AE1MCS: Mathematics for Computer Scientists

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Reading

Kenneth H. Rosen, *Discrete Mathematics and Its Applications*, 7th Edition, 2013.

- Chapter 2, Section 2.1. Sets
- Chapter 2, Section 2.2. Set Operations

Discrete Structures

- Much of discrete mathematics is devoted to the study of **discrete structures**, used to represent discrete objects..
- Many important discrete structures are built using **sets**, which are collections of objects.
  - **combinations**: unordered collections of objects used extensively in counting;
  - **relations**: sets of ordered pairs that represent relationships between objects;
  - **graphs**: sets of vertices and edges that connect vertices;
  - **finite state machines**, used to model computing machines;
  - ...

Set

An intuitive definition (not part of a formal theory of sets)

Definition

A *set* is an unordered collection of objects, called *elements* or *members* of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \notin A$ denotes that a is not an element of the set A.

Uppercase letters are usually used to denote sets. Lowercase letters are usually used to denote elements of sets.

Describe a Set

There are several ways to describe a set.

1. List all the members of a set (if it is possible):
   e.g. $\{a, b, c\}$, $\{1, a\}$, $\{1, 2, 3, ..., 99\}$ (positive integers < 100)
2. Use *set builder* notation: characterize all elements in a set by stating the property or properties they must have.
   - $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$
   - or $O = \{x \in \mathbb{Z}^+ \mid x \text{ is odd and } x < 10\}$

Important Sets

- $\mathbb{N} = \{0, 1, 2, 3, ...\}$, the set of **natural numbers**
- $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$, the set of **integers**
- $\mathbb{Z}^+ = \{1, 2, 3, ...\}$, the set of **positive integers**
- $\mathbb{Q} = \{p/q \mid p \in \mathbb{Z}, q \in \mathbb{Z}$, and $q \neq 0\}$, the set of **rational numbers**
- $\mathbb{R}$, the set of **real numbers**
- $\mathbb{R}^+$ or $\mathbb{R}_{>0}$, the set of **positive real numbers**

Equal Sets

Definition

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x\,(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

- $\{1, 2, 3\}$
- $\{3, 2, 1\}$
- $\{1, 2, 2, 3, 3, 3\}$

Empty Set and Singleton Set

- Empty set: a set that has no element. It is denoted by $\emptyset$ or $\{\}$.
- Singleton set: a set that has only one element.
- $\emptyset$ vs. $\{\emptyset\}$?

Venn Diagram

- Sets can be represented graphically using Venn diagrams [1].
- The universal set U, which contains all the objects under consideration, is represented by a rectangle.
- Inside this rectangle, circles or other geometrical figures are used to represent sets.
- Sometimes points are used to represent the particular elements of the set.
- Venn diagrams are often used to indicate the relationships between sets.

[1] named after the English mathematician John Venn, who introduced their use in 1881.

Subsets

Definition

The set A is a *subset* of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

$A \subseteq B$ iff $\forall x \, (x \in A \rightarrow x \in B)$

Prove or Disprove A is a Subset of B

Showing that A is a Subset of B To show that $A \subseteq B$, show that if x belongs to A then x also belongs to B.

Showing that A is Not a Subset of B To show that $A \not\subseteq B$, find a single $x \in A$ but $x \notin B$.

Proper Subset

A is a *proper subset* of B ($A \subset B$) if and only if

$$\forall x \, (x \in A \rightarrow x \in B) \land \exists x \, (x \in B \land x \notin A)$$

Equal Sets

$A = B$ iff $A \subseteq B$ and $B \subseteq A$.

The Size of a Set

Definition

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a *finite set* and that n is the *cardinality* of S. The cardinality of S is denoted by $|S|$.

Definition

A set is said to be infinite if it is not finite.

Power Sets

Definition

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

- What is the power set of the set $\{0, 1, 2\}$?
- What is the power set of the empty set?
- What is the power set of the set $\{\emptyset\}$?

If a set has n elements, then its power set has 2^n elements.

Ordered n-tuples

Definition

The *ordered n-tuple* $(a_1, a_2, ..., a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its second element,..., and a_n as its nth element.

- We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal.
- Ordered 2-tuples are called *ordered pairs*.

Cartesian products

Definition

Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

- What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
- $A \times B = B \times A$?

