Infinite Hierarchy of Nonequilibrium Criticality in Long-Range Interacting Systems

Theoretical, Computational, and Experimental Studies

Patrick Cook

Advisor: Dr. Mohammad Maghrebi

Michigan State University

September 2022

Abstract

Quantum many-body systems with long-range interactions exhibit a variety of peculiar equilibrium and nonequilibrium phenomena. Foremost of these are critical exponents and their associated universal scaling regimes, where the properties of a system become independent of all but a few general properties such as the dimensionality and the range of interactions. We show—analytically, numerically, and experimentally—that there exists an infinite hierarchy of nonequilibrium critical exponents. To aid in this and future work, we also present a theory-driven (as opposed to data-driven) machine learning approach to efficiently and accurately approximate Hamiltonian and Liouvillian systems.

Outline

Theory

- Phase Transitions
- Criticality
- Lipkin-Meshkov-Glick Model
- Generalized Model
- **Critical Hierarchy**

Computation

- Numerical Linear Algebra
- Matrix-Free Methods
- **Spectral Approximation**

Experiment

- Trapped Ions
- Model
- **Results and Analysis**

Outline

Theory

- Phase Transitions
- Criticality
- Lipkin-Meshkov-Glick Model
- Generalized Model
- **Critical Hierarchy**

Computation

- Numerical Linear Algebra
- Matrix-Free Methods
- **Spectral Approximation**

Experiment

- Trapped Ions
- Model
- **Results and Analysis**

Outline

Theory

- Phase Transitions
- Criticality
- Lipkin-Meshkov-Glick Model
- Generalized Model
- **Critical Hierarchy**

Computation

- Numerical Linear Algebra
- Matrix-Free Methods
- **Spectral Approximation**

Experiment

- Trapped Ions
- Model
- **Results and Analysis**

Main Points

Prediction of Critical Hierarchy

- Distinct nonequilibrium critical exponents

Prediction of Critical Hierarchy

- Distinct nonequilibrium critical exponents
- Generated by quenching the system

Prediction of Critical Hierarchy

- Distinct nonequilibrium critical exponents
- Generated by quenching the system
- $1/2 \longrightarrow 3/4 \longrightarrow 7/8 \longrightarrow 15/16 \longrightarrow \cdots$

Development of Fast Approximate Algorithm

- Approximately simulate any Hamiltonian or Liouvillian system

Development of Fast Approximate Algorithm

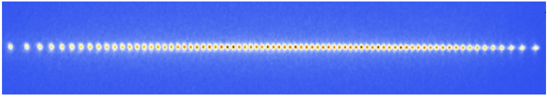
- Approximately simulate any Hamiltonian or Liouvillian system
- Theory-driven machine learning approach

Development of Fast Approximate Algorithm

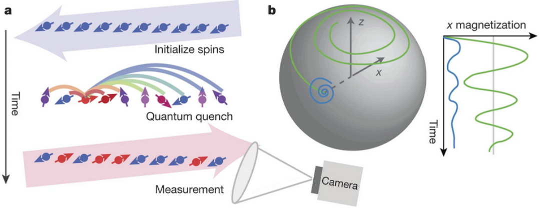
- Approximately simulate any Hamiltonian or Liouvillian system
- Theory-driven machine learning approach
- From $\mathcal{O}\left(n^3\right)$ to $\mathcal{O}\left(n\right)$

Physical Realization with Trapped Ions

- Popular quantum simulation platform



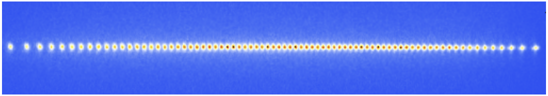
Adapted from Defenu et al. 2021, *arXiv:2109.01063*.



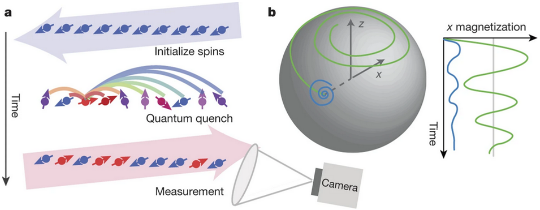
Reproduced from Zhang et al. 2017, *Nature*.

Physical Realization with Trapped Ions

- Popular quantum simulation platform
- Recent paper on observation of dynamical phase transition



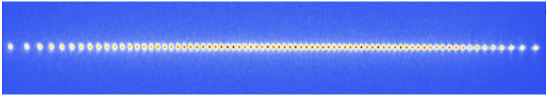
Adapted from Defenu et al. 2021, *arXiv:2109.01063*.



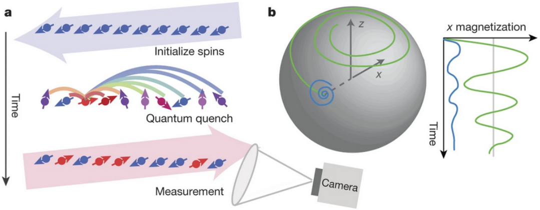
Reproduced from Zhang et al. 2017, *Nature*.

Physical Realization with Trapped Ions

- Popular quantum simulation platform
- Recent paper on observation of dynamical phase transition
- Use to show experimental agreement with our theory



Adapted from Defenu et al. 2021, *arXiv:2109.01063*.



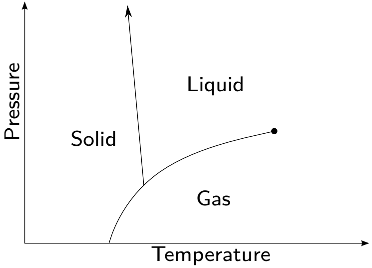
Reproduced from Zhang et al. 2017, *Nature*.

Theory

What interesting physics do we predict?

Phase Transitions

- Change in system state
- Response to changes in some external parameter
- Oftentimes abrupt
- Quantified by an "order parameter", m



Reproduced from Eastman 2015, *Stanford University* under CC BY-ND 4.0.

Criticality and Universality

In General:

- Divergent correlation length
- $\langle s(x)s(y)\rangle \sim 1$

Criticality and Universality

In General:

- Divergent correlation length
- Critical slowdown

- $\langle s(x)s(y)\rangle \sim 1$
- Equilibration is slow

Criticality and Universality

In General:

- Divergent correlation length
- Critical slowdown
- Power-law divergence
  $\longrightarrow$ critical exponents

- $\langle s(x)s(y)\rangle \sim 1$
- Equilibration is slow
- $m \sim \tau^{-a}$ or $m \sim L^{b}$

Criticality and Universality

In General:

- Divergent correlation length
- Critical slowdown
- Power-law divergence
  $\longrightarrow$ critical exponents

- $\langle s(x)s(y)\rangle \sim 1$
- Equilibration is slow
- $m \sim \tau^{-a}$ or $m \sim L^{b}$

Universality

Physically distinct systems have identical critical exponents

Criticality and Universality

In General:

- Divergent correlation length
- Critical slowdown
- Power-law divergence $\longrightarrow$ critical exponents

- $\langle s(x)s(y)\rangle \sim 1$
- Equilibration is slow
- $m \sim \tau^{-a}$ or $m \sim L^{b}$

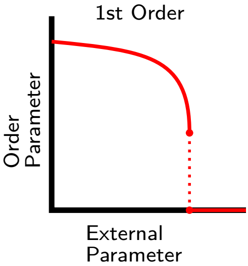
Universality

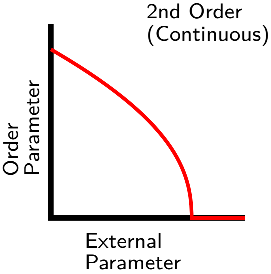
Physically distinct systems have identical critical exponents

Liquid-gas transition (any chemical composition)	3D uniaxial ferromagnet-paramagnet transition

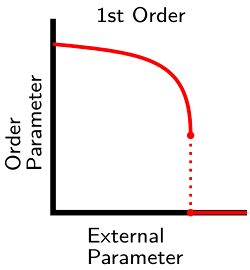
Same critical exponents!

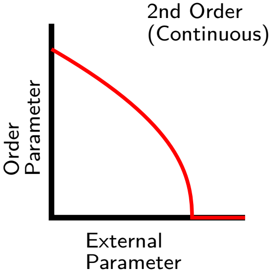
First Order Versus Second Order





First Order Versus Second Order

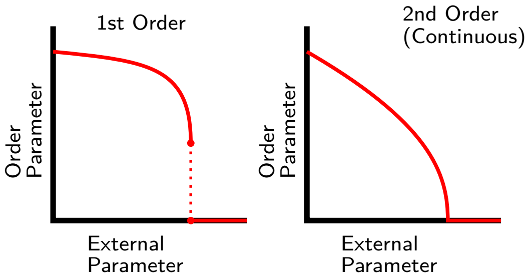




First Order

- Discontinuous order parameter

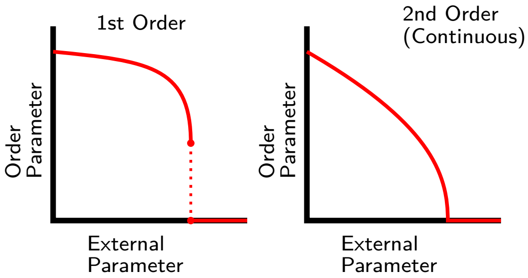
First Order Versus Second Order



First Order

- Discontinuous order parameter
- Involve latent heat

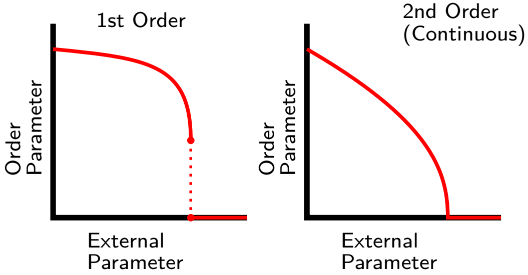
First Order Versus Second Order



First Order

- Discontinuous order parameter
- Involve latent heat
- Water-Ice transition

First Order Versus Second Order



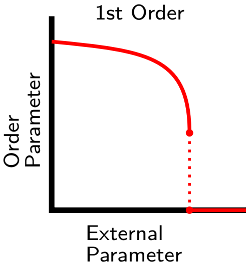
First Order

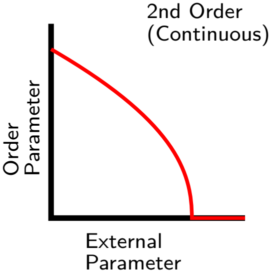
- Discontinuous order parameter
- Involve latent heat
- Water-Ice transition

Second Order (Continuous)

- Continuous order parameter

First Order Versus Second Order





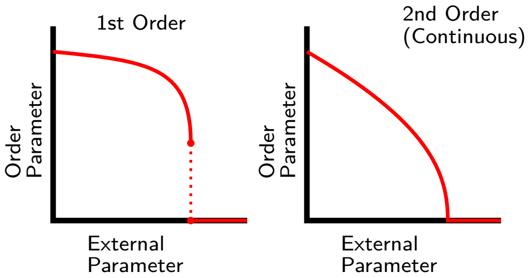
First Order

- Discontinuous order parameter
- Involve latent heat
- Water-Ice transition

Second Order (Continuous)

- Continuous order parameter
- Discontinuous first derivative

First Order Versus Second Order



First Order

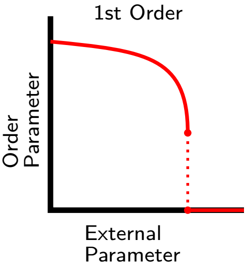
- Discontinuous order parameter
- Involve latent heat
- Water-Ice transition

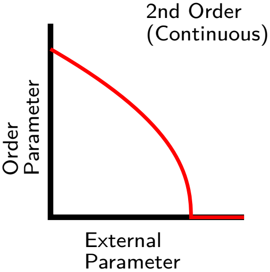
Second Order (Continuous)

- Continuous order parameter
- Discontinuous first derivative
- Criticality and universality[1]

[1] Critical phenomena are not entirely exclusive to continuous phase transitions.

