1   Propositional Logic Questions

1. Suppose that the statement $p \rightarrow \neg q$ is false. Find all combinations of truth values of r and s for which $(\neg q \rightarrow r) \wedge (\neg p \vee s)$ is true.

2. If the statement $q \wedge r$ is true, determine all combinations of truth values for p and s such that the statement $(q \rightarrow [\neg p \vee s]) \wedge [\neg s \rightarrow r]$ is true.

3. Find all combinations of truth values for p, q and r for which the statement $\neg p \leftrightarrow (q \wedge \neg(p \rightarrow r))$ is true.

4. Is $(p \rightarrow q) \rightarrow [(p \rightarrow q) \rightarrow q]$ a tautology? Why or why not?

5.   (a) Show that $(p \rightarrow q) \leftrightarrow (q \rightarrow p)$ is neither a tautology nor a contradiction. What does that tell you about possible relationships between the truth values of a statement and its converse?

    (b) Suppose $\neg[(p \rightarrow q) \leftrightarrow (q \rightarrow p)]$ is false. Can $p \leftrightarrow q$ have both possible truth values? Explain.

6. Show that $[(p \vee q) \wedge (r \vee \neg q)] \rightarrow (p \vee r)]$ is a tautology by making a truth table, and then again by using an argument that considers the two cases "q is true" and "q is false".

7. Write each of the following statements, in English, in the form "if p, then q".

    (a) I go swimming on Mondays.

    (b) In order to be able to go motorcycling on Sunday, the weather must be good.

    (c) Eat your vegetables or you can't have dessert.

    (d) You can ride a bicycle only if you wear a helmet.

    (e) Polynomials are continuous functions.

    (f) A number n that is a multiple of 2 and also a multiple of 3 is a multiple of 6.

    (g) You can't have any pudding unless you eat your meat.

    (h) The cardinality of a set is either finite or infinite.

8. Write in English the converse, contrapositive and negation of each statement.

    The negation here means the inverse of an implication.

    (a) If I had $1,000,000, I'd buy you a fur coat.

    (b) If it is not raining and not windy, then I will go running or cycling.

    (c) A day that's sunny and not too windy is a good day for walking on the waterfront.

    (d) If 11 pigeons live in 10 birdhouses, then there are two pigeons that live in the same birdhouse.

(e) If every domino covers a black square and a white square, then the number of black squares equals the number of white squares.

9. Determine if each statement below is true or false, and explain your reasoning.

   (a) It is possible for an implication and its contrapositive to have different truth values.

   (b) If the statement q is true, then, for any statement p, the statement $p \to q$ is true.

   (c) If $s_1 \to s_2$ is a contradiction, then so is its contrapositive.

   (d) There are truth values for p and q such that $p \to q$ and $q \to p$ are both false.

   (e) $(\neg p \vee q) \wedge \neg(\neg p \vee q)$ is a contradiction.

   (f) If the statement $\mathcal{P}$ is a contradiction, then, for any statement q, the statement $\mathcal{P} \to q$ is a tautology.

   (g) If two statements are logically equivalent, then so are their negations.

10. Consider the statement "if the goods are unsatisfactory, then your money will be refunded". This was an advertising slogan of the T. Eaton Company. Is the given statement logically equivalent to "goods satisfactory or money refunded"? What about to "if your money is not refunded, then the goods are satisfactory"? And what about to "if the goods are satisfactory, then your money will not be refunded".

11. A sign posted outside of Tokyo says "*In order to attack the city, you must be green and related to Godzilla. If you are not green and not related to Godzilla, then you can not attack the city*".

   (a) Render the two statements on the sign in symbols. Start with: *Let a be the assertion "you can attack the city"*, and carry on from there.

   (b) Argue that the two statements on the sign are not logically equivalent, contrary to what the author probably intended. Which is more restrictive on who can attack Tokyo?

   (c) Correct the second statement so that it is logically equivalent to the first one.

12. Show that the two statements $(p \wedge q) \to r$ and $(p \to r) \wedge (q \to r)$ are not logically equivalent.

13. Assume that a and b are integers. Consider the statements:
    $\mathcal{A}$ If c is a prime number such c divides ab,
         then c divides a or c divides b.
    $\mathcal{B}$ If c is a prime number such c divides ab, and c does not divide b,
         then c divides a.

   (a) Write the statements $\mathcal{A}$ and $\mathcal{B}$ in symbolic form and then show that they are logically equivalent.

(b) Write the contrapositive of each statement in English.

14. (a) Use any method to show that $\neg(p \to q) \leftrightarrow (p \wedge \neg q)$ is a tautology.

   (b) Use known logical equivalences to show that $\neg(p \leftrightarrow q) \Leftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$.

15. Use known logical equivalences to show that $(\neg a \to b) \wedge (\neg b \vee (\neg a \vee \neg b))$ is logically equivalent to $\neg(a \leftrightarrow b)$.

16. Use known logical equivalences to show that $\neg(p \leftrightarrow q)$ is logically equivalent to $(p \vee q) \wedge (p \to \neg q)$.

