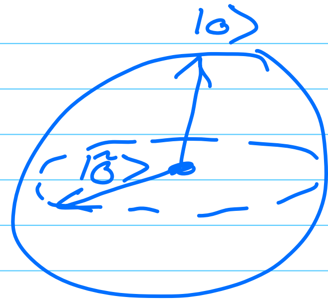
FYS5419, MAY 12, 2023

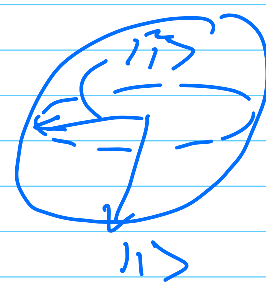
QFT is effectively a change basis from the computational basis to a Fourier basis.

computational $|0\rangle$ and $|1\rangle$

$$\sigma_z |0\rangle = +1 \times |0\rangle$$

$$\sigma_z |1\rangle = -|1\rangle$$





The QFT is a linear operator that transforms an orthonormal

$j: \{ |0\rangle, |1\rangle, \dots |N-1\rangle \}$

n = # qubits, using a binary basis

$$N = 2^n$$

$n = 2$

$$|0\rangle = |0\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|1\rangle = |1\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|2\rangle = |0\rangle_A \otimes |1\rangle_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|3\rangle = |1\rangle_A \otimes |1\rangle_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$n = 2 \Rightarrow N = 4 \quad \{ |0\rangle, |1\rangle, |2\rangle, |3\rangle \}$

$|00\rangle$

QFT transforms a state $|v\rangle$ into another state $|w\rangle$

$$|v\rangle = \sum_{j=0}^{N-1} v_j |j\rangle$$

$$|w\rangle = \sum_{k=0}^{N-1} w_k |k\rangle$$

$k : \{ |0\rangle, |1\rangle, \ldots |N-1\rangle \}$

w_k are given by a discrete

Fourier transform

$$w_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \cdot e^{i2\pi jk/N}$$

$$\sum_{j=0}^{N-1} v_j |j\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} v_j \cdot$$

$$\times \left[\sum_{k=0}^{N-1} e^{2\pi i \cdot jk/N} |k\rangle \right]$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left[\sum_{j=0}^{N-1} v_j \cdot e^{2\pi i \cdot jk/N} \right] |k\rangle$$

$$= \sum_{k=0}^{N-1} w_k |k\rangle$$

Example : 1-qubit case

$$N = 2^1 = 2$$

$$|\tilde{0}\rangle = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} e^{2\pi i \cdot 0 j/2} |j\rangle$$

$$= \frac{1}{\sqrt{2}} \left[|0\rangle + |1\rangle \right]$$

$$|j\rangle = \{ |0\rangle, |1\rangle \}$$

$$|\tilde{1}\rangle = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} e^{2\pi i (1)\cdot j/2} |j\rangle$$

$$= \frac{1}{\sqrt{2}} \left[e^{0\pi i} |0\rangle + e^{\pi i} |1\rangle \right]$$

$$= \frac{1}{\sqrt{2}} \left[|0\rangle - |1\rangle \right]$$

$$|w\rangle = (\widehat{QFT}) |v\rangle$$

$$\swarrow$$

$$H |v\rangle$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \left[|0\rangle - |1\rangle \right]$$

$$H|1\rangle = \frac{1}{\sqrt{2}} \left[- \ \iota \ - \right]$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \left[|0\rangle + |1\rangle \right]$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{2\pi i\, 0\cdot 0/2} & e^{2\pi i\, 0\cdot 1/2} \\ e^{2\pi i\, 1\cdot 0/2} & e^{2\pi i\, 1\cdot 1/2} \end{bmatrix}$$

$$\widehat{(QFT)} =$$

$$\frac{1}{\sqrt{N}}\begin{bmatrix} e^{2\pi i\, 0.0/2} & \cdots & e^{2\pi i\, 0.N-1/2} \\ e^{2\pi i\, 1.0/2} & \cdots & e^{2\pi i\, N-1/2} \\ \vdots & & \\ e^{2\pi i\, N-1.0/2} & \cdots & e^{2\pi i (N-1)(N-1)/2} \end{bmatrix}$$

multiply with $\begin{bmatrix} |0\rangle \\ |1\rangle \\ \vdots \\ |N-1\rangle \end{bmatrix}$

Bit representation:

$$\{ |0\rangle, |1\rangle \cdots \underbrace{|2^n - 1\rangle}_{|N-1\rangle} \}$$

binary basis $N = 2^n$

n = # qubits

$$j = j_0 j_1 j_2 \cdots j_{n-1}$$

$$= j_0 2^{n-1} + j_1 2^{n-2} + \cdots + j_{n-1} 2^0$$

$$j_i = \{0, 1\}$$

(integer representation)

$$= \sum_{m=0}^{n-1} J_m 2^{n-m}$$

alternatively:

