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Charge density on a conducting needle

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We attempt to determine the linear charge density on a finite straight segment of thin charged conducting wire. Several different methods are presented, but none yields entirely convincing results, and it appears that the problem itself may be ill-posed. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

Imagine a straight segment of conducting wire, length $2a$, on which we place an electric charge Q (Fig. 1). *Question*: in the absence of external fields, how will the charge distribute itself along the wire? That is, what is the equilibrium linear charge density, $\lambda(x)$? It seems obvious that Coulomb repulsion will push charge out toward the ends, just as the charge on a solid conductor flows to the surface. However, it is not clear how *much* of the charge goes to the ends; presumably some of it is left spread out along the length of the "needle."

The question sounds simple enough, and it must surely have been considered long ago by Sommerfeld, Smythe, Stratton, or maybe even Maxwell himself. However, we have found no reference to it in the literature. The reason may be that the problem as it stands is ill posed: the answer apparently depends on the particular model used to represent the needle.[1] If we think of it as a solid object, we are obliged to specify its profile: is it (say) an elongated ellipsoid (Sec. II A), or is it perhaps a thin circular cylinder (Sec. II B)? As we shall see, these two cases lead to radically different results, and the differences seem to persist even in the limit as the cross section goes to zero. Alternatively, we might model the system as a collection of charged "beads" on a string stretched between $-a$ and $+a$ (Sec. III A). We can solve for their equilibrium positions, and examine the limit as the number of beads goes to infinity (and the charge on each one goes to zero). Or we might put the beads at fixed locations along the string, and solve for the equilibrium partitioning of the total charge (Sec. III B). Will these two "bead" models yield the same effective linear charge density? To what solid shape (if any) do they correspond? Unfortunately, for more than four beads it is prohibitively difficult to perform the calculations analytically, and we must resort to numerical methods. [However, if we substitute a Hooke's law interaction for the true Coulombic one, the problem is exactly soluble (see the Appendix).]

In this paper we explore each of these models: solids in Sec. II and beads in Sec. III. After that, as a sort of "reality check," it is instructive to compare the two-dimensional analog: the charge density on an infinite conducting "ribbon" (of width $2a$ and infinitesimal thickness). This problem has the virtue that it is well defined and exactly soluble, so the methods applied earlier to the needle can be put to rigorous test (Sec. IV). In the concluding section (Sec. V), we summarize our results, and return to the question of what this problem really *means*.

II. SOLID MODELS

A. Ellipsoidal

The charge density on an ellipsoidal conductor

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{2.1}$$

with total charge Q is[2]

$$\sigma=\frac{Q}{4\pi abc}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-1/2}. \tag{2.2}$$

Here a, b, and c are the three semi-axes (Fig. 2).

To calculate the total charge dQ on a ring of width dx, we note that an element of area dA on the surface is related to its projection in the xz plane by

$$dx\,dz=\cos\theta\,dA, \tag{2.3}$$

where θ is the angle between the y axis and the unit vector $\hat{n}$ normal to the surface:

$$\cos\theta=\hat{j}\cdot\hat{n}. \tag{2.4}$$

Now, $\hat{n}$ can be calculated by taking the gradient of $f(x,y,z)=(x/a)^2+(y/b)^2+(z/c)^2$, and dividing off its length:

$$\hat{n}=\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-1/2}\left(\frac{x}{a^2},\frac{y}{b^2},\frac{z}{c^2}\right). \tag{2.5}$$

Thus

$$\cos\theta=\frac{y}{b^2\sqrt{x^2/a^4+y^2/b^4+z^2/c^4}}. \tag{2.6}$$

Evidently the charge on a patch of surface above $dx\,dz$ is

$$\sigma\,dA=\frac{Qb}{4\pi ac}\frac{1}{y}\,dx\,dz. \tag{2.7}$$

The total charge on the ring is four times the charge on one quadrant:

$$dQ=\frac{Qb}{\pi ac}\,dx\int_0^{c\sqrt{1-x^2/a^2}}\frac{1}{y}\,dz. \tag{2.8}$$

From Eq. (2.1) we have

$$y=b\sqrt{1-(x/a)^2-(z/c)^2}. \tag{2.9}$$

The integral simplifies if we let $u\equiv z/[c\sqrt{1-(x/a)^2}]$:

$$dQ=\frac{Q}{\pi a}\,dx\int_0^1\frac{du}{\sqrt{1-u^2}}=\frac{Q}{2a}\,dx. \tag{2.10}$$

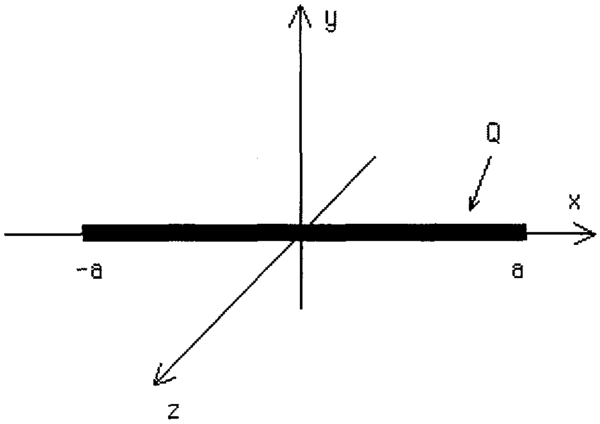


Fig. 1. Conducting "needle."

Astonishingly, the effective line charge $\lambda(x)=dQ/dx$ is *constant*:[3]

$$\lambda(x)=Q/2a. \tag{2.11}$$

Since this result is independent of b and c, it holds in the limit $b,c\rightarrow 0$, when the ellipsoid collapses to a line segment along the x axis. *Conclusion*: If the needle is the limiting case of an *ellipsoid*, then the linear charge density is *constant*. In this case the tapering of the ends exactly cancels the tendency for charge to push out toward the extremities.

B. Cylindrical

The exact charge distribution on a conducting cylinder is not known. In his pioneering study, Smythe[4] remarks that "the literature is blank on this subject;" Taylor[5] adds that the problem "must be regarded as intractable from the point of view of conventional methods." A number of authors have extended and improved Smythe's preliminary results,[6] but the limiting case of zero radius remains elusive.

