Lecture 11 - Eigenvalues and Eigenvectors

COMP1046 - Maths for Computer Scientists

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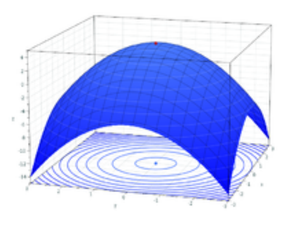


Learning outcomes

By the end of this lecture we will have learned:

- Eigenvalues and Eigenvectors
- Eigenspaces
- Determining Eigenvalues and Eigenvectors

Based on Section 10.5 of text book (Neri 2018).

In order to introduce the new concept of eigenvalues and eigenvectors, let us consider the following example.

Example

Let us consider the following linear mapping $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$f(x, y) = \left(2x - y, 3y\right)$$

corresponding to the matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$. Consider $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so $f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

...continues...

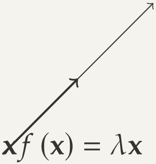
Example

This can be graphically represented as

For analogy, we may think that a linear mapping (at least $\mathbb{R}^2 \rightarrow \mathbb{R}^2$) is represented by a clock where one pointer is the input and the other is the output. Since both the vectors are applied in the origin, a linear mapping varies the input in length and rotates it around the origin.

Example

Now suppose we find an $\mathbf{x}$ which gives no rotation:



Then $\mathbf{x}$ is called an eigenvector of f and scalar λ is the corresponding eigenvalue.

Eigenvalues and Eigenvectors

Definition

Let $f : E \rightarrow E$ be an endomorphism where $(E, +, \cdot)$ is a finite-dimensional vector space defined on the scalar field $\mathbb{K}$ whose dimension is n.

Every vector $\mathbf{x}$ such that $f(\mathbf{x}) = \lambda\mathbf{x}$ with λ scalar and $\mathbf{x} \in E \setminus \{\mathbf{o_E}\}$ is said *eigenvector* of the endomorphism f related to the *eigenvalue* λ.

Perhaps surprisingly eigenvalues and eigenvectors have a wide range of applications in physics, engineering and computer science.

Example

Let us consider the endomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) = 5x.$$

In this case, any vector x (number in this specific case) is a potential eigenvector and $\lambda = 5$ would be the eigenvalue.

In general, for endomorphisms $\mathbb{R} \rightarrow \mathbb{R}$, the detection of eigenvectors and eigenvalues is trivial because the endomorphisms are already in the form $f(\mathbf{x}) = \lambda \mathbf{x}$.

When the endomorphism is between multidimensional vector spaces, the search of eigenvalues and eigenvectors is not trivial.

Example

Let us consider the following endomorphism $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f\left(x, y\right) = \left(x + y, 2x\right).$$

By definition, an eigenvector $\left(x, y\right)$ and an eigenvalue λ, respectively, verify the following equation

$$f\left(x, y\right) = \lambda\left(x, y\right).$$

continued...

Example

By combining the last two equations we have

$$\begin{cases} x + y = \lambda x \\ 2x = \lambda y \end{cases} \Rightarrow \begin{cases} (1 - \lambda)\, x + y = 0 \\ 2x - \lambda y = 0. \end{cases}$$

A scalar λ with a corresponding vector (x, y) that satisfy the homogeneous system of linear equations are an eigenvalue and its eigenvector, respectively.

continued...

Example

- Since the system is homogeneous (i.e. all known terms are zero), the only way for it to be determined is if $(x, y) = (0, 0)$.
- If this situation occurs, regardless of the value of λ, the equations of the system are verified.
- Since by definition an eigenvector $\mathbf{x} \in E \setminus \{\mathbf{o_E}\}$, it follows that $(x, y) = (0, 0) = \mathbf{o_E}$ is not an eigenvector.

continued...

Example

On the other hand, if we fix the value of λ such that the matrix associated with the system is singular, we have infinite eigenvectors associated with λ.

What value of λ will achieve this?
We will return to this question later.

Eigenspace

Eigenspace

Theorem

Let $f : E \longrightarrow E$ be an endomorphism.
The set $V(\lambda) \subset E$ with $\lambda \in \mathbb{K}$ defined as

$$V(\lambda) = \{\mathbf{o_E}\} \cup \left\{\mathbf{x} \in E \mid f(\mathbf{x}) = \lambda\mathbf{x}\right\}$$

with the composition laws is a vector subspace of $(E, +, \cdot)$.

