where the rate of change of the total linear momentum **P** and total angular momentum **L** about a point O is determined by the total force **F** on the body and the total torque **N** about O. These momenta are expressed in terms of the translational **V** and rotational velocity ω by

$$\mathbf{P} = M\mathbf{V} \tag{17.19a}$$

$$\mathbf{L} = \mathcal{I}\omega, \tag{17.19b}$$

where M is the mass, and $\mathcal{I}$ is the moment of inertia tensor. For an unconstrained body, the point O is usually taken to be the center of mass. For a constrained body, such as a spinning top, the point O is usually taken to be the point of support.

Although the translational and rotational equations of motion appear similar, the fact that the inertia tensor is not always constant with respect to axes fixed in space complicates the analysis. To use a constant inertia tensor, we must describe the motion using a noninertial reference frame known as the *body frame* that is fixed in the body. Because we are free to orient the axes within the body, we choose axes for which the moment of inertia tensor is diagonal. These axes are referred to as the body's principal axes and are easy to determine for symmetrical objects. The diagonal elements of the moment of inertia tensor are calculated using the volume integrals

$$I_1 = \int_V \rho(y^2 + z^2)\, dV \tag{17.20a}$$

$$I_2 = \int_V \rho(z^2 + x^2)\, dV \tag{17.20b}$$

$$I_3 = \int_V \rho(x^2 + y^2)\, dV, \tag{17.20c}$$

where ρ is the mass density. The off-diagonal elements are zero in the principal axis coordinate system. The moments of inertia for various simple geometrical objects are shown in Table 17.1.

Given that the general relation between the derivative of a vector **A** in the space frame to the derivative in the body frame is (see Goldstein)

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{space}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{body}} + \omega \times \mathbf{A}, \tag{17.21}$$

it is easy to show that the rotational equation of motion in the body frame can be written as

$$\frac{d\mathbf{L}}{dt} + \omega \times (\mathcal{I}\omega) = \mathbf{N}. \tag{17.22}$$

Equation (17.23) is Euler's equation for the motion of a rigid body. Because the moment of inertia tensor $\mathcal{I}$ is diagonal in the body frame, (17.22) may be written in component form as

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 = N_1 \tag{17.23a}$$

$$I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = N_2 \tag{17.23b}$$

$$I_3\dot{\omega}_3 + (I_2 - I_1)\omega_2\omega_1 = N_3. \tag{17.23c}$$

Table 17.1 The moment of inertia about the center of mass of various geometric shapes. The mass m is assumed to be uniformly distributed.

Body	Axis	Moment of Inertia
ellipsoid, axes $(2a, 2b, 2c)$	axes a	$I = m(b^2 + c^2)/5$
	axes b	$I = m(a^2 + c^2)/5$
	axes c	$I = m(a^2 + b^2)/5$
parallelopiped, sides (a, b, c)	perpendicular to (a, b)	$I = m(a^2 + b^2)/12$
	perpendicular to (b, c)	$I = m(b^2 + c^2)/12$
	perpendicular to (c, a)	$I = m(c^2 + a^2)/12$
sphere, radius r	any diameter	$I = m(2/5)r^2$
thin rod, length l	normal to length at center	$I = ml^2/12$
thin circular sheet, radius r	normal to sheet at center	$I = mr^2/2$
	any diameter	$I = mr^2/4$
thin rectangular sheet, sides (a, b)	normal to sheet at center	$I = m(a^2 + b^2)/2$
	parallel to a at center	$I = mb^2/12$
	parallel to b at center	$I = ma^2/12$

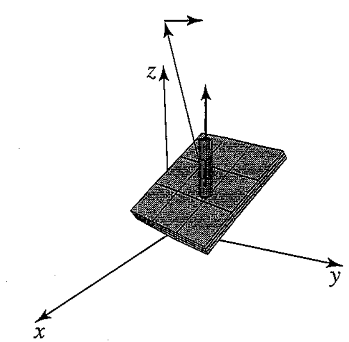

Figure 17.4 `TorqueApp` shows the torque on a rotating shaft with an attached rectangular sheet.

Open source physics elements support the concept of a body frame by providing `toBodyFrame` and `toSpaceFrame` methods in the `Element` class. These methods are used in Listing 17.8 to show the torque on a rectangular sheet rotating on a fixed shaft with uniform angular velocity ω (see Figure 17.4). The mass can be tilted on the shaft to produce an out-of-balance configuration that causes the system to shake unless a torque is applied to the shaft. The program displays the angular momentum vector and the torque vector using color-coded arrows.

Listing 17.8 A mass rotating on a fixed shaft with uniform angular velocity ω.

```
package org.opensourcephysics.sip.ch17;
import org.opensourcephysics.controls.*;
import org.opensourcephysics.display3d.simple3d.*;
import org.opensourcephysics.frames.Display3DFrame;
```