COMP1046 Tutorial 4 : Linear Mappings

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Consider the set $E = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid 3x_1 = x_2 + x_3 + x_4\}$ and the linear mapping $f : E \to \mathbb{R}^4$,

$$f(x_1, x_2, x_3, x_4) = (3x_1 + x_3 + 2x_4, 2x_1 - x_2 + 2x_3 + x_4, x_1 + x_2 - x_3 + x_4, 4x_1 + x_2 + 3x_4).$$

1. Show that $(E, +, \cdot)$ is a vector space, where the internal and external composition laws are the usual real number addition and scalar product.

   Answer:

   Since $E \subset \mathbb{R}^4$ and $(\mathbb{R}^4, +, \cdot)$ is a vector space, then we only need to prove closure of two composition laws:

   - For the internal composition law: consider two arbitrary vectors

     $$\mathbf{x} = (x_1, x_2, x_3, x_4) \in E,$$

     $$\mathbf{x}' \in (x'_1, x'_2, x'_3, x'_4) \in E.$$

     Then, $3x_1 = x_2 + x_3 + x_4$ and $3x'_1 = x'_2 + x'_3 + x'_4$ which implies

     $$3(x_1 + x'_1) = (x_2 + x'_2) + (x_3 + x'_3) + (x_4 + x'_4)$$

     which is the condition for $\mathbf{x} + \mathbf{x}' \in E$.
   - For the external composition law, consider an arbitray scalar λ. Then, since $\mathbf{x} = (x_1, x_2, x_3, x_4) \in E$,

     $$3\lambda x_1 = \lambda x_2 + \lambda x_3 + \lambda x_4$$

     which is the condition for $\lambda \mathbf{x} \in E$.

2. Construct a basis for $(E, +, \cdot)$.

   Answer:

   There are several answers to this, but the general solution is to find a set of three vectors from E that span E and are linearly independent.
   Here is one answer:-

   - Rewrite $E = \{(x_1, x_2, x_3, 3x_1 - x_2 - x_3) \mid (x_1, x_2, x_3) \in \mathbb{R}^3\}$ so we can see that one of the variables can be written as a function of the others. Then a basis is proposed with the last term dependent on the first three:

     $$B = \{\ (1, 0, 0, 3),\ (0, 1, 0, -1),\ (0, 0, 1, -1)\ \}.$$

- Show that B spans E: Take an arbitrary $(x_1, x_2, x_3, 3x_1 - x_2 - x_3)) \in E$ and use scalars $\lambda_1 = x_1$, $\lambda_2 = x_2$ and $\lambda_3 = x_3$. Then,

  $$\lambda_1(1,0,0,3) + \lambda_2(0,1,0,-1) + \lambda_3(0,0,1,-1) = (x_1, x_2, x_3, 3x_1 - x_2 - x_3)$$

  which is the arbitrary vector in E. Hence, the basis is shown to span E.
- Show that B is linearly independent: Suppose

  $$\lambda_1(1,0,0,3) + \lambda_2(0,1,0,-1) + \lambda_3(0,0,1,-1) = (0,0,0,0)$$

  so looking at the first three components, we see that this can only be true if $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 0$. Hence B is linearly independent.

3. Compute $\dim(E)$.

   Answer:
   Since the basis for $(E, +, \cdot)$ has three vectors, $\dim(E) = 3$.

4. Compute $\ker(f)$.

   Answer:
   To compute the kernel, solve for $f(x_1, x_2, x_3, x_4) = (0,0,0,0)$ with constraint in E: $3x_1 = x_2 + x_3 + x_4$.
   The complete matrix is given as:

   $$\mathbf{A^C} = \left(\begin{array}{cccc|c} 3 & 0 & 1 & 2 & 0 \\ 2 & -1 & 2 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \\ 4 & 1 & 0 & 3 & 0 \\ -3 & 1 & 1 & 1 & 0 \end{array}\right).$$

   Use Rouchè-Capelli Theorem to analyse this system of linear equations. By inspection, we notice that the 4th row is the sum of the 1st and 3rd, and the 3rd is the 1st subtract the 2nd. Indeed, no 4 rows together are linearly independent. Hence, we know that any square submatrix of size 4 or 5 must be singular (determinant=0). Take the lower left 3×3 matrix and it is non-singular (determinant$\neq 0$):

   - Use I Laplace Theorem with column 3 to compute determinant:
     $-1(4+3) + 1(1-4) = -10$.

   Hence rank for both incomplete matrix and complete matrix is $\rho = 3$. So it is compatible. However $\rho < n$ so there are ∞^1 solutions $(n - \rho = 4 - 3 = 1)$.

