

Week 4 January 25-29: Metropolis Algorithm and Markov Chains, Importance Sampling, Fokker-Planck and Langevin equations

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Overview of week 4, January 25-29

Topics.

- Metropolis-Hastings sampling (Importance Sampling)
- Fokker-Planck and Langevin equations

Teaching Material, videos and written material.

- Overview video
- Lecture notes and reading assignments
- Recommended background literature

Importance sampling

We need to replace the brute force Metropolis algorithm with a walk in coordinate space biased by the trial wave function. This approach is based on the Fokker-Planck equation and the Langevin equation for generating a trajectory in coordinate space. The link between the Fokker-Planck equation and the Langevin equations are explained, only partly, in the slides below. An

excellent reference on topics like Brownian motion, Markov chains, the Fokker-Planck equation and the Langevin equation is the text by [Van Kampen](#). Here we will focus first on the implementation part first.

For a diffusion process characterized by a time-dependent probability density $P(x, t)$ in one dimension the Fokker-Planck equation reads (for one particle/walker)

$$\frac{\partial P}{\partial t} = D \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} - F \right) P(x, t),$$

where F is a drift term and D is the diffusion coefficient.

Importance sampling

The new positions in coordinate space are given as the solutions of the Langevin equation using Euler's method, namely, we go from the Langevin equation

$$\frac{\partial x(t)}{\partial t} = DF(x(t)) + \eta,$$

with η a random variable, yielding a new position

$$y = x + DF(x)\Delta t + \xi\sqrt{\Delta t},$$

where ξ is gaussian random variable and Δt is a chosen time step. The quantity D is, in atomic units, equal to 1/2 and comes from the factor 1/2 in the kinetic energy operator. Note that Δt is to be viewed as a parameter. Values of $\Delta t \in [0.001, 0.01]$ yield in general rather stable values of the ground state energy.

Importance sampling

The process of isotropic diffusion characterized by a time-dependent probability density $P(\mathbf{x}, t)$ obeys (as an approximation) the so-called Fokker-Planck equation

$$\frac{\partial P}{\partial t} = \sum_i D \frac{\partial}{\partial \mathbf{x}_i} \left(\frac{\partial}{\partial \mathbf{x}_i} - \mathbf{F}_i \right) P(\mathbf{x}, t),$$

where \mathbf{F}_i is the i^{th} component of the drift term (drift velocity) caused by an external potential, and D is the diffusion coefficient. The convergence to a stationary probability density can be obtained by setting the left hand side to zero. The resulting equation will be satisfied if and only if all the terms of the sum are equal zero,

$$\frac{\partial^2 P}{\partial \mathbf{x}_i^2} = P \frac{\partial}{\partial \mathbf{x}_i} \mathbf{F}_i + \mathbf{F}_i \frac{\partial}{\partial \mathbf{x}_i} P.$$

Importance sampling

The drift vector should be of the form $\mathbf{F} = g(\mathbf{x}) \frac{\partial P}{\partial \mathbf{x}}$. Then,

$$\frac{\partial^2 P}{\partial \mathbf{x}_i^2} = P \frac{\partial g}{\partial P} \left(\frac{\partial P}{\partial \mathbf{x}_i} \right)^2 + P g \frac{\partial^2 P}{\partial \mathbf{x}_i^2} + g \left(\frac{\partial P}{\partial \mathbf{x}_i} \right)^2.$$

The condition of stationary density means that the left hand side equals zero. In other words, the terms containing first and second derivatives have to cancel each other. It is possible only if $g = \frac{1}{P}$, which yields

$$\mathbf{F} = 2 \frac{1}{\Psi_T} \nabla \Psi_T,$$

which is known as the so-called *quantum force*. This term is responsible for pushing the walker towards regions of configuration space where the trial wave function is large, increasing the efficiency of the simulation in contrast to the Metropolis algorithm where the walker has the same probability of moving in every direction.

Importance sampling

The Fokker-Planck equation yields a (the solution to the equation) transition probability given by the Green's function

$$G(y, x, \Delta t) = \frac{1}{(4\pi D \Delta t)^{3N/2}} \exp(-(y - x - D \Delta t F(x))^2 / 4D \Delta t)$$

which in turn means that our brute force Metropolis algorithm

$$A(y, x) = \min(1, q(y, x)),$$

with $q(y, x) = |\Psi_T(y)|^2 / |\Psi_T(x)|^2$ is now replaced by the [Metropolis-Hastings algorithm](#) as well as [Hasting's article](#),

$$q(y, x) = \frac{G(x, y, \Delta t) |\Psi_T(y)|^2}{G(y, x, \Delta t) |\Psi_T(x)|^2}$$

