AE1MCS: Tutorial 2
Inference Rules and Proofs

Rule of inferences

Given that $\forall x\big(L(x) \to F(x)\big)$ and $\exists x\big(F(x) \land \neg C(x)\big)$, prove that $\exists x(F(x) \land \neg C(x))$

All lions are fierce.	$\forall x\left(L\left(x\right) \rightarrow F\left(x\right)\right)$
Some lions do not drink coffee.	$\exists x\left(L\left(x\right) \wedge \neg C\left(x\right)\right)$
ome fierce creatures do not drink coffee.	$\exists x\left(F\left(x\right) \wedge \neg C\left(x\right)\right)$

1.	$\exists x\left(L(x)\wedge\neg C(x)\right)$	Premise (given)
2.	$L(Lambert)\wedge\neg C(Lambert)$	Existential Instantiation from (1)
3.	$L(Lambert)$	Simplification from (2)
4.	$\neg C(Lambert)$	Simplification from (2)
5.	$\forall x\left(L(x)\rightarrow F(x)\right)$	Premise (given)
6.	$L(Lambert)\rightarrow F(Lambert)$	Universal Instantiation fro (5)
7.	$F(Lambert)$	Modus Ponens (Law of Detachment) from (3) and (6)
8.	$F(Lambert)\wedge\neg C(Lambert)$	Conjunction from (4) and (7)
9.	$\exists x\left(F(x)\wedge\neg C(x)\right)$	Existential Generalization form (8)

1.	$\exists x\left(L(x)\wedge\neg C(x)\right)$	Premise (given)
2.	$L(Lambert)\wedge\neg C(Lambert)$ $L(a)\wedge\neg C(a)$	Existential Instantiation from (1)
3.	$L(Lambert)$ $L(a)$	Simplification from (2)
4.	$\neg C(Lambert)$ $\neg C(a)$	Simplification from (2)
5.	$\forall x\left(L(x)\to F(x)\right)$	Premise (given)
6.	$L(Lambert)\to F(Lambert)$ $L(a)\to F(a)$ for the same a in (2)	Universal Instantiation fro (5)
7.	$F(Lambert)$	Modus Ponens (Law of Detachment) from (3) and (6)
8.	$F(Lambert)\wedge\neg C(Lambert)$	Conjunction from (4) and (7)
9.	$\exists x\left(F(x)\wedge\neg C(x)\right)$	Existential Generalization form (8)

Proof methods

Proving conditional statements

- 1. Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

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Proof:

p: a and b are real numbers and $a \neq 0$

q: there is a unique real number r such that $ar+b=0$

- First, prove p->q. $r = -\frac{b}{a}$ is a solution of $ar+b=0$ because $a\left(-\frac{b}{a}\right) + b = -b + b = \mathbf{0}$. Consequently, a real number r exists for which $ar+b=0$. (The existence part of the proof)

- Then, prove the uniqueness. Suppose s is a real number such that $as + b = 0$. Then $as + b = ar + b$. Subtracting b, we have $ar = as$. Divide both side by a, where $a \neq 0$, we see that $r = s$. This means if $s \neq r$, then $as + b \neq \mathbf{0}$. (The uniqueness part of the proof)

Prove by contradiction

- 2.Prove that at least one of the real numbers a_1, a_2,⋯, a_n is greater than or equal to the average of these numbers.

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Proof:

- Prove it by contradiction.
- Suppose that a_1, a_2,⋯, a_n are all less than A, where A is the average of these numbers.
- Then $a_1 + a_2 + \cdots + a_n < \mathrm{nA}$
- Divide both sides by n shows that $A = (a_1 + a_2 + \cdots + a_n)/\mathrm{n} < A$,

which is a contradiction.

Prove of Equivalence

- 3. Prove that if n is an integer, these four statements are equivalent: (1) n is even, (2) n+1 is odd, (3) 3n+1 is odd, (4) 3n is even.

