Lecture 5

Simple Linear and Nonlinear System

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This PowerPoint Notes Is Based on the Textbook '***An Introduction to Computer Simulation Methods : Applications to Physical Systems***', 2nd Edition, Harvey Gould and Jan Tobochnik, Addison-Wesley(1996);

"A First Course in Computational Physics"; "Numerical Recipes";

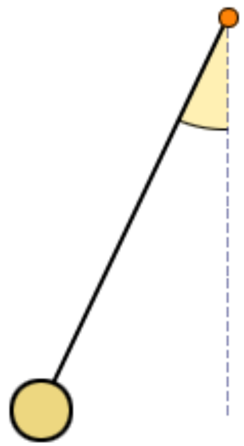
"Elementary Numerical Analysis"; "Computational Methods in Physics and Engineering".

Required for Lecture 5

- SHM, periodic motion, oscillatory motion, frequency, amplitude, phase.
- Conservative / dissipative systems.
- **Similarities between physical systems**.
- **Program: SHO**, **rc**; and **Solution of forced oscillator with damping**
- **Comparison of algorithms**.

Question and Objectives

- Pendulum, ***g***.
- Day, week, month, year, etc.
- Electric circuit and mechanical system, are they the same?

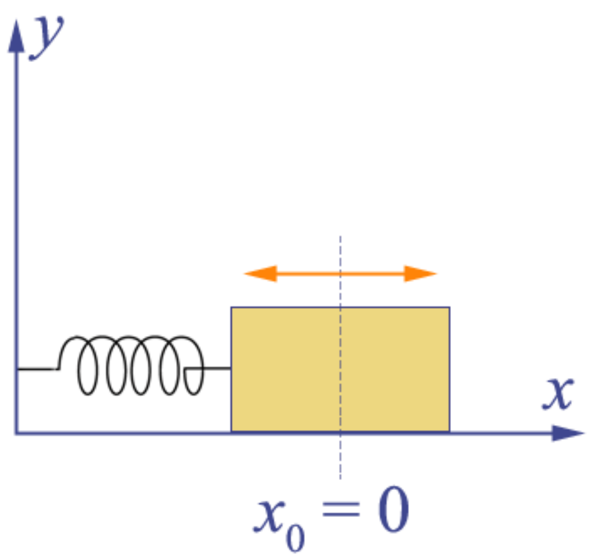


Simple Harmonic Motion (SHM, SHO)

- **Periodic Motion**:
  motion that repeats itself at definite time intervals, e.g., earth arounds the sun.

- **Oscillatory Motion**:
  an object undergoes periodic motion between two limits over the same path.

- **Example**: mass *m* connected to a massless spring.

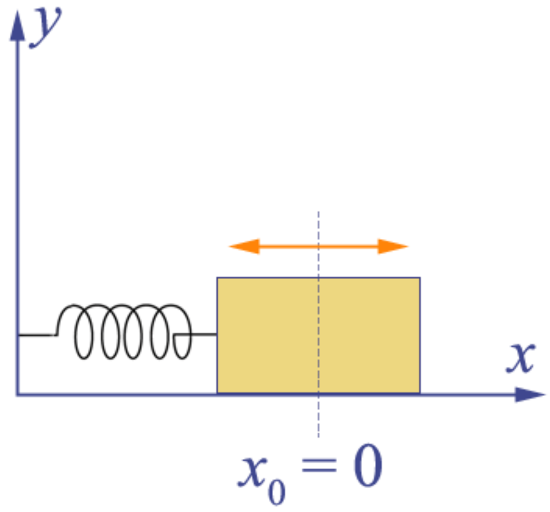


Hooke's Law and Equation of Motion

$F = -k\,(x - x_0)$ ← *Approximation?*

x_0: *equilibrium* position,

k : force constant.



$$\frac{d^2x}{dt^2} = \frac{F}{m} = -\frac{k}{m}\left(x - x_0\right) = -\omega_0^2 x$$

$\omega_0{}^2 = k/m$ **(intrinsic property of the system)**

This is a ***linear*** differential equation describing ***simple harmonic motion***. Its solution can be written as

$x(t) = A\cos(\omega_0 t + \delta)$ (direct check?)

A : amplitude, δ : phase

Harmonic Motion

- **Period *T* :** smallest time for repeated motion

$$x(t+T) = x(t), \quad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{k/m}}$$

- **Frequency *ν* :** number of cycles per second,

  $\nu = 1/T.$

- **Conservation of total energy *E*:**

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 = \text{constant}$$

Numerical Simulation of SHO

The following program computes the time dependence of the position and the velocity of a linear harmonic oscillator using *Euler-Richardson* algorithm. One can put a check for **energy conservation** in easily.

