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QFT and quantum phase estimation (QPE) algorithms:

Discrete Fourier transform

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N}$$

x_j are complex numbers with
$j = 0, 1, \dots N-1$

y_k are complex numbers
$k = 0, 1, \dots N-1$

$x_j = \{1, 2\}$ $\quad N = 2$

$x_0 = 1 \quad x_1 = 2$

$k=0$: $y_0 = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} = \frac{3}{\sqrt{2}}$

$k=1$: $y_1 = \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} e^{i\pi}$

$$= \frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = -1/\sqrt{2}$$

Quantum Fourier transform:

$$|\psi\rangle = \sum_{j=0}^{N-1} a_j |j\rangle$$

$$|j\rangle = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Discrete Fourier transform

$$\sum_{k=0}^{N-1} b_k |k\rangle$$

$$b_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} a_j e^{2\pi i jk/N}$$

$$|\psi\rangle = \sum_{i=0}^{N-1} a_i |i\rangle = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix}$$

with a linear operator

$$\hat{A} \sum a_i |i\rangle = \sum a_i \hat{A} |i\rangle$$

The QFT is a linear operator that transforms an orthonormal basis

$$\{ |0\rangle, |1\rangle, \ldots |k\rangle, \ldots |N-1\rangle \}$$

as follows

$$|j\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i2\pi jk/N} |k\rangle$$

QFT transforms a state $|v\rangle$ of a quantum system into another state $|w\rangle$

$$|v\rangle \Rightarrow |w\rangle$$

$$|v\rangle = \sum_{j=0}^{N-1} v_j |j\rangle$$

$$|w\rangle = \sum_{k=0}^{N-1} w_k |k\rangle$$

Discrete Fourier transform

$$w_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \, e^{2\pi i jk/N}$$

Example 2 qubits

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (|0\rangle)$$

$$|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad (|1\rangle)$$

$$|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad (|2\rangle)$$

$$|11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (|3\rangle)$$

$$|\psi\rangle = \sum_{j=0}^{N-1} v_j \, |j\rangle$$

$$= v_0 |0\rangle + v_1 |1\rangle + v_2 \underbrace{|2\rangle}_{|10\rangle} + v_3 \underbrace{|3\rangle}_{|11\rangle}$$

$$= \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$w_k = \frac{1}{2} \sum_{j=0}^{3} v_j \cdot e^{i 2\pi jk/4} \, |j\rangle$$

$$w_0 = \frac{1}{2}\Big(v_0 |0\rangle + v_1 |1\rangle + v_2 |2\rangle + v_3 |3\rangle \Big)$$

$$w_1 = \frac{1}{2}\Big(v_0 |0\rangle + v_1 e^{i\pi/2} |1\rangle + v_2 e^{i\pi} |2\rangle + v_3 e^{3i\pi/2} |3\rangle \Big)$$

$$w_2 = \frac{1}{2}\Big(v_0 |0\rangle + v_1 e^{i\pi} |1\rangle + v_2 e^{i2\pi} |2\rangle + v_3 e^{3i\pi} |3\rangle \Big)$$

$$w_3 = \frac{1}{2}\Big(v_0|0\rangle + v_1 e^{3i\pi/2}|1\rangle + v_2 e^{3i\pi}|2\rangle + v_3 e^{9i\pi/2}|3\rangle\Big)$$

we can rewrite the mapping in terms of a matrix

$$|v\rangle \Rightarrow |w\rangle$$

$$|v\rangle \Rightarrow F|j\rangle$$

$$= \sum_{k=0}^{N-1=3} F_{jk}|k\rangle$$

$$= \begin{bmatrix} e^{2\pi i\, 0 0/N} & e^{2\pi i\, 0\cdot 1/N} & e^{2\pi i\, 0 2/N} & e^{2\pi i\, 0 3/N} \\ e^{2\pi i\, 1 0/N} & e^{2\pi i\cdot 1\cdot 1/N} & e^{2\pi i\, 1\cdot 2/N} & e^{2\pi i\, 1 3/N} \\ e^{2\pi i\, 2 0\cdot/N} & e^{2\pi i\, 2 1/N} & e^{2\pi\, 2\cdot 2\cdot/N} & e^{2\pi i\, 2 3/N} \\ e^{2\pi i\, 3 0/N} & e^{2\pi i\, 3 1/N} & e^{2\pi\cdot 3 2/N} & e^{2\pi 3 3/N} \end{bmatrix}$$

$$\times \begin{bmatrix} |0\rangle \\ |1\rangle \\ |2\rangle \\ |3\rangle \end{bmatrix}$$

$$F_{jk} = \exp(2\pi i j k/N)$$

Example 2

The Hadamard is a Fourier transform

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i2\pi \cdot 0 \cdot 0/2} & e^{i2\pi \, 0 \cdot 1/2} \\ e^{i2\pi \, 1 \cdot 0/2} & e^{i2\pi \cdot 1 \cdot 1/2} \end{bmatrix}$$

$$(N=2) \qquad = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$|w\rangle = H|v\rangle$$

