Lecture 4

The Two-Body Problem

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This PowerPoint Notes Is Based on the Textbook '***An Introduction to Computer Simulation Methods : Applications to Physical Systems***', 2nd Edition, Harvey Gould and Jan Tobochnik, Addison-Wesley(1996); "A First Course in Computational Physics"; "Numerical Recipes"; "Elementary Numerical Analysis"; "Computational Methods in Physics and Engineering".

Required for Lecture 4

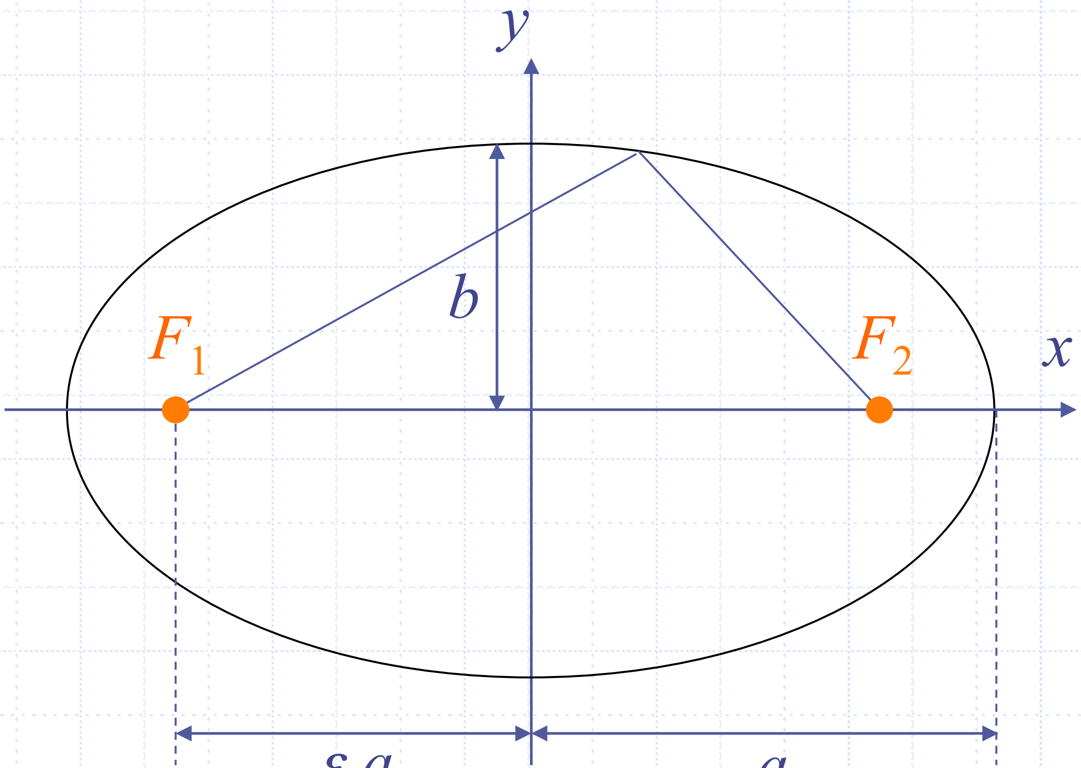
- Simulating solar system (and alike) according to Newton's universal law of gravitation.
- Centre of mass, conservation laws, etc.
- **Physical units and computer simulations**.
- **Log-log plots and data analysis**.
- **Programs: planet** and **planet2**.
- Use functions (e.g., **force**).

Questions and Objective

- One of Newton's great discoveries is the **gravitation law**. How will it be used for simulating planet motion?
- Kepler made great discovery on planets motion, could we repeat his observations by computer simulation?
- Scattering plays an important role in our understanding of matter structures. Can we simulate it?
- We will learn how to simulate solar system, helium atom, and scattering process. At the same time, we will review physical units and its usage.

Kepler's 1st Law of Planets

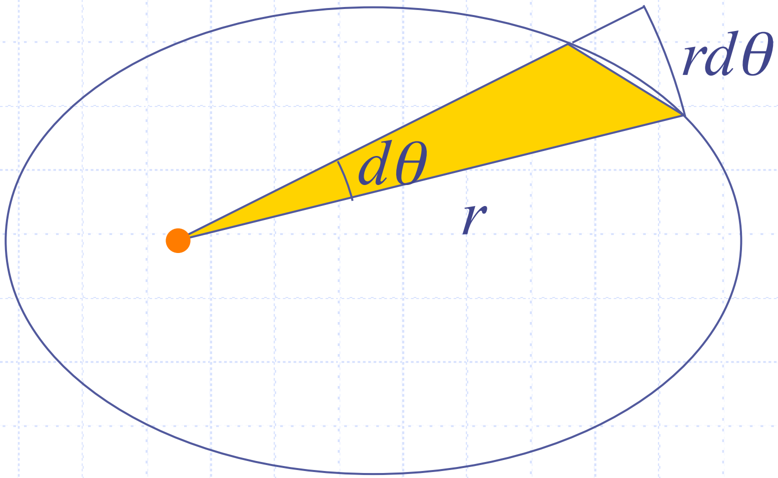
Each planet moves in an **elliptical orbit** with the sun located at one of the **foci** of the ellipse.