Just for Ilustrtion
(no detail discussions)

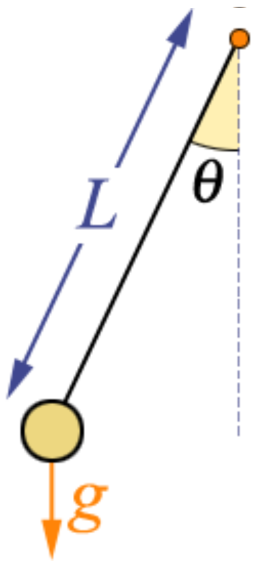
```
PROGRAM SHO
IMPLICIT NONE
real     :: x,v,w2,t,dt,dt2,tmax
integer :: counter,nshow
   CALL initial(x,v,t,w2,dt,dt2,tmax,nshow)
   counter=0
   DO WHILE (t <= tmax)
      CALL update(x,v,w2,t,dt,dt2)
      counter = counter + 1
      if ( mod(counter,nshow) == 0 ) then
         CALL output(x,v,t)
      end if
   END DO
END PROGRAM SHO
```

```
SUBROUTINE initial(x_0,v_0,t,w2,dt,dt2,tmax,nshow)
IMPLICIT NONE
real    :: x_0,v_0,t,w2,dt,dt2,tmax,show_time
integer :: nshow

  write(6,*) 'Enter Initial: x_0,v_0,t,tmax'
  read(5,*) x_0,v_0,t,tmax

  write(6,*) 'Enter k/m, time step, show_time'
  read(5,*) w2,dt,show_time

  dt2=0.5*dt
  nshow=int(show_time/dt)
  write(6,1002)
  1002   format(2x,'Time',5x,'Position',6x,'Velocity'/)

END SUBROUTINE initial
```

```
SUBROUTINE update(x,v,w2,t,dt,dt2)    ! Euler-Richardson
IMPLICIT NONE
real :: x,v,w2,t,dt,dt2
real :: acceleration,vmid,xmid
   acceleration = -w2*x
   vmid = v + acceleration*dt2
   xmid = x + v*dt2
   acceleration = -w2*xmid
   v = v + acceleration*dt
   x = x + vmid*dt
   t = t + dt
END SUBROUTINE update

SUBROUTINE output(x,v,t)
IMPLICIT NONE
real :: x,v,t
   write(6,1004) t,x,v
   1004    format(f6.3,2f13.6)
END SUBROUTINE output
```

The Simple Pendulum

Equation of Motion:



$$v = L\frac{d\theta}{dt} = L\omega, \qquad a = L\frac{d^2\theta}{dt^2} = L\alpha$$

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta, \qquad \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

This is a *nonlinear* equation due to

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \cdots + (-1)^n\frac{\theta^{2n+1}}{(2n+1)!} + \cdots$$

The Simple Pendulum

For sufficiently small θ, $\sin\theta \approx \theta$, we have

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta, \qquad \theta << 1, \text{measured in radians}$$

Thus this is a SHO with $T = 2\pi\sqrt{L/g}$

If θ is not small, then how to solve this equation?

The system is conserved and the total energy *E is*

$$E = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta)$$

The Simple Pendulum

Then

$$\frac{d\theta}{dt} = \sqrt{\frac{2E - 2mgL(1-\cos\theta)}{mL^2}}$$

$$\int \frac{d\theta}{\sqrt{2E - 2mgL(1-\cos\theta)}} = \int \frac{dt}{\sqrt{mL^2}} = \frac{t - t_0}{\sqrt{mL^2}}$$

One can do (numerical) integration to solve this problem.

Dissipative System

SHO belongs to ***conservative*** systems. In nature, we observe ***dissipative*** systems and their motions are damped, e.g., drag force and frictional force.

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma \frac{dx}{dt}$$

where γ is the ***damping coefficient*** measuring the magnitude of dissipative force.

To keep system moving, one needs external force.

