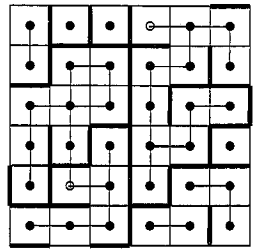

Figure 12.7 Occupied bonds on a bond percolation lattice are shown by heavy dark lines. The dual lattice consists of the open circles. The dashed lines are the occupied bonds on the dual lattice. The original lattice contains a cluster that spans both vertically and horizontally, which prevents the dual lattice from having a spanning cluster.

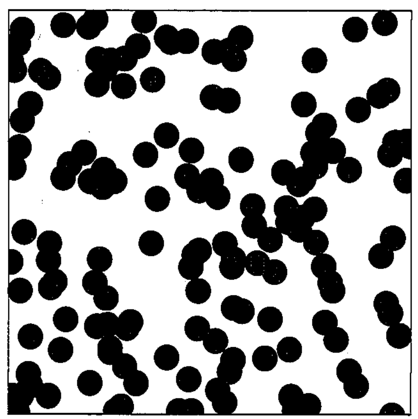

Figure 12.8 A model of continuum (off-lattice) percolation realized by placing disks of unit diameter at random into a square box of linear dimension L. If we concentrate on the voids between the disks rather than the disks, then this model of continuum percolation is known as the Swiss cheese model.

bond percolation on a square lattice and determine the bond percolation threshold. Are your results consistent with the exact result $p_c = 1/2$? ■

We can also consider *continuum* percolation models. For example, we can place disks at random into a two-dimensional box. Two disks are in the same cluster if they touch or overlap. A typical continuum (off-lattice) percolation configuration is depicted in Figure 12.8. One quantity of interest is the quantity ϕ, the fraction of the area (volume in three dimensions) in the system that is covered by disks. In the limit of an infinite size box, it can be shown that

$$\phi = 1 - e^{-\rho\pi r^2}, \tag{12.1}$$

where ρ is the number of disks per unit area, and r is the radius of a disk (see Xia and Thorpe). Equation (12.1) is not accurate for small boxes because disks located near the edge of the box are a significant fraction of the total number of disks.

Problem 12.4 Continuum percolation

(a) Suppose that disks of unit diameter are placed at random on the sites of a square lattice with unit lattice spacing. Define ϕ as the area fraction covered by the disks. Convince yourself that $\phi_c = \pi p_c/4$.

(b) Modify `PercolationApp` to simulate continuum percolation. Instead of placing the disks on regular lattice sites, place their centers at random in a square box of area L^2. The relevant parameter is the density ρ, the number of disks per unit area, instead of the probability p. We can no longer use the `LatticeFrame` class. Instead, two arrays are needed to store the x and y locations of the disks. When the mouse is clicked on a disk, your program will need to determine which disk is at the location of the mouse, and then check all the other disks to see if they overlap or touch the disk you have chosen. This check is recursively continued for all overlapping disks. It is also useful to have an array that keeps track of the `clusterNumber` for each disk. Only disks that have not been assigned a cluster number need to be checked for overlaps.

(c) Estimate the value of the density ρ_c at which a spanning cluster first appears. Given this value of ρ_c, use a Monte Carlo method to estimate the corresponding area fraction ϕ_c (see Section 11.2). Choose points at random in the box and compute the fraction of points that lie within any disk. Explain why ϕ_c is larger for continuum percolation than it is for site percolation. Compare your direct Monte Carlo estimate of ϕ_c with the indirect value of ϕ_c obtained from (12.1) using the value of ρ_c. Explain any discrepancy.

*(d) Consider the simple model of the cookie problem discussed in Section 12.1. Write a program that places disks at random into a square box and chooses their diameter randomly between 0 and 1. Estimate the value of ρ_c at which a spanning cluster first appears and compare its value to your estimate found in part (c)? Is your value for ϕ_c more or less than what was found in part (c)?

*(e) Another variation of the cookie problem is to place disks with unit diameter at random in a box with the constraint that the disks do not overlap. Continue to add disks until the fraction of successful attempts becomes less than 1%, that is, when one hundred successive attempts at adding a disk are not successful. Does a spanning cluster exist? If not, increase the diameters of all the disks at a constant rate (in analogy to the baking of the cookies) until a spanning cluster is attained. How does ϕ_c for this model compare with the value of ϕ_c found in part (d)? ■

A continuum model that is applicable to random porous media is known as the *Swiss cheese* model. In this model the relevant quantity (the cheese) is the space between the disks. For the Swiss cheese model in two dimensions, the cheese area fraction at the percolation threshold ψ_c is given by $\psi_c = 1 - \phi_c$, where ϕ_c is the disk area fraction at the percolation threshold of the disks. Does such a relation hold in three dimensions (see Project 12.14)?

So far, we have emphasized the existence of the percolation threshold p_c and the appearance of a spanning cluster or path for $p \geq p_c$. Another quantity that characterizes percolation is $P_\infty(p)$, the probability that an occupied site belongs to the spanning cluster. The probability P_∞ is defined as

$$P_\infty(p) = \frac{\text{number of sites in the spanning cluster}}{\text{total number of occupied sites}}. \tag{12.2}$$