Lecture 7 - Basis and Dimension

COMP1046- Maths for Computer Scientists

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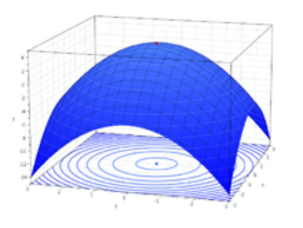

Learning outcomes

By the end of this lecture we will have learned:

- Basis of a Vector Space
- Dimension of a Vector Space

Based on Section 8.5 of the textbook (Neri 2018).

Basis of a Vector Space

Finite-dimensional Vector Spaces

Definition

Let $(E, +, \cdot)$ be a vector space.

The vector space $(E, +, \cdot)$ is said *finite-dimensional* if $\exists$ a finite number of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$, such that the vector space $(L, +, \cdot) = (E, +, \cdot)$ where the span L is $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$.

In this case we say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ span the vector space.

Basis

Definition

Let $(E, +, \cdot)$ be a finite-dimensional vector space.

A *basis* $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ of $(E, +, \cdot)$ is a set of vectors $\in E$ that verify the following properties.

- $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ are linearly independent
- $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$ span E, i.e. $E = L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$.

Span and Basis

Example

Let us consider the vector space $\mathbb{R}^3$. A basis B of $\mathbb{R}^3$ is

$$(0,0,1)\,, \quad (0,1,0)\,, \quad (1,0,0)$$

as they are linearly independent and all vectors in $\mathbb{R}^3$ can be derived by their linear combination.

A set of vectors spanning $\mathbb{R}^3$, i.e. the span L, is given by

$$(0,0,1)\,, \quad (0,1,0)\,, \quad (1,0,0)\,, \quad (1,2,3)$$

as they still generate all vectors in $\mathbb{R}^3$ but are not linearly independent.

Hence, a basis always spans a vector space while a set of vectors spanning a space is not necessarily a basis.

Exercise 1

Consider again the example from Exercise 3 in Lecture 6:

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 4y - 5z = 0\}.$$

Which of these sequences of vectors is a basis and which is not? Explain your reasoning.

1. $\left(1, 0, \frac{3}{5}\right)$, $\left(0, 1, \frac{4}{5}\right)$, $(0, 0, 1)$;
2. $\left(1, 0, \frac{3}{5}\right)$, $\left(0, 1, \frac{4}{5}\right)$, $(4, -3, 0)$;
3. $\left(1, 0, \frac{3}{5}\right)$, $\left(0, 1, \frac{4}{5}\right)$.

Exercise 1: Solution

To be completed.

Lemma

Steinitz's Lemma. *Let $(E, +, \cdot)$ be a finite-dimensional vector space and $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}) = E$ its span.*

Let $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_s}$ be s linearly independent vectors $\in E$.

It follows that $s \leq n$, i.e. the number of a set of linearly independent vectors cannot be higher than the number of vectors spanning the vector space.

Proof.

Let us assume by contradiction that $s > n$.

Since $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_s}$ are linearly independent, using Proposition on Slide 23 they are all different from the null vector $\mathbf{o}$. Hence, we know that $\mathbf{w_1} \neq \mathbf{o}$.

Since $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n})$ spans E and $\mathbf{w_1} \in E$, there exists a tuple $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$\mathbf{w_1} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \ldots + \lambda_n \mathbf{v_n}.$$

Since $\mathbf{w_1} \neq \mathbf{o}$, it follows that one of $\lambda_1, \lambda_2, \ldots, \lambda_n \neq 0, 0, \ldots 0$. Without a loss of generality let us assume that $\lambda_1 \neq 0$.

continued...

Steinitz's Lemma

Proof.

We can now write

$$\mathbf{v_1} = \frac{1}{\lambda_1}\left(\mathbf{w_1} - \lambda_2\mathbf{v_2} - \ldots - \lambda_n\mathbf{v_n}\right).$$

Thus, any vector $\mathbf{u} \in E$ that would be represented as

$$\begin{aligned}
&\mathbf{u} = a_1\mathbf{v_1} + a_2\mathbf{v_2} + \ldots + a_n\mathbf{v_n} = \\
&= \tfrac{a_1}{\lambda_1}\left(\mathbf{w_1} - \lambda_2\mathbf{v_2} - \ldots - \lambda_n\mathbf{v_n}\right) + a_2\mathbf{v_2} + \ldots + a_n\mathbf{v_n} = \\
&= k_1\mathbf{w_1} + k_2\mathbf{v_2} + \ldots + k_n\mathbf{v_n}.
\end{aligned}$$

This means that any vector $\mathbf{u} \in E$ can be represented by a linear combination of $\mathbf{w_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. This means that

$$L\left(\mathbf{w_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\right) = E.$$

continued...

