(d) Compute the time dependence of $-2GMm/E$ for asteroid orbits whose initial position `x(1)` ranges from 2.0 to 5.0 in steps of 0.2. Choose the initial values of `vy(1)` so that circular orbits would be obtained in the absence of Jupiter. Are there any values of `x(1)` for which the time dependence of a is unusual?

(e) Make a histogram of the number of asteroids versus the value of $-2GMm/E$ at $t = 2000$. (You can use the `HistogramFrame` class described on page 206 if you wish.) Assume that the initial value of `x(1)` ranges from 2.0 to 5.0 in steps of 0.02 and choose the initial values of `vy(1)` as before. Use a histogram bin width of 0.1. If you have time, repeat for $t = 5000$ and compare the histogram with your previous results. Is there any evidence for Kirkwood gaps? A resonance occurs when the periods of the asteroid and Jupiter are related by simple fractions. We expect the number of asteroids with values of a corresponding to resonances to be small.

(f) Repeat part (e) with initial velocities that vary from their values for a circular orbit by 1, 3, and 5%. ■

Project 5.19 The classical helium atom

The classical helium atom is a relatively simple example of a three-body problem and is similar to the gravitational three-body problem of a heavy sun and two light planets. The important difference is that the two electrons repel one another, unlike the planetary case where the intraplanetary interaction is attractive. If we ignore the small motion of the heavy nucleus, the equations of motion for the two electrons can be written as

$$\mathbf{a}_1 = -2\frac{\mathbf{r}_1}{r_1^3} + \frac{\mathbf{r}_1 - \mathbf{r}_2}{r_{12}^3} \tag{5.33a}$$

$$\mathbf{a}_2 = -2\frac{\mathbf{r}_2}{r_2^3} + \frac{\mathbf{r}_2 - \mathbf{r}_1}{r_{12}^3}, \tag{5.33b}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are measured from the fixed nucleus at the origin, and r_{12} is the distance between the two electrons. We have chosen units such that the mass and charge of the electron are both unity. The charge of the helium nucleus is two in these units. Because the electrons are sometimes very close to the nucleus, their acceleration can become very large, and a very small time step Δt is required. It is not efficient to use the same small time step throughout the simulation, and instead a variable time step or an *adaptive* step size algorithm is suggested. An adaptive step size algorithm can be used with any standard numerical algorithm for solving differential equations. The RK45 algorithm described in Appendix 3A is adaptive and is a good all-around choice for these types of problems.

(a) For simplicity, we restrict our atom to two dimensions. Modify `Planet2` to simulate the classical helium atom. Choose units such that the electron mass is one and the other constants are absorbed into the unit of charge so that the force between two electrons is

$$|F| = \frac{1}{r^2}. \tag{5.34}$$

Choose the initial value of the time step to be $\Delta t = 0.001$. Some of the possible orbits are similar to those we have seen in our mini-solar system. For example, try the initial condition $\mathbf{r}_1 = (2, 0)$, $\mathbf{r}_2 = (-1, 0)$, $\mathbf{v}_1 = (0, 0.95)$, and $\mathbf{v}_2 = (0, -1)$.

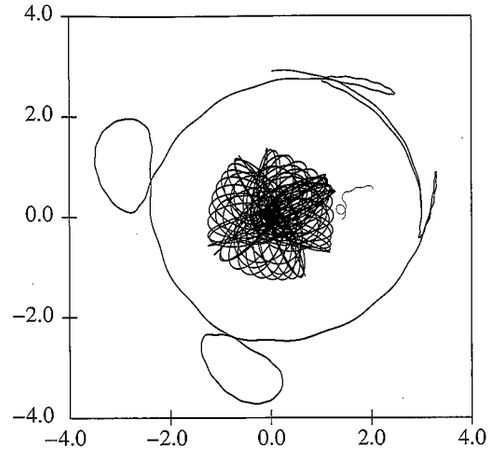

Figure 5.8 Orbits of the two electrons in the classical helium atom with the initial condition $\mathbf{r}_1 = (3, 0)$, $\mathbf{r}_2 = (1, 0)$, $\mathbf{v}_1 = (0, 0.4)$, and $\mathbf{v}_2 = (0, -1)$ (see Project 5.19c).

(b) Most initial conditions result in unstable orbits in which one electron eventually leaves the atom (autoionization). The initial condition $\mathbf{r}_1 = (1.4, 0)$, $\mathbf{r}_2 = (-1, 0)$, $\mathbf{v}_1 = (0, 0.86)$, and $\mathbf{v}_2 = (0, -1)$ gives "braiding" orbits. Make small changes in this initial condition to observe autoionization.

(c) The classical helium atom is capable of very complex orbits (see Figure 5.8). Investigate the motion for the initial condition $\mathbf{r}_1 = (3, 0)$, $\mathbf{r}_2 = (1, 0)$, $\mathbf{v}_1 = (0, 0.4)$, and $\mathbf{v}_2 = (0, -1)$. Does the motion conserve the total angular momentum? Also try $\mathbf{r}_1 = (2.5, 0)$, $\mathbf{r}_2 = (1, 0)$, $\mathbf{v}_1 = (0, 0.4)$, and $\mathbf{v}_2 = (0, -1)$.

(d) Choose the initial condition $\mathbf{r}_1 = (2, 0)$, $\mathbf{r}_2 = (-1, 0)$, and $\mathbf{v}_2 = (0, -1)$. Then vary the initial value of $\mathbf{v}_1$ from $(0.6, 0)$ to $(1.3, 0)$ in steps of $\Delta v = 0.02$. For each set of initial conditions, calculate the time it takes for autoionization. Assume that ionization occurs when either electron exceeds a distance of six from the nucleus. Run each simulation for a maximum time of 2000. Plot the ionization time versus v_{1x}. Repeat for a smaller interval of Δv centered about one of the longer ionization times. These calculations require much computer resources. Do the two plots look similar? If so, such behavior is called "self-similar" and is characteristic of chaotic systems and the geometry of fractals (see Chapters 6 and 13). More discussion on the nature of the orbits can be found in Yamamoto and Kaneko. ■

REFERENCES AND SUGGESTIONS FOR FURTHER READING

Harold Abelson, Andrea diSessa, and Lee Rudolph, "Velocity space and the geometry of planetary orbits," Am. J. Phys. **43**, 579–589 (1975). See also Andrea diSessa, "Orbit: a mini-environment for exploring orbital mechanics," O. Lecarme and R. Lewis, editors, *Computers in Education*, 359, North-Holland (1975). Detailed geometrical rather than calculus-based arguments on the origin of closed orbits for inverse-square forces are presented. Sections 5.7 and 5.8 are based on these papers.