(b) Consider the same questions as in part (a) but with $N = 4$ and $N = 10$. Consider the $n = 2$ mode for $N = 4$ and the $n = 3$ and $n = 8$ modes for $N = 10$. (See (9.10) for the form of the normal mode solutions.) Also consider random initial displacements. ■

Problem 9.4 Different boundary conditions

(a) Modify your program from Problem 9.3 so that periodic boundary conditions are used, that is, $u_0 = u_N$ and $u_1 = u_{N+1}$. Choose $N = 10$ and the initial condition corresponding to the normal mode (9.10) with $n = 2$. Does this initial condition yield a normal mode solution for periodic boundary conditions? (It might be easier to answer this question by plotting u_i versus time for two or more particles.) For fixed boundary conditions, there are $N + 1$ springs, but for periodic boundary conditions, there are N springs. Why? Choose the initial condition corresponding to the $n = 2$ normal mode, but replace $N + 1$ by N in (9.7). Does this initial condition correspond to a normal mode? Now try $n = 3$ and other values of n. Which values of n give normal modes? Only sine functions can be normal modes for fixed boundary conditions (see (9.4)). Can there be normal modes with cosine functions if we use periodic boundary conditions?

(b) Modify your program so that free boundary conditions are used, which means that the masses at the end points are connected to only one nearest neighbor. A simple way to implement this boundary condition is to set $u_0 = u_1$ and $u_N = u_{N+1}$. Choose $N = 10$, and use the initial condition corresponding to the $n = 3$ normal mode found using fixed boundary conditions. Does this condition correspond to a normal mode for free boundary conditions? Is $n = 2$ a normal mode for free boundary conditions? Are the normal modes purely sinusoidal?

(c) Choose free boundary conditions and $N \geq 10$. Let the initial condition be a pulse of the form $u_1 = 0.2$, $u_2 = 0.6$, $u_3 = 1.0$, $u_4 = 0.6$, $u_5 = 0.2$, and all other $u_j = 0$. After the pulse reaches the right end, what is the phase of the reflected pulse; that is, are the displacements in the reflected pulse in the same direction as the incoming pulse (a phase shift of zero degrees) or in the opposite direction (a phase shift of 180 degrees)? What happens for fixed boundary conditions? Choose N to be as large as possible so that it is easy to distinguish the incident and reflected waves.

(d) Choose $N \geq 20$ and let the spring constants on the right half of the system be four times greater than the spring constants on the left half. Use fixed boundary conditions. Set up a pulse on the left side. Is there a reflected pulse at the boundary between the two types of springs? If so, what is its relative phase? Compare the amplitude of the reflected and transmitted pulses. Consider the same questions with a pulse that is initially on the right side. ■

Problem 9.5 Motion of coupled oscillators with external forces

(a) Modify your program from Problem 9.4 so that an external force F_{ext} is exerted on the first particle:

$$F_{\text{ext}}/m = 0.5 \cos \omega t, \tag{9.20}$$

where ω is the angular frequency of the external force. Let the initial displacements and velocities of all N particles be zero. Choose $N = 3$ and consider the response of the system to an external force for $\omega = 0.5$ to 4.0 in steps of 0.5. Record $A(\omega)$, the maximum amplitude of any particle, for each value of ω. Repeat the simulation for $N = 10$.

(b) Choose ω to be one of the normal mode frequencies. Does the maximum amplitude remain constant or does it increase with time? How can you use the response of the system to an external force to determine the normal mode frequencies? Discuss your results in terms of the power input $F_{\text{ext}} v_1$?

(c) In addition to the external force exerted on the first particle, add a damping force equal to $-\gamma v_i$ to all the oscillators. Choose the damping constant $\gamma = 0.05$. How do you expect the system to behave? How does the maximum amplitude depend on ω? Are the normal mode frequencies changed when $\gamma \neq 0$? ■

In Problem 9.5 we saw that a boundary produces reflections that lead to resonances. Although it is not possible to eliminate all reflections, it can be shown analytically that it is possible to absorb waves at a single frequency ω by using a free boundary and a judicious choice of the mass and damping coefficient for the last oscillator (see the text by Main). These conditions are

$$m_{N-1} = \frac{m}{2} \tag{9.21a}$$

$$\gamma_{N-1} = \frac{k}{\omega} \sin qa, \tag{9.21b}$$

where a is the separation between oscillators, q is the wavenumber, and ω is the angular frequency. Problem 9.6 shows that this choice leads to a transparent boundary (at the selected frequency) and is akin to impedance matching in electrical transmission lines enabling us to study traveling waves. In Section 9.7 we will study the classical wave equation in detail.

Problem 9.6 Traveling waves

(a) Modify the oscillator program by imposing a sinusoidal amplitude on the first oscillator:

$$u_1(t) = 0.5 \sin \omega t. \tag{9.22}$$

Run the program with $\omega = 1.0$ and observe the right traveling wave, but note that reflections soon produce a left traveling wave.

(b) Implement transparent boundary conditions on the right-hand side. Is the first wave crest totally transmitted at the end, or is there some residual reflection? Explain. Add a small overall damping to the chain to remove transient effects.

(c) The dispersion relation (9.9) predicts a cutoff frequency:

$$\omega_c^2 = 4\frac{k}{m}. \tag{9.23}$$

What happens if we apply an external driving force above this frequency? ■