$0.J_0 J_1 J_2 \ldots J_{n-1} =$

$J_0 2^{-1} + J_1 {}^{-2} + \ldots J_{n-1} 2^{-(n-1)}$

$$= \sum_{m=0}^{n-1} J_m 2^{-m}$$

New basis

$|k\rangle = k_0 2^{n-1} + k_1 2^{n-2} + \ldots k_{n-1} 2^0$

$|J_0 J_1 J_2 \ldots J_{N-1}\rangle \longrightarrow$

$$\frac{1}{2^{n/2}} \sum_{k_0=(0,1)} \sum_{k_1=(0,1)} \cdots \sum_{k_{n-1}=(0,1)}$$

$$\times\ e^{i 2\pi J \sum_{m=0}^{n-1} k_m 2^{-m}}$$

$$\times\ |k_0 k_1 k_2 \ldots k_{n-1}\rangle$$

$$= \frac{1}{2^{n/2}} \sum_{k_0} \sum_{k_1} \ldots \sum_{k_{n-1}}$$

$$\bigotimes_{m=0}^{n-1} e^{i2\pi j k_m \cdot 2^{-m}} |k_m\rangle$$

$$= \frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ \sum_{k_m=(0,1)} e^{i2\pi j k_m/2^m} |k_m\rangle \right\}$$

the bit k_m can take values 0 and 1

$e^{i2\pi j k_m} = 1$ when $k_m = 0$

$$|j_0 j_1 j_2 \cdots j_{n-1}\rangle \longrightarrow$$

$$\frac{1}{2^{n/2}} \bigotimes_{m=0}^{n-1} \left\{ |0\rangle + e^{i2\pi j/2^m} |1\rangle \right\}$$

$$|0\rangle + e^{i2\pi j/2^m} |1\rangle =$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e^{i2\pi j/2^m} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ e^{i2\pi j/2^m} \end{bmatrix}$$

The transformation is then

$$|j\rangle \rightarrow \frac{1}{2^{n/2}} \begin{bmatrix} 1 \\ e^{i2\pi(j/2^0)} \end{bmatrix} \otimes$$

$$\begin{bmatrix} 1 \\ e^{i2\pi(j/2^1)} \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 1 \\ e^{i2\pi(j/2^{n-1})} \end{bmatrix}$$

Example : $n = 3$

$j = 0$: $|0\rangle = |0\rangle \otimes |0\rangle \otimes |0\rangle$

$= |000\rangle$

$$|000\rangle \rightarrow \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Or

$$|000\rangle \rightarrow \frac{1}{2^{3/2}} \Big[|000\rangle + |001\rangle$$

$$+ |010\rangle + |011\rangle + |100\rangle + |110\rangle$$

$$+ |111\rangle \Big]$$

$$|j=1\rangle = |001\rangle$$

$$|001\rangle \rightarrow \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ e^{i2\pi \cdot 1/2^1} \end{bmatrix}$$

$$\otimes \begin{bmatrix} 1 \\ e^{i2\pi(1/2^2)} \end{bmatrix}$$

$$= \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= \frac{1}{2^{3/2}} \begin{bmatrix} 1 \\ i \\ -1 \\ -i \\ 1 \\ i \\ -1 \\ -i \end{bmatrix}$$

$$|001\rangle \rightarrow \frac{1}{2^{3/2}} \Big[|000\rangle - |010\rangle + |100\rangle - |110\rangle$$

$$+ i\left(|001\rangle - |011\rangle + |101\rangle - |111\rangle\right)\Big]$$

started with

$$|v\rangle = |v_0\rangle \otimes |v_1\rangle \otimes |v_2\rangle \ldots \otimes |v_{n-1}\rangle$$

$$(|0\rangle \otimes |1\rangle \ldots$$

$$|w\rangle = \frac{1}{\sqrt{N}}\left[|0\rangle + e^{2\pi i j/2^0}|1\rangle\right] \otimes$$

2^n

$$\left[|0\rangle + e^{2\pi i j/2^1}|1\rangle\right] \otimes$$

$$n-1 \quad \cdots \quad \left[|0\rangle + e^{2\pi i j/2^{n-1}}|1\rangle\right]$$

$$\bigotimes_{m=0}^{n-1}\left\{|0\rangle + e^{i 2\pi j/2^m}|1\rangle\right\}$$

$$= \prod_{m=0}^{n-1}\left(|0\rangle + e^{i 2\pi j/2^m}|1\rangle\right)$$

$$|j_0 j_1 j_2 \cdots j_{n-1}\rangle \longrightarrow$$

$$\frac{1}{2^{n/2}}\left(|0\rangle + e^{i 2\pi (j/2^0)}|1\rangle\right)$$

$$\times \left(|0\rangle + e^{i 2\pi (j/2^1)}|1\rangle\right)$$

$$\times \cdots \times \left(|0\rangle + e^{i 2\pi (j/2^{n-1})}|1\rangle\right)$$

