, you might have noticed that there are several smooth dark bands in the chaotic region for $r > r_\infty$. Use `BifurcateApp` to generate the bifurcation diagram for $r_\infty \leq r \leq 1$. If we start at $r = 1.0$ and decrease r, we see that there is a band that narrows and eventually splits into two parts at $r \approx 0.9196$. If you look closely, you will see that the band splits into four parts at $r \approx 0.899$. If you look even more closely, you will see many more bands. What type of change occurs near the splitting (merging) of these bands? Use `IterateMap` to look at the time series of (6.5) for $r = 0.9175$. You will notice that although the trajectory looks random, it oscillates back and forth between two bands. This behavior can be seen more clearly if you look at the time series of $x_{n+1} = f^{(2)}(x_n)$. A detailed discussion of the splitting of the bands can be found in Peitgen et al. ■

Project 6.24 Calculation of the Lyapunov spectrum

In Section 6.5 we discussed the calculation of the Lyapunov exponent for the logistic map. If a dynamical system has a multidimensional phase space, for example, the Hénon map and the Lorenz model, there is a set of Lyapunov exponents called the Lyapunov spectrum that characterize the divergence of the trajectory. As an example, consider a set of initial conditions that forms a filled sphere in phase space for the (three-dimensional) Lorenz model. If we iterate the Lorenz equations, then the set of phase space points will deform into another shape. If the system has a fixed point, this shape contracts to a single point. If the system is chaotic, then typically the sphere will diverge in one direction but become smaller in the other two directions. In this case we can define three Lyapunov exponents to measure the deformation in three mutually perpendicular directions. These three directions generally will not correspond to the axes of the original variables. Instead, we must use a Gram–Schmidt orthogonalization procedure.

The algorithm for finding the Lyapunov spectrum is as follows:

(i) Linearize the dynamical equations. If $\mathbf{r}$ is the f-component vector containing the dynamical variables, then define $\Delta\mathbf{r}$ as the linearized difference vector. For example, the linearized Lorenz equations are

$$\frac{d\Delta x}{dt} = -\sigma\Delta x + \sigma\Delta y \tag{6.57a}$$

$$\frac{d\Delta y}{dt} = -x\Delta z - z\Delta x + r\Delta x - \Delta y \tag{6.57b}$$

$$\frac{d\Delta z}{dt} = x\Delta y + y\Delta x - b\Delta z. \tag{6.57c}$$

(ii) Define f orthonormal initial values for $\Delta\mathbf{r}$. For example, $\Delta\mathbf{r}_1(0) = (1, 0, 0)$, $\Delta\mathbf{r}_2(0) = (0, 1, 0)$, and $\Delta\mathbf{r}_3(0) = (0, 0, 1)$. Because these vectors appear in a linearized equation, they do not have to be small in magnitude.

(iii) Iterate the original and linearized equations of motion. One iteration yields a new vector from the original equation of motion and f new vectors $\Delta\mathbf{r}_\alpha$ from the linearized equations.

(iv) Find the orthonormal vectors $\Delta\mathbf{r}'_\alpha$ from the $\Delta\mathbf{r}_\alpha$ using the Gram–Schmidt procedure. That is,

$$\Delta\mathbf{r}'_1 = \frac{\Delta\mathbf{r}_1}{|\Delta\mathbf{r}_1|} \tag{6.58a}$$

$$\Delta\mathbf{r}'_2 = \frac{\Delta\mathbf{r}_2 - (\Delta\mathbf{r}'_1\cdot\Delta\mathbf{r}_2)\Delta\mathbf{r}'_1}{|\Delta\mathbf{r}_2 - (\Delta\mathbf{r}'_1\cdot\Delta\mathbf{r}_2)\Delta\mathbf{r}'_1|} \tag{6.58b}$$

$$\Delta\mathbf{r}'_3 = \frac{\Delta\mathbf{r}_3 - (\Delta\mathbf{r}'_1\cdot\Delta\mathbf{r}_3)\Delta\mathbf{r}'_1 - (\Delta\mathbf{r}'_2\cdot\Delta\mathbf{r}_3)\Delta\mathbf{r}'_2}{|\Delta\mathbf{r}_3 - (\Delta\mathbf{r}'_1\cdot\Delta\mathbf{r}_3)\Delta\mathbf{r}'_1 - (\Delta\mathbf{r}'_2\cdot\Delta\mathbf{r}_3)\Delta\mathbf{r}'_2|}. \tag{6.58c}$$

It is straightforward to generalize the method to higher-dimensional models.

(v) Set the $\Delta\mathbf{r}_\alpha(t)$ equal to the orthonormal vectors $\Delta\mathbf{r}'_\alpha(t)$.

(vi) Accumulate the running sum, S_α as $S_\alpha \to S_\alpha + \log|\Delta\mathbf{r}_\alpha(t)|$.

(vii) Repeat steps (iii)–(vi) and periodically output the approximate Lyapunov exponents $\lambda_\alpha = (1/n)S_\alpha$, where n is the number of iterations.

To obtain a result for the Lyapunov spectrum that represents the steady state attractor, include only data after the transient behavior has ended.