FYS5419, FEB 20, 2023

Spectral decomposition

$$\hat{A} = \sum_a a |a\rangle\langle a|$$

$$f(\hat{A}) = \sum_a f(\hat{A}) |a\rangle\langle a|$$

Trace $\quad tr[ABC] = tr[BAC]$

$$= tr[CBA]$$

$$tr[\hat{A}|\psi\rangle\langle\psi|] = \langle\psi|\hat{A}|\psi\rangle$$

Density matrix/operator

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$$|\psi_i\rangle \rightarrow u|\psi_i\rangle \qquad uu^\dagger = u^\dagger u$$

$$= \mathbb{1}$$

$$Tr\,\rho = 1$$

$$u\rho u^\dagger = \sum_i p_i\, u|\psi_i\rangle\langle\psi_i|u^\dagger$$

$$Tr[u\rho u^\dagger] = tr[\underline{uu^\dagger}\rho]$$

$$= tr\,\rho = 1$$

Let ρ_{AB} be a density of a bipartite system given by $\mathcal{H}_A \otimes \mathcal{H}_B$. $|i\rangle_B$ is an ONB for $\mathcal{H}_B$

The partial trace over the Hilbert space $\mathcal{H}_B$ is given

$$tr_B[\rho_{AB}] = \sum_i [I_A \otimes {}_B\langle i|]$$

$$\rho_{AB} [\mathbb{I}_A \otimes |i\rangle_B]$$

It produces $\rho_A = tr_B(\rho_{AB})$ is called a reduced state or marginal of system A.

$$p(x,y) = p(x)p(y)$$

$$p(x) = \int_{y \in D} p(x)p(y)\,dy$$

$$\int p(y)\,dy = 1$$

$$tr[\rho_A] = 1 = tr[\rho_B]$$

$$\rho = \sum_i p_i |i\rangle\langle i| \qquad \sum_i p_i = 1$$

Example

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}[|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B]$$

$$= \frac{1}{\sqrt{2}} \left[|00\rangle + |11\rangle \right]$$

Schmidt decomposition with

$d = 2 \qquad \lambda_1 = \lambda_2 = \frac{1}{\sqrt{2}}$

$$|\psi\rangle = \sum_i \lambda_i |i\rangle\langle i|$$

$$\lambda_1^2 + \lambda_2^2 = \underline{1}$$

$$\rho = |\phi^+\rangle_{AB}\langle\phi^+|$$

$$= \frac{1}{2}\left[|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \right]$$

$$\frac{1}{2}\left(\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{|00\rangle\langle 00|} + \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

we want to trace out
qubit 0 and 1 of system A.

$$|0\rangle_A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left(I_A \otimes \underset{B}{\langle i|} \right)$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle_A = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$tr_B(\rho_{AB})$$

qubit 0:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

qubit 1 for system

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho_A = \mathrm{tr}_B(\rho_{AB}) = \frac{1}{2}\mathbb{1}$$

$$\mathrm{tr}\,\rho_A = \frac{1}{2} + \frac{1}{2} \left(= \mathrm{tr}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{2} \right)$$

$$= \underline{1}$$

$$\mathrm{tr}(\rho_A^2) = \frac{1}{2} < \underline{1} \quad \text{mixed}$$

$$\mathrm{tr}(\rho_A^2) = 1, \text{ pure state}$$

$$\rho_B = \rho_A = \frac{1}{2}\mathbb{1}$$

$$\rho_{AB} \neq \rho_A \otimes \rho_B$$

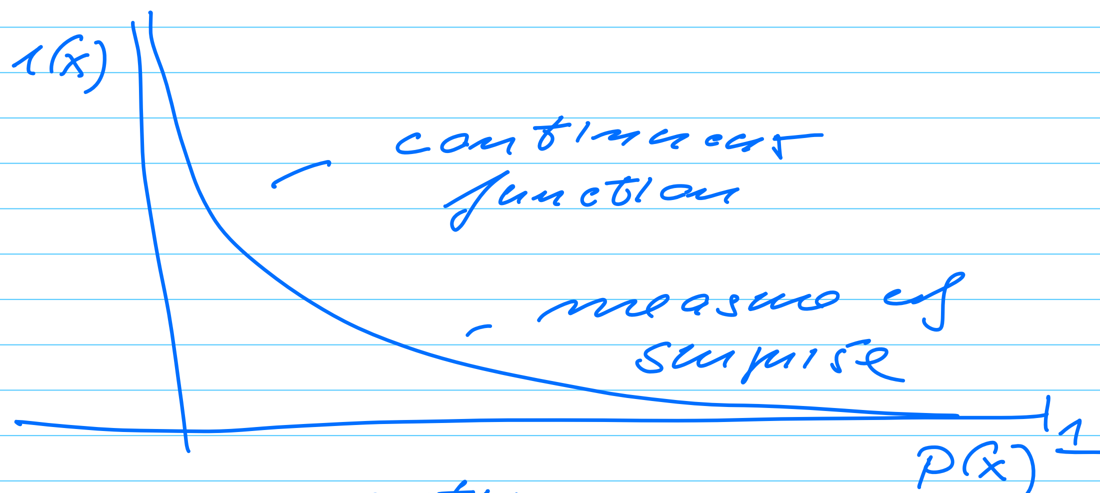
The joint state of 2 qubits (entangled or not) is a pure state, it is known exactly.

