Calculus

Compact Calculus

with Relation to Existence

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1E

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Preface

My vision towards Thailand's calculus pedagogy inspires the creation of this textbook. Thailand's textbook repertoire uses a theory-first approach. It's theoretically viable because it starts from the mathematical fundamental: the limit. However, it doesn't give the full picture of calculus. Calculus is developed from real-life applications of the infinitesimal. The problem of finding area of complex shapes and approximating the star cycle arose since the ancient babylonians. It is more satisfying and rewarding for the student to be able to put themselves back in time, solving problems and "*inventing*" methods used in calculus by themselves. The method used in this book is heavily inspired by SciUS's story-based learning. The writer have taught many calculus students with different approach for each and the results were better using story-based learning, therefore, the writer wants to spread this method hence, the reason why this book was written, putting the reader back in time, making the reader feels like they've invented the whole subject by themselves.

Most of the readers might be questioning why a biochemistry student is writing a calculus book. The whole inspiration and passion dates back to since I was young. My family went on a tour and one of the destination were the anatomy museum in Siriraj. I was fascinated learning about each of the organ in the human body; how it functions, how it connects to each other. At that time, I didn't even understand most of the things spoken by the tour guide but the curiosity sparkled me to learn about how the nature of human came into being. Four to five years later when I was 13, the curiosity expanded to a level that I dedicate my life to understanding the fundamental concept of nature itself. Calculus is one of the ideas that came across my head at that time. I spent a lot of time on it, 2 years to understand Calculus III. I was taught by some really great teachers at a very young age to understand the core of the subject first, then applying it will be a piece of cake. That is what I did with calculus the whole way through from the most basic to Calculus III. I truly appreciate how this whole subject was formed. Calculus might seems like a mathematical riddle, not applicable to the real world at a high level; this is extremely fallacious. Calculus appears in the sole of the universe, Appearing in the heart of biochemistry, the master that the author is writing, in the form of physical chemistry. That physical chemistry also links to quantum mechanics which is the soul of the universe itself. Teaching is one of my passions, to make others also able to see and understand how the universe works. Of course, I cannot separate myself into a million body to teach everyone in this Earth, therefore, this book is written for that purpose, to transfer my knowledge to mankind. Written for 3 years, may it last eternity to come, till when laws and axioms fails, till when thy methods shatter, till when the mankind ends, till when's the dawn of time.

> "*Since Newton, mankind has come to realize that the laws of physics are always expressed in the language of differential equations.*"
>
> - Steven Strogatz

The content of this textbook ranges from the most fundamental part of calculus to the applications of calculus and some extensive mechanics provided by the physical world to help the reader actually understands the core of what each mathematical symbol written out means and what its geometrical and physical interpretation is. I don't want the reader to only see the symbols, I strongly want the reader to see the picture that is constructed with each symbol that is added into the equation. Physical examples of each principle is going to be extensively used. This book can be read by everyone due to the concept-based principles and story-based learning that it carries. The other goal of this is to link the concept between distant parts of the science of calculus. I highly encourage eveyone interested in science and math to give this book a chance.

Acknowledgement

Reading Guide

The abstract contains the reading guide of that chapter and also contains the introduction of the chapter. *Reading the abstract is a necessity.*

The chapter contains the concept fundamental to the topic. A chapter might have a supplementary chapter which is optional to the reader. The supplementary appendix for that chapter is written in the abstract.

Appendices either supplements or extends the chapter. Some goes beyond the chapters. They all vary. Most of them doesn't follow the story of calculus. It's recommended to study them separately. The interludes connects the chapters, not just supplements.

This book is separated into four parts.

I.) The fundamentals
II.) The applications
III.) The extensions
IV.) The foundations, reimagined: real and complex analysis

Part I (The fundamentals) considers the basics of calculus. From derivatives to antiderivatives and some of its applications. Part II (The applications) focuses on applying calculus to real world problems, mostly in physics. Part III (The extensions) explores into the realm of specialized calculus, most of them aren't taught in universities. It includes some of the newer branches of calculus. Finally, Part IV (The foundations, reimagined: real analysis and complex analysis) reconsiders all the basics of calculus and dives deep into the backbone of all symbols that are abstracted away from the physcial world.

At last, a fair warning; not all contents follow accurate historical order.

> *"You'll spend too much time studying everything if you follow through the historical timeline exactly. People think of things independent of each other. It is perhaps better to study on a different story that connects the dot better"*
>
> \- Wiwat Ruenglerdpunyakul

Part I

The fundamentals

Chapter 1

Introduction to calculus and its origin

Abstract

The origin of calculus is a bit weird. It has been developing since ancient times, then it was left off for around a thousand years until, the Europeans came around and revived the concept. This chapter contains just the brief historical background of calculus.

Do not spend too much time reading this chapter. (Ch. 1.2) is completely skippable, (Ch. 1.1) should not be skipped. As the reader guide said, some field isn't impossible to study according to history. This field is one of them; it was left off for so long. The full history of calculus is written in (Interlude **??**) after the main concept were grasped already.

1.1. The themes of calculus

Before diving in, some questions must be answered: "Why does calculus exists?". When a problem arises, it sometimes forces us to invent new tools to solve it. Not every problem can be solved using existing tools. Calculus were invented to solve problems that algebraic tools cannot solve: anything that involves the rate of change and finding areas of certain things.

Calculus separates into two main fields: differential and integral. They all revolve around four things which is quite distinct from each other. However, these four things are connected to each other in the most subtle way, but the subtlety in these connection is what makes calculus sublime.

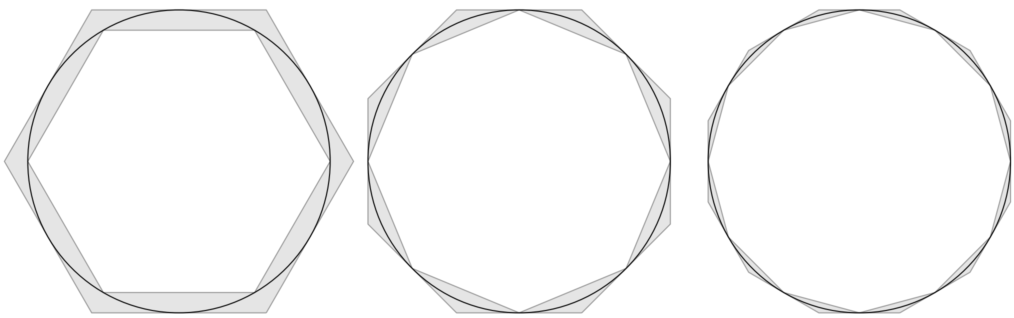
1. The steepness of a curve
2. Dynamics (Motion)
3. The area enclosed by a curve
4. Infinite series

The first two belongs to differential calculus, the last two belongs to integral calculus. As you'd see, these four things will pop up here and there and everywhere in calculus. Always keep these four in mind.

1.2. A brief history of calculus

Isaac Newton invented calculus[1]. He wanted a formal theory to support the existence of the **infinitesimal** to calculate things with it. But why does he need the infinitesimal in the first place? The answer dates back to the ancient times.

Figure 1.1: Method of Exhaustion



Ideas were sketched out since the ancient times but most weren't logically complete. Every thing kicked off in the ancient greek, 400 years before Chirst, Eudoxus, an ancient mathematician, invented the **Method of Exhaustion**. He wanted to find areas and perimeter of complex shapes, including the circle. The thing about

[1] Or Leibniz independently at around the same time.

circle is; its circumference can be used to calculate the constant π which were a big quest for mathematician. Who can calculate more digits of pi will have better social status. He inscribed and circumscribed a circle with polygons; then, he cut the polygons into little triangles. The higher the number of sides, the more the perimeter approaches the real circumference (Fig. 1.1). This method quickly fell out of use because as the name suggest, it exhausts the one calculating. However, this idea will soon be developed into integral calculus.

200 years later, Aristotle, a greek mathematician and philosopher took Eudoxus' idea further. He created the **infinitesimal**: a number that is so close to zero but is *not* zero. One could say, *it approaches zero.* Mathematicians quickly rejected this, because too much questions arose. *"What's the difference between the infinitesimal and zero?" "Why does this have to exist?"* Mathematics back then were the studies of shapes, objects, real bodies that has a clear physical existence. Mathematicians were tied to the reality. They rejected the ideas of infinities, irrationality of numbers ($\sqrt{2}$), and infinitesimal. This leaves the infinitesimal as just a concept floating around in the mind of some mathematicians.

This concept got left off for a really long time until the early seventeenth century, it got revived. There was a mathematical conference in France. A lot of great mathematicians gathered at that conference. The conference was about mathematics in general, but one idea in that conference popped out: functions that can find the rate of change of other functions. Pierre de Fermat found that this function can be used to find the minima and maxima of other functions. He published "*Methodus ad disquirendam maximam et minima*". Also, he invented the method of finding the tangent to any curves, not only circles. With that he published another book called "*De tangentibus linearum curvarum*". These little ideas will evolve and turn into todays' differential calculus.

In the middle of the seventeenth century, at Woolsthorpe, England, a famous mathematician and physicists was borne: Isaac Newton. He strengthens the infinitesimal that were left off. Then, with these infinitesimal, he built the foundations of classical mechanics which were used to accurately predict the trajectory of objects. All of these were published in "*Philosophia Naturalis Principia Mathematica*", or "*Mathematical principles of natural philosophy*" which is considered the bone of calculus. A massive light was shed. After this is the age of rapid development of physics and mathematics. This is why he got the title "*The father of physics*".

However, things weren't so smooth. For some reason, Newton had to delay the publishing of his book by a decade. Another great mathematician came into the scene: Gottfried Wilhelm von Leibniz. Borned in Leipzig, he also laid the foundation of calculus independently in his book "*Acta Euditorium*" before Newton published the Principia.

News spread all over Europe. When it reached Newton, he was furious. Leibniz apparently "stole" his idea. But of course, Leibniz didn't even know who Newton were at that time. 1699 marked the year that this controversy started. It bursts into fire in 1711. Mathematicians were debating about who invented calculus first. Newton invented calculus and applied calculus to physical scenarios: mechanics, orbits, tides, etc. Leibniz on the other hand did not applied calculus anywhere, but his theory was more rigourous. The fight continued on until Leibniz' death.

A group of mathematicians saw the potential of developing calculus further, so they combined both theories into one single "calculus" in the eighteenth century. After that, calculus boomed and evolved into the calculus we know today.

1.3. Why the name "Calculus"?

The reason should be obvious in later chapters, but I shall state it now, because it reflects the main philosophy of calculus. The name "Calculus" came from the Latin language which translates to "pebbles". It's analogous to the infinitesimal. These "pebbles" can be molten together, molded and shaped into a big and resplendent creation.

Chapter 2

Before calculus

Abstract

This chapter covers the basics of math and is skippable. It functions as a reference but is incomplete since studying calculus usually implies that the student is already familiar with basic math. The chapter includes the following contents:

1. Sets
2. Number system
3. Relations and cartesian product
4. Coordinate system
5. Functions and graphs of function
6. Growth of functions

2.1. Set

A **set** is a collective noun that is used to refer to a collection of things. The boundary of a set is written with curly brackets. **Elements** are things in the set. Each element is separated with commas, e.g., if 1, 2, and a Mangosteen, is in A, then, $A = \{1, 2, \text{Mangosteen}\}$. The order of elements in a set does not matter. A set with no elements are called the **empty set**, written with ϕ or a pair of curly bracket $\{\}$.

2.1.1. Set cardinality and the Pigeonhole principle

The **set cardinality**, informally called the size of a set, is a number that represents the amount of elements in a set. Take set $A = \{1, 2, \text{Mangosteen}\}$. The cardinality of A is 3. To write cardinality of a set, use the magnitude value (or absolute value) on a set, or use the cardinality function[1]. E.g. (Eq. 2.1)

$$A = \{1, 2, \text{Mangosteen}\}$$
$$|A| = n(A) = 3 \tag{2.1}$$

There are two methods to compare set cardinalities. Either count the number of elements, then compare it, or use the pigeonhole principle. The pigeonhole principle is handy in combinatorics and graph theory. However, that is far beyond the scope of this book; therefore, only the basics are covered.

The pigeonhole principle states that: if n pigeons are put into m holes, and there are more pigeons than holes, at least one hole must contain more than one pigeon and vice versa. The pigeons represent a set, and the holes another set. If elements from both sets are paired, and no elements are leftover, then both have equal cardinality.

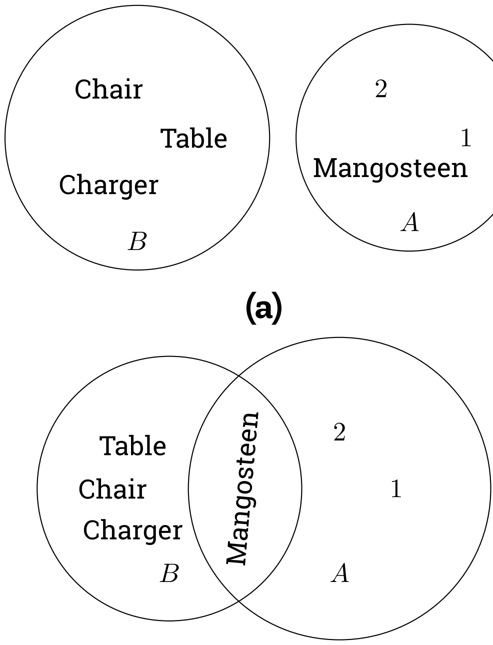
It's uncommon to use the pigeonhole principle on finite sets because it's easy to compare the set cardinality of finite sets. Infinite sets, on the other hand, are a completely different story. Various paradox arises when one tries to compare infinite sets, such as Cantor's Paradox and the Continuum hypothesis. Both paradoxes are written in greater detail in (Appendix A).

2.1.2. Euler diagram and the universe

The Euler diagram is used to visualize sets. Circles represent sets and the set elements are put inside them. E.g. let set $B = \{\text{Chair}, \text{Table}, \text{Charger}\}$. The resulting diagram shown in fig. 2.1a

One element can exist in two sets, such as: if set B also has Mangosteen in it, the diagram will have overlapping areas as shown in fig. 2.1b

[1] Functions are discussed in (Ch. 2.3)



(a)

(b)

Figure 2.1: Examples of Euler diagram

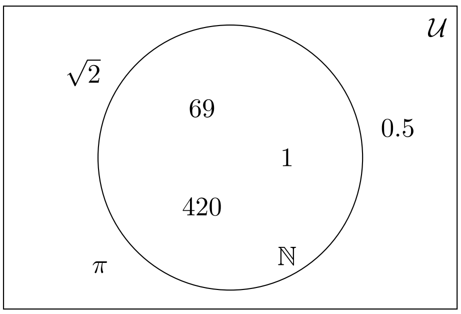


Figure 2.2

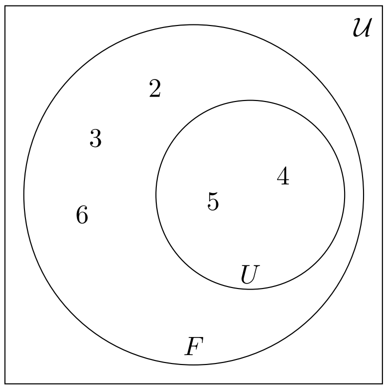


Figure 2.3

In mathematics, every set has to be in a boundary called the **universe**. A **universe** is a collection that contains all the entities one wishes to consider in a given situation. I.e., a set that contains 0.5 is impossible in the positive integers universe. However, in the real numbers universe, it is possible to have a set containing 0.5.

A universe is represented as a square that encloses everything. Since a universe contains things, it itself is also a set. By the standard convention, the universe is named $\mathcal{U}$. E.g., in fig. 2.2, the real number universe. $\mathbb{N}$ represents the set of counting numbers.

Comments:

i.) In some textbooks, the Euler diagram is often referred as Venn-euler diagram. However, there's no "Venn-euler diagram". A Venn diagram is an Euler diagram but with all possible combinations of intersections.

