Exercise 9.14 Natural order

Invoke the `toNaturalOrder` method after performing the FFT in the `FFTApp` program. Modify the print statement so that the natural frequency is shown and repeat Problem 9.13. If N is even, the Fourier components have a frequency separation $\Delta\omega$ given by

$$\Delta\omega = \frac{2\pi}{N\Delta}. \tag{9.40}$$

What is the frequency separation if N is odd? ■

As we have seen, computing Fourier transformations is straightforward but requires a fair amount of bookkeeping. To simplify the process, we have defined the `FFTFrame` class in the frames package to perform a FFT and display the coefficients. This utility class accepts either data arrays or functions as input parameters in the `doFFT` method. The code shown in Listing 9.8 transforms an input array. We use the `FFTFrame` in Problem 9.12.

Listing 9.8 The `FFTCalculationApp` displays the coefficients of $e^{2\pi nx}$.

```
package org.opensourcephysics.sip.ch09;
import org.opensourcephysics.controls.*;
import org.opensourcephysics.frames.FFTFrame;

public class FFTCalculationApp extends AbstractCalculation {
   FFTFrame frame = new FFTFrame("frequency", "amplitude",
                    "FFT Frame Test");

   public void calculate() {
      double xmin = control.getDouble("xmin");
      double xmax = control.getDouble("xmax");
      int n = control.getInt("N");
      double xi = xmin, delta = (xmax-xmin)/n;
      double[] data = new double[2*n];
      int mode = control.getInt("mode");
      for(int i = 0;i<n;i++) {
         data[2*i] = Math.cos(mode*xi);
         data[2*i+1] = Math.sin(mode*xi);
         xi += delta;
      }
      frame.doFFT(data, xmin, xmax);
      frame.showDataTable(true);
   }

   public void reset() {
      control.setValue("mode", 1);
      control.setValue("xmin", 0);
      control.setValue("xmax", "2*pi");
      control.setValue("N", 32);
      calculate();
   }

   public static void main(String[] args) {
      CalculationControl.createApp(new FFTCalculationApp());
   }
}
```


Problem 9.15 Spatial Fourier transforms and phase

So far we have considered only nonnegative values of t for functions $f(t)$. Spatial Fourier transforms are of interest in many contexts, and these transforms usually involve both positive and negative values of x.

(a) Write a program using a `CalculationControl` that computes the real and imaginary parts of the Fourier transform $\phi(q)$ of a complex function $\psi(x) = f(x) + ig(x)$, where $f(x)$ and $g(x)$ are real and x has both positive and negative values. Note that the wavenumber $q = 2\pi/L$ is analogous to the angular frequency $\omega = 2\pi/T$.

(b) Compute the Fourier transform of the Gaussian function $\psi(x) = e^{-bx^2}$ in the interval $[-5, 5]$. Examine $\psi(x)$ and $\phi(q)$ for at least three values of b such that the Gaussian is contained within the interval. Does $\phi(q)$ appear to be a Gaussian? Choose a reasonable criterion for the half-width of $\psi(x)$ and measure its value. Use the same criterion to measure the half-width of $\phi(q)$. How do these widths depend on b? How does the width of $\phi(q)$ change as the width of $\psi(x)$ increases?

(c) Repeat part (b) with the function $\psi(x) = Ae^{-b(x-x_0)^2}$ for various values of x_0. What effect does shifting the peak have on $\phi(q)$?

(d) Repeat part (b) with the function $\psi(x) = Ae^{-bx^2}e^{iq_0x}$ for various values of q_0. What effect does the phase oscillation have on $\phi(q)$? ■

9.4 ■ TWO-DIMENSIONAL FOURIER SERIES

The extension of the ideas of Fourier analysis to two dimensions is simple and direct. We will use two-dimensional FFTs when we study diffraction in Section 9.9.

If we assume a function of two variables $f(x, y)$, then a two-dimensional series is constructed using harmonics of both variables. The basis functions are the products of one-dimensional basis functions $e^{ixq_x}e^{iyq_y}$, and the Fourier series is written as a sum of these harmonics:

$$f(x, y) = \sum_{n=-N/2}^{N/2} \sum_{m=-M/2}^{M/2} c_{n,m}\, e^{iq_nx}e^{iq_my}, \tag{9.41}$$

where

$$q_n = \frac{2\pi n}{X} \quad \text{and} \quad q_m = \frac{2\pi m}{Y}. \tag{9.42}$$

The function $f(x, y)$ is assumed to be periodic in both x and y with periods X and Y, respectively. The Fourier coefficients are again calculated by integrating the product of the function with a basis function:

$$c_{n,m} = \int_{-X/2}^{X/2}\int_{-Y/2}^{Y/2} f(x, y)e^{i(q_nx+q_my)}\, dx\, dy. \tag{9.43}$$