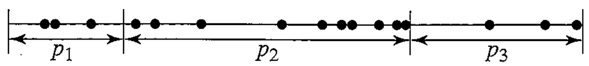

Figure 11.4 The unit interval is divided into three segments of lengths $p_1 = 0.2$, $p_2 = 0.5$, and $p_3 = 0.3$. Sixteen random numbers are represented by the filled circles uniformly distributed on the unit interval. The fraction of circles within each segment is approximately equal to the value of p_i for that segment.

consider several methods for generating random numbers that are not distributed uniformly. In the following, we will denote r as a member of a uniform random number sequence in the unit interval $0 \le r < 1$.

Suppose that two discrete events 1 and 2 occur with probabilities p_1 and p_2 such that $p_1 + p_2 = 1$. How can we choose the two events with the correct probabilities using a uniform probability distribution? For this simple case, it is clear that we choose event 1 if $r < p_1$; otherwise, we choose event 2. If there are three events with probabilities p_1, p_2, and p_3, then if $r < p_1$, we choose event 1; else if $r < p_1 + p_2$, we choose event 2; else we choose event 3. We can visualize these choices by dividing a line segment of unit length into three pieces whose lengths are shown in Figure 11.4.

Now consider n discrete events. How do we determine which event i to choose given the value of r? The generalization of the procedure we have followed for $n = 2$ and $n = 3$ is to find the value of i that satisfies the condition

$$\sum_{j=0}^{i-1} p_j \le r \le \sum_{j=0}^{i} p_j, \tag{11.28}$$

where we have defined $p_0 \equiv 0$. Verify that (11.28) reduces to the correct procedure for $n = 2$ and $n = 3$.

We now consider a continuous nonuniform probability distribution. One way to generate such a distribution is to take the limit of (11.28) and associate p_i with $p(x)\,\Delta x$, where the *probability density* $p(x)$ is defined such that $p(x)\,\Delta x$ is the probability that the event x is in the interval between x and $x + \Delta x$. The probability density $p(x)$ is normalized such that

$$\int_{-\infty}^{+\infty} p(x)\,dx = 1. \tag{11.29}$$

In the continuum limit, the two sums in (11.28) become the same integral, and the inequalities become equalities. Hence, we can write

$$P(x) \equiv \int_{-\infty}^{x} p(x')\,dx' = r. \tag{11.30}$$

From (11.30) we see that the uniform random number r corresponds to the *cumulative probability distribution* function $P(x)$, which is the probability of choosing a value less than or equal to x. The function $P(x)$ should not be confused with the probability density $p(x)$ or the probability $p(x)\,\Delta x$. In many applications the meaningful range of values of x is positive. In that case we have $p(x) = 0$ for $x < 0$.

The relation (11.30) leads to the *inverse transform* method for generating random numbers distributed according to the function $p(x)$. This method involves generating a random

number r and solving (11.30) for the corresponding value of x. As an example of the method, we use (11.30) to generate a random number sequence according to the uniform probability distribution on the interval $a \le x \le b$. The desired probability density $p(x)$ is

$$p(x) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{otherwise.} \end{cases} \tag{11.31}$$

The cumulative probability distribution function $P(x)$ for $a \le x \le b$ can be found by substituting (11.31) into (11.30) and performing the integral. The result is

$$P(x) = \frac{x-a}{b-a}. \tag{11.32}$$

If we substitute the form (11.32) for $P(x)$ into (11.30) and solve for x, we find the desired relation

$$x = a + (b-a)r. \tag{11.33}$$

The variable x given by (11.33) is distributed according to the probability distribution $p(x)$ given by (11.31). Of course, the relation (11.33) is obvious, and we already have used it in our programs.

We next apply the inverse transform method to the probability density function

$$p(x) = \begin{cases} (1/\lambda)\, e^{-x/\lambda} & 0 \le x \le \infty \\ 0 & x < 0. \end{cases} \tag{11.34}$$

If we substitute (11.34) into (11.30) and integrate, we find the relation

$$r = P(x) = 1 - e^{-x/\lambda}. \tag{11.35}$$

The solution of (11.35) for x yields $x = -\lambda \ln(1-r)$. Because $1 - r$ is distributed in the same way as r, we can write

$$x = -\lambda \ln r. \tag{11.36}$$

The variable x found from (11.36) is distributed according to the probability density $p(x)$ given by (11.34). Because the computation of the natural logarithm in (11.36) is relatively slow, the inverse transform method might not be the most efficient method to use in this case.

From the above examples, we see that two conditions must be satisfied in order to apply the inverse transform method: the form of $p(x)$ must allow us to perform the integral in (11.30) analytically, and it must be practical to invert the relation $P(x) = r$ for x.

The Gaussian probability density,

$$p(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} e^{-x^2/2\sigma^2}, \tag{11.37}$$

is an example of a probability density for which the cumulative distribution $P(x)$ cannot be obtained analytically. However, we can generate the two-dimensional probability