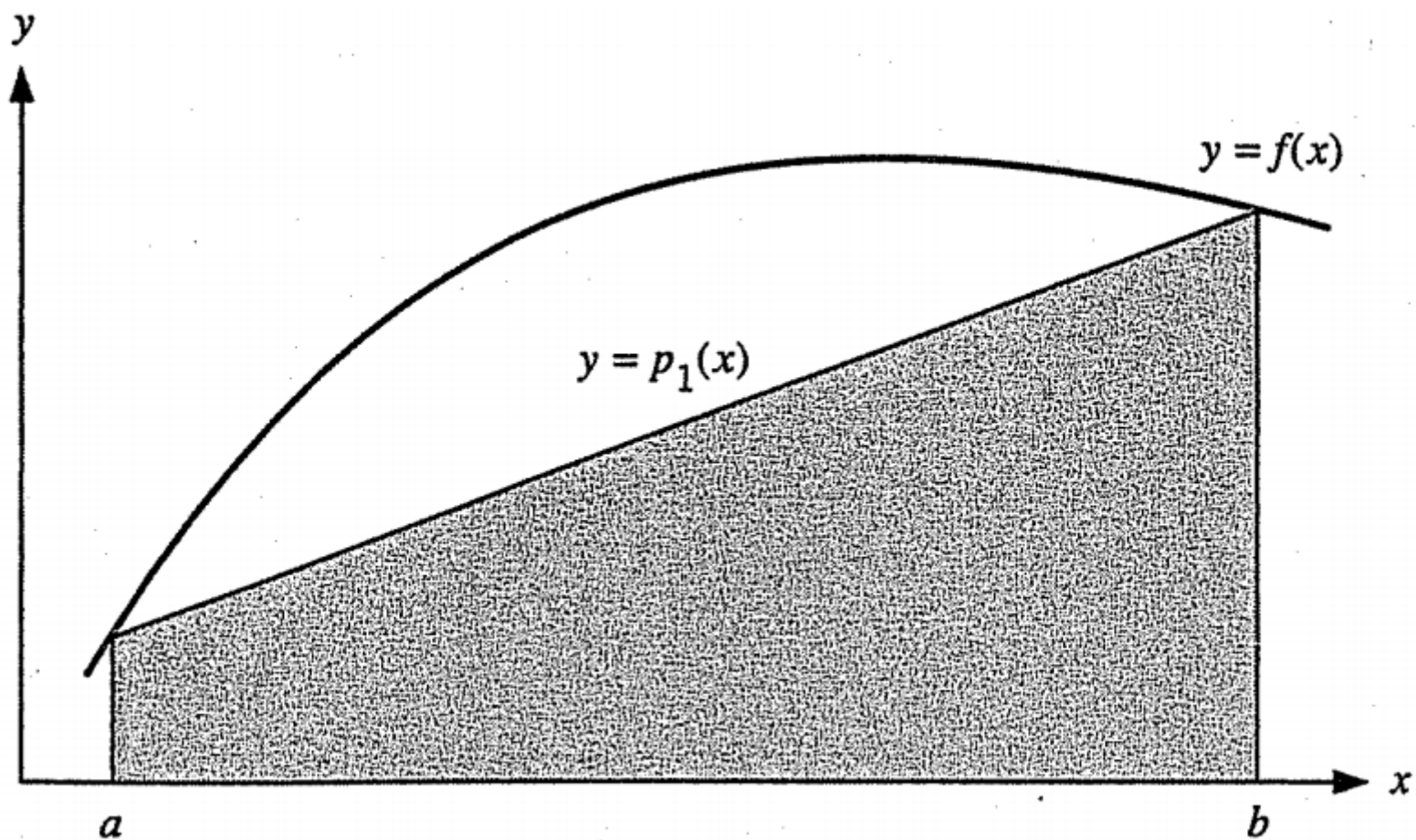
