Broyden's Algorithm for Solving Nonlinear Equations

Broyden's algorithm is a quasi-Newton method used to solve systems of nonlinear equations. Unlike Newton's method, which requires the computation of the Jacobian matrix at each iteration, Broyden's method approximates the Jacobian (or its inverse) to reduce computational cost. This makes it particularly useful for high-dimensional problems where computing the exact Jacobian is expensive.

Problem Formulation

Consider a system of n nonlinear equations:

$$\mathbf{F}(\mathbf{x}) = \mathbf{0},$$

where $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function, and $\mathbf{x} \in \mathbb{R}^n$ is the vector of unknowns. The goal is to find $\mathbf{x}^*$ such that $\mathbf{F}(\mathbf{x}^*) = \mathbf{0}$.

Newton's Method

Newton's method iteratively updates the solution as:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{J}_k^{-1}\mathbf{F}(\mathbf{x}_k),$$

where $\mathbf{J}_k = \mathbf{J}(\mathbf{x}_k)$ is the Jacobian matrix of $\mathbf{F}$ evaluated at $\mathbf{x}_k$. However, computing $\mathbf{J}_k$ and its inverse at each iteration can be computationally expensive.

Broyden's Method

Broyden's method approximates the Jacobian (or its inverse) to avoid recomputing it at every iteration. There are two variants of Broyden's method:

1. **Broyden's Good Method**: Updates an approximation of the Jacobian matrix.
2. **Broyden's Bad Method**: Updates an approximation of the inverse Jacobian matrix.

Broyden's Good Method

Let $\mathbf{B}_k$ be the approximation of the Jacobian $\mathbf{J}_k$ at iteration k. The update rule for $\mathbf{B}_k$ is:

$$\mathbf{B}_{k+1} = \mathbf{B}_k + \frac{(\mathbf{F}(\mathbf{x}_{k+1}) - \mathbf{F}(\mathbf{x}_k) - \mathbf{B}_k\mathbf{s}_k)\mathbf{s}_k^\top}{\mathbf{s}_k^\top\mathbf{s}_k},$$

where $\mathbf{s}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$ is the step vector. The new approximation $\mathbf{B}_{k+1}$ satisfies the **secant equation**:

$$\mathbf{B}_{k+1}\mathbf{s}_k = \mathbf{F}(\mathbf{x}_{k+1}) - \mathbf{F}(\mathbf{x}_k).$$

The solution is updated as:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{B}_k^{-1}\mathbf{F}(\mathbf{x}_k).$$

Broyden's Bad Method

Let $\mathbf{H}_k$ be the approximation of the inverse Jacobian $\mathbf{J}_k^{-1}$ at iteration k. The update rule for $\mathbf{H}_k$ is:

$$\mathbf{H}_{k+1} = \mathbf{H}_k + \frac{(\mathbf{s}_k - \mathbf{H}_k\mathbf{y}_k)\mathbf{s}_k^\top \mathbf{H}_k}{\mathbf{s}_k^\top \mathbf{H}_k \mathbf{y}_k},$$

where $\mathbf{y}_k = \mathbf{F}(\mathbf{x}_{k+1}) - \mathbf{F}(\mathbf{x}_k)$. The new approximation $\mathbf{H}_{k+1}$ satisfies the **inverse secant equation**:

$$\mathbf{H}_{k+1}\mathbf{y}_k = \mathbf{s}_k.$$

The solution is updated as:

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{H}_k\mathbf{F}(\mathbf{x}_k).$$

Algorithm Steps

The steps of Broyden's algorithm (Good Method) are as follows:

1. Initialize $\mathbf{x}_0$ and $\mathbf{B}_0$ (e.g., $\mathbf{B}_0 = \mathbf{I}$, the identity matrix).
2. For $k = 0, 1, 2, \ldots$:
   (a) Compute the step: $\mathbf{s}_k = -\mathbf{B}_k^{-1}\mathbf{F}(\mathbf{x}_k)$.
   (b) Update the solution: $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{s}_k$.
   (c) Compute $\mathbf{y}_k = \mathbf{F}(\mathbf{x}_{k+1}) - \mathbf{F}(\mathbf{x}_k)$.
   (d) Update the Jacobian approximation:
   $$\mathbf{B}_{k+1} = \mathbf{B}_k + \frac{(\mathbf{y}_k - \mathbf{B}_k\mathbf{s}_k)\mathbf{s}_k^\top}{\mathbf{s}_k^\top \mathbf{s}_k}.$$
   (e) Check for convergence: If $\|\mathbf{F}(\mathbf{x}_{k+1})\| < \epsilon$, stop.

Advantages and Limitations

Advantages

- Avoids the need to compute the exact Jacobian at each iteration.
- Suitable for high-dimensional problems where computing the Jacobian is expensive.

Limitations

- Convergence is not guaranteed for all problems. The approximation of the Jacobian may become inaccurate over time, requiring periodic reinitialization.

Applications

Broyden's algorithm is widely used in:

- Optimization problems.
- Solving systems of nonlinear equations in scientific computing.
- Machine learning for training certain types of models.