Kepler's 2nd Law of Planets

The speed of a planet increase as its distance from the sun decrease such that the line from the sun to the planet sweeps out equal areas in equal times.

$$dS = \frac{1}{2} r^2 d\theta$$

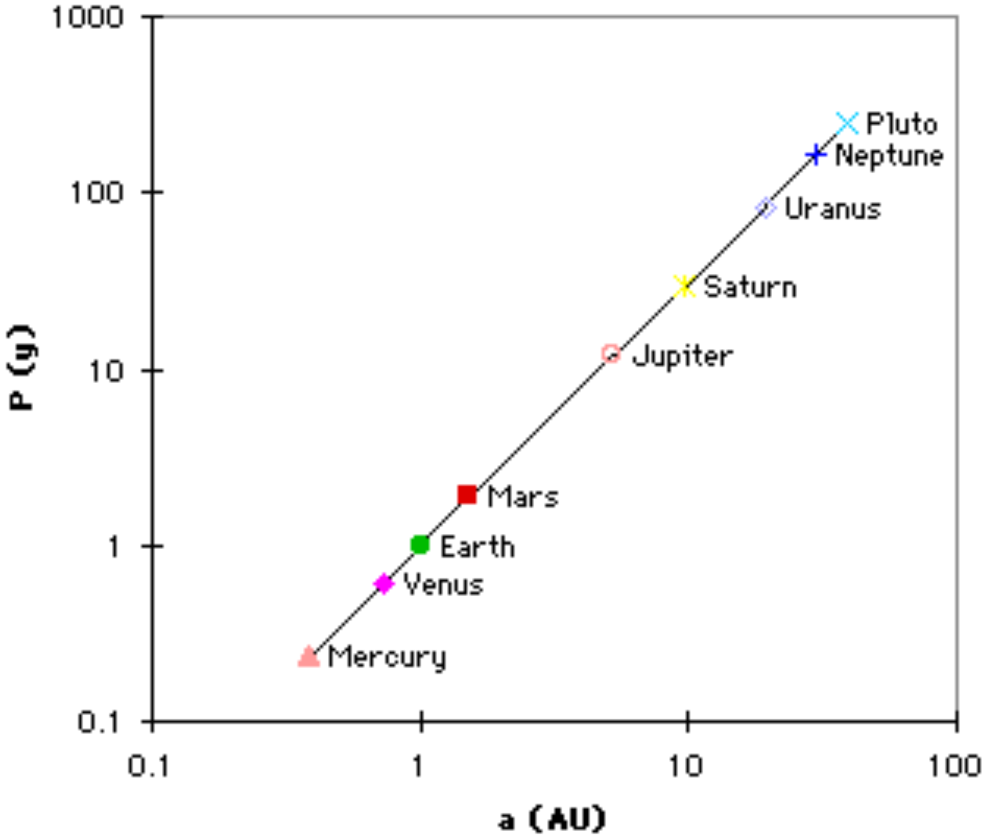


The rate at which area is swept out by the radius is

$$\frac{dS}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2m}$$

Kepler's 3rd Law of Planets

The ratio T^2/a^3 is the same for all planets that orbit the sun, where T is the period of the planet and a is the semi-major axis of the ellipse.



log-log plot

Log-log and Semi-log Plots

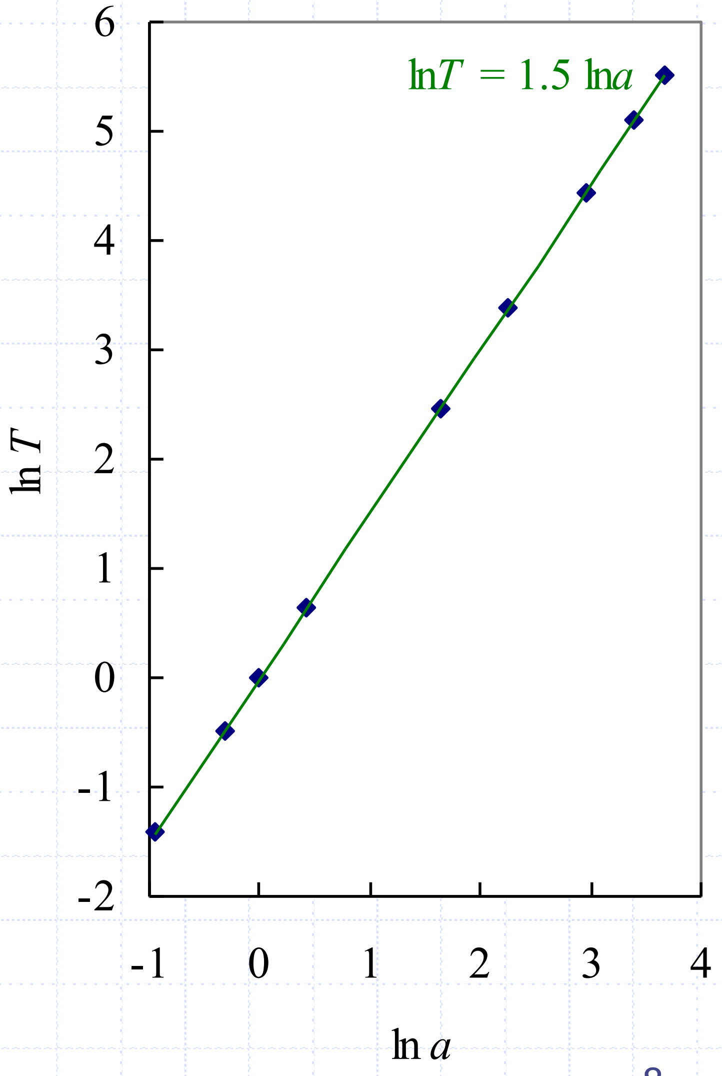
Ways of using graph to analyse data:

- $y = ax + b$
  Plot y versus x.
- $y = Ce^{rx}$, $\ln y = \ln C + rx$
  Plot $\ln y$ versus x (semi-log plots).
- $y = Cx^n$, $\ln y = \ln C + n \ln x$
  Plot $\ln y$ versus $\ln x$ (log-log plots).

Example: Plots of Table 4.1.

Plot of ln*T* vs ln*a*

Planet	T (earth years)	a (AU)
Mercury	0.241	0.387
Venus	0.615	0.723
Earth	1.00	1.00
Mars	1.88	1.523
Jupiter	11.86	5.202
Saturn	29.5	9.539
Uranus	84	19.18
Neptune	165	30.06
Pluto	248	39.44

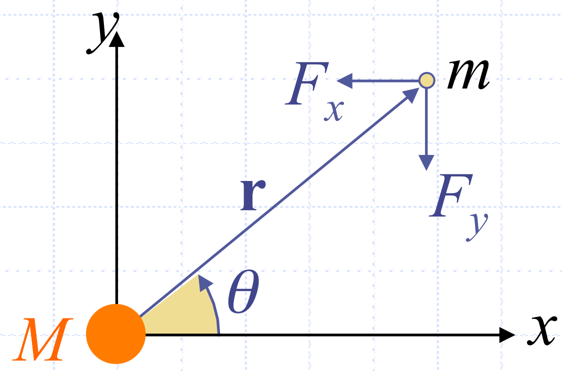


The Equation of Motion

- Reduce two-body problem to one-body problem:

  Centre of mass co-ordinates

  Reduced mass, $\mu = \dfrac{Mm}{M+m}$.