Response to External Force

When the system subjects to an external force (perturbation), the response of the system reveals the nature of the system. Consider a ***driven*** damped linear oscillator, the equation of motion is

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma \frac{dx}{dt} + \frac{1}{m} F(t)$$

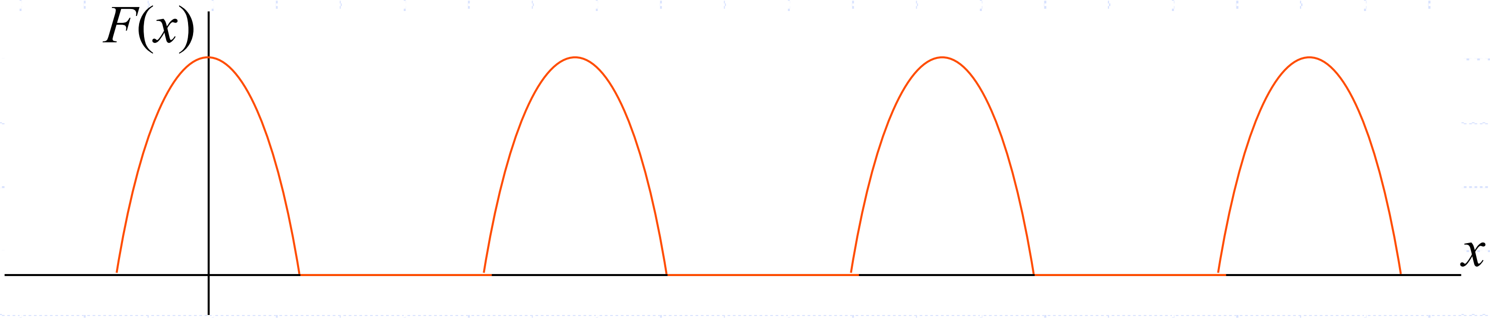
We usually calculate the response of the system in terms of the x rather than the v. Force $F(t)$ could have arbitrary forms and usually we do our analysis by **assuming**

$$F(t) = A_0 \cos(\omega t),$$

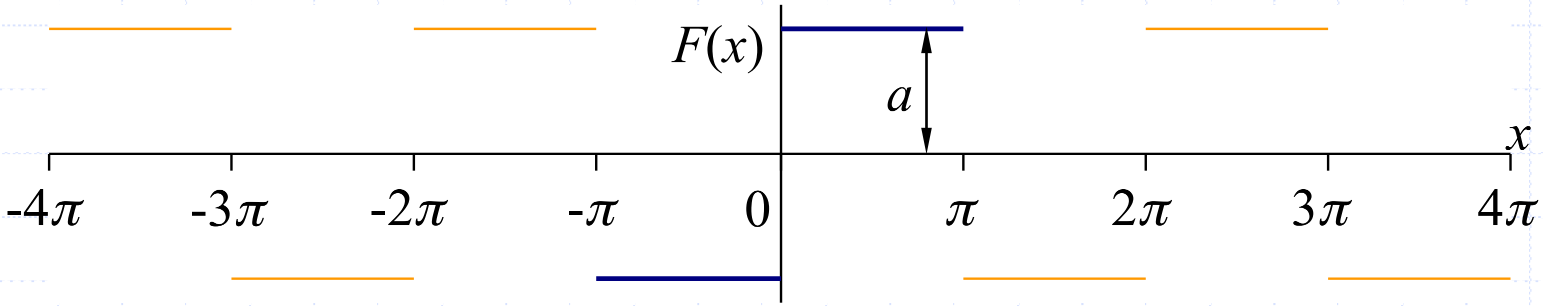
where ω is the angular frequency of the driving force.

Commonly Used External Forces

- Half wave driving force



- Unit step function



Electrical Circuit Oscillations

Physics deals with laws of nature. Many seeming different phenomena can be described by the same set of physical laws (equations), e.g. RLC circuit & SHO.

Electrical Circuit :

Based on *Kirchoff*'s loop rules, came from charge conservation and energy conservation.

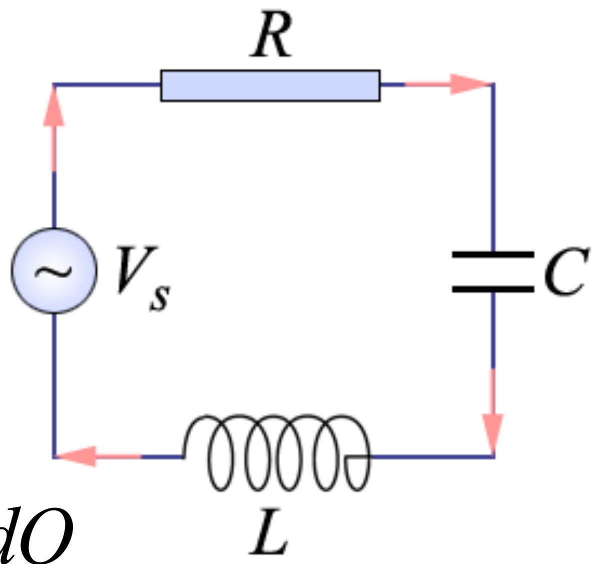
Element	Voltage Drop	Units
Resistor	$V_R = IR$	Resistance R, ohms Ω
Capacitor	$V_C = Q/C$	Capacitance C, farads F
Inductor	$V_L = L\, dI/dt$	Inductance L, henries H

