obtain

$$p_{n+1} - p_n = k \sin \theta_{n+1}. \tag{6.40a}$$

(Remember that p is constant between kicks, and the delta function contributes to the integral only when $t = (n+1)\tau$.) From (6.39b) we have

$$\theta_{n+1} - \theta_n = (\tau/I) p_n. \tag{6.40b}$$

If we choose units such that $\tau/I = 1$, we obtain the standard map

$$\theta_{n+1} = (\theta_n + p_n) \quad \text{modulo } 2\pi \tag{6.41a}$$

$$p_{n+1} = p_n + k \sin \theta_{n+1} \quad \text{(standard map)}. \tag{6.41b}$$

We have added the requirement in (6.41a) that the value of the angle θ is restricted to be between zero and 2π.

Before we iterate (6.41), let us check that (6.41) represents a Hamiltonian system; that is, the area in qp-space is constant as n increases. (Here q corresponds to θ.) Suppose we start with a rectangle of points of length dq_n and dp_n. After one iteration, this rectangle will be deformed into a parallelogram of sides dq_{n+1} and dp_{n+1}. From (6.41) we have

$$dq_{n+1} = dq_n + dp_n \tag{6.42a}$$

$$dp_{n+1} = dp_n + k \cos q_{n+1}\, dq_{n+1}. \tag{6.42b}$$

If we substitute (6.42a) in (6.42b), we obtain

$$dp_{n+1} = (1 + k \cos q_{n+1})\, dp_n + k \cos q_{n+1}\, dq_n. \tag{6.43}$$

To find the area of a parallelogram, we take the magnitude of the cross product of the vectors $d\mathbf{q}_{n+1} = (dq_n, dp_n)$ and $d\mathbf{p}_{n+1} = (1 + k\cos q_n dq_n, k \cos q_n dp_n)$. The result is $dq_n\, dp_n$, and hence the area in phase space has not changed. The standard map is an example of an *area-preserving map* .

The qualitative properties of the standard map are explored in Problem 6.19. You will find that for $k = 0$, the rod rotates with a fixed angular velocity determined by the momentum $p_n = p_0 =$ constant. If p_0 is a rational number times 2π, then the trajectory in phase space consists of a sequence of isolated points lying on a horizontal line (resonant tori). Can you see why? If p_0 is not a rational number times 2π or if your computer does not have sufficient precision, then after a long time, the trajectory will consist of a horizontal line in phase space. As we increase k, these horizontal lines are deformed into curves that run from $q = 0$ to $q = 2\pi$, and the isolated points of the resonant tori are converted into closed loops. For some initial conditions, the trajectories will become chaotic after the transient behavior has ended.

Problem 6.19 The standard map

(a) Write a program to iterate the standard map and plot its trajectory in phase space. Use different colors so that several trajectories can be shown at the same time for the same value of the parameter k. Choose a set of initial conditions that form a rectangle (see Problem 4.10). Does the shape of this area change with time? What happens to the total area?

(b) Begin with $k = 0$ and choose an initial value of p that is a rational number times 2π. What types of trajectories do you obtain? If you obtain trajectories consisting of isolated points, do these points appear to shift due to numerical roundoff errors? How can you tell? What happens if p_0 is an irrational number times 2π? Remember that a computer can only approximate an irrational number.

(c) Consider $k = 0.2$ and explore the nature of the phase space trajectories. What structures appear that do not appear for $k = 0$? Discuss the motion of the rod corresponding to some of the typical trajectories that you find.

(d) Increase k until you first find several chaotic trajectories. How can you tell that they are chaotic? Do these chaotic trajectories fill all of phase space? If there is one trajectory that is chaotic at a particular value of k, are all trajectories chaotic? What is the approximate value for k_c above which chaotic trajectories appear? ■

We now discuss a discrete map that models the rings of Saturn (see Fröyland). The model assumes that the rings of Saturn are due to perturbations produced by Mimas. There are two important forces acting on objects near Saturn. The force due to Saturn can be incorporated as follows. We know that each time Mimas completes an orbit, it traverses a total angle of 2π. Hence, the angle θ of any other moon of Saturn relative to Mimas can be expressed as

$$\theta_{n+1} = \theta_n + 2\pi \frac{\sigma^{3/2}}{{r_n}^{3/2}}, \tag{6.44}$$

where r_n is the radius of the orbit after n revolutions, and $\sigma = 185.7 \times 10^3$ km is the mean distance of Mimas from Saturn. The other important force is due to Mimas and causes the radial distance r_n to change. A discrete approximation to the radial acceleration dv_r/dt is (see (3.16))

$$\frac{\Delta v_r}{\Delta t} \approx \frac{r(t+\Delta t) - 2r(t) + r(t-\Delta t)}{(\Delta t)^2}. \tag{6.45}$$

The acceleration equals the radial force due to Mimas. If we average over a complete period, then a reasonable approximation for the change in r_n due to Mimas is

$$r_{n+1} - 2r_n + r_{n-1} = f(r_n, \theta_n), \tag{6.46}$$

where $f(r_n, \theta_n)$ is proportional to the radial force. (We have absorbed the factor of $(\Delta t)^2$ and the mass into f.)

In general, the form of $f(r_n, \theta_n)$ is very complicated. We make a major simplifying assumption and take f to be proportional to $-(r_n - \sigma)^{-2}$ and to be periodic in θ_n. This form for the force incorporates the fact that for large r_n, the force has the usual form for the gravitational force. For simplicity, we express this periodicity in the simplest possible way, that is, as $\cos \theta_n$. We also want the map to be area conserving. These considerations lead to the following two-dimensional map:

$$\theta_{n+1} = \theta_n + 2\pi \frac{\sigma^{3/2}}{{r_n}^{3/2}} \tag{6.47a}$$

$$r_{n+1} = 2r_n - r_{n-1} - a \frac{\cos \theta_n}{(r_n - \sigma)^2}. \tag{6.47b}$$