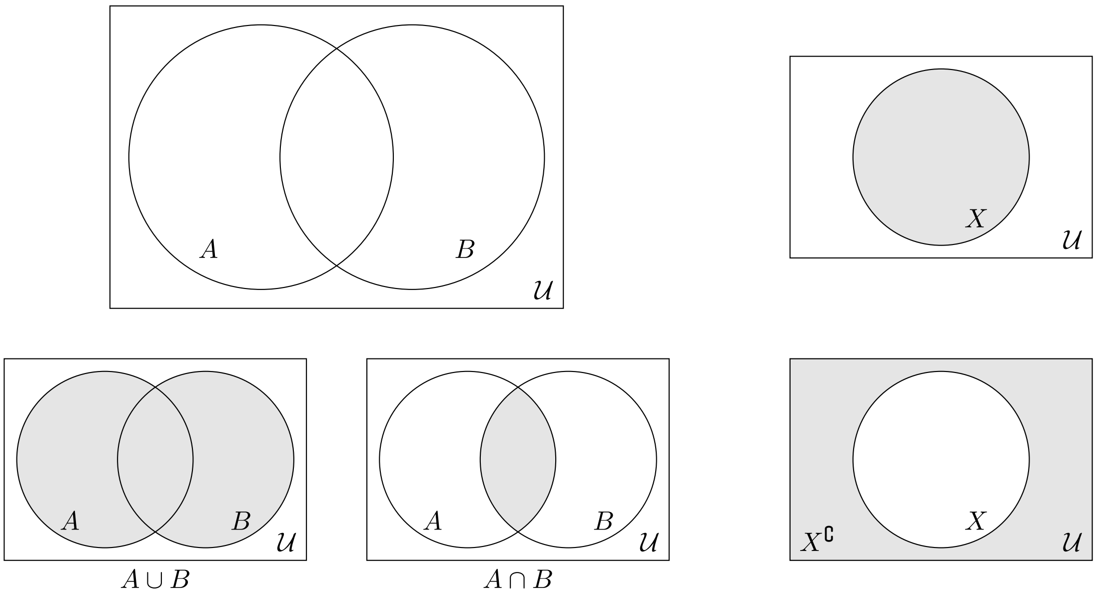


Figure 2.4: Set operations

2.1.3. Subsets and powersets

Set A is a **subset** of set B *if and only if* all elements in set A also exists in set B and a **powerset** of a set is a collection of all subsets of a set.

Let set $F = \{2, 3, 4, 5, 6\}$ and set $U = \{4, 5\}$. Set U is a subset of set F because all elements in set U also exist in set F. This subset relationship is written as $U \subset F$. Notice, the empty set is a subset of every set and every set must be a subset of itself.

2.1.4. Set operations

Set operations operate on set(s), making a new set. They have the same concept as arithmetic operations ($+$, $-$, $\times$,)

The union operator combines elements of two sets together, represented by the symbol $\cup$.

The intersect operator gives a set that only contains the elements that exist in both sets, represented by the symbol $\cap$.

The complement operator gives a set that contains everything except that set, represented by the symbol $^\complement$ floating above the set, e.g., $A^\complement$.

The Euler diagram for each operators are shown in fig. 2.4

Comments:

i.) In some textbooks, the complement operator is also represented with A'. However, the dash is sometimes confused with inverse functions.

ii.) Some textbooks include the subtract operator $A \setminus B$. What it does should be trivial. It isn't included here because it can be written as $A \cap B^\complement$. It's also called the relative complement form.

iii.) The cardinality of $A \cup B$ is not $|A| + |B|$ but rather, $|A| + |B| - |A \cap B|$ because there's overlapping areas.

2.1.5. The number system

Number system is all the sets of numbers. The smallest set is the set of natural numbers $\mathbb{N}$. It includes positive integers except zero; zero is in its own set. The set positive integers is written as $\mathbb{Z}^+$ or $\mathbb{N} \cup \{0\}$ and the set of negative integers as $\mathbb{Z}^-$. All integers combined are written as $\mathbb{Z}$.

Zooming out, there's the set of fractions. All numbers which can be written as

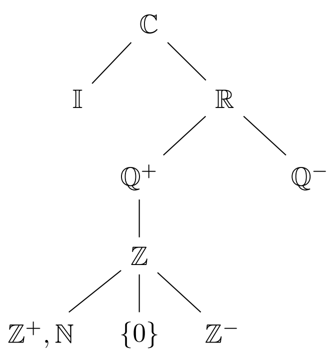


Figure 2.5: The hierarchy of number system

fractions are in the set of rational numbers ($\mathbb{Q}$), including the integers[2]. However, not all numbers can be written as fractions. Numbers that can't be written as fractions are called the irrationals, e.g., π. They belong in the set $\mathbb{Q}^{\complement}$. The union of the rationals and the irrationals gives set of real numbers which are every numbers that can be quantized in the real world.

Still, the set of real numbers doesn't contain every number, e.g., the root of $x^2 + 1 = 0$. A new set of numbers were invented[3]: the imaginaries, i.e., let there be a number i which is defined by $i^2 = -1$. The set of imaginaries $\mathbb{I}$ includes every multiples of i. When an imaginary is added to a real, they form complex number which belongs in a bigger set of numbers: $\mathbb{C}$ which is the largest set of number[4].

$$\begin{aligned} x^2 + 1 &= 0 && (2.2) \\ x^2 &= -1 \\ x &= \sqrt{-1} && (2.3) \end{aligned}$$

2.2. Cartesian product and relations

The concept of **relation** should be trivial. In mathematics, two elements that are related is written as an **ordered pair**, e.g., $(1, 2)$, or $(3, 10)$. An ordered pair is not a set; the ordering matters, i.e., $(a, b) \neq (b, a)$. An ordered pair of two numbers expresses a binary relation. Three numbers relation is also possible and is called trinary relation.

A **cartesian product** between two sets is a set that contains every binary relation possible between the two. E.g., if set $A = \{x, y, z\}$ and set $B = \{a, b, c\}$, the cartesian product between A and B is

$$A \times B = \Big\{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c), (z, a), (z, b), (z, c)\Big\} \tag{2.4}$$

Notice that the binary relation is a subset of the cartesian product between two sets

[2]The name "*rational*" came from the word *ratio* which means fraction.

[3]Take this as an "easy to understand" fact for now. It's later discussed in full details in section 9.3. Rest assure, imaginary numbers aren't *invented*, they arose naturally

[4]Actually, there are more numbers outside of the complex number system, e.g., dual numbers. The dual constant ε is defined as any number that itself squared is zero, but ε is not zero. It has specific usage in two-dimensional algebra but is beyond the scope of this textbook.

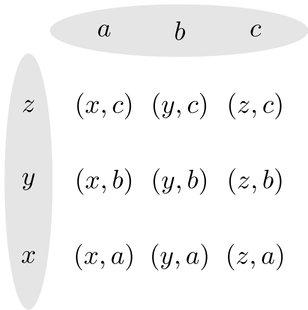


Figure 2.6: The cartesian product between $\{x, y, z\}$ and $\{a, b, c\}$

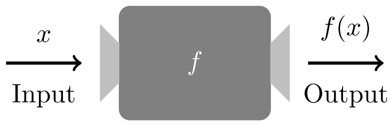


Figure 2.7: An illustration on how function works

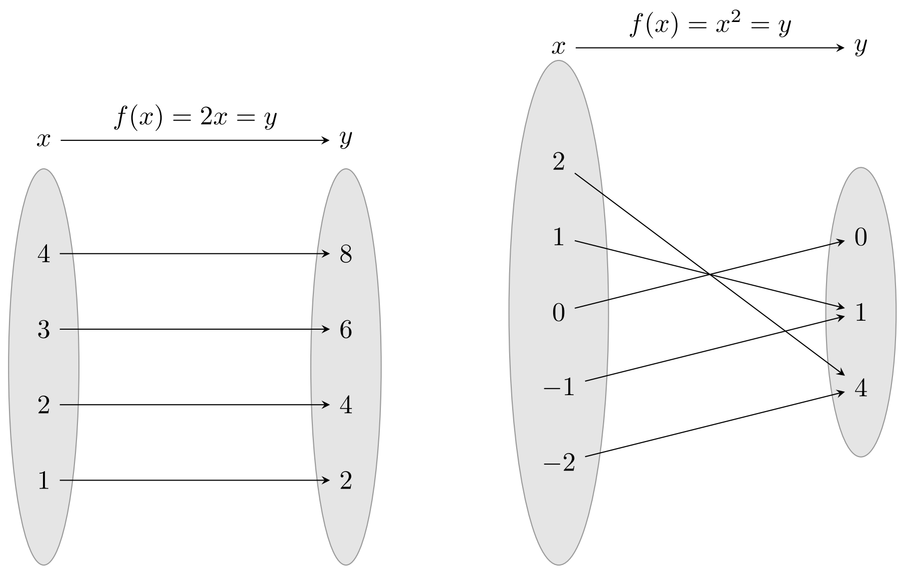
2.3. Functions

A function defines a set that contains deterministic relations. Some function has specific conditions, i.e., one variable is dependent on another variable and vice versa. E.g., $f(x) = y$ means, function f defines a set which contains an infinite amount of relations (x, y) where all x *must* be equal to y.

Functions can have two types of output: one-to-one and many-to-one. E.g., the function $f(x) = 2x$ is a one-to-one function fig. 2.8a and $f(x) = x^2$ is a many-to-one function. However, functions are determisnistic; therefore, a set defined by a function cannot have one to many relations. Sets of relations that contain one-to-many relations is not considered a function.

2.4. Plotting, and graphs of functions

There are ways to visualize functions and relations in general. When a point is given, it defines the value of n variables, usually two. n is the amount of dimension number used to describe a point given by a function. Space is needed to visualize



(a) One-to-one relations **(b)** Many-to-one relations

Figure 2.8: Forms of relations given by a function

these points. In a cartesian plane, a variable is assigned to an axis. Since a point defines two variables, it can be *plotted* on a plane. E.g. point $(3, 4)$, $(4, -2)$ and $(-1, 3)$ would look like as in (Fig. 2.9)

Since functions defines sets of relations, which can be plotted into a plane, a continuous line shows up if every point defined by a function is plotted. E.g. if $f(x) = y = x^2$, a line emerges as shown in (Fig. 2.10).

2.5. Inverse functions

2.6. Functions growth and shapes

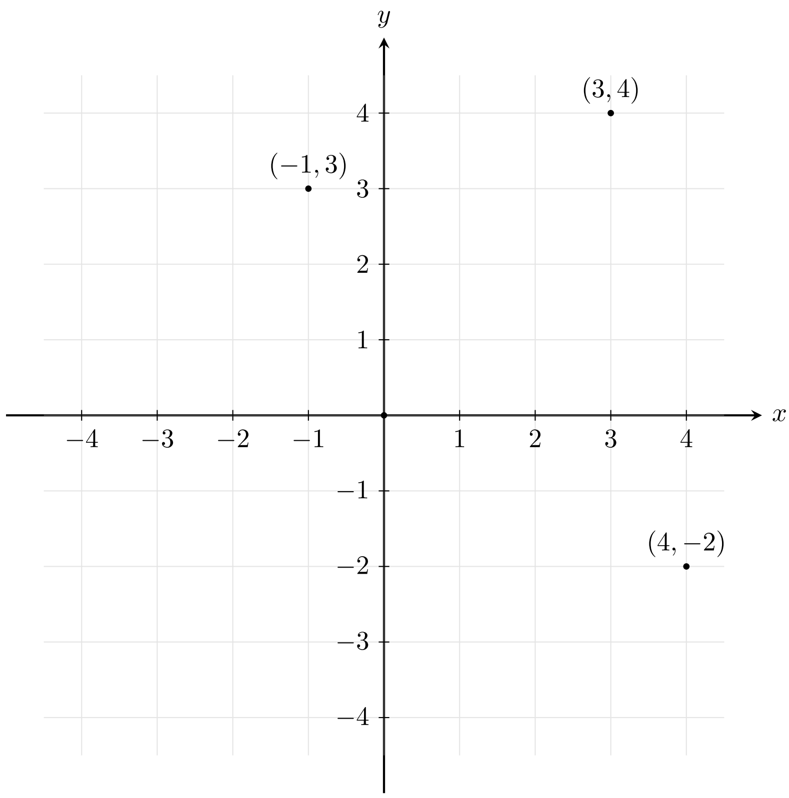


Figure 2.9: Examples of points plotted on a cartesian plane

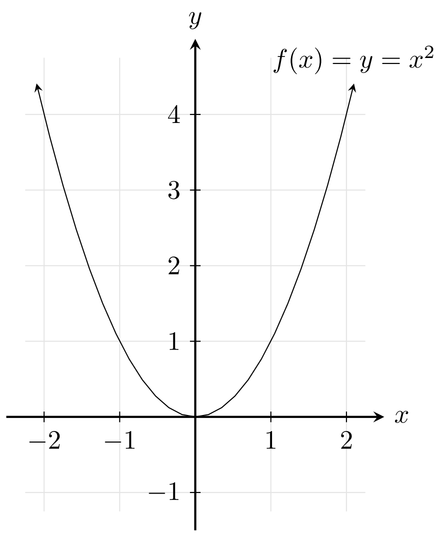


Figure 2.10: Plotting functions on a graph

Chapter 3

Kinematics and derivatives

Abstract

This chapter focuses on one of the two branches of calculus: differential calculus. It deals with "rate of change": the change of one variable depending on another variable. Kinematics are used to illustrate the concept of rate of change. Also, it's not fully based on the historical timeline.

This chapter focuses mostly on the story. The computational part is not focused here, but instead discussed in chapter 5 after the reader has the full concept of calculus.

For the absolute basics of classical mechanics, see appendix B

3.1. Speed, instantaneous rate of change, and the core of differential calculus

Speed, aka. **velocity**, is a physical quantity that quantizes the rate of displacement through time. If a car travels $10\,\mathrm{m}$ in $5\,\mathrm{s}$, its speed is $2\,\mathrm{m\,s^{-1}}$. Formally, this number is the *average* speed. However, if the car is not travelling consistently, the information of speed at a certain timespan is lost. Mostly, it's not a problem. If your friend asked when will you arrive at the party, you'd probably answer $10\,\mathrm{min}$; that answer would suffice. In what situation is the "speed at a certain timespan" useful?

A car crashes into a tree $10\,\mathrm{km}$ away from the starting point. You want to know the damage. If the car travels that distance in $1\,\mathrm{h}$, the damage would be negligible. However, the car doesn't have to travel consistently; it might start slow, then speed into the tree at the last second. The speed at the *instant* of the collision determines the damage. We need the **instantaneous speed** of the car at that instant.

It's not sufficient to use the speed calculated from the beginning of the car's life to display on a speedometer. It must update continuously. It has to display the instantaneous speed of the car.

It might seems paradoxical: instantaneous implies that the timespan *is* zero. If an object move instantaneously from point A to point B, it should get there in no time. The velocity equation $v = \Delta s/\Delta t$ breaks down when $\Delta t = 0$. Dividing by zero isn't possible in general circumstances. Here's where calculus come into play with the idea of **approaching**. The timespan isn't zero, it *approaches* zero. The equation states that v is the ratio between Δs and Δt. If Δt is small, Δs must also be small. This ratio should converge if the Δt is "diminished". Basically, we're avoiding division by zero. *This, is the core of differential calculus.*

This concept of a variable approaching something is a very prominent theme in calculus. Not only to zero, but to constants in general. Approaching zero appears so often that we give it a name: the **infinitesimal**. The rest of the chapter is mostly an exploration of what these theories can do and how to generalize them. Integral calculus on the other hand, frames the infinitesimal in a different way which is further discussed in chapter 4.