However looking at the individual qubits of for example the entangled Bell state, we find they are in a mixed state. We do not have the full knowledge of their states.

Entropy

Consider a random variable $x \in X$ with probability $P_X(x)$. The information content

$$i(x) = -\log_2 P_X(x)$$



non-negative

$$P_X(x_1 x_2) = P_X(x_1) P_X(x_2)$$

i(x) is additive

$$i(x_1, x_2) = -\log_2 (P_X(x_1) P_X(x_2))$$
$$= -\log_2 P_X(x_1) - \log_2 P_X(x_2)$$
$$= i(x_1) + i(x_2)$$

Better measure is Shannon entropy

$$S(X) = -\sum_x P_X(x) \log_2 (P_X(x))$$

$$\lim_{\varepsilon \to 0} \varepsilon \cdot \log_2 \varepsilon = 0$$

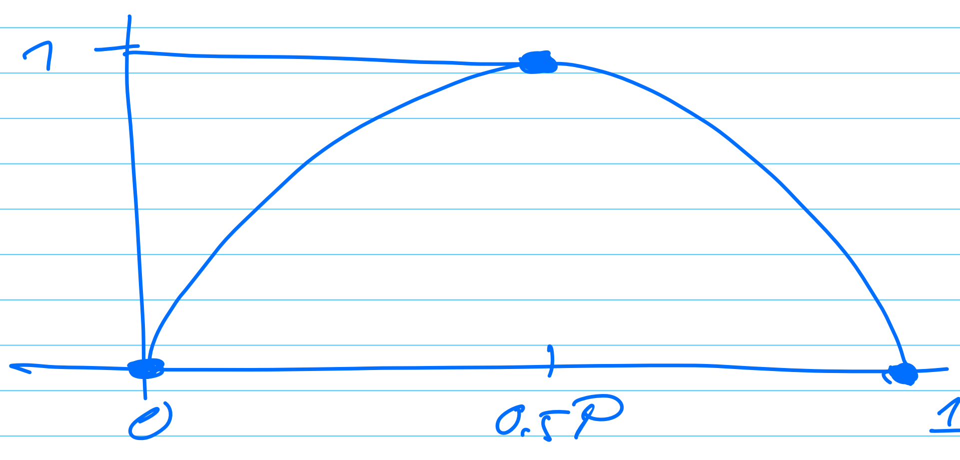
$$0 \cdot \log_2 0 = 0$$

Example of binary system

$$X = [0, 1]$$

$$P_X(0) = p \qquad P_X(1) = 1 - p$$

$$S(X) = -p \log_2 p - (1-p) \log_2 (1-p)$$



(i) $S(X) \geq 0$

(ii) concave function in $P_X(x)$

(iii) $S(X)$ when $p(x) = 1$ or $p(x) = 0$. No uncertainty about deterministic variables.

(iv) $S(X) \geq \log |X|$

Quantum entropies

von Neumann entropy is a generalization of the Shannon entropy to quantum systems

The unit of information is going to be quantum bits.

Definition: Let A be a quantum system that is prepared in a state ρ_A (hint Schmidt decomposition)
$\rho_A \in \mathcal{H}_A$, the von-Neumann entropy is defined as

$$S(A) = -\mathrm{Tr}[\rho_A \log(\rho_A)]$$

Spectral decomposition

$$\rho_A = \sum_j \lambda_j |\psi_j\rangle_A \langle \psi_j|$$

λ_j are the eigenvalues (probabilities) and $|\psi_j\rangle_A$ are the corresponding orthogonal eigenvector.

ρ_A is a semi-positive definite matrix and is always diagonalizable

This means

$$\rho_A = U^{-1} \rho_A' U$$

$$= U^{-1} D_A U$$

$$U^{-1} U = U^{+} U = \mathbb{1}$$

$$U^{-1} \rho_A \underset{U U^{-1}}{\uparrow} \log \rho_A U$$

$$= D_A \log D_A$$

$$\log D_A = \begin{bmatrix} \log \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \log(\lambda_{n-1}) \end{bmatrix}$$

$$\mathrm{tr}\, D_A = \sum_{i=0}^{n-1} \lambda_i$$

$$\mathrm{tr}\left[D_A \log D_A\right]$$

$$= \sum_i \lambda_i \log \lambda_i$$

$$S(A) = -\operatorname{tr}(\rho_A \log \rho_A)$$

$$= -\sum \lambda_i \log \lambda_i$$

probabilities!

Shannon entropy

$$S(\cancel{X}) = -\sum_x P_X(x) \log P_X(x)$$

(i) $S(A) \geq 0$

(ii) $S(A) = 0$ if density matrix is a pure state.