Importance sampling, program elements

The full code is [this link](#). Here we include only the parts pertaining to the computation of the quantum force and the Metropolis update. The program is a modification of our previous c++ program discussed previously. Here we display only the part from the *vmcsolver.cpp* file. Note the usage of the function *GaussianDeviate*.

```
void VMCSolver::runMonteCarloIntegration()
{
    rOld = zeros<mat>(nParticles, nDimensions);
    rNew = zeros<mat>(nParticles, nDimensions);
    QForceOld = zeros<mat>(nParticles, nDimensions);
    QForceNew = zeros<mat>(nParticles, nDimensions);

    double waveFunctionOld = 0;
    double waveFunctionNew = 0;

    double energySum = 0;
    double energySquaredSum = 0;
```

```

double deltaE;

// initial trial positions
for(int i = 0; i < nParticles; i++) {
    for(int j = 0; j < nDimensions; j++) {
        rOld(i,j) = GaussianDeviate(&idum)*sqrt(timestep);
    }
}
rNew = rOld;

```

Importance sampling, program elements

```

for(int cycle = 0; cycle < nCycles; cycle++) {

    // Store the current value of the wave function
    waveFunctionOld = waveFunction(rOld);
    QuantumForce(rOld, QForceOld); QForceOld = QForceOld*h/waveFunctionOld;
    // New position to test
    for(int i = 0; i < nParticles; i++) {
        for(int j = 0; j < nDimensions; j++) {
            rNew(i,j) = rOld(i,j) + GaussianDeviate(&idum)*sqrt(timestep)+QForceOld(i,j)*timestep*D;
        }
        // for the other particles we need to set the position to the old position since
        // we move only one particle at the time
        for (int k = 0; k < nParticles; k++) {
            if ( k != i) {
                for (int j=0; j < nDimensions; j++) {
                    rNew(k,j) = rOld(k,j);
                }
            }
        }
    }
}

```

Importance sampling, program elements

```

// loop over Monte Carlo cycles
// Recalculate the value of the wave function and the quantum force
waveFunctionNew = waveFunction(rNew);
QuantumForce(rNew, QForceNew) = QForceNew*h/waveFunctionNew;
// we compute the log of the ratio of the greens functions to be used in the
// Metropolis-Hastings algorithm
GreensFunction = 0.0;
for (int j=0; j < nDimensions; j++) {
    GreensFunction += 0.5*(QForceOld(i,j)+QForceNew(i,j))*
        (D*timestep*0.5*(QForceOld(i,j)-QForceNew(i,j))-rNew(i,j)+rOld(i,j));
}
GreensFunction = exp(GreensFunction);

// The Metropolis test is performed by moving one particle at the time
if(ran2(&idum) <= GreensFunction*(waveFunctionNew*waveFunctionNew) / (waveFunctionOld*waveFunctionOld)) {
    for(int j = 0; j < nDimensions; j++) {
        rOld(i,j) = rNew(i,j);
        QForceOld(i,j) = QForceNew(i,j);
        waveFunctionOld = waveFunctionNew;
    }
} else {

```

```

    for(int j = 0; j < nDimensions; j++) {
        rNew(i,j) = rOld(i,j);
        QForceNew(i,j) = QForceOld(i,j);
    }
}

```

Importance sampling, program elements

Note numerical derivatives.

```

double VMCSolver::QuantumForce(const mat &r, mat &QForce)
{
    mat rPlus = zeros<mat>(nParticles, nDimensions);
    mat rMinus = zeros<mat>(nParticles, nDimensions);
    rPlus = rMinus = r;
    double waveFunctionMinus = 0;
    double waveFunctionPlus = 0;
    double waveFunctionCurrent = waveFunction(r);

    // Kinetic energy
    double kineticEnergy = 0;
    for(int i = 0; i < nParticles; i++) {
        for(int j = 0; j < nDimensions; j++) {
            rPlus(i,j) += h;
            rMinus(i,j) -= h;
            waveFunctionMinus = waveFunction(rMinus);
            waveFunctionPlus = waveFunction(rPlus);
            QForce(i,j) = (waveFunctionPlus-waveFunctionMinus);
            rPlus(i,j) = r(i,j);
            rMinus(i,j) = r(i,j);
        }
    }
}

```

Importance sampling, program elements

The general derivative formula of the Jastrow factor is (the subscript C stands for Correlation)

$$\frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_k} = \sum_{i=1}^{k-1} \frac{\partial g_{ik}}{\partial x_k} + \sum_{i=k+1}^N \frac{\partial g_{ki}}{\partial x_k}$$

However, with our written in way which can be reused later as

$$\Psi_C = \prod_{i < j} g(r_{ij}) = \exp \left\{ \sum_{i < j} f(r_{ij}) \right\},$$

the gradient needed for the quantum force and local energy is easy to compute. The function $f(r_{ij})$ will depends on the system under study. In the equations below we will keep this general form.