3.2. Speed from graphs, and the method of increment

To visualize an object's movement, one axis of a graph can be assigned to the time, and the other one, the position. If a box moves from a to b in $5\,\mathrm{min}$ with constant speed. It's position and time can be plotted, as shown in fig. 3.1 which looks like a straight line. This graph is called an x-t graph, aka. the position versus time.

Notice, the velocity of the graph can be found from the slope. According to the axis, $\Delta y/\Delta x$ becomes $\Delta s/\Delta t$, which is the velocity. The slope of a linear graph

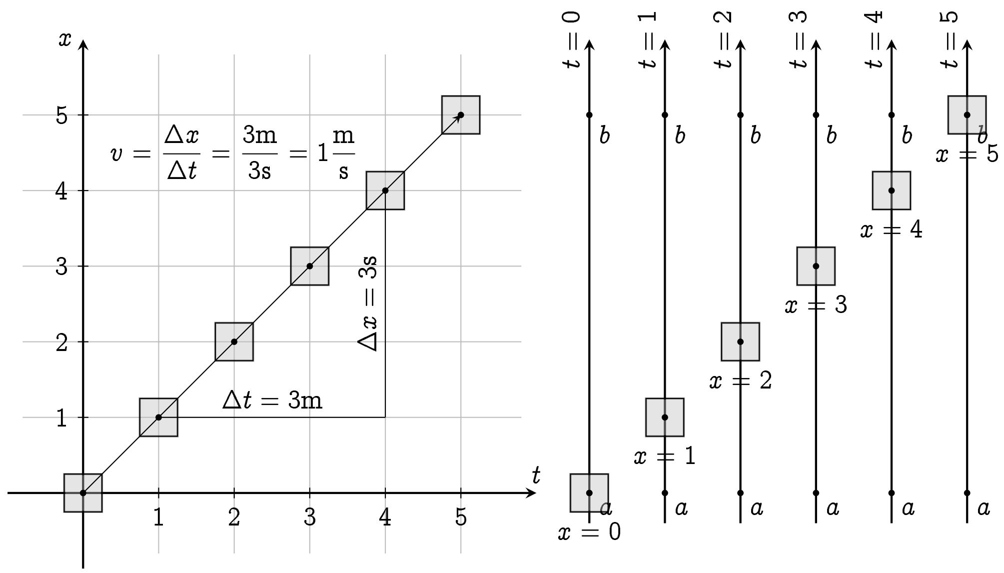


Figure 3.1: Graph of an object that's moving through time with constant velocity

is trivial to find: $\Delta y / \Delta x$. Since the graph is a straight line, how you view it doesn't matter, it's still the same straight line.

But what if the graph is not a line, but you still want to find the instantaneous velocity of a certain point in time? E.g., in fig. 3.2, the velocity of a box when it hits the wall at $t = 5$. From earlier, the formula only supports linear graphs. This graph is a parabola; thus, that formula is unusable. However, if one zooms in on the graph enough, the graph becomes straighter and straighter. With infinite magnification, the graph becomes a straight line. Now, calculating the speed at a certain time from this becomes possible.

Zooming in at $t = 5$, the graph in fig. 3.2 becomes a straight line. Actually, at every point, every graph if zoomed in enough becomes a straight line.[1] Refer back to the velocity equation $v = \Delta s / \Delta t$, our Δt here is a small nudge, let's say, h. The term Δs can be expanded into $x_2 - x_1$.[2] If s is determined by the function $x = f(t) = \frac{t^2}{5}$,

[1] There exists a family of functions that can't be framed as a straight line in whatever magnification, e.g., fractals. Some functions can be zoomed in and be framed as a straight line, but not at all points; they're called "non-differentiable functions". The reason for the name should be obvious later on.

[2] Since only one dimension are worked with, x and s are basically interchangable

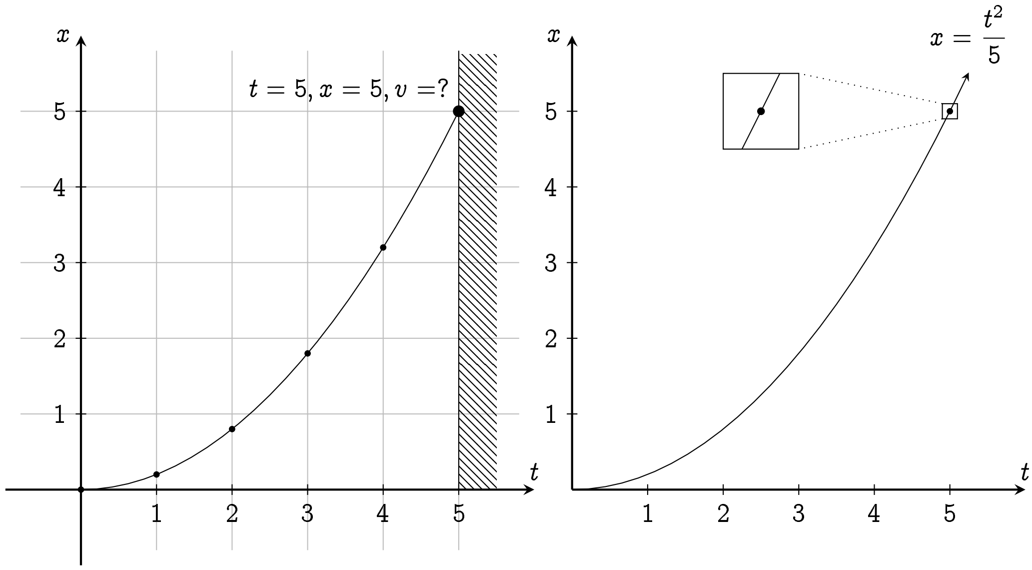


Figure 3.2: A moving object that follows $x = t^2/5$

$s_2 = f(t_2) = (t+h)^2/5$, and $s = f(t) = t^2/5$

$$
\begin{aligned}
v &= \frac{\Delta s}{\Delta t} \\
&= \frac{x_2 - x_1}{h} \\
&= \frac{f(t_2) - f(t_1)}{h} \\
&= \frac{(t+h)^2/5 - t^2/5}{h} \\
&= \frac{1}{5}\frac{\cancel{t^2} + 2th + h^2 \cancel{- t^2}}{h} \\
&= \frac{1}{5}(2t + h) \approx 2t/5 && (3.1) \\
v_{t=5} = 2(5)/5 &= 2 && (3.2)
\end{aligned}
$$

In eq. (3.1), substituting $t = 5$ gives $2(5)/5 = 2$ which *is* the instantaneous velocity at $t = 5$. Notice, the actual result is $(2t+h)/5$ but it's rounded off to $2t$. This is because when $h \to 0$, it's magnitude compared to $2t$ is negligible. For reference, imagine $10 + 0.00000001$ and $10 + 1$. The 0.00000001 trail can be completely ignored, but the 1 is too big to be ignored. In calculus, it's necessary to get the hang of what terms can be ignored and what cannot. Also, this method of calculating slopes by small increments is called "**The method of increment**".

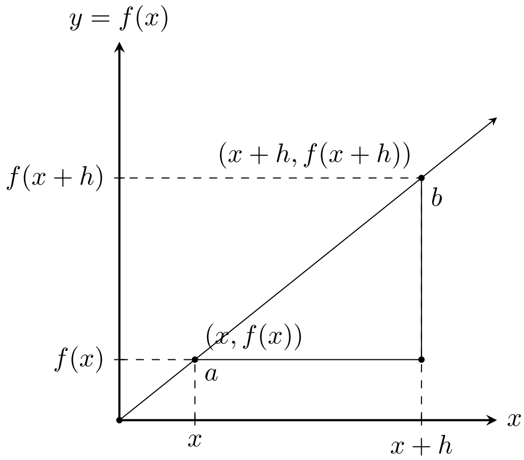


Figure 3.3: Generalizinhe method of increments from finding velocity to the equation for finding slopes of graphs

Writing h in the proof above is highly informal, because it didn't show what h is. **The limit notation** eliminates this ambiguity. If a variable approaches a constant, it's written as $\lim_{a \to b}$, e.g.,[3].

$$v = \lim_{h \to 0} \left(\frac{1}{5} \cdot \frac{2t+h}{5} \right).$$

3.3. Geometric interpretation of velocity

The geometric interpretation of velocity is the just the slope of the graph $x - t$ which we've evaluated using the method of increments. It's also possible to find slopes of other functions. As an exercise, find the slope function of $y = x^3$.

It's a pain if you'd have to go through this everytime; thus, we find the general form for it. Let there be any function $f(x) = y$ as shown in fig. 3.3. The slope of the graph m is given by

$$m = \frac{\Delta y}{\Delta x}. \tag{3.3}$$

Let there be two points, a and b, separated by h along the x-axis. The x component of b is $x + h$. The y component of a is $f(x)$, and at b, $f(x+h)$. Then, eq. (3.3)

$$= \lim_{h \to 0} \frac{y_2 - y_1}{h}$$

[3]Sometimes, the parenthesis is left off to ease writing.

$$= \lim_{h \to 0} \frac{f(x_2) - f(x_1)}{h}.$$

In conclusion,

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3.4}$$

which is the generalized slope formula. This interpretation of velocity calls for the slope of a graph, which is also the rate of change of one variable with respect to the other. As we've seen, these two ideas are tied together. The geometrical interpretation of rate of change is the slope of a graph. In calculus, we call this "rate of change", the derivative.

3.4. The derivative operator

In calculus, we call slope formula: the derivative. What we've done in section 3.2 is, we've calculated the *derivative* of x w.r.t. t. The derivative is an operator that operates on a function and outputs another function that describes how a variable changes depending on another variable. In this case, how much x changes when t changes. The act of computing the derivative is called "differentiating". The definition of derivative is given by eq. (3.4).

In calculus, variables that has d infront means, that variable approaches zero. E.g., dt means $dt \to 0$. $dt = t_2 - t_1 = \lim_{dt \to 0}(t + dt) - t$

The example from section 3.2 can be written as eq. (3.5). We use a language invented by leibniz to express these concepts. dx/dt should be read as "*the ratio between the change in x and the change in t*", or "*the derivative of x w.r.t. t*".

$$v = \frac{dx}{dt} = \frac{df(t)}{dt} = 2t \tag{3.5}$$

In this notation, derivatives are represented as fractions. $\frac{dy}{dx}, \frac{d}{dx}(y)$ reads "rate of change of y w.r.t. x". Sadly, there are no uniform ways of writing derivatives. Multiple great mathematicians have proposed their own notations which is suited for different purposes. This Leibniz language writes derivatives as fractions explicitly, because they behave like fractions. Also, you can cancel terms like fractions. Different notations will be introduced later on when its needed.

Hence, it derivatives does not always mean "the rate of change". Sometimes, derivatives are just ratios between two physical objects. E.g., the ratio between force and area is expressed as $\dfrac{dF}{dA}$

3.5. Accelerations and higher order derivatives

We don't have to stop at just the velocity. There are more terms that can describe the motion of a moving object, e.g., the acceleration: the change of velocity w.r.t. time.

$$a = \frac{dv}{dt} = \frac{d(\frac{dx}{dt})}{dt} = \frac{d^2x}{dt^2} \tag{3.6}$$

The acceleration is called the second order derivative of position w.r.t. time because you've differentiated the position twice. These n-th order derivatives are called higher order derivatives. The notation $\dfrac{d^2x}{dt^2}$ came from the manipulation of symbols.

These higher order derivatives also has its place not only in mechanics, but also in polynomial approximations of functions chapter 7.

3.6. Limits and continuity

So far, limits have been framed to be the formal way to express derivatives. Since this book is themed on real-life problems, some might think, "How are limits useful in real life?". I'd say they're not that useful there. I shall also not attempt to find a practical use for it. Some concepts are useful in real life, some aren't. Most of them that aren't are building blocks that help us understand and formalize other concepts. Although if I have to actually give an example of limits in real life, consider a chemical reaction. The ratio between two substance at its equilibrium, i.e., the time approaches infinity is given by a limit of an equation when $t \to \infty$. Also, not all functions have a limit everywhere. $f(x) = 1/x$ does not have a limit at $x \to 0$. The reason should be obvious (Fig. fig. 3.4)

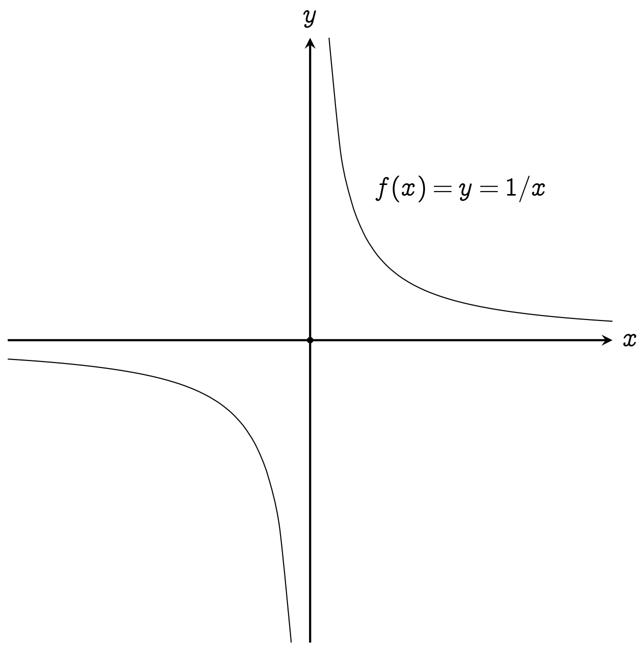


Figure 3.4: $\frac{1}{x}$ does not have a limit as $x \to 0$

Continuity in the other hand is a bit more applicable. Continuity is essentially uninterruption. It's applicable to fluid flow. However, this part of the book will not dive deep into continuity.

Conclusion for Chapter 3

1. When a variable approaches a constant, it is not equal to that constant, but rather, very close to that constant.

2. Derivatives represents the slope, or the rate of change of a function w.r.t. a variable. It can be found using the method of increments.

3. Higher order derivatives can be used to find the rate of change of the rate of change and so on.

4. Limits are used to formalize and represent derivatives.

Chapter 4

Integrals, antiderivatives

Abstract

This chapter develops the basics of integrals and makes the derivative and integral relationship obvious.

1. Integrals are sums of infinitely many parts.
2. Integrals and derivatives are inverses of each other. It's the cumulative effect of the derivative.
3. Integrals are very versatile and can be used to compute many things.

There are three stories in this chapter. The first story continues from (Ch. 3) and develops the intuition of integrals. The second story reinforces the first idea, and the last story shows the versatility of integrals.

4.1. Finding distance from the time-velocity graph

Imagine you're in a windowless car traveling on a straight line. There's nothing in the car except a speedometer. Is there a way to find out how far you've traveled?

If you were to plot the speed of the car on a v-t graph, it might look something like (Fig. 4.1). On section α in the graph, the car moves at $2\,\mathrm{m\,s^{-1}}$ for $2\,\mathrm{s}$, that means, section α represents $4\,\mathrm{m}$ of movement.

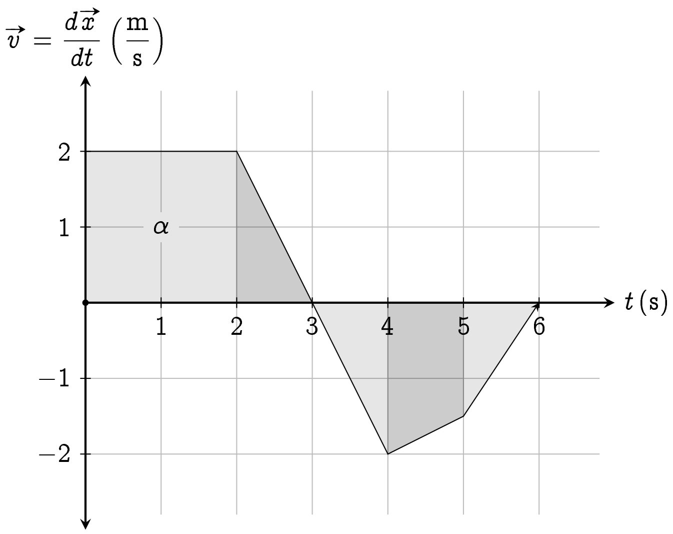


Figure 4.1: Example of an acceleration-time graph

That movement is directly related to the area under the graph, α. We can show this symbolically. From the velocity equation, $\vec{v} = d\vec{x}/dt$, since derivatives are ratios between two quantities, the *dt* can be moved like a fraction. The equation becomes

$$d\vec{x}(t) = \vec{v}(t) \cdot dt\,.$$

This equation relates velocity ($\vec{v}$) and change in time with the displacement ($d\vec{x}$). To clarify this, both $\vec{x}$ and $\vec{v}$ are functions of time; therefore, for the rest of this chapter, it will be written as $\vec{x}(t)$ and $\vec{v}(t)$.

