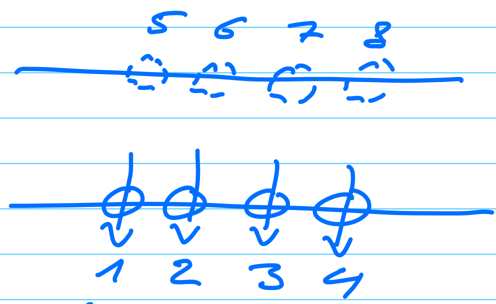
FYS5419/9419, MARCH 13, 2023

Lipkin model and second quantization.

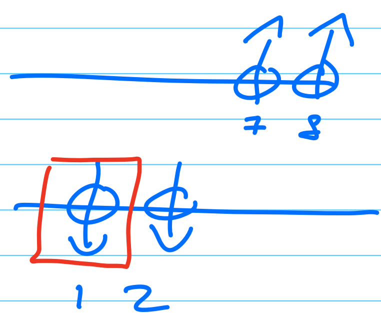
Two level system



$\varepsilon_1 = +\varepsilon/2$

$\varepsilon_0 = -\varepsilon/2$

$N = 4$ fermions



$\varepsilon_1 \rightarrow p$

$\varepsilon_0 \rightarrow p$

Single-particle level p and σ (spin) denote a single-particle state

$p = 1, 2, 3, 4$ $\qquad \sigma = \uparrow = 1/2$ or $\sigma = \downarrow = -1/2$

$$|p\sigma\rangle = a^{\dagger}_{p\sigma}|0\rangle$$

creation operator $a^{\dagger}_{p\sigma}$

annihilation operator $a_{p\sigma}$

anticommunication (FYS4480)

$$\{a^\dagger_{p\sigma}, a^\dagger_{p'\sigma'}\} = a^\dagger_{p\sigma} a^\dagger_{p'\sigma'} + a^\dagger_{p'\sigma'} a^\dagger_{p\sigma}$$
$$= 0$$

$$\{a_{p\sigma}, a_{p'\sigma'}\} = a_{p\sigma} a_{p'\sigma'} + a_{p'\sigma'} a_{p\sigma}$$
$$= 0$$

$$\{a^\dagger_{\underbrace{p\sigma}_{i}}, a_{\underbrace{p'\sigma'}_{j}}\} = \delta_{ij}$$

For fermions we express a many-body state in terms of $a^\dagger_{p\sigma}$

$$|\phi_0\rangle = a^\dagger_{1\downarrow} a^\dagger_{2\downarrow} a^\dagger_{3\downarrow} a^\dagger_{4\downarrow} |0\rangle$$
$$= |(1\downarrow)(2\downarrow)(3\downarrow)(4\downarrow)\rangle$$

computational basis for a many-body state.

a more general state

$$|\phi\rangle = \prod_{i=1}^{N} a^\dagger_i |0\rangle$$

a four particle state

$$|\phi\rangle = a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger |0\rangle$$

