FYS 5419, FEB 6, 2023

- Spectral decomposition
- Measurements and density matrix

- Reminder about QM operators

  - Hermitian (self-adjoint)

$$A = A^{\dagger}$$

  - unitary $AA^{\dagger} = A^{\dagger}A = \mathbb{1}$
  - Normal $[A, A^{\dagger}] = 0$

Defined a basis $|i\rangle = \{|0\rangle, |1\rangle, \dots |n-1\rangle\}$
ONB basis

$$\langle i|j\rangle = \delta_{ij}$$

$$|\psi_a\rangle = \sum_{i=0}^{n-1} \alpha_i |i\rangle$$

$$\alpha_i = \langle i|\psi_a\rangle$$

$$\langle j|\psi_b\rangle = ?$$

$$|\psi_b\rangle = \hat{A}|\psi_a\rangle$$

$$\langle j|\psi_b\rangle = \langle j|\hat{A}|\psi_a\rangle$$

$$= \langle j|\hat{A} \sum_{i=0}^{n-1} \alpha_i |i\rangle =$$

$$\sum_{i=0}^{n-1} \langle j|A|i\rangle\langle i|\psi_a\rangle$$

Example $A = \sigma_x$

$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$

$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(NOT classical $\Rightarrow$ QM NOT $= \sigma_x$)

$|\psi_b\rangle = A|\psi_a\rangle = \sigma_x|\psi_a\rangle$

$= \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)\left(\alpha_0|0\rangle + \alpha_1|1\rangle\right)$

$= \alpha_0 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \alpha_0|1\rangle + \alpha_1|0\rangle$

$= \beta_0|0\rangle + \beta_1|1\rangle$

$\beta_0 = \alpha_1 \quad \beta_1 = \alpha_0$

$\langle\psi_b| = \langle 0|\alpha_1^* + \alpha_0^*\langle 1|$

Projection operators

$|\psi_a\rangle\langle\psi_a| = P_{\psi_a}$

$P_{\psi_a}^2 = P_{\psi_a}$

Two projection operators are orthogonal if

$$P_a P_b |\psi_c\rangle = 0$$

$$P_a P_b = 0$$

$P = |0\rangle\langle 0| \qquad Q = |1\rangle\langle 1|$

$$P + Q = \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Spectral decomposition of an operator

Let $|\psi\rangle$ be a vector in $\mathcal{H}_n$ and A is a normal operator. Let $|\psi\rangle$ be an eigenvector

$$A|\psi\rangle = \lambda|\psi\rangle = \lambda \mathbb{I}|\psi\rangle$$

$$\mathbb{I}|\psi\rangle = 1|\psi\rangle \qquad \mathbb{I} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{bmatrix}$$

$$(A - \lambda\mathbb{I})|\psi\rangle = 0$$

Next with exercises (4x4 mtx) we will express $|\psi\rangle$ in terms of an ONB in $\mathcal{H}_n$

$$|\psi\rangle = \sum_{i=0}^{n-1} \alpha_i |i\rangle$$

$$|i\rangle = \{|0\rangle, |1\rangle, \ldots |n-1\rangle\}$$

eigenvalue equation

$$[A - \lambda I] \sum_i \gamma_i |i\rangle = 0$$

$$A \to a_{ij} \qquad \lambda I \to \lambda \delta_{ij}$$

$$\sum_{i=0}^{n-1} (a_{ij} - \lambda \delta_{ij}) \gamma_i = 0$$

$$\det(A - \lambda \mathbb{I}) = 0$$

spectral decomposition

$|\psi_a\rangle$ are eigenvectors of A in $\mathcal{H}_n$

$$|\psi_a\rangle = \sum_{i=0}^{n-1} \alpha_i |i\rangle$$

new ONB

$$\sum_{i=0}^{n-1} |\alpha_i|^2 = \underline{1}$$

$$A|\psi_a\rangle = \sum_i \alpha_i A |i\rangle$$

$$(A|i\rangle = \lambda_i |i\rangle)$$

$$= \sum_i \alpha_i \lambda_i |i\rangle$$

outer of $|\psi_a\rangle$ label

$$P_{\psi_a} = |\psi_a\rangle\langle\psi_a|$$

$$P_j = |j\rangle\langle j|$$

$$P_j|\psi_a\rangle = |j\rangle\langle j|\sum_{i=0}^{n-1}\alpha_i|i\rangle$$

$$= \sum_{i=0}^{n-1}\alpha_i|j\rangle\underbrace{\langle j|i\rangle}_{\delta_{ij}} = \alpha_j|j\rangle$$

