FYS 5419, MARCH 6, 2023

1-qubit gates

$|\varphi\rangle$ —— $\boxed{G}$ —— $|\varphi'\rangle$

$$|\varphi'\rangle = G|\varphi\rangle$$

$$|\varphi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

$\sigma_x = X$ flips a qubit

$G = X$; $X|0\rangle = |1\rangle$

$X|\varphi\rangle = \alpha_0|1\rangle + \alpha_1|0\rangle$

$\sigma_y = Y$ flips a qubit and multiplies it by $-i$

$\sigma_z = Z$ $\quad Z|0\rangle = |0\rangle$

$Z|1\rangle = -|1\rangle$

S-gate $\quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$$T(\pi/8) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$$

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}$$

Hadamard gate $-H-$

$$H|0\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]$$

$$H|1\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]$$

as 2×2 matrix

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$HXH = Z \;;\; HYH = -Y = XYX$$

$$HZH = X$$

Rotation matrix (gates)

$$R_x(\theta) = \cos\theta/2 \; I - i\sin\theta/2 \; \sigma_x$$

$$= \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_y(\theta) = \cos\theta/2 \; I - i\sin\theta/2 \; \sigma_y$$

$$= \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

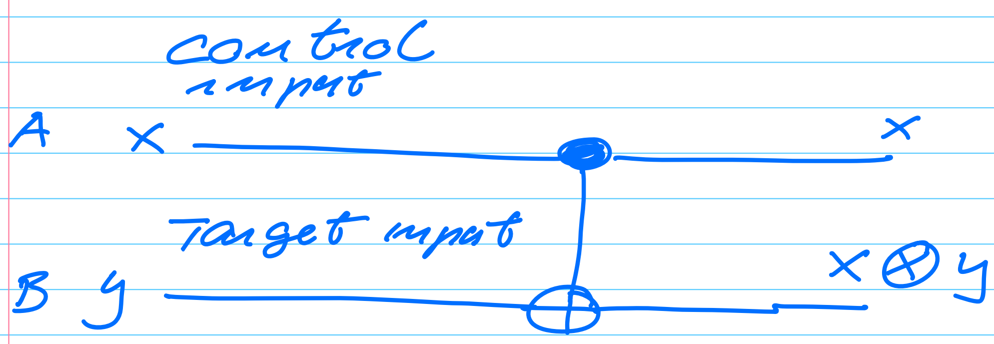
$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

$$HTH = R_x(\pi/4)$$

$$\sigma_x R_y(\theta)\sigma_x = R_y(-\theta)$$

$$\sigma_x R_z(\theta)\sigma_x = R_z(-\theta)$$

The two-qubit gate CNOT



$\oplus$ addition modulo 2

$$\overset{x\,y}{|00\rangle} = \underset{\text{control}}{|0\rangle_A} \otimes \underset{\text{Target}}{|0\rangle_B} \longrightarrow \overset{\text{output}}{|00\rangle}$$

$$|01\rangle \longrightarrow |01\rangle$$

$$|10\rangle \longrightarrow |11\rangle$$

$$|11\rangle \longrightarrow |10\rangle$$

$$G_{CNOT} = |00\rangle\langle 00| + |01\rangle\langle 01|$$

$$+ \underbrace{|10\rangle\langle 11|}_{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}} + |11\rangle\langle 10|$$

$$G_{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

control

$x \to |\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$

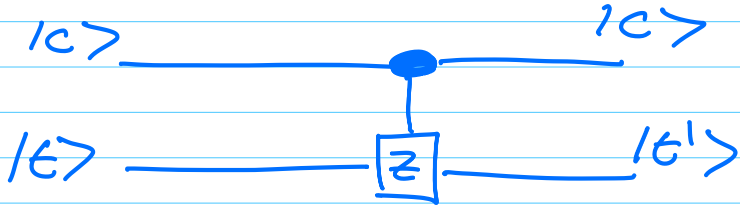
$y \to |\varphi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$

$$|\psi\rangle \otimes |\varphi\rangle = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix} = |\psi\varphi\rangle$$

$$|\psi\varphi'\rangle = G_{CNOT}|\psi\varphi\rangle$$

$$= \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_1 \\ \alpha_\phi\beta_0 \end{bmatrix}$$

