(c) Compute ν from the cell-to-cell transformation discussed in Project 12.13 for $b_1 = 5$ and $b_2 = 4$.

(d) The article by Ziff and Newman discusses the convergence of various estimates of the percolation threshold in two dimensions. Some examples of these estimates include:

(i) The cell-to-site renormalization group fixed point:

$$R_L(p) = p, \tag{12.33}$$

where p^* is the solution to (12.33).

(ii) The average value of p at which spanning first occurs:

$$\langle p \rangle = \int_0^1 p \frac{dR_L(p)}{dp} dp = 1 - \int_0^1 R_L(p)\, dp, \tag{12.34}$$

where we have integrated by parts to obtain the second integral.

(iii) The estimate $p_{\max}$, which is the value of p at which dR_L/dp reaches a maximum:

$$\frac{d^2 R_L(p)}{dp^2} = 0. \tag{12.35}$$

(iv) The cell-to-cell renormalization group fixed point:

$$R_L(p) = R_{L-1}(p), \tag{12.36a}$$

or

$$R_L(p) = R_{L/2}(p). \tag{12.36b}$$

(v) The value of p for which $R_L(p) = R_\infty(p_c)$. For a square lattice, $R_\infty(p_c) = 1/2$.

Verify that the various estimates of the percolation threshold converge to the infinite lattice value p_c either as

$$p_{\text{est}}(L) - p_c \approx cL^{-1/\nu}, \tag{12.37a}$$

or

$$p_{\text{est}}(L) - p_c \approx cL^{-1-1/\nu}, \tag{12.37b}$$

where the constant c is a fit parameter that depends on the criterion and $\nu = 4/3$ for percolation in two dimensions. Determine which estimates converge more quickly. ■

Project 12.14 Percolation in three dimensions

(a) The value of p_c for site percolation on the simple cubic lattice is approximately 0.3112. Do a simulation to verify this value. Compute ϕ_c, the volume fraction occupied at p_c, if a sphere with a diameter equal to the lattice spacing is placed at each occupied site.

(b) Consider continuum percolation in three dimensions where spheres of unit diameter are placed at random in a cubical box of linear dimension L. Two spheres that overlap are in the same cluster. The volume fraction occupied by the spheres is given by

$$\phi = 1 - e^{-\rho 4\pi r^3/3}, \tag{12.38}$$

where ρ is the number density of the spheres, and r is their radius. Write a program to simulate continuum percolation in three dimensions and find the percolation threshold ρ_c. Use the Monte Carlo procedure discussed in Problem 12.4 to estimate ϕ_c and compare its value with the value determined from (12.38). How does ϕ_c for continuum percolation compare with the value of ϕ_c found for site percolation in part (a)? Which do you expect to be larger and why?

(c) In the Swiss cheese model in three dimensions, we are concerned with the percolation of the space between the spheres. This model is appropriate for porous rock with the spheres representing solid material and the space between the spheres representing the pores. Because we need to compute the connectivity properties of the space between the spheres, we superimpose a regular grid with lattice spacing equal to $0.1r$ on the system, where r is the radius of the spheres. If a point on the grid is not within any sphere, it is "occupied." The use of the grid allows us to determine the connectivity between different regions of the pore space. Use a cluster labeling algorithm to label the clusters and determine $\tilde{\phi}_c$, the volume fraction occupied by the pores at threshold. You might be surprised to find that $\tilde{\phi}_c$ is relatively small. If time permits, use a finer grid and repeat the calculation to improve the accuracy of your results.

*(d) Use finite size scaling to estimate the critical percolation exponents for the three models presented in parts (a)–(c). Are they the same within the accuracy of your calculation? ■

Project 12.15 Fluctuations of the stock market

Although the fluctuations of the stock market are believed to be Gaussian for long time intervals, they are not Gaussian for short time intervals. The model of Cont and Bouchaud assumes that percolation clusters act as groups of traders who influence each other. The sites are occupied with probability p as usual. Each occupied site is a trader, and clusters are groups of traders (agents) who buy and sell together an amount proportional to the number s of traders in the cluster. At each time step, each cluster is independently active with probability $2p_a$ and is inactive with probability $1 - 2p_a$. If a cluster is active, it buys with probability p_b and sells with probability $p_s = 1 - p_b$. In the simplest version of the model, the change in the price of a stock is proportional to the difference between supply and demand; that is,

$$R = \sum_{\text{buy}} s n_s - \sum_{\text{sell}} s n_s, \tag{12.39}$$

where the constant of proportionality is taken to be one. If the probability p_a is small, at most one cluster trades at a time, and the distribution $P(R)$ of relative price changes or "returns" scales as $n_s(p)$. In contrast, for large p_a, the relative price variation is the sum