QC + QIT November 5, 2021

orthonormal basis

$|e_i\rangle$

$\langle e_i | e_j \rangle = \delta_{ij}$ $\{ e_i^\dagger e_j = \delta_{ij} \}$

$|\varphi\rangle = \sum_i \alpha_i |e_i\rangle$

$\alpha_i = \varphi_i = \langle \varphi | e_i \rangle$

outer product

$|\varphi\rangle\langle\varphi|$

Example:

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$|1\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\sum_{i=1}^{2} |e_i\rangle\langle e_i| = |0\rangle\langle 0| + |1\rangle\langle 1|$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}$

$$\sum |\varphi_i\rangle\langle\varphi_i| = \mathbb{1}$$

$$\left(\sum_i |e_i\rangle\langle e_i|\right)|\varphi\rangle =$$

$$\sum_i |e_i\rangle\underbrace{\langle e_i|\varphi\rangle}_{\varphi_i} = \sum \varphi_i |e_i\rangle$$

$$= |\varphi\rangle$$

assume

$$|\varphi\rangle = \begin{bmatrix}\alpha \\ \beta\end{bmatrix} = \alpha|0\rangle + \beta|1\rangle$$

$$\begin{bmatrix}1 \\ 0\end{bmatrix}$$

projection operator

$$\hat{P} = |0\rangle\langle 0| = \mathbb{I} - \hat{Q}$$

$$\hat{Q} = |1\rangle\langle 1|$$

$$P = \begin{bmatrix}1 & 0 \\ 0 & 0\end{bmatrix}, \quad P^2 = P$$

idempotent operator

$$P \cdot Q = 0 \qquad |0\rangle\underbrace{\langle 0|1\rangle}\langle 1|$$

$= 0$

$P + Q = \mathbb{1}$

$$\hat{P}|\varphi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

Linear operator

$$A\left(\sum_i \alpha_i |\varphi_i\rangle\right) =$$

$$\sum_i \alpha_i A|\varphi_i\rangle$$

it is bounded

$$\| A|\varphi\rangle \| \leq \|A\| \, \|\varphi\|$$

$$\| |\varphi\rangle \| = \sqrt{\langle \varphi|\varphi\rangle}$$

$$= \|\varphi\|$$

Famous matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

$|1\rangle$ and $|2\rangle$ $\qquad \langle i|j\rangle = \delta_{ij}$

$$H_{11} = \langle 1|H|1\rangle$$

$$H_{ij} = \langle i|H|j\rangle$$

$$|\varphi_i\rangle = \alpha_i |1\rangle + \beta_i |2\rangle$$

$$H = T + V$$

$$T|i\rangle = \varepsilon_i |i\rangle$$

$$H_{ii} = \varepsilon_i + \underbrace{\langle i|V|i\rangle}_{V_{ii}}$$

$$\underset{i \neq j}{H_{ij}} = V_{ij} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T = \frac{\varepsilon_1 + \varepsilon_2}{2} \mathbb{1} + \frac{\varepsilon_1 - \varepsilon_2}{2} Z$$

$$H = \begin{bmatrix} \varepsilon_1 + V_{11} & V_{12} \\ V_{12} & V_{22} + \varepsilon_2 \end{bmatrix}$$

$$V = \frac{V_{11}+V_{22}}{2} \mathbb{I} + \frac{V_{11}-V_{22}}{2} Z$$
$$+ V_{12} X$$

Trace

$$\mathrm{Tr}[A] = \sum_i \langle e_i | A | e_i \rangle$$

$|e_i\rangle$ is an eigen basis for A.

$$A|e_i\rangle = \lambda_i |e_i\rangle$$

1) linear

$$\mathrm{Tr}(A+B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$$
$$\mathrm{Tr}(\gamma A) = \gamma\, \mathrm{Tr}(A)$$
$$\gamma \in \mathbb{C}$$

2) cyclic

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$$
$$= \mathrm{Tr}(CAB)$$

Adjoints

$$A^\dagger = A$$

unitary if

$$A^\dagger A = A A^\dagger = \mathbb{I}$$

$$\left(A^\dagger |\varphi\rangle, |\psi\rangle \right) \qquad \left\{ \langle \varphi | A | \psi \rangle \right.$$

$$= \left(|\varphi\rangle, A|\psi\rangle \right)$$

$$A = \sum_i \lambda_i P_i$$

eigenvalues

orthogonal projection to a subspace

Composite systems

$$\mathcal{H}_A \otimes \mathcal{H}_B$$

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \qquad \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

$$\Big[\alpha_1 \begin{bmatrix} \alpha_2 \end{bmatrix} \Big] \qquad \Big[\alpha_1 \alpha_2 \Big]$$

$$= \begin{bmatrix} \alpha_1 \begin{bmatrix} \; \\ \beta_2 \end{bmatrix} \\ \beta_1 \begin{bmatrix} \alpha_2 \\ \beta_1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \; \\ \alpha_1 \beta_2 \\ \beta_1 \alpha_2 \\ \beta_{\cancel{2}1} \beta_2 \end{bmatrix}$$

$$|\alpha\rangle_A \otimes |\beta\rangle_B = |\alpha\beta\rangle_{AB}$$

Quantum experiments

$$\boxed{P} \longrightarrow \boxed{\mathcal{E}} \longrightarrow \boxed{M}$$

preparation $(|0\rangle, |1\rangle)$

evolution

measurement

computational basis

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \quad \begin{cases} |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

$$= \alpha|0\rangle + \beta|1\rangle$$

$$\langle\psi|\psi\rangle = 1 = |\alpha|^2 + |\beta|^2$$

Hadamard basis

$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$|-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (\mathrm{Tr}\, H = 0)$$

$$H|0\rangle = |+\rangle$$

$$H|1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

$$H^2 = \underline{1} \Rightarrow H^{-1} = H$$

$$X|0\rangle = |1\rangle$$

$$Y|0\rangle = i|1\rangle$$

Exercise: compute the

[illegible]

operators X, Y, Z,
and H on $|0\rangle$ and $|1\rangle$

compute X^2, Y^2, Z^2

$[X, Y]$, $[X, Z]$, $[Z, Y]$