FYS 5419, FEB 13, 2023

Entanglement & Entropies

Projective measurement is a Hermitian M with spectral decomposition

$$M = \sum_x x P_x$$

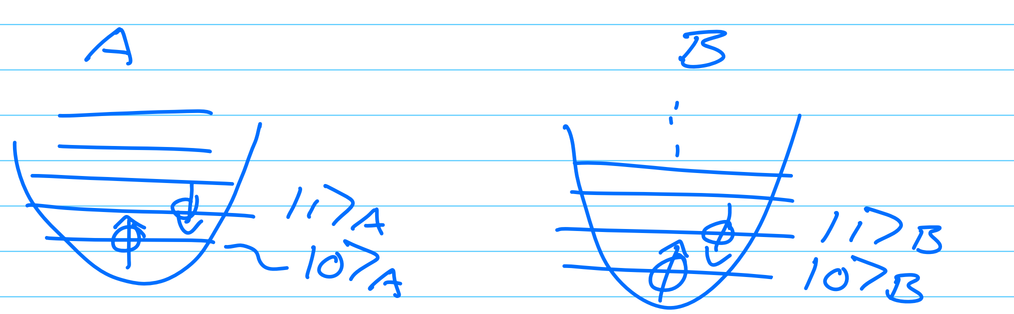
x = eigenvalues of M and P_x form a set of orthogonal projectors $\sum_x P_x = \mathbb{1}$

$$P_x P_{x'} = \delta_{xx'} P_x$$

Composite systems and Entanglement.

$|0\rangle$ and $|1\rangle$

system A and B



$$|0\rangle_A \otimes |0\rangle_B = \begin{bmatrix}1\\0\end{bmatrix}_A \otimes \begin{bmatrix}1\\0\end{bmatrix}_B$$

$$(\mathcal{H}_A \otimes \mathcal{H}_B)$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \quad \{|0\rangle\}$$

$$|0\rangle_A \otimes |1\rangle_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle \quad \{|1\rangle\}$$

$$|1\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle \quad \{|2\rangle\}$$

$$|1\rangle_A \otimes |1\rangle_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle \quad \{|3\rangle\}$$

$$\langle 11|01\rangle = 0 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$= \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

Introduce Bell states

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

$$|\phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi^+\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$|\psi^-\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

Bell states form an ONB of a two-qubit system (Bipartite)

Measuring one of the bits of a Bell state automatically determines the second bit

$|\phi^+\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$. Make a measurement on qubit in A

$$M_0 = |0\rangle_A\langle 0|_A \otimes I_B$$

$$M_1 = |1\rangle_A\langle 1|_A \otimes I_B$$

Probability of 0

$$P_{\phi^+}(0) = {}_{AB}\langle \phi^+ | M_0 | \phi^+\rangle_{AB}$$

$$\frac{1}{2}\left({}_A\langle 0| \langle 0|_B + {}_A\langle 1|{}_B\langle 1| \right) |0\rangle_A\langle 0|_A \otimes I_B$$

$$\left(|0\rangle_A \langle 0|_B + |1\rangle_A \langle 1|_B \right) = \frac{1}{2}$$

$$P_{\phi^+}(1) = 1/2$$

The state after the measurement

$$|\psi_0'\rangle = \sqrt{2}\left(|0\rangle_A\langle 0|_A \otimes I_B \right) |\phi^+\rangle_{AB}$$

$$= |0\rangle_A |0\rangle_B = |00\rangle_{AB}$$

$$= |00\rangle$$

$$|\psi_1'\rangle = \sqrt{2}\left(|1\rangle_A\langle 1|_A \otimes I_B \right)$$

$$\times |\phi^+\rangle_{AB} = |1\rangle_A \otimes |1\rangle_B$$

$$= |11\rangle_{AB} = |11\rangle$$

The state of the second (B) qubit is determined even though the measurement has only taken place locally on system A.

Entanglement

$$|00\rangle = |0\rangle_A \otimes |0\rangle_B = |00\rangle_{AB}$$

pure state

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left[|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right]$$

$$= \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right]$$

We cannot say in which state system A or system B are in. This Bell state is a superposition of $|00\rangle_{AB}$ + $|11\rangle_{AB}$. Cannot determine the individual states. This defines an entangled states.

Definition: A pure bipartite state $|\psi\rangle_{AB}$ is entangled if it cannot be written a a product state $|\varphi\rangle_A \otimes |\psi\rangle_B$. Otherwise we say that it separable

Definition of maximally

entangled state

$$|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$$

Define a density matrix ρ_{AB}, and a system is called separable if and only if its density matrix can be written as

$$\rho_{AB} = \sum_x p_x \psi_A \otimes \varphi_B$$

$\uparrow$ probability p_x

else we call the system ρ_{AB} for entangled.

$$\rho_{AB} = \rho_A \otimes \rho_B$$

$$= \left(\sum_i p_i |\varphi_i\rangle\langle\varphi_i|\right)_A \left(\sum_j q_j |\psi_j\rangle\langle\psi_j|\right)_B$$

Schmidt decomposition:

Let $|\psi\rangle$ be a pure state

in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$

$$|\psi\rangle = \sum_{i=1}^{d} \lambda_i |i\rangle_A |i\rangle_B$$

where the amplitudes λ_i are real $\sum \lambda_i^2 = \underline{1}$

λ_i = Schmidt coeffs.

$|i\rangle_A$ and $|i\rangle_B$ are ONBs

$$d \leq \min \{ \dim(\mathcal{H}_A), \dim(\mathcal{H}_B) \}$$

$$\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$$

Singular value decomposition

$$C = U \Sigma V^{+}$$

Consider an arbitrary bipartite pure state which can be written

$$|\psi\rangle = \sum_{jk} c_{jk} |j\rangle_A |k\rangle_B$$

with orthonormal basis states $|j\rangle_A$ and $|k\rangle_B$

c_{jk} are the matrix elements of C $\quad C \in \mathbb{C}^{d \times d}$

singular value decomp

$$C_{jk} = \sum_i u_{ji} \lambda_i v_{ik}$$

$$\lambda_i \geq 0$$

$$C = U \Sigma V^{+}$$

$$\Sigma = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{bmatrix} \quad \lambda_i \geq 0$$

$$|\psi\rangle = \sum_{jk} \left(\sum_i u_{ji} \lambda_i v_{ik} \right) |j\rangle_A |k\rangle_B$$

$$= \sum_i \lambda_i \underbrace{\left(\sum_j u_{ji} |j\rangle_A \right)}_{|i\rangle_A} \otimes \underbrace{\left(\sum_k v_{ik} |k\rangle_B \right)}_{|i\rangle_B}$$

$$= \sum_i \lambda_i |i\rangle_A |i\rangle_B$$

$|i\rangle_A$ and $|i\rangle_B$ are orthonormal bases for A and B.

Example system

4x4 Hamiltonian H with computational eigenstates

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

$$H|00\rangle \neq E_{00}|00\rangle$$

$$|\psi\rangle_{00} = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$