$$= |1234\rangle$$

$$a_p|\phi\rangle = 0 \text{ if } p \neq 1,2,3,4$$

$$a_p^\dagger a_q|\phi\rangle = 0 \text{ if } q \neq 1,2,3,4$$

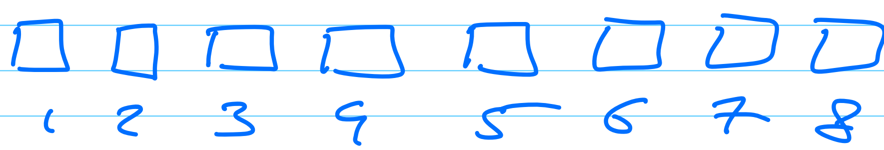
and if $p = 1,2,3,4$

$$a_p^\dagger|1234\rangle = 0 \text{ if } p = 1,2,3,4$$

$$p = \{1,2,3,4,5,\ldots n\}$$

each p is a slot which points to specific quantum number

$n = 8$



$N = 4$

$$|\phi\rangle = a_1^\dagger a_3^\dagger a_6^\dagger a_8^\dagger|0\rangle$$

$$|1368\rangle$$

or as a bit string

$$|\phi\rangle = |10100101\rangle$$

A general two-body Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{H}_I$$

$$\hat{H}_0 = \sum_{pq} \varepsilon_{pq} a_p^\dagger a_q$$

$$\varepsilon_{pq} = \langle p | h_0 | q \rangle$$
$$= \int d\tau \, \varphi_p^*(\tau) \hat{h}_0(\tau) \varphi_q(\tau)$$

(precalculated)

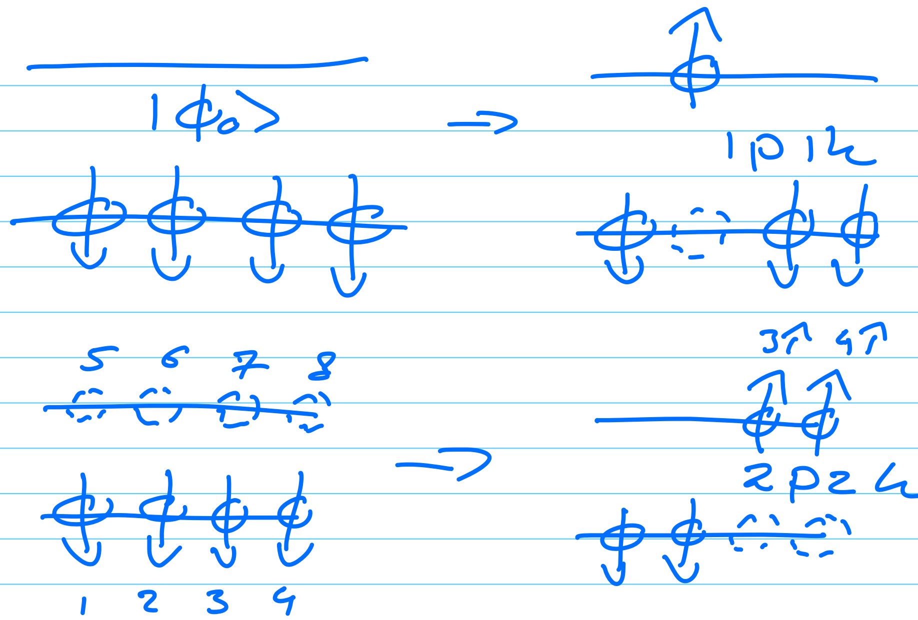
$$\hat{H}_I = \frac{1}{4} \sum_{pqst} \langle pq | v | st \rangle_{AS} \times a_p^\dagger a_q^\dagger a_t a_s$$

$$\langle pq | v | st \rangle_{AS} = \langle pq | v | st \rangle - \langle qp | v | st \rangle$$

$$\langle pq | v | st \rangle = \int_{\tau_1 \in D} d\tau_1 \int_{\tau_2 \in D} d\tau_2 \times \varphi_p^*(\tau_1) \varphi_q^*(\tau_2) v(\tau_1, \tau_2) \times \varphi_s(\tau_1) \varphi_t(\tau_2)$$

(Precalculated)