Smythe begins by expressing the surface charge density on the ends (σ_e) and on the sides (σ_s) in the form of two series:

$$\sigma_s(x)=\sum_{n=0}^{\infty} A_n(a^2-x^2)^{n-1/3}, \tag{2.12}$$

$$\sigma_e(r)=\sum_{n=0}^{\infty} B_n(R^2-r^2)^{n-1/3}, \tag{2.13}$$

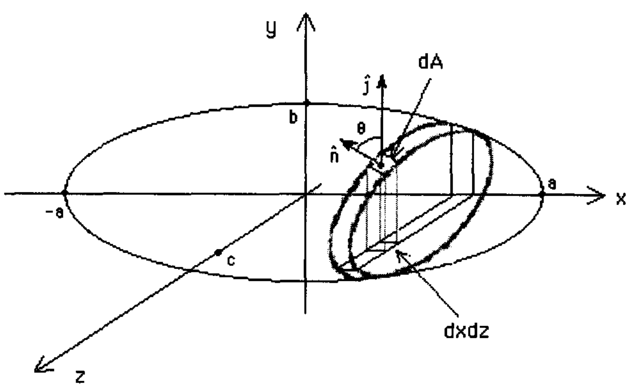


Fig. 2. Conducting ellipsoid.

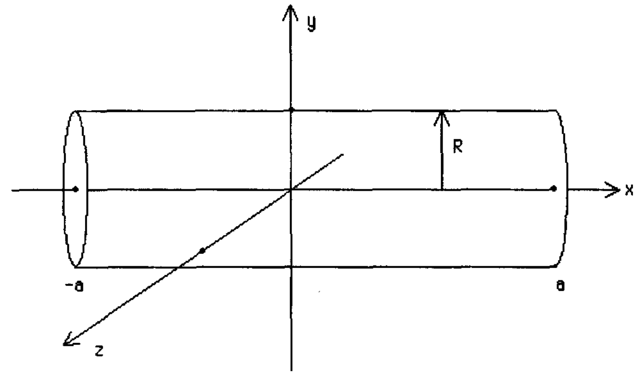


Fig. 3. Conducting cylinder.

where R is the radius of the cylinder (Fig. 3). (The functional form of the singularity at the "corners" is dictated by a general theorem:[7] it goes like $\delta^{-1/3}$, where δ is the distance from the edge. Smythe's *ansatz* simply incorporates this structure in a power series expansion.) An approximate solution is obtained by truncating the sums at $n=r$ and $n=s$, respectively. Smythe provides an algorithm for determining the coefficients A_n and B_n and a criterion for choosing the optimal values of r and s, for a prescribed degree of accuracy. He applies the technique to particular values of the aspect ratio $b\equiv a/R$ ranging from $\frac{1}{4}$ up to 4, but he does not explore the large-b régime that concerns us here.

In truth, Smythe's method is not well suited to our problem. It is pretty clear that we need only one term in the B series, and off-hand one would suppose that the larger r is the better. However, in practice the calculation is wildly unstable, and many of Smythe's own numbers (obtained with the aid of a desk calculator) are incorrect. Taylor[5] refined Smythe's method, using a slightly different series that converges more rapidly. Applying Taylor's technique, with $b=1000$ and $r=10$, we obtained the linear charge density plotted in Fig. 4. This graph appears to confirm our intuition that a portion of the charge pushes out to the two ends, leaving the density relatively flat toward the middle.

However, Djordjević,[8] drawing on extensive numerical studies using a quite different approach, reports that the charge density on a long conducting cylinder is essentially constant $[\lambda(x)=Q/2a]$ over the entire rod, except in the

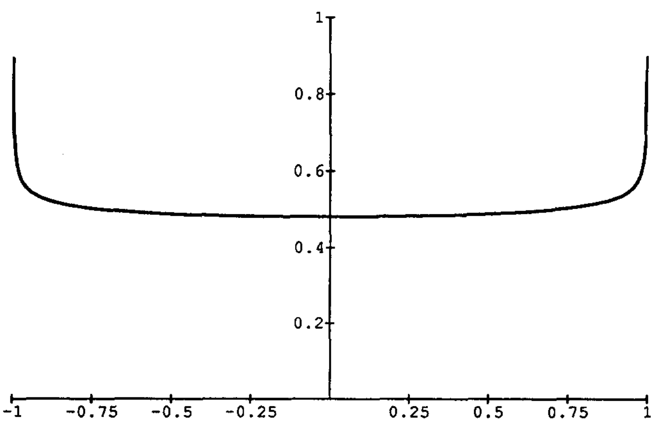


Fig. 4. Linear charge density on a conducting cylinder, using $a=1$, $Q=1$, and $R=0.001$.

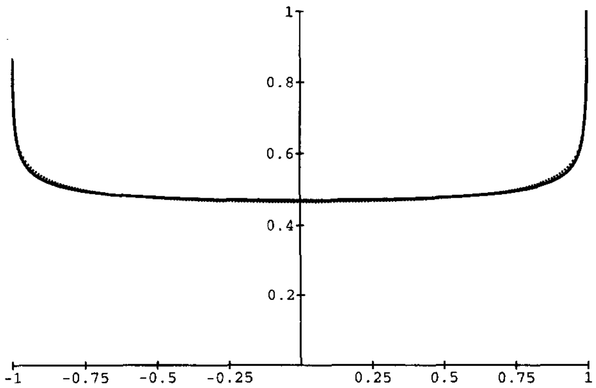


Fig. 7. Graph of Eq. (3.7) superimposed on Fig. 6(c).

B. Fixed position

The previous model is algebraically cumbersome because the variables we seek (the positions of the charges) appear quadratically, and in the denominator. Even for the simplest (nontrivial) case ($n=2$) this led to a quartic equation (3.2). An alternative model, in which we fix the *positions* of the particles and let their *charges* vary yields a system of *linear* equations.

For example, with four evenly spaced beads there are two distinct charges: q_1 at $\pm a/3$, and q_2 at $\pm a$ (Fig. 8). In equilibrium, the net electrical force on q_1 (at $a/3$) is zero—the force to the left (due to q_2 at a) balances the force to the right (due to q_2 at $-a$ and q_1 at $-a/3$):

$$\frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{d^2}\right)=\frac{1}{4\pi\epsilon_0}\left(\frac{q_1q_2}{(2d)^2}+\frac{q_1q_1}{d^2}\right), \tag{3.8}$$

where $d=2a/3$ is the separation between the charges. It follows that

$$q_2=\tfrac{1}{4}q_2+q_1; \tag{3.9}$$

on the other hand, the sum of all the charges is Q, so

$$q_1+q_2=Q/2. \tag{3.10}$$

Solving, we find

$$q_1=\tfrac{3}{14}Q, \quad q_2=\tfrac{2}{7}Q. \tag{3.11}$$

Similarly, for $n=3$ we obtain

$$q_1=\frac{3725}{26242}Q, \quad q_2=\frac{1980}{13121}Q, \quad q_3=\frac{2718}{13121}Q. \tag{3.12}$$

In general, for $2n$ charges a distance $d=2a/(2n-1)$ apart, there are n unknowns, $q_1, q_2,\ldots,q_n$ (Fig. 9). The force on charge q_i is

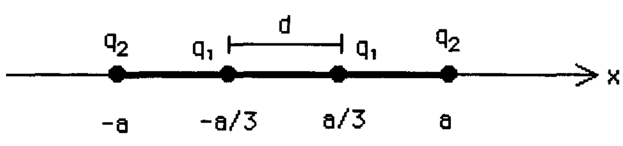


Fig. 8. Four equally spaced charges on a finite wire.

