

Lecture 9 - Linear Mappings Part 2

COMP1046 - Maths for Computer Scientists

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By the end of this lecture we will have learned:

- ⊙ Endomorphisms and Kernel
- ⊙ Injectivity
- ⊙ Rank and Nullity of Linear Mappings

Based on Sections 10.3 and 10.4 of textbook (Neri 2018).

Endomorphism and Kernel

Endomorphism

Definition

Let f be a linear mapping $E \rightarrow F$. If $E = F$, i.e. $f : E \rightarrow E$, the linear mapping is said *endomorphism*.

Example

The linear mapping $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x$ is an endomorphism since both the sets are \mathbb{R} .

Null mapping

Definition

A *null mapping* $O : E \rightarrow F$ is a mapping defined in the following way:

$$\forall \mathbf{v} \in E : O(\mathbf{v}) = \mathbf{o}_F.$$

It can easily be proved that a null mapping is linear.

Example

The linear mapping $f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 0$ is a null mapping.

Identity mapping

Definition

An *identity mapping* $I : E \rightarrow F$ is a mapping defined in the following way:

$$\forall \mathbf{v} \in E : I(\mathbf{v}) = \mathbf{v}.$$

It can easily be proved that an identity mapping is linear and is an endomorphism.

Example

The linear mapping $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x$ is an identity mapping.

Definition

Let $f : E \rightarrow F$ be a linear mapping. The *kernel* of f is the set

$$\ker(f) = \{\mathbf{v} \in E \mid f(\mathbf{v}) = \mathbf{0}_F\}.$$

Example

Let us consider the linear mapping $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $f(x, y) = 5x - y$.

To find the kernel means to find the (x, y) values such that $f(x, y) = 0$, i.e. those (x, y) values that satisfy the equation

$$5x - y = 0.$$

This is an equation in two variables. For the Rouché Capelli Theorem this equation has ∞^1 solutions. These solutions are $(\alpha, 5\alpha)$ for any $\alpha \in \mathbb{R}$.

Therefore the kernel is

$$\ker(f) = \{(\alpha, 5\alpha) \mid \alpha \in \mathbb{R}\}.$$

Example

Consider now the linear mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as

$$f(x, y, z) = (x + y + z, x - y - z, 2x + 2y + 2z).$$

To find the kernel means to solve the following system of linear equations:

$$\begin{cases} x + y + z = 0 \\ x - y - z = 0 \\ 2x + 2y + 2z = 0. \end{cases}$$

Continued...

Example

We can easily verify that

$$\det \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 2 & 2 \end{pmatrix} = 0$$

and the rank of the system is $\rho = 2$. Thus, this system is undetermined and has ∞^1 solutions. If we pose $y = \alpha$ we find out that the infinite solutions of the system are $\alpha (0, 1, -1)$, $\forall \alpha \in \mathbb{R}$. Thus, the kernel of the mapping is

$$\ker(f) = \{\alpha (0, 1, -1) \mid \alpha \in \mathbb{R}\}.$$

Kernel is a vector subspace

Theorem

Let $f : E \rightarrow F$ be a linear mapping.

The triple $(\ker(f), +, \cdot)$ is a vector subspace of $(E, +, \cdot)$.

Proof.

Let us consider two vectors $\mathbf{v}, \mathbf{v}' \in \ker(f)$. If a vector $\mathbf{v} \in \ker(f)$ then $f(\mathbf{v}) = \mathbf{0}_F$. Thus,

$$f(\mathbf{v} + \mathbf{v}') = f(\mathbf{v}) + f(\mathbf{v}') = \mathbf{0}_F + \mathbf{0}_F = \mathbf{0}_F$$

and $\mathbf{v} + \mathbf{v}' \in \ker(f)$. Thus, $\ker(f)$ is closed with respect to the first composition law. *continued...*

Kernel is a vector subspace

Proof.

Let us consider a generic scalar $\lambda \in \mathbb{K}$ and calculate

$$f(\lambda \mathbf{v}) = \lambda f(\mathbf{v}) = \lambda \mathbf{o}_F = \mathbf{o}_F.$$

Hence, $\lambda \mathbf{v} \in \ker(f)$ and $\ker(f)$ is closed with respect to the second composition law.