- Newton's universal law of gravitation (1666,1687):

$$\vec{F} = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}\vec{r},$$

  where **r** is directed from M to m

  $G = 6.67\times10^{-11}\ \mathrm{m^3/\,kg\cdot s^2}$ (too small?)

Central Force

- The force is independent of the direction and only dependent on the separation between M and m: $\vec{F}(\vec{r}) = F(r)\hat{r}$

- Angular momentum **L**, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$
  In component form $L_z = m\,(x\,v_y - y\,v_x)$

- Total energy E, $E = \dfrac{1}{2}mv^2 - \dfrac{GMm}{r}$

Angular momentum and total energy are conserved.

The Equation of Motion

- Equation of motion $m\frac{d^2\vec{r}}{dt^2} = -\frac{GMm}{r^3}\vec{r}$

In component form

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^3}x,$$

$$\frac{d^2y}{dt^2} = -\frac{GM}{r^3}y, \quad \text{where } r^2 = x^2 + y^2$$

Two coupled second-order differential equations .

Circular and Elliptical Orbits

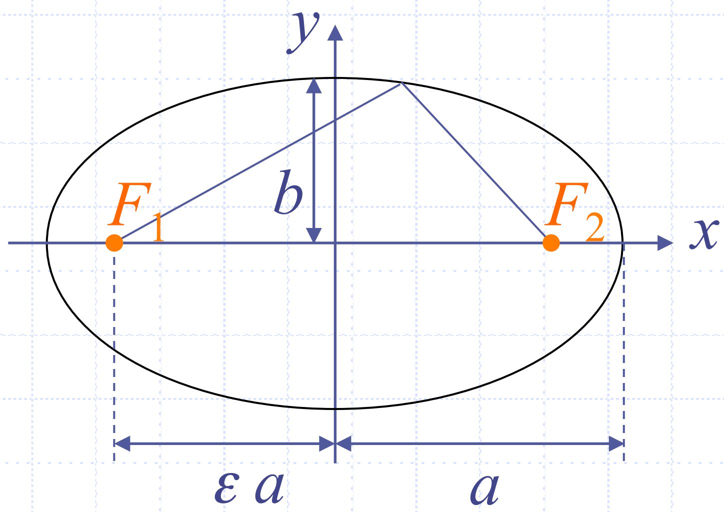
- **Circular Motion**

$$a = \frac{v^2}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$T^2 = \frac{4\pi^2}{GM} r^3$$

- **An Elliptic Orbit**



Eccentricity

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}. \quad (0 < \varepsilon < 1)$$

Remarks

- Principle of Superposition, linear physics.
- Coulomb’ s law in E&M.
- Gauss’ theorem, power 2.
- Potential function $U(r)$:
  - Difference in stating and ending positions
  - $F = -\nabla U(r)$

Circular Orbits

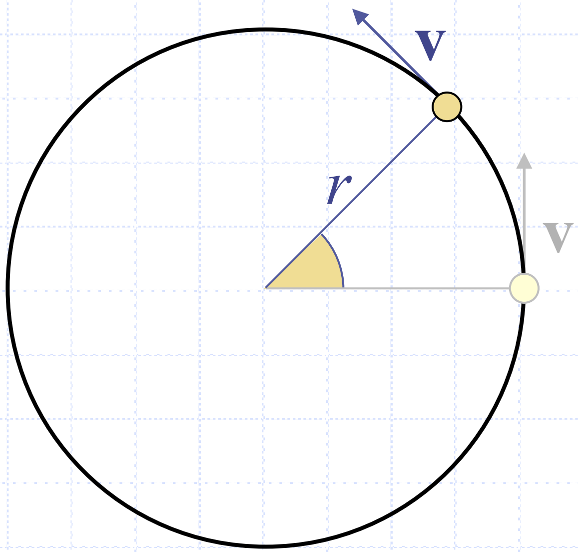
Circular Motion

$$r^2 = R^2$$

$$\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{r} \cdot \mathbf{v} = 0$$

$$\mathbf{r} \cdot \frac{d\mathbf{v}}{dt} + v^2 = 0$$

$$a = \frac{v^2}{r}$$



SI Base Units

- **Length:** *meter*, m, " … the length equal to 1,650,763.73 wavelengths in vacuum of the radiation corresponding to the transition between the level and of the krypton-86-atom." (1960)
- **Mass:** *kilogram*, kg, " … this prototype [a certain platinum-iridium cylinder] shall henceforth be considered to be the unit of mass." (1889)
- **Time:** *second*, s, " … the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium-133 atom." (1967)
- **Thermodynamic Temperature:** *Kelvin*, K, " … the fraction 1/273.16 of the thermodynamic temperature of triple point of water." (1967)

SI Base Units

- **Electric Current:** *ampere*, A, " … that constant current which,if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to Newton per meter of length." (1946)
- **Amount of Substance:** *mole*, mol, " … the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilograms of carbon-12." (1971)
- **Luminous Intensity:** *candela*, cd, " … the luminous intensity, in the perpendicular direction, of a surface of 1/600,000 square meter of a black-body at the temperature of freezing platinum under a pressure of 101,325 Newton per square meter."(1967)