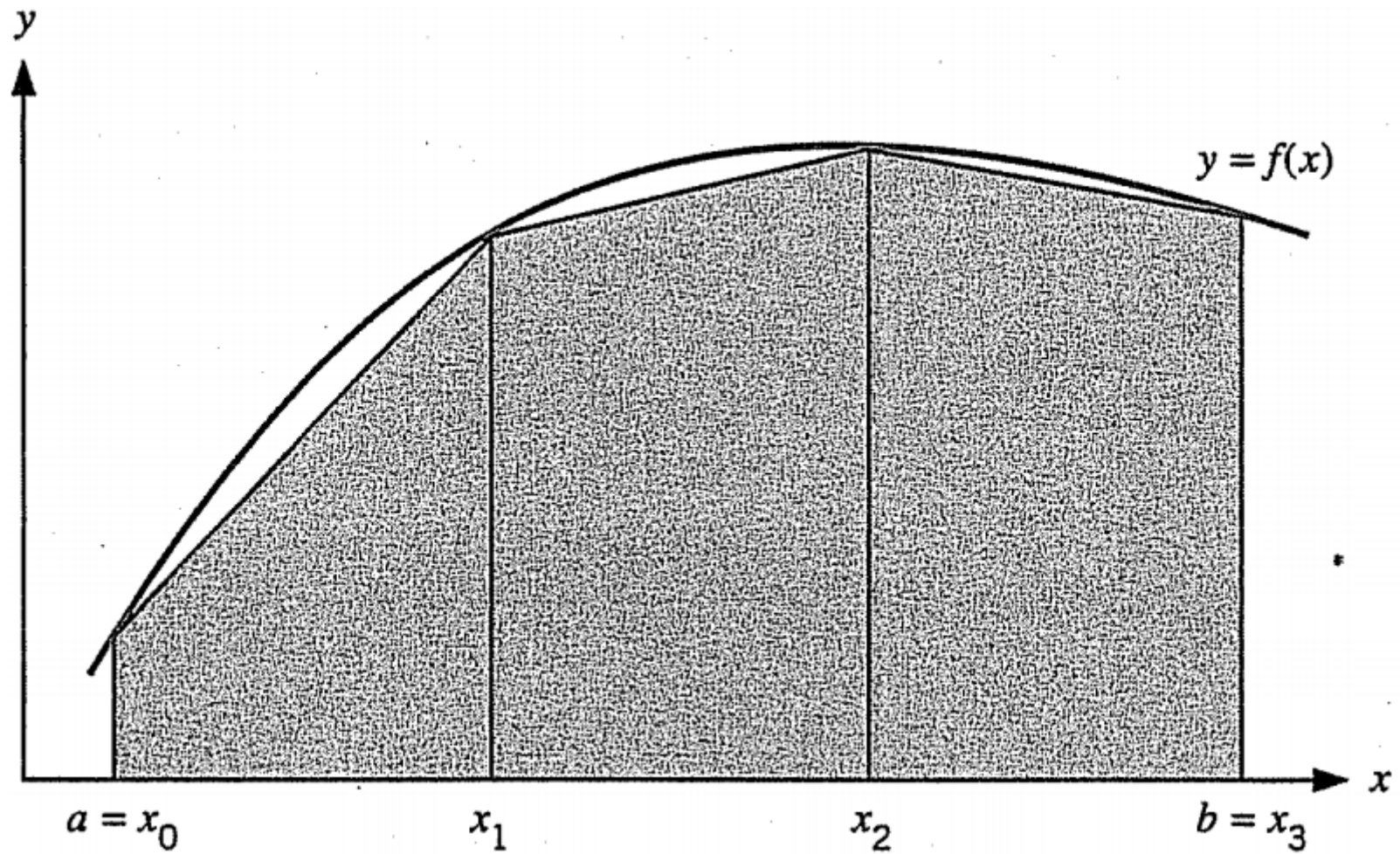