This means that $(\ker(f), +, \cdot)$ is a vector subspace of $(E, +, \cdot)$. □

Theorem

Let $f : E \rightarrow F$ be a linear mapping and $\mathbf{u}, \mathbf{v} \in E$.

It follows that $f(\mathbf{u}) = f(\mathbf{v})$ if and only if $\mathbf{u} - \mathbf{v} \in \ker(f)$.

Proof.

If $f(\mathbf{u}) = f(\mathbf{v})$ then

$$f(\mathbf{u}) - f(\mathbf{v}) = \mathbf{0}_F \Rightarrow f(\mathbf{u}) + f(-\mathbf{v}) = \mathbf{0}_F \Rightarrow f(\mathbf{u} - \mathbf{v}) = \mathbf{0}_F.$$

From the definition of kernel $\mathbf{u} - \mathbf{v} \in \ker(f)$.

If $\mathbf{u} - \mathbf{v} \in \ker(f)$ then

$$f(\mathbf{u} - \mathbf{v}) = \mathbf{0}_F \Rightarrow f(\mathbf{u}) - f(\mathbf{v}) = \mathbf{0}_F \Rightarrow f(\mathbf{u}) = f(\mathbf{v}).$$

□

Exercise 1: Kernels

Consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2 : f(x, y, z) = (2z, x + 2y)$.

1. Compute $\ker(f)$.
2. Show that $\ker(f)$ is a vector space by showing closure with respect to the internal and external composition laws.

Exercise 1: *Solution*

1. The kernel is the set of all (x, y, z) such that

$$f(x, y, z) = (0, 0);$$

i.e. $2z = 0$ and $x + 2y = 0$.

This can easily be solved with solutions in the form

$$\alpha(-2, 1, 0).$$

Hence, $\ker(f) = \{\alpha(-2, 1, 0) : \alpha \in \mathbb{R}\}$.

2. Consider any $\alpha_1(-2, 1, 0) \in \ker(f)$ and $\alpha_2(-2, 1, 0) \in \ker(f)$.

Then $\alpha_1(-2, 1, 0) + \alpha_2(-2, 1, 0) = (\alpha_1 + \alpha_2)(-2, 1, 0)$ which must also be in $\ker(f)$.

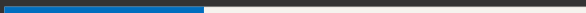
Consider any $\alpha \in \ker(f)$. Then

$$\lambda(\alpha(-2, 1, 0)) = (\lambda\alpha)(-2, 1, 0) \text{ which must also be in } \ker(f).$$

Hence we have proved closure and since $\ker(f) \subset \mathbb{R}^3$,

$\ker(f)$ is a vector (sub)space.

Injectivity



Linear independence and injectivity

Theorem

Let $f : E \rightarrow F$ be a linear mapping. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be n linearly independent vectors $\in E$.

If f is injective then $f(\mathbf{v}_1), f(\mathbf{v}_2), \dots, f(\mathbf{v}_n)$ are also linearly independent vectors $\in F$.

Proof.

Let us assume, by contradiction that

$\exists \lambda_1, \lambda_2, \dots, \lambda_n \neq 0, 0, \dots, 0$ such that

$$\mathbf{0}_F = \lambda_1 f(\mathbf{v}_1) + \lambda_2 f(\mathbf{v}_2) + \dots + \lambda_n f(\mathbf{v}_n).$$

continued...

Linear independence and injectivity

Proof.

From the Proposition on Slide 16 of Lecture 8 and the linearity of f we can write this expression as

$$f(\mathbf{o}_E) = f(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n).$$

Since for hypothesis f is injective, it follows that

$$\mathbf{o}_E = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n$$

with $\lambda_1, \lambda_2, \dots, \lambda_n \neq 0, 0, \dots, 0$.

This is impossible because $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent. Hence we reached a contradiction and $f(\mathbf{v}_1), f(\mathbf{v}_2), \dots, f(\mathbf{v}_n)$ must be linearly independent. \square

Example

Let us consider the injective mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as

$$f(x, y, z) = (x + y + z, x - y - z, x + y + 2z)$$

and the following linearly independent vectors of \mathbb{R}^3 with transformations:

$$\begin{array}{ll} \mathbf{u} = (1, 0, 0) & f(\mathbf{u}) = (1, 1, 1) \\ \mathbf{v} = (0, 1, 0) & f(\mathbf{v}) = (1, -1, 1) \\ \mathbf{w} = (0, 0, 1) & f(\mathbf{w}) = (1, -1, 2). \end{array}$$

Continued...