Definition

The vector subspace $(V(\lambda), +, \cdot)$ defined as above is said *eigenspace* of the endomorphism f related to the eigenvalue λ.
The dimension of the eigenspace is said *geometric multiplicity* of the eigenvalue λ and is indicated with γ_m.

Eigenspace

Proof.

Let us prove the closure of $V(\lambda)$ with respect to the composition laws.

Consider any $\mathbf{x_1}, \mathbf{x_2} \in V(\lambda)$. Then,

$$\mathbf{x_1} \in V(\lambda) \Rightarrow f(\mathbf{x_1}) = \lambda \mathbf{x_1}$$
$$\mathbf{x_2} \in V(\lambda) \Rightarrow f(\mathbf{x_2}) = \lambda \mathbf{x_2}.$$

It follows that

$$f(\mathbf{x_1} + \mathbf{x_2}) = f(\mathbf{x_1}) + f(\mathbf{x_2}) = \lambda \mathbf{x_1} + \lambda \mathbf{x_2} = \lambda(\mathbf{x_1} + \mathbf{x_2}).$$

Hence, since $(\mathbf{x_1} + \mathbf{x_2}) \in V(\lambda)$, the set $V(\lambda)$ is closed with respect to the internal composition law. *continued...*

Proof.

Let us consider a scalar $h \in \mathbb{K}$. From the definition of $V(\lambda)$ we know that

$$\mathbf{x} \in V(\lambda) \Rightarrow f(\mathbf{x}) = \lambda \mathbf{x}.$$

It follows that

$$f(h\mathbf{x}) = hf(\mathbf{x}) = h(\lambda \mathbf{x}) = \lambda(h\mathbf{x}).$$

Hence, since $(h\mathbf{x}) \in V(\lambda)$, the set $V(\lambda)$ is closed with respect to the external composition law.

We can conclude that $(V(\lambda), +, \cdot)$ is a vector subspace of $(E, +, \cdot)$. $\square$

Eigenspace

Example

Let us consider again the endomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$f\left(x, y\right) = \left(x + y, 2x\right).$$

Then eigenvectors are given by

$$\left(x + y, 2x\right) = \lambda(x, y).$$

Notice that a solution is given for $\lambda = -1$ so this is an eigenvalue (*it is not the only one!*).

In particular, $\lambda = -1 \Rightarrow y = -2x$.

This gives solutions of the type $(x, y) = (\alpha, -2\alpha) = \alpha\,(1, -2)$ with the parameter $\alpha \in \mathbb{R}$, forming a set of solutions, $V(-1)$.

continued...

Exercise 1: Eigenspace

Example

The theorem above says that $(V(-1), +, \cdot)$ is a vector space (and referred to as eigenspace) and is a subspace of $\left(\mathbb{R}^2, +, \cdot\right)$.

Confirm that $(V(-1), +, \cdot)$ is indeed a vector space, from first principles.

Exercise 1: *Solution*

Firstly $V(-1)$ is a subset of E, by definition.
Since $(E, +, .)$ is a vector space, it is therefore only necessary to prove closure with respect to the internal and external composition laws.

Take arbitrary $\alpha_1\,(1,-2)\,,\ \alpha_2\,(1,-2)$ from $V(-1)$ and $\lambda \in \mathbb{R}$.

Then,

$$\alpha_1\,(1,-2) + \alpha_2\,(1,-2) = (\alpha_1 + \alpha_2)\,(1,-2) \in V(-1)$$

and

$$\lambda(\alpha_1\,(1,-2)) = (\lambda\alpha_1)\,(1,-2) \in V(-1).$$

This proves closure in both cases.

Determining Eigenvalues and Eigen-vectors

Determining Eigenvalues

This section conceptualizes in a general fashion the method for determining eigenvalues for any $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ endomorphisms.

