

Week 10 March 7-11: Resampling Techniques, Bootstrap and Blocking

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Overview of week 10, March 7-11

Topics.

- Top down approach first, what we need to code
- Resampling Techniques and statistics: Bootstrap and Blocking
- [Video of Lecture](#)
- [Handwritten notes](#)

Teaching Material, videos and written material.

- Overview video on the [Bootstrap method](#)
- These Lecture notes
- Marius Johnson's Master thesis on the Blocking Method

The top-down approach, part 1

Last week we discussed how to implement a gradient descent method like the simplest possible gradient descent with a simple learning rate as parameter to tune. We repeat the codes here.

```
# 2-electron VMC code for 2dim quantum dot with importance sampling  
# Using gaussian rng for new positions and Metropolis- Hastings  
# Added energy minimization  
# Common imports  
from math import exp, sqrt  
from random import random, seed, normalvariate
```

```

import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
import sys

# Trial wave function for the 2-electron quantum dot in two dims
def WaveFunction(r,alpha,beta):
    r1 = r[0,0]**2 + r[0,1]**2
    r2 = r[1,0]**2 + r[1,1]**2
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = r12/(1+beta*r12)
    return exp(-0.5*alpha*(r1+r2)+deno)

# Local energy for the 2-electron quantum dot in two dims, using analytical local energy
def LocalEnergy(r,alpha,beta):
    r1 = (r[0,0]**2 + r[0,1]**2)
    r2 = (r[1,0]**2 + r[1,1]**2)
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = 1.0/(1+beta*r12)
    deno2 = deno*deno
    return 0.5*(1-alpha*alpha)*(r1 + r2) + 2.0*alpha + 1.0/r12+deno2*(alpha*r12-deno2+2*beta*deno-

# Derivate of wave function ansatz as function of variational parameters
def DerivativeWFansatz(r,alpha,beta):
    WfDer = np.zeros((2), np.double)
    r1 = (r[0,0]**2 + r[0,1]**2)
    r2 = (r[1,0]**2 + r[1,1]**2)
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = 1.0/(1+beta*r12)
    deno2 = deno*deno
    WfDer[0] = -0.5*(r1+r2)
    WfDer[1] = -r12*r12*deno2
    return WfDer

# Setting up the quantum force for the two-electron quantum dot, recall that it is a vector
def QuantumForce(r,alpha,beta):
    qforce = np.zeros((NumberParticles,Dimension), np.double)
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = 1.0/(1+beta*r12)
    qforce[0,:] = -2*r[0,:]*alpha*(r[0,:]-r[1,:])*deno*deno/r12
    qforce[1,:] = -2*r[1,:]*alpha*(r[1,:]-r[0,:])*deno*deno/r12
    return qforce

# Computing the derivative of the energy and the energy
def EnergyMinimization(alpha, beta):
    NumberMCcycles= 10000
    # Parameters in the Fokker-Planck simulation of the quantum force
    D = 0.5
    TimeStep = 0.05
    # positions
    PositionOld = np.zeros((NumberParticles,Dimension), np.double)
    PositionNew = np.zeros((NumberParticles,Dimension), np.double)

```

```

# Quantum force
QuantumForceOld = np.zeros((NumberParticles,Dimension), np.double)
QuantumForceNew = np.zeros((NumberParticles,Dimension), np.double)

# seed for rng generator
seed()
energy = 0.0
DeltaE = 0.0
EnergyDer = np.zeros((2), np.double)
DeltaPsi = np.zeros((2), np.double)
DerivativePsiE = np.zeros((2), np.double)
#Initial position
for i in range(NumberParticles):
    for j in range(Dimension):
        PositionOld[i,j] = normalvariate(0.0,1.0)*sqrt(TimeStep)
wfold = WaveFunction(PositionOld,alpha,beta)
QuantumForceOld = QuantumForce(PositionOld,alpha, beta)

#Loop over MC MCcycles
for MCcycle in range(NumberMCcycles):
    #Trial position moving one particle at the time
    for i in range(NumberParticles):
        for j in range(Dimension):
            PositionNew[i,j] = PositionOld[i,j]+normalvariate(0.0,1.0)*sqrt(TimeStep)+\
                QuantumForceOld[i,j]*TimeStep*D
            wfnew = WaveFunction(PositionNew,alpha,beta)
            QuantumForceNew = QuantumForce(PositionNew,alpha, beta)
            GreensFunction = 0.0
            for j in range(Dimension):
                GreensFunction += 0.5*(QuantumForceOld[i,j]+QuantumForceNew[i,j])*\
                    (D*TimeStep*0.5*(QuantumForceOld[i,j]-QuantumForceNew[i,j])-\
                    PositionNew[i,j]+PositionOld[i,j])

            GreensFunction = exp(GreensFunction)
            ProbabilityRatio = GreensFunction*wfnew**2/wfold**2
            #Metropolis-Hastings test to see whether we accept the move
            if random() <= ProbabilityRatio:
                for j in range(Dimension):
                    PositionOld[i,j] = PositionNew[i,j]
                    QuantumForceOld[i,j] = QuantumForceNew[i,j]
                wfold = wfnew
            DeltaE = LocalEnergy(PositionOld,alpha,beta)
            DerPsi = DerivativeWFansatz(PositionOld,alpha,beta)
            DeltaPsi += DerPsi
            energy += DeltaE
            DerivativePsiE += DerPsi*DeltaE

# We calculate mean values
energy /= NumberMCcycles
DerivativePsiE /= NumberMCcycles
DeltaPsi /= NumberMCcycles
EnergyDer = 2*(DerivativePsiE-DeltaPsi*energy)
return energy, EnergyDer

#Here starts the main program with variable declarations
NumberParticles = 2
Dimension = 2
# guess for variational parameters
alpha = 0.95
beta = 0.3