SI Units				
Quantity	**Name**	**Symbol**	**Expression in terms of other units**	**Expression in terms of SI base units**
Frequency	Hertz	Hz		s^{-1}
Force	Newton	N		$m \cdot kg/s^2$
Pressure	Pascal	Pa	N/m^2	$kg/m \cdot s^2$
Energy, work, quantity of heat	Joule	J	$N \cdot m$	$kg \cdot m^2/s^2$
Power radiant flux	Watt	W	J/s	$kg \cdot m^2/s^3$
Quantity of electricity, electric charge	Coulomb	C		$A \cdot s$
Electric potential, potential difference, electromotive force	Volt	V	W/A	$kg \cdot m^2/A \cdot s^2$
Capacitance	Farad	F	C/V	$A^2 \cdot s^4/kg \cdot m^2$
Electric resistance	Ohm	Ω	V/A	$kg \cdot m^2/A \cdot s^3$
Conductance	Siemens	S	A/V	$A^2 \cdot s^3/kg \cdot m^2$
Magnetic flux	Weber	Wb	$V \cdot s$	$kg \cdot m^2/A \cdot s^2$
Magnetic flux	Tesla	T	Wb/m^2	$kg/A \cdot s^2$
Inductance	Henry	H	Wb/A	$kg \cdot m^2/A^2 \cdot s^2$

Comparison of the 1998 and 1986 CODATA recommended values of various constants

Quantity	1998 rel. std. uncert. u_r	1986 rel. std. Uncert. u_r	1986 u_r : 1998 u_r	D_r
α	3.7×10^{-9}	4.5×10^{-8}	12.2	– 1.7
λ_c	7.3×10^{-9}	8.9×10^{-8}	12.2	– 1.7
h	7.8×10^{-8}	6.0×10^{-7}	7.7	– 1.7
N_A	7.9×10^{-8}	5.9×10^{-7}	7.5	1.5
e	3.9×10^{-8}	3.0×10^{-7}	7.8	– 1.8
R	1.7×10^{-6}	8.4×10^{-6}	4.8	– 0.5
k	1.7×10^{-6}	8.5×10^{-6}	4.8	– 0.6
σ	7.0×10^{-6}	3.4×10^{-5}	4.8	– 0.6
G	**1.5×10^{-3}**	**1.3×10^{-4}**	0.1	0.0
R_∞	7.6×10^{-12}	1.2×10^{-9}	157.1	2.7
m_e/m_p	2.1×10^{-9}	2.0×10^{-8}	9.5	0.9
$A_r(e)$	2.1×10^{-9}	2.3×10^{-8}	11.1	0.7

Note: The relative standard uncertainty of a quantity y is defined as $u_r(y) \equiv u(y)/|y|$, if $y \neq 0$, where $u(y)$ is the standard uncertainty of y. D_r is the 1998 value minus the 1986 value divided by the standard uncertainty of the 1986 value. **Aug 2000 Physics Today ©**

CODATA Recommended Values of the Fundamental Physical Constants - 1998

Quantity	Symbol	Value	Unit	Rel. std. uncert. u_r
Speed of light in vacuum	c, c_0	299792458	m s^{-1}	Exact
Magnetic constant	μ_0	$4\pi \times 10^{-7}$	NA^{-2}	Exact
		$=12.566370614\ldots \times 10^{-7}$		
Electric constant $1/\mu_0 c^2$	ε_0	$8.854187817\ldots \times 10^{-12}$	F m^{-1}	Exact
Characteristic impedance	Z_0	$376.730313461\ldots$	Ω	Exact
of vacuum $\mu_0 c$				
Newtonian constant	G	$6.673(10) \times 10^{-11}$	$m^3\,kg^{-1}s^{-2}$	1.5×10^{-3}
of gravitation	G/hc	$6.707(10) \times 10^{-39}$	$(GeV/c^2)^{-2}$	1.5×10^{-3}
Planck constant	h	$6.62606876(52) \times 10^{-34}$	J s	7.8×10^{-8}
in eV s		$4.13566727(16) \times 10^{-15}$	eV s	3.9×10^{-8}
$h/2\pi$	h	$1.054571596(82) \times 10^{-34}$	J s	7.8×10^{-8}
in eV s		$6.58211889(26) \times 10^{-16}$	eV s	3.9×10^{-8}
Planck mass $(hc/G)^{1/2}$	m_p	$2.1767(16) \times 10^{-8}$	kg	7.5×10^{-4}
Planck length $h/m_p c = (hG/c^3)^{1/2}$	l_p	$1.6160(12) \times 10^{-35}$	m	7.5×10^{-4}
Planck time $l_p/c = (hG/c^5)^{1/2}$	t_p	$5.3906(40) \times 10^{-44}$	s	7.5×10^{-4}

Fundamental Physical Constant

- We keep determining precise values of those fundamental physical constants. Examples
  - Gravitation Constant G

    $G = 6.674\ 28(67) \times 10^{-11}$, 1.0×10^{-4}; (2006)

    $G = 6.673\ 28(67) \times 10^{-11}$, 1.5×10^{-3}; (1998)

    $G = 6.673\ xx(xx) \times 10^{-11}$, 1.3×10^{-4}; (1986)
  - Planck Constant h

    $h = 6.626\ 068\ 96(33) \times 10^{-34}$, 5.0×10^{-8}; (2006)

    $h = 6.626\ 068\ 76(52) \times 10^{-34}$, 7.8×10^{-8}; (1998)

Astronomical Units

- Unit Length:
  $1\text{AU} = 1.496 \times 10^{11}\ \text{m}$
- Unit Time:
  $1\text{yr} = 3.15 \times 10^{7}\ \text{s}$
- *Why do we want to use astronomical units?*

$$GM = 4\pi^2$$

$$G = (6.6726 \pm 0.0005) \times 10^{-11}\ \text{m}^3/\ \text{kg}\cdot\text{s}^2$$

Earth Units

- How to simulate motion of satellite in orbit about the earth?
- Same, but chose different unit, the Earth Units (EU).
- Unit Length: $1\text{EU} = 6.37 \times 10^6\ \text{m}$
- Unit Time: 1 hour = 3600 s
- $Gm = 20.0\ \text{EU}^3 / \text{h}^2$

Array Variables

- Array Variables:
  Data structure that consist of more than one variables and is an ordered set of elements that are of the same type, $x(i)$, $y(i)$.

- Useful for multi-dimension and/or many-particle system.

Common Properties of Arrays:

- Arrays are defined by a Dimension(in F90) or DIM(in BASIC), or ***similar*** statements (in other language). Total number of elements of an array is given by its lower and upper limit **(*they must be integers*)**, e.g., $a(30)$, $b(0:3,1:30)$.
- An element of an array is referenced by its subscript, e.g., $a(3)$.
- Array can be passed to a subroutine or function. The entire array is not actually passed, ***only the address of the first element***. Pay attention to the ordering of array elements.

