We next combine (3.75a) and (3.75c) for the momentum coordinate and (3.75b) and (3.75d) for the position and obtain

$$p_i^{(2)} = p_i^{(0)} + (F_i^{(0)} + F_i^{(1)})\delta t \tag{3.76a}$$

$$q_i^{(2)} = q_i^{(0)} + 2p_i^{(1)}\delta t. \tag{3.76b}$$

We take $\delta t = \Delta t/2$ and combine (3.76b) with (3.75a) and find

$$p_i^{(2)} = p_i^{(0)} + \frac{1}{2}(F_i^{(0)} + F_i^{(1)})\Delta t \tag{3.77a}$$

$$q_i^{(2)} = q_i^{(0)} + p_i^{(0)}\Delta t + \frac{1}{2}F_i^{(0)}(\Delta t)^2, \tag{3.77b}$$

which is identical to the Verlet algorithm (3.48), because for unit mass the force and acceleration are equal.

Reversing the order of the updates for the coordinates and the momenta also leads to symplectic algorithms:

$$q_i^{(k+1)} = q_i^{(k)} + b_k\delta t p_i^{(k)} \tag{3.78a}$$

$$p_i^{(k+1)} = p_i^{(k)} + a_k\delta t F_i^{(k+1)}. \tag{3.78b}$$

A third variation uses (3.74) and (3.78) for different values of k in one algorithm. Thus, if $M = 2$, which corresponds to two intermediate calculations per time step, we could use (3.74) for the first intermediate calculation and (3.78) for the second.

Why are these algorithms important? Because of the symplectic property, these algorithms will simulate an exact Hamiltonian, although not the one we started with in general (see Problem 3.1c, for example). However, this Hamiltonian will be close to the one we wish to simulate if the a_k and b_k are properly chosen. Second, these algorithms are frequently more accurate and stable than nonsymplectic algorithms. Finally, for even values of M, the algorithms are time-reversible invariant, which is a property of the actual systems we are trying to simulate. Examples and comparisons for various algorithms are given in the paper by Gray et al.

REFERENCES AND SUGGESTIONS FOR FURTHER READING

F. S. Acton, *Numerical Methods That Work* (The Mathematical Association of America, 1990), Chapter 5.

Robert. K. Adair, *The Physics of Baseball*, 3rd ed. (Harper Collins, 2002).

Byron L. Coulter and Carl G. Adler, "Can a body pass a body falling through the air?," Am. J. Phys. **47**, 841–846 (1979). The authors discuss the limiting conditions for which the drag force is linear or quadratic in the velocity.

Alan Cromer, "Stable solutions using the Euler approximation," Am. J. Phys. **49**, 455–459 (1981). The author shows that a minor modification of the usual Euler approximation yields stable solutions for oscillatory systems including planetary motion and the harmonic oscillator (see Chapter 4).

Paul L. DeVries, *A First Course in Computational Physics* (John Wiley & Sons, 1994).

Denis Donnelly and Edwin Rogers, "Symplectic integrators: An introduction," Am. J. Phys., to be published.

A. P. French, *Newtonian Mechanics* (W. W. Norton & Company, 1971). Chapter 7 has an excellent discussion of air resistance and a detailed analysis of motion in the presence of drag resistance.

Ian R. Gatland, "Numerical integration of Newton's equations including velocity-dependent forces," Am J. Phys. **62**, 259–265 (1994). The author discusses the Euler–Richardson algorithm.

Stephen K. Gray, Donald W. Noid, and Bobby G. Sumpter, "Symplectic integrators for large scale molecular dynamics simulations: A comparison of several explicit methods," J. Chem. Phys. **101** (5), 4062–4072 (1994).

Margaret Greenwood, Charles Hanna, and John Milton, "Air resistance acting on a sphere: Numerical analysis, strobe photographs, and videotapes," Phys. Teacher **24**, 153–159 (1986). More experimental data and theoretical analysis are given for the fall of ping-pong and styrofoam balls. Also see Mark Peastrel, Rosemary Lynch, and Angelo Armenti, "Terminal velocity of a shuttlecock in vertical fall," Am. J. Phys. **48**, 511–513 (1980).

Michael J. Kallaher, editor *Revolutions in Differential Equations: Exploring ODEs with Modern Technology* (The Mathematical Association of America, 1999).

K. S. Krane, "The falling raindrop: Variations on a theme of Newton," Am. J. Phys. **49**, 113–117 (1981). The author discusses the problem of mass accretion by a drop falling through a cloud of droplets.

George C. McGuire, "Using computer algebra to investigate the motion of an electric charge in magnetic and electric dipole fields," Am. J. Phys. **71** (8), 809–812 (2003).

Rabindra Mehta, "Aerodynamics of sports balls," Ann. Rev. Fluid Mech. **17**, 151–189 (1985).

Neville de Mestre, *The Mathematics of Projectiles in Sport* (Cambridge University Press, 1990). The emphasis of this text is on solving many problems in projectile motion, for example, baseball, basketball, and golf in the context of mathematical modeling. Many references to the relevant literature are given.

Tao Pang, *Computational Physics* (Cambridge University Press, 1997).

William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery, *Numerical Recipes*, 2nd ed. (Cambridge University Press, 1992). Chapter 16 discusses the integration of ordinary differential equations.

Emilio Segré, *Nuclei and Particles*, 2nd ed. (W. A. Benjamin, 1977). Chapter 5 discusses decay cascades. The decay schemes described briefly in Problem 3.13 are taken from C. M. Lederer, J. M. Hollander, and I. Perlman, *Table of Isotopes*, 6th ed. (John Wiley & Sons, 1967).

Lawrence F. Shampine, *Numerical Solution of Ordinary Differential Equations* (Chapman and Hall, 1994).