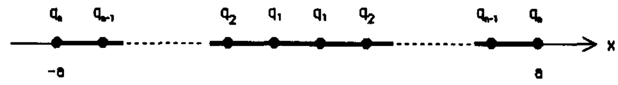


Fig. 9. The $2n$ evenly spaced charges on a finite wire.

$$F_i=\frac{q_i}{4\pi\epsilon_0}\left(\sum_{j=1}^{n}\frac{q_j}{(i+j-1)^2d^2}+\sum_{j=1}^{i-1}\frac{q_j}{(i-j)^2d^2}\right.$$
$$\left.-\sum_{j=i+1}^{n}\frac{q_j}{(j-i)^2d^2}\right). \tag{3.13}$$

In equilibrium $F_i=0$, so

$$\sum_{j=1}^{n}\frac{q_j}{(i+j-1)^2}+\sum_{j=1}^{i-1}\frac{q_j}{(i-j)^2}-\sum_{j=i+1}^{n}\frac{q_j}{(j-i)^2}=0, \tag{3.14}$$

where $i=1,2,3,\ldots,n-1$ (q_n, of course, is subject to the extra constraining force of the wire). Meanwhile, the total charge is Q:

$$\sum_{i=1}^{n}q_i=\frac{Q}{2}. \tag{3.15}$$

Taken together, (3.14) and (3.15) provide n simultaneous linear equations for the n unknowns. The corresponding charge density is

$$\lambda(x_i)=\frac{q_i}{d}=\frac{n-1/2}{a}q_i, \quad i=1,\ldots,n, \tag{3.16}$$

where $x_i\equiv(i-1/2)d$. The resulting plots for $n=5$, 10, and 100 are shown in Fig. 10. Again the graphs appear to converge as n increases, and to the same function as before (compare Figs. 6 and 10).

An interesting variant on this approach is suggested by the work of Ross.[11] Instead of adjusting the charge distribution so as to make the *force* on each charge vanish, we require that the *potentials* be equal at points midway between the charges. The potential at a point $d/2$ to the left of q_i is

$$V_i=\frac{1}{4\pi\epsilon_0}\left(\sum_{j=1}^{n}\frac{q_j}{(i+j-3/2)d}+\sum_{j=1}^{i-1}\frac{q_j}{(i-j-1/2)d}\right.$$
$$\left.+\sum_{j=i}^{n}\frac{q_j}{(j-i+1/2)d}\right]. \tag{3.17}$$

Letting $\phi\equiv4\pi\epsilon_0Vd$, where V is the common potential, we obtain n linear equations,

$$\sum_{j=1}^{n}\frac{q_j}{(i+j-3/2)}+\sum_{j=1}^{i-1}\frac{q_j}{(i-j-1/2)}+\sum_{j=i}^{n}\frac{q_j}{(j-i+1/2)}$$
$$-\phi=0, \quad i=1,\ldots,n, \tag{3.18}$$

which, together with (3.15), determine the $n+1$ unknowns $(q_1,\ldots,q_n,\phi)$. For example, when $n=2$ we find $q_1=(2/9)Q$ and $q_2=(5/18)Q$. In Fig. 11 we graph the charge densities for $n=5$, 10, and 100; the agreement with previous results is excellent, for large n, suggesting again convergence to a common shape.

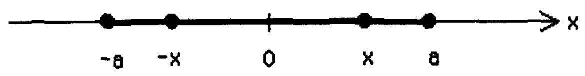


Fig. 5. Four equal charges on a finite wire.

immediate vicinity of the two ends; the length d of the regions over which it deviates significantly is proportional to R (specifically, $d\sim 5R$), and the total charge on these two end caps is proportional to R/a. If Djordjević is right, then the charge density in the limit $R\rightarrow 0$ is *constant* for the entire cylinder, as it is for the ellipsoid.

III. BEAD MODELS[9]

A. Fixed charge

Suppose we place $2n$ equal point charges ($q=Q/2n$) on the line from $x=-a$ to $x=a$. Our task is to find their equilibrium positions. If $n=1$, the two charges will obviously repel out to the ends. However, for $n=2$ the problem is already nontrivial. The outermost pair will be at $\pm a$; let $\pm x$ denote the positions of the other two (Fig. 5). The force on the charge at $+x$ is

$$F=\frac{q^2}{4\pi\epsilon_0}\left(\frac{1}{(x+a)^2}+\frac{1}{(2x)^2}-\frac{1}{(a-x)^2}\right). \tag{3.1}$$

Setting F equal to zero[10] yields a quartic equation:

$$(a^2-x^2)^2=16ax^3, \tag{3.2}$$

to which the numerical solution is

$$x=0.36148a. \tag{3.3}$$

In general, the $2n$ charges are at $\pm(x_1,x_2,\ldots,x_{n-1},x_n\equiv a)$, and the force on the ith charge is

$$F_i=\frac{q^2}{4\pi\epsilon_0}\left(\sum_{j=1}^{n}\frac{1}{(x_i+x_j)^2}+\sum_{j=1}^{i-1}\frac{1}{(x_i-x_j)^2}-\sum_{j=i+1}^{n}\frac{1}{(x_j-x_i)^2}\right). \tag{3.4}$$

Setting $F_i=0$ (for $i=1,2,\ldots,n-1$) yields $n-1$ coupled equations for the equilibrium positions of the charges. For example, with $n=6$ (12 charges), we find

$$x_1=0.10010a,$$

$$x_2=0.29913a,$$

$$x_3=0.49441a,$$

$$x_4=0.68241a,$$

$$x_5=0.85692a.$$

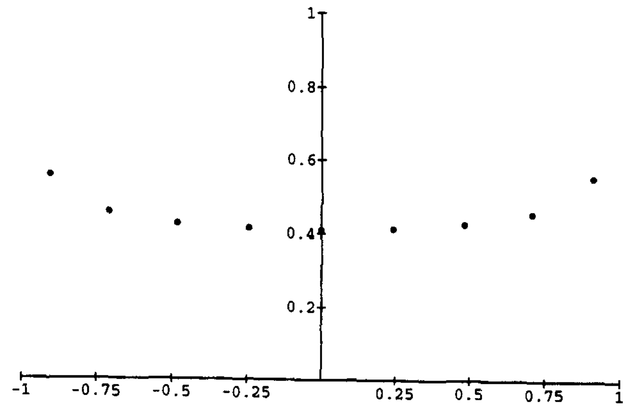
However, what we really want is not so much the *position* of each charge as the effective *charge density*, $\lambda(x)$. To this end we compute

