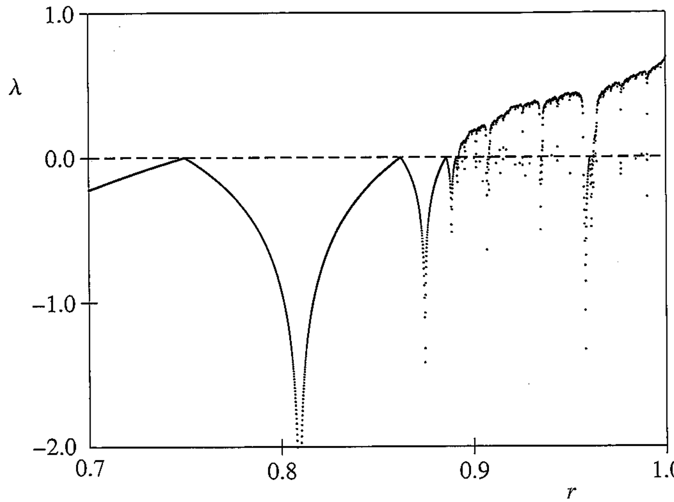


**Figure 6.9** The Lyapunov exponent calculated using the method in (6.19) as a function of the control parameter $r$. Compare the behavior of $\lambda$ to the bifurcation diagram in Figure 6.2. Note that $\lambda < 0$ for $r < 3/4$ and approaches zero at a period doubling bifurcation. A negative spike corresponds to a superstable trajectory. The onset of chaos is visible near $r = 0.892$, where $\lambda$ first becomes positive. For $r \gtrsim 0.892$, $\lambda$ generally increases except for dips below zero whenever a periodic window occurs, for example, the dip due to the period 3 window near $r = 0.96$. For each value of $r$, the first 1000 iterations were discarded, and $10^5$ values of $\ln|f'(x_n)|$ were used to determine $\lambda$.

We have found that nearby trajectories diverge if $\lambda > 0$. For $\lambda < 0$, the two trajectories converge and the system is not chaotic. What happens for $\lambda = 0$? In this case we will see that the trajectories diverge algebraically, that is, as a power of $n$. In some cases a dynamical system is at the "edge of chaos" where the Lyapunov exponent vanishes. Such systems are said to exhibit weak chaos to distinguish their behavior from the strongly chaotic behavior ($\lambda > 0$) that we have been discussing.

If we define $z \equiv |\Delta x_n|/|\Delta x_0|$, then $z$ will satisfy the differential equation

$$\frac{dz}{dn} = \lambda z. \tag{6.20}$$

For weak chaos we do not find an exponential divergence, but instead a divergence that is algebraic and is given by

$$\frac{dz}{dn} = \lambda_q z^q, \tag{6.21}$$

where $q$ is a parameter that needs to be determined. The solution to (6.21) is

$$z = [1 + (1-q)\lambda_q n]^{1/(1-q)}, \tag{6.22}$$

which can be checked by substituting (6.22) into (6.21). In the limit $q \to 1$, we recover the usual exponential dependence.

We can determine the type of chaos using the crude approach of choosing a large number of initial values of $x_0$ and $x_0 + \Delta x_0$ and plotting the average of $\ln z$ versus $n$. If we do not obtain a straight line, then the system does not exhibit strong chaos. How can we check for



the behavior shown in (6.22)? The easiest way is to plot the quantity

$$\frac{z^{1-q} - 1}{1-q} \tag{6.23}$$

versus $n$, which will equal $n\lambda_q$ if (6.22) is applicable. We explore these ideas in the following problem.

***Problem 6.10 Measuring weak chaos**

(a) Write a program that plots $\ln z$, if $q = 1$, or $z_q$, if $q \neq 1$, as a function of $n$. Your program should have $q$, $|\Delta x_0|$, the number of seeds, and the number of iterations as input parameters. To compare with work by Añaños and Tsallis, use a variation of the logistic map given by

$$x_{n+1} = 1 - a x_n^2, \tag{6.24}$$

where $|x_n| \leq 1$ and $0 \leq a \leq 2$. The seeds $x_0$ should be equally spaced in the interval $|x_0| < 1$.

(b) Consider strong chaos at $a = 2$. Choose $q = 1$, 50 iterations, at least 1000 values of $x_0$, and $|\Delta x_0| = 10^{-6}$. Do you obtain a straight line for $\ln z$ versus $n$? Does $z_n$ eventually stop increasing as a function of $n$? If so why? Try $|\Delta x_0| = 10^{-12}$. How do your results differ and how are they the same? Also iterate $\Delta x$ directly:

$$\begin{aligned} \Delta x_{n+1} &= x_{n+1} - \tilde{x}_{n+1} = -a\left(x_n^2 - \tilde{x}_n^2\right) \\ &= -a(x_n - \tilde{x}_n)(x_n + \tilde{x}_n) = -a\Delta x_n(x_n + \tilde{x}_n), \end{aligned} \tag{6.25}$$

where $x_n$ is the iterate starting at $x_0$, and $\tilde{x}_n$ is the iterate starting at $x_0 + \Delta x_0$. Show that straight lines are not obtained for your plot if $q \neq 1$.

(c) The edge of chaos for this map is at $a = 1.401155189$. Repeat part (a) for this value of $a$ and various values of $q$. Simulations with $10^5$ values of $x_0$ points show that linear behavior is obtained for $q \approx 0.36$. ■

A system of fixed energy (and number of particles and volume) has an equal probability of being in any microstate specified by the positions and velocities of the particles (see Section 15.2). One way of measuring the ability of a system to be in any state is to measure its entropy defined by

$$S = -\sum_i p_i \ln p_i, \tag{6.26}$$

where the sum is over all states, and $p_i$ is the probability or relative frequency of being in the $i$th state. For example, if the system is always in only one state, then $S = 0$, the smallest possible entropy. If the system explores all states equally, then $S = \ln \Omega$, where $\Omega$ is the number of possible states. (You can show this result by letting $p_i = 1/\Omega$.)

***Problem 6.11 Entropy of the logistic map**

(a) Write a program to compute $S$ for the logistic map. Divide the interval [0, 1] into bins or subintervals of width $\Delta x = 0.01$ and determine the relative number of times