Cartesian products

Definition

The *Cartesian product* of the sets A_1, A_2,..., A_n, denoted by $A_1 \times A_2 \times ... \times A_n$, is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$, where a_i belongs to A_i for $i = 1, 2, ..., n$. In other words,

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) \mid a_i \in A_i \ for \ i = 1, 2, ..., n\}$$

$$A^n = \{(a_1, a_2, ..., a_n) \mid a_i \in A \ for \ i = 1, 2, ..., n\}$$

An example

Example Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3\}$ and $D = \{7, 8, 9\}$. Determine which of the following are true, false, or meaningless.

1. $A \subset B$.
2. $B \subset A$.
3. $B \in C$.
4. $\emptyset \in A$.
5. $\emptyset \subset A$.
6. $A < D$.
7. $3 \in C$.
8. $3 \subset C$.
9. $\{3\} \subset C$.

Using Set Notation with Quantifiers

- Sometimes we restrict the domain of a quantified statement explicitly by making use of a particular notation.
- For example, $\forall x \in S\ P(x)$ denotes the universal quantification of $P(x)$ over all elements in the set S.
- $\forall x \in S\ P(x) \equiv \forall x\ (x \in S \rightarrow P(x))$
- $\exists x \in S\ P(x)$ denotes the existential quantification of $P(x)$ over all elements in S.
- $\exists x \in S\ P(x) \equiv \exists x\ (x \in S \wedge P(x))$

Truth Sets and Quantifiers

- We will now tie together concepts from set theory and from predicate logic.
- Given a predicate P, and a domain D, we define the truth set of P to be the set of elements x in D for which $P(x)$ is true.
- The truth set of $P(x)$ is denoted by $\{x \in D \mid P(x)\}$.
- $\forall x\ P(x)$ is true over the domain U if and only if the truth set of P is the set U.
- $\exists x\ P(x)$ is true over the domain U if and only if the truth set of P is nonempty.

Set Operations

- Union
- Intersection
- Difference
- Complement

Union

Definition

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Intersection

Definition

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Disjoint

Definition

Two sets are called *disjoint* if their intersection is the empty set.

Difference

Definition

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\}$$

Remark: The difference of sets A and B is sometimes denoted by $A \setminus B$.

Complement

Once the universal set U has been specified, the complement of a set can be defined.

Definition

Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

$$\overline{A} = \{x \in U \mid x \notin A\}$$

Difference and Complement

$$A - B = A \cap \overline{B}$$

Set Identities

	Identity	Name
1	$A \cap U = A$	Identity laws
2	$A \cup \emptyset = A$	
3	$A \cup U = U$	Domination laws
4	$A \cap \emptyset = \emptyset$	
5	$A \cup A = A$	Idempotent laws
6	$A \cap A = A$	
7	$\overline{(\overline{A})} = A$	Complementation law
8	$A \cup B = B \cup A$	Commutative laws
9	$A \cap B = B \cap A$	

Set Identities

	Identity	Name
10	$A \cup (B \cup C) = (A \cup B) \cup C$	Associative laws
11	$A \cap (B \cap C) = (A \cap B) \cap C$	
12	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
13	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
14	$\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
15	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	
16	$A \cup (A \cap B) = A$	Absorption laws
17	$A \cap (A \cup B) = A$	
18	$A \cup \overline{A} = U$	Complement laws
19	$A \cap \overline{A} = \emptyset$	

Exercise

Let A, B and C be sets. Show that

- $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

Exercise

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement} \\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol} \\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection} \\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences} \\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol} \\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement} \\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union} \\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

Note that besides the definitions of complement, union, set membership, and set builder notation, this proof uses the second De Morgan law for logical equivalences. ◀

Generalized Unions and Intersections

Definition

The *union* of a collection of sets is the set that contains those elements that are members of *at least one* set in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

Definition

The *intersection* of a collection of sets is the set that contains those elements that are members of *all* the sets in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$

An example

For $i = 1, 2, \ldots$, let $A_i = \{i, i+1, i+2, \ldots\}$. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{1, 2, 3, \ldots\},$$

and

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, \ldots\} = \{n, n+1, n+2, \ldots\} = A_n.$$

Homework: Proving a Theorem

Theorem

For every set S, $\emptyset \subseteq S$ and $S \subseteq S$.

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