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Proof:

- Let us show that four statements are equivalent by showing that (1) implies (2), (2) implies (3), (3) implies (4), (4) implies (1).
- First, assume n is even. Hence there exists an integer k such that $n= 2k$ Then $n + 1 = 2k + 1$, so $n + 1$ is odd. This shows (1) implies (2).
- Second, assume $n + 1$ is odd. Hence there exists an integer k such that $n + 1 = 2k + 1$. Then 3n+1 $= 2n + (n + 1) = 2\mathrm{n} + 2\mathrm{k} + 1 = 2(\mathrm{n} + \mathrm{k}) + 1$, thus 3$n$+1 is odd and (2) implies (3).
- Third, suppose 3n+1 is odd. Hence there exists an integer k such that 3n+1 = $2k + 1$. Then $3n = (2k + 1) - 1 = 2k$, so $3n$ is even. (3) implies (4)
- Finally, suppose that n is not even. Then n is odd, so there exists an integer k such that $n= 2k$+1. $3n = 3(2k + 1) = 6k + 3 = 2(3k + 1) + 1$, so $3n$ is odd. By contraposition, (4) implies (1).

Proof by (counter) Example

- 4. For all integers a and b, prove or disprove if a is odd or b is odd, then a+b is odd.

Proof by (counter) Example

4. For all integers a and b, prove or disprove if a is odd or b is odd, then a+b is odd.

- The statement is false. To prove the it is false we need to find two integers a and b so that a is odd or b is odd, but a+b is not odd (so even). That's easy: 1 and 3. (remember, "or" means one or the other or both). Both of these are odd, but 1+3=4 is not odd.

Mathematical Induction

5. Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

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Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n.

Solution: To construct the proof, let $P(n)$ denote the proposition: "$7^{n+2} + 8^{2n+1}$ is divisible by 57."

BASIS STEP: To complete the basis step, we must show that $P(0)$ is true, because we want to prove that $P(n)$ is true for every nonnegative integer. We see that $P(0)$ is true because $7^{0+2} + 8^{2\cdot 0+1} = 7^2 + 8^1 = 57$ is divisible by 57. This completes the basis step.

INDUCTIVE STEP: For the inductive hypothesis we assume that $P(k)$ is true for an arbitrary nonnegative integer k; that is, we assume that $7^{k+2} + 8^{2k+1}$ is divisible by 57. To complete the inductive step, we must show that when we assume that the inductive hypothesis $P(k)$ is true, then $P(k+1)$, the statement that $7^{(k+1)+2} + 8^{2(k+1)+1}$ is divisible by 57, is also true.

The difficult part of the proof is to see how to use the inductive hypothesis. To take advantage of the inductive hypothesis, we use these steps:

$$\begin{aligned}
7^{(k+1)+2} + 8^{2(k+1)+1} &= 7^{k+3} + 8^{2k+3} \\
&= 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} \\
&= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\
&= 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.
\end{aligned}$$

We can now use the inductive hypothesis, which states that $7^{k+2} + 8^{2k+1}$ is divisible by 57. We will use parts (i) and (ii) of Theorem 1 in Section 4.1. By part (ii) of this theorem, and the inductive hypothesis, we conclude that the first term in this last sum, $7(7^{k+2} + 8^{2k+1})$, is divisible by 57. By part (ii) of this theorem, the second term in this sum, $57 \cdot 8^{2k+1}$, is divisible by 57. Hence, by part (i) of this theorem, we conclude that $7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} = 7^{k+3} + 8^{2k+3}$ is divisible by 57. This completes the inductive step.

Because we have completed both the basis step and the inductive step, by the principle of mathematical induction we know that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer n. ◀

Strong Induction

6. Show that if n is an integer greater than 1, then n can be written as the product of primes.

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Solution: Let $P(n)$ be the proposition that n can be written as the product of primes.

BASIS STEP: $P(2)$ is true, because 2 can be written as the product of one prime, itself. (Note that $P(2)$ is the first case we need to establish.)

INDUCTIVE STEP: The inductive hypothesis is the assumption that $P(j)$ is true for all integers j with $2 \leq j \leq k$, that is, the assumption that j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k. To complete the inductive step, it must be shown that $P(k+1)$ is true under this assumption, that is, that $k+1$ is the product of primes.

There are two cases to consider, namely, when $k+1$ is prime and when $k+1$ is composite. If $k+1$ is prime, we immediately see that $P(k+1)$ is true. Otherwise, $k+1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k+1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if $k+1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b. ◀

More Exercises in the Textbook

- Section 1.6
  - 3, 5, 7, 13, 15, 17-20, 23-29, 33, 34-35*
- Section 1.7
  - 13, 14, 16, 19-25, 34, 35, 38-40
- Section 1.8
  - 3, 4, 7, 15, 29-32
- Section 5.1
  - 3-17, 18, 19
- Section 5.2
  - 1-4
- Supplementary questions: Q23-Q32