Proof.

We can now express $\mathbf{w_2} \in E$ as

$$\mathbf{w_2} = \mu_1 \mathbf{w_1} + \mu_2 \mathbf{v_2} + \ldots + \mu_n \mathbf{v_n}$$

where $\mu_1, \mu_2, \ldots, \mu_n \neq 0, 0, \ldots, 0$ (it would happen that $\mathbf{w_2} = \mathbf{o}$ and hence from Proposition on Slide 23 again, the vectors would be linearly dependent).

Furthermore, $\mathbf{w_2}$ cannot be expressed as $\mathbf{w_2} = \mu_1 \mathbf{w_1}$ (they are linearly independent). Hence, there exists $j \in \{2, 3, \ldots, n\}$ such that $\mu_j \neq 0$. We can then assume that $\mu_2 \neq 0$ and state that any vector $\mathbf{u} \in E$ can be expressed as

$$\mathbf{u} = l_1 \mathbf{w_1} + l_2 \mathbf{w_2} + l_3 \mathbf{v_3} \ldots + {}_n \mathbf{v_n}.$$

continued...

Proof.

This means that $L(\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{v_n}) = E$.

We repeat... At the generic k^{th} step we have

$$\mathbf{w_k} = \gamma_1 \mathbf{w_1} + \gamma_2 \mathbf{w_2} + \ldots + \gamma_{k-1} \mathbf{w_{k-1}} + \gamma_k \mathbf{v_k} \cdots + \gamma_n \mathbf{v_n}.$$

Since $\mathbf{w_k}$ cannot be expressed as the linear combination of $\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_{k-1}}$ then $\exists j \in \{k, k+1, \ldots n\}$ such that $\gamma_j \neq 0$. Assuming that $\gamma_j = \gamma_k$ we have

$$L(\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}, \ldots, \mathbf{v_n}) = E.$$

continued...

Proof.

Reiterating until the n^{th} step, we have that any vector $\mathbf{u} \in E$ can be expressed as

$$\mathbf{u} = h_1\mathbf{w_1} + h_2\mathbf{w_2} + h_3\mathbf{w_3} \ldots + h_n\mathbf{w_n}.$$

This means that $L(\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}) = E$.

Since by contradiction $s > n$ there are no more $\mathbf{v}$ vectors while still there are $s - n$ $\mathbf{w}$ vectors.

In particular, $\mathbf{w_{n+1}} \in E$ and hence can be written as

$$\mathbf{w_{n+1}} = \delta_1\mathbf{w_1} + \delta_2\mathbf{w_2} + \delta_3\mathbf{w_3} \ldots + \delta_n\mathbf{w_n}.$$

continued...

Proof.

Since $\mathbf{w_{n+1}}$ has been expressed as linear combination of the others then the vectors are linearly dependent. This is against the hypothesis and a contradiction has been reached. $\square$

Exercise 2 (homework)

These questions are based on the method of proof for Steinitz's Lemma.

Consider the vector space $(\mathbb{R}^3, +, \cdot)$ with vectors

$$\mathbf{v_1} = (1, 0, 0), \quad \mathbf{v_2} = (0, 1, 0), \quad \mathbf{v_3} = (0, 0, 1)$$

and

$$\mathbf{w_1} = (1, -1, 0), \quad \mathbf{w_2} = (-1, 2, 1), \quad \mathbf{w_3} = (2, 1, -1)$$

in $\mathbb{R}^3$.

...continued...

Exercise 2 (homework)

Q1. Show that $L(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}) = \mathbb{R}^3$.

Q2. Show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are linearly independent.

Q3. Using the substitution method used in the proof of Steinitz's Lemma, show that each of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ can be expressed as a linear combination of $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$.

Q4. Use the answers to Q1 and Q3 to show $L(\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}) = \mathbb{R}^3$.

Q5. Suppose some $\mathbf{w_4} \in \mathbb{R}^3$. Use the answer to Q4 to show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}, \mathbf{w_4}$ must be linearly dependent.

Span and Basis

Theorem

Let $(E, +, \cdot)$ be a finite-dimensional vector space and $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}) = E$ its span and $B = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_s}\}$ be its basis. It follows that $s \leq n$.