First Order Versus Second Order





First Order

- Discontinuous order parameter
- Involve latent heat
- Water-Ice transition

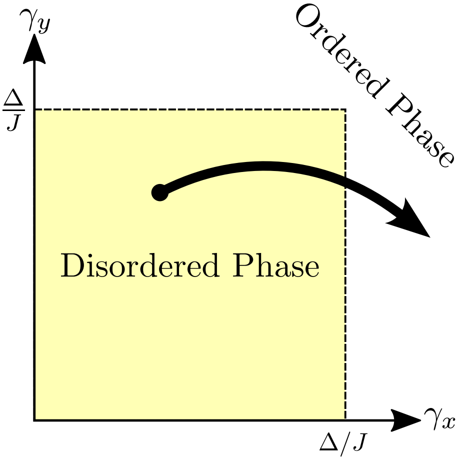
Second Order (Continuous)

- Continuous order parameter
- Discontinuous first derivative
- Criticality and universality[1]
- Ferromagnetic transition

[1] Critical phenomena are not entirely exclusive to continuous phase transitions.

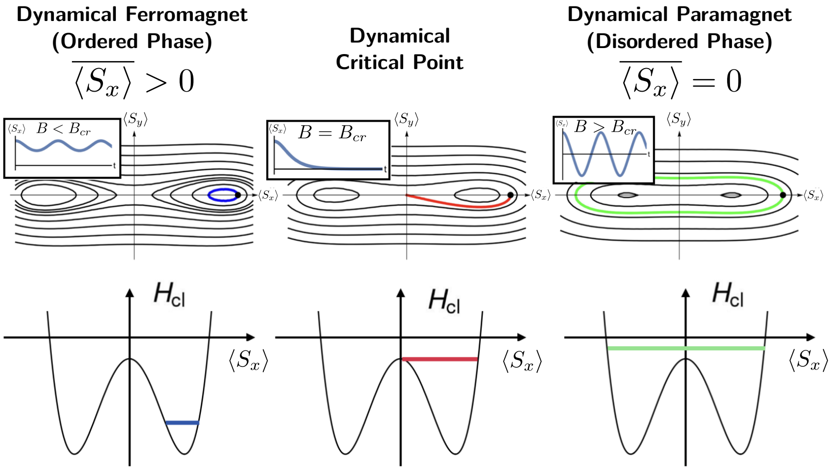
Dynamical Phase Transitions I

- Nonequilibrium
- Manifests in dynamics
- Can be associated with pre-thermalization[1]



[1] Marino et al. 2022, *arXiv:2101.09894*

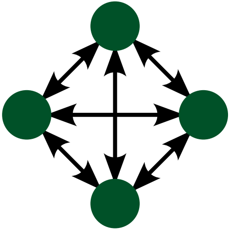
Dynamical Phase Transitions II



Adapted from Marino et al. 2022, *arXiv:2101.09894*.

Lipkin-Meshkov-Glick (LMG) Model

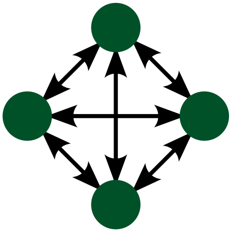
- N spin-$1/2$ particles
- Infinite range couplings
- Transverse field B



$$H_{LMG} = -\frac{J}{2N}\sum_{i\neq j}^{N}\left(\gamma_x\sigma_i^x\sigma_j^x + \gamma_y\sigma_i^y\sigma_j^y\right) - B\sum_{i}^{N}\sigma_i^z$$

Lipkin-Meshkov-Glick (LMG) Model

- N spin-$1/2$ particles
- Infinite range couplings
- Transverse field B



$$H_{LMG} = -\frac{J}{2N}\sum_{i\neq j}^{N}\left(\gamma_x\sigma_i^x\sigma_j^x + \gamma_y\sigma_i^y\sigma_j^y\right) - B\sum_{i}^{N}\sigma_i^z$$

or

$$H_{LMG} = -\frac{2J}{N}\left(\gamma_x S_x^2 + \gamma_y S_y^2\right) - 2BS_z + \text{const.}$$

LMG: Previous Work

Equilibrium

- Order parameter
- Phase diagram
- Scaling

LMG: Previous Work

Equilibrium

- Order parameter
- Phase diagram
- Scaling

Non-Equilibrium

- Quenches
- Dynamical phase diagram
- Divergent fluctuations

Magnetization and Fluctuations

In-plane magnetization

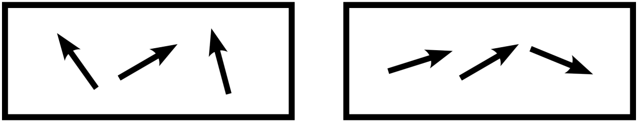
$\langle S_x \rangle /N$ or $\langle S_y \rangle /N$

Magnetization and Fluctuations

In-plane magnetization

$\langle S_x \rangle / N$ or $\langle S_y \rangle / N$

Disordered Paramagnet | Ordered Ferromagnet



$J\gamma_x < B$ | $J\gamma_x > B$

$\langle S_x \rangle = 0$ | $\langle S_x \rangle \neq 0$

Magnetization and Fluctuations

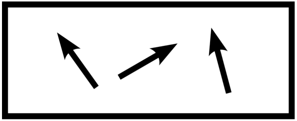
In-plane magnetization

$\langle S_x \rangle / N$ or $\langle S_y \rangle / N$

Fluctuations

$\langle S_x^2 \rangle / N$ or $\left\langle S_y^2 \right\rangle / N$

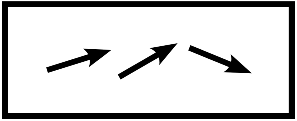
Disordered Paramagnet



$$J\gamma_x < B$$

$$\langle S_x \rangle = 0$$

Ordered Ferromagnet



$$J\gamma_x > B$$

$$\langle S_x \rangle \neq 0$$

Ground State Phase Diagram

Mean-field yields

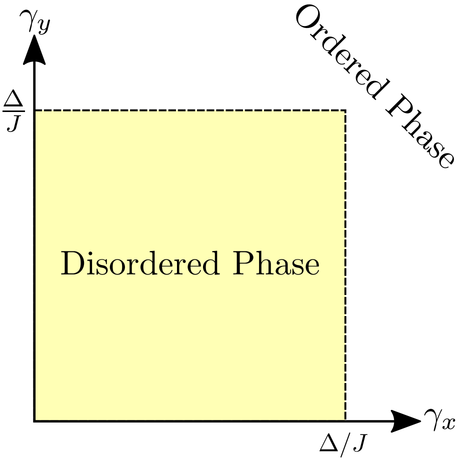
$$\langle S_{x,y}\rangle /N = \begin{cases} 0 & \text{if } J\gamma_{x,y} \leq B \\ \pm\sqrt{\gamma_{x,y}^2 - \frac{B^2}{J^2}} & \text{if } J\gamma_{x,y} > B \end{cases}$$

Click for Details

Ground State Phase Diagram

Mean-field yields

$$\langle S_{x,y} \rangle / N = \begin{cases} 0 & \text{if } J\gamma_{x,y} \le B \\ \pm\sqrt{\gamma_{x,y}^2 - \frac{B^2}{J^2}} & \text{if } J\gamma_{x,y} > B \end{cases}$$



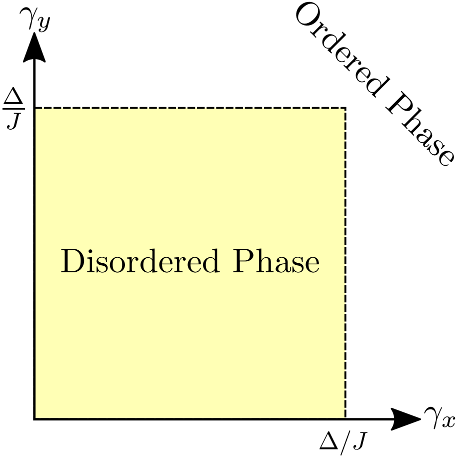
Click for Details

Ground State Phase Diagram

Mean-field yields

$$\langle S_{x,y} \rangle / N = \begin{cases} 0 & \text{if } J\gamma_{x,y} \leq B \\ \pm\sqrt{\gamma_{x,y}^2 - \frac{B^2}{J^2}} & \text{if } J\gamma_{x,y} > B \end{cases}$$

Fails to describe fluctuations



Click for Details

Holstein-Primakoff

Scaling of fluctuations in the ground state

- Disordered phase
- Critical point

Click for Details

Holstein-Primakoff

Scaling of fluctuations in the ground state

- Disordered phase
- Critical point

Map to bosons with spin operators

$$S_{\pm} = S_x \pm iS_y = \begin{matrix}(+)\\(-)\end{matrix}\begin{cases}\sqrt{N - a^\dagger a}a = \sqrt{N}\left(1 - \frac{a^\dagger a}{2N}\right)a + \mathcal{O}\left(N^{-2}\right)\\ a^\dagger\sqrt{N - a^\dagger a} = \sqrt{N}a^\dagger\left(1 - \frac{a^\dagger a}{2N}\right) + \mathcal{O}\left(N^{-2}\right)\end{cases}$$

$$S_z = N/2 - a^\dagger a$$

Click for Details

Holstein-Primakoff

$$a = (x + ip)/\sqrt{2} \qquad S_x \leftrightarrow \sqrt{\frac{N}{2}}x,\ S_y \leftrightarrow -\sqrt{\frac{N}{2}}p$$

Click for Details

Holstein-Primakoff

$$a = (x + ip)/\sqrt{2} \qquad S_x \leftrightarrow \sqrt{\frac{N}{2}}x,\ S_y \leftrightarrow -\sqrt{\frac{N}{2}}p$$

Reduces to harmonic oscillator with corrections

$$\boxed{H = \frac{1}{2m}p^2 + \frac{1}{2}m\Omega^2 x^2 + \frac{1}{N}\left[u_x x^4 + \cdots\right] + \mathcal{O}\left(N^{-2}\right)}$$

Click for Details

Holstein-Primakoff

$$a = (x + ip)/\sqrt{2} \qquad S_x \leftrightarrow \sqrt{\frac{N}{2}}x,\ S_y \leftrightarrow -\sqrt{\frac{N}{2}}p$$

Reduces to harmonic oscillator with corrections

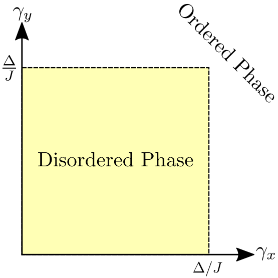
$$\boxed{H = \frac{1}{2m}p^2 + \frac{1}{2}m\Omega^2 x^2 + \frac{1}{N}\left[u_x x^4 + \cdots\right] + \mathcal{O}\left(N^{-2}\right)}$$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

Click for Details

Ground State Scaling

Disordered	Critical	Ordered
$(J\gamma_{x,y} < B)$	$(J\gamma_x = B,\ \gamma_x > \gamma_y)$	$(J\gamma_{x,y} > B)$
$\left\langle S_x^2 \right\rangle /N \sim 1$	$\left\langle S_x^2 \right\rangle /N \sim N^{1/3}$	$\left\langle S_x^2 \right\rangle /N \sim N$
$\left\langle S_y^2 \right\rangle /N \sim 1$	$\left\langle S_y^2 \right\rangle /N \sim N^{-1/3}$	$\left\langle S_y^2 \right\rangle /N \sim N$

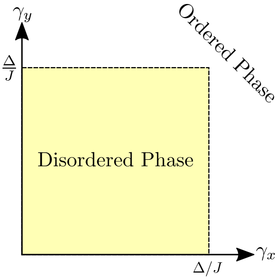


Click for Details

Ground State Scaling

Disordered	Critical	Ordered
$(J\gamma_{x,y} < B)$	$(J\gamma_x = B,\ \gamma_x > \gamma_y)$	$(J\gamma_{x,y} > B)$
$\left\langle S_x^2 \right\rangle /N \sim 1$	$\left\langle S_x^2 \right\rangle /N \sim N^{1/3}$	$\left\langle S_x^2 \right\rangle /N \sim N$
$\left\langle S_y^2 \right\rangle /N \sim 1$	$\left\langle S_y^2 \right\rangle /N \sim N^{-1/3}$	$\left\langle S_y^2 \right\rangle /N \sim N$

