COMP1046 Mathematics for Computer Scientists
Coursework 2

1 Instructions.

Use NumPy in Python to implement a Markov transition matrix and compute a Markov chain.

- This coursework has a total of 100 marks and represents 10% of assessment for the COMP1046 module.
- Answer all the questions below.
- Show all your working and use the same notation that is used in the lecture notes. The examples and exercises in the lectures and tutorials are a model for how you should present your answers.
- This must be your own individual work so do not work with other students on this coursework. Do not use AI tools. Please ensure you know the University policy on academic misconduct:
  https://www.nottingham.ac.uk/qualitymanual/ assessment-awards-and-deg-classification/pol-academic-misconduct.aspx.
- You must submit your work as a single PDF document `MCS_CW2_`<*student-id*>`.pdf` and single `.ipynb` file on Moodle. You can use Word or Latex to type up your answer and save as PDF. Alternatively, you can handwrite your answer on A4 paper and scan to PDF. Do not use a mobile phone to take a photo of your coursework and submit the photo, since the quality is often poor. For the Python code, submit as a file `MCS_CW2_`<*student-id*>`.ipynb`.
- Your Python code should run stand-alone on Python3 notebook. You should use the NumPy package but **no other packages or external libraries should be used**. If your code does not run or has errors, you could get zero marks for the coding part.
- If your submitted coursework is illegible, you will receive a zero mark for the parts that cannot be read clearly. You will not be offered a second chance to resubmit, so ensure the good quality of your submission.

- Submit by the given deadline **16:00 on 4th December 2024**. You will lose 10% marks for each day your submission is late.

2 Background.

Consider a problem where we have a finite number of states n and we need to model the transition from state to state over time. We can use a probability to express the transition between states. For example, $\mathbb{P}(s_t = 2 \mid s_{t-1} = 3)$ is the probabilty of moving to state 2 from state 3. A Markov tranisition matrix is the square of all possible transition probabilities. In particular, the Markov transition matrix $\mathbf{P}$ is such that each of it's elements

$$p_{i,j} = \mathbb{P}(s_t = j \mid s_{t-1} = i).$$

Since $\mathbf{P}$ is a matrix of conditional probabilities over n states it must have the following three properties:

1. $\mathbf{P}$ is a $n \times n$ matrix.
2. For all elements $p_{i,j}$ of $\mathbf{P}$, $0 \leq p_{i,j} \leq 1$.
3. For all $i \in \{1, \cdots, n\}$,

$$\sum_{j=1}^{n} p_{i,j} = 1.$$

A Markov transition model can be used to estmate *state probability vectors* forward from a given initial state. Hence, given an *initial state* at time $t = 0$, $\mathbf{S}_{[0]}$, then future probabilities of states can be computed using the rule,

$$\mathbf{S}_{[t]} = \mathbf{S}_{[t-1]}\mathbf{P}.$$

Note that the index in parentheses $[\cdot]$ *indexes time, not elements of the matrix.*

2.1 Example.

Suppose $n = 2$, $\mathbf{P} = \begin{pmatrix} 0.6 & 0.4 \\ 0.8 & 0.2 \end{pmatrix}$ and $\mathbf{S}_{[0]} = (0\ 1)$. Then,

- $\mathbf{S}_{[1]} = \mathbf{S}_{[0]}\mathbf{P} = (0.8\ 0.2)$ and
- $\mathbf{S}_{[2]} = \mathbf{S}_{[1]}\mathbf{P} = (0.48 + 0.16 \quad 0.32 + 0.04) = (0.64 \quad 0.36)$.

Notice that $\mathbf{S}_{[0]}$ has probability 1 for state 2, hence we know it starts in state 2 with certainty. $\mathbf{S}_{[2]}$ has value 0.64 for state 1, which means there is a probability of 0.64 it is in state 1 at time $t = 2$.

2.2 Further reading.

The information given above is sufficient to complete this project. However, it may help you to know more about Markov models. Therefore you are encouraged to do your own **self-learning** and find futher resources to understand about how they work and the many application areas they are useful. As a help, and to get you started, please read the following article:

- *Markov models—Markov chains* by Grewal, Krzywinski and Altman in *Nature Methods* 16, August 2019, 661–664.

A pdf copy is available on Moodle.

3 Questions.

Marks are shown against each question part. 5 marks is available for reports with answers presented clearly and neatly. For coding parts, marks will be deducted for poorly formatted code, lack of comments or poor choices of variable and function names.

Q1. Let $\mathbf{P} = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.1 & 0.9 & 0.0 \\ 0.0 & 0.5 & 0.5 \end{pmatrix}$ and $\mathbf{S}_{[0]} = (1\ 0\ 0)$.

(a) Compute $\mathbf{S}_{[1]}$ and $\mathbf{S}_{[2]}$.

[20 marks]

(b) Suppose $\mathbf{S}_{[0]} = \mathbf{S}_{[1]}\mathbf{X}$. Find a matrix $\mathbf{X}$ that satisfies this equation. Is it a Markov transition matrix?

[15 marks]

Q2. If $\mathbf{P}$ and $\mathbf{Q}$ are both Markov transition matrices for n states, show that $\mathbf{PQ}$ is also a Markov transition matrix.
Hint: Prove the three properties for $\mathbf{PQ}$.

[15 marks]

Q3. Construct the matrix,

$$\mathbf{M} = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \\ 0.5 & 0.4 & 0.1 & 0.0 \\ 0.0 & 0.2 & 0.5 & 0.3 \\ 1.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

where for each $k \in \{1, \cdots, 4\}$, $v_k = (d_k + 1)/(d_1 + d_2 + d_3 + d_4 + 4)$
and d_1, d_2, d_3, d_4 are the last four digits of your student ID.

- For example, if my student ID is `2124039` then $d_1 = 4, d_2 = 0, d_3 = 3, d_4 = 9$ and my $\mathbf{M}$ will be

$$\mathbf{M} = \begin{pmatrix} 0.25 & 0.05 & 0.2 & 0.5 \\ 0.5 & 0.4 & 0.1 & 0.0 \\ 0.0 & 0.2 & 0.5 & 0.3 \\ 1.0 & 0.0 & 0.0 & 0.0 \end{pmatrix}$$

  But you must use your own student ID.

In this question, use the initial state $\mathbf{S}_{[0]} = (1\ 0\ 0\ 0)$.

(a) Construct your $\mathbf{M}$ in NumPy, print it and include in your report.

[5 marks]

(b) Write a user-defined function `check_transition_matrix(Mat, n)` that checks if the matrix `Mat` is a Markov transition matrix for `n` states. It should return a boolean value. Use it to verify that your $\mathbf{M}$ is a Markov transition matrix.

[10 marks]

(c) What is special about the fourth row of $\mathbf{M}$?

[5 marks]

(d) Write Python code to print the sequence of state probability vectors $\mathbf{S}_{[0]}, \mathbf{S}_{[1]}, \cdots, \mathbf{S}_{[T]}$ up to time $t = T$ where T is input by the user. Show the output for $T = 10$ in your report.

[25 marks]

- For example, if my student ID is `2124039` then output for just $T = 5$ would be:

```
What is T? 5
Initial state = [1 0 0 0]
Time 1: [0.25 0.05 0.2  0.5 ]
Time 2: [0.5875 0.0725 0.155  0.185 ]
Time 3: [0.368125 0.089375 0.20225  0.34025 ]
Time 4: [0.47696875 0.09460625 0.1836875  0.2447375 ]
Time 5: [0.41128281 0.09842844 0.19669813 0.29359062]
```