For example a two dimension array defined as $a(3,5)$

$$
\begin{array}{ccccc}
a(1,1) & a(1,2) & a(1,3) & a(1,4) & a(1,5) \\
a(2,1) & a(2,2) & a(2,3) & a(2,4) & a(2,5) \\
a(3,1) & a(3,2) & a(3,3) & a(3,4) & a(3,5)
\end{array}
$$

Column

```
program vector          ! illustrate the use of arrays
    use common
    real, dimension (3) :: a,b
    real :: dot
    call initial(a,b)
    dot = dot_product(a,b)
    print *, "dot product = ", dot
    call cross(a,b)
end program vector

module common
  public :: initial,cross
  contains   !all subroutines here
end module common

subroutine initial(a,b)
    real, dimension (:), intent(out) :: a,b
    a(1:3) = (/ 2.0, -3.0, -4.0 /)
    b(1:3) = (/ 6.0, 5.0, 1.0 /)
end subroutine initial
```

```
subroutine cross(r,s)
   real, dimension (:), intent(in) :: r,s
   real, dimension (3) :: cross_product
   ! note use of dummy variables
   integer :: component,i,j
   do component = 1,3
     i = modulo(component,3) + 1        ! Cyclic
     j = modulo(i,3) + 1
     cross_product(component) = r(i)*s(j) - s(i)*r(j)
   end do
   print *, ""
   print *, "three components of the vector product:"
   print "(a,t10,a,t16,a)", "x","y","z"
   print *, cross_product
 end subroutine cross
```

Simulation of the Orbit

- Differential equations to solve:

$$\frac{d^2x}{dt^2} = -\frac{GM}{r^3}x$$

$$\frac{d^2y}{dt^2} = -\frac{GM}{r^3}y, \quad \text{where } r^2 = x^2 + y^2$$

- Units: Astronomical Units
- Initial conditions:

$x_1 = \text{(read in)}, x_2 = 0; v_1 = 0, v_2 = \text{(read in)}$

```
program planet                              ! planetary motion
   use common
   integer :: nshow,counter
   call initial(nshow)
!  call energy(eoverm0)                     ! conservation quantities
   call output()
   counter = 0
   do
   if (t > 2) then
      exit
   end if
      call euler()                          ! or Euler-Richardson
      counter = counter + 1
      if (modulo(counter,nshow) == 0) then
         call output()                      ! coordinates of "earth“
!        call energy(eoverm)                ! Note: define quantities
      end if
   end do
end program planet
```

```
module common
public :: initial,euler,output

real (selected_real_kind(15,307)), public :: t,dt
real (selected_real_kind(15,307)), public, dimension (2) ::
    pos,vel

real (selected_real_kind(15,307)), public, parameter ::
    pi = 3.141569265358979323846   ! or pi =

real (selected_real_kind(15,307)), public, parameter ::
    gm = 4.0*pi*pi                          ! astronomical units

contains
!all subroutines here
end module common
```

```
subroutine initial(nshow)
  integer, intent (out) :: nshow
  t = 0.0
  print *, "time step = "
  read *, dt
  print *, "number of time steps between output = "
  read *, nshow
  print *, "initial x position = "
  read *, pos(1)
  pos(2) = 0                    ! initial y-position (read in)
  vel(1) = 0                    ! initial x-velocity
  print *, "initial y-velocity = "
  read *, vel(2)
end subroutine initial
```

```
subroutine euler()    ! Euler-Cromer algorithm
   real (selected_real_kind(15,307)), dimension (2) :: accel
   real (selected_real_kind(15,307)):: r2,r3
   integer :: i
   r2 = pos(1)*pos(1) + pos(2)*pos(2)
   r3 = r2*sqrt(r2)
   do i = 1,2
      accel(i) = -gm*pos(i)/r3
      vel(i) = vel(i) + accel(i)*dt
      pos(i) = pos(i) + vel(i)*dt
   end do
   t = t + dt
end subroutine euler

subroutine output()
   print *, pos(1),pos(2), vel(1), vel(2)
end subroutine output
```

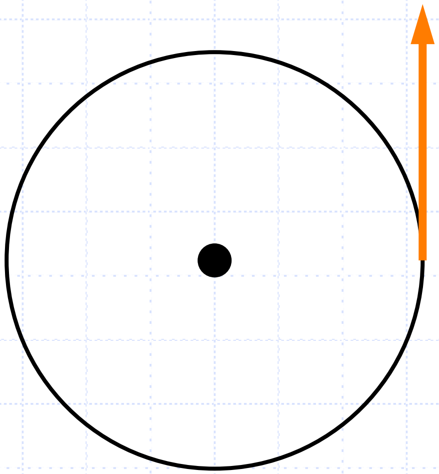
Conservation Laws

- You can also write sub programs to check ***conserved*** quantities like

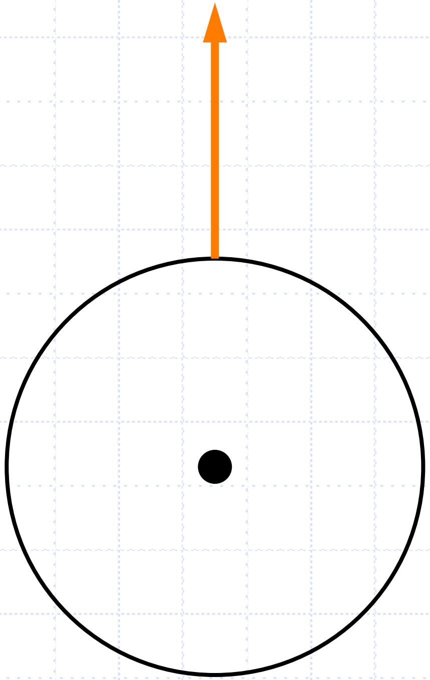
$$\frac{E}{m}, \quad \frac{L}{m}, \quad \frac{T^2}{a^3}, \ \dots$$

```
subroutine conserve(eoverm)
   real,intent(in,out) :: eoverm
   r2=pos(1)*pos(1)+pos(2)*pos(2)
   v2=vel(1)*vel(1)+vel(2)*vel(2)
   eoverm=v2/2.0-GM/sqrt(r2)
end subroutine conserve
```

Perturbation



An impulse applied in the ***tangential*** direction.