RLC Circuit

$$V_L + V_R + V_C = V_s(t)$$

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V_s(t)$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_s(t), \quad I = \frac{dQ}{dt}$$



This equation is identical to the previous equation for driven damped linear oscillator.

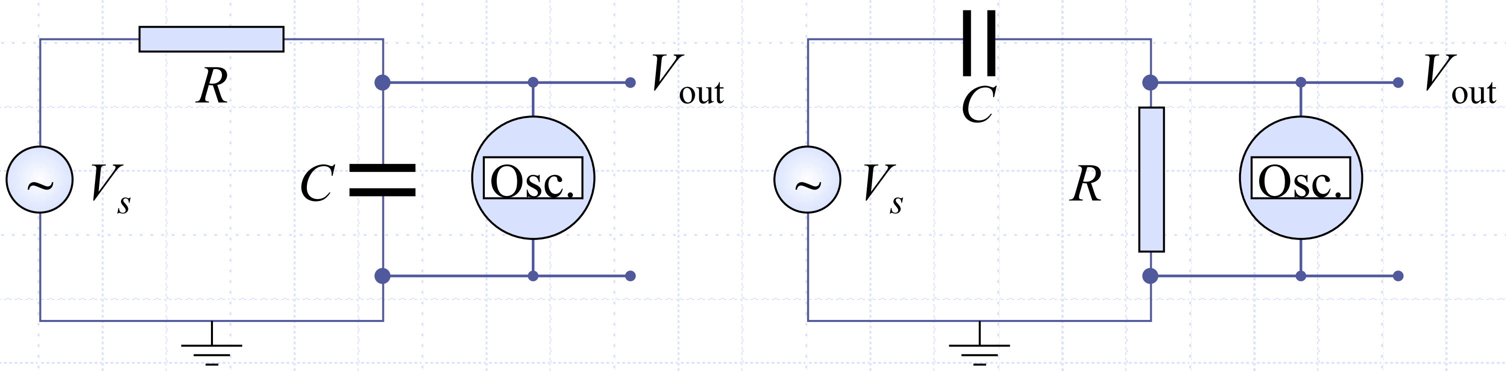
$$\frac{d^2x}{dt^2} + \omega_0^2 x + \gamma\frac{dx}{dt} = \frac{1}{m}F(t)$$

Comparisons of Electric Circuit & Mechanical System

Electric Circuit	**Mechanical System**
Charge Q	Displacement x
Current $I = dQ/dt$	Velocity $v = dx/dt$
Inductance L	Mass m
Inverse capacitance $1/C$	Spring constant k
Resistance R	Damping γ

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_s(t) \qquad \frac{d^2x}{dt^2} + \omega_0^2 x + \gamma\frac{dx}{dt} = \frac{1}{m}F(t)$$

Real Measurement vs Computer Simulation



Note that although the loop equations is identical regardless of the order of placement of the electrical elements, the output voltage measured by the oscilloscope could be **different**.

An advantage of a computer simulation of an electrical circuit is that the measurement of a voltage drop across a circuit element does **not affect** the properties of the circuit. In fact, digital computers are often used to optimize the design of circuits for special applications.

Simulating RC Circuit

Program **rc** simulates an **RC** circuit with an alternating current (ac) voltage source of the form $V_s(t) = \cos \omega t$.

```
PROGRAM rc   !simulation of RC circuit with ac voltage source
LIBRARY "csgraphics"
CALL initial(Q,R,tau,V0,omega,tmax,t,dt)
CALL set_up_windows(#1,#2,V0,tmax)
DO  while t <= tmax
      CALL scope(Q,I,R,tau,V0,omega,dt)
      LET t = t + dt
      CALL source_voltage(#1,V0,omega,t)
      CALL output_voltage(#2,I,R,t)
LOOP
CLOSE #1
CLOSE #2
END
```

