FYS5419/9419, FEB 27, 2023

$$H = H_0 + H_I$$

computational basis

$$|00\rangle = |0\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|01\rangle = |0\rangle_A \otimes |1\rangle_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = |1\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|11\rangle = |1\rangle_A \otimes |1\rangle_A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

ONB : $\{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$

$$H_0 |00\rangle = \varepsilon_0 |00\rangle = \varepsilon_0 |0\rangle$$

$$H_0 |01\rangle = \varepsilon_1 |01\rangle = \varepsilon_1 |1\rangle$$

$$H_0 |10\rangle = \varepsilon_2 |10\rangle = \varepsilon_2 |2\rangle$$

$$H_0 |11\rangle = \varepsilon_3 |11\rangle = \varepsilon_3 |3\rangle$$

Eigenstates of $H = H_0 + H_1$

$$|\psi_0\rangle = \alpha_0 |00\rangle + \alpha_1 |01\rangle + \alpha_2 |10\rangle + \alpha_3 |11\rangle$$

$$H = \begin{bmatrix} \varepsilon_0 + \lambda H_I(00) & \lambda H_1(01) & \lambda H_1(10) & \lambda H_1(11) \\ \lambda H_1(10) & - & - & - \\ \lambda H_1(20) & - & - & - \\ \lambda H_1(30) & - & - & \varepsilon_3 + \lambda_I(33) \end{bmatrix}$$

$$H = H(\lambda) = H_0 + \lambda H_1$$

$$\lambda \in [0, 1]$$

$$\lambda H_1 = \lambda(\sigma_x \otimes \sigma_x) + \lambda(\sigma_z \otimes \sigma_z)$$

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$|\psi_0\rangle \to |\psi_0(\lambda)\rangle$$

$$= \alpha_0(\lambda)|00\rangle + \alpha_1(\lambda)|01\rangle + \alpha_2(\lambda)|10\rangle + \alpha_3(\lambda)|11\rangle$$

$$H\psi_0 = E_0 \psi_0$$

$H\psi_1 = E_1\psi_1$, $H\psi_2 = E_2\psi_2$

$H\psi_3 = E_3\psi_3$

$\rho_0 = |\psi_0\rangle\langle\psi_0| = |\psi_0\rangle_{AB}\langle\psi_0|$

$\rho_A = Tr_B(\rho_0) = \rho_B = Tr_A(\rho_0)$

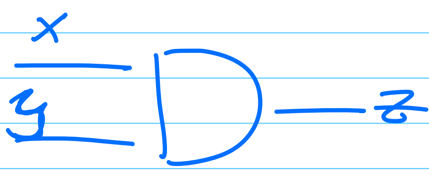
Classical (logic) gates & circuits

NOT GATE

reversible

x	y
0	1
1	0

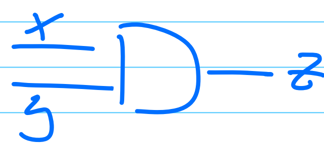
XOR



irreversible or non-invertible

x	y	z
0	0	0
0	1	1
1	0	1
1	1	0

NAND



non-invertible

(NOR, OR, AND)

x	y	z
0	0	1
0	1	1
1	0	1
1	1	0

Quantum computing gates

- input & output are n-qubit states
- Two-qubit is $|\varphi\rangle_A \otimes |\psi\rangle_B$
- The ONBs for the analysis
  1) one-qubit gates $|0\rangle$ and $|1\rangle$
  2) two-qubit gates, $|00\rangle, |01\rangle$ $|10\rangle$ and $|11\rangle$ (4 ONBs)
  3) three-qubit gates, eight vectors

$$|000\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C$$

$$|001\rangle, |010\rangle \quad = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$|011\rangle, |100\rangle, |101\rangle$

$|110\rangle$ and $|111\rangle$

initial state $|\psi\rangle$
output state $|\varphi\rangle$

$$|\varphi\rangle = \hat{G}|\psi\rangle$$

- one-qubit gates $\hat{G}$
- two-qubit gates - CNOT
- three-qubit gates, Fredkin and Toffoli.

— One qubit gates

initial state $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

output state $|\psi'\rangle = \alpha_0'|0\rangle + \alpha_1'|1\rangle$

$$\hat{G} = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix}$$

$$\hat{G}^{\dagger}\hat{G} = \mathbb{1} = \hat{G}\hat{G}^{\dagger}$$

$$|\psi'\rangle = G|\psi\rangle$$

$$\begin{bmatrix} \alpha_0' \\ \alpha_1' \end{bmatrix} = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$\alpha_0' = g_{00}\alpha_0 + g_{01}\alpha_1$$

$$\alpha_1' = g_{10}\alpha_0 + g_{11}\alpha_1$$

1) identity gate $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2) X or NOT gate, σ_x

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ (classical NOT gate)

3) Y

$$Y = \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

4) Z gates (changes the phase of a qubit, flip the sign)

$$Z = \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

5) Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

simple algo : Random numbers

$$H|0\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

on measurement the state will collapse to $|0\rangle$ or $|1\rangle$, with the same probability, 50%.

$$H \times H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \mathbb{1}$$

$$|\psi\rangle = H \times H |\psi\rangle = \mathbb{1} |\psi\rangle$$

$$= |\psi\rangle$$

$$|\varphi\rangle = H \times H \cdots H|\psi\rangle = H^{2k}|\psi\rangle$$
$$= (H^2)^k|\psi\rangle = I^k|\psi\rangle$$
$$= |\psi\rangle$$

$n = 2k+1$, $\quad H|\psi\rangle$

$$|\varphi\rangle = (H)^{2k+1} = H|\psi\rangle$$

Two-qubit gate CNOT

$|00\rangle, |01\rangle, |10\rangle, |11\rangle$

input state $|\varphi\rangle \otimes |\psi\rangle$

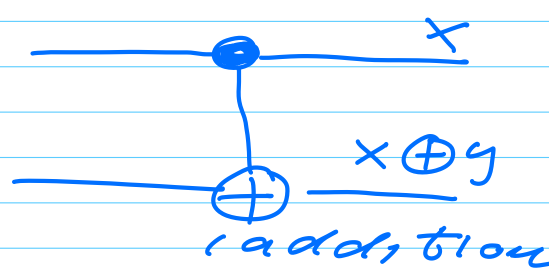
$$= \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$|\varphi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$
$$|\psi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

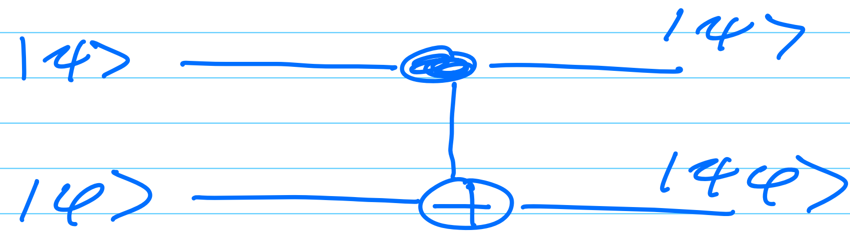
$$\begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix} = |\varphi\rangle \otimes |\psi\rangle$$

control/input x

Target input y



modulo 2



The components of the input vector

$|00\rangle \Rightarrow |00\rangle$

$|01\rangle \Rightarrow |01\rangle$

$|10\rangle \Rightarrow |11\rangle$

$|11\rangle \Rightarrow |10\rangle$

classical equivalent is the XOR gate

x	y	z
0	0	0
0	1	1
1	0	1
1	1	0

The CNOT gate

$$G_{CNOT} = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$