An impulse applied in the ***radial*** direction.

Perturbation (Read Textbook)

```
subroutine EulerRichardson(...)
  use common
  define quantities
  if (s .eq. 2) then
    pos_save = pos
    kick_flag = "on"
  end if

  do while (kick_flag = "on")
    diff=abs(pos-pos_save)
    if( diff(1) < 0.02) and ( diff(2) < 0.02 ) then
      v2 = vel(1)*vel(1)+vel(2)*vel(2)
      vel(2) = vel(2) + 0.1*sqrt(v2)
      kick_flag = "off"
    end if
  end do
```

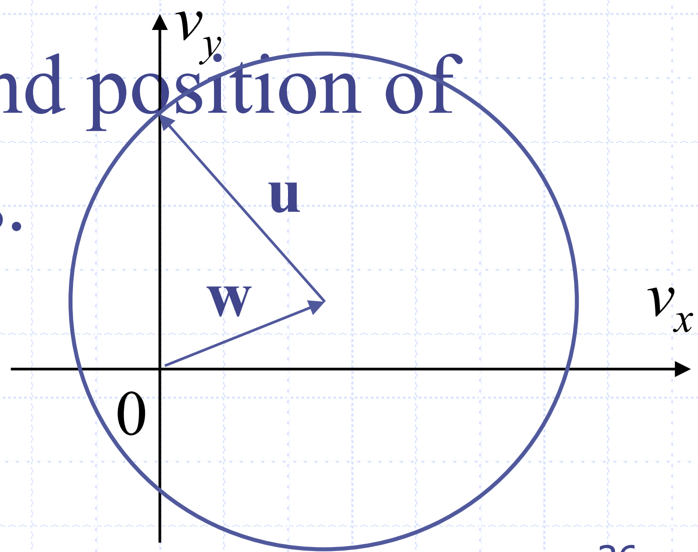
```
call acceleration(pos(),vel(),accel(),GM)

velmid = vel + 0.5*dt*accel
posmid = pos + 0.5*dt*vel

call acceleration(posmid(),velmid(),accel(),GM)

vel = vel + dt*accel
pos = pos + dt*velmid
t   = t   + dt

end subroutine EulerRichardson
```

Check Problems 4.6 and 4.7 for more examples.

Drag Force?

Velocity Space

- *Forces act on the path of particle by changing its **velocity, NOT position.***
- Plot (v_x, v_y) instead of (x, y), what is its shape?
- Consider both velocity and position of particle on an equal basis.
- Phase space graph
  Advanced Mechanics
  Quantum Mechanics

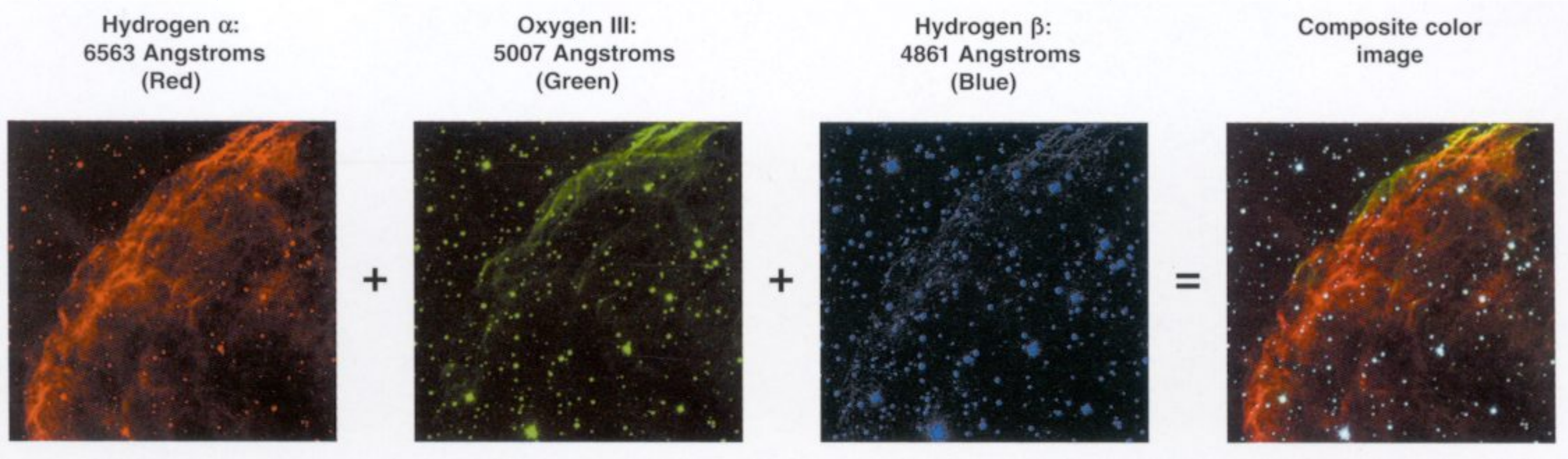


Supernova Remnant IC 443

The filamentary nebula IC 443 lies in the constellation Gemini. It is the remnant of a supernova that exploded thousands of years ago; it lies about 5000 light-years form the Earth. The arches of gas in this image are part of a much larger bubble of gas that is still expanding outward into interstellar space. As it does so, it sweeps up interstellar gas and dust, churning it up, mixing in the heavy elements produced in the explosion, and producing the filamentary structure.

The supernova that produced this nebula was very close to a molecular cloud complex – a region of interstellar space thick with dust and gas, much of which is cool enough for molecules to form. Molecular clouds are the sites of current star formation. The death of this nearby star, and the resulting compression of the surrounding material form the blast wave, may contribute to formation of new stars in the future.

