Master Thesis Project in Computational Physics

Parametric Matrix Models for Solving the Black–Scholes Equation

Duration: One Year (60 ECTS)

1. Introduction

The Black–Scholes partial differential equation (PDE) is one of the central equations in quantitative finance. It describes the fair price of European options as a function of time, volatility, and underlying asset price. Solutions to the PDE can be obtained analytically in certain cases, but more complex derivatives or modified market assumptions require numerical methods such as finite differences or finite elements.

Machine learning methods have recently gained attention as surrogate approximators for PDE solutions, offering the potential for efficient evaluation and parameter sensitivity exploration. However, most neural-network approaches are not constructed to respect the underlying analytical properties of the PDE. They often lack interpretability, extrapolation reliability, and stability.

In contrast, *Parametric Matrix Models* (PMMs), introduced by Cook *et al.* (Nature Communications 16, 5929 (2025)), provide a physics-inspired framework rooted in matrix equations and operator theory. The PMM architecture replaces traditional neural networks with parametrized matrices whose spectra and eigenvectors generate outputs. These models:

- are universal function approximators,
- allow analytic continuation,
- incorporate physical constraints directly into their structure,
- have excellent extrapolation properties,
- often require far fewer trainable parameters than neural networks.

This thesis aims to combine the theory of PMMs with the financial mathematics of the Black–Scholes equation to build a reduced-order surrogate model for option pricing. The goal is to investigate whether PMMs can serve as highly efficient and interpretable models for PDE-based pricing in computational finance.

2. Scientific Motivation

The Black–Scholes PDE is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

with terminal condition $V(S,T) = \max(S-K,0)$.

Traditional numerical solvers compute the full PDE solution on a discretized spatial grid. This is computationally expensive when evaluating the price repeatedly for different parameter values (σ, r, K). A PMM surrogate model may achieve:

- reduced computational complexity,
- efficient multi-parameter interpolation,
- automatic incorporation of analyticity of the solution,
- excellent performance across a compact domain of input features.

Applying PMMs to the Black–Scholes equation also offers a test case for more complex PDEs in physics, finance, and engineering.

3. Objectives of the Thesis

The main objective of this thesis is to develop, implement, and test a parametric matrix model (PMM) for solving the Black–Scholes PDE. The work involves theoretical analysis, numerical implementation, and extensive benchmarking.

Objective 1: Understanding PMMs and Reduced-Order Methods

- Study and summarize the PMM framework introduced by Cook *et al.*
- Analyze connections to eigenvector continuation, reduced basis methods, and implicit-function models.

Objective 2: Numerical Foundation of the Black–Scholes PDE

- Derive the Black–Scholes equation and its analytical solution.
- Implement a stable finite-difference solver as the reference solution.

Objective 3: Parametric Matrix Model for PDE Surrogate Learning

- Construct PMMs where the input features are (S, t, σ, r, K) or suitable lower-dimensional encodings.
- Explore Hermitian vs. unitary PMM formulations.
- Investigate how boundary conditions and payoff constraints can be embedded into secondary matrices.

Objective 4: Training, Optimization, and Stability

- Implement gradient-based learning using the analytic gradients described in the PMM paper.
- Train models on option–price datasets generated from the PDE solver.
- Study extrapolation behavior with respect to large or small volatility, deep in/out-of-the-money regimes, and long maturities.

Objective 5: Benchmarking and Comparisons

- Compare PMMs with:
  - feed-forward neural networks,
  - physics-informed neural networks,
  - standard surrogate models (Gaussian processes, Kernel Ridge Regression).
- Evaluate:
  - accuracy,
  - computational efficiency,
  - number of trainable parameters,
  - extrapolation quality.

Objective 6: Extensions

Possible extensions depending on time:

- PMM-based solution of the time-dependent heat equation (via Black–Scholes transformation).
- Multi-asset options (two-dimensional PDE).
- American option approximation via penalty methods.

4. Work Plan and Milestones (One-Year Timeline)

Semester 1 (Months 1–6)

Month 1–2: Literature Study

- Read the PMM paper and supporting literature on reduced-basis methods.
- Study the derivation and numerical properties of the Black–Scholes PDE.
- Begin preliminary numerical experiments.

Deliverable: A short report summarizing PMMs and the Black–Scholes PDE.

Month 3–4: Numerical Solvers and Data Generation

- Implement a finite-difference solver for Black–Scholes.
- Generate datasets for a range of volatilities, interest rates, and maturities.
- Test grid-convergence and accuracy.

Deliverable: Verified PDE solver and dataset.

Month 5–6: PMM Model Construction

- Construct primary and secondary matrices.
- Test small-scale PMMs (e.g. 5x5, 7x7) on simplified payoff functions.
- Begin training using analytic gradients.

Deliverable: First working PMM surrogate for one-parameter models.

Semester 2 (Months 7–12)

Month 7–9: Full PMM Training and Analysis

- Train PMMs across full parameter domain.
- Assess accuracy, convergence, and extrapolation.
- Optimize hyperparameters and matrix sizes.

Deliverable: Fully trained PMM for Black–Scholes pricing.

Month 10–11: Benchmarking and Comparisons

- Compare PMM with neural networks and other regression models.
- Evaluate computational speed and parameter efficiency.

Deliverable: Benchmarking report with figures and tables.

Month 12: Thesis Writing and Finalization

- Complete thesis manuscript.
- Prepare defense presentation.

Deliverable: Final thesis and presentation.

5. Expected Outcomes

- A full numerical implementation of PMMs applied to the Black–Scholes equation.
- Demonstration of PMM capabilities as surrogate PDE solvers.
- Quantitative comparisons with traditional machine learning and numerical PDE methods.
- A master thesis document summarizing:
  - theoretical background,
  - numerical implementations,
  - parameter studies,
  - performance analysis.

6. References

1. P. Cook, D. Jammooa, M. Hjorth-Jensen, D. D. Lee, *Parametric Matrix Models*, *Nature Communications*, 16, 5929 (2025).
2. F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81 (1973).
3. J. C. Hull, *Options, Futures, and Other Derivatives*, Pearson.
4. Additional references to be added