

# Tutorial 3

# Sets and Functions

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10/12/2023

# Sets

# Power sets

How many elements does each of these sets have where  $a$  and  $b$  are distinct elements?

- a)  $\mathcal{P}(\{a, b, \{a, b\}\})$
- b)  $\mathcal{P}(\{\emptyset, a, \{a\}, \{\{a\}\}\})$
- c)  $\mathcal{P}(\mathcal{P}(\emptyset))$

Let  $A$  be a set, and the elements of  $A$  be sets.

Define  $\bigcup A = \{x \mid \exists y \in A, x \in y\}$ .  
 $A = \{\{a, b\}, \{b, c\}\}$   
 $\bigcup A = \{a, b, c\}$

1) Calculate  $\bigcup \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\}$ ;

2) **Calculate**  $\bigcup P(A)$

$$A = \{\{a\}, \{b\}\}, \{b, c\}$$

3) Whether  $P(\bigcup A) = A$ ?

$$\begin{aligned} \bigcup A &= \{\{a\}, \{b\}\} \cup \{b, c\} \\ &= \{\{a\}, \{b\}, \underline{c}\} \end{aligned}$$

$$\bigcup A = \{x \mid \exists y \in A, x \in y\}.$$

Answer:

1). Calculate  $\bigcup \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\}$ ;

$$\begin{aligned}\text{Solution: } & \bigcup \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\} \\ &= \{a, b, c\} \cup \{a, d, e\} \cup \{a, f\} \\ &= \{a, b, c, d, e, f\}\end{aligned}$$

$$(2) \quad A = \{ \{a, b, c\}, \{a, d, e\}, \{a, f\} \}$$

$$\underline{P(A)} = \{ \underline{\emptyset}, \{ \{a, b, c\} \}, \{ \{a, d, e\} \}, \{ \{a, f\} \}, \\ \{ \{a, b, c\}, \{a, d, e\} \}, \{ \{a, b, c\}, \{a, f\} \}, \\ \{ \{a, d, e\}, \{a, f\} \}, \underline{A} \}$$

$$\underline{UP(A)} = \{ \{a, b, c\}, \{a, d, e\}, \{a, f\} \}$$

$$\bigcup A = \{x \mid \exists y \in A, x \in y\}.$$

3). Whether  $P(\bigcup A) = A$ ?

Answer: No!

Counter example:


$$A = \{\{a, b, c\}, \{a, d, e\}, \{a, f\}\}$$

$$\bigcup A = \{a, b, c, d, e, f\}$$

$$\{a, e\} \in P(\bigcup A), \text{ but } \{a, e\} \notin A$$

$$\text{so } P(\bigcup A) \neq A$$

- Let A and B be sets, prove


$$A \cap B = \emptyset \leftrightarrow A \subseteq \bar{B}$$



We split task into two subtasks.

The **first** is to prove that  $A \cap B = \emptyset \rightarrow A \subseteq \overline{B}$ .

By way of contradiction, suppose  $A \cap B = \emptyset$  and  $A \not\subseteq \overline{B}$ .

If  $A \not\subseteq \overline{B}$ , we can find an  $x$ , such that  $x \in A$  and  $x \notin \overline{B}$ .

But,

$$x \in A \wedge x \notin \overline{B}$$

therefore,  $x \in A \wedge x \in B$  (by definition of complement)

therefore,  $x \in A \cap B$  (by definition of intersection)  $\rightarrow A \cap B \neq \emptyset$

This leads to a contradiction.

The **second** task is to prove that  $A \subseteq \overline{B} \rightarrow A \cap B = \emptyset$

Again, by way of contradiction, suppose  $A \subseteq \overline{B}$  and  $A \cap B \neq \emptyset$

If  $A \cap B \neq \emptyset$  there exists an  $x$ , such that,  $x \in A \cap B$ .

But,

$$x \in A \cap B$$

therefore,  $x \in A \wedge x \in B$  (by definition of intersection)

therefore,  $x \in A \wedge x \notin \overline{B}$  (by definition of complement)  $\rightarrow A \not\subseteq \overline{B}$

This leads to a contradiction.