Let $f : E \longrightarrow E$ be an endomorphism defined over $\mathbb{K}$ and let $(E, +, \cdot)$ be a finite-dimensional vector space having dimension n. A matrix $\mathbf{A} \in \mathbb{R}_{n,n}$ is associated with the endomorphism:

$$\mathbf{y} = f(\mathbf{x}) = \mathbf{A}\mathbf{x}$$

Let us impose the constraint for eigenvectors, $f(x) = \lambda \mathbf{x}$, so then we can write

$$\begin{aligned} \mathbf{A}\mathbf{x} = \lambda \mathbf{x} \quad &\Rightarrow \quad \mathbf{A}\mathbf{x} - \lambda \mathbf{I}\mathbf{x} = \mathbf{o} \\ &\Rightarrow \quad (\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{o} \end{aligned}$$

Determining Eigenvalues

If $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{o}$ is solved (for $\mathbf{x}$), it must have multiple solutions. Hence, by Cramer's Theorem, $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.
The determinant can be characterized as a polynomial in λ with the form,

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} k_{n-1}\lambda^{n-1} + \ldots + (-1)\,k_1\lambda + k_0.$$

called the *characteristic polynomial* of the endomorphism f.

Solving for $p(\lambda) = 0$ where $\lambda \in \mathbb{K}$ will yield the eigenvalues.

In general, for $n > 2$, computational methods are needed to solve this. In practice, an iterative method such as QR Algorithm is used to find eigenvalues and eigenvectors (not covered here).

Determining Eigenvalues

Example

Let us consider again the endomorphism $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$f\left(x, y\right) = \left(x + y, 2x\right).$$

The identifying matrix is $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ and $\lambda \mathbf{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.
Hence we need to solve

$$\det\left(\mathbf{A} - \lambda \mathbf{I}\right) = \det \begin{pmatrix} (1-\lambda) & 1 \\ 2 & -\lambda \end{pmatrix} = 0.$$

This means that $(1-\lambda)(-\lambda) - 2 = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$.

The solutions $\lambda_1 = -1$ and $\lambda_2 = 2$ are the eigenvalues.
continued...

Determining Eigenvalues

Example

- Choose $\lambda_1 = -1$ for the homogeneous system above:

$$\begin{cases} (1-\lambda_1)\,x + y = 0 \\ 2x - \lambda_1 y = 0 \end{cases} \quad \Rightarrow \begin{cases} 2x + y = 0 \\ 2x + y = 0. \end{cases}$$

  This gives eigenvectors of the type $(\alpha, -2\alpha) = \alpha\,(1, -2)$ with the parameter $\alpha \in \mathbb{R}$.

- Choose $\lambda_2 = 2$ for the homogeneous system above:

$$\begin{cases} (1-\lambda_1)\,x + y = 0 \\ 2x - \lambda_1 y = 0 \end{cases} \quad \Rightarrow \begin{cases} y - x = 0 \\ 2x - 2y = 0. \end{cases}$$

  This gives eigenvectors of the type $(\alpha, \alpha) = \alpha\,(1, 1)$ with the parameter $\alpha \in \mathbb{R}$.

Exercise 2: Eigenvalues and Eigenvectors

Consider the endomorphism $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$f\left(x, y\right) = \left(2x, x + 3y\right).$$

Find the eigenvalues and eigenvectors of f.

Exercise 2: *Solutions*

The identifying matrix is $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ and $\lambda \mathbf{I} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

Hence we need to solve

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \det \begin{pmatrix} (2-\lambda) & 0 \\ 1 & (3-\lambda) \end{pmatrix} = 0.$$

This means that $(2-\lambda)(3-\lambda) = 0$. Therefore $\lambda = 2$ or $\lambda = 3$.

- When $\lambda = 2$, $2y = x + 3y \Rightarrow x + y = 0$.
  Therefore eigenvectors are $\alpha(1, -1)$ for $\alpha \in \mathbb{R} \setminus \{0\}$.
- When $\lambda = 3$, $3x = 2x \Rightarrow x = 0$ and $3y = 3y$.
  Therefore eigenvectors are $\alpha(0, 1)$ for $\alpha \in \mathbb{R} \setminus \{0\}$.

Example

Let us analyse a case with three variables, i.e. the following linear mapping $f:\mathbb{R}^3 \rightarrow \mathbb{R}^3$,

$$f\left(x,y,z\right)=\left(x+z,2y,-x+3z\right).$$

By applying the definition of eigenvalue we write

$$\begin{cases} x+z=\lambda x \\ 2y=\lambda y \\ -x+3z=\lambda z \end{cases} \Rightarrow \begin{cases} \left(1-\lambda\right)x+z=0 \\ \left(2-\lambda\right)y=0 \\ -x+\left(3-\lambda\right)z=0. \end{cases}$$

...continued...