The Color-imaging Process



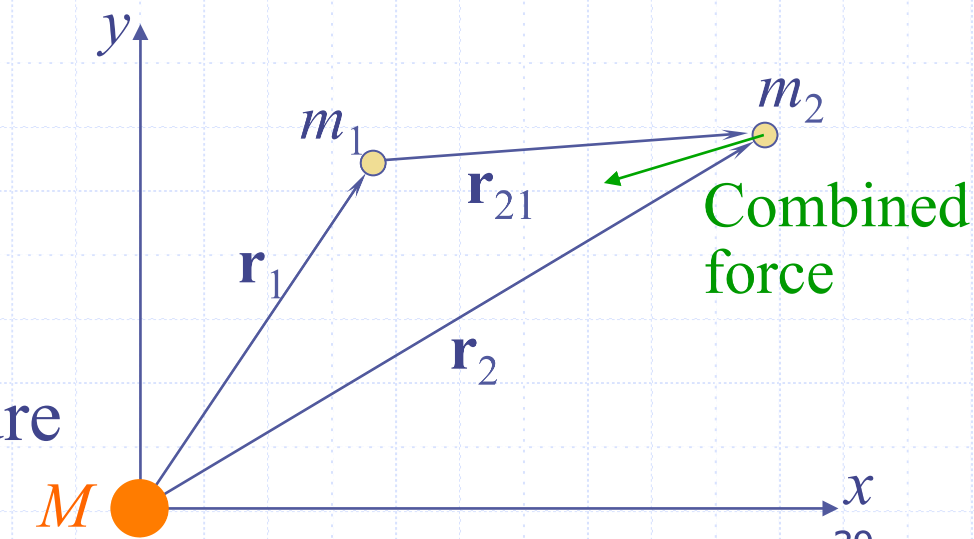
http://www.phy.cuhk.edu.hk/astroworld

http://www.nasa.gov/vision/starsgalaxies/index.html

A Mini-Solar System

The existence of other planets in the solar system means:

- Kepler's three laws are no longer true.
- The total force on a given planet is not a central force.
- The motion is mostly 3D.



Examples: (2D, 3 bodies on the same plane)

Equation of motion:

$$m_1 \frac{d^2\mathbf{r}_1}{dt^2} = -\frac{GMm_1}{r_1^3}\mathbf{r}_1 + \frac{Gm_1m_2}{r_{21}^3}\mathbf{r}_{21}$$

$$m_2 \frac{d^2\mathbf{r}_2}{dt^2} = -\frac{GMm_2}{r_2^3}\mathbf{r}_2 + \frac{Gm_1m_2}{r_{21}^3}\mathbf{r}_{21}$$

$m_1/M = 10^{-3}$, $m_2/M = 4 \times 10^{-3}$,

ratio(1) = $(m_2/M)\ GM = 0.004 \times GM$,

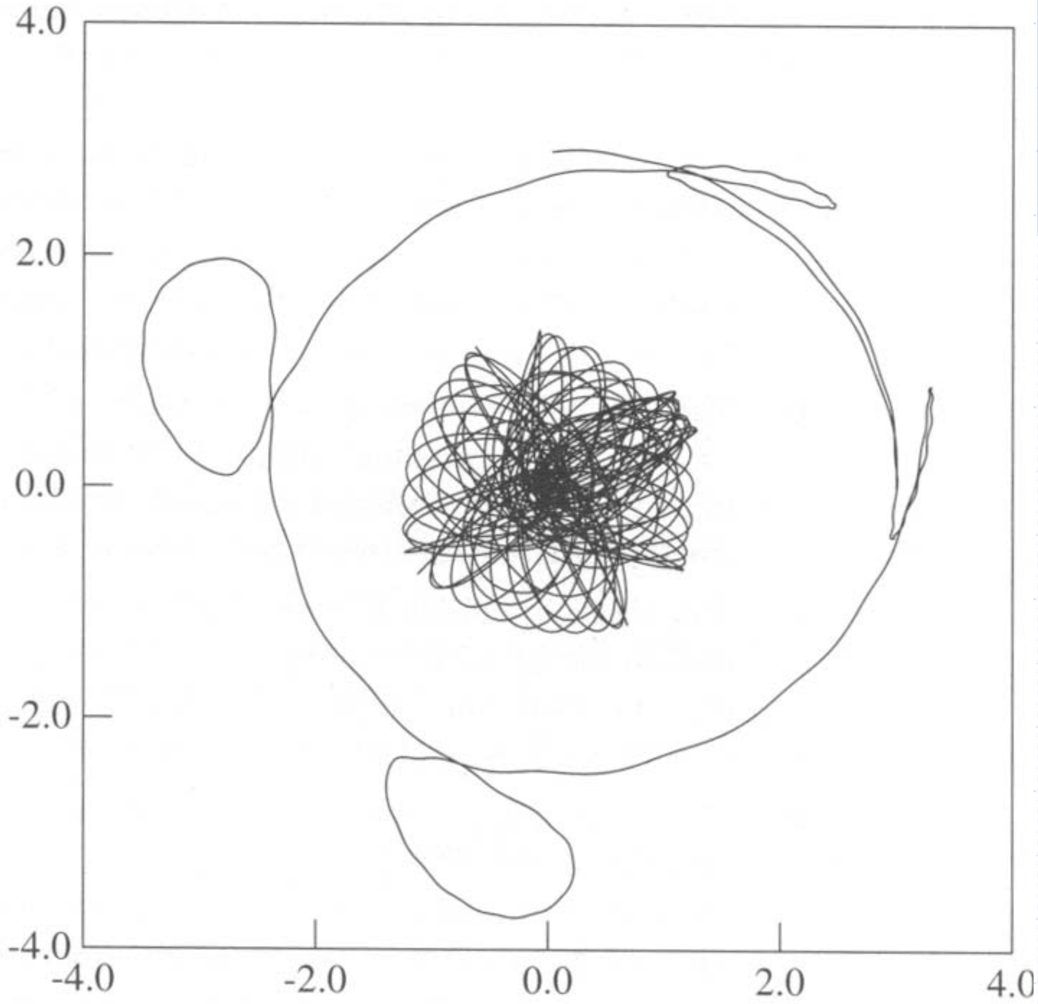
ratio(2) = $-(m_1/M)\ GM = -0.001 \times GM$.