Linear independence and injectivity

Example

Let us check their linear dependence by finding, if they exist, the values of λ, μ, ν such that

$$\mathbf{o} = \lambda f(\mathbf{u}) + \mu f(\mathbf{v}) + \nu f(\mathbf{w}).$$

This is equivalent to solving the following homogeneous system of linear equations:

$$\begin{cases} \lambda + \mu + \nu = 0 \\ \lambda - \mu - \nu = 0 \\ \lambda + \mu + 2\nu = 0. \end{cases}$$

The system is determined; thus, its only solution is $(0, 0, 0)$. It follows that the vectors are linearly independent.

Injectivity and the Kernel

Theorem

Let $f : E \rightarrow F$ be a linear mapping. The mapping f is injective if and only if

$$\ker(f) = \{\mathbf{0}_E\}.$$

Proof.

Let us assume that f is injective and, by contradiction, let us assume that $\exists \mathbf{v} \in \ker(f)$ with $\mathbf{v} \neq \mathbf{0}_E$.

For definition of kernel

$$\forall \mathbf{v} \in \ker(f) : f(\mathbf{v}) = \mathbf{0}_F.$$

On the other hand, $f(\mathbf{0}_E) = \mathbf{0}_F$continued

Injectivity and the Kernel

Proof.

Thus,

$$f(\mathbf{v}) = f(\mathbf{o}_E).$$

Since f is injective, for definition of injective mapping this means that $\mathbf{v} = \mathbf{o}_E$. We have reached a contradiction.

Hence, every vector \mathbf{v} in the kernel is \mathbf{o}_E , i.e.

$$\ker(f) = \{\mathbf{o}_E\}.$$

Let us assume that $\ker(f) = \{\mathbf{o}_E\}$ and let us consider two vectors $\mathbf{u}, \mathbf{v} \in E$ such that $f(\mathbf{u}) = f(\mathbf{v})$. It follows that

$$f(\mathbf{u}) = f(\mathbf{v}) \Rightarrow f(\mathbf{u}) - f(\mathbf{v}) = \mathbf{o}_F.$$

...continued

Injectivity and the Kernel

Proof.

It follows from the linearity of f that $f(\mathbf{u} - \mathbf{v}) = \mathbf{0}_F$.

For the definition of kernel, $\mathbf{u} - \mathbf{v} \in \ker(f)$.

However, since for hypothesis, $\ker(f) = \{\mathbf{0}_E\}$, $\mathbf{u} - \mathbf{v} = \mathbf{0}_E$.

Hence, $\mathbf{u} = \mathbf{v}$.

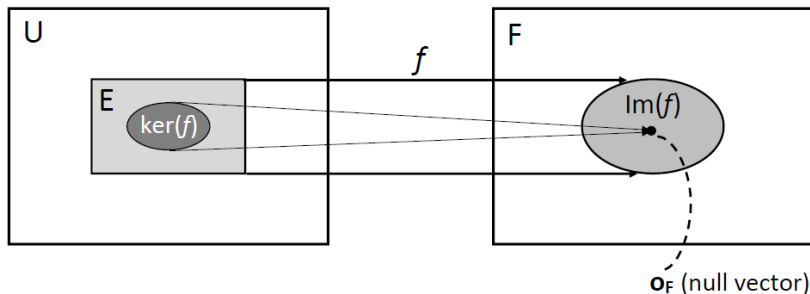
Since, $\forall \mathbf{u}, \mathbf{v} \in E$ such that $f(\mathbf{u}) = f(\mathbf{v})$ it follows that $\mathbf{u} = \mathbf{v}$ then f is injective. \square

The curious null vector space

- ⊙ Notice that $\{\mathbf{o}_E\}$ is a vector subspace, since $\mathbf{o}_E + \mathbf{o}_E = \mathbf{o}_E$ and $\lambda \mathbf{o}_E = \mathbf{o}_E$ shows closure.
- ⊙ The basis of $\{\mathbf{o}_E\}$ is empty since the only vector in $\{\mathbf{o}_E\}$ is \mathbf{o}_E and this is linearly dependent (by definition).
- ⊙ It immediately follows that $\dim(\{\mathbf{o}_E\}) = 0$.