Example

This system is undetermined when

$$\det\begin{pmatrix} (1-\lambda) & 0 & 1 \\ 0 & (2-\lambda) & 0 \\ -1 & 0 & (3-\lambda) \end{pmatrix} = (2-\lambda)^3 = 0$$

that is when $\lambda = 2$.
(*Exercise: confirm for yourself that* $(2-\lambda)^3$ *is the determinant using the method in Lecture 3*).

This means that only one eigenvector can be calculated, i.e. the three eigenvectors are linearly dependent. *...continued...*

Example

By substituting into the system we have

$$\begin{cases} -x + z = 0 \\ 0y = 0 \\ -x + z = 0. \end{cases}$$

The second equation is always verified while the first and the third say that $x = z$. Equivalently, we can see that this system has rank $\rho = 1$ and thus ∞^2 solutions. If we pose $x = \alpha$ and $y = \beta$ for $\alpha, \beta \in \mathbb{R}$, we have that the generic solution $(\alpha, \beta, \alpha) = \alpha (1, 0, 1) + \beta (0, 1, 0)$. The eigenspace is thus spanned by the vectors $(1, 0, 1)$ and $(0, 1, 0)$.

Exercise 3: Eigenvalues and eigenspaces

Find eigenvalues and corresponding eigenspaces for the following linear mapping $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$,

$$f\left(x, y, z\right) = \left(5x + 2y + 3z, z - 2y, 6z\right).$$

Exercise 3: *Solutions*

This system is undetermined when

$$\det\begin{pmatrix} (5-\lambda) & 2 & 3 \\ 0 & (-2-\lambda) & 1 \\ 0 & 0 & (6-\lambda) \end{pmatrix} = (5-\lambda)(-2-\lambda)(6-\lambda) = 0;$$

that is when $\lambda = 5$ or -2 or 6.

- When $\lambda = 5$,
  - from last row, $(6-5)z = 0 \Rightarrow z = 0$;
  - from second row, $-7y = 0 \Rightarrow y = 0$;
  - and from first row, $0x = 0$.

  So eigenvectors are given as $\alpha(1, 0, 0)$ for $\alpha \in \mathbb{R} \setminus \{0\}$.

...continued...

Exercise 3: *Solutions*

- When $\lambda = -2$,
  - from last row, $(-2-5)z = 0 \Rightarrow z = 0$;
  - from second row, $0y = 0$;
  - and from first row, $7x + 2y = 0$.

  So eigenvectors are given as $\alpha(2, -7, 0)$ for $\alpha \in \mathbb{R} \setminus \{0\}$.
- When $\lambda = 6$,
  - from last row, $0z = 0$;
  - from second row, $-8y + z = 0 \Rightarrow z = 8y$;
  - and from first row, $-x + 2y + 3z = 0 \Rightarrow x = 26y$.

  So eigenvectors are given as $\alpha(26, 1, 8)$ for $\alpha \in \mathbb{R} \setminus \{0\}$.

Historical Note

Why are they called "eigenvalues", "eigenvectors" and "eigenspaces"?

- The earliest known use is by Cauchy 1829, where he called eigenvalues, "variables princpales" and termed the phrase "équation caractéristique" for $p(\lambda)$:

Les équations (6) deviendront

$$
(9)\quad \begin{cases} A_{xx}x + A_{xy}y + A_{xz}z + \ldots = sx, \\ A_{xy}x + A_{yy}y + A_{yz}z + \ldots = sy, \\ A_{xz}x + A_{yz}y + A_{zz}z + \ldots = sz; \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases}
$$

et pourront s'écrire comme il suit :

$$
(10)\quad \begin{cases} (A_{xx} - s)x + A_{xy}y + A_{xz}z + \ldots = 0, \\ A_{xy}x + (A_{yy} - s)y + A_{yz}z + \ldots = 0, \\ A_{xz}x + A_{yz}y + (A_{zz} - s)z + \ldots = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}
$$

Historical Note

- Poincaré followed in 1894 and called eigenvalues, “nombre caractéristique”.
- The terms “eigen...” first appear in 1904 by David Hilbert who, working in German, translated “characteristic number” into “eigenwert”.
- The term “eigenvalue”, which is a half-translation seems to have been popularized later by mathematics students in North America.

Reference source:
`https://www.maa.org/press/periodicals/convergence/math-origins-eigenvectors-and-eigenvalues`

Summary

Summary and next lecture

Summary

- Eigenvalues and Eigenvectors
- Eigenspaces
- Determining Eigenvalues and Eigenvectors

The next lecture

Univariate calculus and optimization.