Examples: (2D, 3 bodies on the same plane)

```
PROGRAM  planet2
! D = 2 solar system with major and minor planet
DIM x(2),y(2),vx(2),vy(2),ratio(2)
LIBRARY "csgraphics"
CALL initial (x(),y(),vx(),vy(),t,GM,ratio(),dt, nshow)
CALL output(x(),y(),t)
LET counter = 0
DO
    CALL Euler(x(),y(),vx(),vy(),t,GM,ratio(),dt)
        LET counter = counter + 1
    IF mod(counter,nshow) = 0 then
        CALL output(x(),y(),t)
    END IF
LOP until key input
END
```

Orbits of the two e^-s using the initial condition $\mathbf{r}_1 = (3,0)$, $\mathbf{r}_2 = (1,0)$, $\mathbf{v}_1 = (0, 0.4)$, and $\mathbf{v}_2 = (0, -1)$

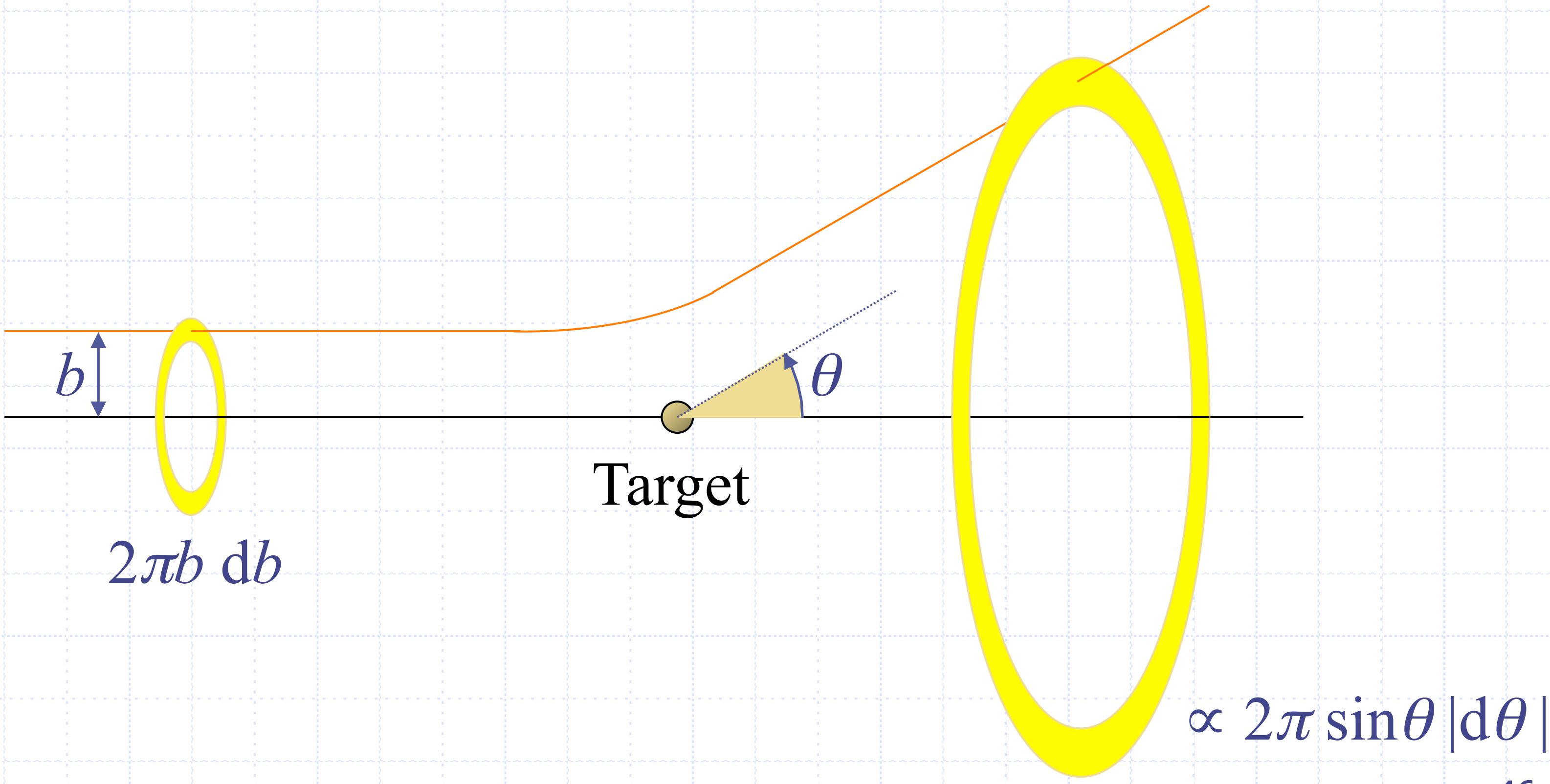
Program details omitted.



Two-Body Scattering

A tool for understanding the structure of matter

Differential cross section: (**classical**)



Two-Body Scattering

$$\frac{dN}{N} = n\sigma(\theta)d\Omega$$

N: total no. of particles in the beam,

dN: the no. of particles scattered into $d\Omega$,

n: target density (number of targets per unit area).

$$\sigma(\theta) = \frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

In real experiments, we only know dN/N as function of θ.

Lecture 4 Review and Required

- Simulating solar system (and alike) according to Newton's universal law of gravitation.
- **Physical units and computer simulations**.
- Centre of mass, conservation laws, etc.
- **Log-log plots and data analysis**.
- Two-body scattering, differential cross section.
- **Programs: planet**, **planet2**, etc.
- Use arrays and functions (e.g., **force**).

Non-inverse Square Forces

- Einstein's theory of gravitation predicts corrections to Newton's law:

$$m\frac{d^2\vec{r}}{dt^2} = -\frac{GMm}{r^3}\left[1-\frac{\alpha GM}{c^2 r}\right]\vec{r}$$

- The constant α is dimensionless. In AU units, α/c^2 is a maximum for the planet Mercury, smaller than $\mathbf{10^{-3}}$
- What would the orbit look?