November 19

Composite system

$$|0\rangle_A \otimes |0\rangle_B = |00\rangle$$

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Bell states

$$|\phi^+\rangle = |\psi^{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\phi^-\rangle = |\psi^{01}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi^+\rangle = |\psi^{10}\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$|\psi^-\rangle = |\psi''\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

Density matrix for a composite system

$$\rho_A = \sum_i p_i |\phi_i\rangle_A \langle \phi_i|$$

$$\rho_{AB} = \sum_{i,j} p_i |\phi_i\rangle_A \langle\phi_i| \, q_j |\psi_j\rangle_B \langle\psi_j|$$

probabilities

$$\sum_i p_i^2 = 1$$

$$= \sum_{i,j} p_i q_j \left(|\phi_i\rangle_A \otimes |\psi_j\rangle_B\right)$$

$$\left({}_A\langle\phi_i| \otimes {}_B\langle\psi_j|\right)$$

$$= \sum_i p_i |\phi_i\rangle_A \langle\phi_i| \otimes \sum_j q_j |\psi_j\rangle_B \langle\psi_j|$$

$$= \rho_A \otimes \rho_B$$

Two-qubit gates

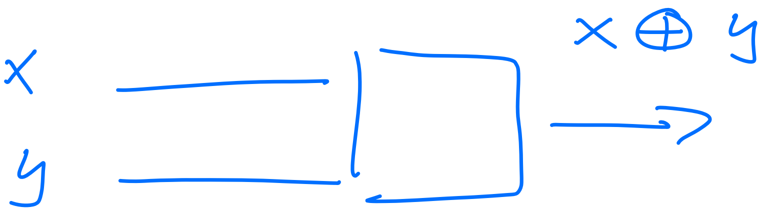
NOT, X, Y, Z

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$X \begin{bmatrix} 1 \\ 0 \end{bmatrix} = X|0\rangle = |1\rangle$$

$$X \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |0\rangle$$

XOR-gate (classical)



x	y	output
0	0	0
0	1	1
1	0	1
1	1	0

irreversible. cannot tell x or y from the output

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cdots \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11|$$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + |11\rangle\langle 10|$$

$$CNOT \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle$$

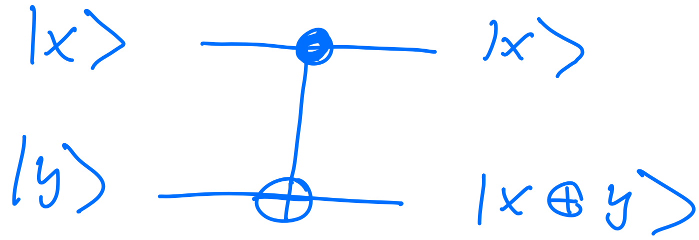
$$= |0\rangle_A \otimes |0\rangle_B$$

$$CNOT\, |01\rangle = |01\rangle$$

$$CNOT \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |11\rangle = |1\rangle_A \otimes |1\rangle_B$$

$$CNOT\, |10\rangle = |11\rangle$$

input		output	
x	y	x	$x \oplus y$
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

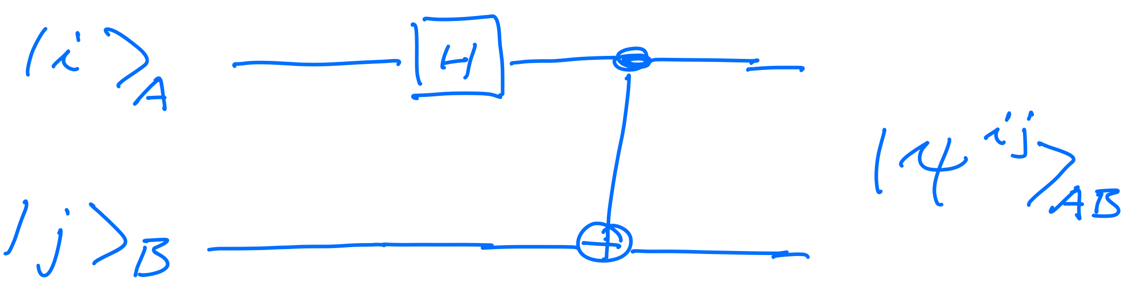


reversible operation.