Importance sampling, program elements

In the Metropolis/Hasting algorithm, the *acceptance ratio* determines the probability for a particle to be accepted at a new position. The ratio of the trial wave functions evaluated at the new and current positions is given by (*OB* for the onebody part)

$$R \equiv \frac{\Psi_T^{new}}{\Psi_T^{old}} = \frac{\Psi_{OB}^{new} \Psi_C^{new}}{\Psi_{OB}^{old} \Psi_C^{old}}$$

Here Ψ_{OB} is our onebody part (Slater determinant or product of boson single-particle states) while Ψ_C is our correlation function, or Jastrow factor. We need to optimize the $\nabla\Psi_T/\Psi_T$ ratio and the second derivative as well, that is the $\nabla^2\Psi_T/\Psi_T$ ratio. The first is needed when we compute the so-called quantum force in importance sampling. The second is needed when we compute the kinetic energy term of the local energy.

$$\frac{\nabla\Psi}{\Psi} = \frac{\nabla(\Psi_{OB} \Psi_C)}{\Psi_{OB} \Psi_C} = \frac{\Psi_C \nabla\Psi_{OB} + \Psi_{OB} \nabla\Psi_C}{\Psi_{OB} \Psi_C} = \frac{\nabla\Psi_{OB}}{\Psi_{OB}} + \frac{\nabla\Psi_C}{\Psi_C}$$

Importance sampling

The expectation value of the kinetic energy expressed in atomic units for electron i is

$$\langle \hat{K}_i \rangle = -\frac{1}{2} \frac{\langle \Psi | \nabla_i^2 | \Psi \rangle}{\langle \Psi | \Psi \rangle},$$

$$\hat{K}_i = -\frac{1}{2} \frac{\nabla_i^2 \Psi}{\Psi}.$$

Importance sampling

The second derivative which enters the definition of the local energy is

$$\frac{\nabla^2 \Psi}{\Psi} = \frac{\nabla^2 \Psi_{OB}}{\Psi_{OB}} + \frac{\nabla^2 \Psi_C}{\Psi_C} + 2 \frac{\nabla \Psi_{OB}}{\Psi_{OB}} \cdot \frac{\nabla \Psi_C}{\Psi_C}$$

We discuss here how to calculate these quantities in an optimal way,

Importance sampling

We have defined the correlated function as

$$\Psi_C = \prod_{i < j} g(r_{ij}) = \prod_{i < j}^N g(r_{ij}) = \prod_{i=1}^N \prod_{j=i+1}^N g(r_{ij}),$$

with $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}$ in three dimensions or $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$ if we work with two-dimensional systems.

In our particular case we have

$$\Psi_C = \prod_{i < j} g(r_{ij}) = \exp \left\{ \sum_{i < j} f(r_{ij}) \right\}.$$

Importance sampling

The total number of different relative distances r_{ij} is $N(N-1)/2$. In a matrix storage format, the relative distances form a strictly upper triangular matrix

$$\mathbf{r} \equiv \begin{pmatrix} 0 & r_{1,2} & r_{1,3} & \cdots & r_{1,N} \\ \vdots & 0 & r_{2,3} & \cdots & r_{2,N} \\ \vdots & \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & r_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

This applies to $\mathbf{g} = \mathbf{g}(r_{ij})$ as well.

In our algorithm we will move one particle at the time, say the k th-particle. This sampling will be seen to be particularly efficient when we are going to compute a Slater determinant.

Importance sampling

We have that the ratio between Jastrow factors R_C is given by

$$R_C = \frac{\Psi_C^{\text{new}}}{\Psi_C^{\text{cur}}} = \prod_{i=1}^{k-1} \frac{g_{ik}^{\text{new}}}{g_{ik}^{\text{cur}}} \prod_{i=k+1}^N \frac{g_{ki}^{\text{new}}}{g_{ki}^{\text{cur}}}.$$

For the Pade-Jastrow form

$$R_C = \frac{\Psi_C^{\text{new}}}{\Psi_C^{\text{cur}}} = \frac{\exp U_{\text{new}}}{\exp U_{\text{cur}}} = \exp \Delta U,$$

where

$$\Delta U = \sum_{i=1}^{k-1} (f_{ik}^{\text{new}} - f_{ik}^{\text{cur}}) + \sum_{i=k+1}^N (f_{ki}^{\text{new}} - f_{ki}^{\text{cur}})$$

Importance sampling

One needs to develop a special algorithm that runs only through the elements of the upper triangular matrix \mathbf{g} and have k as an index.

The expression to be derived in the following is of interest when computing the quantum force and the kinetic energy. It has the form

$$\frac{\nabla_i \Psi_C}{\Psi_C} = \frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_i},$$

for all dimensions and with i running over all particles.

