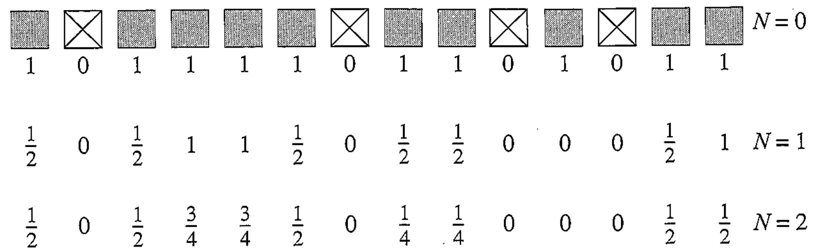

Figure 7.4 Example of the exact enumeration of walks on a given configuration of traps. The filled and empty squares denote regular and trap sites, respectively. At step $N = 0$, a walker is placed at each regular site. The numbers at each site i represent the number of walkers w_i. Periodic boundary conditions are used. The initial number of walkers in this example is $w_0 = 10$. The mean survival probability at step $N = 1$ and $N = 2$ is found to be 0.6 and 0.475, respectively.

with probability 1/2 to each neighbor, the number of walkers $w_i(N+1)$ on site i at step $N+1$ is given by

$$w_i(N+1) = \frac{1}{2}[w_{i+1}(N) + w_{i-1}(N)]. \tag{7.25}$$

(Compare the relation (7.25) to the relation that you found in Problem 7.5d.) The survival probability S_N after N steps for a given configuration of traps is given exactly by

$$S_N = \frac{1}{w_0}\sum_i w_i(N), \tag{7.26}$$

where w_0 is the initial number of walkers and the sum is over all sites in the lattice. Explain the relation (7.26) and write a program that computes S_N using (7.25) and (7.26). Then obtain $\langle S_N \rangle$ by averaging over several configurations of traps. Choose $\rho = 0.5$ and determine S_N for $N = 32$, 64, 128, 512, and 1024. Choose periodic boundary conditions and as large a lattice as possible. How well can you estimate the exponent α? For comparison Havlin et al. consider a lattice of $L = 50{,}000$ and values of N up to 10^7. ■

One reason that random walks are very useful in simulating many physical processes is that they are closely related to solutions of the *diffusion* equation. The one-dimensional diffusion equation can be written as

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}, \tag{7.27}$$

where D is the self-diffusion coefficient, and $P(x,t)\,\Delta x$ is the probability of a particle being in the interval between x and $x + \Delta x$ at time t. In a typical application $P(x,t)$ might represent the concentration of ink molecules diffusing in a fluid. In three dimensions the second derivative $\partial^2/\partial x^2$ is replaced by the Laplacian ∇^2. In Appendix 7B we show that the solution to the diffusion equation with the boundary condition $P(x = \pm\infty, t) = 0$ yields

$$\langle x(t) \rangle = 0, \tag{7.28}$$

and

$$\langle x^2(t) \rangle = 2Dt \quad \text{(one dimension)}. \tag{7.29}$$

If we compare the form of (7.29) with (7.10), we see that the random walk on a one-dimensional lattice and the diffusion equation give the same time dependence if we identify t with $N\Delta t$ and D with $a^2/\Delta t$.

The relation of discrete random walks to the diffusion equation is an example of how we can approach many problems in several ways. The traditional way to treat diffusion is to formulate the problem as a partial differential equation as in (7.27). The usual method for solving (7.27) numerically is known as the Crank–Nicholson method (see Press et al.). One difficulty with this approach is the treatment of complicated boundary conditions. An alternative is to formulate the problem as a random walk on a lattice for which it is straightforward to incorporate various boundary conditions. We will consider random walks in many contexts (see, for example, Section 10.5 and Chapter 16).

7.4 ■ THE POISSON DISTRIBUTION AND NUCLEAR DECAY

As we have seen, we can often change variable names and consider a seemingly different physical problem. Our goal in this section is to discuss the decay of unstable nuclei, but we first discuss a conceptually easier problem related to throwing darts at random. Related physical problems are the distribution of stars in the sky and the distribution of photons on a photographic plate.

Suppose we randomly throw $N = 100$ darts at a board that has been divided into $M = 1000$ equal size regions. The probability that a dart hits a given region or cell in any one throw is $p = 1/M$. If we count the number of darts in the different regions, we would find that most cells are empty, some cells have one dart, and other cells have more than one dart. What is the probability $P(n)$ that a given cell has n darts?

Problem 7.18 Throwing darts
Write a program that simulates the throwing of N darts at random into M cells in a dart board. Throwing a dart at random at the board is equivalent to choosing an integer at random between 1 and M. Determine $H(n)$, the number of cells with n darts. Average $H(n)$ over many trials and then compute the probability distribution

$$P(n) = \frac{H(n)}{M}. \tag{7.30}$$

As an example, choose $N = 50$ and $M = 500$. Choose the number of trials to be sufficiently large, so that you can determine the qualitative form of $P(n)$. What is $\langle n \rangle$? ■

In this case the probability p that a dart lands in a given cell is much less then unity. The conditions $N \gg 1$ and $p \ll 1$ with $\langle n \rangle = Np$ fixed and the independence of the events (the presence of a dart in a particular cell) satisfy the requirements for a *Poisson distribution.* The form of the Poisson distribution is

$$P(n) = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle}, \tag{7.31}$$