Lecture 2 - Matrices

COMP1046 - MCS2

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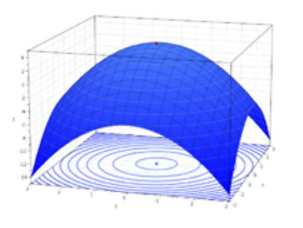

Learning outcomes

By the end of this lecture we will have learned:

- Numeric vectors
- Matrices
- Matrix operations: sum and product

Learning Resources

Chapter 2, Section 2.1-2.3: F. Neri, Linear Algebra for Computational Sciences and Engineering

Numeric vectors

Definition of a Numeric Vector

Definition

Numeric Vector

- Let $n \in \mathbb{N}$ and $n > 0$.
- The set generated by the Cartesian product of $\mathbb{R}$ by itself n times ($\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ldots$) is indicated with $\mathbb{R}^n$ and is a set of ordered n-tuples of real numbers.
- The generic element $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ of this set is named *numeric vector* or simply vector of order n on the real field.
- The generic a_i $\forall i$ from 1 to n is said the i^{th} *component* of the vector $\mathbf{a}$.

Example: Vector

Example

The n-tuple

$$\mathbf{a} = \left(1, 0, 56.3, \sqrt{2}\right)$$

is a vector of $\mathbb{R}^4$.

Sum of Vectors

Definition

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two numeric vectors $\in \mathbb{R}^n$. The *sum* of these two vectors is the vector

$$\mathbf{c} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

generated by the sum of the corresponding components.

We write the sum of two vectors as $\mathbf{a} + \mathbf{b}$.
Similarly, we can subtract one vector from another, $\mathbf{a} - \mathbf{b}$, component-wise.

Example: Adding two vectors

Example

Let us consider the following vectors of $\mathbb{R}^3$

$$\begin{aligned}\mathbf{a} &= (1, 0, 3) \\ \mathbf{b} &= (2, 1, -2) \,.\end{aligned}$$

The sum of these two vectors is

$$\mathbf{a} + \mathbf{b} = (3, 1, 1) \,.$$

Scalars

Definition

Scalar A numeric vector $\lambda \in \mathbb{R}^1$ is said *scalar*.

Definition

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be a numeric vector $\in \mathbb{R}^n$ and λ a number $\in \mathbb{R}$. The *product of a vector by a scalar* is the vector

$$\lambda\mathbf{a} = (\lambda a_1, \lambda a_2, \ldots, \lambda a_n)$$

generated by the product of λ by each corresponding component.

Example: Product of a scalar and a vector

Example

Let us consider the vector $\mathbf{a} = (1, 0, 4)$ and the scalar $\lambda = 2$.

The product of this scalar by this vector is

$$\lambda \mathbf{a} = (2, 0, 8)\,.$$

Scalar product

Definition

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be two numeric vectors $\in \mathbb{R}^n$. The *scalar product* of $\mathbf{a}$ by $\mathbf{b}$ is a real number

$$\mathbf{ab} = a_1 b_1 + a_2 b_2, \ldots, a_n b_n$$

generated by the sum of the products of each pair of corresponding components.

Note: The scalar product is also referred to as the *dot product* written as $\mathbf{a} \cdot \mathbf{b}$ (just different notation).

Scalar Product

Example

Let us consider again

$$\mathbf{a} = (1, 0, 3)$$
$$\mathbf{b} = (2, 1, -2)\,.$$

The scalar product of these two vectors is

$$\mathbf{ab} = (1 \cdot 2) + (0 \cdot 1) + (3 \cdot (-2)) = 2 + 0 - 6 = -4.$$

Properties of the scalar product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. It can be proved that the following properties of the scalar product are valid.

- symmetry: $\mathbf{ab} = \mathbf{ba}$
- associativity: $\lambda(\mathbf{ba}) = (\lambda\mathbf{a})\,\mathbf{b} = \mathbf{a}\,(\lambda\mathbf{b})$
- distributivity: $\mathbf{a}\,(\mathbf{b} + \mathbf{c}) = \mathbf{ab} + \mathbf{ac}$

Matrices

Matrix definition

Definition

Matrix

Let $m, n \in \mathbb{N}$ and both $m, n > 0$. A matrix $(m \times n)$ $\mathbf{A}$ is a generic table of the kind:

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{pmatrix}$$

where each *matrix element* $a_{i,j} \in \mathbb{R}$.