Proof.

The vectors composing the basis are linearly independent. For the Steinitz's Lemma, it follows immediately that $s \leq n$. $\square$

Order of a Basis

Definition

The number of vectors composing a basis is said *order of a basis*.

Theorem

Let $(E, +, \cdot)$ be a finite-dimensional vector space. All the bases of the vector space have the same order.

Order of a Basis

Proof.

Let $(E, +, \cdot)$ be a finite-dimensional vector space with bases $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_b}\}$ and $C = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_c}\}$.

By Steinitz's Lemma,

- Since B spans E and vectors in C are linearly independent, $c \leq b$;
- Since C also spans E and vectors in B are linearly independent, $b \leq c$.

Hence $b = c$; i.e. the two bases must have the same order. □

Dimension of a Vector Space

Dimension of a Vector Space

Definition

Let $(E, +, \cdot)$ be a finite-dimensional vector space.

The order of a basis of $(E, +, \cdot)$ is said *dimension* of $(E, +, \cdot)$ and is indicated with $\dim(E, +, \cdot)$ or simply with $\dim(E)$.

Dimension of a Vector Space

Theorem

Let $(E, +, \cdot)$ *be a finite-dimensional vector space.*

The dimension $\dim(E, +, \cdot) = n$ *of a vector space (or simply* $\dim(E)$*) is*

- *the maximum number of linearly independent vectors of* E*;*
- *the minimum number of vectors spanning* E

Proof.

If $\dim(E, +, \cdot) = n$, then $\exists$ a basis $B = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$.

The basis, by definition, contains n linearly independent vectors spanning E and the order of the basis is the number of vectors n.

Let us assume, by contradiction, that n is not the maximum number of linearly independent vectors.

Let us assume that there exist $n+1$ linearly independent vectors in E:

$$\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_n}, \mathbf{w_{n+1}}.$$

Since B is a basis, its elements span the vector space $(E, +, \cdot)$, i.e. $L(\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}) = E$.

continued...

Proof.

From Steinitz's Lemma, the number of linearly independent vectors $n+1$ cannot be higher than the number of vectors n that span the vector space $(E, +, \cdot)$. Thus, $n+1 \leq n$, that is a clear contradiction.

This means that the maximum number of linearly independent vectors is n. □

In order to prove that n is also the minimum number of vectors spanning a space, let us assume, by contradiction, that $n-1$ vectors span the vector space $(E, +, \cdot)$, i.e.

$$L(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_{n-1}}) = E.$$

□

Proof.

Since, for the hypotheses $\dim(E,+,\cdot) = n$, the order of a basis B is n, i.e. $\exists\, n$ linearly independent vectors:

$$\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}.$$

From Steinitz's Lemma, the number of linearly independent vectors n cannot be higher than the number of vectors $n-1$ that generate the vector subspace,

$$n \le n-1,$$

that is a clear contradiction. This means that n is the minimum number of vectors spanning a space. □

Dimension of a Vector Space

Example

In the case of $\mathbb{R}^3$, we know that each basis is composed of three vectors (e.g. $(0,0,1), \quad (0,1,0), \quad (1,0,0) \quad$).

For definition of dimension $\dim\left(\mathbb{R}^3\right) = 3$.

We already know that in $\mathbb{R}^3$ at most three linearly independent vectors exist and at least three vectors are needed to span the vector space.

Thus, the dimension of the vector space is the maximum number of linearly independent vectors and the minimum number of spanning vectors.

Exercise 3

Consider

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 4y - 5z = 0\}$$

(from slide 10) and

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x = 2y = 3z\}.$$

Compute $\dim(U)$ and $\dim(V)$.

Exercise 3: Solution

To be completed.

Dimension of a vector subspace

Theorem

Let $(E, +, .)$ *be a vector space with vector subspace* $(F, +, .)$*. Then,* $\dim(F) \leq \dim(E)$.

Theorem

Let $(E, +, .)$ *be a vector space with vector subspace* $(F, +, .)$*. Then,* $\dim(F) = \dim(E)$ *if and only if* $F = E$.

Exercise 4

Can you prove these two theorems?

Hard

Exercise 4: Proof of first theorem

To be completed.

Exercise 4: Proof of second theorem

To be completed.

Exercise 4: Proof of second theorem

To be completed.

Summary and next lecture

Summary

- Basis of a Vector Space
- Dimension of a Vector Space

The next lecture

We will learn about linear mappings.