- Non-trivial divergence at the critical point
- Power-law scaling

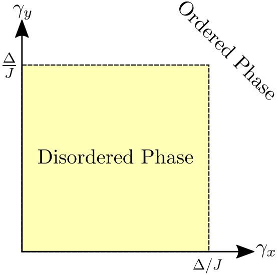


Click for Details

Ground State Scaling

Disordered	Critical	Ordered
$(J\gamma_{x,y} < B)$	$(J\gamma_x = B,\ \gamma_x > \gamma_y)$	$(J\gamma_{x,y} > B)$
$\left\langle S_x^2 \right\rangle /N \sim 1$	$\left\langle S_x^2 \right\rangle /N \sim N^{1/3}$	$\left\langle S_x^2 \right\rangle /N \sim N$
$\left\langle S_y^2 \right\rangle /N \sim 1$	$\left\langle S_y^2 \right\rangle /N \sim N^{-1/3}$	$\left\langle S_y^2 \right\rangle /N \sim N$

- Non-trivial divergence at the critical point
- Power-law scaling
- Exponent $1/3$ signifies quantum phase transition



Click for Details

Dynamical Phase Transitions

$$\alpha = \frac{1-\alpha_0}{2}$$

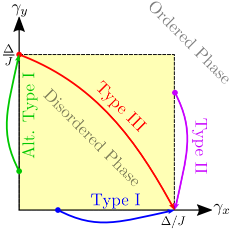
$$\zeta = \frac{1+\alpha_0}{4}$$

Type-I

$$\left\langle S_x^2(0)\right\rangle /N \sim N^{\alpha_0}$$

$$\left\langle S_x^2(t)\right\rangle /N \sim N^{\alpha} f\left(\frac{t}{N^{\zeta}}\right)$$

Type-II



Type-III

Click for Details

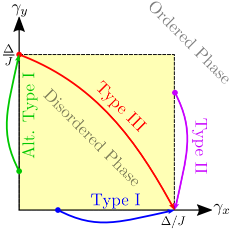
Dynamical Phase Transitions

$$\alpha = \frac{1-\alpha_0}{2}$$

$$\zeta = \frac{1+\alpha_0}{4}$$

$$\left\langle S_x^2(0) \right\rangle /N \sim N^{\alpha_0}$$

$$\left\langle S_x^2(t) \right\rangle /N \sim N^{\alpha} f\left(\frac{t}{N^{\zeta}}\right)$$



Type-I Type-II Type-III

$$\alpha_0 = 0$$

Click for Details

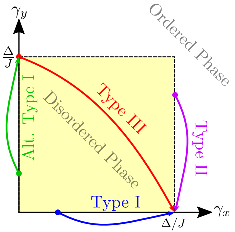
Dynamical Phase Transitions

$$\alpha = \frac{1-\alpha_0}{2}$$

$$\zeta = \frac{1+\alpha_0}{4}$$

$$\left\langle S_x^2(0)\right\rangle / N \sim N^{\alpha_0}$$

$$\left\langle S_x^2(t)\right\rangle / N \sim N^{\alpha} f\left(\frac{t}{N^{\zeta}}\right)$$



Type-I

Type-II

Type-III

$$\alpha_0 = 0$$
$$\alpha = 1/2$$
$$\zeta = 1/4$$

Click for Details

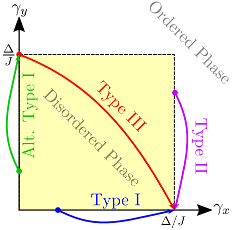
Dynamical Phase Transitions

$$\alpha = \frac{1-\alpha_0}{2}$$

$$\zeta = \frac{1+\alpha_0}{4}$$

$$\left\langle S_x^2(0) \right\rangle / N \sim N^{\alpha_0}$$

$$\left\langle S_x^2(t) \right\rangle / N \sim N^{\alpha} f\left(\frac{t}{N^{\zeta}}\right)$$



Type-I

$$\alpha_0 = 0$$
$$\alpha = 1/2$$
$$\zeta = 1/4$$

Type-II

$$\alpha_0 = 1/3$$

Type-III

Click for Details

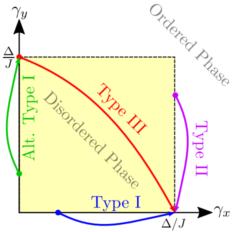
Dynamical Phase Transitions

$$\alpha = \frac{1-\alpha_0}{2}$$

$$\zeta = \frac{1+\alpha_0}{4}$$

$$\left\langle S_x^2(0)\right\rangle/N \sim N^{\alpha_0}$$

$$\left\langle S_x^2(t)\right\rangle/N \sim N^{\alpha} f\left(\frac{t}{N^{\zeta}}\right)$$



Type-I

$$\alpha_0 = 0$$
$$\alpha = 1/2$$
$$\zeta = 1/4$$

Type-II

$$\alpha_0 = 1/3$$
$$\alpha = 1/3$$
$$\zeta = 1/3$$

Type-III

Click for Details

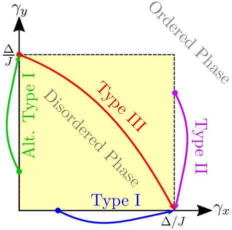
Dynamical Phase Transitions

$$\alpha = \frac{1-\alpha_0}{2}$$

$$\zeta = \frac{1+\alpha_0}{4}$$

$$\left\langle S_x^2(0) \right\rangle /N \sim N^{\alpha_0}$$

$$\left\langle S_x^2(t) \right\rangle /N \sim N^{\alpha} f\left(\frac{t}{N^{\zeta}}\right)$$



Type-I	Type-II	Type-III
$\alpha_0 = 0$	$\alpha_0 = 1/3$	$\alpha_0 = -1/3$
$\alpha = 1/2$	$\alpha = 1/3$	
$\zeta = 1/4$	$\zeta = 1/3$	

Click for Details

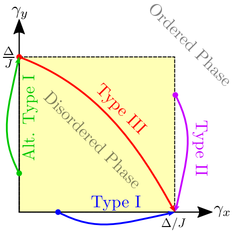
Dynamical Phase Transitions

$$\alpha = \frac{1-\alpha_0}{2}$$

$$\zeta = \frac{1+\alpha_0}{4}$$

$$\left\langle S_x^2(0)\right\rangle/N \sim N^{\alpha_0}$$

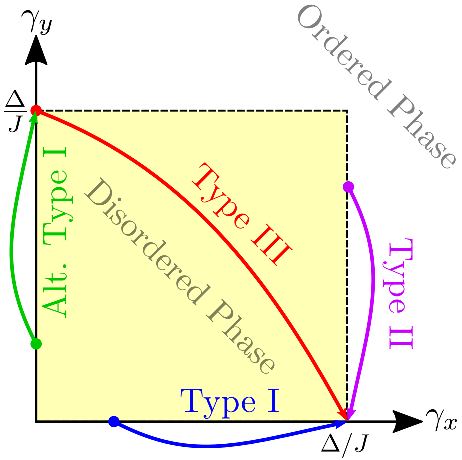
$$\left\langle S_x^2(t)\right\rangle/N \sim N^{\alpha} f\left(\frac{t}{N^{\zeta}}\right)$$



Type-I	Type-II	Type-III
$\alpha_0 = 0$	$\alpha_0 = 1/3$	$\alpha_0 = -1/3$
$\alpha = 1/2$	$\alpha = 1/3$	$\alpha = 2/3$
$\zeta = 1/4$	$\zeta = 1/3$	$\zeta = 1/6$

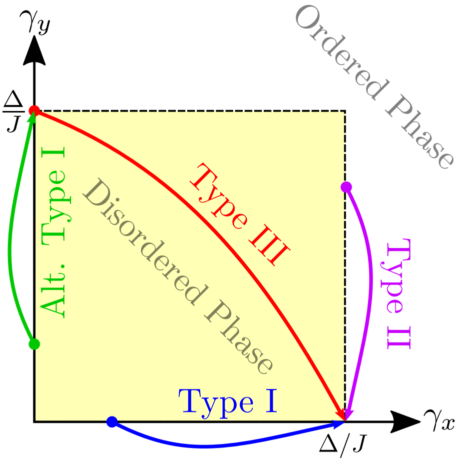
Click for Details

Dynamical Phase Transitions



Dynamical Phase Transitions

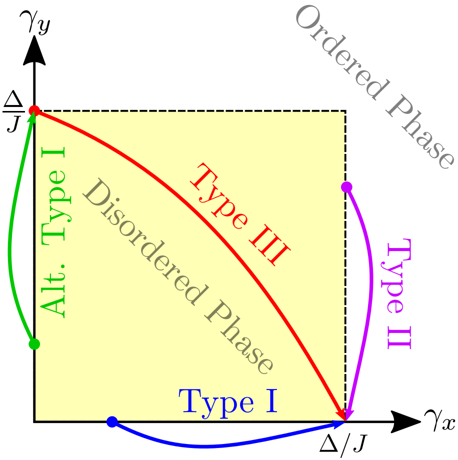
- $\langle S_{x,y} \rangle / N \equiv 0$
- Dynamics lie within the fluctuations



(Titum and Maghrebi 2020, *Phys. Rev. Lett.*)

Dynamical Phase Transitions

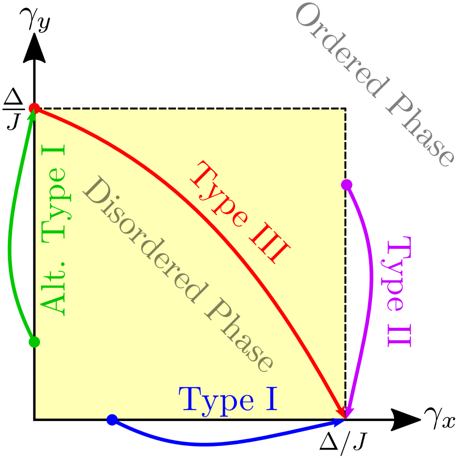
- $\langle S_{x,y} \rangle / N \equiv 0$
- Dynamics lie within the fluctuations
- Type-I $\longrightarrow$ Thermal



(Titum and Maghrebi 2020, *Phys. Rev. Lett.*)

Dynamical Phase Transitions

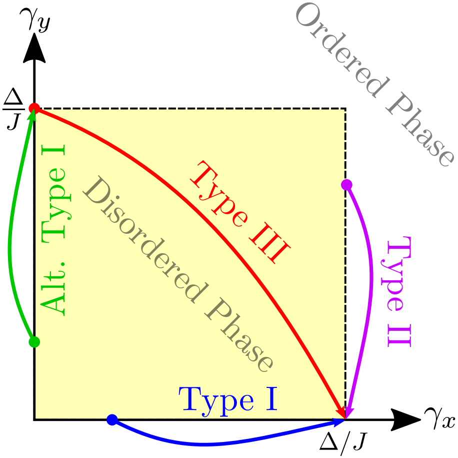
- $\langle S_{x,y} \rangle / N \equiv 0$
- Dynamics lie within the fluctuations
- Type-I $\longrightarrow$ Thermal
- Type-II $\longrightarrow$ Quantum Critical



(Titum and Maghrebi 2020, *Phys. Rev. Lett.*)

Dynamical Phase Transitions

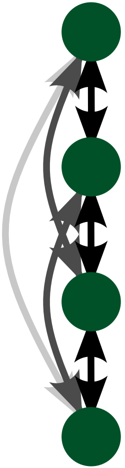
- $\langle S_{x,y} \rangle / N \equiv 0$
- Dynamics lie within the fluctuations
- Type-I $\longrightarrow$ Thermal
- Type-II $\longrightarrow$ Quantum Critical
- Type-III $\longrightarrow$ Nonequilibrium