Diagram showing Kernel (ker) and Image (Im)

Suppose the set of vectors $E \subset U$ form a vector space $(E, +, \cdot)$.
Consider any linear mapping $f : E \rightarrow F$.
Then f , $\ker(f)$ and $\text{Im}(f)$ can be represented by this diagram.



Rank and Nullity of Linear Mappings

Definition

Let $f : E \rightarrow F$ be a linear mapping and $\text{Im}(f)$ its image. The dimension of the image, $\dim(\text{Im}(f))$ is said *rank* of a mapping.

Definition

Let $f : E \rightarrow F$ be a linear mapping and $\ker(f)$ its kernel. The dimension of the kernel, $\dim(\ker(f))$ is said *nullity* of a mapping.

Rank-Nullity Theorem

Theorem

Let $f : E \rightarrow F$ be a linear mapping where $(E, +, \cdot)$ and $(F, +, \cdot)$ are vector spaces defined on the same scalar field \mathbb{K} . Let $(E, +, \cdot)$ be a finite-dimensional vector space whose dimension is $\dim(E) = n$.

Under these hypotheses the sum of rank and nullity of a mapping is equal to the dimension of the vector space $(E, +, \cdot)$:

$$\dim(\ker(f)) + \dim(\operatorname{Im}(f)) = \dim(E).$$

Usually, $\dim(\operatorname{Im}(f))$ is the hardest to calculate directly. This theorem allows an easy way to compute it as

$$\dim(E) - \dim(\ker(f)).$$

Rank-Nullity Theorem

Proof.

The proof is long (11 slides!) and structured into three parts:

- ⊙ Well-posedness of the equality
- ⊙ Special (degenerate) cases
- ⊙ General case

Well-posedness of the equality.

At first, let us prove that the equality considers only finite numbers. In order to prove this fact, since

$$\dim(E) = n$$

is a finite number we have to prove that also $\dim(\ker(f))$ and $\dim(\operatorname{Im}(f))$ are finite numbers. ...continued

Rank-Nullity Theorem

Proof.

Since, by definition of kernel, the $\ker(f)$ is a subset of E , then

$$\dim(\ker(f)) \leq \dim(E) = n.$$

Hence, $\dim(\ker(f))$ is a finite number.

Since $(E, +, \cdot)$ is finite-dimensional,

$$\exists \text{ a basis } B = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$$

such that every vector $\mathbf{v} \in E$ can be expressed as

$$\mathbf{v} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \dots + \lambda_n \mathbf{e}_n.$$

...continued

Rank-Nullity Theorem

Proof.

Let us apply the linear transformation f to both the terms in the equation

$$\begin{aligned} f(\mathbf{v}) &= f(\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \dots + \lambda_n \mathbf{e}_n) = \\ &= \lambda_1 f(\mathbf{e}_1) + \lambda_2 f(\mathbf{e}_2) + \dots + \lambda_n f(\mathbf{e}_n). \end{aligned}$$

Thus, remembering L denotes linear span,

$$\text{Im}(f) = L(f(\mathbf{e}_1), f(\mathbf{e}_2), \dots, f(\mathbf{e}_n)).$$

It follows (from Steinitz's Lemma) that

$$\dim(\text{Im}(f)) \leq n.$$

Hence, the equality contains only finite numbers.

...continued

Rank-Nullity Theorem

Proof.

Special cases.

Let us consider now two special cases:

1. $\dim(\ker(f)) = 0$
2. $\dim(\ker(f)) = n$

If $\dim(\ker(f)) = 0$, i.e. $\ker(f) = \{\mathbf{0}_E\}$, then f injective.

Hence, if a basis of $(E, +, \cdot)$ is $B = \{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, also the vectors

$$f(\mathbf{e}_1), f(\mathbf{e}_2), \dots, f(\mathbf{e}_n) \in \text{Im}(f)$$

are linearly independent from Theorem on Slide 16. Since these vectors also span $(\text{Im}(f), +, \cdot)$, they compose a basis.

...continued

Rank-Nullity Theorem

Proof.

It follows that $\dim(\operatorname{Im}(f)) = n$ and
 $\dim(\ker(f)) + \dim(\operatorname{Im}(f)) = \dim(E)$.

