13.4 ■ FRACTALS AND CHAOS

In Chapter 6 we explored dynamical systems that exhibited chaos under certain conditions. We found that after an initial transient, the trajectory of such a dynamical system consists of a set of points in phase space called an attractor. For chaotic motion this attractor is often an object that can be described as a fractal. Such attractors are called *strange attractors*.

We first consider the familiar logistic map (see (6.1)) $x_{n+1} = 4rx_n(1 - x_n)$. For most values of the control parameter $r > r_\infty = 0.892486417967\ldots$, the trajectories are chaotic. Are these trajectories fractals?

To calculate the fractal dimension for dynamical systems, we use the *box counting* method introduced in Section 13.2 in which space is divided into d-dimensional boxes of length ℓ. Let $N(\ell)$ equal the number of boxes that contain a piece of the trajectory. The fractal dimension is defined by the relation

$$N(\ell) \sim \lim_{\ell\to 0} \ell^{-D} \quad \text{(box dimension).} \tag{13.17}$$

Equation (13.17) holds only when the number of boxes is much larger than $N(\ell)$ and the number of points on the trajectory is sufficiently large. If the trajectory moves through many dimensions, that is, the phase space is very large, box counting becomes too memory intensive because we need an array of size $\propto \ell^{-d}$. This array becomes very large for small ℓ and large d.

A more efficient approach is to compute the *correlation dimension*. In this approach we store in an array the position of N points on the trajectory. We compute the number of points $N_i(r)$ and the fraction of points $f_i(r) = N_i(r)/(N-1)$ within a distance r of the point i. The correlation function $C(r)$ is defined by

$$C(r) \equiv \frac{1}{N}\sum_i f_i(r), \tag{13.18}$$

and the *correlation dimension* D_c is defined by

$$C(r) \sim \lim_{r\to 0} r^{D_c} \quad \text{(correlation dimension).} \tag{13.19}$$

From (13.19) we see that the slope of a log-log plot of $C(r)$ versus r yields an estimate of the correlation dimension. In practice, small values of r must be discarded because we cannot sample all of the points on the trajectory, and hence there is a cutoff value of r below which $C(r) = 0$. In the large r limit, $C(r)$ saturates to unity if the trajectory is localized as it is for chaotic trajectories. We expect that for intermediate values of r, there is a scaling regime where (13.19) holds.

In Problems 13.12–13.14, we consider the fractal properties of some of the dynamical systems that we considered in Chapter 6.

Problem 13.12 Strange attractor of the logistic map

(a) Write a program that uses box counting to determine the fractal dimension of the attractor for the logistic map. Compute $N(\ell)$, the number of boxes of length ℓ that have been visited by the trajectory. Test your program for $r < r_\infty$. How does the number of boxes containing a piece of the trajectory change with ℓ? What does this dependence tell you about the dimension of the trajectory for $r < r_\infty$?

(b) Compute $N(\ell)$ for $r = 0.9$ using at least five different values of ℓ, for example, $1/\ell = 100, 300, 1000, 3000, \ldots$. Iterate the map at least 10,000 times before determining $N(\ell)$. What is the fractal dimension of the attractor? Repeat for $r \approx r_\infty$, $r = 0.95$, and $r = 1$.

(c) Generate points at random in the unit interval and estimate the fractal dimension using the same method as in part (b). What do you expect to find? Use your results to estimate the accuracy of the fractal dimension that you found in part (b).

(d) Write a program to compute the correlation dimension for the logistic map and repeat the calculations for parts (b) and (c). ■

Problem 13.13 Strange attractor of the Hénon map

(a) Use two-dimensional boxes of linear dimension ℓ to estimate the fractal dimension of the strange attractor of the Hénon map (see (6.32)) with $a = 1.4$ and $b = 0.3$. Iterate the map at least 1000 times before computing $N(\ell)$. Does it matter what initial condition you choose?

(b) Compute the correlation dimension for the same parameters used in part (a) and compare D_c with the box dimension computed in part (a).

(c) Iterate the Hénon map and view the trajectory on the screen by plotting x_{n+1} versus x_n in one window and y_n versus x_n in another window. Do the two ways of viewing the trajectory look similar? Estimate the correlation dimension, where the ith data point is defined by (x_i, x_{i+1}) and the distance r_{ij} between the ith and jth data point is given by $r_{ij}{}^2 = (x_i - x_j)^2 + (x_{i+1} - x_{j+1})^2$.

(d) Estimate the correlation dimension with the ith data point defined by x_i and $r_{ij}{}^2 = (x_i - x_j)^2$. What do you expect to obtain for D_c? Repeat the calculation for the ith data point given by (x_i, x_{i+1}, x_{i+2}) and $r_{ij}{}^2 = (x_i - x_j)^2 + (x_{i+1} - x_{j+1})^2 + (x_{i+2} - x_{j+2})^2$. What do you find for D_c? ■

*Problem 13.14 Strange attractor of the Lorenz model

(a) Use three-dimensional graphics or three two-dimensional plots of $x(t)$ versus $y(t)$, $x(t)$ versus $z(t)$, and $y(t)$ versus $z(t)$ to view the structure of the Lorenz attractor. Use $\sigma = 10$, $b = 8/3$, $r = 28$, and the time step $\Delta t = 0.01$. Compute the correlation dimension for the Lorenz attractor.

(b) Repeat the calculation of the correlation dimension using $x(t)$, $x(t+\tau)$, and $x(t+2\tau)$ instead of $x(t)$, $y(t)$, and $z(t)$. Choose the delay time τ to be at least ten times greater than the time step Δt.

(c) Compute the correlation dimension in two dimensions using $x(t)$ and $x(t+\tau)$. Do the same calculation in four dimensions using $x(t)$, $x(t+\tau)$, $x(t+2\tau)$, and $x(t+3\tau)$. What can you conclude about the results for the correlation dimension using two-, three-, and four-dimensional spaces? What do you expect to see for $d > 4$? ■

Problems 13.13 and 13.14 illustrate a practical method for determining the underlying structure of systems when, for example, the data consists only of a single time series, that is,