Program details not shown

Damped Oscillator

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

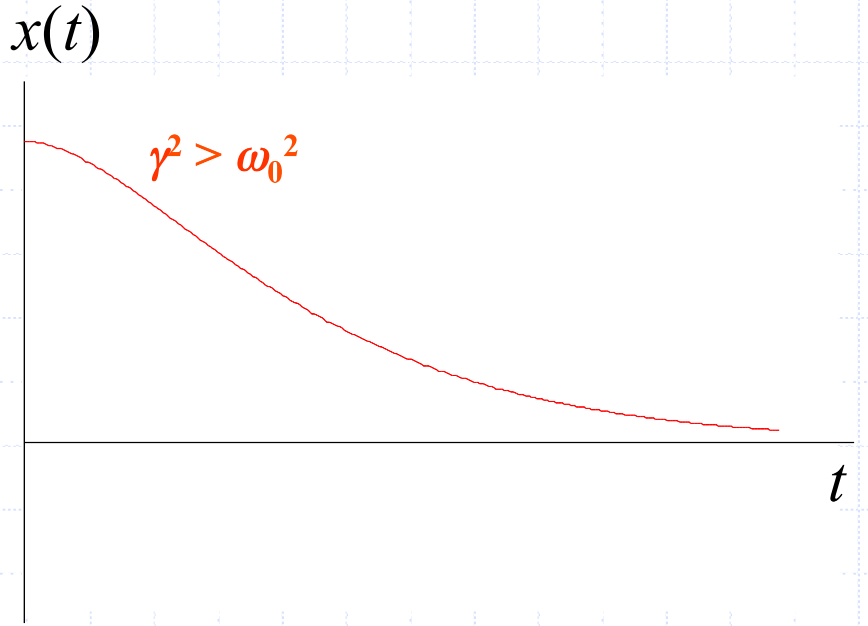
$x'' + 2\gamma x' + \omega_0^2 x = 0$ with $\beta/m = 2\gamma$, $k/m = \omega_0^2$

Over-Damped

$\gamma^2 > \omega_0^2$, i.e. $\beta^2 > 4km$

$x = e^{-\gamma t}(Ae^{\alpha t} + Be^{-\alpha t})$

where $\alpha^2 = \gamma^2 - \omega_0^2$



Damped Oscillator

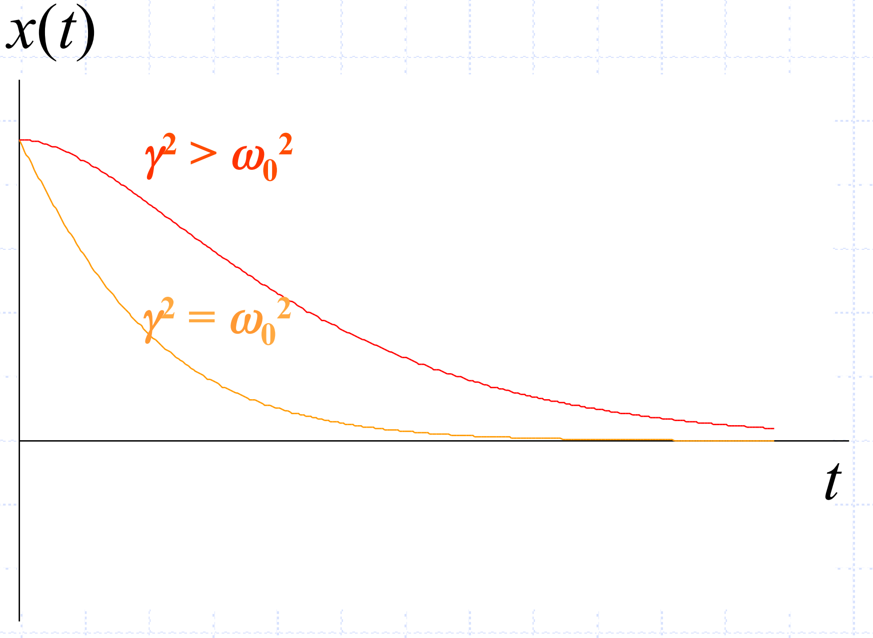
$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

$$x'' + 2\gamma x' + \omega_0^2 x = 0 \quad \text{with } \beta/m = 2\gamma,\ k/m = \omega_0^2$$