```

```

# Set up iteration using stochastic gradient method
Energy = 0
EDerivative = np.zeros((2), np.double)
# Learning rate eta, max iterations, need to change to adaptive learning rate
eta = 0.01
MaxIterations = 50
iter = 0

Energies = np.zeros(MaxIterations)
EnergyDerivatives1 = np.zeros(MaxIterations)
EnergyDerivatives2 = np.zeros(MaxIterations)
AlphaValues = np.zeros(MaxIterations)
BetaValues = np.zeros(MaxIterations)

while iter < MaxIterations:
    Energy, EDerivative = EnergyMinimization(alpha,beta)
    alphagradient = EDerivative[0]
    betagradient = EDerivative[1]
    alpha -= eta*alphagradient
    beta -= eta*betagradient
    Energies[iter] = Energy
    EnergyDerivatives1[iter] = EDerivative[0]
    EnergyDerivatives2[iter] = EDerivative[1]
    AlphaValues[iter] = alpha
    BetaValues[iter] = beta
    iter += 1

#nice printout with Pandas
import pandas as pd
from pandas import DataFrame
pd.set_option('max_columns', 6)
data ={'Alpha':AlphaValues,'Beta':BetaValues,'Energy':Energies,'Alpha Derivative':EnergyDerivatives1,
        'Beta Derivative':EnergyDerivatives2}

frame = pd.DataFrame(data)
print(frame)

```

What have we done?

The exact energy is 3.0 for an oscillator frequency $\omega = 1$ (with $\hbar = 1$). We note however that with this learning rate and number of iterations, the energies and the derivatives are not yet converged.

We can improve upon this by using the algorithms provided by the **optimize** package in Python. One of these algorithms is Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm.

The optimization problem is to minimize $f(\mathbf{x})$ where \mathbf{x} is a vector in R^n , and f is a differentiable scalar function. There are no constraints on the values that \mathbf{x} can take.

The algorithm begins at an initial estimate for the optimal value \mathbf{x}_0 and proceeds iteratively to get a better estimate at each stage.

The search direction p_k at stage k is given by the solution of the analogue of the Newton equation

$$B_k \mathbf{p}_k = -\nabla f(\mathbf{x}_k),$$

where B_k is an approximation to the Hessian matrix, which is updated iteratively at each stage, and $\nabla f(\mathbf{x}_k)$ is the gradient of the function evaluated at x_k . A line search in the direction p_k is then used to find the next point x_{k+1} by minimising

$$f(\mathbf{x}_k + \alpha \mathbf{p}_k),$$

over the scalar $\alpha > 0$.

Code part 2

The modified code here uses the BFGS algorithm but performs now a production run and writes to file all average values of the energy.

```
# 2-electron VMC code for 2dim quantum dot with importance sampling
# Using gaussian rng for new positions and Metropolis- Hastings
# Added energy minimization
from math import exp, sqrt
from random import random, seed, normalvariate
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D
from matplotlib import cm
from matplotlib.ticker import LinearLocator, FormatStrFormatter
from scipy.optimize import minimize
import sys
import os

# Where to save data files
PROJECT_ROOT_DIR = "Results"
DATA_ID = "Results/EnergyMin"

if not os.path.exists(PROJECT_ROOT_DIR):
    os.mkdir(PROJECT_ROOT_DIR)

if not os.path.exists(DATA_ID):
    os.makedirs(DATA_ID)

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

outfile = open(data_path("Energies.dat"), 'w')

# Trial wave function for the 2-electron quantum dot in two dims
def WaveFunction(r,alpha,beta):
    r1 = r[0,0]**2 + r[0,1]**2
    r2 = r[1,0]**2 + r[1,1]**2
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = r12/(1+beta*r12)
    return exp(-0.5*alpha*(r1+r2)+deno)

# Local energy for the 2-electron quantum dot in two dims, using analytical local energy
def LocalEnergy(r,alpha,beta):
    r1 = (r[0,0]**2 + r[0,1]**2)
    r2 = (r[1,0]**2 + r[1,1]**2)
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = 1.0/(1+beta*r12)
```

```

deno2 = deno*deno
return 0.5*(1-alpha*alpha)*(r1 + r2) +2.0*alpha + 1.0/r12+deno2*(alpha*r12-deno2+2*beta*deno-

# Derivate of wave function ansatz as function of variational parameters
def DerivativeWFansatz(r,alpha,beta):

    WfDer = np.zeros((2), np.double)
    r1 = (r[0,0]**2 + r[0,1]**2)
    r2 = (r[1,0]**2 + r[1,1]**2)
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = 1.0/(1+beta*r12)
    deno2 = deno*deno
    WfDer[0] = -0.5*(r1+r2)
    WfDer[1] = -r12*r12*deno2
    return WfDer

# Setting up the quantum force for the two-electron quantum dot, recall that it is a vector
def QuantumForce(r,alpha,beta):

    qforce = np.zeros((NumberParticles,Dimension), np.double)
    r12 = sqrt((r[0,0]-r[1,0])**2 + (r[0,1]-r[1,1])**2)
    deno = 1.0/(1+beta*r12)
    qforce[0,:] = -2*r[0,:]*alpha*(r[0,:]-r[1,:])*deno*deno/r12
    qforce[1,:] = -2*r[1,:]*alpha*(r[1,:]-r[0,:])*deno*deno/r12
    return qforce

# Computing the derivative of the energy and the energy
def EnergyDerivative(x0):

    # Parameters in the Fokker-Planck simulation of the quantum force
    D = 0.5
    TimeStep = 0.05
    # positions
    PositionOld = np.zeros((NumberParticles,Dimension), np.double)
    PositionNew = np.zeros((NumberParticles,Dimension), np.double)
    # Quantum force
    QuantumForceOld = np.zeros((NumberParticles,Dimension), np.double)
    QuantumForceNew = np.zeros((NumberParticles,Dimension), np.double)

    energy = 0.0
    DeltaE = 0.0
    alpha = x0[0]
    beta = x0[1]
    EnergyDer = 0.0
    DeltaPsi = 0.0
    DerivativePsiE = 0.0
    #Initial position
    for i in range(NumberParticles):
        for j in range(Dimension):
            PositionOld[i,j] = normalvariate(0.0,1.0)*sqrt(TimeStep)
    wfold = WaveFunction(PositionOld,alpha,beta)
    QuantumForceOld = QuantumForce(PositionOld,alpha, beta)

    #Loop over MC MCcycles
    for MCcycle in range(NumberMCcycles):
        #Trial position moving one particle at the time
        for i in range(NumberParticles):
            for j in range(Dimension):
                PositionNew[i,j] = PositionOld[i,j]+normalvariate(0.0,1.0)*sqrt(TimeStep)+\

