FYS5419, JANUARY 23, 2023

Define basis states/computational basis.

Example systems:

Harmonic oscillator

$$H_0 \Rightarrow H_0(x_i) = -\frac{\hbar^2}{2m_i}\nabla_i^2 + \frac{1}{2}m_i\omega^2 x_i^2$$

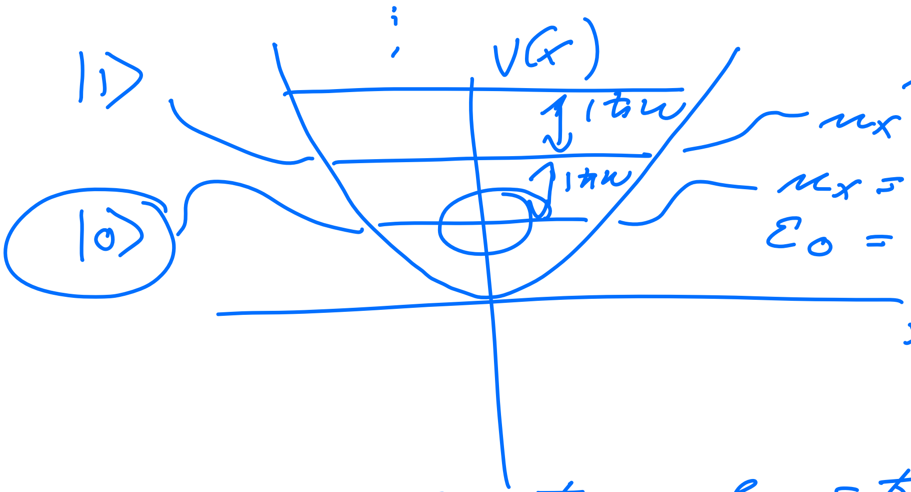
Eigenbasis $\varphi_\alpha(x_i)$

$$x_i = \{\vec{r}_i, \vec{\sigma}_i\} \qquad \vec{r}_i \rightarrow \tilde{x}_i \text{ 1-Dim}$$

$$\varphi_\alpha(x_i) \rightarrow \varphi_{n_x}(x_i)$$

$$H_0(x_i)\varphi_{n_x}(x_i) = \hbar\omega(n_x + 1/2) \times \varphi_{n_x}(x_i)$$

$n_x = 0, 1, 2, \ldots \infty$



n_x $\varepsilon_1 = \hbar\omega \cdot 3/2$

$n_x = 0$

$\varepsilon_0 = \hbar\omega/2$

an infinity of states

$\varphi_{nx}(x_i)$ tensor product of spacial and spin degrees of freedom

$\sigma_i = 1/2$; $m_{s_i} = +1/2, -1/2$

$$\chi_{\sigma_i m_{s_i}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

spin up $\uparrow$ $m_{s_i} = 1/2$

spin down $\Downarrow$ $m_{s_i} = -1/2$

Discrete single-particle positions :

$$\psi_{nx}(x) = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$x : \{x_0, x_1, \ldots x_{n-1}\}$

$$\varphi_{nx_i}(x, \sigma_i) = \psi_{nx_i}(x) \otimes \chi_{\sigma_i m_{s_i}}$$

$$= \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \otimes X_{r_i, m_{s_i}}$$

$$= \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \vdots \\ x_{n-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = |n_{x_i}, r_i, m_{s_i}\rangle$$

More general basis

$$|x\rangle = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} = x \in \mathbb{C}^n$$

$$\langle x| = \begin{bmatrix} x_0^* & x_1^* & x_2^* & \cdots & x_{n-1}^* \end{bmatrix}$$

$$= \left(|x\rangle\right)^{\dagger} = \langle x|$$

$$\left(|x\rangle^{\dagger}\right)^{\dagger} = |x\rangle$$

Example

$$|x\rangle = \begin{bmatrix} 1-i \\ 2+i \end{bmatrix}$$

$$|x\rangle^{\dagger} = \langle x| = \left[\overline{1+i}, 2-i\right]$$

First postulate

Every quantum system is described completely by a set of state vectors. All properties of the system can be deducted from the state vectors

Set of state vectors

$|\phi_0\rangle, |\phi_1\rangle, \ldots |\phi_{n-1}\rangle$

$$\langle \phi_i | \phi_j \rangle = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

can define a new state

$$|\psi\rangle = a_0 |\phi_0\rangle + a_1 |\phi_1\rangle + \dots a_{n-1} |\phi_{n-1}\rangle$$

$$= \sum_{i=0}^{n-1} a_i |\phi_i\rangle$$

$$a_i = \langle \psi | \phi_i \rangle$$

The basis $\{ |\phi_i\rangle \}$ allows us to describe any point in the state space of the system.