$$P_j|\psi_a\rangle = \alpha_j|j\rangle$$

$$A|\psi_a\rangle = \sum_{i=0}^{n-1}\alpha_i\lambda_i|i\rangle$$

$$= \left(\sum_{i=0}^{n-1}\lambda_i P_i\right)|\psi_a\rangle$$

$$\Rightarrow A = \sum_{i=0}^{n-1}\lambda_i P_i$$

This is the spectral decomposition of a normal operator A.

Observables are specified by measurements described by the spectral decomposition by the relevant operator

Example

A has λ_a and λ_b as eigen values and $|a\rangle$ and $|b\rangle$ as eigenvectors

$$|a\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$|b\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$P_a = |a\rangle\langle a| = \begin{bmatrix} |\alpha_0|^2 & \alpha_0 \alpha_1^* \\ \alpha_0^* \alpha_1 & |\alpha_1|^2 \end{bmatrix}$$

$$P_b = |b\rangle\langle b|$$

$$A = \lambda_a |a\rangle\langle a| + \lambda_b |b\rangle\langle b|$$

$$= \lambda_a P_a + \lambda_b P_b$$

Measurement of observables

Define projection operators

$$\{ P_0, P_1 \, - - - \, P_{m-1} \}$$

$$m \leq n \qquad \mathcal{H}_n$$

$$\sum_{i=0}^{m-1} P_i = \mathbb{1}$$

Example : $|0\rangle$ and $|1\rangle$

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = |0\rangle\langle 0|$$

$$P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = |1\rangle\langle 1|$$

$$P_0 + P_1 = \mathbb{1}$$

$$P_0 |0\rangle = |0\rangle \qquad P_0 = |0\rangle\langle 0|$$

$$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \quad \text{(mixed state)}$$

$$P_0|\psi_a\rangle = \alpha_0|0\rangle \quad \text{(pure state)}$$

$$\langle\psi_a|P_0^{\dagger} P_0|\psi_a\rangle = |\alpha_0|^2$$

Probability of the i-th measurement (outcome λ_i)

$$\text{Prob}(\lambda_i) = |P_i|\psi_a\rangle|^2$$

$$= \langle\psi_a|P_i^{\dagger} P_i|\psi_a\rangle$$

$$= \langle\psi_a|P_i^2|\psi_a\rangle$$

$$= \langle\psi_a|P_i|\psi_a\rangle$$

$$\sum_{i=0}^{n-1} \text{Prob}(\lambda_i) = 1$$

Post measurement normalized pure quantum state

$$\frac{P_i |\psi_a\rangle}{\sqrt{\langle\psi_a|P_i|\psi_a\rangle}}$$

Example $\langle x|y\rangle = 0$

$|\psi_a\rangle = \alpha_x|x\rangle + \alpha_y|y\rangle$

$|\psi_b\rangle = \beta_x|x\rangle + \beta_y|y\rangle$

$P_x = |x\rangle\langle x|$ $P_y = |y\rangle\langle y|$

$\text{Prob}(\psi_a) = p$ $\text{Prob}(\psi_b) = 1-p$

$\text{Prob}(\lambda_x|\psi_a)$

probability of λ_x given ψ_a

$\langle\psi_a|P_x|\psi_a\rangle = |\alpha_x|^2$

$\text{Prob}(\lambda_x|\psi_b) = \langle\psi_b|P_x|\psi_b\rangle$

$= |\beta_x|^2$

Example

$|0\rangle$ and $|1\rangle$

$P_0 = |0\rangle\langle 0|$ $P_1 = |1\rangle\langle 1|$

$$\sum P_i^\dagger P_i = \sum P_i^2 = \sum P_i$$

$$= \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$P_\psi(0) = \langle\psi|P_0^\dagger P_0|\psi\rangle$$