(Titum and Maghrebi 2020, *Phys. Rev. Lett.*)

Generalized Long-Range Model

- 1D spin chain
- Arbitrary coupling strength
- Usually (approximately) power-law
- Nonintegrable and nontrivial



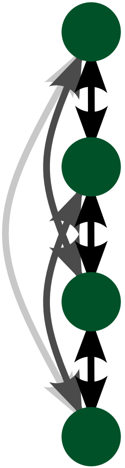
$$H = -\frac{1}{2\mathcal{N}} \sum_{i \neq j}^{N} J_{ij} \left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y \right) - B \sum_{i}^{N} \sigma_i^z$$

Generalized Long-Range Model

- 1D spin chain
- Arbitrary coupling strength
- Usually (approximately) power-law
- Nonintegrable and nontrivial
- Kac normalization
  (Defenu et al. 2021, *arXiv:2109.01063*)

$$H = -\frac{1}{2\mathcal{N}} \sum_{i \neq j}^{N} J_{ij} \left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y \right) - B \sum_{i}^{N} \sigma_i^z$$

$$\mathcal{N} = \frac{1}{N-1} \sum_{i \neq j}^{N} J_{ij}$$



"Long Range"

Following the prescription in Defenu et al. 2021, *arXiv:2109.01063*

"Long Range"

Following the prescription in Defenu et al. 2021, *arXiv:2109.01063*

- $V(r) \overset{r\to\infty}{\sim} r^{-\alpha}$

"Long Range"

Following the prescription in Defenu et al. 2021, *arXiv:2109.01063*

- $V(r) \overset{r\to\infty}{\sim} r^{-\alpha}$

Weak Long Range

$$d < \alpha < \alpha_*$$

"Long Range"

Following the prescription in Defenu et al. 2021, *arXiv:2109.01063*

- $V(r) \overset{r\to\infty}{\sim} r^{-\alpha}$

Weak Long Range

$$d < \alpha < \alpha_*$$

Strong Long Range

$$\alpha < d$$

"Long Range"

Following the prescription in Defenu et al. 2021, *arXiv:2109.01063*

- $V(r) \overset{r\to\infty}{\sim} r^{-\alpha}$

Weak Long Range	$\boxed{\text{Strong Long Range}}$
$d < \alpha < \alpha_*$	$\alpha < d$

Strong long-range interacting systems typically fall into the same universality class as the LMG model

Spin-Wave Analysis

- Similar to Holstein-Primakoff

Spin-Wave Analysis

- Similar to Holstein-Primakoff
- Introduce bosonic operators **for each site**

$$\sigma_i^{\pm} = \sigma_i^x \pm i\sigma_i^y = \begin{matrix}(+)\\(-)\end{matrix} \begin{cases}\sqrt{1 - a_i^\dagger a_i}\, a_i \sim a_i \\ a_i^\dagger \sqrt{1 - a_i^\dagger a_i} \sim a_i^\dagger\end{cases}$$

$$\sigma_i^z = 1 - a_i^\dagger a_i$$

Spin-Wave Analysis

- Similar to Holstein-Primakoff
- Introduce bosonic operators **for each site**

$$\sigma_i^{\pm} = \sigma_i^x \pm i\sigma_i^y = \begin{matrix}(+)\\(-)\end{matrix} \begin{cases} \sqrt{1 - a_i^\dagger a_i}\, a_i \sim a_i \\ a_i^\dagger \sqrt{1 - a_i^\dagger a_i} \sim a_i^\dagger \end{cases}$$

$$\sigma_i^z = 1 - a_i^\dagger a_i$$

- Which yields a suggestive Hamiltonian

$$H = \sum_{i,j} a_i^\dagger A_{ij} a_j + \frac{1}{2}\left(a_i B_{ij} a_j + a_i^\dagger B_{ij} a_j^\dagger\right)$$

Spin-Wave Analysis

- Similar to Holstein-Primakoff
- Introduce bosonic operators **for each site**

$$\sigma_i^{\pm} = \sigma_i^x \pm i\sigma_i^y = \begin{matrix} (+) \\ (-) \end{matrix} \begin{cases} \sqrt{1 - a_i^\dagger a_i} a_i \sim a_i \\ a_i^\dagger \sqrt{1 - a_i^\dagger a_i} \sim a_i^\dagger \end{cases}$$

$$\sigma_i^z = 1 - a_i^\dagger a_i$$

- Which yields a suggestive Hamiltonian

$$H = \sum_{i,j} a_i^\dagger A_{ij} a_j + \frac{1}{2}\left(a_i B_{ij} a_j + a_i^\dagger B_{ij} a_j^\dagger\right)$$

- Diagonalized by a Bogoliubov transformation $\{a_i, a_i^\dagger\} \to \{b_i, b_i^\dagger\}$

$$H = \sum_k \Lambda_k b_k^\dagger b_k$$

Spin-Wave Analysis

$$H = \sum_k \Lambda_k b_k^\dagger b_k$$

Spin-Wave Analysis

$$H = \sum_k \Lambda_k b_k^\dagger b_k$$

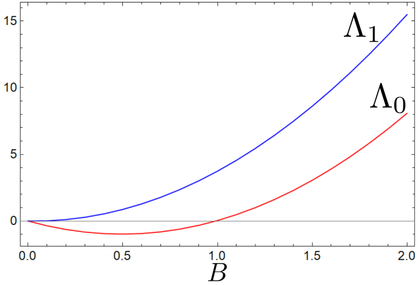
Key point: $k = 0$ represents the LMG mode

Spin-Wave Analysis

$$H = \sum_k \Lambda_k b_k^\dagger b_k$$

Key point: $k = 0$ **represents the LMG mode**

- All other modes remain gapped $(\Lambda_k \neq 0)$ at critical point

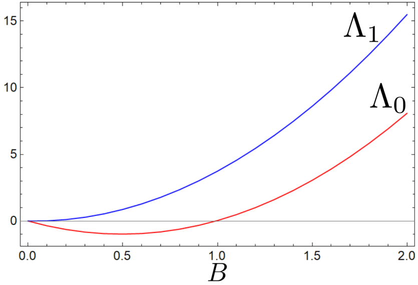


Spin-Wave Analysis

$$H = \sum_k \Lambda_k b_k^\dagger b_k$$

Key point: $k = 0$ **represents the LMG mode**

- All other modes remain gapped ($\Lambda_k \neq 0$) at critical point
- Only the LMG mode contributes to the critical behavior

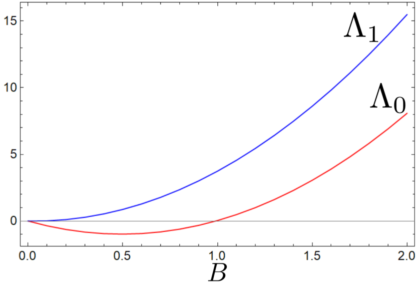


Spin-Wave Analysis

$$H = \sum_k \Lambda_k b_k^\dagger b_k$$

Key point: $k = 0$ **represents the LMG mode**

- All other modes remain gapped ($\Lambda_k \neq 0$) at critical point
- Only the LMG mode contributes to the critical behavior
- Same universality class as the LMG model

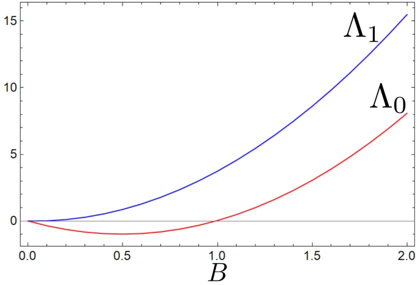


Spin-Wave Analysis

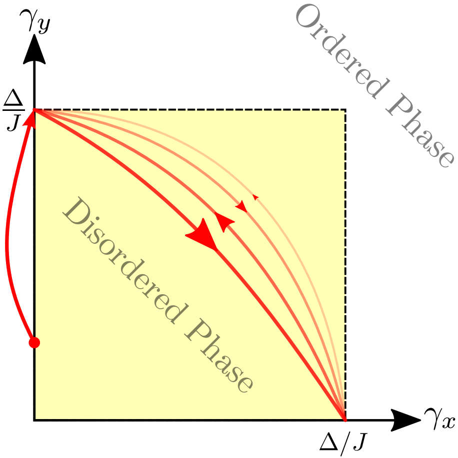
$$H = \sum_k \Lambda_k b_k^\dagger b_k$$

Key point: $k = 0$ **represents the LMG mode**

- All other modes remain gapped ($\Lambda_k \neq 0$) at critical point
- Only the LMG mode contributes to the critical behavior
- Same universality class as the LMG model
- Does **not** say anything about non-critical behavior

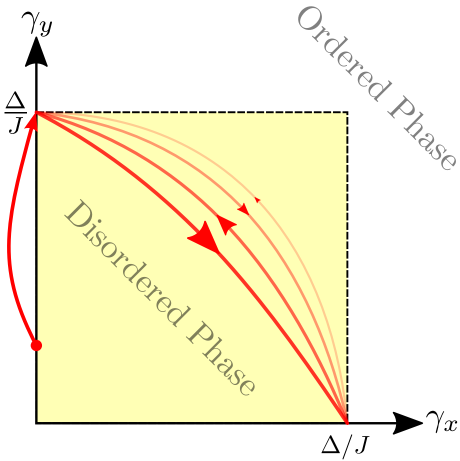


Multiple Quenches



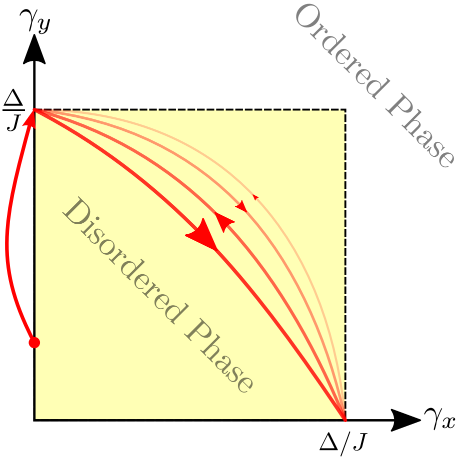
Multiple Quenches

- Subvert critical preparation



Multiple Quenches

- Subvert critical preparation
- New critical exponents?



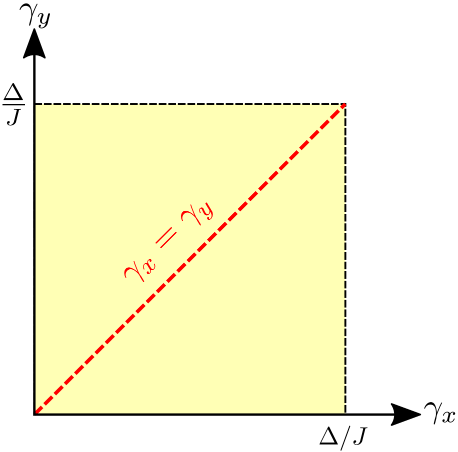
Critical Hierarchy

$$\alpha_D^{(n)} = 1 - \frac{1-\alpha_D^{(0)}}{2^{k-1}}$$

$$\alpha_S^{(n)} = 1 - \frac{1-\alpha_D^{(0)}}{2^{k-2}}$$

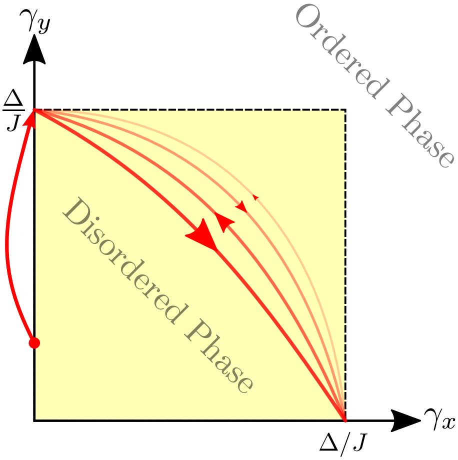
$$\zeta^{(n)} = \frac{1-\alpha_D^{(0)}}{2^{k}}$$

k is the number of times $\gamma_x \lesseqgtr \gamma_y$



Critical Hierarchy

	Type-I-III-III-...		Type-III-III-III-...	
k	$\alpha_D^{(k)}$	$\zeta^{(k)}$	$\alpha_D^{(k)}$	$\zeta^{(k)}$
0	$1/2$	$1/4$	$1/3$	$1/3$
1	$3/4$	$1/8$	$2/3$	$1/6$
2	$7/8$	$1/16$	$5/6$	$1/12$



Experiment

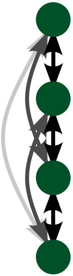
Can we observe the hierarchy of critical exponents?