Each sections of the graph represents a certain displacement. If v goes below 0, that means, the car is traveling backwards. The sum of all areas under the graph represents the total displacement, i.e., the distance traveled. In this graph, it turns out to be 3.5m. Solving that numerically should be an exercise. It shouldn't be too hard to do.

There's a catch, in real life v-t graphs are not always subdividable into simple sections. A car traveling on a road definitely have to accelerate around. If that's the case, is there a way to find the total displacement given a v-t graph?

Before that, notice that $\vec{v}(t)$ is the derivative of the position. From the example above, we found the displacement from the graph. We've essentially reversed the derivative. Therefore, this question can be reframed to fit the calculus context as *to*

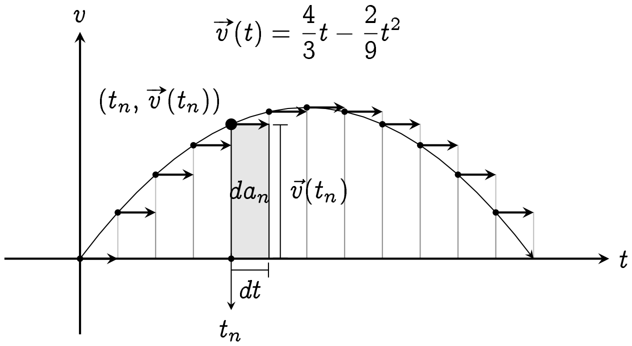


Figure 4.2: An example to illustrate Riemann sum

find the function that is the anti-derivative of the velocity.

4.2. Riemann sum, antiderivative, and integrals

An example would be the v-t graph in (Fig. 4.2). To find the total displacement, slice the graph into multiple vertical rectangles width dt. The area of the graph can be approximated by adding up all the little rectangles. The total approximated sum is called the **Riemann sum**. The more little you slice up the graph, the more precise the approximation. If you let $dt \to 0$[1], the approximated area will be equal to the actual area. With that, you can neatly find the displacement with just the graph.

This procedure of finding the area under the graph is called **integration**, i.e., finding the **antiderivative** of a function. Specifically, **integrals** are Riemann sum of functions where the rectangles' width approaches zero. The thing you're trying to integrate is called the **integrand**. We use the $\int$ sign to represent integrals. E.g.,

$$\int \vec{v}(t)\, dt = \sum_n \vec{v}(t_n) \cdot dt = \vec{x}(t) \tag{4.1}$$

which also reads that integrals are sums of small rectangles.

To generalize integrals to any functions, we write

$$\int_0^x da = A(x) \tag{4.2}$$

which reads as: *"The area under the graph of da from 0 to x is $A(x)$."*. da is the **integrand**, and $A(x)$ **integral** of da evaluated from 0 to x. The number on the bottom and

[1]Or what Newton would say, dt diminishes

top of the integral sign is called the **integral bounds**. Specifically, 0 is the **lower bound** and x, the **upper bound**. This concept will later be formalized into the fundamental theorem of calculus, discussed in (Ch. 4.4).

Now, do not worry about the evaluating process yet. Calculus is packed with tools to help with these problems. You'll learn these tools in (Ch. 5). They're not relevant right now. For now, we'll focus on establishing a clear relationship between integrals and derivatives. The next example will help further reinforces this concept.

4.3. Infections and integrals-derivative relationship

In a pandemic[2], more and more people are getting infected daily. The amount of newly infected people $\dot{n}(t)$ is tracked daily (Tab. 4.1), but not the total number people infected. How do you find the total number of people infected ΣN?

Obviously, ΣN is the sum of all $\dot{n}(t)$ which turns out to be 30. How's about the total amount of infected people *each day* $N(t)$? $N(t)$ is formerly called the **partial sum**. Here, it represents the amount of people infected each day which is quite easy to find. Just take the number of people infected before and add it to the newly infected.

t	$\dot{n}(t)$	$N(t)$
1	1	1
2	3	$1+3=4$
3	7	$4+7=11$
4	15	$11+15=26$
5	3	$26+3=29$
6	1	$1+29=30$
7	0	$30+0=30$

Table 4.1: Newly infected people tracked daily

But what does $\dot{n}(t)$ and $N(t)$ have to do in calculus? (Fig. 4.3) plot both variables against time. $\dot{n}(t)$ is actually the derivative of $N(t)$ w.r.t. time. It can be written as $\dot{n}(t) = \dfrac{dN(t)}{dt}$. $N(t)$ is the antiderivative of $\dot{n}(t)$ and it's the sum of all the $\dot{n}(t)$'s and it is indeed the integral of $\dot{n}(t)$.

At day 1, a person is infected; the change is 1. At day 2, three more people are infected; the change is 3, and the partial sum is $1+3$. The change added to the previous partial sum, i.e., the derivative is added to the riemann sum. How's about the other way? On day 4, $N(t)$ is 26. $N(t)$ comes from all the people that's infected

[2] Totally not COVID-19

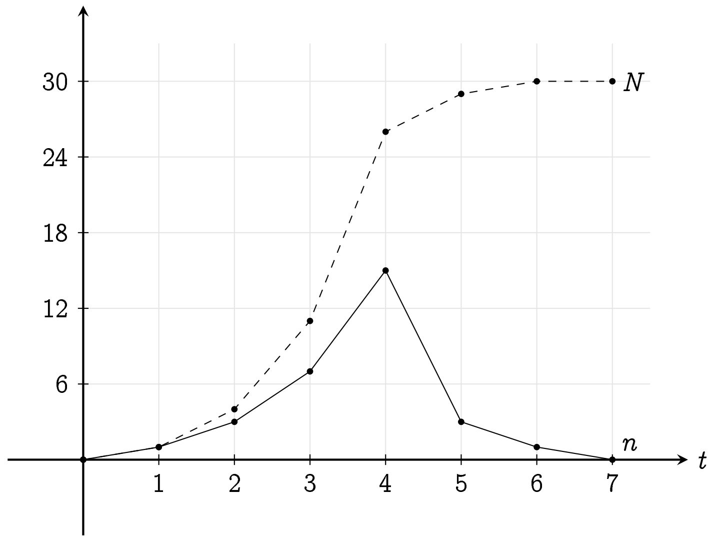


Figure 4.3: Newly infected people and partial sum graph

earlier: $26 = 1 + 3 + 7 + 15$. That number is the area that's under the graph of the derivative. We see that, an integral *is* the cumulative effect of the derivative[3].

4.4. The fundamental theorem of calculus

(Ch. 4.2) and (Ch. 4.3) leads to the final conclusion: **the fundamental theorem of calculus**. It's actually two theorems which links derivative with integrals; slopes with areas under the graph.

From (Fig. 4.4), let there be a function $A(x)$ which outputs the total area under the graph $f(x)$ from 0 to x ,i.e., $\int_0^x f(x)\,dx = A(x)$. The actual area under a graph from a certain x to $x+dx$ is $A(x+dx)-A(x)$. The area approximated by a strip shown in the figure is $f(x)\,dx$. The two areas are related by (Eq. 4.3). The approximation sign can be removed by adding a correction term ϵ (Eq. 4.4).

$$A(x + dx) - A(x) \approx f(x)\,dx \tag{4.3}$$

$$A(x + dx) - A(x) = f(x)\,dx + \epsilon \tag{4.4}$$

[3]This example is a discrete example: the timestep is in discrete values. Technically, this requires discrete calculus, discussed in (Ch. 17), but it still retains its core concept: changes and cumulative effects.

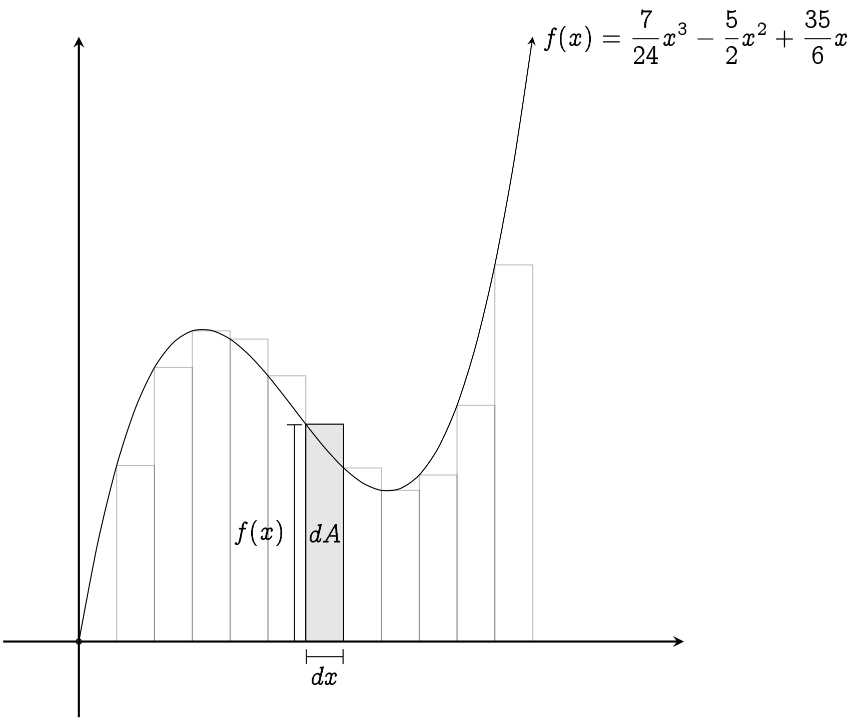


Figure 4.4: The geometrical interpretation of the fundamental theorem of calculus

If we let dx diminish, the ϵ also diminishes and is negligible; therefore, (Eq. 4.4) simplifies to (Eq. 4.5). Note the limit that's added into the equation.

$$\lim_{dx \to 0} A(x + dx) - A(x) = \lim_{dx \to 0} f(x)\,dx \tag{4.5}$$

Rearranging the equation into its final form, (Eq. 4.6). Notice that the right hand side actually has the same form as the derivative (Eq. 3.4). Since the left hand side is the function, we can conclude that the derivative of the area of the function is in fact, the function itself. That proves that derivative and integrals are inverses of each other, and that is why integrals are sometimes called **antiderivatives**.

$$\begin{gathered} \lim_{dx \to 0} A(x + dx) - A(x) = f(x) \lim_{dx \to 0} dx \\ f(x) = \lim_{dx \to 0} \frac{A(x + dx) - A(x)}{dx} = \frac{dA}{dx} \end{gathered} \tag{4.6}$$

This also enables us to calculate the integral from any value, not just zero.

$$\int_a^b f(x)\,dx = \int_0^b f(x)\,dx - \int_0^a f(x)\,dx = A(b) - A(a) \tag{4.7}$$

4.5. The area of a circle

All intuitions of integrals have been built. In this chapter, I'd like to reinforce the versatility of integrals and develop the spirit of mathematics. With careful reasonings, it's possible to turn a hard problem into an easy one. We turn ourselves to a problem that seems unrelated to integrals: finding the area of a circle.

For communication purpose, a circle is a set of all points that has the same distance from the origin.[4] The line that's formed by all those points is called the **circumference** of the circle. There are three main components of a circle:

1. **The radius** (R): a line from the origin to any point on the circumference
2. **The diameter** (d): a line from one point on the circumference to another with passing the origin
3. **The tangent** (T): a line outside the circle that intersects the circle *once*

With that out of the way, pause and think how you can find the area of a circle. We know how to find the area of rectangles, triangles, and simple geometric shapes. If we could dissect the circle into simple geometric shapes, that would be sublime. What's the simplest way to dissect a circle? Dissect it into multiple rings! (Fig. 4.6)

If you think about it, it might make our problem worse. We don't know the area of the circle yet, but we're trying to find the area of a ring. However, these rings can actually be turned into simple geometric shape: the rectangle.

[4] You could say that a circle is a **locus** of points that has the same distance from the origin, but there's practically no use to use the locus definition.

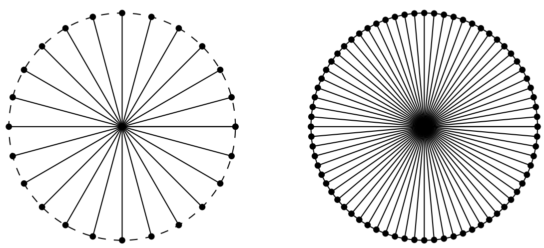


Figure 4.5: The definition of a circle

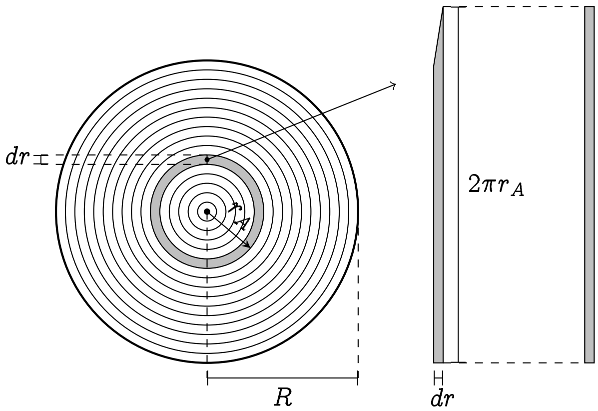


Figure 4.6: Dissecting a circle into rings with various sizes

Imagine a thick ring paper ring. You can't really stretch it without ripping. However, a thin ring is much easier to stretch. I encourage you to grab a sheet of paper, cut a ring, then try stretching it.

(Fig. 4.6) dissects the circle into multiple rings. To stretch the ring into a rectangle, we need a really small ring. Since we're in calculus, we have a notation for that: let each ring has the width dr. The ring is then stretched out into a rectangle. However, it's not *quite* a rectangle; it's a bit distorted. Recall from (Ch. 4.4). The same concept of approximation and taking the limit as dr goes to 0 can be applied here. For any strip r_A away from the origin, if $dr \to 0$, the strip can be approximated with a rectangle width dr. Since it's stretched from a circle, it's height must be the same as the circle's circumference: $2\pi r_A$.

Currently, these little rectangles are everywhere. Then, how to sum them all up? At first sight, it might seem impossible. Since $dr \to 0$, there are infinitely many little rectangles. There's actually a neat way to do this. In (Fig. 4.7), all the rectangles are put on a graph. It's possible to draw a line governed by $f(r) = 2\pi r$ to represent the height of all these rectangles. And if $dr \to 0$, the shape of all these rectangles combined actually becomes a triangle under the graph.