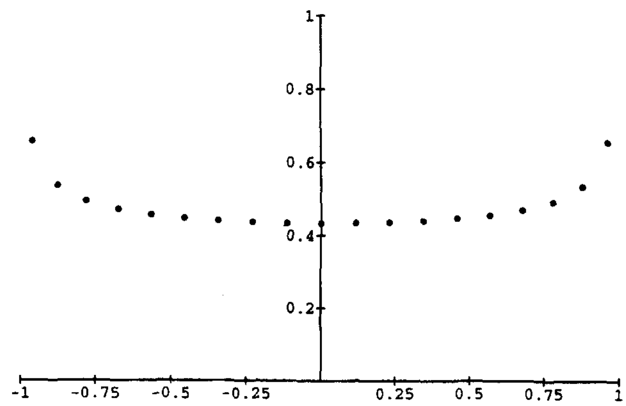
$$\lambda(\bar{x}_i)=\frac{q}{x_{i+1}-x_i}, \tag{3.5}$$

where

$$\bar{x}_i\equiv\tfrac{1}{2}(x_{i+1}+x_i) \tag{3.6}$$

is the center of the interval. The resulting plots (using $Q=1$ and $a=1$) for $n=5$, 10, and 100 are shown in Fig. 6. The





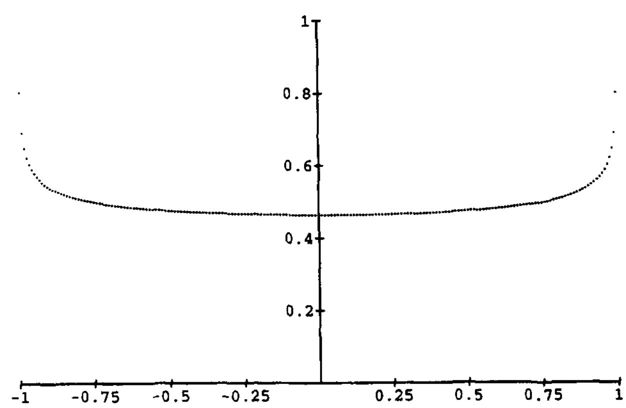


Fig. 6. Charge densities for (a) $n=5$, (b) $n=10$, and (c) $n=100$.

graphs appear to converge as n increases, suggesting that there is a well-defined limit function as $n\rightarrow\infty$. Indeed, an expression of the form

$$\lambda(x)=A+\frac{B}{(a^2-x^2)^{1/3}} \tag{3.7}$$

(inspired by Smythe's treatment of the finite cylinder) fits the results quite well (see Fig. 7), if $A=0.384985$ and $B=0.083684$. However, we have no theoretical justification for this form, no *ab initio* means for calculating the parameters A and B, and, in fact, no real assurance that the graphs converge at all.

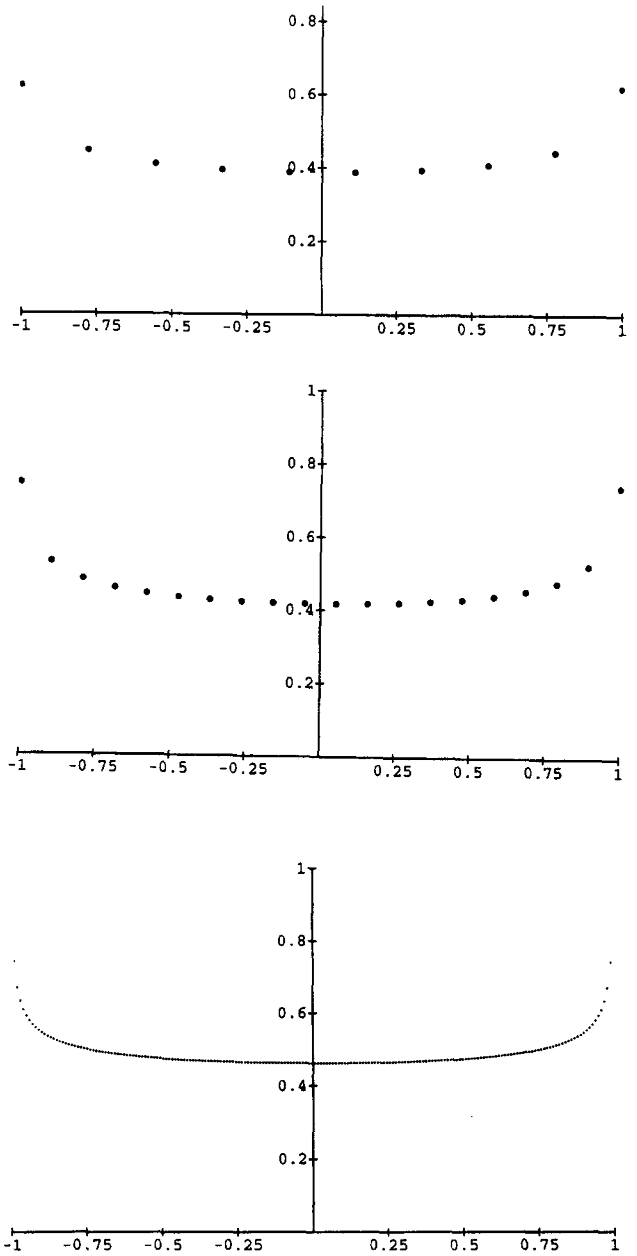


Fig. 10. Charge densities using the fixed-position method: (a) $n=5$, (b) $n=10$, and (c) $n=100$.