```

```

        QuantumForceOld[i,j]*TimeStep*D
wfnew = WaveFunction(PositionNew,alpha,beta)
QuantumForceNew = QuantumForce(PositionNew,alpha, beta)
GreensFunction = 0.0
for j in range(Dimension):
    GreensFunction += 0.5*(QuantumForceOld[i,j]+QuantumForceNew[i,j])*\  

        (D*TimeStep*0.5*(QuantumForceOld[i,j]-QuantumForceNew[i,j])-\  

        PositionNew[i,j]+PositionOld[i,j])

GreensFunction = exp(GreensFunction)
ProbabilityRatio = GreensFunction*wfnew**2/wfold**2
#Metropolis-Hastings test to see whether we accept the move
if random() <= ProbabilityRatio:
    for j in range(Dimension):
        PositionOld[i,j] = PositionNew[i,j]
        QuantumForceOld[i,j] = QuantumForceNew[i,j]
    wfold = wfnew
DeltaE = LocalEnergy(PositionOld,alpha,beta)
DerPsi = DerivativeWFansatz(PositionOld,alpha,beta)
DeltaPsi += DerPsi
energy += DeltaE
DerivativePsiE += DerPsi*DeltaE

# We calculate mean values
energy /= NumberMCcycles
DerivativePsiE /= NumberMCcycles
DeltaPsi /= NumberMCcycles
EnergyDer = 2*(DerivativePsiE-DeltaPsi*energy)
return EnergyDer

# Computing the expectation value of the local energy
def Energy(x0):
    # Parameters in the Fokker-Planck simulation of the quantum force
    D = 0.5
    TimeStep = 0.05
    # positions
    PositionOld = np.zeros((NumberParticles,Dimension), np.double)
    PositionNew = np.zeros((NumberParticles,Dimension), np.double)
    # Quantum force
    QuantumForceOld = np.zeros((NumberParticles,Dimension), np.double)
    QuantumForceNew = np.zeros((NumberParticles,Dimension), np.double)

    energy = 0.0
    DeltaE = 0.0
    alpha = x0[0]
    beta = x0[1]
    #Initial position
    for i in range(NumberParticles):
        for j in range(Dimension):
            PositionOld[i,j] = normalvariate(0.0,1.0)*sqrt(TimeStep)
    wfold = WaveFunction(PositionOld,alpha,beta)
    QuantumForceOld = QuantumForce(PositionOld,alpha, beta)

    #Loop over MC MCcycles
    for MCcycle in range(NumberMCcycles):
        #Trial position moving one particle at the time
        for i in range(NumberParticles):
            for j in range(Dimension):
                PositionNew[i,j] = PositionOld[i,j]+normalvariate(0.0,1.0)*sqrt(TimeStep)+\  

                    QuantumForceOld[i,j]*TimeStep*D

```

```

wfnew = WaveFunction(PositionNew,alpha,beta)
QuantumForceNew = QuantumForce(PositionNew,alpha, beta)
GreensFunction = 0.0
for j in range(Dimension):
    GreensFunction += 0.5*(QuantumForceOld[i,j]+QuantumForceNew[i,j])*\  

        (D*TimeStep*0.5*(QuantumForceOld[i,j]-QuantumForceNew[i,j])\  

        PositionNew[i,j]+PositionOld[i,j])

GreensFunction = exp(GreensFunction)
ProbabilityRatio = GreensFunction*wfnew**2/wfold**2
#Metropolis-Hastings test to see whether we accept the move
if random() <= ProbabilityRatio:
    for j in range(Dimension):
        PositionOld[i,j] = PositionNew[i,j]
        QuantumForceOld[i,j] = QuantumForceNew[i,j]
    wfold = wfnew
DeltaE = LocalEnergy(PositionOld,alpha,beta)
energy += DeltaE
if Printout:
    outfile.write('%f\n' %(DeltaE))# ('%f\n' %(energy/(MCcycle+1.0)))
# We calculate mean values
energy /= NumberMCcycles
return energy

#Here starts the main program with variable declarations
NumberParticles = 2
Dimension = 2
# seed for rng generator
seed()
# Monte Carlo cycles for parameter optimization
Printout = False
NumberMCcycles= 10000
# guess for variational parameters
x0 = np.array([0.9,0.2])
# Using Broydens method to find optimal parameters
res = minimize(Energy, x0, method='BFGS', jac=EnergyDerivative, options={'gtol': 1e-4,'disp': True})
x0 = res.x
# Compute the energy again with the optimal parameters and increased number of Monte Cycles
NumberMCcycles= 2**19
Printout = True
FinalEnergy = Energy(x0)
EResult = np.array([FinalEnergy,FinalEnergy])
outfile.close()
#nice printout with Pandas
import pandas as pd
from pandas import DataFrame
data ={'Optimal Parameters':x0, 'Final Energy':EResult}
frame = pd.DataFrame(data)
print(frame)