Critically Damped

$\gamma^2 = \omega_0^2$, i.e. $\beta^2 = 4km$

$x = e^{-\gamma t}(A + Bt)$



Damped Oscillator

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0$$

$x'' + 2\gamma\, x' + \omega_0^2 x = 0$ with $\beta/m = 2\gamma,\ k/m = \omega_0^2$

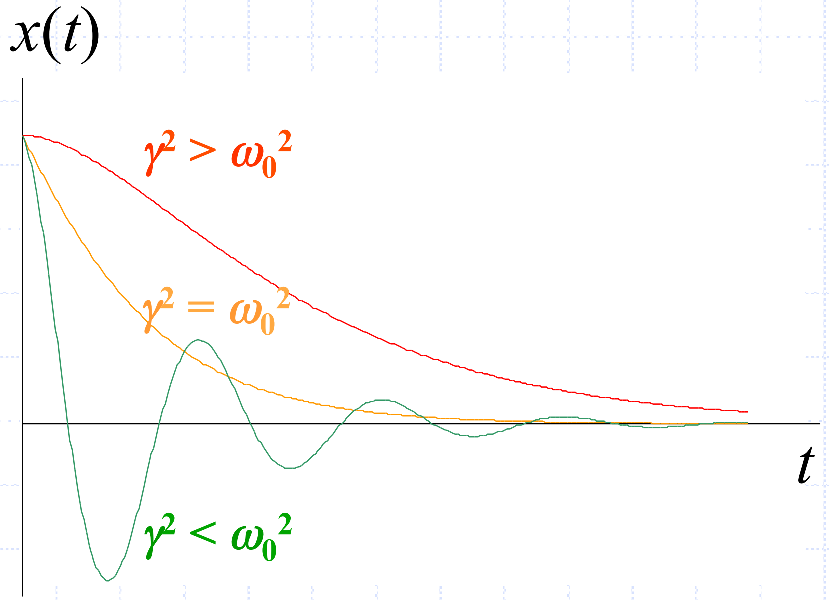
Under-Damped

$\gamma^2 < \omega_0^2$, i.e. $\beta^2 < 4km$

$$x = e^{-\gamma t}(A\sin\lambda t + B\cos\lambda t)$$

$$= Ce^{-\gamma t}\cos(\lambda t - \phi)$$

where $\lambda^2 = \omega_0^2 - \gamma^2$



Forced Vibrations, $F(t) = F_0 \cos \alpha t$

$x'' + 2\gamma x' + \omega_0^2 x = f_0 \cos \alpha t$

with $\beta/m = 2\gamma$, $k/m = \omega_0^2$, $F_0/m = f_0$

General solution of $x'' + 2\gamma x' + \omega_0^2 x = f_0 \cos \alpha t$

$=$ general solution of $x'' + 2\gamma x' + \omega_0^2 x = 0$

(***transient/homogeneous*** **solution**)

$+$ a particular solution of $x'' + 2\gamma x' + \omega_0^2 x = f_0 \cos \alpha t$

(***steady-state/inhomogeneous*** **solution**)

$$x = \frac{f_0}{\sqrt{(\alpha^2 - \omega_0^2)^2 + 4\gamma^2\alpha^2}} \cos(\alpha t - \phi)$$

$$\text{where} \quad \tan\phi = \frac{2\gamma\alpha}{\alpha^2 - \omega_0^2}, \quad 0 \le \phi \le \pi$$

Amplitude Resonance

- Amplitude A is maximum at resonance frequency α_r by setting

$$\left.\frac{dA}{d\alpha}\right|_{\alpha=\alpha_r} = 0, \qquad \text{where } A = \frac{f_0}{\sqrt{(\alpha^2 - \omega_0{}^2)^2 + 4\gamma^2\alpha^2}}$$

