The difference $|p - p_c|$ in (12.14) is the "distance" from the percolation threshold point at which finite size effects occur. Hence, if ξ and L are approximately the same size, we can replace (12.11) by the relation

$$P_\infty(p = p_c) \sim L^{-\beta/\nu} \quad (L \to \infty). \tag{12.15}$$

The relation (12.15) between P_∞ and L at $p = p_c$ is consistent with the fact that a phase transition is defined only for infinite systems.

One implication of (12.15) is that we can use it to determine the ratio β/ν. This method of analysis is known as *finite size scaling*. Suppose that we generate percolation configurations at $p = p_c$ for different values of L and analyze P_∞ as a function of L. If our values of L are sufficiently large, we can use the asymptotic relation (12.15) to estimate the ratio β/ν. A similar analysis can be used for $S(p)$ and other quantities of interest. We use this method in Problem 12.8.

Problem 12.8 Finite size scaling analysis of critical exponents

(a) Compute P_∞ at $p = p_c$ for at least 100 configurations for $L = 10$, 20, 40, and 80. Include in your average only those configurations that have a spanning cluster. Best results are obtained using the value of p_c for the infinite square lattice, $p_c \approx 0.5927$. Plot $\ln P_\infty$ versus $\ln L$ and estimate the ratio β/ν.

(b) Use finite size scaling to determine the dependence of the mean cluster size S on L at $p = p_c$. Average S over the same configurations considered in part (a). Remember that S is the mean number of sites in the nonspanning clusters.

(c) Find the size (number of particles) M in the spanning cluster at $p = p_c$ as a function of L. Use the same configurations as in part (a). Determine an exponent from the slope of a plot of $\ln M$ versus $\ln L$. This exponent is called the fractal dimension and is discussed in Chapter 13. ■

Finite size scaling is particularly useful at the percolation threshold in comparison to thermal critical points where, as we will learn in Chapter 15, *critical slowing down* occurs. Critical slowing down makes it very time consuming to sample statistically independent configurations. No such slowing down occurs at the percolation threshold because we can easily create new configurations at any value of p by simply using a new set of random numbers.

We found in Section 12.2 that the numerical value of the percolation threshold p_c depends on the symmetry and dimension of the lattice, for example, $p_c \approx 0.5927$ for the square lattice and $p_c = 1/2$ for the triangular lattice. A remarkable feature of the power law dependencies summarized in Table 12.1 is that the values of the critical exponents do not depend on the symmetry of the lattice and are independent of the existence of the lattice itself; for example, they are identical for site percolation, bond percolation, and continuum percolation. Moreover, it is not necessary to distinguish between the exponents for site and bond percolation. In the vocabulary of critical phenomena, we say that site, bond, and continuum percolation all belong to the same *universality class* and that their critical exponents are identical for the same spatial dimension.

Another important idea in critical phenomena is the existence of relations between the critical exponents. An example of such a *scaling law* is

$$2\beta + \gamma = \nu d, \tag{12.16}$$

where d is the spatial dimension of the lattice. The scaling law (12.16) indicates that the universality class depends on the spatial dimension. A more detailed discussion of finite size scaling and the scaling laws can be found in Chapter 15 and in the references.

12.5 ■ THE RENORMALIZATION GROUP

In Section 12.4 we studied the properties of various quantities on different length scales to determine the values of the critical exponents. The idea of examining physical quantities near the critical point on different length scales can be extended beyond finite size scaling and is the basis of the *renormalization group* method, one of the more important new methods in theoretical physics developed in the past several decades.[2] Although the method originated in the theory of elementary particles and was first applied to thermodynamic critical points, it is simpler to understand the method in the context of the percolation transition. We will find that the renormalization group method yields the critical exponents directly and, in combination with Monte Carlo methods, is more powerful than Monte Carlo methods alone.

The basic idea of the renormalization group method is the following. Imagine a percolation configuration generated at $p = p_0$. What would happen if we average the configuration over groups of sites to obtain a configuration of occupied and empty cells? For example, the cells could be groups of four sites such that each cell is occupied or empty according to a mapping rule from the sites to the cell. If the original group of 2×2 sites spans, the cell would be occupied; otherwise, the cell would be empty. What value of $p = p_1$ would describe the new configuration of cells? If $p_0 < p_c$, we would expect $p_1 < p_0$. To understand why, consider a value of p_0 near $p = 0$ where almost all the clusters are of size one. Clearly, the occupied sites would be mapped into empty cells, and there would be a lower percentage of occupied cells than before. For $p_0 > p_c$, we would find $p_1 > p_0$ because the rare isolated unoccupied sites would be grouped into occupied cells. At $p = p_c$, we might expect that this blocking procedure would lead to configurations that look like they were generated at the same value of p because of the existence of clusters of all length scales.

Given the new configuration of cells at probability p_1, we can group the cells according to the same mapping rule, leading to a new $p = p_2$. The sequence $p_0, p_1, p_2, \ldots$ is called a renormalization group flow. We expect for $p_0 < p_c$ the flow will move to the trivial *fixed point* $p = 0$, and for $p > p_c$ the flow will move to the other trivial fixed point $p = 1$. At $p = p_c$, there is a *nontrivial* fixed point. We will see that by analyzing the renormalization group flow, we can determine the location of the critical point and the critical exponent ν.

We now consider a way of using a computer to change the configurations in a way that is similar to the procedure that we have just described. Consider a square lattice that is partitioned into *cells* or *blocks* that cover the lattice (see Figure 12.11). Note that we have defined the cells so that the new lattice of cells has the same symmetry as the original lattice. However, the replacement of sites by the cells has changed the length scale—all distances are now smaller by a factor of b, where b is the linear dimension of the cell. Hence, the effect of a "renormalization" is to replace each group of sites by a single renormalized site and to rescale the connectedness length for the renormalized lattice by a factor of b.

[2]Kenneth Wilson was honored with the Nobel prize in physics in 1981 for his contributions to the development of the renormalization group method.