```

Note that the **minimize** function returns the final values for the variable $\alpha = x0[0]$ and $\beta = x0[1]$ in the array x .

When we have found the minimum, we use these optimal parameters to perform a production run of energies. The output is in turn written to file and is used, together with resampling methods like the **blocking method**, to obtain the best possible estimate for the standard deviation. The optimal minimum is, even with our guess, rather close to the exact value of 3.0 a.u.

The [sampling functions](#) can be used to perform both a blocking analysis, or a standard bootstrap and jackknife analysis.

How do we proceed?

There are several paths which can be chosen. One is to extend the brute force gradient descent method with an adaptive stochastic gradient. There are several examples of this. A recent approach based on [the Langevin equations](#) seems like a promising approach for general and possibly non-convex optimization problems.

Here we would like to point out that our next step is now to use the optimal values for our variational parameters and use these as inputs to a production run. Here we would output values of the energy and perform for example a blocking analysis of the results in order to get a best possible estimate of the standard deviation.

Resampling analysis

The next step is then to use the above data sets and perform a resampling analysis, either using say the Bootstrap method or the Blocking method. Since the data will be correlated, we would recommend to use the non-iid Bootstrap code here. The theoretical background for these resampling methods is found in the [statistical analysis lecture notes](#)

Here we have tailored the codes to the output file from the previous example. We present first the bootstrap resampling with non-iid stochastic event.

```
# Common imports
import os

# Where to save the figures and data files
DATA_ID = "Results/EnergyMin"

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

infile = open(data_path("Energies.dat"), 'r')

from numpy import std, mean, concatenate, arange, loadtxt, zeros, ceil
from numpy.random import randint
from time import time

def tsboot(data, statistic, R, l):
    t = zeros(R); n = len(data); k = int(ceil(float(n)/l));
    inds = arange(n); t0 = time()

    # time series bootstrap
    for i in range(R):
        # construct bootstrap sample from
        # k chunks of data. The chunksize is l
        _data = concatenate([data[j:j+l] for j in randint(0, n-l, k)])[0:n];
        t[i] = statistic(_data)
```

```

# analysis
print ("Runtime: %g sec" % (time()-t0)); print ("Bootstrap Statistics :")
print ("original      bias      std. error")
print ("%8g %14g %15g" % (statistic(data), \
                           mean(t) - statistic(data), \
                           std(t) ))

return t
# Read in data
X = loadtxt(infile)
# statistic to be estimated. Takes two args.
# arg1: the data
def stat(data):
    return mean(data)
t = tsboot(X, stat, 2**12, 2**10)

```

The blocking code, based on the article of [Marius Jonsson](#) is given here

```

# Common imports
import os

# Where to save the figures and data files
DATA_ID = "Results/EnergyMin"

def data_path(dat_id):
    return os.path.join(DATA_ID, dat_id)

infile = open(data_path("Energies.dat"), 'r')

from numpy import log2, zeros, mean, var, sum, loadtxt, arange, array, cumsum, dot, transpose, diag
from numpy.linalg import inv

def block(x):
    # preliminaries
    n = len(x)
    d = int(log2(n))
    s, gamma = zeros(d), zeros(d)
    mu = mean(x)

    # estimate the auto-covariance and variances
    # for each blocking transformation
    for i in arange(0,d):
        n = len(x)
        # estimate autocovariance of x
        gamma[i] = (n)**(-1)*sum( (x[0:(n-1)]-mu)*(x[1:n]-mu) )
        # estimate variance of x
        s[i] = var(x)
        # perform blocking transformation
        x = 0.5*(x[0::2] + x[1::2])

    # generate the test observator  $M_k$  from the theorem
    M = (cumsum( ((gamma/s)**2*2**arange(1,d+1)[::-1])[:-1] ) )[:-1]

    # we need a list of magic numbers
    q = array([6.634897, 9.210340, 11.344867, 13.276704, 15.086272, 16.811894, 18.475307, 20.090235])

    # use magic to determine when we should have stopped blocking
    for k in arange(0,d):
        if (M[k] < q[k]):
            break
    if (k >= d-1):
        print("Warning: Use more data")

```

```

    return mu, s[k]/2**(d-k)

x = loadtxt(infile)
(mean, var) = block(x)
std = sqrt(var)
import pandas as pd
from pandas import DataFrame
data = {'Mean': [mean], 'STDev': [std]}
frame = pd.DataFrame(data, index=['Values'])
print(frame)

```

Why resampling methods ?

Statistical analysis.