Note that the Fourier transform of an arbitrary state is given by

$$\sum_{j=0}^{N-1} x_j |j\rangle = \sum_{j=0}^{N-1} x_j \left[\sum_{k=0}^{N-1} \frac{e^{2\pi i jk/N}}{\sqrt{N}} |k\rangle \right]$$

$$= \sum_{k=0}^{N-1} \underbrace{\left[\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i jk/N} \right]}_{} |k\rangle$$

$$= \sum_{k=0}^{N-1} y_k |k\rangle$$

It is useful when working with qubits to use a binary basis

$$\{ |0\rangle, |1\rangle, \ldots |2^n - 1\rangle \}, \quad N = 2^n$$

we can write $|j\rangle$ in binary representation

$$j = j_1 . j_2 \ldots j_n \quad \text{which means}$$

$$j_1 . j_2 \ldots j_n = j_1 \cdot 2^{n-1} + j_2 2^{n-2} + \ldots + j_n \cdot 2^0$$

$$= \sum_{m=1}^{n} j_m 2^{n-m}$$

where $j_i = 0, 1$

$$0.j_1 . j_2 \ldots j_n 2^{-n} = \sum_{m=1} j_m 2^{-m}$$

We can use this notation to show an equivalent representation of the Fourier transform called the product representation,

$$|j\rangle \rightarrow \frac{1}{2^{n/2}} \sum_{k=0}^{2^n - 1} e^{2\pi i jk/2^n} |k\rangle$$

$$\left(\frac{1}{\sqrt{N}} = \frac{1}{\sqrt{2^n}} = \frac{1}{2^{n/2}} \right)$$

$$= \frac{1}{2^{n/2}} \sum_{k_1 k_2 \ldots k_n \in \{0,1\}} e^{2\pi i j \sum_{m=1}^{n} k_m 2^{n-m}/2^n} |k_1\rangle |k_2\rangle \ldots |k_n\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_1 k_2 \ldots k_n \in \{0,1\}} \bigotimes_{m=1}^{n} e^{2\pi i j k_m 2^{-m}} |k_m\rangle$$

$$= \frac{1}{2^{n/2}} \bigotimes_{m=1}^{n} \left[\sum_{k_m \in \{0,1\}} e^{2\pi i j k_m 2^{-m}} |k_m\rangle \right]$$

$$= \frac{1}{2^{n/2}} \bigotimes_{m=1}^{n} \left[|0\rangle + e^{2\pi i j 2^{-m}} |1\rangle \right]$$

note that

$$\exp\left(2\pi i \sum_{l=1}^{n} j_l 2^{n-l-m} \right)$$

$$= \exp\left(2\pi i \sum_{t=1}^{m} j_{n-m+t} 2^{-t} \right)$$

$$= \exp\left(2\pi i \, 0.j_{n-m+1} j_{n-m+2} \ldots j_n \right)$$

$$|j\rangle \to \frac{1}{2^{n/2}} \bigotimes_{m=1}^{n} \Big[|0\rangle + \exp\left(2\pi i\, 0.j_{n-m+1}\, j_{n-m+2} \ldots j_n\right] |1\rangle \Big]$$

$$= \left[|0\rangle + e^{2\pi i 0.j_n} |1\rangle\right]$$

$$\times \left[|0\rangle + e^{2\pi i 0.j_{n-1} j_n} |1\rangle\right]$$

$$\times \ldots \left[|0\rangle + e^{2\pi i 0.j_1 j_2 \ldots j_n} |1\rangle\right]$$

$$\times \frac{1}{2^{n/2}}$$

Example two qubit with binary rep and standard rep.

$|j\rangle = |3\rangle$ (standard)

$(= |11\rangle$ binary$)$

in binary $|j_1 j_2\rangle = |11\rangle$

$N = 3$ $\quad |0\rangle \otimes |0\rangle \otimes |0\rangle$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |0\rangle$$

in binary $|j_1 . j_2 . j_3\rangle = |000\rangle$