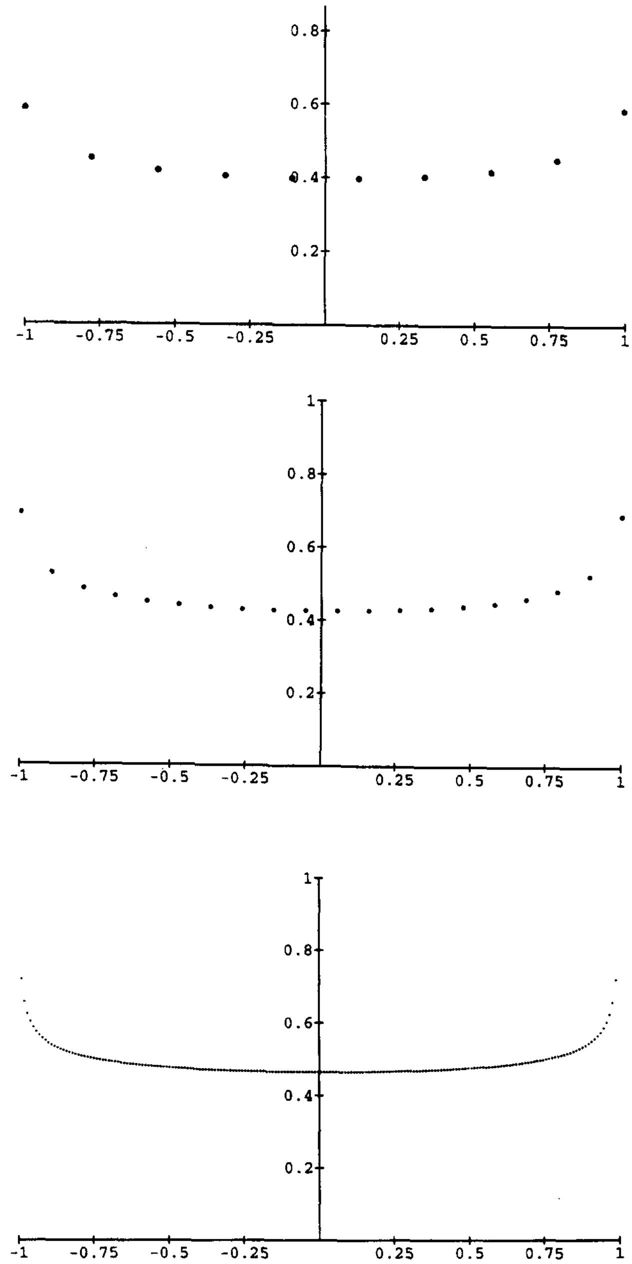


Fig. 11. Charge densities using Ross' method: (a) $n=5$, (b) $n=10$, and (c) $n=100$.

IV. INFINITE CONDUCTING RIBBON

A. Exact solution

The analogous case of an infinite conducting *ribbon*, of width $2a$ and infinitesimal thickness (Fig. 12), can be solved exactly. For this is a strictly *two-dimensional* problem (the potential is plainly independent of z), and hence accessible to the method of conformal mapping. If Λ is the net charge per unit length (in the z direction), the potential $V(x,y)$ is given (implicitly) by the equation[12]

$$\frac{x^2}{a^2\cosh^2(V/V_0)}+\frac{y^2}{a^2\sinh^2(V/V_0)}=1, \tag{4.1}$$

where

$$V_0\equiv\frac{\Lambda}{2\pi\epsilon_0}. \tag{4.2}$$

Equipotentials are ellipses, with semi-axes $a\cosh(V/V_0)$ and $a\sinh(V/V_0)$. As $V\rightarrow 0$ they collapse to a line, $-a<x<+a$, along the x axis. (In this problem it is convenient to set the potential equal to 0 on the conductor.)

The charge density on the ribbon is determined by the discontinuity in the normal derivative of V:

$$\sigma(x)=-\epsilon_0\left(\frac{\partial V_{\text{above}}}{\partial y}\bigg|_{y=0}-\frac{\partial V_{\text{below}}}{\partial y}\bigg|_{y=0}\right). \tag{4.3}$$

For very small y, on the interval $-a<x<a$ (where V is also very small), Eq. (4.1) reduces to

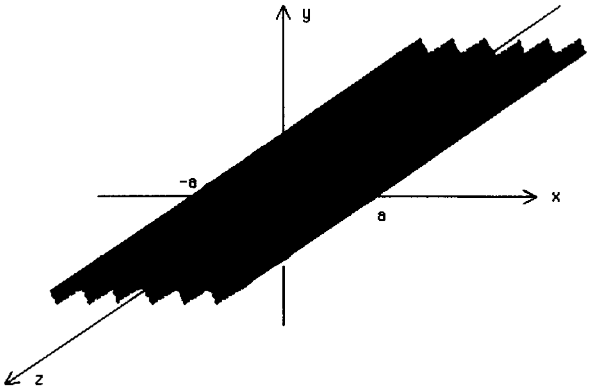


Fig. 12. Infinite conducting ribbon.

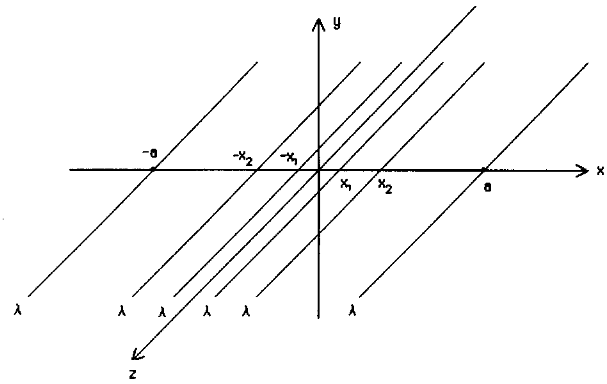


Fig. 14. Array of parallel wires with equal charges.

$$V(x,y) \approx -\frac{V_0}{\sqrt{a^2-x^2}}|y|, \tag{4.4}$$

so

$$\left.\frac{\partial V_{\text{above}}}{\partial y}\right|_{y=0} = -\left.\frac{\partial V_{\text{below}}}{\partial y}\right|_{y=0} = -\frac{V_0}{\sqrt{a^2-x^2}}, \tag{4.5}$$

and hence[13]

$$\sigma(x) = \frac{\Lambda}{\pi\sqrt{a^2-x^2}}. \tag{4.6}$$

The is the *exact* charge density on a conducting ribbon—the analog to the formula that has remained so elusive in the case of the needle. It is plotted in Fig. 13 (for $\Lambda=1$ and $a=1$). As expected, the charge is repelled out toward the edges of the ribbon.[14]

B. Wire models

1. Fixed charge

Consider an array of $2n$ infinite wires running parallel to the z axis in the xz plane, free to move in the x direction between $-a$ and $+a$ (Fig. 14). If each wire carries the same linear charge density λ, what are their equilibrium positions, $\pm(x_1,x_2,\ldots,x_{n-1},x_n \equiv a)$? The force per unit length on the ith wire is

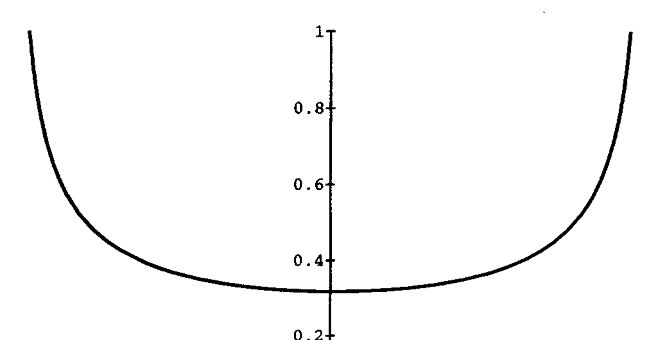


Fig. 13. Charge density on a ribbon.

$$f_i = \frac{\lambda^2}{2\pi\epsilon_0}\left(\sum_{j=1}^{n}\frac{1}{x_i+x_j} + \sum_{j=1}^{i-1}\frac{1}{x_i-x_j} - \sum_{j=i+1}^{n}\frac{1}{x_j-x_i}\right). \tag{4.7}$$

Setting f_i equal to 0 (for $i=1,2,\ldots,n-1$) yields $n-1$ equations for the equilibrium positions of the wires. For example, if $n=2$ we find $x_1 = a/\sqrt{5} = 0.447a$; if $n=3$, then $x_1 = (a/\sqrt{3})(1-2/\sqrt{7})^{1/2} = 0.285a$ and $x_2 = (a/\sqrt{3})(1+2/\sqrt{7})^{1/2} = 0.765a$.