- Our simulations can be treated as *computer experiments*. This is particularly the case for Monte Carlo methods
- The results can be analysed with the same statistical tools as we would use analysing experimental data.
- As in all experiments, we are looking for expectation values and an estimate of how accurate they are, i.e., possible sources for errors.

Statistical analysis

- As in other experiments, many numerical experiments have two classes of errors:
 - Statistical errors
 - Systematical errors
- Statistical errors can be estimated using standard tools from statistics
- Systematical errors are method specific and must be treated differently from case to case.

Statistics

The *probability distribution function (PDF)* is a function $p(x)$ on the domain which, in the discrete case, gives us the probability or relative frequency with which these values of X occur:

$$p(x) = \text{prob}(X = x)$$

In the continuous case, the PDF does not directly depict the actual probability. Instead we define the probability for the stochastic variable to assume any value

on an infinitesimal interval around x to be $p(x)dx$. The continuous function $p(x)$ then gives us the *density* of the probability rather than the probability itself. The probability for a stochastic variable to assume any value on a non-infinitesimal interval $[a, b]$ is then just the integral:

$$\text{prob}(a \leq X \leq b) = \int_a^b p(x)dx$$

Qualitatively speaking, a stochastic variable represents the values of numbers chosen as if by chance from some specified PDF so that the selection of a large set of these numbers reproduces this PDF.

Statistics, moments

A particularly useful class of special expectation values are the *moments*. The n -th moment of the PDF p is defined as follows:

$$\langle x^n \rangle \equiv \int x^n p(x) dx$$

The zero-th moment $\langle 1 \rangle$ is just the normalization condition of p . The first moment, $\langle x \rangle$, is called the *mean* of p and often denoted by the letter μ :

$$\langle x \rangle = \mu \equiv \int x p(x) dx$$

Statistics, central moments

A special version of the moments is the set of *central moments*, the n -th central moment defined as:

$$\langle (x - \langle x \rangle)^n \rangle \equiv \int (x - \langle x \rangle)^n p(x) dx$$

The zero-th and first central moments are both trivial, equal 1 and 0, respectively. But the second central moment, known as the *variance* of p , is of particular interest. For the stochastic variable X , the variance is denoted as σ_X^2 or $\text{var}(X)$:

$$\sigma_X^2 = \text{var}(X) = \langle (x - \langle x \rangle)^2 \rangle = \int (x - \langle x \rangle)^2 p(x) dx \quad (1)$$

$$= \int (x^2 - 2x\langle x \rangle + \langle x \rangle^2) p(x) dx \quad (2)$$

$$= \langle x^2 \rangle - 2\langle x \rangle \langle x \rangle + \langle x \rangle^2 \quad (3)$$

$$= \langle x^2 \rangle - \langle x \rangle^2 \quad (4)$$

The square root of the variance, $\sigma = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$ is called the *standard deviation* of p . It is clearly just the RMS (root-mean-square) value of the deviation of the PDF from its mean value, interpreted qualitatively as the *spread* of p around its mean.

Statistics, covariance

Another important quantity is the so called covariance, a variant of the above defined variance. Consider again the set $\{X_i\}$ of n stochastic variables (not necessarily uncorrelated) with the multivariate PDF $P(x_1, \dots, x_n)$. The *covariance* of two of the stochastic variables, X_i and X_j , is defined as follows:

$$\begin{aligned} \text{cov}(X_i, X_j) &\equiv \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle \\ &= \int \cdots \int (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) P(x_1, \dots, x_n) dx_1 \dots dx_n \end{aligned} \quad (5)$$

with

$$\langle x_i \rangle = \int \cdots \int x_i P(x_1, \dots, x_n) dx_1 \dots dx_n$$

Statistics, more covariance

If we consider the above covariance as a matrix $C_{ij} = \text{cov}(X_i, X_j)$, then the diagonal elements are just the familiar variances, $C_{ii} = \text{cov}(X_i, X_i) = \text{var}(X_i)$. It turns out that all the off-diagonal elements are zero if the stochastic variables are uncorrelated. This is easy to show, keeping in mind the linearity of the expectation value. Consider the stochastic variables X_i and X_j , ($i \neq j$):

$$\text{cov}(X_i, X_j) = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle \quad (6)$$

$$= \langle x_i x_j - x_i \langle x_j \rangle - \langle x_i \rangle x_j + \langle x_i \rangle \langle x_j \rangle \rangle \quad (7)$$

$$= \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle - \langle \langle x_i \rangle x_j \rangle + \langle \langle x_i \rangle \langle x_j \rangle \rangle \quad (8)$$

$$= \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle - \langle x_i \rangle \langle x_j \rangle + \langle x_i \rangle \langle x_j \rangle \quad (9)$$

$$= \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle \quad (10)$$

Statistics, independent variables

If X_i and X_j are independent, we get $\langle x_i x_j \rangle = \langle x_i \rangle \langle x_j \rangle$, resulting in $\text{cov}(X_i, X_j) = 0$ ($i \neq j$).

