COMP1046 Tutorial 6 : Eigenvalues and Eigenvectors

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QUESTION 1.
Find the eigenvalues, eigenvectors and eigenspaces for the endomorphism, $f : \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(x, y, z) = (2x + 3y - z, 4y + 2z, z - y).$$

Answer:

Eigenvalues:

Solve $f(x, y, z) = \lambda(x, y, z)$. This forms a system of three linear equations:

$$\begin{aligned} 2x + 3y - z &= \lambda x \\ 4y + 2z &= \lambda y \\ z - y &= \lambda z \end{aligned}$$

Equivalently,

$$\begin{aligned} (2-\lambda)x + 3y - z &= 0 \\ (4-\lambda)y + 2z &= 0 \\ -y + (1-\lambda)z &= 0 \end{aligned} \tag{1}$$

and as a complete matrix:

$$\mathbf{B^C} = \left(\begin{array}{ccc|c} (2-\lambda) & 3 & -1 & 0 \\ 0 & (4-\lambda) & 2 & 0 \\ 0 & -1 & (1-\lambda) & 0 \end{array} \right).$$

Since we know there will be multiple solutions, for $\mathbf{B}$ is the incomplete matrix, $\det(\mathbf{B}) = 0$. Use I Laplace Theorem, choosing the first column (since it has only one non-zero value):

$$\begin{aligned} \det(\mathbf{B}) &= (2-\lambda)((4-\lambda)(1-\lambda) - (-2)) \\ &= (2-\lambda)(\lambda^2 - 5\lambda + 6) \\ &= (2-\lambda)(\lambda - 2)(\lambda - 3) = 0 \end{aligned}$$

so eigenvalues are $\lambda = 2$ and 3.

Eigenvectors:

- Take $\lambda = 2$.
  Substitute into first and second linear equation of (1): $3y - z = 0$ and $2y + 2z = 0$, hence $y = z = 0$ is the only solution. There is no constraint on x, hence the general form of eigenvector is
  $$(\alpha, 0, 0)$$
  for $\alpha \in \mathbb{R}$, except $\alpha \neq 0$.
- Take $\lambda = 3$.
  Substitute into second and third linear equation of (1): $y + 2z = 0$ and $-y - 2z = 0$, hence $y = -2z$.
  Pose $z = -\alpha$, then $y = 2\alpha$.
  Then from the first linear equation of (1): $-x + 3y - z = 0 \Rightarrow x = 7\alpha$.
  Therefore, the general form of eigenvector is
  $$\alpha(7, 2, -1)$$
  for $\alpha \in \mathbb{R}$, except $\alpha \neq 0$.

Eigenspace:

- For the general form of eigenvector $(\alpha, 0, 0)$, the eigenspace is
  $$\{(\alpha, 0, 0) \mid \alpha \in \mathbb{R}\}.$$
- For the general form of eigenvector $\alpha(7, 2, -1)$, the eigenspace is
  $$\{\alpha(7, 2, -1) \mid \alpha \in \mathbb{R}\}.$$

Note that in both cases, the eigenspace contains the null vector when $\alpha = 0$.

QUESTION 2.
Find the eigenvalues, eigenvectors and eigenspaces for the endomorphism, $f : \mathbb{R}^4 \to \mathbb{R}^4$,

$$f(w, x, y, z) = (-2w - y,\ 4w + 2x - 2y + z,\ w,\ 4w + 4y + 2z).$$

Answer:

Eigenvalues:

Solve $f(w, x, y, z) = \lambda(w, x, y, z)$. This forms a system of four linear equations:

$$\begin{aligned} -2w - y &= \lambda w \\ 4w + 2x - 2y + z &= \lambda x \\ w &= \lambda y \\ 4w + 4y + 2z &= \lambda z \end{aligned}$$

Equivalently,

$$\begin{aligned} (-2 - \lambda)w - y &= 0 \\ 4w + (2 - \lambda)x - 2y + z &= 0 \\ w - \lambda y &= 0 \\ 4w + 4y + (2 - \lambda)z &= 0 \end{aligned} \tag{2}$$

and as a complete matrix:

$$\mathbf{B^C} = \left(\begin{array}{rrrr|r} (-2-\lambda) & 0 & -1 & 0 & 0 \\ 4 & (2-\lambda) & -2 & 1 & 0 \\ 1 & 0 & -\lambda & 0 & 0 \\ 4 & 0 & 4 & (2-\lambda) & 0 \end{array} \right).$$

Since we know there will be multiple solutions, for $\mathbf{B}$ is the incomplete matrix, $\det(\mathbf{B}) = 0$.
Use I Laplace Theorem, choosing the second column (since it has only one non-zero value):

$$\det(\mathbf{B}) = (2 - \lambda)\det \begin{pmatrix} (-2-\lambda) & -1 & 0 \\ 1 & -\lambda & 0 \\ 4 & 4 & (2-\lambda) \end{pmatrix}$$

Use I Laplace Theorem again on third column:

$$\begin{aligned} \det(\mathbf{B}) &= (2 - \lambda)(2 - \lambda)\det \begin{pmatrix} (-2-\lambda) & -1 \\ 1 & -\lambda \end{pmatrix} \\ &= (2 - \lambda)(2 - \lambda)((-2 - \lambda) \times -\lambda + 1) = 0 \end{aligned}$$

so eigenvalues are $\lambda = 2$ and -1.

Eigenvectors:

- Take $\lambda = 2$:
  Equation 3 reduces to:

$$\begin{aligned} -4w - y &= 0 \\ 4w - 2y + z &= 0 \\ w + 2y &= 0 \\ 4w + 4y &= 0 \end{aligned} \tag{3}$$

  Taking first and third equation, both $w = y = 0$, then second equation gives $z = 0$. There is no constraint on x, hence the general form of eigenvector is

$$(\alpha, 0, 0, 0)$$

  for $\alpha \in \mathbb{R}$, except $\alpha \neq 0$.

- Take $\lambda = -1$.
  Equation 3 reduces to:

$$\begin{aligned} -w - y &= 0 \\ 4w + 3x - 2y + z &= 0 \\ w + y &= 0 \\ 4w + 4y + 3z &= 0 \end{aligned}$$

  The first and third equation both give $y = -w$. Then substitute in fourth equation gives $z = 0$. Finally, the third equation gives $x = -2w$. Posing $x = \alpha$ gives the general form of eigenvector as

$$\alpha(1, -2, -1, 0)$$

  for $\alpha \in \mathbb{R}$, except $\alpha \neq 0$.

Eigenspace:

- For the general form of eigenvector $(\alpha, 0, 0, 0)$, the eigenspace is

$$\{(\alpha, 0, 0, 0) \mid \alpha \in \mathbb{R}\}.$$

- For the general form of eigenvector $\alpha(1, -2, -1, 0)$, the eigenspace is

$$\{\alpha(1, -2, -1, 0) \mid \alpha \in \mathbb{R}\}.$$

Note that in both cases, the eigenspace contains the null vector when $\alpha = 0$.