$\Rightarrow$ all operations in computation can be done with a NOT gate and a CNOT gate.

Bell states

$$|\psi^{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

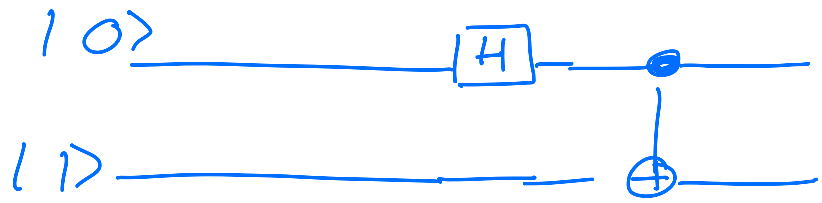


$|i\rangle$ and $|j\rangle$ are either $|0\rangle$ or $|1\rangle$

$H = \frac{1}{\;} \begin{bmatrix} 1 & 1 \end{bmatrix}$

$$\sqrt{2}\ \left[1 \ \ -1\right]$$

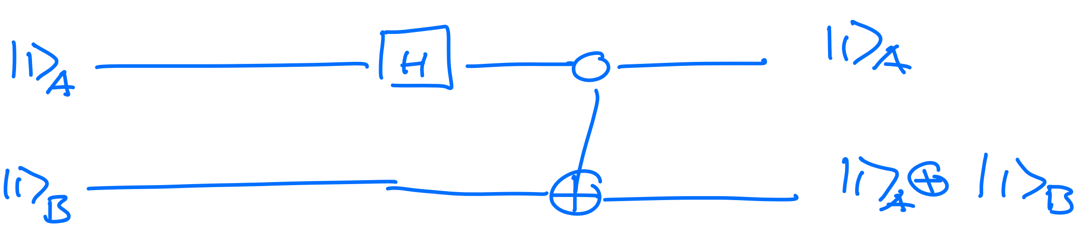
$$|01\rangle = |i\rangle_A \otimes |j\rangle_B = |0\rangle_A \otimes |1\rangle_B$$



$$H\,|01\rangle = \frac{|01\rangle + |11\rangle}{\sqrt{2}}$$

$$\mathrm{CNOT}\left(\frac{|01\rangle + |11\rangle}{\sqrt{2}}\right) = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$|11\rangle$ = initial



$$H\,|1\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$$

$$CNOT\,(|01\rangle - |10\rangle)/\sqrt{2} = \frac{|01\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi^-\rangle = |\psi^{11}\rangle$$

Entanglement

pure state: two separate systems share the two-qubit (two-qubit system) quantum state

$$= |0\rangle_A |0\rangle_B$$

(separable)

Bell states are superpositions of $|0\rangle$ and $|1\rangle$

$$|\psi^{00}\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

$$= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

cannot determine the indivi-
dual states of A & B.

We say the states are
entangled.

Think of probabilities

if iid

$$P(x,y) = p(x)p(y)$$

Maximally entangled

$$\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B)$$

$$\dim(\mathcal{H}_A) = d$$

orthonormal basis sets

$|i\rangle_A$ and $|i\rangle_B$

Max entangled state

$$|\Omega\rangle = \frac{1}{d}\sum_{i=1}^{d} |i\rangle_A \otimes |i\rangle_B$$

$$= \frac{1}{d}\sum_{i=1}^{d} |ii\rangle$$

→ Schmidt decomposition

density matrix

+ density matrix

$$\rho_{AB} \rightarrow \rho_A$$

Say whether system is entangled or not.