We are interested in the effective surface charge $\sigma(x)$, in the limit $n\to\infty$. As before,

$$\sigma(\bar{x}_i) = \frac{\lambda}{x_{i+1}-x_i}, \tag{4.8}$$

with $\lambda = \Lambda/2n$ and

$$\bar{x}_i \equiv \tfrac{1}{2}(x_{i+1}+x_i). \tag{4.9}$$

In Fig. 15 we plot numerical solutions for $n=5$, 10, and 100. The results fit the exact solution (4.6) very well.

2. Fixed position

Suppose now that the wires are *evenly* spaced, a distance $d=2a/(2n-1)$ apart, but their *charges* $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ are variable (Fig. 16). The force per unit length on the ith wire is

$$f_i = \frac{\lambda_i}{2\pi\epsilon_0 d}\left(\sum_{j=1}^{n}\frac{\lambda_j}{i+j-1} + \sum_{j=1}^{i-1}\frac{\lambda_j}{i-j} - \sum_{j=i+1}^{n}\frac{\lambda_j}{j-i}\right). \tag{4.10}$$

At equilibrium $f_i=0$, from which it follows that

$$\lambda_i = (2i-1)^2\sum_{j\neq i}^{n}\frac{\lambda_j}{(j-i)(j+i-1)}, \quad i=1,2,\ldots,n-1. \tag{4.11}$$

This gives us $n-1$ equations in n unknowns—the remaining constraint is

$$\sum_{i=1}^{n}\lambda_i = \frac{\Lambda}{2}. \tag{4.12}$$

For example, if $n=2$, we obtain $\lambda_1=\Lambda/6=0.167\Lambda$, $\lambda_2=\Lambda/3=0.333\Lambda$; if $n=3$, $\lambda_1=(31/290)\Lambda=0.107\Lambda$, $\lambda_2=(18/145)\Lambda=0.124\Lambda$, $\lambda_3=(39/145)\Lambda=0.269\Lambda$. In Fig. 17 the resulting surface charge $\sigma(x)$ is plotted for $n=5$, 10, and 100; again, the graphs are in close agreement with the exact answer.

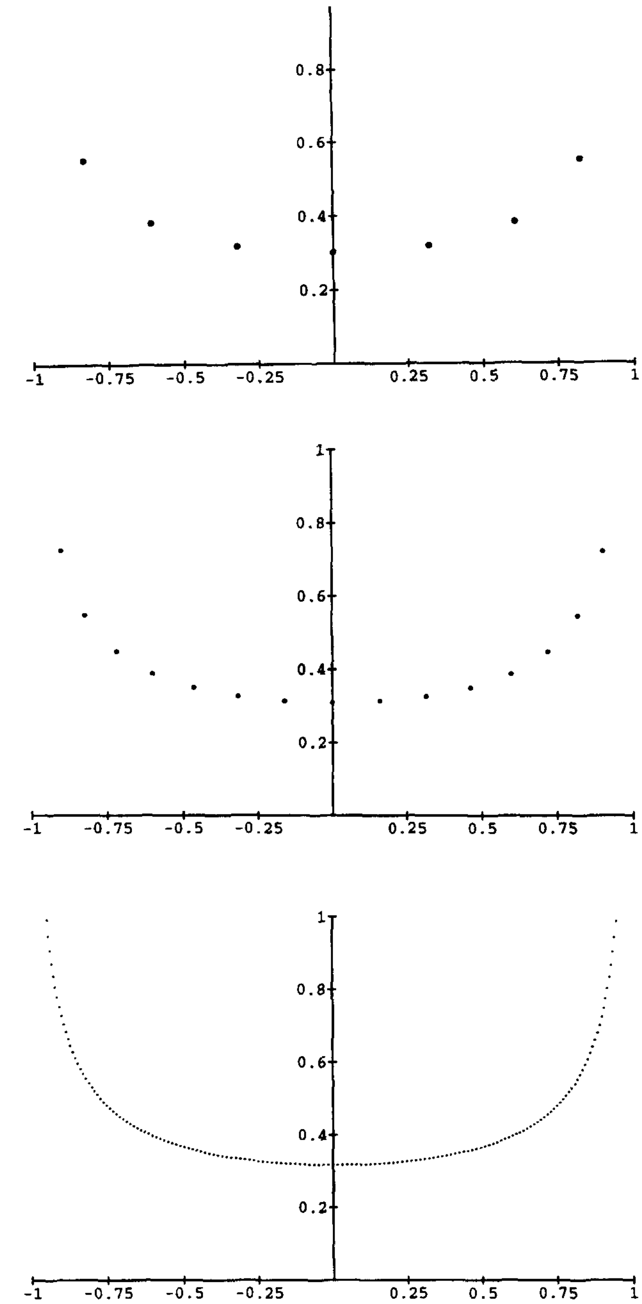


Fig. 15. Surface charge by constant charge method: (a) $n=5$, (b) $n=10$, and (c) $n=100$.