We've turned the area of the circle is the same as the area under the graph. Well that graph, it looks exactly like a triangle. The area of a triangle is $\frac{1}{2} \times$ Base $\times$ Height. In this case, it turns into (Eq. 4.8) which is the area of the circle.

$$\int_0^R f(r)dr = \frac{1}{2} \times (\pi R) \times (R) = \pi R^2 \tag{4.8}$$

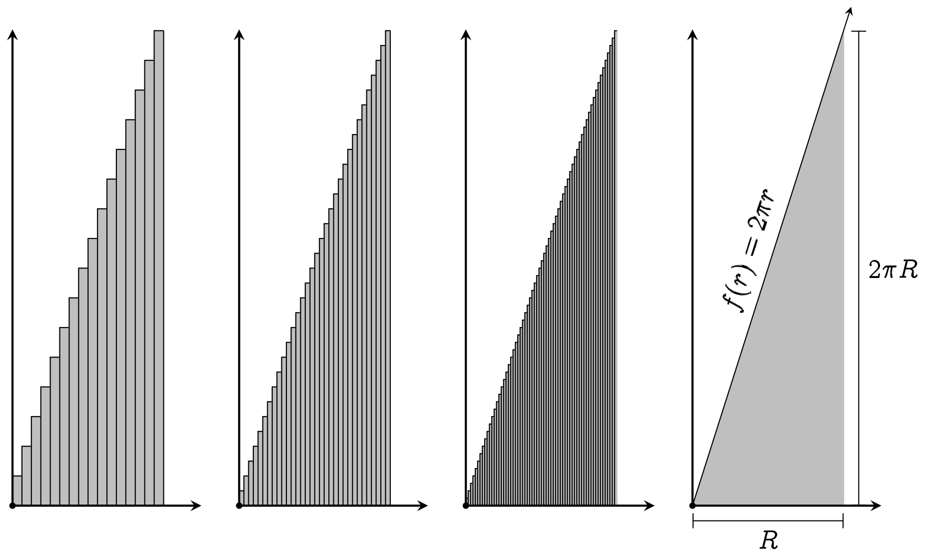


Figure 4.7: Rectangles by circle area on a graph

4.6. The spirit of mathematics

I'd like to end off this chapter by mentioning the spirit of mathematics. Sometimes, you can't solve the problems directly, e.g., the circle example. Most of the time, you have to re-frame the problem into another more-solvable problem. Problems like these often have the most sublime connections to be foundations of mathematics. This is a common theme in most of mathematics, especially calculus. So, be sure to keep this in mind while reading through this textbook.

Conclusion for Chapter 4

1. Riemann sum can be used to approximate the area under the graph of a function by slicing the area into little rectangles and summing it.

2. Integrals are functions which finds the area under the graph of other functions.

3. Integrals can be written as Riemann sum that uses infinitely many rectangles: the rectangles width approaches zero.

4. Integrals and derivatives are inverses of each other: integrals are cumulative effects of derivative. They're related by the fundamental theorem of calculus.

5. Integrals can be used in various ways by reframing questions and using the spirit of mathematics.

Intermission

Now you've learnt the basic concepts of calculus. I'd recommend you to go back to (Ch. 1) again to gain further understanding on how the subject were developed.

Chapter 5

Basic derivatives, integrals and differential equations

Abstract

This chapter focuses on calculating derivatives and integrals. One thing to keep in mind. Derivatives have rules of differentiation, but integrals doesn't. To integrate something an art of mathematical manipulation. The techniques used in integrations are further discussed in (Ch. 9). Also, keep in mind the actual meaning of these expressions to gain further intuition.

Binomial theorems (Appendix D) are going to be used in this chapter aswell. Make sure to familiarize yourself with it.

5.1. The trivial rules

5.1.1. The equity rule

As seen in (Ch. 3), derivatives can be interpreted as a function that takes in other functions as input. If two function are equal to each other, its derivative must also be equal to each other[1]. Note, this doesn't work the other way around because

[1] I can't find a good name for this property. I tried, trust me.

the integration constant (discussed in (Ch. 5.3)). This rule is used in implicit differentiation (Ch. 5.9).

$$f = g \implies \frac{df}{dt} = \frac{dg}{dt} \tag{5.1}$$

This idea also works for integrals:

$$f = g \implies \int f = \int g \tag{5.2}$$

5.1.2. The derivative of a constant

A constant stays constant; therefore, the derivative of any constant is 0.

$$\frac{d}{dx}(c) = 0 \tag{5.3}$$

5.1.3. The sum rule

The rate of change of a function that came from the sum of two functions is equal to the rate of change of both functions combined (Eq. 5.4).

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \tag{5.4}$$

This idea should be pretty easy to illustrate using limits. Convert the derivatives to the limit form. It shouldn't be too hard to interpret this rule geometrically. It should be left off as a thought exercise.

$$\begin{aligned} \frac{d}{dx}(f(x) + g(x)) &= \lim_{h\to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \\ &= \lim_{h\to 0} \left(\frac{f(x) - f(x+h)}{h} + \frac{g(x) - g(x+h)}{h} \right) \\ &= \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \end{aligned}$$

This idea also works for integrals. The sum of the area under the graph two function should be equal to the sum of the area under the graph when those two functions are combined.

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx \tag{5.5}$$

5.1.4. The constant multiple rule

The rate of change of "a function multiplied by a constant" is the rate of change of that function mutliplied by that constant (Eq. 5.6). Also, the area under the graph of "a function multiplied by a constant" is equal to the "area under the graph of that function" multiplied by that constant (Eq. 5.7)[2]

$$\frac{d}{dx}(cf(x)) = c \cdot \frac{d}{dx}(f(x)) \tag{5.6}$$

$$\int c \cdot f(x)dx = c \int f(x)dx \tag{5.7}$$

5.2. The power rule: derivatives and antiderivatives of polynomials

We then proceed to the most comprehensive family of functions: polynomials. As a recap, they're a family of functions that has the form $a_0 + a_1x + a_2x^2 + \ldots$. It's possible to breakdown polynomials into multiple monomials (Eq. 5.8). Thus, if we know the derivative of x^n, we also know the derivative of the entire family. But first, some geometric intuition must be built. Start simple: $f(x) = y = x^2$

$$\begin{aligned}\frac{d}{dx}(a_0 + a_1x + a_2x^2 + \ldots) &= \frac{d}{dx}(a_0) + \frac{d}{dx}(a_1x) + \frac{d}{dx}(a_2x^2) + \ldots \\ &= a_1 + a_2\frac{d}{dx}(x^2) + a_3\frac{d}{dx}(x^3) + \ldots\end{aligned} \tag{5.8}$$

Try solving dx^2/dx geometrically. Take a square with sidelength x. Then, increase its side length by dx. How much area has changed in proportion with dx?

According to (Fig. 5.1), the derivative of x^2 is the ratio between *the change of area after its side length has been increased by a small amount* dx and dx. It turns out to be (Eq. 5.9). When simplified there's a dx leftover. It can be safely left out[3] because it's is going to approach 0, leaving us with (Eq. 5.10): the derivative of x^2.

$$\frac{d}{dx}x^2 = \frac{x \cdot dx + x \cdot dx + dx^2}{dx^2}$$

[2]For the proofreader, do not correct this. This is solely for the meme.

[3]The same concept as what we've done to obtain (Eq. 3.1) in (Ch. 3.2)

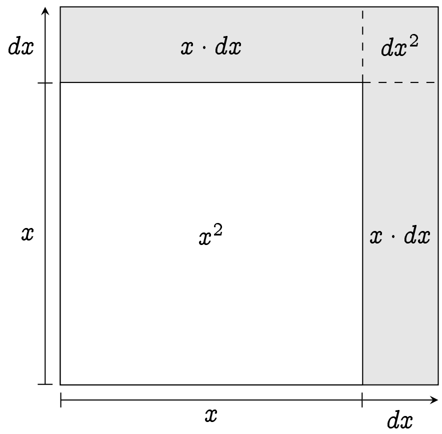


Figure 5.1: The rate of change illustration: $y = x^2$

$$= \frac{2x \cdot dx + dx^2}{dx} \tag{5.9}$$

$$= 2x + dx$$

$$= 2x \tag{5.10}$$

The dx that we left off came from the little square dx^2. When $dx \to 0$, the area of that square will be negligible. In conclusion, it's safe to leave it off.

The derivative that we got geometrically also matches with the calculation. The geometrical interpretation of the derivative of x^3 shall be a thought exercise. The answer should be equal to $3x^2$.

$$\begin{aligned}
\frac{d}{dx}(x^2) &= \lim_{h\to 0}\left(\frac{(x+h)^2 - x^2}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{2xh + h^2}{h}\right)\\
&= \lim_{h\to 0}(2x + h)\\
&= 2x
\end{aligned}$$

The intuition for the derivative of simple polynomials have been built (x^2 and x^3). However, that still doesn't cover the whole family of polynomial functions. How's about x^4, or x^5? or x^n. We don't have the glory of higher dimensions in our world.

Thus, we turn ourselves to the mighty definition of derivatives.

$$\frac{d}{dx}(x^n) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

We then use the binomial expansion (Appendix D) to expand the $(x+h)^n$ out

$$= \lim_{h\to 0} \frac{\left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k} \cdot h^n\right) - x^n}{h}$$
$$= \lim_{h\to 0} \frac{\left(\binom{n}{0} x^n h^0 + \binom{n}{1} x^{n-1} h^1 + \binom{n}{2} x^{n-2} h^2 + \dots \binom{n}{n} h^n\right) - x^n}{h}$$
$$= \lim_{h\to 0} \cancel{\frac{x^n}{h}} + nx^{n-1} + \cancel{\left(\binom{n}{2} x^{n-2} h^1 + \dots + 1 \cdot h^{n-1}\right)} - \cancel{\frac{x^n}{h}}$$

Rearrange the terms, then notice all terms in the parenthesis have at least one h in it. When $x \to 0$, all these terms will approach 0 and become negligible. They can be left out safely.

$$= nx^{n-1} \tag{5.11}$$

At the end we're left with (Eq. 5.11): the **power rule**. Taking derivatives of polynomials becomes trivial. E.g., $\frac{d}{dx}(x^2 + 2x + 1)$ is $2x + 2$. However, the proof for this uses the binomial theorem which only works for integer n[4]. What's about the real numbers? In the end, (Ch. 5.8) will redefinie the power rule and generalize it further to any number n.

Comments:

i.) Technically, it's possible to create a geometrical interpretation of derivatives of polynomial higher than the third degree. Hypercube, penteract, hexeract, and so on. However, that'd be too cumbersome to visualize with a simple paper diagram.

ii.) Higher dimensional visualization will be discussed in (Ch. 15)

Reversing the power rule should also be easy. You're asking "*What function when taken derivative is equal to* x^n" or, "*What function has the area under graph equal to* x^n." The following is the thought process of how to derive this.

[4]Technically, with the complex extension, binomial theroem works for every integer n

Start with simple, we know that the power n must decrease by one. The derivative of x^{n+1} is $(n+1)x^n$. Then, get rid of the $n+1$ by dividing it with $n+1$ giving us (Eq. 5.12): the **reversed power rule.**

$$\int x^n dx = \frac{x^{n+1}}{n+1} \tag{5.12}$$

However, the reversed power rule does not work for $n = -1$. If $n = -1$, the denominator is 0, and you cannot divide by zero. This does not mean that the integral of x^{-1} does not exist. The power rule and the reversed power rule is rule, not a definition. It's used to quickly calculate the derivatives and antiderivatives of polynomials. The definition is given by the fundamental theorem of calculus. The answer to the antiderivative of x^{-1} is in (Eq. 5.23) and will be discussed later on.

5.3. Motion of a falling object and the integral constant

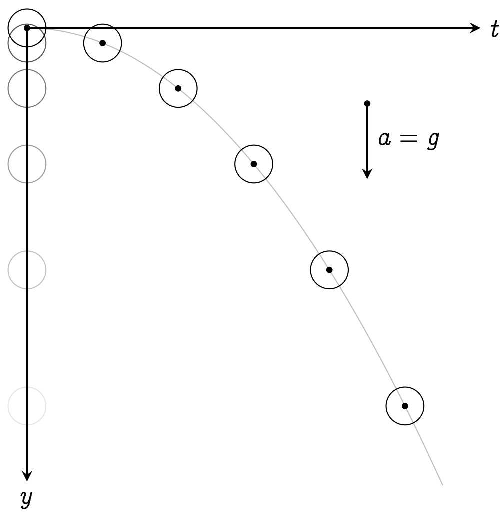


Figure 5.2: Motion of a falling object

Here is an example of where the power rule is used in physics. I shall also use this to introduce the integral constant.

Assume a falling mass. Gravity accelerates the mass downward with acceleration g. Our goal is to find the equation of motion for this object at any time t: $y = f(t)$.

We start with what we know: the acceleration of this object is g. The second derivative of the position with respect through time is g. We then create an equation to describe, or model this.

$$\frac{d^2}{dt^2}y = g \text{ or } \frac{d}{dt}\left(\frac{d}{dt}y\right) = g \tag{5.13}$$

(Eq. 5.13) is our equation of motion. We're trying to find a function $y(t)$ which satisfies the equation which contains the derivative. We call these types of equation "Differential equations".

We can move a dt to the right and integrate both sides twice, giving us the result: $y(t) = gt^2/2$ Notice the axis, I point $y+$ downwards.

$$\begin{aligned}
\int d\left(\frac{d}{dt}y(t)\right) &= \int g dt \\
\frac{d}{dt}y(t) &= g\int dt \\
\frac{dy(t)}{dt} &= gt \\
\int dy(t) &= \int gt\,dt \\
y(t) &= g\int t\,dt = gt^2/2
\end{aligned} \tag{5.14}$$

If we take the derivative of $y(t) = gt^2/2$ twice, we get g back. However, something is lost. If the whole function $y(t)$ is shifted down by 1, it still has the same derivative: $y(t) = gt^2/2$ and $y(t) = gt^2/2 + 1$ have the same derivative. Information about the original function is lost in the derivative process. Since integrals are supposed to be reversing derivatives, we have to add this piece of information back. Thus, after evaluating the integral, we must add a constant term $+\mathrm{C}$: the **integral constant**. Acknowledging that, we do the integral again:

$$\begin{aligned}
\int d\left(\frac{d}{dt}y(t)\right) &= \int g dt \\
\frac{d}{dt}y(t) &= gt + \mathrm{C}_1 \\
\int dy(t) &= \int (gt + \mathrm{C}_1)dt \\
y(t) &= \frac{1}{2}gt^2 + \mathrm{C}_1 t + \mathrm{C}_2
\end{aligned} \tag{5.15}$$

(Eq. 5.15) is the full form of our solution. The $+C_1 t$ represents the initial velocity of the mass, and $+C_2$ the position. The information that is lost in (Eq. 5.14) has been added in. In conclusion, whenever you're doing integrals, don't forget to add the integral constant $+C$.

5.4. Escape velocity: the chain rule

The chain rule (Eq. 5.16) from all of my calculus knowledge distilled, is "*The art of multiplying by one*". Its simplicity holds one of the greatest power of calculus.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{5.16}$$

It holds the intuition to how rate of changes are combined. E.g., the cheetah's speed is 10 times the bicycle's speed. The bicycle's speed is 4 times the walking speed. The ratio between the speed of the cheetah compared to the walking speed would obviously be $10 \cdot 4$ which is 40:

$$\frac{d\text{Cheetah}}{d\text{Walking}} = \frac{d\text{Cheetah}}{d\text{Bicycle}} \cdot \frac{d\text{Bicycle}}{d\text{Walking}}$$

We use this rule to simplify or find a derivative with composite functions. It adds an intermediate step. We can also rewrite it as

$$\frac{d}{dx} f(g(x)) = \frac{d}{dg(x)} f(g(x)) \cdot \frac{d}{dx} g(x)$$

Example 5.4.1: $\frac{d}{dx}(x^3 + 2x + 1)^{69}$

ne could expand out all that and use the power rule. But we're definitely not doing that. We turn ourselves to the chain rule. Let $u = x^3 + 2x + 1$

$$\frac{d}{dx}\left(x^3 + 2x + 1\right)^{69} = \frac{d}{dx} u^{69}$$

Then, multiply everything by 1, or du/du.

$$\begin{aligned} \frac{d}{dx} u^{69} &= \frac{d}{du} u^{69} \cdot \frac{du}{dx} \\ &= 68(x^3 + 2x + 1)^{69} \frac{d}{dx}(x^3 + 2x + 1) \end{aligned}$$

$$= 68(x^3 + 2x + 1)^{69} \cdot (3x^2 + 2)$$

To illustrate the usage of chain rule even more, let's consider the problem of escape velocity. This is a very beautiful case where the chain rule is used to change variables around into something solvable.