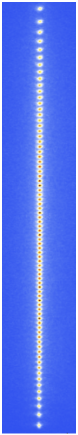
Trapped Ions

Collaboration with experimentalists at the Joint Quantum Institute of the University of Maryland and IonQ Inc.

- Dr. Chris Monroe
- Dr. Guido Pagano
- Dr. Alexey Gorshkov
- Arinjoy De
- Dr. William Morong

Able to tune power-law interactions
$p \sim 0\text{–}3$
(Defenu et al. 2021, *arXiv:2109.01063*)





Adapted from Defenu et al. 2021, *arXiv:2109.01063*.

Experimental Hamiltonian

$$H = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} J_{i,j} \left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y \right) - B \sum_{i}^{N} \sigma_i^z .$$

Experimental Hamiltonian

$$H = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} J_{i,j} \left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y \right) - B \sum_i^N \sigma_i^z .$$

Some important differences:

- Antiferromagnetic

Experimental Hamiltonian

$$H = \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}}^{N} J_{i,j}\left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y\right) - B\sum_{i}^{N}\sigma_i^z.$$

Some important differences:

- Antiferromagnetic
- Not Kac normalized

Experimental Hamiltonian

$$H = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} J_{i,j} \left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y \right) - B \sum_i^N \sigma_i^z .$$

Some important differences:

- Antiferromagnetic
- Not Kac normalized
  - Moves the critical point $B_{cr} = \mathcal{N}$

Experimental Hamiltonian

$$H = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}}^{N} J_{i,j} \left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y \right) - B \sum_i^N \sigma_i^z .$$

Some important differences:

- Antiferromagnetic
- Not Kac normalized
  - Moves the critical point $B_{cr} = \mathcal{N}$
  - Characteristic timescale $\mathcal{N} t / N^{\zeta}$

Experimental Hamiltonian

$$H = \frac{1}{2}\sum_{\substack{i,j \\ i\neq j}}^{N} J_{i,j}\left(\gamma_x \sigma_i^x \sigma_j^x + \gamma_y \sigma_i^y \sigma_j^y\right) - B\sum_i^N \sigma_i^z.$$

Some important differences:

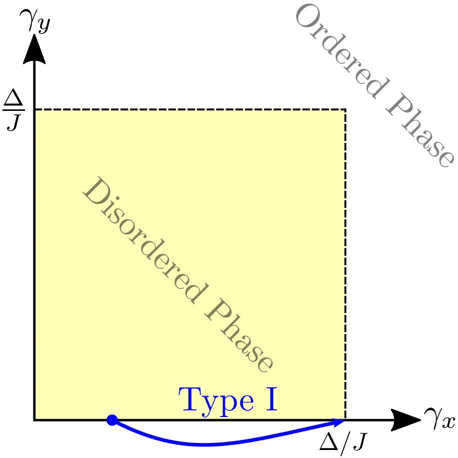
- Antiferromagnetic
- Not Kac normalized
  - Moves the critical point $B_{cr} = \mathcal{N}$
  - Characteristic timescale $\mathcal{N}t/N^{\zeta}$
- J_{ij} is only *approximately* power-law

$$J(r) \approx \frac{J_0}{r^p} e^{-K(r-1)}$$

Results

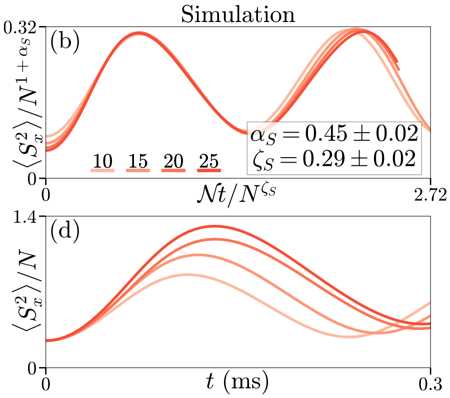
Single Type I Quench

$$\alpha = 1/2$$
$$\zeta = 1/4$$



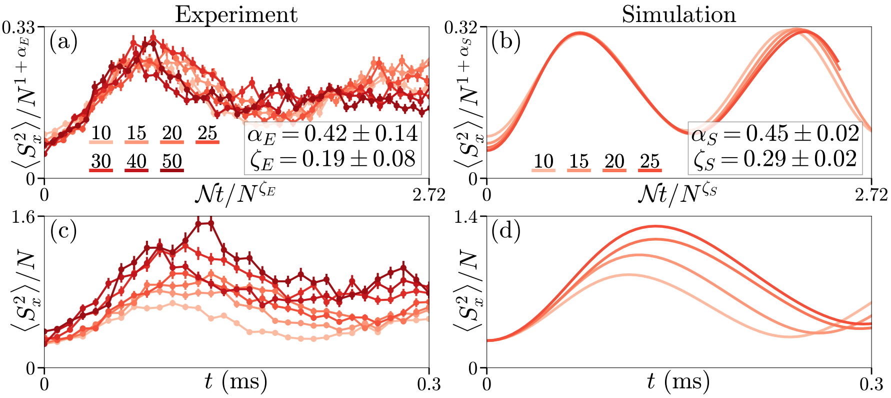
Results

Single Type I Quench



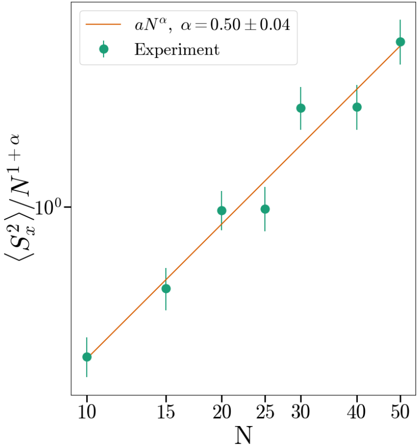
Results

Single Type I Quench



Results

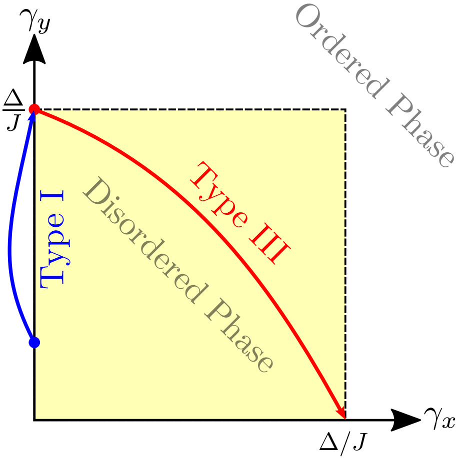
Single Type I Quench



Results

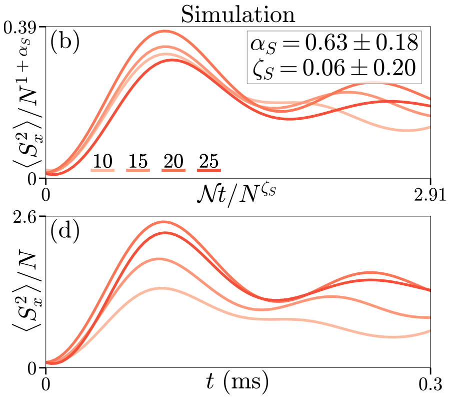
Type I–III Quench

$$\alpha = 3/4$$
$$\zeta = 1/8$$



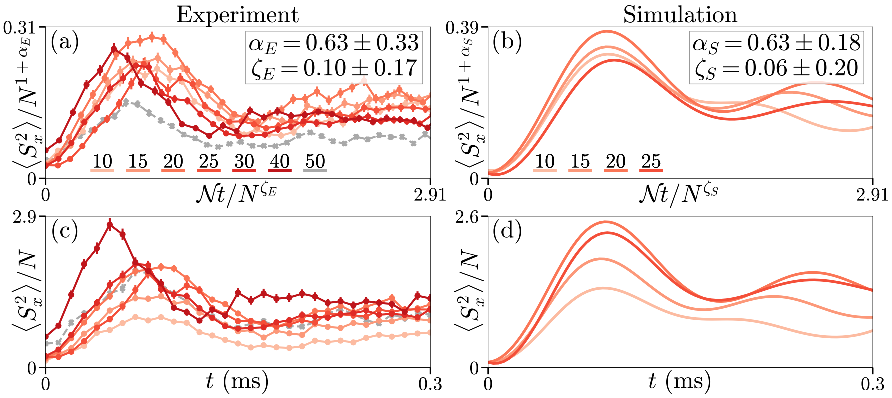
Results

Type I–III Quench



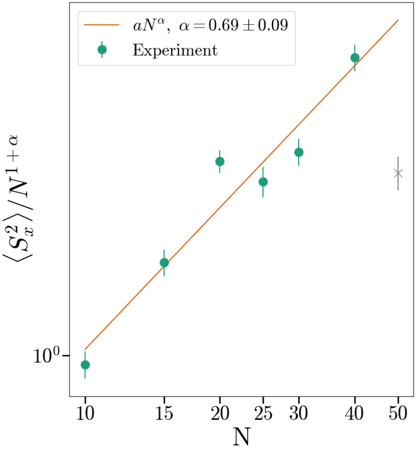
Results

Type I–III Quench



Results

Type I–III Quench



Computation

What novel techniques are required?

Matrices, Vectors, and Decompositions

- Operators $H, S_x, S^2, \sigma_i^y, \ldots$
- States $|\psi\rangle, |{\uparrow\uparrow\downarrow}\rangle, \langle b|, \ldots$
- Operator actions $H|\psi\rangle, \sigma_i^y |{\uparrow\uparrow\downarrow}\rangle, \ldots$
- Hilbert space $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$

Matrices, Vectors, and Decompositions

- Operators $H, S_x, S^2, \sigma_i^y, \ldots$
- States $|\psi\rangle, |\uparrow\uparrow\downarrow\rangle, \langle b|, \ldots$
- Operator actions $H|\psi\rangle, \sigma_i^y|\uparrow\uparrow\downarrow\rangle, \ldots$
- Hilbert space $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$

- Matrices

$$M_{ij} = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \times & \cdots & \times \end{bmatrix}$$

Matrices, Vectors, and Decompositions

- Operators $H, S_x, S^2, \sigma_i^y, \ldots$
- States $|\psi\rangle, |\uparrow\uparrow\downarrow\rangle, \langle b|, \ldots$
- Operator actions $H|\psi\rangle, \sigma_i^y|\uparrow\uparrow\downarrow\rangle, \ldots$
- Hilbert space $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$

- Matrices
$$M_{ij} = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \times & \cdots & \times \end{bmatrix}$$
- Vectors $v_i = \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix}$

Matrices, Vectors, and Decompositions

- Operators $H, S_x, S^2, \sigma_i^y, \ldots$
- States $|\psi\rangle, |\uparrow\uparrow\downarrow\rangle, \langle b|, \ldots$
- Operator actions $H|\psi\rangle, \sigma_i^y|\uparrow\uparrow\downarrow\rangle, \ldots$
- Hilbert space $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$

- Matrices
$$M_{ij} = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \times & \cdots & \times \end{bmatrix}$$
- Vectors $v_i = \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix}$
- Inner product

Matrices, Vectors, and Decompositions

- Operators $H, S_x, S^2, \sigma_i^y, \ldots$
- States $|\psi\rangle, |\uparrow\uparrow\downarrow\rangle, \langle b|, \ldots$
- Operator actions $H|\psi\rangle, \sigma_i^y|\uparrow\uparrow\downarrow\rangle, \ldots$
- Hilbert space $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$