$$0.j_m j_{m+1} \cdots j_{n-1} =$$

$$j_m 2^{-1} + j_{m+1} 2^{-2} + + \cdots j_{n-1} 2^{-(n-m)}$$

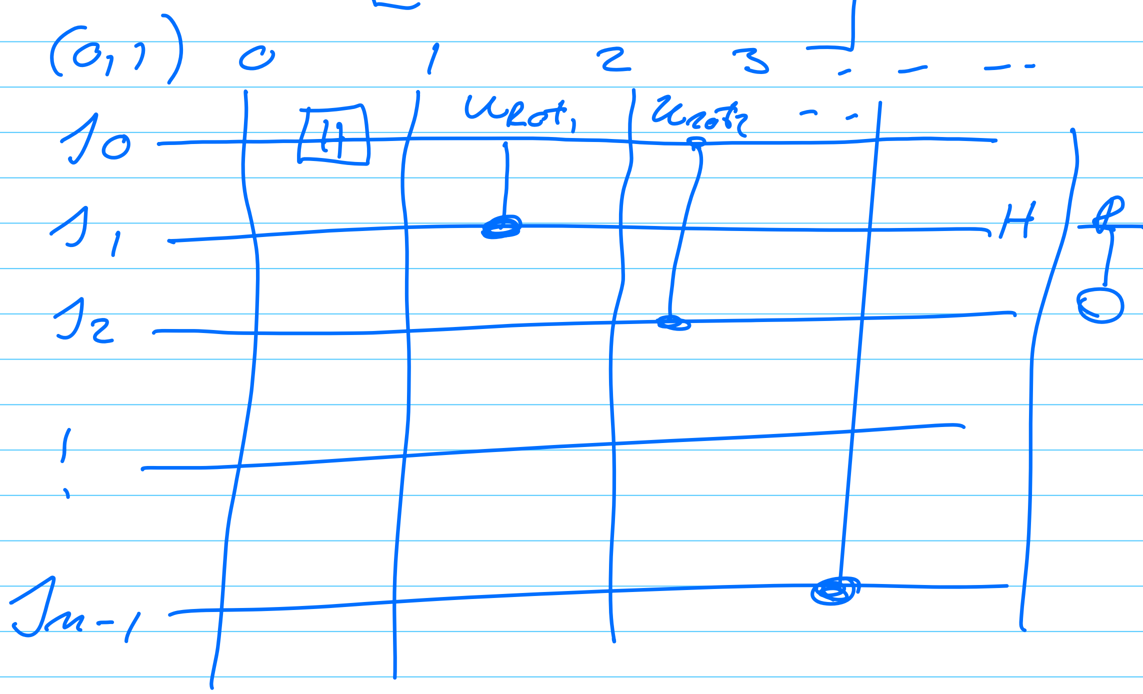
$|j_0 j_1 \cdots j_{n-1}\rangle \longrightarrow$

$$\frac{1}{2^{n/2}} \left(|0\rangle + e^{i2\pi\, 0.j_{n-1}} |1\rangle\right)$$

$$\times \left(|0\rangle + e^{i2\pi\, 0.j_{n-2} j_{n-1}} |1\rangle\right)$$

$$\cdots \left(|0\rangle + e^{i2\pi\, 0.j_0 j_1 \ldots j_{n-1}} |1\rangle\right)$$

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{i2\pi/2^k} \end{bmatrix}$$



Step 0 : $|j_0 j_1 j_2 \ldots j_{n-1}\rangle$

$|0\,0\,0 \ldots 0\rangle$

Example 2 qubits

$|00\rangle = |0\rangle \otimes |0\rangle$

Step 1 : $\left[|0\rangle + e^{\frac{2\pi i}{2^0} j_0} |1\rangle \right]$

$\times |j_1 j_2, \ldots j_{n-1}\rangle$

Step 2 ; $\left[|0\rangle + e^{2\pi i j_0} e^{i 2\pi/2 \, j_1} |1\rangle \right]$

$\otimes |j_1 j_2 \ldots j_{n-1}\rangle$

Step 3 ; $\left[|0\rangle + e^{i 2\pi j_0} e^{i 2\pi j_1/2^1}\right.$

$\left. \times e^{i 2\pi j_2/2^2} |1\rangle \right]$

$\otimes |j_1 \ldots j_{n-1}\rangle$

⋮

Step $n-1$

repeat this on qubit j_1 . . .

all way till the final,

Example $n=2$

QFT applied to Quantum phase estimation in order (QPE) to find eigenvalues.

QPE : Basic idea is that a unitary matrix has eigen-values of the form $e^{i\phi}$

$$UU^{+} = \mathbb{1}$$

$$UHU^{+} = D = \begin{bmatrix} \lambda_0 & & \\ & \ddots & \\ & & \lambda_{n-1} \end{bmatrix}$$