# Numerical Integration



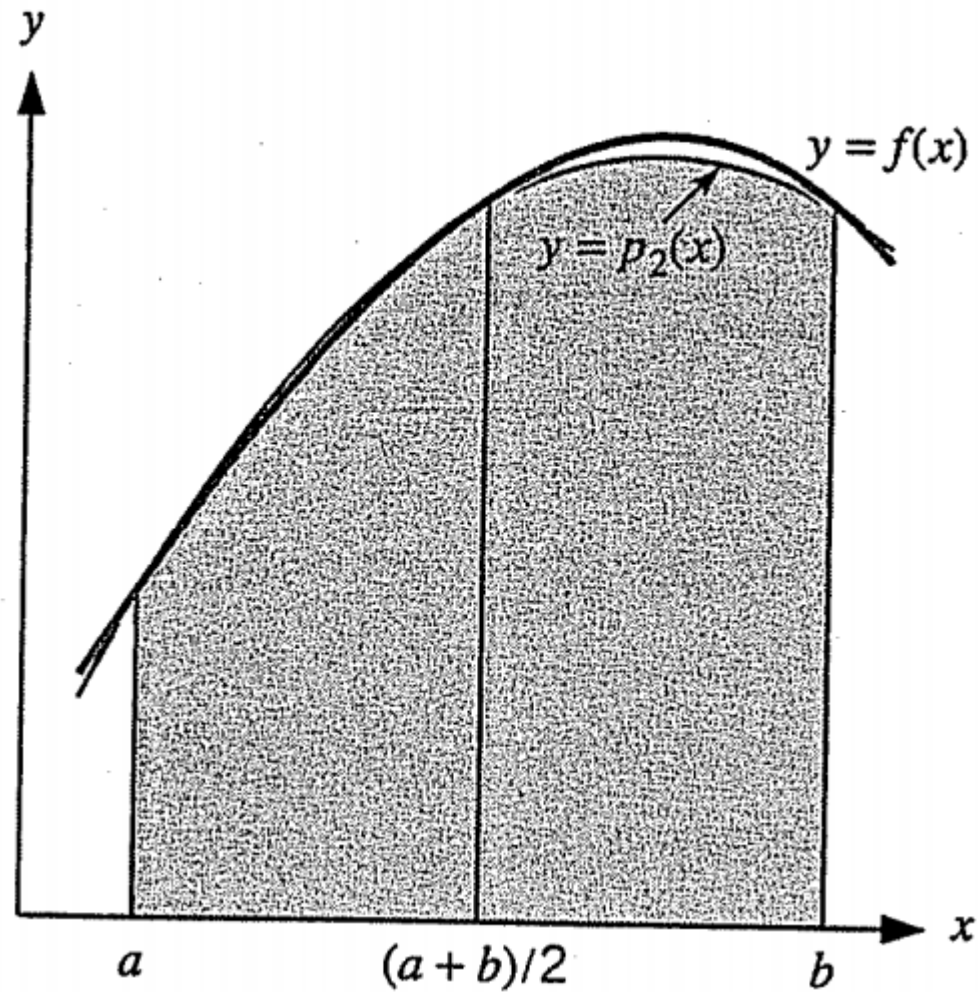
**Figure 5.1.** An illustration of the trapezoidal rule (5.4)



**Figure 5.2.** An illustration of  $T_3(f)$

$n$	$I^{(1)}$		$I^{(2)}$		$I^{(3)}$	
	Error	Ratio	Error	Ratio	Error	Ratio
2	$1.55\text{E} - 2$		$-1.33\text{E} - 1$		$-5.61\text{E} - 1$	
4	$3.84\text{E} - 3$	4.02	$-3.59\text{E} - 3$	37.0	$-3.76\text{E} - 2$	14.9
8	$9.59\text{E} - 4$	4.01	$5.64\text{E} - 4$	-6.37	$-1.93\text{E} - 4$	195.0
16	$2.40\text{E} - 4$	4.00	$1.44\text{E} - 4$	3.92	$-5.19\text{E} - 9$	37,600.0
32	$5.99\text{E} - 5$	4.00	$3.60\text{E} - 5$	4.00	*	
64	$1.50\text{E} - 5$	4.00	$9.01\text{E} - 6$	4.00	*	
128	$3.74\text{E} - 6$	4.00	$2.25\text{E} - 6$	4.00	*	

**Table 5.1.** Examples of the Trapezoidal Rule



**Figure 5.3.** An illustration of Simpson's rule (5.18)

$n$	$I^{(1)}$		$I^{(2)}$		$I^{(3)}$	
	Error	Ratio	Error	Ratio	Error	Ratio
2	$-3.56\text{E} - 4$		$8.66\text{E} - 2$		$-1.26$	
4	$-3.12\text{E} - 5$	11.4	$3.95\text{E} - 2$	2.2	$1.37\text{E} - 1$	-9.2
8	$-1.99\text{E} - 6$	15.7	$1.95\text{E} - 3$	20.3	$1.23\text{E} - 2$	11.2
16	$-1.25\text{E} - 7$	15.9	$4.02\text{E} - 6$	485.0	$6.43\text{E} - 5$	191.0
32	$-7.79\text{E} - 9$	16.0	$2.33\text{E} - 8$	172.0	$1.71\text{E} - 9$	37,600.0
64	$-4.87\text{E} - 10$	16.0	$1.46\text{E} - 9$	16.0	*	
128	$-3.04\text{E} - 11$	16.0	$9.15\text{E} - 11$	16.0	*	