Every mass has gravity. Gravity is a force that attracts other masses towards itself. Earth is a huge mass; Earth's gravity pulls you toward the ground. If one wants to escape the Earth, one must move fast. The minimum speed required to escape gravity is called **the escape velocity**. We want $\vec{v}$ it to be a function of r, the distance away from the Earth's radius. The goal is, find $\vec{v}(r)$ that outputs the velocity required at r away from the Earth.

From (Fig. 5.3) Earth's gravity is given by the Newton's law of gravitation:

$$\vec{F_E} = G\frac{Mm}{r^2}$$

To combat gravity, the rocket must accelerate outwards. The force is given by the rocket's acceleration.

$$\vec{F_r} = m\vec{a}$$

To not accelerate down into Earth, $\vec{F_r}$ must be greater than $\vec{F_E}$. If you'd like to find the minimum velocity, i.e., no acceleration, let $\vec{F_r} = \vec{F_E}$.[5]

$$\cancel{m}\vec{a} = G\frac{M\cancel{m}}{r^2}$$

[5]No acceleration does not mean velocity is zero. No acceleration means no change in velocity.

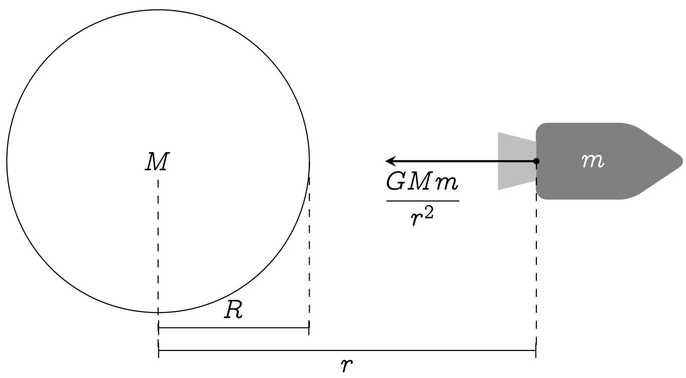


Figure 5.3: Escape velocity of a rocket

$$\vec{a} = G\frac{M}{r^2} \tag{5.17}$$

Acceleration is the derivative of the velocity

$$\frac{d\vec{v}}{dt} = G\frac{M}{r^2}$$

The L.H.S. is dependent on t, but the R.H.S. is dependent on r. Then, is there any way to express the dt in terms of r? The chain rule comes into play:

$$\frac{d\vec{v}}{dr} \cdot \frac{dr}{dt} = G\frac{M}{r^2}$$

Recall that dr/dt is $\vec{v}$. Substitute that in, then isolate variables. Now, we can solve the differential equation.

$$\begin{aligned} G\frac{M}{r^2} &= \frac{d\vec{v}}{dr} \cdot \vec{v} \\ \int G\frac{M}{r^2}\, dr &= \int \vec{v}\, d\vec{v} \\ -\frac{GM}{r} + \mathrm{C} &= \frac{\vec{v}^2}{2} \\ \vec{v}(r) &= \sqrt{\frac{2GM}{r} + \mathrm{C}} \end{aligned} \tag{5.18}$$

And finally, we've arrived at (Eq. 5.18), the escape velocity at any distance r away from the Earth. The $+\mathrm{C}$ represents the initial velocity of the rocket at $r = R$. In a realistic scenario, this term equals to zero because the rocket have to start at $\vec{v} = 0$.

5.5. Bank interests: exponentials and the Euler's number

From here, we turn our attention to the most important functions in calculus: exponentials. They're a family of functions in the form $c \cdot a^x$. Growths can also be modelled using exponentials, these growths are called exponential growths. To illustrate exponential growths, I'll use the problem of bank interests.

There are two types of interests. 1.) Fixed interest is the type of interest you don't really want. If you have $1000 with 3% annual interests, you'll get $30 every year. It's very trivial to calculate the total money at year n: $1000 + (3\% \cdot n)$. It doesn't seem that bad, until you consider compound interest.

2.) Compound interest calculate the interest based on how much there is before the interest is calculated originally. On the first year, you have \$1000. The second year it becomes $\$1000 + (3\% \cdot \$1000) = \$1030$. The third year; take the amount from the second year and calculate the interest: $\$1030 + (3\% \cdot \$1030) = \$1060.9$ and so on and so forth. For the amount of money calculated till year ten, check (Table 5.1). The formula can also be easily formalized by induction: $a_n = \$1000 \cdot (1 + 3\%)^{n-1}$. But what if the interest is halved, but you get it two times more often?

Comments:

i.) The interest that's calculated in (Table 5.1) is discrete. When we're in calculus, we don't use discrete intervals, we use an infinitesimal interval *dt*.

For the sake of generalization, let the interest rate be 100%, the money doubles every unit interval and start with a unit money, \$1. If the interest is halved but it's calculated twice, at year two, you'll have \$2.25. If the interest is reduced by $1/3$, at year two, you'll have \$2.37. This number seems to be increasing ever. Will it ever converge?

The questions (Eq. 5.19). This expression actually has a limit: $2.71\ldots$ This is what we called the number e. It's a constant of growth, or what we call the "Euler

Year	Interest	Total	Change of interest
1	-	\$1000.00	-
2	$\$1000.00 \cdot 3\% = \$30.00$	\$1030.00	-
3	$\$1030.00 \cdot 3\% = \$30.90$	\$1060.90	\$0.90
4	$\$1060.90 \cdot 3\% = \$31.83$	\$1092.73	\$0.93
5	$\$1092.73 \cdot 3\% = \$32.78$	\$1125.51	\$0.95
6	$\$1125.51 \cdot 3\% = \$33.77$	\$1159.27	\$0.99
7	$\$1159.27 \cdot 3\% = \$34.78$	\$1194.05	\$1.01
8	$\$1194.05 \cdot 3\% = \$35.82$	\$1229.87	\$1.04
9	$\$1229.87 \cdot 3\% = \$36.89$	\$1266.77	\$1.07
10	$\$1304.77 \cdot 3\% = \$38.00$	\$1304.77	\$1.11

Table 5.1: 10 years 3% compound interest starting with \$1000

constant", or the natural growth constant.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e \tag{5.19}$$

Most of the time in calculus, we're interested about the growth of a function, aka., the derivative of a function. In this case, the derivative of an exponential function: how it grows over time.

The function that's modelled using this constant has a special property: it's rate of change is the function itself! (Eq. 5.20)

$$\frac{d}{dx}e^x = e^x \tag{5.20}$$

This is because of how we've derived this constant. To build the intuition for this, start with any exponential function and use discrete changes in time. Plotting $y = f(t) = 2^t$ in (Table 5.2a), we quickly see that the *discrete* rate of change of the function is actually itself but shifted to the right. This is because the function is self-referential: the rate of change of the function came from the value of the original function.

Try doing this for other bases, e.g., 3^x (Table 5.2b), or 4^x (Table 5.2c). Δf might seems weird at first, but upon closer inspection, it's actually itself times a constant. For 3^x, $\Delta f = \frac{2}{3}\cdot 3^x$, for 4^x; $\Delta f = \frac{3}{4}\cdot 4^x$. Seems like 2 is the magic number where the discrete rate of change of 2^x is itself. The constant is 1.

t_n	$f(t_n)$	Δf
0	1	0.5
1	2	1
2	4	2
3	8	4
4	16	8
5	32	16
6	64	32
7	128	64
8	256	128
9	512	256
10	1024	512

(a) $f(t) = 2^t$

t_n	$f(t_n)$	Δf
0	1	0.6
1	3	2
2	9	6
3	27	18
4	81	54
5	243	162
6	729	486
7	2187	1458
8	6561	4374
9	19683	13122
10	59049	39366

(b) $f(t) = 3^t$

t_n	$f(t_n)$	Δf
0	1	0.75
1	4	3
2	16	12
3	64	48
4	256	192
5	1024	768
6	4096	3072
7	16384	12288
8	65536	49152
9	262144	196608
10	1048576	786432

(c) $f(t) = 4^t$

Table 5.2: Plotting exponential functions with different bases and its discrete rate of change.

In conclusion, this is one of the defining features of exponents. Its rate of change is itself! But would that still hold true using an infinitesimal change dt? We turn to the definition of derivative:

$$
\begin{aligned}
\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} \\
&= \lim_{h\to 0}\frac{a^x a^h - a^x}{h} \\
&= a^x\left(\lim_{h\to 0}\frac{a^h-1}{h}\right)
\end{aligned}
$$

The term a^x actually comes out in the exponents like what we've expected. But now, we actually have a formula for the constant multiple: $(a^h - 1)/h$ where $h \to 0$.

It should be expected that this term actually converges to something. For 2^x, it's about 0.69, for 3^x it's about 1.09. The obvious question would be: is there a base that that constant is exactly 1. If we could find that number, we would have a function that its derivative is itself and that'd be extremely useful because it'd be the key to so much more.

That base turns out to be the Euler's constant e! because it's the constant of continuous rate of change, defined in (Eq. 5.19), giving (Eq. 5.20). If we move the dx and take the integral on both sides, we also see that:

$$\int e^x \, dx = e^x + \mathrm{C} \tag{5.21}$$

This also means that we've found the equation that the slope is e^x, and the area from 0 to x is e^x. It's usefulness will show up later on and is very prominent throughout calculus.

5.6. Logarithms

Logarithms are inverses of exponentiation. In order to gain more intuition, I shall compare it with a simple language: roots.

Roots try to find the base given the exponents:

$$x^a : \sqrt[a]{x^a} = x$$

but, logarithms try to find the exponents, given the base:

$$a^x : \log_a a^x = x$$

In a similar fashion to exponents, logarithms has some properties that's listed in (Table 5.3). All of which should be relatively trivial to prove.

Since we're in calculus, we're interested in the rate of change of this function; especially, logarithms of base e: abbreviated as $\ln(x)$.[6] Start with the assumption from (Ch. 5.5): the derivative of e^x equals e^x.

Let $y = \ln(x)$. The derivative of $\ln(x)$ is now dy/dx. Since logarithms and exponentials are inverses of each other, exponentiate both sides of the equation.

$$\begin{aligned} y &= \ln(x) \\ e^y &= x \end{aligned}$$

Then, use the property from (Ch. 5.1.1). Differentiate both sides with respect to y. Notice that on the right hand side we're left with dx/dy. That is the reciprocal of dy/dx; therefore, we take the reciprocal of both sides to get the answer.

$$\begin{aligned} \frac{de^y}{dy} &= \frac{dx}{dy} \\ e^y &= \frac{dx}{dy} \\ \frac{1}{e^y} &= \frac{dy}{dx} \end{aligned}$$

Substituting $y = \ln(x)$, we then get the derivative of the natural logarithm: (Eq. 5.22).

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{e^{\ln(x)}} \\ \frac{d}{dx}\ln(x) &= \frac{1}{x} \end{aligned} \tag{5.22}$$

It also means that:

$$\int \frac{1}{x} dx = \ln(x) + C \tag{5.23}$$

[6] $\log_e(x) = \ln(x)$ It's read as "Natural log of x"

Product property	$\log_a(m \cdot n) = \log_a(m) + \log_a(n)$
Quotient property	$\log_a(m/n) = \log(m) - \log(n)$
Power property	$\log_a(m^n) = n \log_a(m)$

Table 5.3: Properties of logarithms

Comments:

i.) Actually, the indefinite integral of $1/x$ isn't actually just $\ln(x) + C$. It's actually a piecewise function with two integral constants. This topic is discussed further in (Appendix E). But for now, this is enough to proceed.

With the power of logarithms and the chain rule, it also enables us to differentiate any exponential functions a^x:

$$\begin{aligned}\frac{d}{dx}a^x &= \frac{d}{dx}e^{\ln(a^x)}\\ &= \frac{d}{dx}e^{x\ln(a)}\\ &= \frac{d}{du}e^u\frac{d}{dx}x\ln(a)\\ &= e^{x\ln(a)}\ln(a)\\ &= a^x\ln(a)\end{aligned}$$

By using the equity rule, it's also possible to differentiate logarithms of any bases.

$$\begin{aligned}\log_a(x) &= y\\ x &= a^y\\ \ln(x) &= y\ln(a)\\ \frac{d\ln(x)}{dx} &= \ln(a)\frac{dy}{dx}\\ \frac{d\log_a(x)}{dx} &= \frac{1}{x\ln(a)}\end{aligned}$$

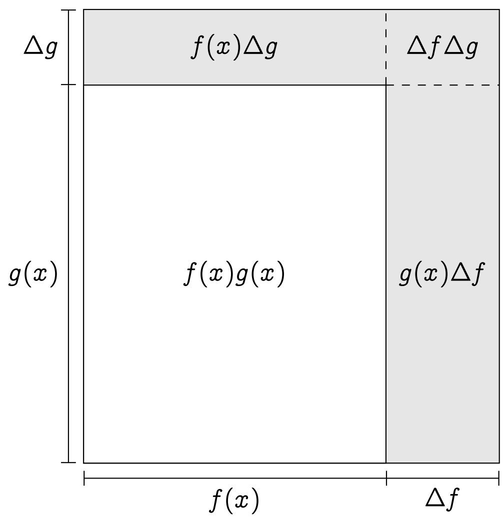


Figure 5.4: The geometrical interpretation of the product rule

5.7. The product rule and the quotient rule

Most of the time, we're required to differentiate two functions that's multiplied together, $f(x)g(x)$. It might not be prominent yet, but this set of rules will be used quite often later on.

The geometrical interpretation of this product rule is very similar to the one discussed in (Ch. 5.2) Let there be a rectangle with side length $f(x)$ and $g(x)$ (Fig. 5.4). Our objective is to find the change in area if x is increased by dx. I would also like to point out that these two functions doesn't have to change at the same rate.

The change in area is $f(x)\Delta g + g(x)\Delta f + \Delta f \Delta g$. The $\Delta f \Delta g$ is negligible and can be dropped out. We then expand out the Δf and Δg into $f(x + dx) - f(x)$ and $g(x + dx) - g(x)$. The final result is the product rule (Eq. 5.24).

$$\begin{aligned}
\frac{d}{dx} f(x)g(x) &= \lim_{dx \to} \frac{f(x)(g(x+dx) - g(x)) + g(x)(f(x+dx) - f(x))}{dx} \\
&= \lim_{dx \to 0} \frac{f(x)(g(x+dx) - g(x))}{dx} + \lim_{dx \to 0} \frac{g(x)(f(x+dx) - f(x))}{dx} \\
&= f(x) \lim_{dx \to 0} \frac{g(x+dx) - g(x)}{dx} + \lim_{dx \to 0} \frac{f(x+dx) - f(x)}{dx} \\
&= f(x) \frac{d}{dx} g(x) + g(x) \frac{d}{dx} f(x) \qquad (5.24)
\end{aligned}$$

The quotient rule on the other hand deals with the derivative of $f(x)/g(x)$. Plug in $1/g(x)$ instead of $g(x)$, then you'll get the quotient rule (Eq. 5.25).

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{g(x)^2} \qquad (5.25)$$

But what about the derivative of $f(x)g(x)h(x)$? You could use either the geometrical interpretation or the method of increment. However, it's too cumbersome. We'd need a better way. We then turn ourselves to one of the properties of logarithms: $\ln(ab) = \ln(a) + \ln(b)$.