Lipkin model



Total number states = 8

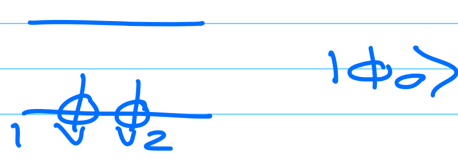
— " — particle = 4

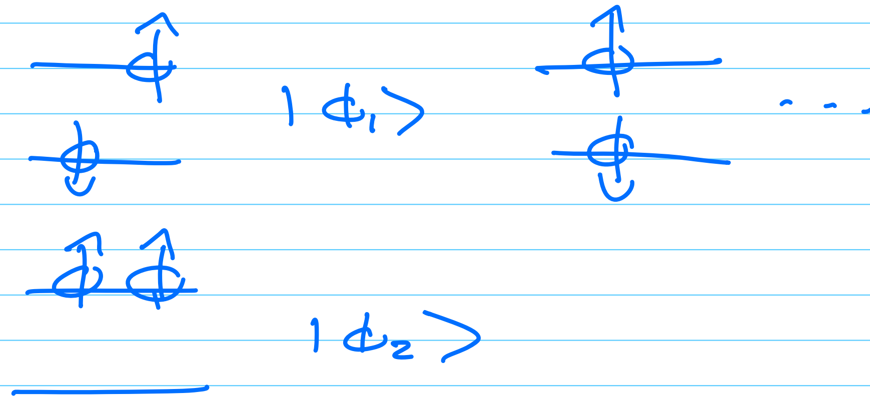
Total number of configurations

$$= \binom{8}{4} = \frac{8!}{4!\,4!} = \frac{5\cdot \cancel{6}^{\,2}\cdot 7 \cdot 8}{\cancel{2\cdot 3\cdot 4}} = 70 \text{ states}$$

$N=2$

$$H = \varepsilon J_z - \frac{1}{2}V\left[J_+^2 + J_-^2\right]$$





$J = 1$ $\qquad$ $J = J_1 + J_2 \quad J_{1,2} = 1/2$

$J = 1$

$J_z = -1, 0, 1$

$$|\phi_1\rangle = |J=1, J_z=0\rangle = J_+ |J=1, J_z=-1\rangle$$

$$|\phi_2\rangle = J_+ |J=1, J_z=0\rangle$$

Variational Quantum Eigensolver (VQE)

Example

$$H = 0.2\,\sigma_x + 0.5\,\sigma_y + 0.6\,\sigma_z$$
$$= 0.2\,X + 0.5\,Y + 0.6\,Z$$

VQE algorithm

- ansatz: prepare a parametrized initial state $|\psi\rangle$ (called the ansatz)

- Measurement: Measure the expectation value
$\langle \psi | H | \psi \rangle$

- Minimize: Tune the parameter of the ansatz to minimize the expectation value.

σ_x (X) computational basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

σ_y (Y)

$$|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

σ_z (Z) $\quad |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Z|0\rangle = \sigma_z|0\rangle = |0\rangle$$

$$\sigma_z|1\rangle = -|1\rangle$$

$$\sigma_x|+\rangle = |+\rangle$$

$$\sigma_y|i\rangle = |i\rangle$$

$$\sigma_z |1\rangle = -|1\rangle$$

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$\langle \psi | \sigma_z | \psi \rangle =$$

$$(\alpha_0^* \langle 0| + \alpha_1^* \langle 1|) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$\sigma_z = Z$

$$\times \left(\alpha_0 |0\rangle + \alpha_1 |1\rangle \right)$$

$$= |\alpha_0|^2 - |\alpha_1|^2$$

we perform N measurements

$$|\alpha_0|^2 = \frac{n_0}{N}$$

$$|\alpha_1|^2 = \frac{n_1}{N}$$

$$\langle \psi | Z | \psi \rangle = \frac{n_0 - n_1}{N}$$

suppose we start with $|0\rangle$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$= |+\rangle$$

our measurements give

$|\alpha_0|^2 = \frac{1}{2} \quad |\alpha_1|^2 = \frac{1}{2} \Rightarrow$

$\langle \psi | \sigma_z | \psi \rangle = 0.$

$H \cdot H |0\rangle = \mathbb{I} |0\rangle = |0\rangle \Rightarrow$

$|\alpha_0|^2 = 1 \quad |\alpha_1|^2 = 0$

$\langle \psi | \sigma_z | \psi \rangle = 1.$

$\langle \psi | H | \psi \rangle =$

$\langle \psi | 0.2\sigma_x + 0.5\sigma_y + 0.6\sigma_z | \psi \rangle$