Control - Z operation



$|0\rangle \to |0\rangle \qquad |1\rangle \to |-1\rangle$

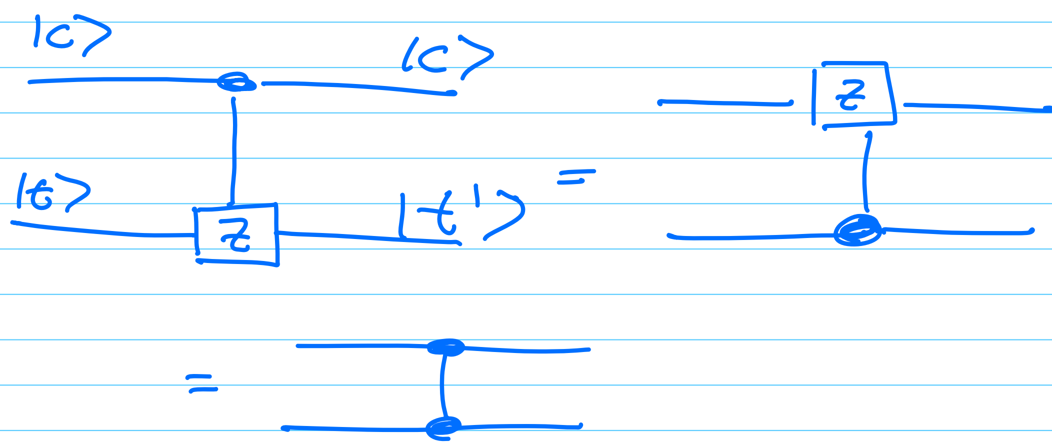
$$|00\rangle \to |00\rangle \;; \quad |01\rangle \to |01\rangle$$

$$|10\rangle \to |10\rangle \qquad |11\rangle \to -|11\rangle$$

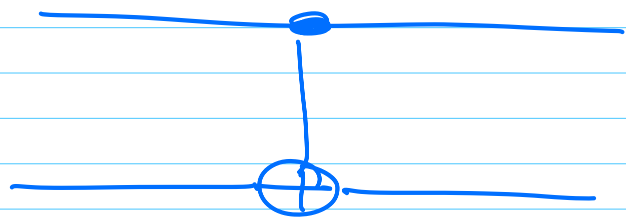
$$G_{cz} = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| - |11\rangle\langle 11|$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{1}_2 & \mathbb{O}_2 \\ \mathbb{O}_2 & \sigma_z \end{bmatrix}$$



We can construct a CNOT



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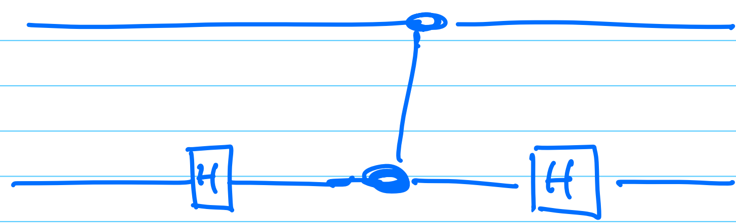
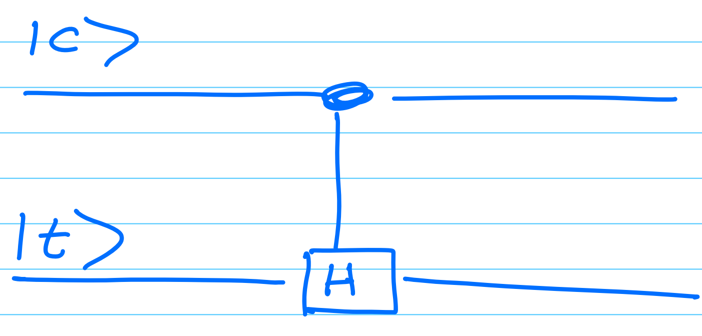


Controlled - H



$|00\rangle \rightarrow |00\rangle$; $|01\rangle \rightarrow |01\rangle$

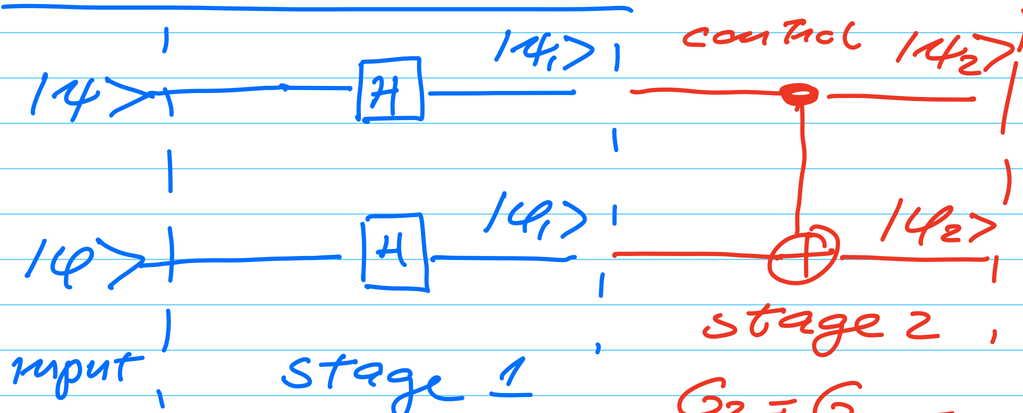
$|10\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|10\rangle + |11\rangle\right]$