Importance sampling

For the first derivative only $N - 1$ terms survive the ratio because the g -terms that are not differentiated cancel with their corresponding ones in the denominator. Then,

$$\frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_k} = \sum_{i=1}^{k-1} \frac{1}{g_{ik}} \frac{\partial g_{ik}}{\partial x_k} + \sum_{i=k+1}^N \frac{1}{g_{ki}} \frac{\partial g_{ki}}{\partial x_k}.$$

An equivalent equation is obtained for the exponential form after replacing g_{ij} by $\exp(f_{ij})$, yielding:

$$\frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_k} = \sum_{i=1}^{k-1} \frac{\partial g_{ik}}{\partial x_k} + \sum_{i=k+1}^N \frac{\partial g_{ki}}{\partial x_k},$$

with both expressions scaling as $\mathcal{O}(N)$.

Importance sampling

Using the identity

$$\frac{\partial}{\partial x_i} g_{ij} = -\frac{\partial}{\partial x_j} g_{ij},$$

we get expressions where all the derivatives acting on the particle are represented by the *second* index of g :

$$\frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_k} = \sum_{i=1}^{k-1} \frac{1}{g_{ik}} \frac{\partial g_{ik}}{\partial x_k} - \sum_{i=k+1}^N \frac{1}{g_{ki}} \frac{\partial g_{ki}}{\partial x_i},$$

and for the exponential case:

$$\frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_k} = \sum_{i=1}^{k-1} \frac{\partial g_{ik}}{\partial x_k} - \sum_{i=k+1}^N \frac{\partial g_{ki}}{\partial x_i}.$$

Importance sampling

For correlation forms depending only on the scalar distances r_{ij} we can use the chain rule. Noting that

$$\frac{\partial g_{ij}}{\partial x_j} = \frac{\partial g_{ij}}{\partial r_{ij}} \frac{\partial r_{ij}}{\partial x_j} = \frac{x_j - x_i}{r_{ij}} \frac{\partial g_{ij}}{\partial r_{ij}},$$

we arrive at

$$\frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_k} = \sum_{i=1}^{k-1} \frac{1}{g_{ik}} \frac{\mathbf{r}_{ik}}{r_{ik}} \frac{\partial g_{ik}}{\partial r_{ik}} - \sum_{i=k+1}^N \frac{1}{g_{ki}} \frac{\mathbf{r}_{ki}}{r_{ki}} \frac{\partial g_{ki}}{\partial r_{ki}}.$$

Importance sampling

Note that for the Pade-Jastrow form we can set $g_{ij} \equiv g(r_{ij}) = e^{f(r_{ij})} = e^{f_{ij}}$ and

$$\frac{\partial g_{ij}}{\partial r_{ij}} = g_{ij} \frac{\partial f_{ij}}{\partial r_{ij}}.$$

Therefore,

$$\frac{1}{\Psi_C} \frac{\partial \Psi_C}{\partial x_k} = \sum_{i=1}^{k-1} \frac{\mathbf{r}_{ik}}{r_{ik}} \frac{\partial f_{ik}}{\partial r_{ik}} - \sum_{i=k+1}^N \frac{\mathbf{r}_{ki}}{r_{ki}} \frac{\partial f_{ki}}{\partial r_{ki}},$$

where

$$\mathbf{r}_{ij} = |\mathbf{r}_j - \mathbf{r}_i| = (x_j - x_i)\mathbf{e}_1 + (y_j - y_i)\mathbf{e}_2 + (z_j - z_i)\mathbf{e}_3$$

is the relative distance.

Importance sampling

The second derivative of the Jastrow factor divided by the Jastrow factor (the way it enters the kinetic energy) is

$$\left[\frac{\nabla^2 \Psi_C}{\Psi_C} \right]_x = 2 \sum_{k=1}^N \sum_{i=1}^{k-1} \frac{\partial^2 g_{ik}}{\partial x_k^2} + \sum_{k=1}^N \left(\sum_{i=1}^{k-1} \frac{\partial g_{ik}}{\partial x_k} - \sum_{i=k+1}^N \frac{\partial g_{ki}}{\partial x_i} \right)^2$$

Importance sampling

But we have a simple form for the function, namely

$$\Psi_C = \prod_{i < j} \exp f(r_{ij}),$$

and it is easy to see that for particle k we have

$$\frac{\nabla_k^2 \Psi_C}{\Psi_C} = \sum_{ij \neq k} \frac{(\mathbf{r}_k - \mathbf{r}_i)(\mathbf{r}_k - \mathbf{r}_j)}{r_{ki}r_{kj}} f'(r_{ki})f'(r_{kj}) + \sum_{j \neq k} \left(f''(r_{kj}) + \frac{2}{r_{kj}} f'(r_{kj}) \right)$$

Use the C++ random class for random number generations

```
// Initialize the seed and call the Mersienne algo
std::random_device rd;
std::mt19937_64 gen(rd());
// Set up the uniform distribution for x \in [[0, 1]
std::uniform_real_distribution<double> UniformNumberGenerator(0.0,1.0);
std::normal_distribution<double> Normaldistribution(0.0,1.0);
```

