***Problem 10.15 Vector potential and magnetic fields**

The magnetic field from arbitrary currents can also be obtained using Poisson's equation. The field is generated from a vector potential **A** that satisfies

$$\nabla^2 \mathbf{A} = \mu \mathbf{j}, \tag{10.18}$$

where **j** is the current density in the wires and μ is the magnetic permeability. If current flows only in the z direction, then $\mathbf{j} = (0, 0, j_z(x, y))$ and $\mathbf{A} = (0, 0, A_z(x, y))$, and we again have a two-dimensional problem that can be solved using the relaxation method.

Do a simulation that models the magnetic field from an arbitrary number of wires. Combine features of the `ElectricFieldApp` and the `LaplaceApp` programs. The program should read the control and create a current carrying wire when a custom button is clicked. The computation is performed using the animation's `doStep` method to perform a Gauss–Seidel relaxation step. Compute the magnetic field after the computation converges by computing the curl of the vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{10.19}$$

See (10.52) for how to compute the curl when only discrete values are available. ■

Dielectrics can be added to the solution of Laplace's equation by adding an array to store the dielectric constant k at every grid site and imposing the condition

$$D_{1n} = D_{2n}, \tag{10.20}$$

where $\mathbf{D} = k\mathbf{E}$ and k is the dielectric susceptibility. This condition is equivalent to

$$0 = \oint_l k\nabla V \cdot d\mathbf{l} = \oint_l k\frac{\partial V}{\partial n} dl, \tag{10.21}$$

where $\partial V/\partial n$ denotes the derivative of V parallel to $d\mathbf{l}$. The vector $d\mathbf{l}$ is the two-dimensional equivalent of a surface vector. Its magnitude is the length of a line segment and its direction is perpendicular to the tangent to the line segment. If we approximate (10.21) along each edge of length $2h$ and dielectric susceptibility k, using a finite difference for the derivative, we obtain (see Figure 10.3)

$$0 = k_1\frac{V_1 - V_0}{h}2h + k_2\frac{V_2 - V_0}{h}2h + k_3\frac{V_3 - V_0}{h}2h + k_4\frac{V_4 - V_0}{h}2h. \tag{10.22}$$

We rearrange terms in (10.22) and find a modified form of (10.12) that includes the dielectric:

$$V_0 = \frac{1}{4(k_1 + k_2 + k_3 + k_4)}[k_1V_1 + k_2V_2 + k_3V_3 + k_4V_4], \tag{10.23}$$

where k_i is the average dielectric constant at a site where the electric potential is V_i.

Problem 10.16 Capacitor with dielectric

(a) Modify your Laplace program to include a dielectric medium. That is, create an array of dielectric susceptibilities and implement (10.23) using a relaxation algorithm. Be sure to set the dielectric array elements to unity in free space and inside conductors.

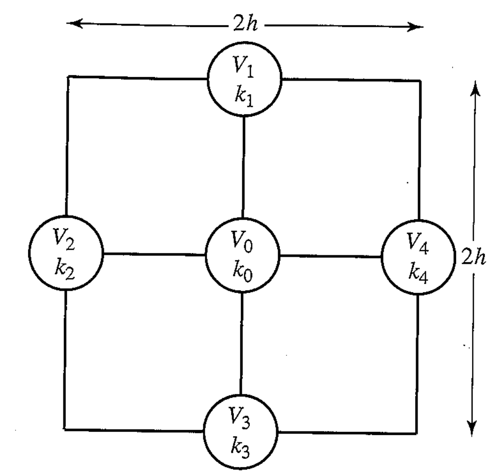

Figure 10.3 The grid sites used to compute the integral in (10.21) based on Gauss's law for the electric field.

(b) Test your algorithm by creating a capacitor consisting of +10 and −10 potential plates near the center of the grid. Initialize the dielectric susceptibility to two in half the capacitor and run the program. Use a `Scalar2DFrame` to display the electric potential, but note that some representations of the scalar field are more appropriate than others. Compare the spacing between the contour lines inside and outside the dielectric. Why does the spacing change?

(c) The bound charge on the surface of a dielectric can be computed by subtracting $V(x, y)$ from the average of the potential at the four nearest neighbor sites. You are, in effect, using (10.17) to solve for the charge. Implement this calculation and describe the bound charge on the surface of the dielectric. ■

10.6 ■ RANDOM WALK SOLUTION OF LAPLACE'S EQUATION

In Section 10.5 we found that the solution to Laplace's equation in two dimensions at the point (x, y) is given by

$$V(x, y) = \frac{1}{4}\sum_{i=1}^{4} V(i), \tag{10.24}$$

where $V(i)$ is the value of the potential at the ith neighbor. A generalization of this result is that the potential at any point equals the average of the potential on a circle (or sphere in three dimensions) centered about that point.

The relation (10.24) can be given a probabilistic interpretation in terms of random walks (see Problem 10.10d). Suppose that many random walkers are at the site (x, y), and each walker "jumps" to one of its four neighbors (on a square grid) with equal probability $p = 1/4$. From (10.24) we see that the average potential found by the walkers after jumping one step is the potential at (x, y). This relation generalizes to walkers that visit a site on a