COMP1046 Tutorial 2: Systems of Linear Equations

Anthony Bellotti

For these questions, consider Rouchè-Capelli Theorem and the cases for different systems of linear equations in Lecture 5.

Consider the system of linear equations for variables x_1, x_2, x_3, x_4 represented by this complete matrix:

$$\mathbf{B^c} = \left(\begin{array}{cccc|c} 2 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 3 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 & 0 \end{array}\right).$$

1. Use Cramer's Method to compute solutions for just x_1 and x_2.
   Show your working.
   Hint: Be strategic in your choice of computing determinants.

   Answer: First, compute $\det\mathbf{B}$. Since the first column has two zeros, use I Laplace Theorem (the sum of elements times their cofactors):

   $$\det\mathbf{B} = 2\det\begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + 0 + 0 + (-1)\det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix}.$$

   The first 3×3 deteriment is computed by applying I Laplace Theorem recursively using the third column, since this has only one non-zero element:

   $$\det\begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} = -1 \times \det\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} + 0 + 0 = 4$$

   whilst for the second 3×3 matrix, we notice that the rows are linear dependent ($\mathbf{a_1} + \mathbf{a_2} - \mathbf{a_3} = \mathbf{o}$), hence its determinant is zero. Hence

   $$\det\mathbf{B} = 2 \times 4 = 8.$$

   Now compute the determinant of hybrid matrix, $\det\mathbf{B_1}$. Again, use I Laplace Theorem with the first column (*since there are only two zeroes and we can* *reuse calculations*):

$$\begin{aligned}
\det\mathbf{B_1} &= \det\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ -1 & 3 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix} \\
&= \det\begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} + 0 - 1 \times \det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} + 0.
\end{aligned}$$

We already know the determinant of the first 3×3 matrix is 4. For the second 3×3 matrix, use I Laplace Theorem again with row 1:

$$\det\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} + 0 + \det\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = -1 - 3 = -4$$

so

$$\det\mathbf{B_1} = 4 - 1 \times -4 = 8.$$

Then by Cramer's Method,

$$x_1 = \frac{\det\mathbf{B_1}}{\det\mathbf{B}} = \frac{8}{8} = 1.$$

Now compute the determinant of hybrid matrix, $\det\mathbf{B_2}$. Again, use I Laplace Theorem with the first column:

$$\begin{aligned}
\det\mathbf{B_2} &= \det\begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix} \\
&= 2\det\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} + 0 + 0 + (-1)\det\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}.
\end{aligned}$$

Apply I Laplace Theorem recursively with first column and first row for each 3×3 matrix:

$$\begin{aligned}
\det\mathbf{B_2} &= 2\left[(-1)\det\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}\right] - \left[\det\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} + (-1)\det\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\right] \\
&= 2 \times -1 - [1 + 1] \\
&= -4.
\end{aligned}$$

Then by Cramer's Method,

$$x_2 = \frac{\det\mathbf{B_2}}{\det\mathbf{B}} = \frac{-4}{8} = -\frac{1}{2}.$$

2. Which case of system of linear equations does $\mathbf{B^c}$ represent?

   Answer: From Q1, det$\mathbf{B} = 8$. Since $\mathbf{B}$ is the largest submatrix of both $\mathbf{B}$ and $\mathbf{B^c}$ and it is order 4, $\rho_{\mathbf{B}} = \rho_{\mathbf{B^c}} = 4$. Additionally, $m = n = 4$, hence $\mathbf{B^c}$ represents Case 1.

3. Is this system of linear equations compatible or incompatible?

$$\mathbf{C^c} = \left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 1 \end{array}\right)$$

   Answer: Since det$\mathbf{C} = 0$, $\rho_{\mathbf{C}} = 1$.
   However, since $\det \begin{pmatrix} 2 & 0 \\ 4 & 1 \end{pmatrix} = 2$ (non-zero), $\rho_{\mathbf{C^c}} = 2$, therefore $\rho_{\mathbf{C}} < \rho_{\mathbf{C^c}}$ and this system is incompatible (no solution).

4. Show that the system of linear equations represented by the following complete matrix is Case 2? Can you point out where the *redundancy* is?

$$\mathbf{D^c} = \left(\begin{array}{rrr|r} 1 & 2 & -2 & -5 \\ 3 & 0 & 1 & 8 \\ 2 & -1 & -1 & 9 \\ -2 & -4 & 4 & 10 \end{array}\right)$$

   Answer: Notice row 4 is -2 times row 1. This means all 4×4 matrices have determinant 0. To compute the rank, try a 3×3 matrix. Try top left, since this is a submatrix for both $\mathbf{D}$ and $\mathbf{D^c}$:

$$\det \begin{pmatrix} 1 & 2 & -2 \\ 3 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix} > 0.$$

   Therefore, ranks $\rho_{\mathbf{D}} = \rho_{\mathbf{D^c}} = 3$. However, since $m > n = 3$, this is Case 2.

   Redundancy means that either row 1 or row 4 can be removed and the system can still be solved with the same unique solution.

5. Show that the system of linear equations in x_1, x_2, x_3, x_4 represented by the following complete matrix is Case 3? Can you show how x_1 and x_2 can be expressed in terms of x_3 and x_4 hence giving ∞^2 possible solutions?

$$\mathbf{E^c} = \left(\begin{array}{rrrr|r} 1 & 2 & 2 & 1 & 0 \\ 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

   Answer: The determinant for the leftmost 2×2 submatrix is non-zero, hence $\rho_{\mathbf{E}} = \rho_{\mathbf{E^c}} = 2$, but $m = 2$ and $n = 4$, so this is Case 3 with $\infty^{n-\rho} = \infty^2$ solutions.

From row 2,

$$x_1 = 1 + x_3$$

then from row 1,

$$x_1 + 2x_2 + 2x_3 + x_4 = 0 \quad \Rightarrow \quad x_2 = \frac{-1}{2}(1 + 3x_3 + x_4).$$