Use the C++ random class for RNGs, the Mersenne twister class

Finding the new position for importance sampling

```
for (int cycles = 1; cycles <= NumberMCsamples; cycles++){
    // new position
    for (int i = 0; i < NumberParticles; i++) {
        for (int j = 0; j < Dimension; j++) {
            // gaussian deviate to compute new positions using a given timestep
            NewPosition(i,j) = OldPosition(i,j) + Normaldistribution(gen)*sqrt(timestep)+OldQuantumFor
        }
    }
}
```

Use the C++ random class for RNGs, the Metropolis test

Using the uniform distribution for the Metropolis test

```
// Metropolis-Hastings algorithm
double GreensFunction = 0.0;
for (int j = 0; j < Dimension; j++) {
    GreensFunction += 0.5*(OldQuantumForce(i,j)+NewQuantumForce(i,j))*
        (D*timestep*0.5*(OldQuantumForce(i,j)-NewQuantumForce(i,j))-NewPosition(i,j)+OldPosition
}
GreensFunction = exp(GreensFunction);
// The Metropolis test is performed by moving one particle at the time
if(UniformNumberGenerator(gen) <= GreensFunction*NewWaveFunction*NewWaveFunction/OldWaveFun
    for (int j = 0; j < Dimension; j++) {
        OldPosition(i,j) = NewPosition(i,j);
        OldQuantumForce(i,j) = NewQuantumForce(i,j);
    }
    OldWaveFunction = NewWaveFunction;
}
```

Importance sampling, Fokker-Planck and Langevin equations

A stochastic process is simply a function of two variables, one is the time, the other is a stochastic variable X , defined by specifying

- the set $\{x\}$ of possible values for X ;
- the probability distribution, $w_X(x)$, over this set, or briefly $w(x)$

The set of values $\{x\}$ for X may be discrete, or continuous. If the set of values is continuous, then $w_X(x)$ is a probability density so that $w_X(x)dx$ is the probability that one finds the stochastic variable X to have values in the range $[x, x + dx]$.

Importance sampling, Fokker-Planck and Langevin equations

An arbitrary number of other stochastic variables may be derived from X . For example, any Y given by a mapping of X , is also a stochastic variable.

The mapping may also be time-dependent, that is, the mapping depends on an additional variable t

$$Y_X(t) = f(X, t).$$

The quantity $Y_X(t)$ is called a random function, or, since t often is time, a stochastic process. A stochastic process is a function of two variables, one is the time, the other is a stochastic variable X . Let x be one of the possible values of X then

$$y(t) = f(x, t),$$

is a function of t , called a sample function or realization of the process. In physics one considers the stochastic process to be an ensemble of such sample functions.

Importance sampling, Fokker-Planck and Langevin equations

For many physical systems initial distributions of a stochastic variable y tend to equilibrium distributions: $w(y, t) \rightarrow w_0(y)$ as $t \rightarrow \infty$. In equilibrium detailed balance constrains the transition rates

$$W(y \rightarrow y')w(y) = W(y' \rightarrow y)w_0(y),$$

where $W(y' \rightarrow y)$ is the probability, per unit time, that the system changes from a state $|y\rangle$, characterized by the value y for the stochastic variable Y , to a state $|y'\rangle$.

Note that for a system in equilibrium the transition rate $W(y' \rightarrow y)$ and the reverse $W(y \rightarrow y')$ may be very different.

Importance sampling, Fokker-Planck and Langevin equations

Consider, for instance, a simple system that has only two energy levels $\epsilon_0 = 0$ and $\epsilon_1 = \Delta E$.

For a system governed by the Boltzmann distribution we find (the partition function has been taken out)

$$W(0 \rightarrow 1) \exp(-\epsilon_0/kT) = W(1 \rightarrow 0) \exp(-\epsilon_1/kT)$$

We get then

$$\frac{W(1 \rightarrow 0)}{W(0 \rightarrow 1)} = \exp(-(\Delta E/kT)),$$

which goes to zero when T tends to zero.

Importance sampling, Fokker-Planck and Langevin equations

If we assume a discrete set of events, our initial probability distribution function can be given by

$$w_i(0) = \delta_{i,0},$$

and its time-development after a given time step $\Delta t = \epsilon$ is

$$w_i(t) = \sum_j W(j \rightarrow i) w_j(t=0).$$

The continuous analog to $w_i(0)$ is

$$w(\mathbf{x}) \rightarrow \delta(\mathbf{x}),$$

where we now have generalized the one-dimensional position x to a generic-dimensional vector \mathbf{x} . The Kroenecker δ function is replaced by the δ distribution function $\delta(\mathbf{x})$ at $t = 0$.

