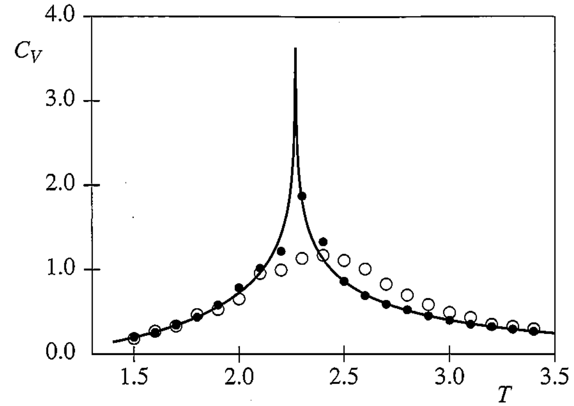

Figure 15.3 The temperature dependence of the specific heat C (per spin) of the Ising model on a square lattice with periodic boundary conditions for $L = 8$ and $L = 16$. One thousand Monte Carlo steps per spin were used for each value of the temperature. The continuous line represents the temperature dependence of C in the limit of an infinite lattice. (Note that C is infinite at $T = T_c$ for an infinite lattice.)

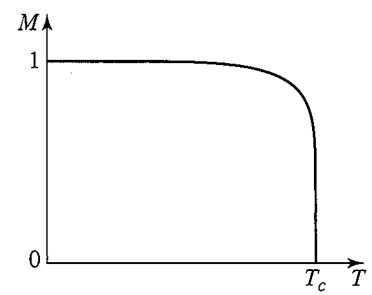

Figure 15.4 The temperature dependence of $m(T)$, the mean magnetization per spin, for the Ising model in two dimensions in the thermodynamic limit.

How can we characterize a continuous magnetic phase transition? Because a nonzero m implies that a net number of spins are spontaneously aligned, we designate m as the *order parameter* of the system. Near T_c, we can characterize the behavior of many physical quantities by power law behavior just as we characterized the percolation threshold (see Table 12.1). For example, we can write m near T_c as

$$m(T) \sim (T_c - T)^{\beta}, \tag{15.31}$$

where β is a *critical* exponent (not to be confused with the inverse temperature). Various thermodynamic derivatives such as the susceptibility and specific heat diverge at T_c and are characterized by critical exponents. We write

$$\chi \sim |T - T_c|^{-\gamma}, \tag{15.32}$$

and

$$C \sim |T - T_c|^{-\alpha}, \tag{15.33}$$

where we have introduced the critical exponents γ and α. We have assumed that χ and C are characterized by the same critical exponents above and below T_c.

Another measure of the magnetic fluctuations is the linear dimension $\xi(T)$ of a typical magnetic domain. We expect the *correlation length* $\xi(T)$ to be the order of a lattice spacing for $T \gg T_c$. Because the alignment of the spins becomes more correlated as T approaches T_c from above, $\xi(T)$ increases as T approaches T_c. We can characterize the divergent behavior of $\xi(T)$ near T_c by the critical exponent ν:

$$\xi(T) \sim |T - T_c|^{-\nu}. \tag{15.34}$$

As we found in our discussion of percolation in Chapter 12, a finite system cannot exhibit a true phase transition. We expect that if $\xi(T)$ is less than the linear dimension L of the system, our simulations will yield results comparable to an infinite system. In contrast, if T is close to T_c, our simulations will be limited by finite-size effects. Because we can simulate only finite lattices, it is difficult to obtain estimates for the critical exponents α, β, and γ by using the definitions (15.31)–(15.33) directly. We learned in Section 12.4 that we can use *finite-size scaling* to extrapolate finite L results to $L \to \infty$. For example, from Figure 15.3 we see that the temperature at which C exhibits a maximum becomes better defined for larger lattices. This behavior provides a simple definition of the transition temperature $T_c(L)$ for a finite system. According to finite-size scaling theory, $T_c(L)$ scales as

$$T_c(L) - T_c(L = \infty) \sim aL^{-1/\nu}, \tag{15.35}$$

where a is a constant and ν is defined in (15.34). The finite size of the lattice is important when the correlation length is comparable to the linear dimension of the system:

$$\xi(T) \sim L \sim |T - T_c|^{-\nu}. \tag{15.36}$$

As in Section 12.4, we can set $T = T_c$ and consider the L-dependence of M, C, and χ:

$$m(T) \sim (T_c - T)^{\beta} \to L^{-\beta/\nu} \tag{15.37}$$

$$C(T) \sim |T - T_c|^{-\alpha} \to L^{\alpha/\nu} \tag{15.38}$$

$$\chi(T) \sim |T - T_c|^{-\gamma} \to L^{\gamma/\nu}. \tag{15.39}$$

In Problem 15.17 we use the relations (15.37)–(15.39) to estimate the critical exponents β, γ, and α.

Problem 15.17 Finite size scaling for the two-dimensional Ising model

(a) Use the relation (15.35) together with the exact result $\nu = 1$ to estimate the value of T_c for an infinite square lattice. Because it is difficult to obtain a precise value for T_c with small lattices, we will use the exact result $kT_c/J = 2/\ln(1 + \sqrt{2}) \approx 2.269$ for the infinite lattice in the remaining parts of this problem.

(b) Determine the mean value of the absolute value of the magnetization per spin $|m|$, the specific heat C, and the susceptibility χ at $T = T_c$ for $L = 4, 8, 16$, and 32. Compute χ using (15.21) with $\langle |M| \rangle$ instead of $\langle M \rangle$. Use as many Monte Carlo steps per spin as possible. Plot the logarithm of $|m|$ and χ versus L and use the scaling relations