   Pose $x_1 = \alpha$ for $\alpha \in \mathbb{R}$.
   Then subtracting the fifth row of the system of linear equations from the third,

   $$4\alpha - 2x_3 = 0 \Rightarrow x_3 = 2\alpha$$

and then from the first row,

$$3\alpha + 2\alpha + 2x_4 = 0 \Rightarrow x_4 = -\frac{5}{2}\alpha$$

and then from the third row,

$$x_2 = 2\alpha + 4\alpha - \frac{5}{2}\alpha = \frac{7}{2}\alpha.$$

Therefore a general solution is

$$\alpha\left(1, \frac{7}{2}, 2, -\frac{5}{2}\right).$$

and

$$\ker(f) = \left\{ \alpha\left(1, \frac{7}{2}, 2, -\frac{5}{2}\right) \mid \alpha \in \mathbb{R} \right\}.$$

5. Compute $\dim(\ker(f))$.

Answer:

We see that $(2, 7, 4, -5)$ spans $\ker(f)$ (note: the vector has just been rescaled by 2) and it is non-null, hence it is the basis.
Then, dimension is number of vectors in the basis:

$$\dim(\ker(f)) = 1.$$

6. Compute $\mathrm{Im}(f)$.

Answer:

Using the basis B from question part 2, each input vector can be expressed as

$$\mathbf{v} = (\lambda_1, \lambda_2, \lambda_3, 3\lambda_1 - \lambda_2 - \lambda_3) \in E.$$

Then

$$\begin{aligned} f(\mathbf{v}) &= (\quad 3\lambda_1 + \lambda_3 + 2(3\lambda_1 - \lambda_2 - \lambda_3), \\ &\qquad 2\lambda_1 - \lambda_2 + 2\lambda_3 + (3\lambda_1 - \lambda_2 - \lambda_3), \\ &\qquad \lambda_1 + \lambda_2 - \lambda_3 + (3\lambda_1 - \lambda_2 - \lambda_3), \\ &\qquad 4\lambda_1 + \lambda_2 + 3(3\lambda_1 - \lambda_2 - \lambda_3) \quad) \\ &= (\quad 9\lambda_1 - 2\lambda_2 - \lambda_3, \\ &\qquad 5\lambda_1 - 2\lambda_2 + \lambda_3, \\ &\qquad 4\lambda_1 - 2\lambda_3, \\ &\qquad 13\lambda_1 - 2\lambda_2 - 3\lambda_3 \quad). \end{aligned}$$

So

$$\begin{aligned} \mathrm{Im}(f) &= \{\mathbf{w} \in \mathbb{R}^4 \mid \exists \mathbf{v} \in E, f(\mathbf{v}) = \mathbf{w}\} \\ &= \left\{ \begin{pmatrix} 9\lambda_1 - 2\lambda_2 - \lambda_3, \\ 5\lambda_1 - 2\lambda_2 + \lambda_3, \\ 4\lambda_1 - 2\lambda_3, \\ 13\lambda_1 - 2\lambda_2 - 3\lambda_3. \end{pmatrix} \in \mathbb{R}^4 \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\} \end{aligned}$$

7. Compute $\dim(\mathrm{Im}(f))$ based on your result for $\mathrm{Im}(f)$.

Answer:

Firstly, construct a basis for $\mathrm{Im}(f)$. Rewrite as

$$\mathrm{Im}(f) = \{\lambda_1(9,5,4,13)+\lambda_2(-2,-2,0,-2)+\lambda_3(-1,1,-2,-3) \mid \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\}.$$

Hence the three vectors in this set span $\mathrm{Im}(f)$, but we need to check if they are linearly independent:

- Solve for the case when their linear combination is $(0,0,0,0)$, a system of linear equations.
- By inspection, I spot that values $\lambda_1 = 2, \lambda_2 = 7, \lambda_3 = 4$ give $(0,0,0,0)$, hence they are linearly dependent and cannot form a basis.

Now, to form a basis, drop the third vector. Do the remaining vectors $B_{\mathrm{Im}} = \{(9,5,4,13), (-2,-2,0,-2)\}$ form a basis?

- Since the third vector can be constructed as a linear combination of B_{Im}, since the three are linearly dependent, any linear combination of all three can be written as a linear combination of just these two, hence B_{Im} also spans $\mathrm{Im}(f)$.
- They are linearly independent: consider

$$\lambda_1(9,5,4,13) + \lambda_2(-2,-2,0,-2) = (0,0,0,0).$$

  Then from the third components, $\lambda_1 = 0$, and then from any other, $\lambda_2 = 0$.

Hence, B_{Im} is a basis for $\mathrm{Im}(f)$ and since it has two vectors, $\dim(\mathrm{Im}(f)) = 2$.

8. Are your results from question parts 5 and 7 confirmed by the Rank-Nullity Theorem?

Answer:

The Rank-Nullity Theorem states that $\dim(\ker(f)) + \dim(\mathrm{Im}(f)) = \dim(E)$, so, yes, $1 + 2 = 3$.