Importance sampling, Fokker-Planck and Langevin equations

The transition from a state j to a state i is now replaced by a transition to a state with position \mathbf{y} from a state with position \mathbf{x} . The discrete sum of transition probabilities can then be replaced by an integral and we obtain the new distribution at a time $t + \Delta t$ as

$$w(\mathbf{y}, t + \Delta t) = \int W(\mathbf{y}, t + \Delta t | \mathbf{x}, t) w(\mathbf{x}, t) d\mathbf{x},$$

and after m time steps we have

$$w(\mathbf{y}, t + m\Delta t) = \int W(\mathbf{y}, t + m\Delta t | \mathbf{x}, t) w(\mathbf{x}, t) d\mathbf{x}.$$

When equilibrium is reached we have

$$w(\mathbf{y}) = \int W(\mathbf{y} | \mathbf{x}, t) w(\mathbf{x}) d\mathbf{x},$$

that is no time-dependence. Note our change of notation for W

Importance sampling, Fokker-Planck and Langevin equations

We can solve the equation for $w(\mathbf{y}, t)$ by making a Fourier transform to momentum space. The PDF $w(\mathbf{x}, t)$ is related to its Fourier transform $\tilde{w}(\mathbf{k}, t)$ through

$$w(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\mathbf{k} \exp(i\mathbf{k}\mathbf{x}) \tilde{w}(\mathbf{k}, t),$$

and using the definition of the δ -function

$$\delta(\mathbf{x}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\mathbf{k} \exp(i\mathbf{k}\mathbf{x}),$$

we see that

$$\tilde{w}(\mathbf{k}, 0) = 1/2\pi.$$

Importance sampling, Fokker-Planck and Langevin equations

We can then use the Fourier-transformed diffusion equation

$$\frac{\partial \tilde{w}(\mathbf{k}, t)}{\partial t} = -D\mathbf{k}^2 \tilde{w}(\mathbf{k}, t),$$

with the obvious solution

$$\tilde{w}(\mathbf{k}, t) = \tilde{w}(\mathbf{k}, 0) \exp[-(D\mathbf{k}^2 t)] = \frac{1}{2\pi} \exp[-(D\mathbf{k}^2 t)].$$

Importance sampling, Fokker-Planck and Langevin equations

With the Fourier transform we obtain

$$w(\mathbf{x}, t) = \int_{-\infty}^{\infty} d\mathbf{k} \exp[i\mathbf{k}\mathbf{x}] \frac{1}{2\pi} \exp[-(D\mathbf{k}^2 t)] = \frac{1}{\sqrt{4\pi Dt}} \exp[-(\mathbf{x}^2/4Dt)],$$

with the normalization condition

$$\int_{-\infty}^{\infty} w(\mathbf{x}, t) d\mathbf{x} = 1.$$

Importance sampling, Fokker-Planck and Langevin equations

The solution represents the probability of finding our random walker at position \mathbf{x} at time t if the initial distribution was placed at $\mathbf{x} = 0$ at $t = 0$.

There is another interesting feature worth observing. The discrete transition probability W itself is given by a binomial distribution. The results from the central limit theorem state that transition probability in the limit $n \rightarrow \infty$ converges to the normal distribution. It is then possible to show that

$$W(il - jl, n\epsilon) \rightarrow W(\mathbf{y}, t + \Delta t | \mathbf{x}, t) = \frac{1}{\sqrt{4\pi D \Delta t}} \exp[-((\mathbf{y} - \mathbf{x})^2 / 4D \Delta t)],$$

and that it satisfies the normalization condition and is itself a solution to the diffusion equation.

Importance sampling, Fokker-Planck and Langevin equations

Let us now assume that we have three PDFs for times $t_0 < t' < t$, that is $w(\mathbf{x}_0, t_0)$, $w(\mathbf{x}', t')$ and $w(\mathbf{x}, t)$. We have then

$$w(\mathbf{x}, t) = \int_{-\infty}^{\infty} W(\mathbf{x}.t|\mathbf{x}'.t')w(\mathbf{x}', t')d\mathbf{x}',$$

and

$$w(\mathbf{x}, t) = \int_{-\infty}^{\infty} W(\mathbf{x}.t|\mathbf{x}_0.t_0)w(\mathbf{x}_0, t_0)d\mathbf{x}_0,$$

and

$$w(\mathbf{x}', t') = \int_{-\infty}^{\infty} W(\mathbf{x}'.t'|\mathbf{x}_0.t_0)w(\mathbf{x}_0, t_0)d\mathbf{x}_0.$$

Importance sampling, Fokker-Planck and Langevin equations

We can combine these equations and arrive at the famous Einstein-Smoluchenski-Kolmogorov-Chapman (ESKC) relation

$$W(\mathbf{x}t|\mathbf{x}_0t_0) = \int_{-\infty}^{\infty} W(\mathbf{x}, t|\mathbf{x}', t')W(\mathbf{x}', t'|\mathbf{x}_0, t_0)d\mathbf{x}'.$$

We can replace the spatial dependence with a dependence upon say the velocity (or momentum), that is we have

$$W(\mathbf{v}, t|\mathbf{v}_0, t_0) = \int_{-\infty}^{\infty} W(\mathbf{v}, t|\mathbf{v}', t')W(\mathbf{v}', t'|\mathbf{v}_0, t_0)d\mathbf{x}'.$$