If $\dim(\ker(f)) = n$, i.e. $\ker(f) = E$ (from Lecture 7, slide 29).
Hence,

$$\forall \mathbf{v} \in E : f(\mathbf{v}) = \mathbf{0}_F$$

and

$$\operatorname{Im}(f) = \{\mathbf{0}_F\}.$$

Thus,

$$\dim(\operatorname{Im}(f)) = 0$$

and $\dim(\ker(f)) + \dim(\operatorname{Im}(f)) = \dim(E)$.

...continued

Rank-Nullity Theorem

Proof.

General case.

In the remaining cases, $\dim(\ker(f)) \neq 0$ and $\neq n$.

We can write

$$\dim(\ker(f)) = r \Rightarrow \exists B_{\ker} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r\}$$

$$\dim(\operatorname{Im}(f)) = s \Rightarrow \exists B_{\operatorname{Im}} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s\}$$

with $0 < r < n$ and $0 < s < n$ where B_{\ker} and B_{Im} are bases for $\ker(f)$ and $\operatorname{Im}(f)$ respectively.

We have $\mathbf{w}_i = f(\mathbf{v}_i)$ for some $\mathbf{v}_i \in E$.

...continued

Rank-Nullity Theorem

Proof.

$\forall \mathbf{x} \in E$, express the linear mapping $f(\mathbf{x})$ as linear combination of the elements of B_{Im} by means of the scalars h_1, h_2, \dots, h_s ,

$$\begin{aligned} f(\mathbf{x}) &= h_1 \mathbf{w}_1 + h_2 \mathbf{w}_2 + \dots + h_s \mathbf{w}_s = \\ &= h_1 f(\mathbf{v}_1) + h_2 f(\mathbf{v}_2) + \dots + h_s f(\mathbf{v}_s) = \\ &= f(h_1 \mathbf{v}_1 + h_2 \mathbf{v}_2 + \dots + h_s \mathbf{v}_s). \end{aligned}$$

We know that f is not injective because $r \neq 0$. On the other hand, for the Theorem on Slide 12,

$$\mathbf{u} = \mathbf{x} - h_1 \mathbf{v}_1 - h_2 \mathbf{v}_2 - \dots - h_s \mathbf{v}_s \in \ker(f).$$

...continued

Rank-Nullity Theorem

Proof.

If we express \mathbf{u} as a linear combination of the elements of B_{\ker} by means of the scalars l_1, l_2, \dots, l_r , we can rearrange the equality as

$$\mathbf{x} = h_1 \mathbf{v}_1 + h_2 \mathbf{v}_2 + \dots + h_s \mathbf{v}_s + l_1 \mathbf{u}_1 + l_2 \mathbf{u}_2 + \dots + l_r \mathbf{u}_r.$$

Since \mathbf{x} has been arbitrarily chosen, we can conclude that the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$ span E :

$$E = L(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r).$$

...continued

Rank-Nullity Theorem

Proof.

Let us check the linear independence of these vectors.

Consider scalars $a_1, a_2, \dots, a_s, b_1, b_2, \dots, b_r$ and let us express the null vector as linear combination of the other vectors

$$\mathbf{o}_E = a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_s \mathbf{v}_s + b_1 \mathbf{u}_1 + b_2 \mathbf{u}_2 + \dots + b_r \mathbf{u}_r.$$

Then apply the linear properties, $f(\mathbf{o}_E) = \mathbf{o}_F =$

$$\begin{aligned} &= f(a_1 \mathbf{v}_1 + a_2 \mathbf{v}_2 + \dots + a_s \mathbf{v}_s + b_1 \mathbf{u}_1 + b_2 \mathbf{u}_2 + \dots + b_r \mathbf{u}_r) \\ &= a_1 f(\mathbf{v}_1) + a_2 f(\mathbf{v}_2) + \dots + a_s f(\mathbf{v}_s) \\ &\quad + b_1 f(\mathbf{u}_1) + b_2 f(\mathbf{u}_2) + \dots + b_r f(\mathbf{u}_r) \\ &= a_1 \mathbf{w}_1 + a_2 \mathbf{w}_2 + \dots + a_s \mathbf{w}_s \\ &\quad + b_1 f(\mathbf{u}_1) + b_2 f(\mathbf{u}_2) + \dots + b_r f(\mathbf{u}_r). \end{aligned}$$

...continued

Rank-Nullity Theorem

Proof.

