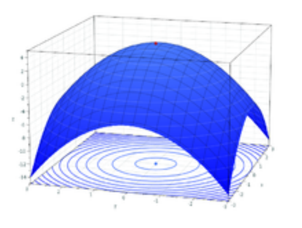
Lecture 6 - Vector Spaces (supplement)

COMP1046- Maths for Computer Scientists



Vector Subspace

Proposition from slide 8 of main lecture notes:

Proposition

Let $(E, +, \cdot)$ *be a vector space,* $U \subset E$*, and* $U \neq \emptyset$.

The triple $(U, +, \cdot)$ *is a vector subspace of* $(E, +, \cdot)$ *if and only if* U *is closed with respect to both the composition laws* $+$ *and* $\cdot$*, i.e.*

- $\forall \mathbf{u}, \mathbf{v} \in U : \mathbf{u} + \mathbf{v} \in U$
- $\forall \lambda \in \mathbb{K}$ *and* $\forall \mathbf{u} \in U : \lambda \mathbf{u} \in U$.

This supplement shows that all 10 vector space axioms follow from closure on the two composition laws.

Vector Subspace

Proof.

- Axioms 1 & 2 follow directly from the proposition.
- Axioms 3 & 4: whatever is true for elements in E, is true for U.
- Axiom 5: need to show $\mathbf{o} \in U$.
  - Firstly, for any $\mathbf{u} \in E$, $0\mathbf{u} = \mathbf{o}$.
    Homework: can you prove this just from the 10 axioms (for E)? It is harder than it looks it should be.
  - By axiom 2, take $\lambda = 0$, then for any $\mathbf{u} \in U$, $0\mathbf{u} \in U$ and $0\mathbf{u} = \mathbf{o}$, hence $\mathbf{o} \in U$.
- Axiom 6: Need to show $\exists ! \mathbf{u} \in U$. Let's come back to this.

continued...

Vector Subspace

Proof.

- Axioms 7 to 10: whatever is true for elements in E, is true for U.
- Axiom 6: Consider Axiom 9, distributivity 2, with $\lambda = 1$ & $\mu = -1$. For any $\mathbf{u} \in U$:

$$
\begin{aligned}
& (1 + -1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u} \\
=> \quad & 0\mathbf{u} = \mathbf{u} + (-1)\mathbf{u} \\
=> \quad & \mathbf{o} = \mathbf{u} + (-1)\mathbf{u}
\end{aligned}
$$

So $(-1)\mathbf{u} = -\mathbf{u}$ for Axiom 6.

$\square$