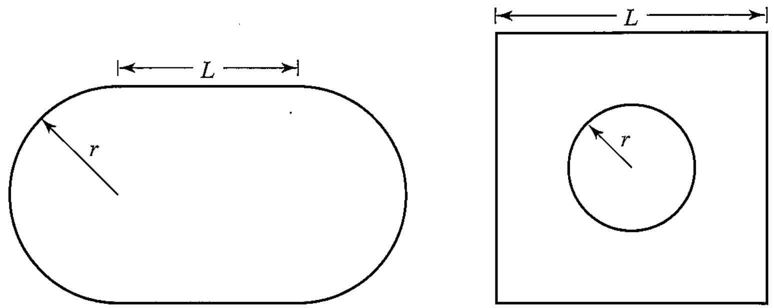

(a) (b)

Figure 6.14 (a) Geometry of the stadium billiard model. (b) Geometry of the Sinai billiard model.

(a) Compute the Lyapunov spectrum for the Lorenz model for $\sigma = 16$, $b = 4$, and $r = 45.92$. Try other values of the parameters and compare your results.

(b) Linearize the equations for the Hénon map and find the Lyapunov spectrum for $a = 1.4$ and $b = 0.3$ in (6.32). ■

Project 6.25 A spinning magnet

Consider a compass needle that is free to rotate in a periodically reversing magnetic field which is perpendicular to the axis of the needle. The equation of motion of the needle is given by

$$\frac{d^2\phi}{dt^2} = -\frac{\mu}{I} B_0 \cos \omega t \sin \phi, \tag{6.59}$$

where ϕ is the angle of the needle with respect to a fixed axis along the field, μ is the magnetic moment of the needle, I its moment of inertia, and B_0 and ω are the amplitude and the angular frequency of the magnetic field, respectively. Choose an appropriate numerical method for solving (6.59) and plot the Poincaré map at time $t = 2\pi n/\omega$. Verify that if the parameter $\lambda = \sqrt{2B_0\mu/I/\omega^2} > 1$, then the motion of the needle exhibits chaotic motion. Briggs (see references) discusses how to construct the corresponding laboratory system and other nonlinear physical systems. ■

Project 6.26 Billiard models

Consider a two-dimensional planar geometry in which a particle moves with constant velocity along straight line orbits until it elastically reflects off the boundary. This straight line motion occurs in various "billiard" systems. A simple example of such a system is a particle moving with fixed speed within a circle. For this geometry the angle between the particle's momentum and the tangent to the boundary at a reflection is the same for all points.

Suppose that we divide the circle into two equal parts and connect them by straight lines of length L as shown in Figure 6.14a. This geometry is called a *stadium billiard*. How does the motion of a particle in the stadium compare to the motion in the circle? In both cases we can find the trajectory of the particle by geometrical considerations. The stadium billiard model and a similar geometry known as the Sinai billiard model (see Figure 6.14b) have been used as model systems for exploring the foundations of statistical mechanics. There is also much interest in relating the behavior of a classical particle in various billiard models to the solution of Schrödinger's equation for the same geometries.

(a) Write a program to simulate the stadium billiard model. Use the radius r of the semicircles as the unit of length. The algorithm for determining the path of the particle is as follows:

(i) Begin with an initial position (x_0, y_0) and momentum (p_{x0}, p_{y0}) of the particle such that $|\mathbf{p}_0| = 1$.

(ii) Determine which of the four sides the particle will hit. The possibilities are the top and bottom line segments and the right and left semicircles.

(iii) Calculate the next position of the particle from the intersection of the straight line defined by the current position and momentum, and also determine the equation for the segment where the next reflection occurs.

(iv) Determine the new momentum, (p'_x, p'_y), of the particle after reflection such that the angle of incidence equals the angle of reflection. For reflection off the line segments we have $(p'_x, p'_y) = (p_x, -p_y)$. For reflection off a circle we have

$$p'_x = [y^2 - (x - x_c)^2]p_x - 2(x - x_c)yp_y \tag{6.60a}$$

$$p'_y = -2(x - x_c)yp_x + [(x - x_c)^2 - y^2]p_y, \tag{6.60b}$$

where $(x_c, 0)$ is the center of the circle. (Note that the momentum p_x rather than p'_x is on the right-hand side of (6.60b). Remember that all lengths are scaled by the radius of the circle.)

(v) Repeat steps (ii)–(iv).

(b) Determine if the particle dynamics is chaotic by estimating the largest Lyapunov exponent. One way to do so is to start two particles with almost identical positions and/or momenta (varying by say 10^{-5}). Compute the difference Δs of the two phase space trajectories as a function of the number of reflections n, where Δs is defined by

$$\Delta s = \sqrt{|\mathbf{r}_1 - \mathbf{r}_2|^2 + |\mathbf{p}_1 - \mathbf{p}_2|^2}. \tag{6.61}$$

Choose $L = 1$ and $r = 1$. The Lyapunov exponent can be found from a semilog plot of Δs versus n. Repeat your calculation for different initial conditions and average your values of Δs before plotting. Repeat the calculation for $L = 0.5$ and 2.0 and determine if your results depend on L.

(c) Another test for the existence of chaos is the reversibility of the motion. Reverse the momentum after the particle has made n reflections, and let the drawing color equal the background color so that the path can be erased. What limitation does roundoff error place on your results? Repeat this simulation for $L = 1$ and $L = 0$.

(d) Place a small hole of diameter d in one of the circular sections of the stadium so that the particle can escape. Choose $L = 1$ and set $d = 0.02$. Give the particle a random position and momentum, and record the time when the particle escapes through the hole. Repeat for at least 10^4 particles and compute the fraction of particles $S(n)$ remaining after a given number of reflections n. The function $S(n)$ will decay with n. Determine the functional dependence of S on n and calculate the characteristic decay time if $S(n)$ decays exponentially. Repeat for $L = 0.1$, 0.5, and 2.0. Is the decay time a function of L? Does $S(n)$ decay exponentially for the circular billiard model ($L = 0$) (see Bauer and Bertsch)?