Logarithms can separate products into sums. Since logarithms and exponents are inverses of each other, we can place the product inside logarithms.

$$\frac{d}{dx} f_1(x) f_2(x) \ldots f_n(x) = \frac{d}{dx} e^{\ln(f_1(x) f_2(x) \ldots f_n(x))}$$

$$= \frac{d}{dx} e^{\ln(f_1(x)) + \ln(f_2(x)) + \cdots + \ln(f_n(x))}$$

Let $u = \ln(f_1(x)) + \ln(f_2(x)) + \cdots + \ln(f_n(x))$. Then, use the chain rule.

$$\begin{aligned}
&= \frac{d}{du} e^u \cdot \frac{du}{dx} \\
&= e^{\ln(f_1(x)) + \ln(f_2(x)) + \cdots + \ln(f_n(x))} \cdot \frac{d}{dx}(\ln(f_1(x)) + \ln(f_2(x)) + \cdots + \ln(f_n(x))) \\
&= f_1(x) f_2(x) \dots f_n(x) \left(\frac{d}{dx} \ln(f_1(x)) + \frac{d}{dx} \ln(f_2(x)) + \cdots + \frac{d}{dx} \ln(f_n(x)) \right)
\end{aligned}$$

Then, use the chain rule again: $\frac{d}{dx} \ln(f_n(x)) = \frac{d}{du} \ln(u) \cdot \frac{du}{dx} = \frac{1}{f_n(x)} \cdot \frac{d}{dx} f_n(x)$. Substituting that in gives us the generalized product rule (Eq. 5.26)

$$\begin{aligned}
&f_1(x) f_2(x) \dots f_n(x) \left(\frac{d}{dx} \ln(f_1(x)) + \frac{d}{dx} \ln(f_2(x)) + \cdots + \frac{d}{dx} \ln(f_n(x)) \right) \\
&= f_1(x) f_2(x) \dots f_n(x) \left(\frac{1}{f_1(x)} \frac{d}{dx} f_1(x) + \frac{1}{f_2(x)} \frac{d}{dx} f_2(x) + \cdots + \frac{1}{f_n} \frac{d}{dx} f_n(x) \right)
\end{aligned} \tag{5.26}$$

5.8. Redefining the power rule

It's very tempting to apply the concept we used to derive the product rule to the power rule. If it's possible, we'd have the generalized power rule to every n.

$$\begin{aligned}
\frac{d}{dx} x^n &= \frac{d}{dx} e^{\ln(x^n)} \\
&= \frac{d}{dx} e^{n \ln x}
\end{aligned}$$

Let $u = n \ln x$. Then, use the chain rule.

$$\begin{aligned}
&= \frac{d}{du} e^u \cdot \frac{du}{dx} \\
&= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \\
&= x^{\cancel{n}^{n-1}} \cdot \left(n \cdot \frac{1}{\cancel{x}} \right) = n x^{n-1}
\end{aligned}$$

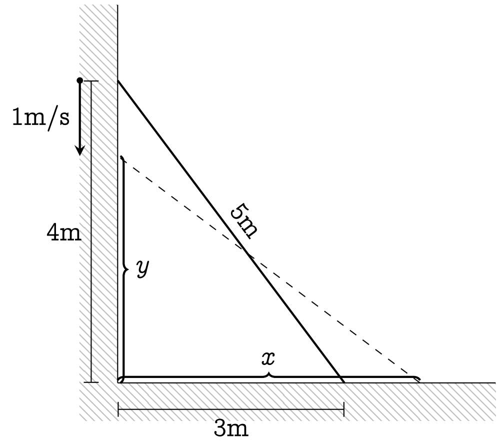


Figure 5.5: A sliding ladder

5.9. Implicit differentiation

This section is a peek into the world of multivariable calculus where one variable is related to another. I'd like to introduce this idea by the problem of the sliding ladder.

With the equity rule, it's possible to differentiate functions that can't be easily expressed with the form $f(x) = y$. Functions that aren't expressed in the form $f(x) = y$ are called "**implicit functions**".

A ladder has constant length, say 5m. As in (Fig. 5.5), on the y-axis, the ladder is sliding down with the speed of 1m/s. Our objective is to find the speed of sliding on the x-axis. This is the problems of related rates. One rate of change relates to the other. To rephrase, our objective is to find dx/dy.

From the pythagorean theroem (Assume everything is in meters and seconds):

$$x^2 + y^2 = 5$$

See what happens if we differentiate everything with respect to time, t:

$$\begin{aligned} \frac{d}{dt}\left(x^2 + y^2\right) &= \frac{d}{dt}(5) \\ \frac{dx^2}{dt} + \frac{dy^2}{dt} &= 0 \\ \frac{dx^2}{dt} &= -\frac{dy^2}{dt} \end{aligned} \tag{5.27}$$

We now have two derivatives, but we can't differentiate it yet. It's differentiable with respect to time. We then use the chain rule to solve this:

$$\begin{aligned}\frac{dx^2}{dx} \cdot \frac{dx}{dt} &= -\frac{dy^2}{dy} \cdot \frac{dy}{dt} \\ \cancel{2}x \cdot \frac{dx}{dt} &= -\cancel{2}y \cdot \frac{dy}{dt}\end{aligned}$$

As the Leibniz' interpretation of derivatives, the dts on both sides can cancel each other. We then isolate the derivative term to get our solution in (Eq. 5.28)

$$\begin{aligned}x \cdot dx &= -y \cdot dy \\ \frac{dx}{dy} &= \frac{-y}{x}\end{aligned} \tag{5.28}$$

Each step can be interpretted geometrically. One might asks why a zero appears in (Eq. 5.27). We've created a related rates equation. Notice, if one variable increases, the other must decrease. If the projected height increases, the projected length must decrease. The increase and decrease cancel each other out, and we see that in the final equation even. The change in x depending on y is the negative of its reciprocal.

5.10. But why is the integral of the reciprocal the natural logarithm?

From the writer's, the fact that the integral of $1/x$ is $\ln(x)$ is very weird. The pattern given by the inverse power rule (Eq. 5.12) works for every x. However, at $x = 0$, there's a logarithm out of nowhere. It seems baffling. Thus, I came up with a way to evaluate the integral for the integral by using the inverse power rule. Enjoy the transformation!

$$\begin{aligned}\int \frac{1}{x}\,dx &= \int \lim_{h\to 0}\left(\frac{1}{2}x^{-1+h} + \frac{1}{2}x^{-1-h}\right) \\ &= \lim_{h\to 0}\int \left(\frac{1}{2}x^{-1+h} + \frac{1}{2}x^{-1-h}\right) dx \\ &= \lim_{h\to 0}\left(\frac{1}{2}\frac{x^{-1+h+1}}{(-1+h+1)} + \frac{1}{2}\frac{x^{-1-h+1}}{(-1-h+1)}\right)\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0}\left(\frac{1}{2}\frac{x^h}{h} - \frac{1}{2}\frac{x^{-h}}{-h}\right) = \lim_{h\to 0}\left(\frac{1}{2}\frac{x^h}{h}\frac{x^h}{x^h} - \frac{1}{2}\frac{1}{hx^h}\right) \\
&= \lim_{h\to 0}\left(\frac{x^{2h}-1}{2hx^h}\right) = \lim_{h\to 0}\left(\frac{e^{2h\ln(x)}-1}{2h\ln(x)}\cdot\frac{\ln(x)}{x^h}\right) \\
&= \lim_{h\to 0}\left(\frac{e^{2h\ln(x)}}{2h\ln(x)}\right)\cdot\lim_{h\to 0}\left(\frac{\ln(x)}{x^h}\right) \\
&= \lim_{h\to 0}\left(\frac{e^{2h\ln(x)}}{2h\ln(x)}\right)\cdot\ln(x)
\end{aligned} \tag{5.29}
$$

Then, we have to evaluate the limit at the front. Let $u = 2h\ln(x)$. When $h \to 0$, $u \to 0$ aswell. Then, use the definition of constant e from (Eq. 5.19).

$$
\begin{aligned}
\lim_{h\to 0}\left(\frac{e^{2h\ln(x)}}{2h\ln(x)}\right) &= \lim_{u\to 0}\left(\frac{e^u-1}{u}\right) \\
&= \lim_{u\to 0}\left(\frac{\left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^u}{u}\right)
\end{aligned}
$$

Change from $n\to\infty$ into $n \to 0$. Notice, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to 0}\left(1+n\right)^{1/n}$. If $n \to 0$ and $u \to 0$, that means $n = u$. Substitute $n = u$ into the limit.

$$
\begin{aligned}
&= \lim_{u\to 0}\left(\frac{\left(\lim_{u\to 0}(1+u)^{1/u}\right)^u - 1}{u}\right) \\
&= \lim_{u\to 0}\left(\frac{1+u-1}{u}\right) \\
&= \lim_{u\to 0}\left(\frac{\cancel{u}}{\cancel{u}}\right) = \lim_{u\to 0}(1) \\
&= 1
\end{aligned}
$$

Then, substitute that back into (Eq. 5.29). Finally, we're left with the integral of $1/x$.

$$
\begin{aligned}
\lim_{h\to 0}\left(\frac{e^{2h\ln(x)}}{2h\ln(x)}\right)\cdot\ln(x) &= 1\cdot\ln(x) \\
&= \ln(x)
\end{aligned}
$$

Conclusion for Chapter 5

Formulas in Chapter 5

1. $x = y \implies \frac{d}{dt}x = \frac{d}{dt}y$
2. $\frac{d}{dx}x^n = nx^{n-1}$
3. $\int x^n dx = \frac{x^{n+1}}{n+1}; n \neq -1$
4. $\frac{d}{dx}e^x = e^x$: $\int e^x \, dx = e^x$
5. $\frac{d}{dx}a^x = a^x \ln(a)$
6. $\frac{d}{dx}\ln(x) = \frac{1}{x}$: $\int \frac{1}{x} = \ln(x)$

Chapter 6

The essence of trigonometry

Interlude I

Abstract

At this point along the book is where trigonometry breaches into calculus. Trigonometry is the study of triangles. Most of the readers probably have studied trigonometry before. This interlude is dedicated to the history and development of trigonometry; not the theorems of trigonometry. We'll not be discussing much of the theories. After this, the intuition gained here will be used in chapter 8.

This interlude is skippable if the reader already has all the foundations and *development* of trigonometry laid out, otherwise, I'd advise the reader to study this.

Part of this interlude is going to be shortened, e.g., the circle's circumference because it shares the same origin as calculus, written in chapter 1

6.1. The simplest shape: the circle

Back then, mathematics was not as we've seen. Mathematics was the study of two-dimensional shapes, as mentioned earlier in chapter 1. We shall start off this interlude by discussing the simplest shapes of all: the circle, defined in section 4.5 and also discussed in chapter 1. A circle is the simplest shape because it has no vertices, edges, and sides.

Now, is there a way to define a circle with only points?

People in the ancient times decided to divide the length of the circumference of the circle by its diameter. Turns out, for every circle in a flat two dimensional space, that ratio is π.

6.2. The right triangle

Chapter 7

Geometrical significants of calculus

Abstract

Here is where we explore more about calculus using various problems and examples.

7.1. Tangent and to a curve

7.2. Rocket equation: calculus and the rate of change

A rocket is a device that expel masses out at a certain rate to push itself forward. The problem here is to find the velocity of the rocket after a given time.

7.3. Snell's law: optimization problem, maxima and minima

Light travels at light speed, $c = 299\,792.458\,\mathrm{km\,s^{-1}}$. However, it does not always travel at this speed. Light travels with different speed in different substances aka. mediums. In a vacuum, light travels at c: $299\,792.458\,\mathrm{km\,s^{-1}}$. In water, light slows down to around $225000\mathrm{km\,s^{-1}}$; in diamonds, $125\,000\,\mathrm{km\,s^{-1}}$, and so on.

There's also law that governs how light travels: the snell's law. Light always travels between two points by using the shortest path possible.

What if, light have to travel from point a to point b that's in a different medium?

7.4. Newton's approximation

What's the value of $\sqrt{2}$?

7.5. The perimeter of a circle: arc length formula

In (Ch. 4), we've calculated the area of a circle assuming the perimeter of a circle radius r is $2\pi r$. But, where does that formula comes from? Integrals could probably help us derive that.

A circle is a set of points that's r unit away from the origin. (Fig. 7.1), take the pythagorean theorem.

7.6. Harmonic motion: the need of approximation of functions

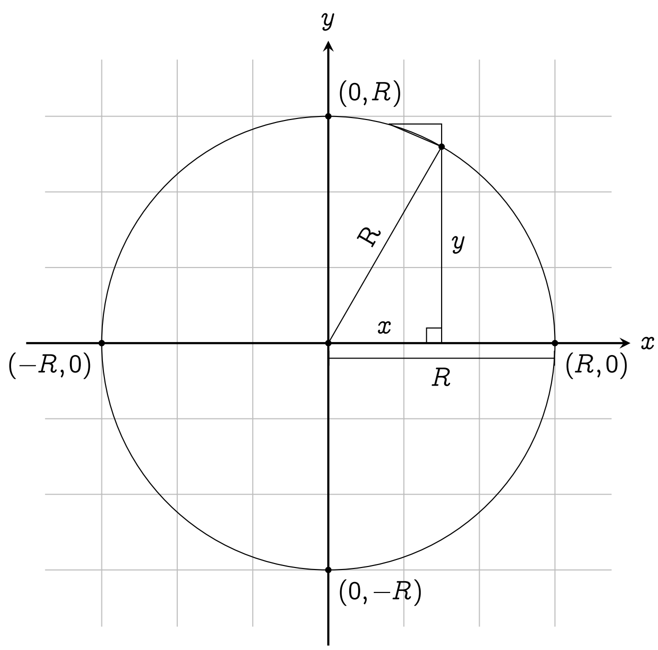


Figure 7.1: Relation between the circle and the pythagorean theorem.

Chapter 8

Calculus and trigonometry

Abstract

Before entering the chapter, make sure to have all the foundations of trigonometry laid out, otherwise, study (Ch. 6)

Chapter 9

Techniques of integrations

Abstract

The whole purpose of this chapter is to show that you can integrate virtually anything. There's not much "story" to be seen here. For the physics student, these integration techniques will be useful when solving differential equations. Differential equations are discussed in chapter 12

As said in chapter 5, evaluating integrals is an art form. One must learn the patterns hidden in the integrals. This chapter only covers the core concepts and some examples. The exercises and solutions are written in appendix F.

I shall also assume that the reader is already familiar with trigonometric functions, its inverses, and its derivative.

9.1. Change of variables

Change of variables or u-substitution is a technique to simplify problems in which the original variables are replaced with functions of other variables. The problem might become simpler after.

Integrals outputs the anti-derivative of the integrand. Consider the function $x^2 \cos\left(x^3\right)$. The form of the function suggests that it might come from the chain rule. The derivative of sine is cosine, and the derivative of x^3, the inner function, is

x^2, which is conveniently multiplied on the outside.

$$\begin{aligned}\frac{d\sin(x^3)}{dx} &= \frac{d\sin(u)}{du}\cdot\frac{dx^3}{dx}\\ &= 3x^2\cdot\cos\left(x^3\right).\end{aligned}$$

To solve $\int x^2\cos\left(x^3\right)$, substitute $u = x^3$, we then replace all x's with u's, including dx to du.

$$\begin{aligned}u &= x^3\\ \frac{du}{dx} &= \frac{dx^3}{dx}\\ du &= 3x^2\,dx\end{aligned}$$

Then, rearrange the terms to fit the du

$$\begin{aligned}\int x^2\cos\left(x^3\right)dx &= \frac{1}{3}\int\cos\left(x^3\right)\cdot 3x^2\,dx\\ &= \frac{1}{3}\int\cos(u)\,du\\ &= \frac{1}{3}\sin(u)+\mathrm{C} = \frac{1}{3}\sin\left(x^3\right)+\mathrm{C}\end{aligned}$$

Or, we can directly replace all x's with u:

$$\begin{aligned}u &= x^3\\ x &= \sqrt[3]{u}\\ \frac{dx}{du} &= \frac{d\sqrt[3]{u}}{du}\\ dx &= \frac{1}{3\sqrt[3]{u^2}}\,du\,.\end{aligned}$$

Then,

$$\begin{aligned}\int x^2\cos\left(x^3\right)dx &= \int(\sqrt[3]{u})^2\cos\left((\sqrt[\cancel{3}]{u})^{\cancel{3}}\right)\cdot\frac{1}{3\sqrt[3]{u^2}}\,du\\ &= \int\left(\cancel{\sqrt[3]{u^2}}\right)\cos(u)\cdot\frac{1}{3\cancel{\sqrt[3]{u^2}}}\,du\\ &= \frac{1}{3}\int\cos(u)\,du\\ &= \frac{1}{3}\sin(u)+\mathrm{C} = \frac{1}{3}\sin\left(x^3\right)+\mathrm{C}.\end{aligned}$$

For the rest of change of variables method in this book, we'll rearrange the integrands to get the desired result. To get the hang of these substitution, one must practice and learn from a lot of examples. For the following examples, try to observe the substitution patterns.