Finally, we have proved:  $A \cap B = \emptyset$  if and only if  $A \not\subseteq \overline{B}$

- Let  $A$ ,  $B$ , and  $C$  be any sets, show that

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$\{(a, b) \mid a \in A, b \in (B \cup C)\}$$

$$A = B : \textcircled{1} A \subseteq B, B \subseteq A$$

$$\textcircled{2} \forall x \in A, \rightarrow x \in B$$

$$\forall x \in B \rightarrow x \in A$$



$$\forall x \notin A \rightarrow x \notin B$$

*We can prove  $X = Y$  by showing that if  $a \in X$  then  $a \in Y$ , and if  $a \notin X$  then  $a \notin Y$ .*

*Let  $a = (x, y)$ , and suppose  $a \in A \times (B \cup C)$ .*

$$(x, y) \in A \times (B \cup C)$$

*therefore,  $x \in A \wedge y \in (B \cup C)$  (by definition of Cartesian Products)*

*therefore,  $x \in A \wedge (y \in B \vee y \in C)$  (by definition of union)*

*therefore,  $(x \in A \wedge y \in B) \vee (x \in A \wedge y \in C)$  (by distributive law)*

*therefore,  $(x, y) \in A \times B \vee (x, y) \in A \times C$  (by definition of Cartesian Products)*

*therefore,  $(x, y) \in (A \times B) \cup (A \times C)$  (by definition of union)*

*Therefore,  $a \in (A \times B) \cup (A \times C)$*

*Let  $a = (x, y)$ , and suppose  $a \notin A \times (B \cup C)$ .*

$$= (x, y) \notin A \times (B \cup C)$$

*therefore,  $x \notin A \vee y \notin (B \cup C)$  (by definition of Cartesian Products)*

*therefore,  $x \notin A \vee (y \notin B \wedge y \notin C)$  (by definition of union)*

*therefore,  $(x \notin A \vee y \notin B) \wedge (x \notin A \vee y \notin C)$  (by distributive law)*

*therefore,  $(x, y) \notin A \times B \wedge (x, y) \notin A \times C$  (by definition of Cartesian Products)*

*therefore,  $(x, y) \notin (A \times B) \cup (A \times C)$  (by definition of union)*

*Therefore,  $a \notin (A \times B) \cup (A \times C)$*

*We have finally proved that  $A \times (B \cup C) = (A \times B) \cup (A \times C)$*

# Functions and Sequences

# Functions

- Show that the function  $f(x) = |x|$  from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of non-negative real numbers, the resulting function is invertible.

$$\mathbb{R} \rightarrow \mathbb{R}^+$$

$$\mathbb{R}^+ \rightarrow \mathbb{R}^+$$

## Answer:

For a function to be invertible, it needs to be bijective. Therefore, we need to check if this is one-to-one (injective) and onto (surjective).

(a) Injective: No:

For some  $x_1 \neq x_2$ , say  $x_1 = -x_2$ , we have  $f(x_1) = f(x_2)$ , by definition of injective, the function is not injective.

If the domain is restricted to the set of nonnegative real numbers,  $f(x)$  is injective because  $\mathbb{R}^+ \cup \{0\}$

Assume  $f(x_1) = f(x_2)$ , we have  $x_1 = x_2$

Therefore, on the restricted domain  $f(x)$  is injective.

(b). Surjective

$\text{range} = \text{codomain}$

For some element  $b \in \text{rng}(f)$ , that  $b = |a|$ , with  $a \in \mathbb{R}$ ,  $b$  must be positive. Thus, the range is the set of all nonnegative real numbers. Because the range and codomain are the same, we can conclude that  $f$  is surjective.

(c) Bijective: No, because it is not injective. Though, on the restricted domain, it is bijective because it is both injective and surjective.

d) Invertible: Again, only on the restricted domain.

- How can we produce the terms of a sequence if the first 10 terms are 1,3,4,7,11,18,29,47,76,123?



- How can we produce the terms of a sequence if the first 10 terms are 1,3,4,7,11,18,29,47,76,123?
- Solution:

$$L_n = L_{n-1} + L_{n-2}, \text{ with initial condition } L_1 = 1, L_2 = 3$$



# Additional exercises on the textbook

- Section 2.1 : 7 11 13 21-27 33 39 43
- Section 2.2 : 5-10 21-23 29-31 37-43
- Section 2.3: 7,15,23-27, 33,35,41,45-47,53,59,71,73
- Section 2.4: 1-4,6-7,9