Problem 9.24 Nonlinear oscillators

(a) Modify your program so that cubic forces between the particles are added to the linear spring forces. That is, let the force on particle i due to particle j be

$$F_{ij} = -(u_i - u_j) - \alpha(u_i - u_j)^3, \tag{9.48}$$

where α is the amplitude of the nonlinear term. Choose the masses of the particles to be unity. Consider $N = 10$ and choose initial displacements corresponding to a normal mode of the linear ($\alpha = 0$) system. Compute the power spectrum over a time $T = 51.2$ with $\Delta = 0.1$ for $\alpha = 0$, 0.1, 0.2, and 0.3. For what value of α does the system become ergodic; that is, for what value of α are the heights of all the normal mode peaks approximately the same?

(b) Repeat part (a) for the case where the displacements of the particles are initially random. Use the same set of random displacements for each value of α.

*(c) We now know that the number of oscillators is not as important as the magnitude of the nonlinear interaction. Repeat parts (a) and (b) for $N = 20$ and 40 and discuss the effect of increasing the number of particles. ■

9.7 ■ WAVE MOTION

Our simulations of coupled oscillators have shown that the microscopic motion of the individual oscillators leads to macroscopic wave phenomena. To understand the transition between microscopic and macroscopic phenomena, we reconsider the oscillations of a linear chain of N particles with equal spring constants k and equal masses m. As we found in Section 9.1, the equations of motion of the particles can be written as (see (9.1))

$$\frac{d^2u_j(t)}{dt^2} = -\frac{k}{m}[2u_j(t) - u_{j+1}(t) - u_{j-1}(t)] \quad (j = 1, \ldots, N). \tag{9.49}$$

We consider the limits $N \to \infty$ and $a \to 0$ with the length of the chain Na fixed. We will find that the discrete equations of motion (9.49) can be replaced by the continuous *wave equation*

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{9.50}$$

where c has the dimension of velocity.

We obtain the wave equation (9.50) as follows. First we replace $u_j(t)$, where j is a discrete variable, by the function $u(x,t)$, where x is a continuous variable, and rewrite (9.49) in the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{ka^2}{m}\frac{1}{a^2}[u(x+a,t) - 2u(x,t) + u(x-a,t)]. \tag{9.51}$$

We have written the time derivative as a partial derivative because the function u depends on two variables. If we use the Taylor series expansion,

$$u(x \pm a) = u(x) \pm a\frac{du}{dx} + \frac{a^2}{2}\frac{d^2u}{dx^2} + \cdots, \tag{9.52}$$

it is easy to show that as $a \to 0$, the quantity

$$\frac{1}{a^2}[u(x+a,t) - 2u(x,t) + u(x-a,t)] \to \frac{\partial^2 u(x,t)}{\partial x^2}. \tag{9.53}$$

(We have written a spatial derivative as a partial derivative for the same reason as before.) The wave equation (9.50) is obtained by substituting (9.53) into (9.51) with $c^2 = ka^2/m$. If we introduce the linear mass density $\mu = M/a$ and the tension $T = ka$, we can express c in terms of μ and T and obtain the familiar result $c^2 = T/\mu$.

It is straightforward to show that any function of the form $f(x \pm ct)$ is a solution to (9.50). Among these many solutions to the wave equation are the familiar forms:

$$u(x,t) = A\cos\frac{2\pi}{\lambda}(x \pm ct) \tag{9.54a}$$

$$u(x,t) = A\sin\frac{2\pi}{\lambda}(x \pm ct). \tag{9.54b}$$

Because the wave equation is linear and hence satisfies a superposition principle, we can understand the behavior of a wave of arbitrary shape by representing its shape as a sum of sinusoidal waves.

One way to solve the wave equation (9.50) numerically is to retrace our steps back to the discrete equations (9.49) to find a discrete form of the wave equation that is convenient for numerical calculations. The conversion of a continuum equation to a physically motivated discrete form frequently leads to useful numerical algorithms. From (9.53) we see how to approximate the second derivative by a finite difference. If we replace a by Δx and take Δt to be the time step, we can rewrite (9.49) by

$$\begin{aligned}\frac{1}{(\Delta t)^2}[u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)] &= \\ \frac{c^2}{(\Delta x)^2}[u(x+\Delta x,t) &- 2u(x,t) + u(x-\Delta x,t)].\end{aligned} \tag{9.55}$$

The quantity Δx is the spatial interval. The result of solving (9.55) for $u(x, t + \Delta t)$ is

$$\begin{aligned}u(x,t+\Delta t) &= 2(1-b)u(x,t) \\ &+ b[u(x+\Delta x,t) + u(x+\Delta x,t)] - u(x,t-\Delta t),\end{aligned} \tag{9.56}$$

where $b = (c\Delta t/\Delta x)^2$. Equation (9.56) expresses the displacements at time $t + \Delta t$ in terms of the displacements at the current time t and at the previous time $t - \Delta t$.

Problem 9.25 Solution of the discrete wave equation

(a) Write a program to compute the numerical solutions of the discrete wave equation (9.56). Three spatial arrays corresponding to $u(x)$ at times $t + \Delta t$, t, and $t - \Delta t$ are needed. Denote the displacement $u(j\Delta x)$ by the array element `u[j]` where $j = 0, \ldots, N+1$. Use periodic boundary conditions so that $u_0 = u_N$ and $u_1 = u_{N+1}$. Draw lines between the displacements at neighboring values of x. Note that the initial conditions require the specification of u at $t = 0$ and at $t = -\Delta t$. Let the waveform at $t = 0$ and $t = -\Delta t$ be $u(x, t = 0) = \exp(-(x-10)^2)$ and $u(x, t = -\Delta t) = \exp(-(x - 10 + c\Delta t)^2)$, respectively. What is the direction of motion implied by these initial conditions?