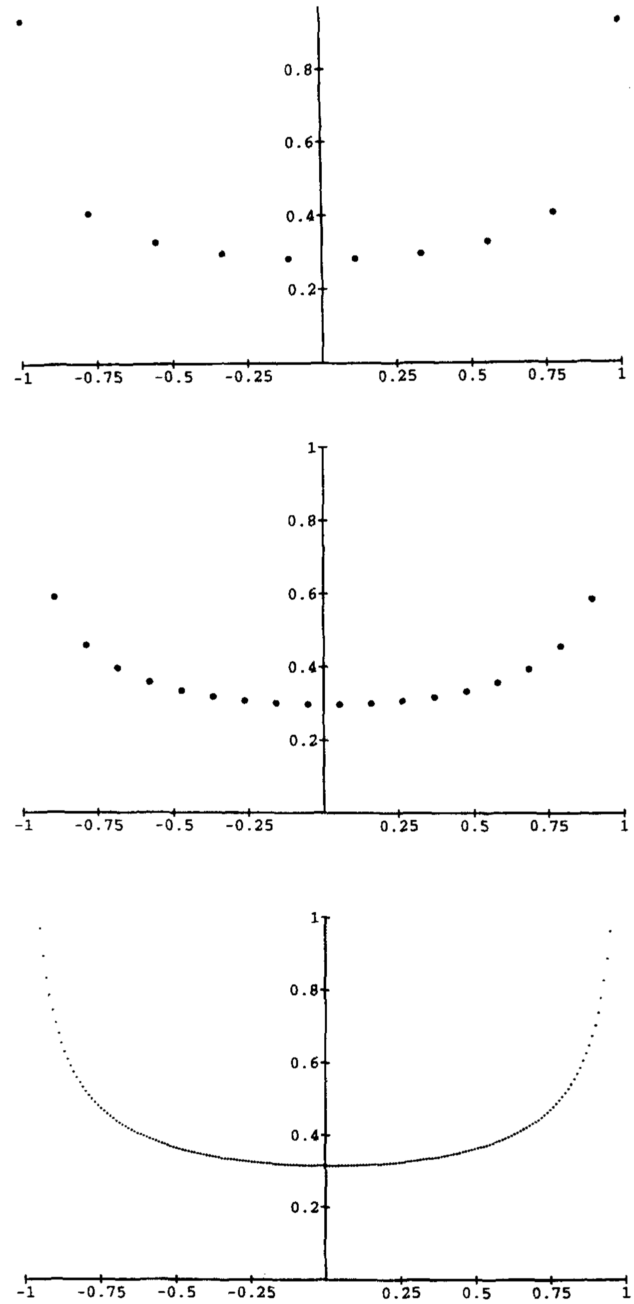


Fig. 17. Surface charge by constant spacing method: (a) $n=5$, (b) $n=10$, and (c) $n=100$.

V. CONCLUSION

What, then, *is* the charge density on a conducting needle? All our numerical studies support one's intuition that the function is fairly flat in the center, with "rabbit ears" at the two ends. And yet the ellipsoid model and Djordjević's analysis of the cylinder model indicate that the charge density should be *constant*, in the limit of infinitesimal cross section. Is it conceivable that we have been fooled by very slow convergence of the graphs, and that, actually, as n goes to infinity, Figs. 6, 10, and 11 would all flatten out to $\lambda(x) = 0.5$? This would resolve the awkward paradox—and yet, in the case of the ribbon the analogous graphs (Figs. 15 and 17) give a reasonably accurate representation of the exact answer, even for quite small n. Moreover, if the charge density *were* constant, on a truly one-dimensional needle, how could the force on an off-center point be zero, considering that the charge out to the near end is exactly balanced, leaving, inevitably, some extra uncanceled charge at the far end?

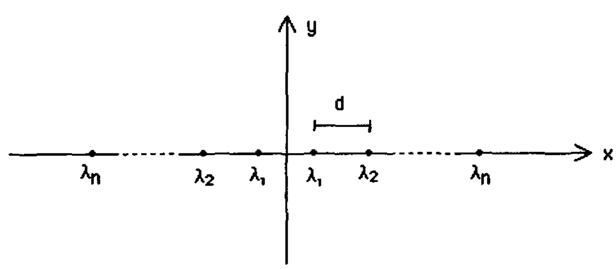


Fig. 16. Array of $2n$ parallel wires with equal spacing.

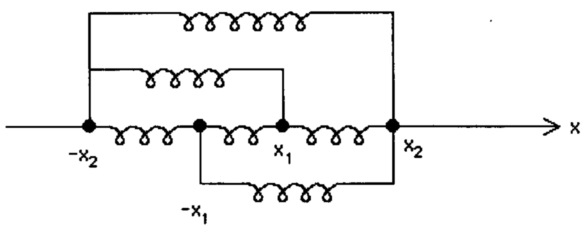


Fig. 18. Four beads with interconnecting springs.

It is embarrassing to conclude that we still do not know what the charge density on a conducting needle is. We suspect that the problem is ill posed (in the sense that the answer depends on the model adopted), and even for those ("bead") models that seem to be approaching a common limit we cannot tell what the functional form of that limit might be[15]—nor can we absolutely exclude the counterintuitive possibility that it is in fact a *constant*.

APPENDIX: HOOKE'S LAW ANALOG

We consider here an exactly soluble "toy" model, in which the true Coulombic interaction is replaced by a Hooke's law force. Specifically, we suppose there are $2n$ beads spread out along the x axis, and each one is connected to every other one by a spring of force constant k and natural length a (Fig. 18). Our problem is to determine the equilibrium spacing of the beads.

Clearly, they will arrange themselves symmetrically about the center, at positions we may as well call $-x_n,\ldots,-x_2$, $-x_1, x_1, x_2,\ldots,x_n$. The force on the ith bead is

$$\begin{aligned}F_i&=-k\left(\sum_{j=1}^{n}(x_i+x_j-a)+\sum_{j=1}^{i-1}(x_i-x_j-a)\right.\\&\qquad\left.-\sum_{j=i+1}^{n}(x_j-x_i-a)\right)\\&=-k[2nx_i-(2i-1)a].\end{aligned}\tag{A1}$$

At equilibrium this force is zero, so

$$x_i=\frac{(2i-1)}{2n}a.\tag{A2}$$

Evidently the beads are *evenly spaced*, a distance a/n apart. In the limit $n\to\infty$ the "charge" density is *uniform* on the interval from $-a$ to $+a$.

In this "toy" model we can also handle the continuum case exactly. Imagine that the "beads" are smeared out, with linear density $\lambda(x)$. The bead at x will experience forces from both sides; at equilibrium the force due to all the beads to its left will balance the force due to all the beads to its right:[16]

$$\int_{-\infty}^{x}(x-x'-a)\lambda(x')dx'=\int_{x}^{\infty}(x'-x-a)\lambda(x')dx'.\tag{A3}$$

Combining terms,

$$\begin{aligned}&x\int_{-\infty}^{\infty}\lambda(x')dx'-\int_{-\infty}^{\infty}x'\lambda(x')dx'-a\int_{-\infty}^{x}\lambda(x')dx'\\&\quad+a\int_{x}^{\infty}\lambda(x')dx'=0.\end{aligned}\tag{A4}$$

Now

$$\int_{-\infty}^{\infty}\lambda(x')dx'\equiv Q\tag{A5}$$

is the total "charge." And since the configuration is symmetrical about the origin, $\lambda(x)$ must be an even function, so

$$\int_{-\infty}^{\infty}x'\lambda(x')dx'=0.\tag{A6}$$

Therefore,

$$(x+a)Q-2a\int_{-\infty}^{x}\lambda(x')dx'=0.\tag{A7}$$

Differentiating with respect to x, we find that

$$Q-2a\lambda(x)=0,\tag{A8}$$

or

$$\lambda(x)=Q/2a.\tag{A9}$$

Again, the "charge" density is *uniform* on the interval $-a<x<a$ (and 0 otherwise).

The virtue of the Hooke's law model is that we can solve it analytically, in both the discrete and continuum regimes. However, it is hardly a *realistic* model, and it is not clear what, if anything, it tells us about the true Coulombic case.