Quantum notation for qubits

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$

$$\langle 0|1\rangle = 0$$

inner product between $|x\rangle$ and $|y\rangle \in \mathbb{C}^n$

$$\langle y|x\rangle = \begin{bmatrix} y_0^* & y_1^*, & \cdots & y_{n-1}^* \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$= \sum_{i=0}^{n-1} y_i^* x_i$$

A general qubit state

$$|\psi\rangle = a_0|0\rangle + a_1|1\rangle$$

ket-state

$$= a_0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

$$\langle\psi| = a_0^* \langle 0| + a_1^* \langle 1|$$

$$= \begin{bmatrix} a_0^* & a_1^* \end{bmatrix}$$

Bra-state (are row vectors)

General inner product

$|\psi\rangle$, $|\beta\rangle$

computational basis $\{|\phi_i\rangle\}$

$$|\psi\rangle = \sum_{i=0}^{n-1} a_i |\phi_i\rangle$$

$$|\beta\rangle = \sum_{i=0}^{n-1} b_i |\phi_i\rangle$$

inner product

$$\langle\psi|\beta\rangle = \left(\sum_i a_i^* \langle\phi_i|\right) \times \left(\sum_j b_j |\phi_j\rangle\right)$$

$$= \sum_{ij} a_i^* b_j \underbrace{\langle\phi_i|\phi_j\rangle}_{\delta_{ij}}$$

ONB

$$= \sum_i a_i^* b_i$$

For the qubit basis

$$\langle 0|0\rangle = \langle 1|1\rangle = 1$$

$$|\psi_1\rangle = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, \quad |\psi_2\rangle = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

$$\Rightarrow \langle\psi_1|\psi_2\rangle = a_0^* b_0 + a_1^* b_1$$

Note that if we have two different vectors $|\psi_1\rangle$ and $|\psi_2\rangle$

$$\langle\psi_1|\psi_2\rangle \neq \langle\psi_2|\psi_1\rangle$$

Example

$$|\psi_1\rangle = \begin{bmatrix} -1 \\ 2i \\ 1 \end{bmatrix} \quad |\psi_2\rangle = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

$$\langle\psi_1|\psi_2\rangle = -1 + i$$

$$\langle\psi_2|\psi_1\rangle = -1 - i$$

This leads to

$$\langle\psi_1|\psi_2\rangle^* = \langle\psi_2|\psi_1\rangle$$

Outer product

$$|x\rangle = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$|x\rangle\langle x| = \begin{bmatrix} x_0 x_0^* & x_0 x_1^* & \cdots & x_0 x_{n-1}^* \\ x_1 x_0^* & - & - & \cdot \\ \vdots & & & \\ x_{n-1} x_0^* & - & - & x_{n-1} x_{n-1}^* \end{bmatrix}$$

$$|x\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$|x\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow |x\rangle\langle x| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|\phi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \wedge |\phi_1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbb{1} = \sum_{i=0}^{n-1=1} |\phi_i\rangle\langle\phi_i| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

in more general terms

$$\mathbb{1} = \sum_{i=0}^{n-1} |\phi_i\rangle\langle\phi_i|$$

$|\phi_0\rangle$ and $|\phi_1\rangle$ case

$$\mathbb{1} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{|\phi_0\rangle\langle\phi_0|} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{|\phi_1\rangle\langle\phi_1|}$$

$$\hat{P} = \hat{P}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{Does } P^{-1} \text{ exist?}$$

$$|\phi_1\rangle\langle\phi_1| = \hat{Q} = \hat{Q}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\hat{P}, \hat{Q}] = 0 \qquad \hat{P}\hat{Q} = 0$$

Define general qubit state

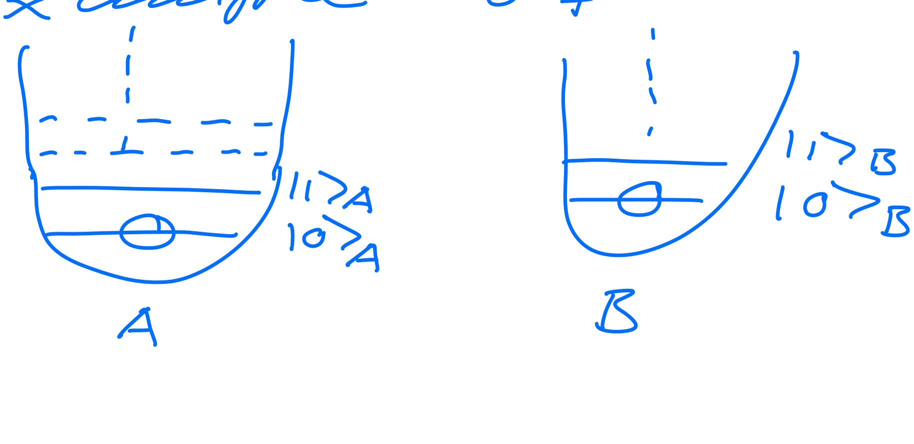
$$|\psi\rangle = a_0 \underbrace{|\phi_0\rangle}_{|0\rangle} + a_1 \underbrace{|\phi_1\rangle}_{|1\rangle}$$

$$\hat{P}|\psi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_0|0\rangle$$

$$\hat{Q}|\psi\rangle = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = a_1|1\rangle$$

What more complicated qubit states?

Example 2 quantum dots



Two qubit state

$$|0\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = |00\rangle \quad (|0\rangle)$$

$$|0\rangle_A \otimes |1\rangle_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = |01\rangle \quad (|1\rangle)$$

$$|1\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = |10\rangle \quad (|2\rangle)$$

$$|1\rangle_A \otimes |1\rangle_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = |11\rangle$$

$$(|3\rangle)$$