Also useful for us is the covariance of linear combinations of stochastic variables. Let $\{X_i\}$ and $\{Y_i\}$ be two sets of stochastic variables. Let also $\{a_i\}$ and $\{b_i\}$ be two sets of scalars. Consider the linear combination:

$$U = \sum_i a_i X_i \quad V = \sum_j b_j Y_j$$

By the linearity of the expectation value

$$\text{cov}(U, V) = \sum_{i,j} a_i b_j \text{cov}(X_i, Y_j)$$

Statistics, more variance

Now, since the variance is just $\text{var}(X_i) = \text{cov}(X_i, X_i)$, we get the variance of the linear combination $U = \sum_i a_i X_i$:

$$\text{var}(U) = \sum_{i,j} a_i a_j \text{cov}(X_i, X_j) \quad (11)$$

And in the special case when the stochastic variables are uncorrelated, the off-diagonal elements of the covariance are as we know zero, resulting in:

$$\text{var}(U) = \sum_i a_i^2 \text{cov}(X_i, X_i) = \sum_i a_i^2 \text{var}(X_i)$$

$$\text{var}\left(\sum_i a_i X_i\right) = \sum_i a_i^2 \text{var}(X_i)$$

which will become very useful in our study of the error in the mean value of a set of measurements.

Statistics and stochastic processes

A *stochastic process* is a process that produces sequentially a chain of values:

$$\{x_1, x_2, \dots, x_k, \dots\}.$$

We will call these values our *measurements* and the entire set as our measured *sample*. The action of measuring all the elements of a sample we will call a stochastic *experiment* since, operationally, they are often associated with results of empirical observation of some physical or mathematical phenomena; precisely an experiment. We assume that these values are distributed according to some PDF $p_X(x)$, where X is just the formal symbol for the stochastic variable whose PDF is $p_X(x)$. Instead of trying to determine the full distribution p we are often only interested in finding the few lowest moments, like the mean μ_X and the variance σ_X .

Statistics and sample variables

In practical situations a sample is always of finite size. Let that size be n . The expectation value of a sample, the *sample mean*, is then defined as follows:

$$\bar{x}_n \equiv \frac{1}{n} \sum_{k=1}^n x_k$$

The *sample variance* is:

$$\text{var}(x) \equiv \frac{1}{n} \sum_{k=1}^n (x_k - \bar{x}_n)^2$$

its square root being the *standard deviation of the sample*. The *sample covariance* is:

$$\text{cov}(x) \equiv \frac{1}{n} \sum_{kl} (x_k - \bar{x}_n)(x_l - \bar{x}_n)$$

Statistics, sample variance and covariance

Note that the sample variance is the sample covariance without the cross terms. In a similar manner as the covariance in Eq. (5) is a measure of the correlation between two stochastic variables, the above defined sample covariance is a measure of the sequential correlation between succeeding measurements of a sample.

These quantities, being known experimental values, differ significantly from and must not be confused with the similarly named quantities for stochastic variables, mean μ_X , variance $\text{var}(X)$ and covariance $\text{cov}(X, Y)$.

Statistics, law of large numbers

The law of large numbers states that as the size of our sample grows to infinity, the sample mean approaches the true mean μ_X of the chosen PDF:

$$\lim_{n \rightarrow \infty} \bar{x}_n = \mu_X$$

The sample mean \bar{x}_n works therefore as an estimate of the true mean μ_X .

What we need to find out is how good an approximation \bar{x}_n is to μ_X . In any stochastic measurement, an estimated mean is of no use to us without a measure of its error. A quantity that tells us how well we can reproduce it in another experiment. We are therefore interested in the PDF of the sample mean itself. Its standard deviation will be a measure of the spread of sample means, and we will simply call it the *error* of the sample mean, or just sample error, and denote it by err_X . In practice, we will only be able to produce an *estimate* of the sample error since the exact value would require the knowledge of the true PDFs behind, which we usually do not have.

Statistics, more on sample error

Let us first take a look at what happens to the sample error as the size of the sample grows. In a sample, each of the measurements x_i can be associated with its own stochastic variable X_i . The stochastic variable \bar{X}_n for the sample mean \bar{x}_n is then just a linear combination, already familiar to us:

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

All the coefficients are just equal $1/n$. The PDF of \bar{X}_n , denoted by $p_{\bar{X}_n}(x)$ is the desired PDF of the sample means.

Statistics

The probability density of obtaining a sample mean \bar{x}_n is the product of probabilities of obtaining arbitrary values x_1, x_2, \dots, x_n with the constraint that the mean of the set $\{x_i\}$ is \bar{x}_n :

$$p_{\bar{X}_n}(x) = \int p_X(x_1) \cdots \int p_X(x_n) \delta\left(x - \frac{x_1 + x_2 + \cdots + x_n}{n}\right) dx_n \cdots dx_1$$

And in particular we are interested in its variance $\text{var}(\bar{X}_n)$.

The Central Limit Theorem

Suppose we have a PDF $p(x)$ from which we generate a series N of averages $\mathbb{E}[x_i]$. Each mean value $\mathbb{E}[x_i]$ is viewed as the average of a specific measurement, e.g., throwing dice 100 times and then taking the average value, or producing a certain amount of random numbers. For notational ease, we set $\mathbb{E}[x_i] = x_i$ in the discussion which follows. We do the same for $\mathbb{E}[z] = z$.

If we compute the mean z of m such mean values x_i

$$z = \frac{x_1 + x_2 + \cdots + x_m}{m},$$

the question we pose is which is the PDF of the new variable z .

Finding the Limit

The probability of obtaining an average value z is the product of the probabilities of obtaining arbitrary individual mean values x_i , but with the constraint that the average is z . We can express this through the following expression

$$\tilde{p}(z) = \int dx_1 p(x_1) \int dx_2 p(x_2) \cdots \int dx_m p(x_m) \delta\left(z - \frac{x_1 + x_2 + \cdots + x_m}{m}\right),$$

where the δ -function embodies the constraint that the mean is z . All measurements that lead to each individual x_i are expected to be independent, which in turn means that we can express \tilde{p} as the product of individual $p(x_i)$. The independence assumption is important in the derivation of the central limit theorem.