$$A_{\max} = \frac{f_0}{2\gamma\sqrt{\omega_0{}^2 - \gamma^2}}$$

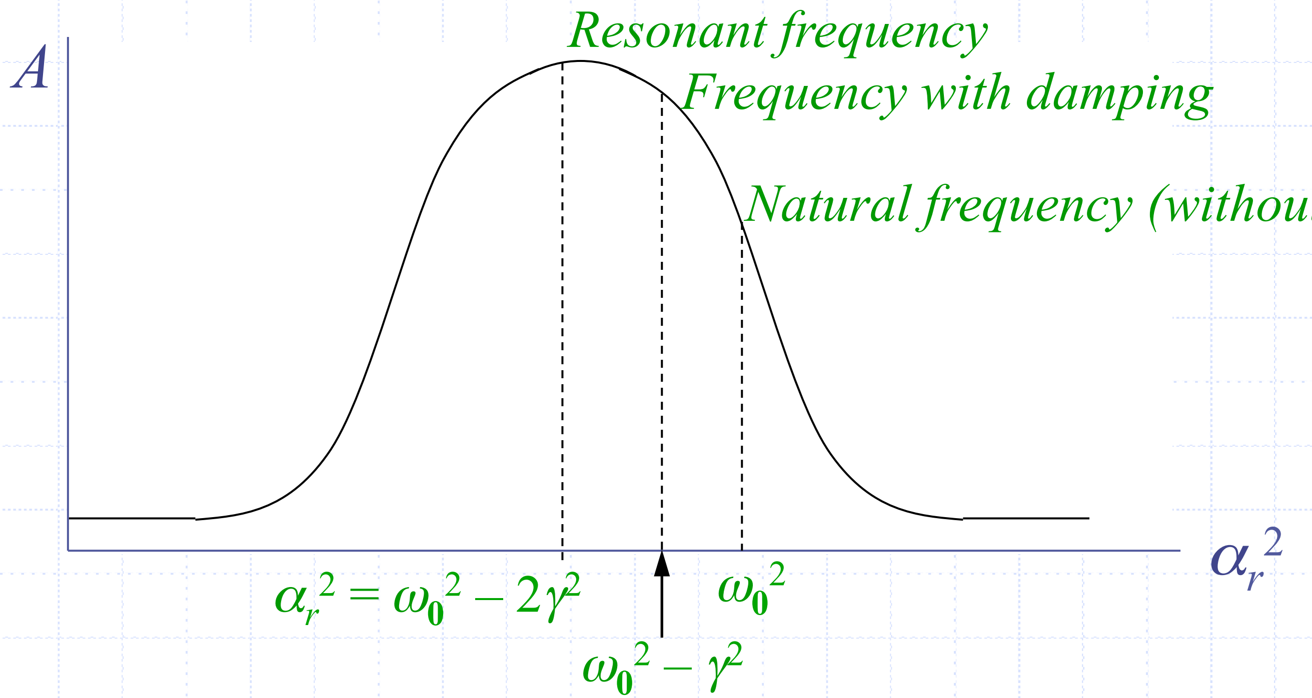
$$\alpha_r = \sqrt{\omega_0{}^2 - 2\gamma^2}$$

- For small damping, $\gamma \to 0$ $A_0 \approx f_0/2\gamma\omega_0 \to \infty$

Plot of A vs α^2

- Write the amplitude in terms of α_r

$$A = \frac{f_0}{\sqrt{(\alpha^2 - \alpha_r^2)^2 + 4\gamma^2(\omega_0^2 - \gamma^2)}}$$



Lecture 5 Review and Required

- SHM, periodic motion, oscillatory motion, frequency, amplitude, phase.
- Conservative / dissipative systems.
- Similarities between physical systems.
- Program: **SHO**, **rc**.
- Comparison of algorithms.
- **Damped oscillator**.

Chemical Oscillations

$$A + 2B \rightarrow 3B + C$$

Rate equations are:

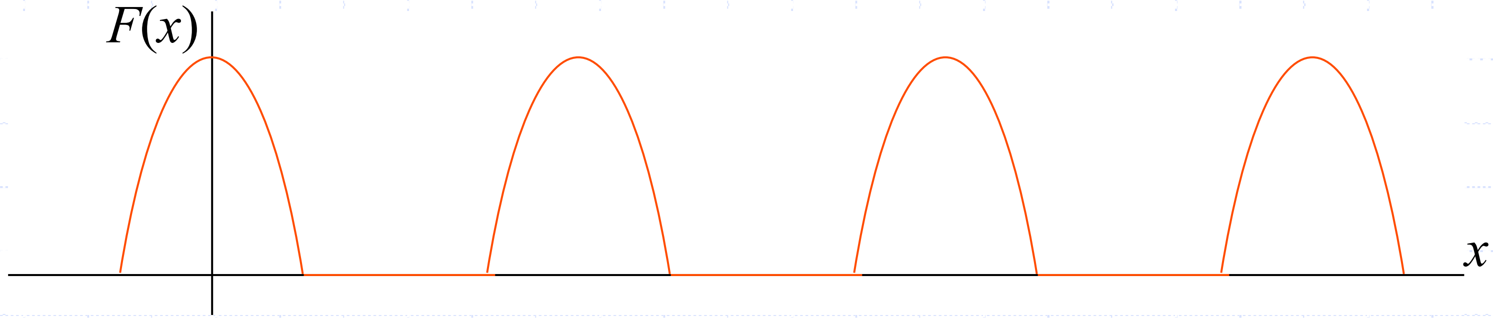
$$\frac{dA}{dt} = -kAB^2$$

$$\frac{dB}{dt} = kAB^2$$

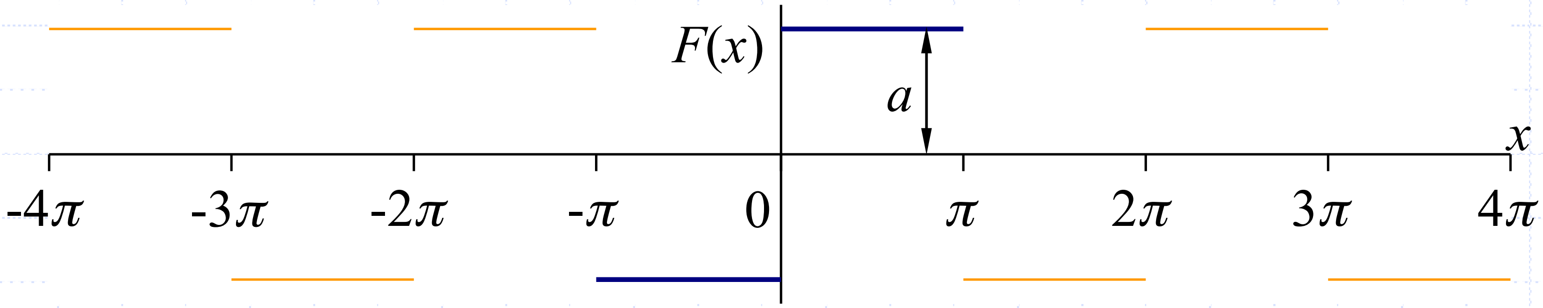
$$\frac{dC}{dt} = kAB^2$$

Commonly Used External Forces

- Half wave driving force (voltage source)

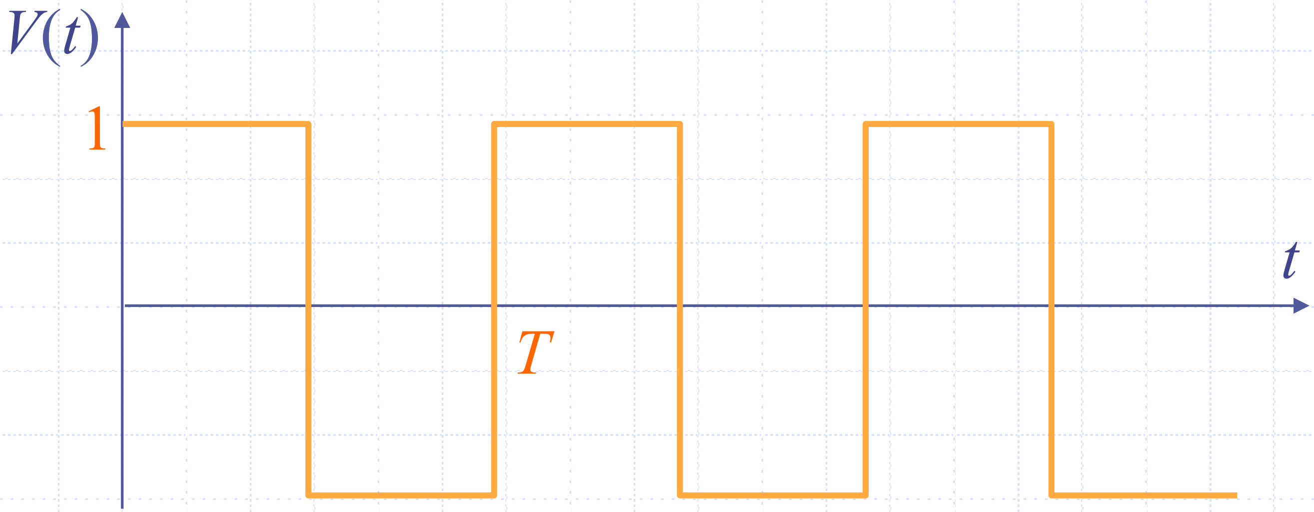


- Unit step function



Square Wave Input

Voltage source is a periodic square wave with period T and unit amplitude, how to simulate?



We will discuss these in Fourier Transform