Lecture November 12

Computational basis (1-qubit)

$$|0\rangle \quad \left(\begin{bmatrix}1\\0\end{bmatrix}\right) \qquad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Hadamard basis

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$H|0\rangle = |+\rangle$$

$$H|1\rangle = |-\rangle$$

NOT-gate

$$X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \qquad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$X|0\rangle = |1\rangle$$

Density matrix

set of states $|\psi_i\rangle$ with

probabilities $p_i \geq 0$

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$$\sum p_i = 1$$

Density matrix

$$\rho = \sum_{i=1}^{d} p_i \, |\varphi_i\rangle\langle\varphi_i|$$

Maximally mixed state

$$\pi = \frac{1}{d} \underbrace{\sum_x |x\rangle\langle x|}_{\mathbb{I}} = \frac{\mathbb{I}}{d}$$

$$\pi = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1| \right)$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\mathbb{I}}{2}$$

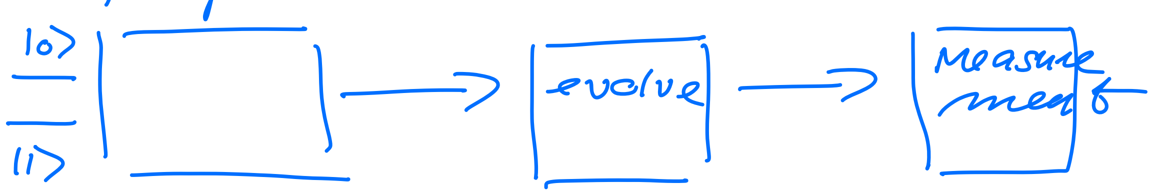
purity of a state

$$P(\rho) = \mathrm{Tr}\left[\rho^\dagger \rho\right] = \mathrm{Tr}\left[\rho^2\right]$$

if pure state $P(\rho) = 1$

$P(\rho) < 1$, mixed state.

preparation



Measurement

x possible outcomes in a finite set X $\quad x \in X$

M_x = measurement operator

$$\sum_{x \in X} M_x^{+} M_x = \mathbb{1}$$

probability:

$$P_\psi(x) = \langle \psi | M_x^{+} M_x | \psi \rangle$$

$|0\rangle$ $\wedge$ $|1\rangle$

$M_0 = |0\rangle\langle 0| \qquad M_1 = |1\rangle\langle 1|$

$$M_0^{+} M_0 = M_0^2 = M_0 \quad (|0\rangle \underbrace{\langle 0|0\rangle}\langle 0|$$
$$= |0\rangle\langle 0|$$

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

$$P_\psi(0) = \langle \psi | M_0^{+} M_0 | \psi \rangle =$$
$$\langle \psi | M_0 | \psi \rangle$$
$$\left\{ \left(\langle 0| \alpha^* + \langle 1| \beta^* \right) |0\rangle\langle 0| \left(\alpha |0\rangle + \beta |1\rangle \right) \right\}$$
$$= |\alpha|^2$$

$$P_{\psi_1}(1) = |\beta|^2$$

$$|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \Rightarrow P_\psi(0) = P_\psi(1) = 1/2$$

After the measurement

$$|\psi'\rangle = \frac{M_x |\psi\rangle}{\sqrt{\langle\psi| M_x^\dagger M_x |\psi\rangle}}$$

$$|\psi_0'\rangle = \frac{M_0 |\psi\rangle}{\sqrt{\langle\psi| M_0^\dagger M_0 |\psi\rangle}} = \frac{\alpha}{|\alpha|} |0\rangle$$

$$|\psi_0'\rangle = e^{i\phi} |0\rangle \quad \phi \in \mathbb{R}$$

$$|\psi_1'\rangle = \frac{\beta}{|\beta|} |1\rangle$$

suppose the system is described by a collection of states $\{p_i, |\psi_i\rangle\}$

$$P_{\psi_i}(x) = \langle\psi_i| M_x^\dagger M_x |\psi_i\rangle$$

$$= Tr\left[M_x^\dagger M_x |\psi_i\rangle\langle\psi_i| \right]$$

Example

$$M_0 = |0\rangle\langle 0|$$
$$M_1 = |1\rangle\langle 1|$$

$$P_{\psi_i}(x) = \langle\psi_i| M_0^\dagger M_0 |\psi_i\rangle$$
$$+ \langle\psi_i| M_1^\dagger M_1 |\psi_i\rangle$$