Matrix characteristics

- The set containing all the matrices of real numbers having m rows and n columns is indicated with $\mathbb{R}_{m,n}$.
- A matrix $\mathbf{A} \in \mathbb{R}_{n,n}$ is said n *order square matrix*.
- If the matrix is not square ($m \neq n$), it is said *rectangular*.
- The numeric vector $\mathbf{a_i} = (a_{i,1}, a_{i,2}, \dots, a_{i,n})$ is said generic i^{th} *row vector*, and
- $\mathbf{a^j} = \left(a_{1,j}, a_{2,j}, \dots, a_{m,j}\right)$ is said generic j^{th} *column vector*.

Example: Matrix

Example

Let

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 3 \\ 2 & -2 & 0 \end{pmatrix}$$

- Then $m = 3$ and $n = 3$. Hence $\mathbf{A}$ is a square matrix and $\mathbf{A} \in \mathbb{R}_{3,3}$.
- The 2nd row vector is $\mathbf{a}_2 = (0, 1, 3)$ and the 3rd column vector is $\mathbf{a}^3 = (-1, 3, 0)$.

Matrix transpose

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$. The *transpose* matrix of $\mathbf{A}$ is a matrix $\mathbf{A}^{\mathbf{T}}$ whose elements are the same as $\mathbf{A}$ but $\forall i, j$: $a_{j,i} = a^{T}_{i,j}$.

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 7 & 3.4 & \sqrt{2} \\ 5 & 0 & 4 & 1 \end{pmatrix}$$

$$\mathbf{A}^{\mathbf{T}} = \begin{pmatrix} 2 & 5 \\ 7 & 0 \\ 3.4 & 4 \\ \sqrt{2} & 1 \end{pmatrix}$$

It can be easily proved that the transpose of the transpose of a matrix is the matrix itself: $\mathbf{A} = \left(\mathbf{A}^{\mathbf{T}}\right)^{\mathbf{T}}$.

Symmetry

Definition

A matrix $\mathbf{A} \in \mathbb{R}_{n,n}$ is said *symmetric* when $\forall i, j : a_{i,j} = a_{j,i}$.

Example

The following matrix is symmetric:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 0 \\ 3 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

Proposition

Let $\mathbf{A}$ be a symmetric matrix. It follows that $\mathbf{A}^\mathrm{T} = \mathbf{A}$.

Diagonal and trace

Definition

Let $\mathbf{A} \in \mathbb{R}_{n,n}$. The *diagonal of a matrix* is the ordered n-tuple that displays the same index twice: $\forall i$ from 1 to n $a_{i,i}$.

Definition

Let $\mathbf{A} \in \mathbb{R}_{n,n}$. The *trace of a matrix* $\operatorname{tr}(\mathbf{A})$ is the sum of the diagonal elements: $\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{i,i}$.

Example: Diagonal and trace

Example

The diagonal of the matrix

$$\begin{pmatrix} 1 & 3 & 0 \\ 9 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

is $(1, 2, 2)$ and the trace is $1 + 2 + 2 = 5$.

Null Matrices

Definition

A matrix is said *null* $\mathbf{O}$ if all its elements are zeros.

Example

The null matrix of $\mathbb{R}_{2,3}$ is

$$\mathbf{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Identity Matrices

Definition

An *identity matrix* $\mathbf{I}$ is a square matrix whose diagonal elements are all ones while all the other extra-diagonal elements are zeros.

Example

The identity matrix of $\mathbb{R}_{3,3}$ is

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix operations: Sum and Product

Matrix sum

Definition

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}_{m,n}$. The *matrix sum* $\mathbf{C}$ is defined as:

$$\forall i, j : c_{i,j} = a_{i,j} + b_{i,j}.$$

Written as $\mathbf{C} = \mathbf{A} + \mathbf{B}$.

Important: Both matrices must be the same size.

Similarly, one matrix can be subtracted from another by $c_{i,j} = a_{i,j} - b_{i,j}$.

Example: Matrix addition and subtraction

Example

The sum of two matrices is shown below.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 5 & 1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 7 & 4 \\ 2 & 5 & 1 \\ 3 & 0 & 2 \end{pmatrix}$$

Example

Matrix subtraction is shown below.

$$\begin{pmatrix} 2 & 1 & 0 \\ 4 & 4 & -1 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 1 \\ 1 & 3 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & -1 \\ 3 & 1 & 1 \end{pmatrix}$$

Properties of the matrix sum

The following properties can be easily proved for the sum operation amongst matrices.

- commutativity: $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- associativity: $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- neutral element: $\mathbf{A} + \mathbf{O} = \mathbf{A}$
- opposite element:
  $\forall \mathbf{A} \in \mathbb{R}_{m,n} : \exists! \mathbf{B} \in \mathbb{R}_{m,n}$ such that $\mathbf{A} + \mathbf{B} = \mathbf{O}$

Product of a scalar and a matrix

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,n}$ and $\lambda \in \mathbb{R}$. The *product of a scalar by a matrix* is a matrix $\mathbf{C}$ defined as: $\forall i, j : c_{i,j} = \lambda a_{i,j}$.