$$= \left(\alpha^*\langle 0| + \beta^*\langle 1|\right)|0\rangle\langle 0|$$

$$\times\left(\alpha|0\rangle + \beta|1\rangle\right) = |\alpha|^2$$

$$P_\psi(1) = |\beta|^2$$

if $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$$|\alpha|^2 = |\beta|^2 = 1/2$$

$$|\psi_0'\rangle = \frac{P_0|\psi\rangle}{\sqrt{\langle\psi|P_0^2|\psi\rangle}} = \frac{\alpha}{|\alpha|}|0\rangle$$

$$|\psi_1'\rangle = \frac{\beta}{|\beta|}|1\rangle$$

Take a general state $\psi_1(x)$

$$P_{\psi_1}(x) = \langle \psi_1 | P_x^{\dagger} P_x | \psi_1 \rangle$$

we can rewrite this as

$$P_{\psi_1}(x) = \mathrm{Tr}\left[P_x^{\dagger} P_x | \psi_1 \rangle \langle \psi_1 | \right]$$

Example

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$P_{\psi}(0) = \langle \psi | P_0^{\dagger} P_0 | \psi \rangle = |\alpha|^2$$

$$|\psi\rangle\langle\psi| = \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}$$

$$P_0^{\dagger} P_0 = P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix} =$$

$$\begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ 0 & 0 \end{bmatrix}$$

$$\mathrm{Tr}\left[\; - \; 1 \; - \; \right] = |\alpha|^2$$

Spectral decomposition

$$p(x) = \sum_i p_i \hat{P}_{\psi_1}(x)$$

- Digression : link with statistics

  - joint probability $x \in \mathcal{X}$, $y \in \mathcal{Y}$

    $P_{XY}(x,y)$

    if iid

    $P_{XY}(x,y) = p(x)\,p(y)$

    if independent

    $P_{XY}(x,y) = P_X(x)\,P_Y(y)$

  - marginal probability

    $P_X(x) = \sum_{y \in \mathcal{Y}} P_{XY}(x,y) \quad \forall x \in \mathcal{X}$

  - conditional probability

    $$P_{Y|X}(y|x) = \frac{P_{XY}(x,y)}{P_X(x)}$$

    (if $P_X(x) = 0$, $P_{XY}(x,y) = 0$)

    if independent

    $P_{Y|X} = P_Y(y)$

$$P_X(x) = \sum_{y \in \mathcal{Y}} P_{X|Y}(x|y)\, P_Y(y)$$

$\Rightarrow$ Bayes' theorem

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)\, P_X(x)}{P_Y(y)}$$

Density matrix (operator)

$$p(x) = \sum_i p_i \underbrace{P_{\psi_i}(x)}$$

$$= \sum_i p_i \, \mathrm{Tr}\left[P_X^\dagger P_X |\psi_i\rangle\langle\psi_i| \right]$$

$$= \mathrm{Tr}\left[P_X^\dagger P_X \rho \right]$$

$$\boxed{\rho = \sum_{i=0}^{n-1} p_i |\psi_i\rangle\langle\psi_i|}$$

Schumacher, section 2.3

The density operator ρ on Hilbert space $\mathcal{H}_n$ has the following properties

(i) There is $p_i \in \mathbb{R}$ that satisfies

$$p_i \geq 0$$

$$\sum_{i \in D} p_i = 1$$

and an ONB $|\psi_i\rangle$ in $\mathcal{H}_n$ such that

$$\rho = \sum_{i \in \mathcal{H}_n} p_i |\psi_i\rangle\langle\psi_i|$$

$$= \sum_i p_i P_{\psi_i}$$

(ii) $0 \leq \rho^2 \leq \rho$

(iii) $\|\rho\|_2 \leq \underline{1}$

After measurement

$$\rho_x' = \frac{P_x \rho P_x^+}{\mathrm{Tr}\left[P_x^+ P_x \rho\right]}$$

if we have ρ for a state

$$P_\rho(x) = \mathrm{Tr}(\rho P_x^\dagger P_x)$$

= probability of outcome

Entropy and ρ $\Rightarrow$

Entanglement and Schmidt decomposition.