17. Use known logical equivalences to do each of the following.

   (a) Show $p \to (q \vee r) \Leftrightarrow (p \wedge \neg q) \to r$.

   (b) Show $\neg(p \vee q) \vee (\neg p \wedge q) \vee \neg(\neg p \vee \neg q) \Leftrightarrow \neg(p \wedge \neg q)$.

   (c) Find an expression logically equivalent to $\neg(p \leftrightarrow q)$ that involves only $\neg$ and $\vee$.

18. Let s be the statement whose truth table is given below.

p	q	r	s
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

   (a) Express the statement s in terms of p, q and r in such a way that only negation ($\neg$) and the logical connectives $\vee$ and $\wedge$ are used.

   (b) Find an equivalent formulation of s that uses only $\neg$ and $\vee$.

   (c) Find an equivalent formulation of s that uses only $\neg$ and $\wedge$.

19. Define the logical connective "nand" (not and) by $p \barwedge q \Leftrightarrow \neg(p \wedge q)$.

   (a) Find a representation of each of the following statements using only the logical connective nand.

      i. $\neg p$

      ii. $p \wedge q$

      iii. $p \vee q$

      iv. $p \to q$

      v. $p \leftrightarrow q$

(b) Explain why every statement has a representation using only the logical connective nand.

20. Repeat question 19 using the logical connective "nor" (not or) defined by $p \triangledown q \Leftrightarrow \neg(p \vee q)$.

21. Referring to questions 19 and 20, prove that $\neg(p \triangledown q) \Leftrightarrow \neg p \barwedge \neg q$. Guess and prove a similar logical equivalence for $\neg(p \barwedge q)$.

22. (a) Argue that "logically implies" has the property (called *transitivity*) that if a, b and c are statements such that $a \Rightarrow b$ and $b \Rightarrow c$, then $a \Rightarrow c$.

(b) Suppose a, b, c and d are statements such that $a \Rightarrow b$, $b \Rightarrow c$, $c \Rightarrow d$, and $d \Rightarrow a$. Argue that any two of these statements are logically equivalent.

23. Determine whether each statement is true or false, and briefly explain your reasoning.

(a) If an argument is valid then it is possible the conclusion to be false when all premises are true.

(b) If the premises can't all be true, then the argument is valid.

(c) If $p \Leftrightarrow q$ and $q \Leftrightarrow r$, then $p \Leftrightarrow r$.

24. Show that the argument

$$p \leftrightarrow q$$
$$q \to r$$
$$r \vee \neg s$$
$$\underline{\neg s \to q}$$
$$\therefore \quad s$$

is invalid by providing a counterexample.

25. Use basic inference rules to establish the validity of the argument

$$p \to \neg q$$
$$q \vee r$$
$$p \vee u$$
$$\underline{\neg r}$$
$$\therefore \quad u$$

26. If the argument below is valid, then use any method to prove it. Otherwise, give a counterexample to show that the argument is invalid.

$$\neg r \to p$$
$$q \to \neg p$$
$$\underline{\neg(r \vee t)}$$
$$\therefore \quad q$$

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27. Use any method to show the following argument is valid.

$$\frac{\begin{array}{c} p \\ \neg q \leftrightarrow \neg p \end{array}}{\therefore \quad q}$$

28. Show that the following argument is not valid.

$$\frac{\begin{array}{c} p \vee r \\ p \vee q \end{array}}{\therefore \quad q \vee r}$$

29. Use logical equivalences and the rules of inference to determine whether the following argument is valid.

$$\frac{\begin{array}{c} \neg(\neg p \vee q) \\ \neg z \rightarrow \neg s \\ (p \wedge \neg q) \rightarrow s \\ \neg z \vee r \end{array}}{\therefore \quad r}$$

30. Write the argument below in symbolic form. If the argument is valid, prove it. If the argument is not valid, give a counterexample:

$$\frac{\begin{array}{l} \text{If I watch football, then I don't do mathematics} \\ \text{If I do mathematics, then I watch hockey} \end{array}}{\therefore \quad \text{If I don't watch hockey, then I watch football}}$$

31. Write the argument below in symbolic form. If the argument is valid, prove it. If the argument is not valid, give a counterexample:

$$\frac{\begin{array}{l} \text{If you are pregnant or have a heart condition, then you can not use the hot tub.} \\ \text{You do not have a heart condition.} \\ \text{You can use the hot tub.} \end{array}}{\therefore \quad \text{You are not pregnant.}}$$

32. (a) Show that $p \rightarrow (q \rightarrow r)$ is logically equivalent to $(p \wedge q) \rightarrow r$.

    (b) Establish the validity of the argument

$$\frac{\begin{array}{c} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ q \end{array}}{\therefore \quad p}$$

(c) Consider the argument

$$u \to r$$
$$(r \wedge s) \to (p \vee t)$$
$$q \to (u \wedge s)$$
$$\neg t$$
$$\therefore \quad q \to p$$

Use part (a), and the fact that if b is true then $a \to b$ is true, to explain why the validity of this argument is established by your proof in (b).