- Matrices
$$M_{ij} = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \times & \cdots & \times \end{bmatrix}$$
- Vectors $v_i = \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix}$
- Inner product
- Hilbert Space

Matrices, Vectors, and Decompositions

- Operators $H, S_x, S^2, \sigma_i^y, \ldots$
- States $|\psi\rangle, |\uparrow\uparrow\downarrow\rangle, \langle b|, \ldots$
- Operator actions $H|\psi\rangle, \sigma_i^y |\uparrow\uparrow\downarrow\rangle, \ldots$
- Hilbert space $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$

- Matrices
$$M_{ij} = \begin{bmatrix} \times & \cdots & \times \\ \vdots & \ddots & \vdots \\ \times & \cdots & \times \end{bmatrix}$$
- Vectors $v_i = \begin{bmatrix} \times \\ \vdots \\ \times \end{bmatrix}$
- Inner product
- Hilbert Space

Eigenvalue Decomposition

Matrix-Free Methods

- Represent any linear operator
- Defined by its action

$$Mv \longrightarrow M(v)$$

Matrix-Free Methods

- Represent any linear operator
- Defined by its action

$$Mv \longrightarrow M(v)$$

```
[[0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0]
 [0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0]
 [1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
 [0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0]
 [0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 0]
 [0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0]
 [0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0]
 [0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1]
 [0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0]
 [0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0]]
```

$\longrightarrow$

```python
def Sx(N, site):
    # the action of sigma^x_i is to flip the spin at site i
    # NOTE: SITE IS 0-INDEXED
    def matvec(v):
        res = np.zeros(v.shape, dtype=complex)
        for i, coeff in enumerate(v):
            if (i >> site) & 1:  # check if the spin at `site` is up
                res[i - 2 ** site] = coeff
            else:  # or down
                res[i + 2 ** site] = coeff

        return res

    return LinearOperator((2 ** N, 2 ** N), matvec=matvec, dtype=complex)
```

Matrix-Free Example: Two-Spin System

"Traditional" approach:

$$|{\uparrow\uparrow}\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \; |{\uparrow\downarrow}\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \; |{\downarrow\uparrow}\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \; |{\downarrow\downarrow}\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Matrix-Free Example: Two-Spin System

"Traditional" approach:

$$|\uparrow\uparrow\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \; |\uparrow\downarrow\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \; |\downarrow\uparrow\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \; |\downarrow\downarrow\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\sigma_1^x = \mathbb{I}_{2\times 2} \otimes \sigma^x = \begin{bmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\end{bmatrix}$$

Matrix-Free Example: Two-Spin System

Matrix-Free approach:

$$|\uparrow\uparrow\rangle \equiv |3\rangle = |11_2\rangle \qquad |\downarrow\uparrow\rangle \equiv |1\rangle = |01_2\rangle$$

$$|\uparrow\downarrow\rangle \equiv |2\rangle = |10_2\rangle \qquad |\downarrow\downarrow\rangle \equiv |0\rangle = |00_2\rangle$$

Matrix-Free Example: Two-Spin System

Matrix-Free approach:

$$|\uparrow\uparrow\rangle \equiv |3\rangle = |11_2\rangle \qquad |\downarrow\uparrow\rangle \equiv |1\rangle = |01_2\rangle$$

$$|\uparrow\downarrow\rangle \equiv |2\rangle = |10_2\rangle \qquad |\downarrow\downarrow\rangle \equiv |0\rangle = |00_2\rangle$$

$$\begin{aligned}
\sigma_1^x |\psi\rangle &= \sigma_1^x \begin{bmatrix} \psi_{11_2} \\ \psi_{10_2} \\ \psi_{01_2} \\ \psi_{00_2} \end{bmatrix} \\
&= \sigma_1^x \left(\psi_{11_2} |11_2\rangle + \psi_{10_2} |10_2\rangle + \psi_{01_2} |01_2\rangle + \psi_{00_2} |00_2\rangle\right) \\
&= \psi_{11_2}\sigma_1^x |11_2\rangle + \psi_{10_2}\sigma_1^x |10_2\rangle + \psi_{01_2}\sigma_1^x |01_2\rangle + \psi_{00_2}\sigma_1^x |00_2\rangle
\end{aligned}$$

Matrix-Free Example: Two-Spin System

Matrix-Free approach:

$$|\uparrow\uparrow\rangle \equiv |3\rangle = |11_2\rangle \qquad |\downarrow\uparrow\rangle \equiv |1\rangle = |01_2\rangle$$

$$|\uparrow\downarrow\rangle \equiv |2\rangle = |10_2\rangle \qquad |\downarrow\downarrow\rangle \equiv |0\rangle = |00_2\rangle$$

$$\begin{aligned}
\sigma_1^x |\psi\rangle &= \sigma_1^x \begin{bmatrix} \psi_{11_2} \\ \psi_{10_2} \\ \psi_{01_2} \\ \psi_{00_2} \end{bmatrix} \\
&= \sigma_1^x \left(\psi_{11_2} |11_2\rangle + \psi_{10_2} |10_2\rangle + \psi_{01_2} |01_2\rangle + \psi_{00_2} |00_2\rangle \right) \\
&= \psi_{11_2} \sigma_1^x |11_2\rangle + \psi_{10_2} \sigma_1^x |10_2\rangle + \psi_{01_2} \sigma_1^x |01_2\rangle + \psi_{00_2} \sigma_1^x |00_2\rangle
\end{aligned}$$

$$\sigma_1^x |b_2\rangle = \sigma_1^x |(xy)_2\rangle = ?$$

Matrix-Free Example: Two-Spin System

"σ_1^x flips the bit in the first position:"

Matrix-Free Example: Two-Spin System

"σ_1^x flips the bit in the first position:"

$$\sigma_1^x \left|(xy)_2\right\rangle = \left|[(\neg x)y]_2\right\rangle$$

Matrix-Free Example: Two-Spin System

"σ_1^x flips the bit in the first position:"

$$\sigma_1^x \left|(xy)_2\right\rangle = \left|[(\neg x)y]_2\right\rangle$$

or

$$\sigma_1^x \left|b_2\right\rangle = \left|(b \oplus 10_2)\right\rangle$$

Matrix-Free Example: Two-Spin System

"σ_1^x flips the bit in the first position:"

$$\sigma_1^x \left|(xy)_2\right\rangle = \left|[(\neg x)y]_2\right\rangle$$

or

$$\sigma_1^x \left|b_2\right\rangle = \left|(b \oplus 10_2)\right\rangle$$

$\oplus$ is the bitwise XOR operator:

A	B	A $\oplus$ B
0	0	0
0	1	1
1	0	1
1	1	0

Matrix-Free Example: Two-Spin System

"σ_i^x flips the bit in the i^{th} position:"

Matrix-Free Example: Two-Spin System

"σ_i^x flips the bit in the i^{th} position:"

$$\sigma_i^x \left|b_2\right\rangle = \left|b \oplus (1 \ll i)\right\rangle$$

Matrix-Free Example: Two-Spin System

"σ_i^x flips the bit in the i^{th} position:"

$$\sigma_i^x \left|b_2\right\rangle = \left|b \oplus (1 \ll i)\right\rangle$$

$\ll$ is the left bit-shift operator:

n	$1 \ll n$
0	1, or 01_2
1	2, or 10_2
2	4, or 100_2

Matrix-Free Example: Two-Spin System

"σ_i^x flips the bit in the i^{th} position:"

$$\sigma_i^x \left| b_2 \right\rangle = \left| b \oplus (1 \ll i) \right\rangle$$

$\ll$ is the left bit-shift operator:

n	$1 \ll n$
0	1, or 01_2
1	2, or 10_2
2	4, or 100_2

- Can depend on the spin(s) in basis states

Matrix-Free Example: Two-Spin System

"σ_i^x flips the bit in the i^{th} position:"

$$\sigma_i^x \left|b_2\right\rangle = \left|b \oplus (1 \ll i)\right\rangle$$

$\ll$ is the left bit-shift operator:

n	$1 \ll n$
0	1, or 01_2
1	2, or 10_2
2	4, or 100_2

- Can depend on the spin(s) in basis states
- Spin i is up: $(b \gg i)\&1$

Matrix-Free Method Advantages

- Intuitive to implement

Matrix-Free Method Advantages

- Intuitive to implement
- Quantum mechanics **is** linear operators

Matrix-Free Method Advantages

- Intuitive to implement
- Fast
- Quantum mechanics **is** linear operators

Matrix-Free Method Advantages

- Intuitive to implement
- Fast

- Quantum mechanics **is** linear operators
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(n\right)$[1]

[1] Not true in general, but true for all of the operators we are interested in.

Matrix-Free Method Advantages

- Intuitive to implement
- Fast
- Memory-efficient

- Quantum mechanics **is** linear operators
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(n\right)$[1]

[1] Not true in general, but true for all of the operators we are interested in.

Matrix-Free Method Advantages

- Intuitive to implement
- Fast
- Memory-efficient

- Quantum mechanics **is** linear operators
- $\mathcal{O}(n^2) \longrightarrow \mathcal{O}(n)$[1]
- $\mathcal{O}(n^2) \longrightarrow \mathcal{O}(1)$[2]

[1] Not true in general, but true for all of the operators we are interested in.
[2] For the operator only, not the matrix-vector product.

Matrix-Free Method Advantages

- Intuitive to implement
- Fast
- Memory-efficient
- Already supported

- Quantum mechanics **is** linear operators
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(n\right)$[1]
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(1\right)$[2]

[1] Not true in general, but true for all of the operators we are interested in.
[2] For the operator only, not the matrix-vector product.

Matrix-Free Method Advantages

- Intuitive to implement
- Fast
- Memory-efficient
- Already supported

- Quantum mechanics **is** linear operators
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(n\right)$[1]
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(1\right)$[2]
- ARPACK, BLAS, etc.

[1] Not true in general, but true for all of the operators we are interested in.
[2] For the operator only, not the matrix-vector product.

Matrix-Free Method Advantages

- Intuitive to implement
- Fast
- Memory-efficient
- Already supported
- Versatile

- Quantum mechanics **is** linear operators
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(n\right)$[1]
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(1\right)$[2]
- ARPACK, BLAS, etc.

[1] Not true in general, but true for all of the operators we are interested in.
[2] For the operator only, not the matrix-vector product.

Matrix-Free Method Advantages

- Intuitive to implement
- Fast
- Memory-efficient
- Already supported
- Versatile

- Quantum mechanics **is** linear operators
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(n\right)$[1]
- $\mathcal{O}\left(n^2\right) \longrightarrow \mathcal{O}\left(1\right)$[2]
- ARPACK, BLAS, etc.
- AoT and JIT compilation, parallel, etc.

[1] Not true in general, but true for all of the operators we are interested in.
[2] For the operator only, not the matrix-vector product.