Rewriting the δ -function

If we use the integral expression for the δ -function

$$\delta\left(z - \frac{x_1 + x_2 + \cdots + x_m}{m}\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \exp\left(iq\left(z - \frac{x_1 + x_2 + \cdots + x_m}{m}\right)\right),$$

and inserting $e^{i\mu q - i\mu q}$ where μ is the mean value we arrive at

$$\tilde{p}(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dq \exp(iq(z - \mu)) \left[\int_{-\infty}^{\infty} dx p(x) \exp(iq(\mu - x)/m) \right]^m,$$

with the integral over x resulting in

$$\int_{-\infty}^{\infty} dx p(x) \exp(iq(\mu - x)/m) = \int_{-\infty}^{\infty} dx p(x) \left[1 + \frac{iq(\mu - x)}{m} - \frac{q^2(\mu - x)^2}{2m^2} + \dots \right].$$

Identifying Terms

The second term on the rhs disappears since this is just the mean and employing the definition of σ^2 we have

$$\int_{-\infty}^{\infty} dx p(x) e^{iq(\mu - x)/m} = 1 - \frac{q^2 \sigma^2}{2m^2} + \dots,$$

resulting in

$$\left[\int_{-\infty}^{\infty} dx p(x) \exp(iq(\mu - x)/m) \right]^m \approx \left[1 - \frac{q^2 \sigma^2}{2m^2} + \dots \right]^m,$$

and in the limit $m \rightarrow \infty$ we obtain

$$\tilde{p}(z) = \frac{1}{\sqrt{2\pi}(\sigma/\sqrt{m})} \exp\left(-\frac{(z - \mu)^2}{2(\sigma/\sqrt{m})^2}\right),$$

which is the normal distribution with variance $\sigma_m^2 = \sigma^2/m$, where σ is the variance of the PDF $p(x)$ and μ is also the mean of the PDF $p(x)$.

Wrapping it up

Thus, the central limit theorem states that the PDF $\tilde{p}(z)$ of the average of m random values corresponding to a PDF $p(x)$ is a normal distribution whose mean is the mean value of the PDF $p(x)$ and whose variance is the variance of the PDF $p(x)$ divided by m , the number of values used to compute z .

The central limit theorem leads to the well-known expression for the standard deviation, given by

$$\sigma_m = \frac{\sigma}{\sqrt{m}}.$$

The latter is true only if the average value is known exactly. This is obtained in the limit $m \rightarrow \infty$ only. Because the mean and the variance are measured quantities we obtain the familiar expression in statistics (the so-called Bessel correction)

$$\sigma_m \approx \frac{\sigma}{\sqrt{m-1}}.$$

In many cases however the above estimate for the standard deviation, in particular if correlations are strong, may be too simplistic. Keep in mind that we have assumed that the variables x are independent and identically distributed. This is obviously not always the case. For example, the random numbers (or better pseudorandom numbers) we generate in various calculations do always exhibit some correlations.

The theorem is satisfied by a large class of PDFs. Note however that for a finite m , it is not always possible to find a closed form /analytic expression for $\tilde{p}(x)$.

Resampling methods: Bootstrap steps

The independent bootstrap works like this:

1. Draw with replacement n numbers for the observed variables $\mathbf{x} = (x_1, x_2, \dots, x_n)$.
2. Define a vector \mathbf{x}^* containing the values which were drawn from \mathbf{x} .
3. Using the vector \mathbf{x}^* compute $\hat{\beta}^*$ by evaluating $\hat{\beta}$ under the observations \mathbf{x}^* .
4. Repeat this process k times.

When you are done, you can draw a histogram of the relative frequency of $\hat{\beta}^*$. This is your estimate of the probability distribution $p(t)$. Using this probability distribution you can estimate any statistics thereof. In principle you never draw the histogram of the relative frequency of $\hat{\beta}^*$. Instead you use the estimators corresponding to the statistic of interest. For example, if you are interested in estimating the variance of $\hat{\beta}$, apply the estimator $\hat{\sigma}^2$ to the values $\hat{\beta}^*$.

Code example for the Bootstrap method and demonstration of central limit theorem

The following code starts with a Gaussian distribution with mean value $\mu = 100$ and variance $\sigma = 15$. We use this to generate the data used in the bootstrap analysis. The bootstrap analysis returns a data set after a given number of bootstrap operations (as many as we have data points). This data set consists of estimated mean values for each bootstrap operation. The histogram generated by the bootstrap method shows that the distribution for these mean values is also a Gaussian, centered around the mean value $\mu = 100$ but with standard deviation σ/\sqrt{n} , where n is the number of bootstrap samples (in this case the same as the number of original data points). The value of the standard deviation is what we expect from the central limit theorem.

```
import numpy as np
from time import time
from scipy.stats import norm
import matplotlib.pyplot as plt
```

```

# Returns mean of bootstrap samples
# Bootstrap algorithm
def bootstrap(data, datapoints):
    t = np.zeros(datapoints)
    n = len(data)
    # non-parametric bootstrap
    for i in range(datapoints):
        t[i] = np.mean(data[np.random.randint(0,n,n)])
    # analysis
    print("Bootstrap Statistics :")
    print("original      bias      std. error")
    print("%8g %8g %14g %15g" % (np.mean(data), np.std(data), np.mean(t), np.std(t)))
    return t

# We set the mean value to 100 and the standard deviation to 15
mu, sigma = 100, 15
datapoints = 10000
# We generate random numbers according to the normal distribution
x = mu + sigma*np.random.randn(datapoints)
# bootstrap returns the data sample
t = bootstrap(x, datapoints)

```

We see that our new variance and from that the standard deviation, agrees with the central limit theorem.

