FYS 5419/9419, JAN 30, 2023

Trace of density matrix

$\rho = |\psi\rangle\langle\psi|$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$\rho = \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{bmatrix}$

$\rho|0\rangle = \begin{bmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} \alpha\alpha^* \\ \beta\alpha^* \end{bmatrix}$

$Tr(A) = \sum_{i=0}^{n-1} a_{ii}$

$Tr(cA) = c\,Tr(A)$

$Tr(AB) = Tr(BA)$

$Tr(A \otimes B) = Tr(A)\,Tr(B)$

$Tr(\rho) = \alpha\alpha^* + \beta\beta^* = 1$

Quantum gates/operations

$$|\varphi\rangle = U|\phi\rangle$$

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$$

$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix}$$

$$|\varphi\rangle = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} u_{00}\alpha + u_{01}\beta \\ u_{10}\alpha + u_{11}\beta \end{bmatrix}$$

$$UU^{\dagger} = \mathbb{1} \qquad U^{\dagger}_{ij} = U^{*}_{ji}$$

$$= U^{\dagger}U \implies \qquad U^{\dagger} = U^{-1}$$

$$U^{-1}|\varphi\rangle = |\phi\rangle$$

$$\hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \det P = 0$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad \det Q = 0$$

$$\langle \phi | \phi \rangle = \alpha\alpha^* + \beta\beta^* = 1$$

$$\langle \varphi | \varphi \rangle = \langle \phi | \underbrace{U^\dagger U}_{I} | \phi \rangle = 1$$

Pauli matrices

$$\sigma_x = X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_x^{-1} = \frac{1}{\det \sigma_x} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\det \sigma_x = -1$$

$$\sigma_x^{-1} \sigma_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_x^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbb{1}$$

$$\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_y^2 = \mathbb{1}$$

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \det \sigma_z = -1$$

$$\sigma_z^{-1} = \sigma_z \qquad \sigma_z^2 = \mathbb{1}$$

$$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\sigma_x|\varphi\rangle = \alpha\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

$$= \alpha\begin{bmatrix}0\\1\end{bmatrix} + \beta\begin{bmatrix}1\\0\end{bmatrix}$$

$$= \alpha|1\rangle + \beta|0\rangle$$

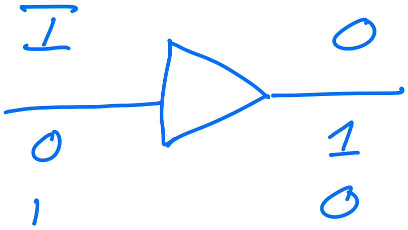
$$\sigma_x\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

bit 0 associate with $\begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$

can think of this as a bit flip operation.

$$\sigma_x\begin{bmatrix}0\\1\end{bmatrix} = \sigma_x|1\rangle = |0\rangle$$

Classical not gate



I	O
0	1
1	0

Quantum mechanical equivalent is QNOT

given $\sigma_x |\varphi\rangle$

$$\sigma_z |\varphi\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \left[\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

$$= \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \alpha |0\rangle - \beta |1\rangle$$

$$[\sigma_x, \sigma_y] = i \sigma_z \cdot 2$$

short reminder on Hermitian operators

$$(\alpha A)^\dagger = \alpha^* A^\dagger$$

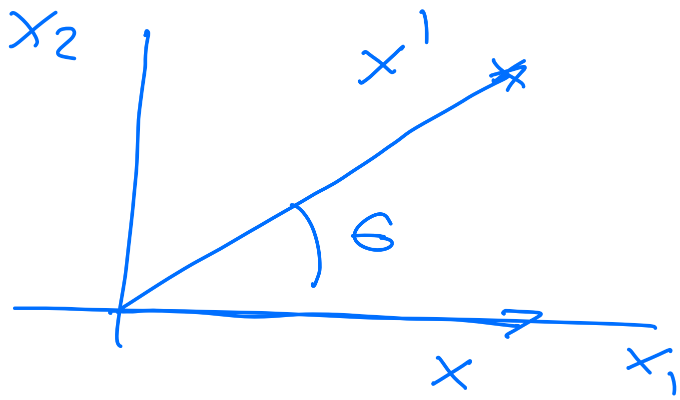
$$(AB)^\dagger = B^\dagger A^\dagger$$

$$(A|\varphi\rangle)^\dagger = \langle\varphi| A^\dagger$$

$$(AB|\varphi\rangle)^\dagger = \langle\varphi| B^\dagger A^\dagger$$

$$A = |\psi\rangle\langle\varphi| \qquad A^\dagger = |\varphi\rangle\langle\psi|$$

Rotations



$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = |x\rangle \qquad x' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = |x'\rangle$$

$$|x'\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$|x\rangle \rightarrow |\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$|\psi'\rangle = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{U} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$U^T U = U^\dagger U = \mathbb{1}$$

$\alpha = 1 \quad \wedge \quad \beta = 0$

$$|\psi'\rangle = \alpha \cos\theta\,|0\rangle + \alpha \sin\theta\,|1\rangle$$

in more general terms

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Define a rotation around the x, y, and z axes

$$R_x(\theta) = e^{i\,\theta/2\,\sigma_x}$$

$$= \cos\theta/2\,\mathbb{1} - i\sin(\theta/2)\,\sigma_x$$

$$R_y(\theta) = e^{i\,\theta/2\,\sigma_y}$$

$$R_z(\theta) = e^{i\,\theta/2\,\sigma_x}$$

$$e^{i\theta A} = \cos\theta\, I + i\sin\theta\, A$$

$A^2 = \mathbb{1}$

$$R_x(\theta) = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ +i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$$

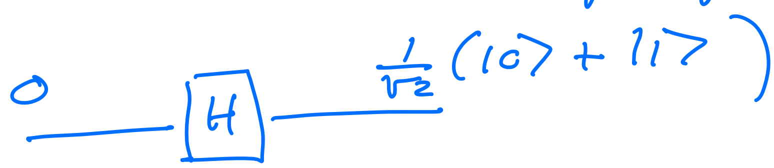
Hadamard matrix

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

created a superposition.



$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} [|0\rangle - |1\rangle]$$

Hadamard matrices

$$H_1 = 1$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2^k} = \begin{bmatrix} H_{2^{k-1}} & H_{2^{k-1}} \\ H_{2^{k-1}} & -H_{2^{k-1}} \end{bmatrix}$$

$$\mathrm{Tr}\left[H_{2^k}\right] = 0$$

$$H H^T = n I_n \qquad n = 2^k$$

Two level system:

$|\psi_1\rangle$

$|\psi_0\rangle$

unknown

$$H|\psi_0\rangle = (H_0 + \lambda H_1)|\psi_0\rangle = E_0|\psi_0\rangle$$

$$H|\psi_1\rangle = (H_0 + \lambda H_1)|\psi_1\rangle = E_1|\psi_1\rangle$$

known

$$\left.\begin{aligned} H_0|0\rangle &= \varepsilon_0|0\rangle \\ H_0|1\rangle &= \varepsilon_1|1\rangle \end{aligned}\right] \text{ comp. basis}$$

$$|\psi_0\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|\psi_1\rangle = \gamma|0\rangle + \delta|1\rangle$$

$$\langle 0|H|\psi_1\rangle \quad \langle 1|H|\psi_1\rangle \quad \Rightarrow$$

$$\langle 0|H|\psi_0\rangle \quad \langle 1|H|\psi_0\rangle$$

$$\begin{bmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle \end{bmatrix} C$$

$$= CE$$