Spectral Approximation

Setting:

- We have a specific $|\Psi_0\rangle$ at $t = 0$

Spectral Approximation

Setting:

- We have a specific $|\Psi_0\rangle$ at $t = 0$
- We have a specific Hamiltonian $H \in \mathbb{C}^{n \times n}$ with $n^2 \gg 1$

Spectral Approximation

Setting:

- We have a specific $|\Psi_0\rangle$ at $t = 0$
- We have a specific Hamiltonian $H \in \mathbb{C}^{n\times n}$ with $n^2 \gg 1$
- We want $\langle X(t)\rangle = \langle\Psi(t)|X|\Psi(t)\rangle$ for an arbitrary X

Spectral Approximation

Setting:

- We have a specific $|\Psi_0\rangle$ at $t = 0$
- We have a specific Hamiltonian $H \in \mathbb{C}^{n\times n}$ with $n^2 \gg 1$
- We want $\langle X(t)\rangle = \langle \Psi(t)|X|\Psi(t)\rangle$ for an arbitrary X

Goal: Find an approximation such that

Spectral Approximation

Setting:

- We have a specific $|\Psi_0\rangle$ at $t = 0$
- We have a specific Hamiltonian $H \in \mathbb{C}^{n\times n}$ with $n^2 \gg 1$
- We want $\langle X(t)\rangle = \langle \Psi(t)|X|\Psi(t)\rangle$ for an arbitrary X

Goal: Find an approximation such that

- $n \longrightarrow k$ with $k \lll n$

Spectral Approximation

Setting:

- We have a specific $|\Psi_0\rangle$ at $t = 0$
- We have a specific Hamiltonian $H \in \mathbb{C}^{n\times n}$ with $n^2 \gg 1$
- We want $\langle X(t)\rangle = \langle \Psi(t)|X|\Psi(t)\rangle$ for an arbitrary X

Goal: Find an approximation such that

- $n \longrightarrow k$ with $k \lll n$
- $|\psi(t)\rangle \in \mathbb{C}^k$, $x \in \mathbb{C}^{k\times k}$, $h \in \mathbb{C}^{k\times k}$

Spectral Approximation

Setting:

- We have a specific $|\Psi_0\rangle$ at $t = 0$
- We have a specific Hamiltonian $H \in \mathbb{C}^{n \times n}$ with $n^2 \gg 1$
- We want $\langle X(t)\rangle = \langle \Psi(t)|X|\Psi(t)\rangle$ for an arbitrary X

Goal: Find an approximation such that

- $n \longrightarrow k$ with $k \lll n$
- $|\psi(t)\rangle \in \mathbb{C}^k$, $x \in \mathbb{C}^{k\times k}$, $h \in \mathbb{C}^{k\times k}$
- $\langle x(t)\rangle = \langle \psi(t)|x|\psi(t)\rangle \approx \langle X(t)\rangle$

Spectral Approximation

Recall

$$|\Psi_0\rangle = \sum_j^n c_j |\Phi_j\rangle$$

$$|\Psi(t)\rangle = \sum_j^n c_j e^{-iE_j t} |\Phi_j\rangle$$

- $H|\Phi_j\rangle = E_j |\Phi_j\rangle$

Spectral Approximation

Recall

$$|\Psi_0\rangle = \sum_j^n c_j |\Phi_j\rangle$$

$$|\Psi(t)\rangle = \sum_j^n c_j e^{-iE_j t} |\Phi_j\rangle$$

- $H|\Phi_j\rangle = E_j |\Phi_j\rangle$

Order $|c_1|^2 \geq |c_2|^2 \geq \cdots \geq |c_n|^2$

Spectral Approximation

Recall

$$|\Psi_0\rangle = \sum_j^n c_j |\Phi_j\rangle$$

$$|\Psi(t)\rangle = \sum_j^n c_j e^{-iE_j t} |\Phi_j\rangle$$

- $H|\Phi_j\rangle = E_j |\Phi_j\rangle$

Order $|c_1|^2 \geq |c_2|^2 \geq \cdots \geq |c_n|^2$

New basis

$$\{|\Phi_1\rangle, |\Phi_2\rangle, \ldots, |\Phi_k\rangle\}$$

Spectral Approximation

Recall

$$|\Psi_0\rangle = \sum_j^n c_j |\Phi_j\rangle$$

$$|\Psi(t)\rangle = \sum_j^n c_j e^{-iE_j t} |\Phi_j\rangle$$

- $H|\Phi_j\rangle = E_j |\Phi_j\rangle$

Order $|c_1|^2 \geq |c_2|^2 \geq \cdots \geq |c_n|^2$

New basis

$$\{|\Phi_1\rangle, |\Phi_2\rangle, \ldots, |\Phi_k\rangle\}$$

$$|\psi_0\rangle = c_1 |\Phi_1\rangle + c_2 |\Phi_2\rangle + \cdots + c_k |\Phi_k\rangle \cancel{+ c_{k+1} |\Phi_{k+1}\rangle + \cdots + c_n |\Phi_n\rangle}$$

Spectral Approximation

- Can we find only $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ ***without*** completely diagonalizing H?

Spectral Approximation

- Can we find only $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ ***without*** completely diagonalizing H?
- Lanczos and Arnoldi methods

Spectral Approximation

- Can we find only $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ ***without*** completely diagonalizing H?
- ~~Lanczos and Arnoldi methods~~
- Modified Jacobi–Davidson method (Tackett and Ventra 2002, *Phys. Rev. B*)

Spectral Approximation

- Can we find only $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ ***without*** completely diagonalizing H?
- ~~Lanczos and Arnoldi methods~~
- Modified Jacobi–Davidson method
  (Tackett and Ventra 2002, *Phys. Rev. B*)
  - Arbitrary selection criteria

Spectral Approximation

- Can we find only $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ ***without*** completely diagonalizing H?
- ~~Lanczos and Arnoldi methods~~
- Modified Jacobi–Davidson method
  (Tackett and Ventra 2002, *Phys. Rev. B*)
  - Arbitrary selection criteria

$H, X, |\Psi\rangle$

- Time $\mathcal{O}(n^3)$
- Memory $\mathcal{O}(n^2)$

Spectral Approximation

- Can we find only $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ ***without*** completely diagonalizing H?
- ~~Lanczos and Arnoldi methods~~
- Modified Jacobi–Davidson method
  (Tackett and Ventra 2002, *Phys. Rev. B*)
  - Arbitrary selection criteria

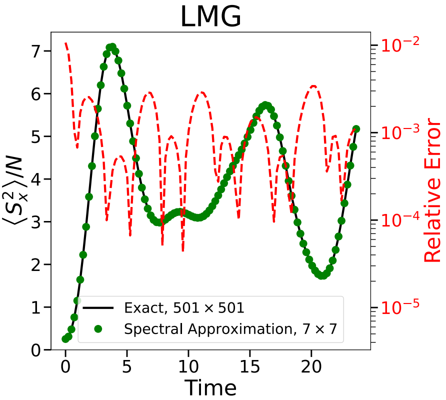
$H, X, \|\Psi\rangle$	$h, x, \|\psi\rangle$
Time $\mathcal{O}(n^3)$	Time $\mathcal{O}(kn^2)$
Memory $\mathcal{O}(n^2)$	Memory $\mathcal{O}(n^2)$

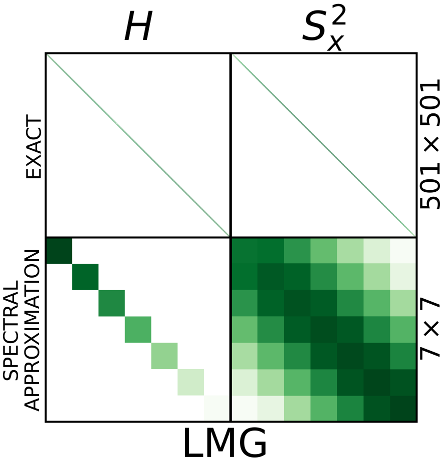
Spectral Approximation

- Can we find only $|\Phi_1\rangle, \ldots, |\Phi_k\rangle$ ***without*** completely diagonalizing H?
- ~~Lanczos and Arnoldi methods~~
- Modified Jacobi–Davidson method (Tackett and Ventra 2002, *Phys. Rev. B*)
  - Arbitrary selection criteria

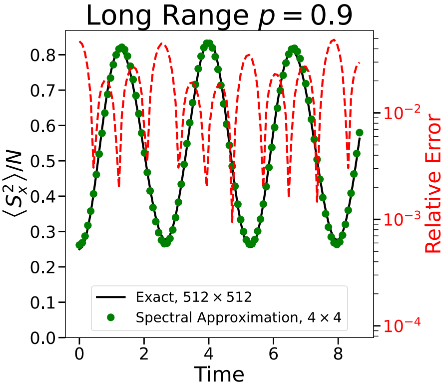
$H, X, \|\Psi\rangle$	$h, x, \|\psi\rangle$	Matrix-Free & $h, x, \|\psi\rangle$
Time $\mathcal{O}(n^3)$	Time $\mathcal{O}(kn^2)$	Time $\mathcal{O}(kn)$
Memory $\mathcal{O}(n^2)$	Memory $\mathcal{O}(n^2)$	Memory $\mathcal{O}(\max\{n, k^2\})$

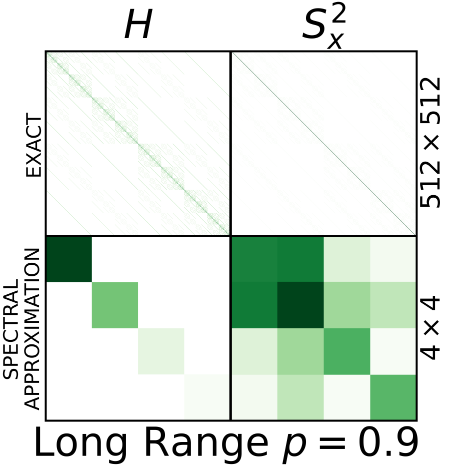
Spectral Approximation





Spectral Approximation





Summary

Summary

- Predicted the existence of an infinite hierarchy of nonequilibrium critical exponents

Summary

- Predicted the existence of an infinite hierarchy of nonequilibrium critical exponents
- Developed a novel approximation technique for Hamiltonian and Liouvillian dynamics

Summary

- Predicted the existence of an infinite hierarchy of nonequilibrium critical exponents
- Developed a novel approximation technique for Hamiltonian and Liouvillian dynamics
- Experimentally observed the hierarchy of critical exponents

Summary

- Predicted the existence of an infinite hierarchy of nonequilibrium critical exponents
- Developed a novel approximation technique for Hamiltonian and Liouvillian dynamics
- Experimentally observed the hierarchy of critical exponents
  - And simulated the experimental system

Future Work

Future Work

- Submit manuscripts

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG
  - $d < p < p_*$

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG
  - $d < p < p_*$
  - $d = 1,\ p = 2$ is exactly solvable

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG
  - $d < p < p_*$
  - $d = 1,\ p = 2$ is exactly solvable
  - $p > p_*$ is nearest-neighbor Defenu et al. 2021, *arXiv:2109.01063*

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG
  - $d < p < p_*$
  - $d = 1,\ p = 2$ is exactly solvable
  - $p > p_*$ is nearest-neighbor Defenu et al. 2021, *arXiv:2109.01063*
- Implement a proper Modified Jacobi–Davidson algorithm

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG
  - $d < p < p_*$
  - $d = 1,\ p = 2$ is exactly solvable
  - $p > p_*$ is nearest-neighbor Defenu et al. 2021, *arXiv:2109.01063*
- Implement a proper Modified Jacobi–Davidson algorithm
  - Currently using a naive implementation

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG
  - $d < p < p_*$
  - $d = 1,\ p = 2$ is exactly solvable
  - $p > p_*$ is nearest-neighbor Defenu et al. 2021, *arXiv:2109.01063*
- Implement a proper Modified Jacobi–Davidson algorithm
  - Currently using a naive implementation
  - Better preconditioning is needed

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \le p < d$ is LMG
  - $d < p < p_*$
  - $d = 1,\ p = 2$ is exactly solvable
  - $p > p_*$ is nearest-neighbor Defenu et al. 2021, *arXiv:2109.01063*
- Implement a proper Modified Jacobi–Davidson algorithm
  - Currently using a naive implementation
  - Better preconditioning is needed
  - Issues with losing orthogonality

Future Work

- Submit manuscripts
  - Experimental paper to *Nature*
  - Theory paper to *Physical Review Letters*
- Continue exploring long-range interacting systems
  - $0 \leq p < d$ is LMG
  - $d < p < p_*$
  - $d = 1, \ p = 2$ is exactly solvable
  - $p > p_*$ is nearest-neighbor Defenu et al. 2021, *arXiv:2109.01063*
- Implement a proper Modified Jacobi–Davidson algorithm
  - Currently using a naive implementation
  - Better preconditioning is needed
  - Issues with losing orthogonality
  - Optimize implementation (e.g. lower level languages)

Acknowledgements

My Committee

- **Dr. Mohammad Maghrebi** (*MSU-PA*)
- Dr. Scott Bogner (*FRIB, MSU-PA*)
- Dr. Johannes Pollanen (*MSU-PA, MSU-Q*)
- **Dr. Yang Yang** (*MSU-CMSE*)
- Dr. Alexei Bazavov (*MSU-CMSE, MSU-PA*)

My Colleagues

- **Daniel Paz** (*MSU-PA*)
- Dr. Sanku Paul (*MSU-PA*)
- Dr. Paraj Titum (*Johns Hopkins APL*)
- Arinjoy De (*Univ. Maryland, JQI*)
- Dr. William Morong (*Univ. Maryland, JQI*)
- Dr. Guido Pagano (*Rice Univ.*)
- Dr. Alexey Gorshkov (*Univ. Maryland, JQI*)
- Dr. Chris Monroe (*Univ. Maryland, JQI, IonQ Inc.*)

References

Adesso, Gerardo et al. (2004). In: *Phys. Rev. A* 70 (2), p. 022318. DOI: 10.1103/PhysRevA.70.022318.