Plotting the Histogram

```

# the histogram of the bootstrapped data (normalized data if density = True)
n, binsboot, patches = plt.hist(t, 50, density=True, facecolor='red', alpha=0.75)
# add a 'best fit' line
y = norm.pdf(binsboot, np.mean(t), np.std(t))
lt = plt.plot(binsboot, y, 'b', linewidth=1)
plt.xlabel('x')
plt.ylabel('Probability')
plt.grid(True)
plt.show()

```

Statistics, more technicalities

The desired variance $\text{var}(\overline{X}_n)$, i.e. the sample error squared err_X^2 , is given by:

$$\text{err}_X^2 = \text{var}(\overline{X}_n) = \frac{1}{n^2} \sum_{ij} \text{cov}(X_i, X_j) \quad (12)$$

We see now that in order to calculate the exact error of the sample with the above expression, we would need the true means μ_{X_i} of the stochastic variables X_i . To calculate these requires that we know the true multivariate PDF of all the X_i . But this PDF is unknown to us, we have only got the measurements of one sample. The best we can do is to let the sample itself be an estimate of the PDF of each of the X_i , estimating all properties of X_i through the measurements of the sample.

Statistics

Our estimate of μ_{X_i} is then the sample mean \bar{x} itself, in accordance with the central limit theorem:

$$\mu_{X_i} = \langle x_i \rangle \approx \frac{1}{n} \sum_{k=1}^n x_k = \bar{x}$$

Using \bar{x} in place of μ_{X_i} we can give an *estimate* of the covariance in Eq. (12)

$$\text{cov}(X_i, X_j) = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle \approx \langle (x_i - \bar{x})(x_j - \bar{x}) \rangle,$$

resulting in

$$\frac{1}{n} \sum_l^n \left(\frac{1}{n} \sum_k^n (x_k - \bar{x}_n)(x_l - \bar{x}_n) \right) = \frac{1}{n} \frac{1}{n} \sum_{kl} (x_k - \bar{x}_n)(x_l - \bar{x}_n) = \frac{1}{n} \text{cov}(x)$$

Statistics and sample variance

By the same procedure we can use the sample variance as an estimate of the variance of any of the stochastic variables X_i

$$\text{var}(X_i) = \langle x_i - \langle x_i \rangle \rangle \approx \langle x_i - \bar{x}_n \rangle,$$

which is approximated as

$$\text{var}(X_i) \approx \frac{1}{n} \sum_{k=1}^n (x_k - \bar{x}_n) = \text{var}(x) \quad (13)$$

Now we can calculate an estimate of the error err_X of the sample mean \bar{x}_n :

$$\begin{aligned} \text{err}_X^2 &= \frac{1}{n^2} \sum_{ij} \text{cov}(X_i, X_j) \\ &\approx \frac{1}{n^2} \sum_{ij} \frac{1}{n} \text{cov}(x) = \frac{1}{n^2} n^2 \frac{1}{n} \text{cov}(x) \\ &= \frac{1}{n} \text{cov}(x) \end{aligned} \quad (14)$$

which is nothing but the sample covariance divided by the number of measurements in the sample.

Statistics, uncorrelated results

In the special case that the measurements of the sample are uncorrelated (equivalently the stochastic variables X_i are uncorrelated) we have that the off-diagonal elements of the covariance are zero. This gives the following estimate of the sample error:

$$\text{err}_X^2 = \frac{1}{n^2} \sum_{ij} \text{cov}(X_i, X_j) = \frac{1}{n^2} \sum_i \text{var}(X_i),$$

resulting in

$$\text{err}_X^2 \approx \frac{1}{n^2} \sum_i \text{var}(x) = \frac{1}{n} \text{var}(x) \quad (15)$$

where in the second step we have used Eq. (13). The error of the sample is then just its standard deviation divided by the square root of the number of measurements the sample contains. This is a very useful formula which is easy to compute. It acts as a first approximation to the error, but in numerical experiments, we cannot overlook the always present correlations.

Statistics, computations

For computational purposes one usually splits up the estimate of err_X^2 , given by Eq. (14), into two parts

$$\text{err}_X^2 = \frac{1}{n} \text{var}(x) + \frac{1}{n} (\text{cov}(x) - \text{var}(x)),$$

which equals

$$\frac{1}{n^2} \sum_{k=1}^n (x_k - \bar{x}_n)^2 + \frac{2}{n^2} \sum_{k < l} (x_k - \bar{x}_n)(x_l - \bar{x}_n) \quad (16)$$

The first term is the same as the error in the uncorrelated case, Eq. (15). This means that the second term accounts for the error correction due to correlation between the measurements. For uncorrelated measurements this second term is zero.

Statistics, more on computations of errors

Computationally the uncorrelated first term is much easier to treat efficiently than the second.

$$\text{var}(x) = \frac{1}{n} \sum_{k=1}^n (x_k - \bar{x}_n)^2 = \left(\frac{1}{n} \sum_{k=1}^n x_k^2 \right) - \bar{x}_n^2$$

We just accumulate separately the values x^2 and x for every measurement x we receive. The correlation term, though, has to be calculated at the end of the experiment since we need all the measurements to calculate the cross terms. Therefore, all measurements have to be stored throughout the experiment.