ACKNOWLEDGMENTS

We thank Ray Mayer for many useful discussions, and Matt Kleban for extensive computer exploration of the Smythe approach (Sec. II B) and for drawing many of the figures. We thank an anonymous referee for supplying Ref. 3. Above all, we thank A. R. Djordjević of Belgrade University for illuminating e-mail correspondence regarding the charge density on a conducting cylinder; we have not begun to do justice (in Sec. II B) to his results and his insights, which we hope he will see fit to publish.

[1]This is already a somewhat surprising conclusion. The charge density on a thin conducting disk, for example, can be obtained by a variety of limiting procedures, all of which yield the same result [for a particularly ingenious method, see R. Friedberg, "The electrostatics and magnetostatics of a conducting disk," Am. J. Phys. **61**, 1084–1096, (1993)]. Evidently, there is some deep pathology in the reduction to a one-dimensional object that does not infect the reduction from three to two.

[2]William R. Smythe, *Static and Dynamic Electricity*, 3rd ed., revised printing (Hemisphere, New York, 1989), Sec. 5.02.

[3]This was noted by Max Abraham and Richard Becker, *Classical Theory of Electricity and Magnetism*, 2nd ed. (Hafner, New York, 1949); our derivation is based on Jeffrey Gima's unpublished Reed College senior thesis (1993).

[4]W. R. Smythe, "Charged Right Circular Cylinder," J. Appl. Phys. **27**, 917–920 (1956). See also Ref. 2, Sec. 5.39.

[5]T. T. Taylor, "Electric Polarizability of a Short Right Circular Conducting Cylinder," J. Res. NBS B **64**, 135–143 (1960). Taylor is concerned with the problem of an *un*charged cylinder in an external electric field, but it is straightforward to adapt his method to the case of a *charged* cylinder with *no* external field. See also T. T. Taylor, "Magnetic Polarizability of a Short Right Circular Conducting Cylinder," J. Res. NBS B **64**, 199–210 (1960).

[6]W. R. Smythe, "Charged Right Circular Cylinder," J. Appl. Phys. **33**, 2966–2967 (1962); P. C. Waterman, "Matrix Methods in Potential Theory and Electromagnetic Scattering," J. Appl. Phys. **50**, 4550–4566 (1979); B. D. Popović, M. B. Dragović, and A. R. Djordjević, *Analysis and Synthesis of Wire Antennas* (Research Studies Press, Chichester, 1982); P. K. Wang, "Calculation of Electrostatic Fields Surrounding Finite Circular Cylindrical Conductors," IEEE Trans. Antennas Propag. **32**, 956–962 (1984); A. R. Djordjević, "Comments on 'Calculation of Electrostatic Fields Surrounding Finite Circular Cylindrical Conductors,'" IEEE Trans. Antennas Propag. **33**, 683–684 (1985); P. K. Wang, C. H. Chuang, and N. L. Miller, "Electrostatic, Thermal and Vapor Density Fields Surrounding Stationary Columnar Ice Crystals," J. Atmos. Sci. **42**, 2371–2379 (1985).

[7]W. R. Smythe, Ref. 2, p. 209.

[8]A. R. Djordjević, personal communication.

[9]Material in this section is based on Ye Li's unpublished Reed College senior thesis (1994).

[10]The electrical force on the *outermost* charges (at $\pm a$) is *not* zero, of course, because they are subject to the extra constraining force holding the charges on the wire. By the way, one can (equivalently) calculate the total *potential energy* of the configuration, and minimize it to determine the positions of the charges.

[11]J. B. Ross, "Plotting the charge distribution of a closed-loop conducting wire using a microcomputer," Am. J. Phys. **55**, 948–950 (1987). Ross used the potential *at* each charge (due to all the others), but for open segments it is better to use the midpoints, since the end charges are subject to nonelectrostatic forces.

[12]Smythe, Ref. 2, Sec. 4.22.

[13]This result can also be obtained by treating the ribbon as the limiting case of an elongated ellipsoid. With $\Lambda \equiv Q/2c$ and $c \to \infty$, Eqs. (2.1) and (2.2) yield

$$\sigma(x)=\frac{\Lambda}{\pi}\left[a^2-x^2+\left(\frac{bx}{a}\right)^2\right]^{-1/2}.$$

This is the net surface charge density (both sides) for an infinite cylinder of elliptical cross section. In the limit $b \to 0$ it reduces to a ribbon, and we recover Eq. (4.6). [In the case $c=\infty$, the analog to the theorem in Sec. II A states that the charge per unit length (in the z direction) on a strip of width dx is $\Lambda dx/(\pi\sqrt{a^2-x^2})$ for all b—but (unlike the *finite* ellipsoid) it is *not* independent of x.]

[14]It is easy to check that the total linear charge density $[\int \sigma(x)dx]$ is Λ.

[15]We have pushed the fixed-position bead model up to $n=300$ with barely detectable changes in the curve; the best fit of the form (3.7) occurs for $A=0.384049$, $B=0.088295$.

[16]This method cannot be applied to the Coulomb problem, of course, because of the nasty singularity in the integrand.

Electric field line diagrams don't work

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Electric fields produced by coplanar point charges have often been represented by field line diagrams that depict two-dimensional slices of the three-dimensional field. Serious problems with these "conventional" field line diagrams (CFLDs) have been overlooked. Two of these problems, "equatorial clumping" and "false monopole moment," occur because a two-dimensional slice lacks information vital to the accurate representation of an inherently three-dimensional field. Equatorial clumping causes most CFLDs to exhibit unphysical behavior such as irregular spacing between field lines terminating on negative charges. CFLDs can also mistakenly indicate that a neutral charge distribution has a significant monopole moment. Such phenomena make the visual estimation of local field strengths impossible and render CFLDs of little utility for representing three-dimensional fields. While these "projection" problems can be avoided by using two-dimensional field line diagrams to represent two-dimensional ($1/r$) electric fields, or by using three-dimensional field line diagrams to represent three-dimensional fields, other forms of distortion generally remain. © *1996 American Association of Physics Teachers.*

I. INTRODUCTION

Elementary physics textbooks generally attempt some form of two-dimensional graphical representation of the three-dimensional electric field produced by coplanar point charges. The most common approach, the conventional field line diagram (CFLD), employs continuous electric field lines, or "lines of force," which are everywhere tangent to the electric field. Each field line is traced from a positive charge until it terminates on a negative charge or at infinity (i.e., "far" from all charges). An individual field line conveys information about the local direction of the electric field, but not about the field's magnitude. The latter property requires consideration of the local density of field lines. Figure 1(a), for example, illustrates the CFLD for a simple dipole. Field lines near the dipole are generally spaced close together, which is thought to indicate a high field strength, while more distant lines are spaced farther apart. The tendency of the field line density to decrease with distance from the dipole is apparently consistent with the asymptotic $1/r^3$ decay of the field strength. Field lines are, of course, creatures of fiction.[1] However CFLDs such as Fig. 1(a) are gen-