$|11\rangle \rightarrow \frac{1}{\sqrt{2}}\left[|10\rangle - |11\rangle\right]$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

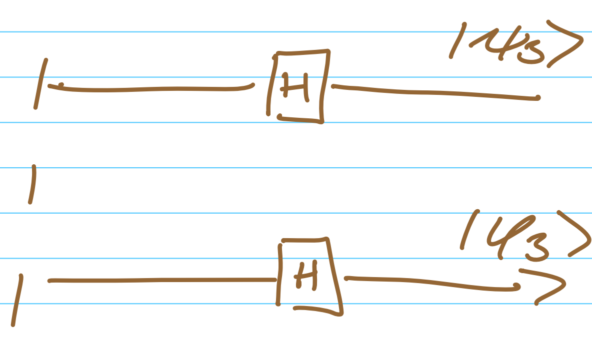
$$= \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & H_2 \end{bmatrix}$$

Circuits (2-qubits)



$G_2 = G_{CNOT}$

$G_1 = H \otimes H$



$G_3 = H \otimes H$

$$G = G_3 G_2 G_1$$

stage 1 input

$$|\psi\rangle \otimes |\varphi\rangle = |\xi\rangle$$

$$|\xi_1\rangle = G_1 |\xi\rangle$$

with $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

$$|\varphi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

$$G_1 = H \otimes H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$|\xi_1\rangle = G_1 |\xi\rangle$$

stage 2

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$G_3 = G_1$$

$$|\xi_2\rangle = |\psi_2\rangle \otimes |\varphi_2\rangle$$

$$= G_2 \cdot G_1 |\xi\rangle$$

$$|\xi_3\rangle = \underbrace{G_3 \cdot G_2 G_1}\, |\xi\rangle$$

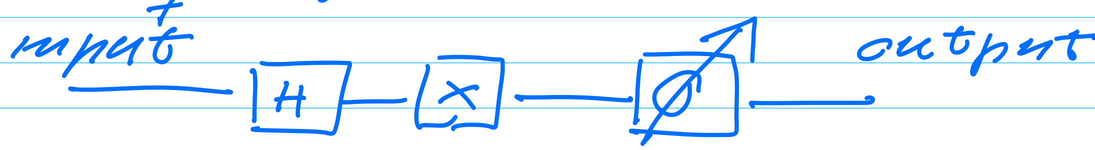
$$\frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} |\xi\rangle$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$$

Note : no measurement,

with measurement

1-qubit



Hamiltonian (aka eigenvalue problems) matrix

$$H \in \mathbb{C}^{2\times 2}$$

$$H = \begin{bmatrix} \varepsilon_0 + V_{00} & V_{01} \\ V_{10} & \varepsilon_1 + V_{11} \end{bmatrix}$$

$$V_{01} = V_{10} = \alpha$$

$$\sigma_x \cdot \alpha = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$$

$H = C_1 Z$, $\quad |\psi(\vec{\theta})\rangle$

$\langle E\rangle = \langle\psi(\vec{\theta})|H|\psi(\vec{\theta})\rangle$

$\sigma_z|0\rangle = |0\rangle$

$\sigma_z|1\rangle = -|1\rangle$

$|\psi(\vec{\theta})\rangle \quad$ 0 - 400

1 - 600

$$\langle z\rangle = \frac{1\cdot 400 + (-1)\cdot 600}{1000}$$

$\langle H\rangle = C_1\langle Z\rangle$

$\langle H\rangle(\vec{\theta})$

$H = C_1 Z + C_2 Z$

$\langle H\rangle = C_1\langle Z\rangle + C_2\langle Z\rangle$

$$\sigma_x\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) = 1\cdot\frac{(|0\rangle+|1\rangle)}{\sqrt{2}} = |+\rangle$$

$$\sigma_x\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = -1\cdot\frac{(|0\rangle-|1\rangle)}{\sqrt{2}} = |-\rangle$$

$H|+\rangle = |0\rangle$

$H|-\rangle = |1\rangle$

σ_y HS^+

$$H = \begin{bmatrix} \varepsilon_0 + V_{00} & V_{01} \\ V_{10} & \varepsilon_1 + V_{11} \end{bmatrix}$$

$$= H_0 + H_I$$

$$H_0 = \begin{bmatrix} \varepsilon_0 + V_{00} & 0 \\ 0 & \varepsilon_1 + V_{11} \end{bmatrix}$$

$|0\rangle$ and $|1\rangle$

$$H_0 |0\rangle = (\varepsilon_0 + V_{00}) |0\rangle$$

$$H_0 |1\rangle = (\varepsilon_1 + V_{11}) |1\rangle$$

$$H_I = \begin{bmatrix} 0 & V_{01} \\ V_{10} & 0 \end{bmatrix} = V_{01} \sigma_x$$

$$|\psi_0\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$|\psi_1\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$$