We know that since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r \in \ker(f)$ then

$$f(\mathbf{u}_1) = \mathbf{0}_F, \quad f(\mathbf{u}_2) = \mathbf{0}_F, \quad \dots \quad f(\mathbf{u}_r) = \mathbf{0}_F.$$

It follows that $f(\mathbf{o}_E) = \mathbf{0}_F = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \dots + a_s\mathbf{w}_s$.

Since $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_s$ compose a basis, they are linearly independent. It follows that $a_1, a_2, \dots, a_s = 0, 0, \dots, 0$ and that

$$\begin{aligned} \mathbf{o}_E &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \dots + a_s\mathbf{v}_s + b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \dots + b_r\mathbf{u}_r = \\ &= b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \dots + b_r\mathbf{u}_r. \end{aligned}$$

Since $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$ compose a basis, they are linearly independent. Hence, also $b_1, b_2, \dots, b_r = 0, 0, \dots, 0$.

...continued

Rank-Nullity Theorem

Proof.

It follows that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_r$ are linearly independent.

Since these vectors also span E , they compose a basis.

We know, for the hypothesis, that $\dim(E) = n$ and we know that this basis is composed of $r + s$ vectors, that is $\dim(\ker(f)) + \dim(\operatorname{Im}(f))$.

Hence,

$$\dim(\ker(f)) + \dim(\operatorname{Im}(f)) = r + s = n = \dim(E).$$

□

Example

Consider the following mapping $f_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f_2(x, y) = x + y.$$

The kernel is calculated as

$$x + y = 0 \Rightarrow (x, y) = \alpha(1, -1), \alpha \in \mathbb{R}$$

so $\ker(f_2) = \alpha(1, -1) \Rightarrow \dim(\ker(f_2)) = 1$.

Since $\dim(\mathbb{R}^2) = 2$, it follows that $\dim(\operatorname{Im}(f_2)) = 1$.

This means that the mapping f_2 transforms the points of the plane (\mathbb{R}^2) into the points of a line in the plane.

Rank-Nullity Theorem

Example

Let us consider the linear mapping $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as

$$f(x, y, z) = (x + 2y + z, 3x + 6y + 3z, 5x + 10y + 5z).$$

The kernel of this linear mapping is the set of points (x, y, z) such that

$$\begin{cases} x + 2y + z = 0 \\ 3x + 6y + 3z = 0 \\ 5x + 10y + 5z = 0. \end{cases}$$

It can be checked that the rank of this homogeneous system of linear equations is $\rho = 1$. Thus ∞^2 solutions exists.

Continued...

Rank-Nullity Theorem

Example

If we pose $x = \alpha$ and $z = \gamma$ with $\alpha, \gamma \in \mathbb{R}$ we have that the solution of the system of linear equations is

$$(x, y, z) = \left(\alpha, -\frac{\alpha + \gamma}{2}, \gamma \right),$$

that is also the kernel of the mapping:

$$\ker(f) = \left(\alpha, -\frac{\alpha + \gamma}{2}, \gamma \right).$$

It follows that $\dim(\ker(f), +, \cdot) = 2$. Since $\dim(\mathbb{R}^3, +, \cdot) = 3$, it follows from the rank-nullity theorem that $\dim(\operatorname{Im}(f)) = 1$. We can conclude that the mapping f transforms the points of the space (\mathbb{R}^3) into the points of a line of the space.

Exercise 2: Rank-Nullity Theorem

Consider $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $f(x, y, z) = (2x - y + z, x + y + z, x - 2y)$.

Compute $\dim(\ker(f))$ and $\dim(\operatorname{Im}(f))$.

Exercise 2: *Solution*

The kernel forms the system of linear equations,

$$2x - y + z = 0$$

$$x + y + z = 0$$

$$x - 2y = 0$$

Since they are linearly dependent, there are ∞^1 solutions. Pose $y = \alpha$. Then, solutions are given by $\alpha(2, 1, -3)$.

This shows the basis of the kernel is 1 (i.e. only one vector is needed to span all elements): $\dim(\ker(f)) = 1$.

Since $\dim(\mathbb{R}^3) = 3$, use the Rank-Nullity Theorem to show $\dim(\operatorname{Im}(f)) = 3 - 1 = 2$.

Summary and next lecture

Summary

- ⊙ Endomorphisms and Kernel
- ⊙ Injectivity
- ⊙ Rank and Nullity of Linear Mappings

The next lecture

We will learn about Geometric Mappings.