Importance sampling, Fokker-Planck and Langevin equations

We will now derive the Fokker-Planck equation. We start from the ESKC equation

$$W(\mathbf{x}, t|\mathbf{x}_0, t_0) = \int_{-\infty}^{\infty} W(\mathbf{x}, t|\mathbf{x}', t')W(\mathbf{x}', t'|\mathbf{x}_0, t_0)d\mathbf{x}'.$$

Define $s = t' - t_0$, $\tau = t - t'$ and $t - t_0 = s + \tau$. We have then

$$W(\mathbf{x}, s + \tau|\mathbf{x}_0) = \int_{-\infty}^{\infty} W(\mathbf{x}, \tau|\mathbf{x}')W(\mathbf{x}', s|\mathbf{x}_0)d\mathbf{x}'.$$

Importance sampling, Fokker-Planck and Langevin equations

Assume now that τ is very small so that we can make an expansion in terms of a small step xi , with $\mathbf{x}' = \mathbf{x} - \xi$, that is

$$W(\mathbf{x}, s|\mathbf{x}_0) + \frac{\partial W}{\partial s} \tau + O(\tau^2) = \int_{-\infty}^{\infty} W(\mathbf{x}, \tau|\mathbf{x} - \xi) W(\mathbf{x} - \xi, s|\mathbf{x}_0) d\mathbf{x}'.$$

We assume that $W(\mathbf{x}, \tau|\mathbf{x} - \xi)$ takes non-negligible values only when ξ is small. This is just another way of stating the Master equation!!

Importance sampling, Fokker-Planck and Langevin equations

We say thus that \mathbf{x} changes only by a small amount in the time interval τ . This means that we can make a Taylor expansion in terms of ξ , that is we expand

$$W(\mathbf{x}, \tau|\mathbf{x} - \xi) W(\mathbf{x} - \xi, s|\mathbf{x}_0) = \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} \frac{\partial^n}{\partial x^n} [W(\mathbf{x} + \xi, \tau|\mathbf{x}) W(\mathbf{x}, s|\mathbf{x}_0)].$$

Importance sampling, Fokker-Planck and Langevin equations

We can then rewrite the ESKC equation as

$$\frac{\partial W}{\partial s} \tau = -W(\mathbf{x}, s|\mathbf{x}_0) + \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} \frac{\partial^n}{\partial x^n} \left[W(\mathbf{x}, s|\mathbf{x}_0) \int_{-\infty}^{\infty} \xi^n W(\mathbf{x} + \xi, \tau|\mathbf{x}) d\xi \right].$$

We have neglected higher powers of τ and have used that for $n = 0$ we get simply $W(\mathbf{x}, s|\mathbf{x}_0)$ due to normalization.

Importance sampling, Fokker-Planck and Langevin equations

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Importance sampling, Fokker-Planck and Langevin equations

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$$\frac{\partial W(\mathbf{x}, s|\mathbf{x}_0)}{\partial s} \tau = -W(\mathbf{x}, s|\mathbf{x}_0) + \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} \frac{\partial^n}{\partial x^n} \left[W(\mathbf{x}, s|\mathbf{x}_0) \int_{-\infty}^{\infty} \xi^n W(\mathbf{x} + \xi, \tau|\mathbf{x}) d\xi \right].$$

We have neglected higher powers of τ and have used that for $n = 0$ we get simply $W(\mathbf{x}, s|\mathbf{x}_0)$ due to normalization.

Importance sampling, Fokker-Planck and Langevin equations

We simplify the above by introducing the moments

$$M_n = \frac{1}{\tau} \int_{-\infty}^{\infty} \xi^n W(\mathbf{x} + \xi, \tau|\mathbf{x}) d\xi = \frac{\langle [\Delta x(\tau)]^n \rangle}{\tau},$$

resulting in

$$\frac{\partial W(\mathbf{x}, s|\mathbf{x}_0)}{\partial s} = \sum_{n=1}^{\infty} \frac{(-\xi)^n}{n!} \frac{\partial^n}{\partial x^n} [W(\mathbf{x}, s|\mathbf{x}_0) M_n].$$

Importance sampling, Fokker-Planck and Langevin equations

When $\tau \rightarrow 0$ we assume that $\langle [\Delta x(\tau)]^n \rangle \rightarrow 0$ more rapidly than τ itself if $n > 2$. When τ is much larger than the standard correlation time of system then M_n for $n > 2$ can normally be neglected. This means that fluctuations become negligible at large time scales.