Defenu, Nicolò et al. (2021). In: *arXiv:2109.01063*. DOI: 10.48550/ARXIV.2109.01063.

Demmel, James et al. (2007). In: *Numerische Mathematik* 108.1, pp. 59–91. DOI: 10.1007/s00211-007-0114-x.

Eastman, Peter (2015).

Gardiner, Crispin W. and Paul Zoller (2004). In:

Marino, Jamir et al. (2022). In: *arXiv:2101.09894*. DOI: 10.48550/ARXIV.2201.09894.

Paz, D. A. et al. (2022). unpublished.

Paz, Daniel A. and Mohammad F. Maghrebi (2021). In: *Physical Review A* 104.2. DOI: 10.1103/physreva.104.023713.

Pumplin, J. et al. (2001). In: *Physical Review D* 65.1. DOI: 10.1103/physrevd.65.014011.

Tackett, Alan R. and Massimiliano Di Ventra (2002). In: *Phys. Rev. B* 66 (24), p. 245104. DOI: 10.1103/PhysRevB.66.245104.

Titum, Paraj and Mohammad F. Maghrebi (2020). In: *Phys. Rev. Lett.* 125 (4), p. 040602. DOI: 10.1103/PhysRevLett.125.040602.

Zhang, J. et al. (2017). In: *Nature* 551.7682, pp. 601–604. DOI: 10.1038/nature24654.

Thank You

Questions

Mean Field Derivation of Ground State Phase Diagram

Mean-field:

$$H_{\mathrm{LMG-MF}} = -\frac{J}{2N}\sum_{i\neq j}^{N}\left(\sigma_i^x \left\langle \sigma_j^x \right\rangle + \left\langle \sigma_i^x \right\rangle \sigma_j^x\right) - \Delta \sum_i^N \sigma_i^z$$

Translational invariance:

$$H_{\mathrm{LMG-MF}} = -Jm\sum_i^N \sigma_i^x - \Delta \sum_i^N \sigma_i^z$$

Uncoupled Hamiltonians:

$$H_{\mathrm{LMG-MF}} = \begin{bmatrix} -\Delta & -Jm \\ -Jm & \Delta \end{bmatrix}$$

$$E_0 = -\sqrt{\Delta^2 + J^2 m^2}$$

$$|0\rangle = \frac{1}{\sqrt{\left(\Delta - E_0\right)^2 + J^2 m^2}} \begin{bmatrix} \Delta - E_0 \\ Jm \end{bmatrix}$$

Parent Slide

Holstein-Primakoff

$$a = (x + ip)/\sqrt{2} \qquad S_x \leftrightarrow \sqrt{\tfrac{N}{2}}x,\ S_y \leftrightarrow -\sqrt{\tfrac{N}{2}}p$$

$$\begin{aligned}\frac{1}{N}\left[\gamma_x S_x^2 + \gamma_y S_y^2\right] =& \frac{1}{4}\left[\gamma_x\left(a+a^\dagger\right)^2 - \gamma_y\left(a^\dagger - a\right)^2\right] \\ &- \frac{1}{4N}\left[\gamma_x a^\dagger a\left(a+a^\dagger\right)^2 - \gamma_y a^\dagger a\left(a^\dagger - a\right)^2\right] \\ &+ \mathcal{O}\left(a^2/N^2\right)\end{aligned}$$

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\Omega^2 x^2 + \frac{1}{N}\left[u_x x^4 + u_y p^4 + u_x p^2 x^2 + u_y x^2 p^2\right] + \mathcal{O}\left(1/N^2\right)$$

- $m^{-1} \equiv 2\left(\Delta - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(\Delta - J\gamma_x\right)\left(\Delta - J\gamma_y\right)$
- $u_x = J\gamma_x/2$
- $u_y = J\gamma_y/2$

Parent Slide

Disordered Phase Scaling

$$J\gamma_{x,y} < B$$

- $m^{-1} \equiv 2\,(B - J\gamma_y)$
- $\Omega^2 \equiv 4\,(B - J\gamma_x)\,(B - J\gamma_y)$

Disordered Phase Scaling

$$J\gamma_{x,y} < B$$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

- Quartic terms suppressed by $1/N$

Disordered Phase Scaling

$$J\gamma_{x,y} < B$$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

- Quartic terms suppressed by $1/N$
- H becomes purely quadratic

Disordered Phase Scaling

$$J\gamma_{x,y} < B$$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

- Quartic terms suppressed by $1/N$
- H becomes purely quadratic
- Gaussian Wigner function

$$W(x,p) = 2\exp\left\{-m\Omega x^2 - \frac{1}{m\Omega}p^2\right\}$$

Disordered Phase Scaling

$$J\gamma_{x,y} < B$$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

- Quartic terms suppressed by $1/N$
- H becomes purely quadratic
- Gaussian Wigner function

$$W(x,p) = 2\exp\left\{-m\Omega x^2 - \frac{1}{m\Omega}p^2\right\}$$

$$\left\langle S_{x,y}^2 \right\rangle \sim N$$

Critical Point Scaling

W.L.O.G. $\gamma_x > \gamma_y$

$$J\gamma_x = B$$

- $m^{-1} \equiv 2\,(B - J\gamma_y)$
- $\Omega^2 \equiv 4\,(B - J\gamma_x)\,(B - J\gamma_y)$

Critical Point Scaling

W.L.O.G. $\gamma_x > \gamma_y$

$$J\gamma_x = B$$

- $\Omega \to 0$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

Critical Point Scaling

W.L.O.G. $\gamma_x > \gamma_y$

$$J\gamma_x = B$$

- $\Omega \to 0$
- $H = \frac{1}{2m}p^2 + \frac{u_x}{N}x^4$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

Critical Point Scaling

W.L.O.G. $\gamma_x > \gamma_y$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

$$J\gamma_x = B$$

- $\Omega \to 0$
- $H = \frac{1}{2m}p^2 + \frac{u_x}{N}x^4$
- Form of the Wigner function given by Gardiner and Zoller 2004

$$W(x,p) = \widetilde{W}\left[\left(\frac{N}{mu_x}\right)^{-1/3} x, \left(\frac{N}{mu_x}\right)^{1/3} p\right]$$

Critical Point Scaling

W.L.O.G. $\gamma_x > \gamma_y$

- $m^{-1} \equiv 2\left(B - J\gamma_y\right)$
- $\Omega^2 \equiv 4\left(B - J\gamma_x\right)\left(B - J\gamma_y\right)$

$$J\gamma_x = B$$

- $\Omega \to 0$
- $H = \frac{1}{2m}p^2 + \frac{u_x}{N}x^4$
- Form of the Wigner function given by Gardiner and Zoller 2004

$$W(x,p) = \widetilde{W}\left[\left(\frac{N}{mu_x}\right)^{-1/3} x, \left(\frac{N}{mu_x}\right)^{1/3} p\right]$$

$$\left\langle S_x^2 \right\rangle / N \sim N^{1/3}$$
$$\left\langle S_y^2 \right\rangle / N \sim N^{-1/3}$$

Ordered Phase Scaling

Parent Slide

Ordered Phase Scaling

Fluctuations not too large, so

Parent Slide

Ordered Phase Scaling

Fluctuations not too large, so

$$\left\langle S_x^2 \right\rangle / N \approx \left\langle S_x \right\rangle^2 / N$$

Parent Slide

Ordered Phase Scaling

Fluctuations not too large, so

$$\left\langle S_x^2 \right\rangle /N \approx \left\langle S_x \right\rangle^2 /N$$

$$\left\langle S_x^2 \right\rangle /N \sim N$$

Parent Slide

Scaling After a Quench

From the disordered phase[1]

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\Omega^2 x^2 + \frac{u_x}{N}x^4$$

Parent Slide

[1] Possibly arbitrarily close to the critical point

Scaling After a Quench

From the disordered phase[1]

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\Omega^2 x^2 + \frac{u_x}{N}x^4$$

With $r \equiv B - J\gamma_x$,

$$\ddot{x} + rx + \frac{x^3}{N} = 0$$

Parent Slide

[1] Possibly arbitrarily close to the critical point

Scaling After a Quench

From the disordered phase[1]

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\Omega^2 x^2 + \frac{u_x}{N}x^4$$

With $r \equiv B - J\gamma_x$,

$$\ddot{x} + rx + \frac{x^3}{N} = 0$$

Define scaling exponents

$$[x] = \frac{\alpha}{2}[N] \text{ and } [t] = \zeta[N]$$

Parent Slide

[1] Possibly arbitrarily close to the critical point

Scaling After a Quench

From the disordered phase[1]

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\Omega^2 x^2 + \frac{u_x}{N}x^4$$

With $r \equiv B - J\gamma_x$,

$$\ddot{x} + rx + \frac{x^3}{N} = 0$$

Define scaling exponents

$$[x] = \frac{\alpha}{2}[N] \text{ and } [t] = \zeta[N]$$

Each term in the EOM must have the same scaling dimensions

Parent Slide

[1] Possibly arbitrarily close to the critical point

Scaling After a Quench

$$\zeta = \frac{1-\alpha}{2}$$

Gaussian Wigner function requires

$$\left\langle x^2(t) \right\rangle \sim N^{\alpha-\alpha_0}$$
$$\left\langle \dot{x}^2(t) \right\rangle \sim N^{\alpha-2\zeta+\alpha_0}$$

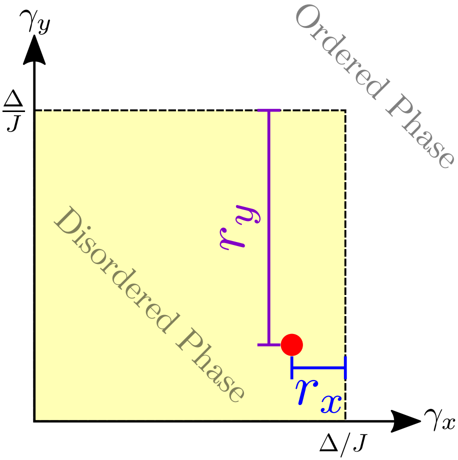
$$\alpha = \frac{1-\alpha_0}{2}$$
$$\zeta = \frac{1+\alpha_0}{4}$$

Parent Slide

Critical Exponents

$$\left\langle S_x^2 \right\rangle / N \sim \sqrt{\frac{r_y^{(0)}}{r_x^{(0)}}} + \frac{r_y}{r_x}\sqrt{\frac{r_x^{(0)}}{r_y^{(0)}}}$$

$$\left\langle S_y^2 \right\rangle / N \sim \sqrt{\frac{r_x^{(0)}}{r_y^{(0)}}} + \frac{r_x}{r_y}\sqrt{\frac{r_y^{(0)}}{r_x^{(0)}}}$$



Click for Details

Scaling With r After a Quench

Parent Slide

Spin-Wave Analysis

Parent Slide

Effective Temperature

Entanglement Entropy using HP

Coarsening

Black-Box Optimization

Uncertainty Propagation

Sources of Error