**Table 5.2.** Examples of the Simpson Rule

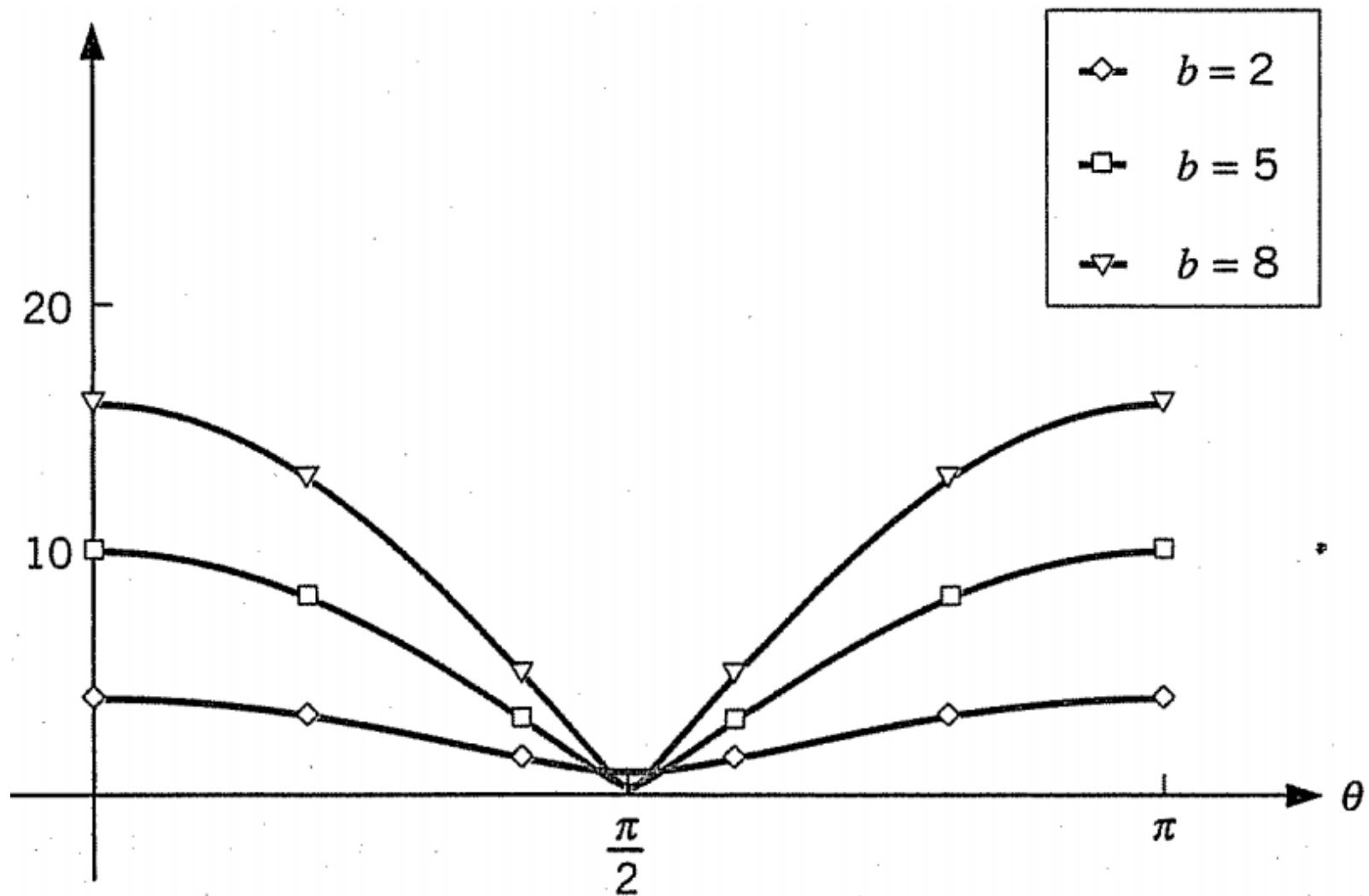
$n$	$I - T_n(f)$	$\tilde{E}_n(f)$	$CT_n(f)$	$I - CT_n(f)$	Ratio
2	1.545E - 2	1.533E - 2	0.746698561877	1.26E - 4	
4	3.840E - 3	3.832E - 3	0.746816175313	7.96E - 6	15.8
8	9.585E - 4	9.580E - 4	0.746823634224	4.99E - 7	16.0
16	2.395E - 4	2.395E - 4	0.746824101633	3.12E - 8	16.0
32	5.988E - 5	5.988E - 5	0.746824130863	1.95E - 9	16.0
64	1.497E - 5	1.497E - 5	0.746824132690	2.22E - 10	16.0