If we neglect such terms we can rewrite the ESKC equation as

$$\frac{\partial W(\mathbf{x}, s|\mathbf{x}_0)}{\partial s} = -\frac{\partial M_1 W(\mathbf{x}, s|\mathbf{x}_0)}{\partial x} + \frac{1}{2} \frac{\partial^2 M_2 W(\mathbf{x}, s|\mathbf{x}_0)}{\partial x^2}.$$

Importance sampling, Fokker-Planck and Langevin equations

In a more compact form we have

$$\frac{\partial W}{\partial s} = -\frac{\partial M_1 W}{\partial x} + \frac{1}{2} \frac{\partial^2 M_2 W}{\partial x^2},$$

which is the Fokker-Planck equation! It is trivial to replace position with velocity (momentum).

Importance sampling, Fokker-Planck and Langevin equations

Langevin equation. Consider a particle suspended in a liquid. On its path through the liquid it will continuously collide with the liquid molecules. Because on average the particle will collide more often on the front side than on the back side, it will experience a systematic force proportional with its velocity, and directed opposite to its velocity. Besides this systematic force the particle will experience a stochastic force $\mathbf{F}(t)$. The equations of motion are

- $\frac{d\mathbf{r}}{dt} = \mathbf{v}$ and
- $\frac{d\mathbf{v}}{dt} = -\xi\mathbf{v} + \mathbf{F}$.

Importance sampling, Fokker-Planck and Langevin equations

Langevin equation. From hydrodynamics we know that the friction constant ξ is given by

$$\xi = 6\pi\eta a/m$$

where η is the viscosity of the solvent and a is the radius of the particle .

Solving the second equation in the previous slide we get

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-\xi t} + \int_0^t d\tau e^{-\xi(t-\tau)} \mathbf{F}(\tau).$$

Importance sampling, Fokker-Planck and Langevin equations

Langevin equation. If we want to get some useful information out of this, we have to average over all possible realizations of $\mathbf{F}(t)$, with the initial velocity as a condition. A useful quantity for example is

$$\begin{aligned} \langle \mathbf{v}(t) \cdot \mathbf{v}(t) \rangle_{\mathbf{v}_0} &= v_0^{-\xi 2t} + 2 \int_0^t d\tau e^{-\xi(2t-\tau)} \mathbf{v}_0 \cdot \langle \mathbf{F}(\tau) \rangle_{\mathbf{v}_0} \\ &+ \int_0^t d\tau' \int_0^t d\tau e^{-\xi(2t-\tau-\tau')} \langle \mathbf{F}(\tau) \cdot \mathbf{F}(\tau') \rangle_{\mathbf{v}_0}. \end{aligned}$$

Importance sampling, Fokker-Planck and Langevin equations

Langevin equation. In order to continue we have to make some assumptions about the conditional averages of the stochastic forces. In view of the chaotic character of the stochastic forces the following assumptions seem to be appropriate

$$\langle \mathbf{F}(t) \rangle = 0,$$

and

$$\langle \mathbf{F}(t) \cdot \mathbf{F}(t') \rangle_{\mathbf{v}_0} = C_{\mathbf{v}_0} \delta(t - t').$$

We omit the subscript \mathbf{v}_0 , when the quantity of interest turns out to be independent of \mathbf{v}_0 . Using the last three equations we get

$$\langle \mathbf{v}(t) \cdot \mathbf{v}(t) \rangle_{\mathbf{v}_0} = v_0^2 e^{-2\xi t} + \frac{C_{\mathbf{v}_0}}{2\xi} (1 - e^{-2\xi t}).$$

For large t this should be equal to $3kT/m$, from which it follows that

$$\langle \mathbf{F}(t) \cdot \mathbf{F}(t') \rangle = 6 \frac{kT}{m} \xi \delta(t - t').$$

This result is called the fluctuation-dissipation theorem .

Importance sampling, Fokker-Planck and Langevin equations

Langevin equation. Integrating

$$\mathbf{v}(t) = \mathbf{v}_0 e^{-\xi t} + \int_0^t d\tau e^{-\xi(t-\tau)} \mathbf{F}(\tau),$$

we get

$$\mathbf{r}(t) = \mathbf{r}_0 + \mathbf{v}_0 \frac{1}{\xi} (1 - e^{-\xi t}) + \int_0^t d\tau \int_0^\tau \tau' e^{-\xi(\tau-\tau')} \mathbf{F}(\tau'),$$

from which we calculate the mean square displacement

$$\langle (\mathbf{r}(t) - \mathbf{r}_0)^2 \rangle_{\mathbf{v}_0} = \frac{v_0^2}{\xi} (1 - e^{-\xi t})^2 + \frac{3kT}{m\xi^2} (2\xi t - 3 + 4e^{-\xi t} - e^{-2\xi t}).$$

Importance sampling, Fokker-Planck and Langevin equations

Langevin equation. For very large t this becomes

$$\langle (\mathbf{r}(t) - \mathbf{r}_0)^2 \rangle = \frac{6kT}{m\xi} t$$

from which we get the Einstein relation

$$D = \frac{kT}{m\xi}$$

where we have used $\langle (\mathbf{r}(t) - \mathbf{r}_0)^2 \rangle = 6Dt$.