*Problem 13.10 Laplacian growth models

(a) Solve the discrete Laplace equation (13.10) by hand for the growth probabilities of a DLA cluster of mass 1, 2, and 3. Set $P = 1$ on the boundary and $P = 0$ on the cluster. Compare your results to your results in Problem 13.9b for mass 1 and 2.

(b) You are probably familiar with the random nature of electrical discharge patterns that occur in atmospheric lightning. Although this phenomenon, known as *dielectric breakdown*, is complicated, we will see that a simple model leads to discharge patterns that are similar to those that are observed in nature. Because lightning occurs in an inhomogeneous medium with differences in the density, humidity, and conductivity of air, we will develop a model of electrical discharge in an inhomogeneous insulator. We know that when an electrical discharge occurs, the electrical potential ϕ satisfies Laplace's equation $\nabla^2\phi = 0$. One version of the model (see Family et al.) is specified by the following steps:

(i) Consider a large boundary circle of radius R and place a charge source at the origin. Choose the potential $\phi = 0$ at the origin (an occupied site) and $\phi = 1$ for sites on the circumference of the circle. The radius R should be larger than the radius of the growing pattern.

(ii) Use the relaxation method (see Section 10.5) to compute the values of the potential ϕ_i for (empty) sites within the circle.

(iii) Assign a random number r to each empty site within the boundary circle. The random number r_i at site i represents a breakdown coefficient and the random inhomogeneous nature of the insulator.

(iv) The growth sites are the nearest neighbor sites of the discharge pattern (the occupied sites). Form the product $r_i\phi_i^a$ for each growth site i, where a is an adjustable parameter. Because the potential for the discharge pattern is zero, ϕ_i for growth site i can be interpreted as the magnitude of the potential gradient at site i.

(v) The perimeter site with the maximum value of the product $r\phi^a$ breaks down; that is, set ϕ for this site equal to zero.

(vi) Use the relaxation method to recompute the values of the potential at the remaining unoccupied sites and repeat steps (iv) and (v).

Choose $a = 1/4$ and analyze the structure of the discharge pattern. Does the pattern appear qualitatively similar to lightning? Does the pattern appear to have a fractal geometry? Estimate the fractal dimension by counting $M(b)$, the average number of sites belonging to the discharge pattern that are within a $b \times b$ box. Consider other values of a, for example, $a = 1/6$ and $a = 1/3$, and show that the patterns have a fractal structure with a tunable fractal dimension that depends on the parameter a. Published results (Family et al.) are for patterns with 800 occupied sites.

(c) Another version of the dielectric breakdown model associates a growth probability $p_i = \phi_i^a / \sum_j \phi_j^a$ with each growth site i, where the sum is over all the growth sites. One of the growth sites is occupied with probability p_i. That is, choose a growth site at random and generate a random number r between 0 and 1. If $r \leq p_i$, the growth site i is occupied. As before, the exponent a is a free parameter. Convince yourself that $a = 1$ corresponds to diffusion limited aggregation. (The boundary condition used in the latter corresponds to a zero potential at the growth sites.) To what type of cluster does $a = 0$ correspond? Consider $a = 1/2$, 1, and 2 and explore the dependence of the visual appearance of the clusters on a. Estimate the fractal dimension of the clusters.

(d) Consider a deterministic growth model for which *all* growth sites are tested for occupancy at each growth step. Adopt the same geometry and boundary conditions as in part (b) and use the relaxation method to solve Laplace's equation for ϕ_i. Then find the perimeter site with the largest value of ϕ and set $\phi_{\max}$ equal to this value. Only those perimeter sites for which the ratio $\phi_i/\phi_{\max}$ is larger than a parameter p become part of the cluster; ϕ_i is set equal to unity for these sites. After each growth step, the new growth sites are determined and the relaxation method is used to recompute the values of ϕ_i at each unoccupied site. Choose $p = 0.35$ and determine the nature of the regular fractal pattern. What is the fractal dimension? Consider other values of p and determine the corresponding fractal dimension. These patterns have been termed *Laplace fractal carpets* (see Family et al.). ■

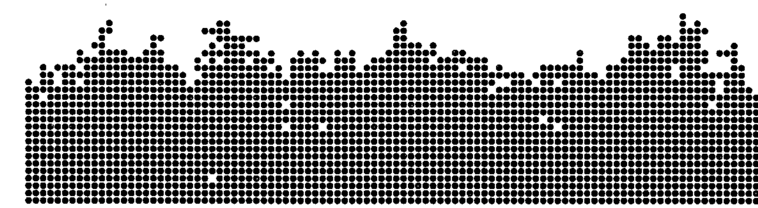

Figure 13.12 Example of surface growth according to the Eden model. The surface site in column i is the perimeter site with the maximum value of h_i. In the figure the average height of the surface is 20.46 and the width is 2.33.

Surface growth models. The fractal objects we have discussed so far are self-similar; that is, if we look at a small piece of the object and magnify it isotropically to the size of the original, the original and the magnified object look similar (on the average). In the following, we introduce some simple models that generate a class of fractals that are self-similar only for scale changes in certain directions.

Suppose that we have a flat surface at time $t = 0$. How does the surface grow as a result of vapor deposition and sedimentation? For example, consider a surface that is initially a line of L occupied sites. Growth is in the vertical direction only (see Figure 13.12).

As before, we simply choose a growth site at random and occupy it (the Eden model again). The average height of the surface is given by

$$\bar{h} = \frac{1}{N_s}\sum_{i=1}^{N_s} h_i, \tag{13.11}$$

where h_i is the distance of the ith surface site from the substrate, and the sum is over all surface sites N_s. (The precise definition of a surface site is discussed in Problem 13.11.)