**Table 5.3.** Example of  $CT_n(f)$  and  $\tilde{E}_n(f)$

$n$	Error	Ratio
2	$2.860\text{E} - 2$	
4	$1.014\text{E} - 2$	2.82
8	$3.587\text{E} - 3$	2.83
16	$1.268\text{E} - 3$	2.83
32	$4.485\text{E} - 4$	2.83

**Table 5.4.** Simpson's Rule for  $\sqrt{x}$

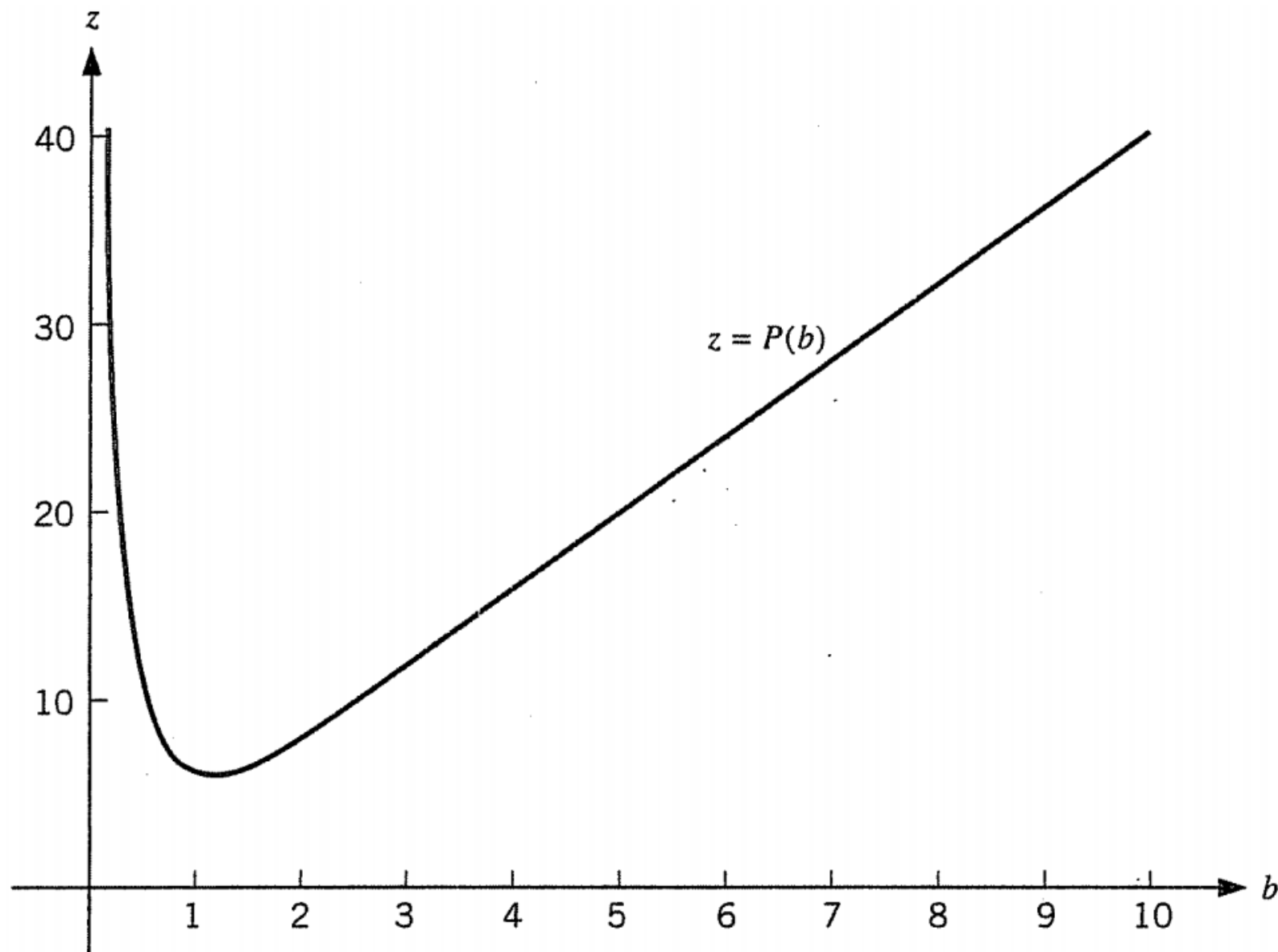




**Figure 5.4.** The graph of integrand  $f(\theta) : b=2, 5, 8$

$n$	$b = 2$	$b = 5$	$b = 8$
8	8.575517	19.918814	31.690628
16	8.578405	20.044483	31.953632
32	8.578422	20.063957	32.008934
64	8.578422	20.065672	32.018564
128	8.578422	20.065716	32.019660
256	8.578422	20.065717	32.019709

**Table 5.5.** Trapezoidal Rule Approximations of  
(5.46)



**Figure 5.5.** The graph of perimeter function  $P(b)$  for ellipse

$n$	$\rho_n(f)$	$n$	$\rho_n(f)$
1	5.30E - 2	6	7.82E - 6
2	1.79E - 2	7	4.62E - 7
3	6.63E - 4	8	9.64E - 8
4	4.63E - 4	9	8.05E - 9
5	1.62E - 5	10	9.16E - 10

**Table 5.6.** Minimax Errors for  $e^{-x^2}$ ,  $0 \leq x \leq 1$

$n$	$x_i$	$w_i$
2	$\pm 0.5773502692$	1.0
3	$\pm 0.7745966692$	0.5555555556
	0.0	0.8888888889
4	$\pm 0.8611363116$	0.3478548451
	$\pm 0.3399810436$	0.6521451549