Example

The product of a scalar $\lambda = 2$ by the matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 4 \end{pmatrix}$$

is

$$\lambda \mathbf{A} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & -2 & 8 \end{pmatrix}.$$

Properties of multiplying by a scalar

The following properties can be easily proved for the product of a scalar by a matrix.

- associativity: $\forall \mathbf{A} \in \mathbb{R}_{m,n}$ and $\forall \lambda, \mu \in \mathbb{R}$:

$$(\lambda\mu)\,\mathbf{A} = (\mathbf{A}\mu)\,\lambda = (\mathbf{A}\lambda)\,\mu$$

- distributivity of the product of a scalar by the sum of two matrices: $\forall \mathbf{A}, \mathbf{B} \in \mathbb{R}_{m,n}$ and $\forall \lambda \in \mathbb{R}$:

$$\lambda\,(\mathbf{A} + \mathbf{B}) = \lambda\mathbf{A} + \lambda\mathbf{B}$$

- distributivity of the product of a matrix by the sum of two scalars: $\forall \mathbf{A} \in \mathbb{R}_{m,n}$ and $\forall \lambda, \mu \in \mathbb{R}$:

$$(\lambda + \mu)\,\mathbf{A} = \lambda\mathbf{A} + \mu\mathbf{A}$$

Matrix product

Definition

Let $\mathbf{A} \in \mathbb{R}_{m,r}$ and $\mathbf{B} \in \mathbb{R}_{r,n}$. The *product of matrices* **A** *and* **B** is a matrix $\mathbf{C} = \mathbf{AB}$ whose generic element $c_{i,j}$ is defined in the following way:

$$c_{i,j} = \mathbf{a_i} \mathbf{b^j} = \sum_{k=1}^{r} a_{i,k} b_{k,j} = a_{i,1} b_{1,j} + a_{i,2} b_{2,j} + \ldots + a_{i,n} b_{m,j}.$$

- This is just the scalar product of row vectors of **A** with column vectors of **B**.
- Hence the number of columns in the first matrix must be the same as the number of rows in the second.
- The product of a $(m \times r)$ and a $(r \times n)$ matrix always yields a $(m \times n)$ matrix. That is, $\mathbf{C} \in \mathbb{R}_{m,n}$.

Example: Matrix product

Example

$$\mathbf{A} = \begin{pmatrix} 2 & 7 & 3 & 1 \\ 5 & 0 & 4 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 8 & 0 \\ 2 & 2 \end{pmatrix}$$

Compute by taking scalar products of rows and columns:
e.g. $c_{2,1} = \mathbf{a_2 b^1} = 5 \times 1 + 0 \times 2 + 4 \times 8 + 1 \times 2 = 39$.
Repeat for all $c_{i,j}$ then

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} \mathbf{a_1 b^1} & \mathbf{a_1 b^2} \\ \mathbf{a_2 b^1} & \mathbf{a_2 b^2} \end{pmatrix} = \begin{pmatrix} 42 & 41 \\ 39 & 12 \end{pmatrix}$$

Properties of matrix product

The following properties can be easily proved for the product between two matrices.

- left distributivity: $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$
- right distributivity: $(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$
- associativity: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- transpose of the product: $(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$
- neutral element: $\forall\mathbf{A} : \mathbf{AI} = \mathbf{A}$
- absorbing element: $\forall\mathbf{A} : \mathbf{AO} = \mathbf{O}$

Commutativity is *not* a general property

- It must be observed that commutativity with respect to the matrix product is generally not valid.
- It may happen in some cases that $\mathbf{AB} = \mathbf{BA}$. In these cases the matrices are said *commutable* (one with respect to the other).
- Every matrix $\mathbf{A}$ is commutable with $\mathbf{O}$ (and the result is always $\mathbf{O}$) and with $\mathbf{I}$ (and the result is always $\mathbf{A}$).

Example: Non-commutativity

Example

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 2 & 4 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

The products **AB** and **BA** are

$$\mathbf{AB} = \begin{pmatrix} 4 & 3 & 4 \\ 4 & 4 & 9 \\ 6 & 6 & 13 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} 18 & 6 & 2 \\ 3 & 1 & 0 \\ 4 & 1 & 2 \end{pmatrix}.$$

Hence, $\mathbf{AB} \neq \mathbf{BA}$ in this case.

Summary and next lecture

Summary

- Numeric vectors
- Matrices
- Matrix operations: sum and product

The next lecture

We will learn more about matrices: Determinants and Inversion.

THANK
YOU