$$|\psi_i\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|0\rangle\langle 0| \left(\alpha|0\rangle + \beta|1\rangle \right)\left(\langle 0|\alpha^* + \langle 1|\beta^* \right)$$
$$+ \beta\alpha^* \cancel{|1\rangle}\langle 0| + \alpha|0\rangle\langle 1|\beta^*$$

$$+ |1\rangle\langle 1| \left(\underline{\qquad\qquad} \; 1 \; \underline{\qquad} \right)$$
$$+ \alpha^*|1\rangle\beta|0\rangle \quad \alpha^*\beta|1\rangle\langle 0|$$

$$= |0\rangle\langle 0| \left(|\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1| \right)$$
$$+ |1\rangle\langle 1| \left(\underline{\qquad\qquad} \; 1 \; \underline{\qquad} \right)$$

$$|0\rangle\langle 1| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$|1\rangle\langle 0| = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$|0\rangle\langle 0| |\alpha|^2 + \alpha\beta^* |0\rangle\langle 1|$$

$$|0\rangle\langle 0|\,|\alpha|^2 + \alpha\beta^*\,|0\rangle\langle 1|$$

$$+\ |1\rangle\langle 1|\,|\beta|^2 + \alpha^*\beta\,|1\rangle\langle 0|$$

$$= \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}$$

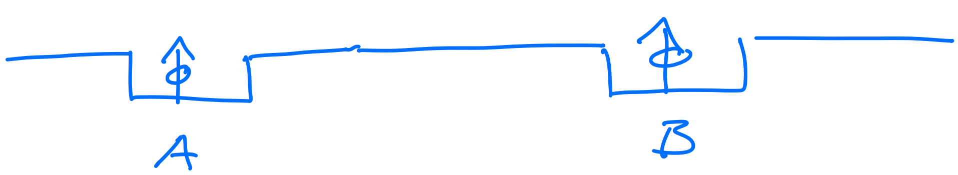
$$\mathrm{Tr}\left[\quad\quad\right] = |\alpha|^2 + |\beta|^2$$

composite systems + Entanglement

computational basis

dim = 2 $\{|0\rangle, |1\rangle\}$

$$\mathcal{H}_A \otimes \mathcal{H}_B$$



$$|0\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= |00\rangle_{AB} = |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\langle 01 | 11 \rangle = 0$$

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$= \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix}$$

Bell states

$$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$|\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Psi^+\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$|\Psi^-\rangle = \frac{|10\rangle - |01\rangle}{\sqrt{2}}$$

Measurement on one of these bits, automagically determines the other

$$M_0 = |0\rangle_A\langle 0| \otimes I_B$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & a_{12}\begin{bmatrix} \ \end{bmatrix} \\ a_{21}\begin{bmatrix} \ \end{bmatrix} & a_{22}\begin{bmatrix} \ \end{bmatrix} \end{bmatrix}$$

$$\Big[\ 1 \quad 0 \quad 0$$

$$\begin{bmatrix} 0 & 1 & & \\ 0 & & 0 & \end{bmatrix}$$

$$P_{\phi^+}(0) = {}_{AB}\langle \phi^+ | \underbrace{M_0^+ M_0}_{M_0} | \phi^+ \rangle_{AB}$$

$$= \frac{1}{2} \Big(\underbrace{{}_A\langle 0| \, {}_B\langle 0|}_{[1\ 0\ 0\ 0]} + \underbrace{{}_A\langle 1| \, {}_B\langle 1|}_{[0\ 0\ 0\ 1]} \Big) M_0$$

$$\Big(\underbrace{|0\rangle_A \, |0\rangle_B}_{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + \underbrace{|1\rangle_A \, |1\rangle_B}_{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}} \Big)$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1/2$$

$$|\psi_0'\rangle = |0\rangle_A \, |0\rangle_B$$

$$|\psi_1'\rangle = |1\rangle_A \, |1\rangle_B$$

Entanglement