5	$\pm 0.9061798459$	0.2369268851
	$\pm 0.5384693101$	0.4786286705
	0.0	0.5688888889
6	$\pm 0.9324695142$	0.1713244924
	$\pm 0.6612093865$	0.3607615730
	$\pm 0.2386191861$	0.4679139346
7	$\pm 0.9491079123$	0.1294849662
	$\pm 0.7415311856$	0.2797053915
	$\pm 0.4058451514$	0.3818300505
	0.0	0.4179591837
8	$\pm 0.9602898565$	0.1012285363
	$\pm 0.7966664774$	0.2223810345
	$\pm 0.5255324099$	0.3137066459
	$\pm 0.1834346425$	0.3626837834

**Table 5.7.** Nodes and Weights of Gaussian Quadrature Formulas

$n$	Error in $I^{(1)}$	Error in $I^{(2)}$	Error in $I^{(3)}$
2	2.29E - 4	-2.33E - 2	8.23E - 1
3	9.55E - 6	-3.49E - 2	-4.30E - 1
4	-3.35E - 7	-1.90E - 3	1.77E - 1
5	6.05E - 9	1.70E - 3	-8.12E - 2
6	-7.77E - 11	2.74E - 4	3.55E - 2
7	7.89E - 13	-6.45E - 5	-1.58E - 2
10	*	1.27E - 6	1.37E - 3
15	*	7.40E - 10	-2.33E - 5
20	*	*	3.96E - 7

**Table 5.8.** Gaussian Numerical Integration Examples

$n$	$I - I_n$	Ratio
2	$-7.22\text{E} - 3$	
4	$-1.16\text{E} - 3$	6.2
8	$-1.69\text{E} - 4$	6.9
16	$-2.30\text{E} - 5$	7.4
32	$-3.00\text{E} - 6$	7.6
64	$-3.84\text{E} - 7$	7.8

**Table 5.9.** Gaussian Integration of (5.64)