Example 9.1.1: $\int \frac{x+1}{\sqrt{x^2+2x-5}}\,dx$

We see that the derivative of x^2+2x-5 *w.r.t.* x *is* $2x+2$, *or* $2(x+1)$*; thus, let*

$$\begin{aligned} u &= x^2+2x-5 \\ \frac{du}{dx} &= \frac{d}{dx}x^2+2x-5 \\ du &= 2(x+1)\,dx\,. \end{aligned}$$

Then, we rearrange the integrand:

$$\begin{aligned} \int \frac{x+1}{\sqrt{x^2+2x-5}}\,dx &= \frac{1}{2}\int \frac{2(x+1)}{\sqrt{x^2+2x-5}}\,dx \\ &= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\ &= \frac{1}{2}\cdot\frac{u^{1/2}}{\frac{1}{2}}+\mathrm{C} \\ &= \sqrt{x^2+2x-5}+\mathrm{C} \end{aligned}$$

Notice, the power rule outputs a power that's one below it. If you see consecutive powers, it might be a good sign to substitute u as the higher powers. Also, if u is in the form of x^n+a: $a \in \mathbb{C}$ and $n \in \mathbb{Z}$, the a will disappear when differentiated. Also, the lower power must be mutliplied with dx: $\int x^3\cos\left(x^2\right)\,dx$ would not work.

Another common theme in change of variables is multiplying some number in and dividing out by the same number in order to make the terms fit to the variables. You can also use trigonometric function inside, such as:

Example 9.1.2: $\int \tan(x)\,dx$

From basic trigonometry, $\tan(x) = \sin(x)/\cos(x)$. *Notice, the derivative of* $\cos(x)$ *is* $-\sin(x)\,dx$*; hence, it's reasonable to let* $u = \cos(x)$, du *then equals to* $-\sin(x)\,dx$.

$$\begin{aligned}\int \tan(x)\,dx &= \int \frac{\sin(x)}{\cos(x)}\,dx \\ &= -\int \frac{1}{u}\,du \\ &= -\ln|u| = -\ln|\cos(x)|.\end{aligned}$$

Example 9.1.3: $\int \frac{\cos(\ln(x))}{x}\,dx$

We see that the derivative of $\ln(x)$, *the inside function is* $1/x$. *We then let* $u = \ln(x)$, *then* $du = 1/x$.

$$\begin{aligned}\int \frac{\cos(\ln(x))}{x}\,dx &= \int \cos(u)\,du \\ &= \sin(u) = \sin(\ln(x)).\end{aligned}$$

Of course, some integrals are not this straightforward to solve. One must also simplify the integrand in some cases. The following are exercises which might help the reader with evaluating integrals using this method.

- $\int \frac{x-4}{x^2-8x+3}\,dx$ $\quad(u = x^2 - 8x + 3)$
- $\int \frac{e^x-1}{e^x+1}\,dx$ $\quad(u = e^x + 1)$
- $\int x^2 e^{x^3}\,dx$ $\quad(u = x^3)$
-

9.1.1. Change of variables with definite integrals

Indefinite integrals have bounds. If x is changed to u, the bound must also be expressed in u. Consider $\int_0^{\sqrt[3]{\pi}} x^2\cos\left(x^3\right)\,dx$, or for clarity: $\int_{x=0}^{x=\sqrt[3]{\pi}} x^2\cos\left(x^3\right)\,dx$.

Writing this is cumbersome. Normally, the "$x =$" is omitted. However, here, I shall use it for clarity.

There are two methods for evaluating this indefinite integral.

The first one changes the integral bound. From earlier, let $u = x^3$. Thus, the lower bound $x = 0$ must be changed to $u = \cos(0) = 1$ and the upper bound, $u = \sqrt[3]{\pi}^3 = \pi$. The integral then becomes $\int_{u=0}^{u=\pi} \cos(u)\, du$. This one can be evaluated directly in u.

$$\begin{aligned} \int_{u=0}^{u=\pi} \cos(u)\, du &= \sin(u)\big|_{u=0}^{u=\pi} \\ &= \sin(\pi) - \sin(0) \\ &= 0 - 0 = 0 \end{aligned}$$

Or, the second one[1]; turn the integral back to x, but not change the bound.

9.2. Integration by part

Integrals are inverses of derivatives. Sometimes you have to integrate a product of two functions, e.g., $\int xe^x\, dx$. The product rule suggests a really strong hint that this might lead to some new discovery. Notice, the equation of the product rule has a common factor in all terms: dx. We could cancel that out, and maybe get something useful.

$$\begin{aligned} \frac{df(x) \cdot g(x)}{dx} &= f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx} \\ d(f(x)g(x)) &= f(x)\, dg(x) + g(x)\, df(x) \\ f(x)g(x) &= \int f(x)\, dg(x) + \int g(x)\, df(x) \\ \int f(x)\, dg(x) &= f(x)g(x) - \int g(x)\, df(x) \end{aligned}$$

Voila! we found a way to separate functions in integrals. We call this technique **integration by part**. This is the integral version of the product rule. Integration by part is often used to transform antiderivative of a product of functions into an antiderivative for which a solution can be more easily found.

[1] This is more preferred by physicists.

Example 9.2.1: $\int xe^x \, dx$

et $f(x) = x$ *and* $dg(x) = e^x \, dx$. *Firstly, find* $g(x)$ *out of* $dg(x)$.

$$\int dg(x) = \int e^x \, dx$$
$$g(x) = e^x$$

Then, $df(x)$ *out of* $f(x)$

$$f(x) = x$$
$$\frac{df(x)}{dx} = \frac{dx}{dx}$$
$$df(x) = dx$$

$$\int f(x) \, dg(x) = f(x)g(x) - \int g(x) \, df(x)$$
$$\int (x) \, (e^x \, dx) = (x)(e^x) - \int (e^x)(dx)$$
$$= xe^x - e^x = e^x(x-1) + \mathrm{C}$$

9.2.1. The DI method for integration by part

9.2.2. Parametric curves: geometrical intepretation of integration by part

9.3. Complex numbers

We briefly turn our attention to polynomic equations. People have been able to solve the quadratics ($ax^2 + bx + c = 0$) for thousands of years via.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

However, no one has been able to solve the cubics ($ax^3 + bx^2 + cx + d = 0$) yet. Back then, mathematics was very competitive. Mathematicians would duel each other to gain fame. Since cubics used to be unsolvable, finding the general solution for the cubics would mean instant fame. This is where the hunt begins.

Scipione del Ferro, a quite unknown mathematician, found a way to solve the depressed cubic ($ax^3 + cx + d = 0$) but not the full cubic. He kept the formula to himself. However, on his deathbed, he gave the formula to his student, Antonio Fior: an ambitious mathematician. Fior bragged about his ability to solve the cubic equation. He challenged a quite well known mathematician, Niccolò Fontana Tartaglia. Tartaglia gave Fior a bunch of asserted problems, none of which Fior can solve. Fior gave Tartaglia thirty problems, all of which were depressed cubics. Tartaglia was anxious he'd lose his fame, but at the last night, he figured out the solution of the depressed cubic on his own (Eq. 9.1)[2].

$$x^3 + px + q = 0 \implies x = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \tag{9.1}$$

In the end, Fior miserably lost the duel. News started spreading around. When Gerolamo Cardano heard this, he was shocked. He rushed to Tartaglia. Cardano begged Tartaglia to give him the formula, but Tartaglia was secretive; he didn't give it. However, because of annoyance, Tartaglia finally gave it to Cardano but at a cost. Cardano had to sign a solemn oath that he would not publish it anywhere.

> *"I swear to you by the Sacred Gospel, and on my faith as a gentleman, not only never to publish your discoveries, if you tell them to me, but I also promise and pledge my faith as a true Christian to put them down in cipher so that after my death no one shall be able to understand them."*
>
> - Gerolamo Cardano

Then, Cardano found a way to transform any cubic into a depressed cubic. He taught his method to his student Ludovico Ferrari despite vowing to Tartaglia. After which, Ferrari then was able to reduce any quartic ($ax^4 + bx^3 + cx^2 + d$) into a cubic. Cardano wanted to publish it so badly, but because the solemn oath, he'd have to keep it to himself. A decade later, Cardano found a note in Scipione del Ferro's house with the depressed cubic formula that predates Tartaglia's discovery. He quickly published a book named "*Ars Magna*" which contains the solution to the cubic in it.

[2]Yes, this equation is quite ugly

However, something seems a bit off with his equation. $x^3 - 15x - 4 = 0$ has a root at $x = 4$, its factorization is $(x-4)(x^2-4x-1)$. If you plug in $p = -15$ and $q = -4$ into (Eq. 9.1), you quickly ran into a problem.

$$\begin{aligned} x &= \sqrt[3]{-\frac{-4}{2} + \sqrt{\frac{(-4)^2}{4} + \frac{(-15)^3}{27}}} - \sqrt[3]{-\frac{-15}{2} - \sqrt{\frac{(-4)^2}{4} + \frac{(-15)^3}{27}}} \\ &= \sqrt[3]{2 + \sqrt{-124}} - \sqrt[3]{\frac{15}{4} - \sqrt{-124}} \end{aligned}$$

Look what has happened, there's a negative inside a square root. Does that mean that the cubic equation is broken? It certainly does not because the cubic certainly has a root at $x = 4$.

It took a while for mathematicians to figure this out. About a century later, Rafael Bombelli, an Italian mathematician came along and shed us an incredibly useful insight. If the square root asks for a number that when squared gives that the number inside the root and there is none for negative numbers, maybe there's a new type of number. Voila! the imaginary number were discovered. We use i to designate imaginary unit, also, $i^2 = -1$. Bombelli continued this using i. Finally, the i cancels off at and a real solution is given.

The name "imaginary" reflects the lack of physical interpretation and its absent on the number line. This number lives on its own line. When an imaginary number is added to a real number, it forms a complex number, i.e., $a + bi$. The resulting "complex number" is a two-dimensional number. Still, it's worth mentioning that it is still a single number; a scalar, not a vector.

But, aren't we supposed to be talking about techniques of integration in this chapter? Yes, we will return to that in the following section. Unsurprisingly[3], Euler, the one that appears everywhere in mathematics, found a connection between trigonometric functions and exponentials via. imaginary numbers. To be honest, doing calculus with exponentials are way easier than with trigonometric functions. These sublime connections are going to help us a lot in integration.

[3] Euler appears everywhere, it might be unsurprising now that he's the one that found this sublime connecion. I can hear a student in the background screaming "Euler again!?" to me from my experience teaching.

9.4. Complex forms of trigonometric functions

Much of the fiddling in number theory in the 1800's were done to ease numerical computations. As we've seen in (Ch. 5), logarithms' can convert multiplication into addition and division into addition. What about the trigonometric functions? Can we rewrite trigonometric functions into something easier to work with?

Take a look at the taylor series of e^x, $\sin(x)$, and $\cos(x)$.

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + \dots \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \dots \end{aligned}$$

Notice that both sine and cosine has an alternating plus and minus signs, but e^x does not. How's about we make the signs in the taylor series of e^x alternate? By the definition of i: $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on and so forth. If instead, we calculate the taylor series for e^{ix}, we get an alternating plus and minus signs.

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \dots$$

If we group the terms with i at the front and pull all the i's out, the result is just spectacular:

$$\begin{aligned} e^{ix} &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) \\ &= \cos(x) + i\sin(x) \end{aligned}$$

This formula is called the **Euler's formula**. Here, we see that exponentials and trigonometric functions is interchangable. The next step is to isolate the trignometric functions on its own. To do that, try writing down the taylor series of e^{-ix} and compare it with e^{ix}.

$$\begin{array}{ccc} e^{ix} & = & \cos(x) + i\sin(x) \\ e^{-ix} & = & \cos(x) - i\sin(x) \\ \hline e^{ix} + e^{-ix} & = & 2\cos(x) \\ e^{ix} - e^{-ix} & = & 2i\sin(x) \end{array}$$

From that, we can conclude that

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \tag{9.2}$$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \tag{9.3}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} \tag{9.4}$$

Then, various integrals of trigonometric functions become trivial.

This method should be used very sparringly, because it isn't this straightforward sometimes.

9.4.1. An alternative proof for Euler's formula

Let $t = \cos(x) + \imath\sin(x)$. Then,

$$\begin{aligned}
\frac{dt}{dx} &= -\sin(x) + \imath\cos(x) \\
&= \imath^2\sin(x) + \imath\cos(x) \\
&= \imath(\cos(x) + \imath\sin(x)) = \imath t. \\
&\therefore \frac{1}{\imath t}\, dt = dx\,.
\end{aligned}$$

Integrate both sides and get

$$\begin{aligned}
\frac{1}{\imath}\ln(t) &= x \\
\ln(t) &= \imath x \\
t &= e^{\imath x} \\
\cos(x) + \imath\sin(x) &= e^{\imath x}
\end{aligned}$$

9.5. Even and odd functions

- A function is even if $f(x) = +f(-x)$
- A function is odd if $f(x) = -f(-x)$

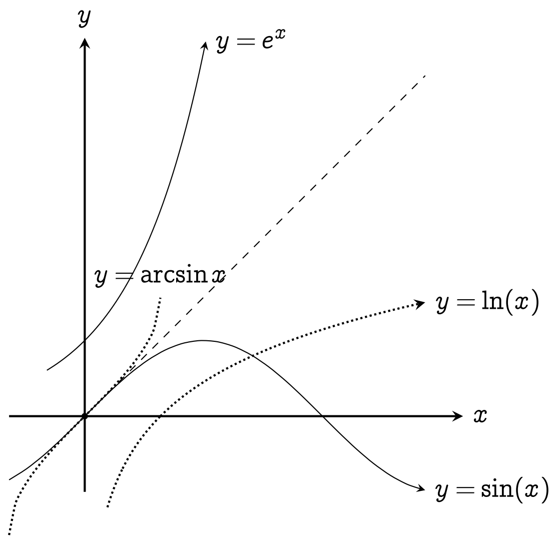


Figure 9.1: e^x with its inverse function $\ln(x)$ and $\sin(x)$ with its inverse function $\arcsin(x)$

In some integrals, we can use just these properties of functions to evaluate them.

Example 9.5.1: [a] $\int_{-2}^{2} \left(x^3 \cos\left(\frac{1}{2}\right) + \frac{1}{2} \right) \sqrt{4 - x^2}\, dx$

[a] This used to be a problem for the Wi-Fi password at a restaurant. The solution to this integral is the Wi-Fi password.

e can separate the integrals into two parts:

$$\int_{-2}^{2} \left(x^3 \cos\left(\frac{1}{2}\right) \right) \sqrt{4 - x^2}\, dx \;\; \textit{and} \;\; \frac{1}{2} \int_{-2}^{2} \sqrt{4 - x^2}\, dx\,.$$

9.6. Definite integrals using inverse functions

Some definite integrals are daunting to evaluate, e.g., $\ln(x)$, or $\arcsin(x)$ but the integral of its inverse, e^x and $\sin(x)$, are easy to. There are two concepts that's used. First, integrals are areas under the graph. Second, inverse function mirrors the original function around the line $x = y$.

The geometrical interpretation of inverse functions are very useful in evaluating integrals. It mirrors the function around the line $x = y$, i.e., the line that runs 45°

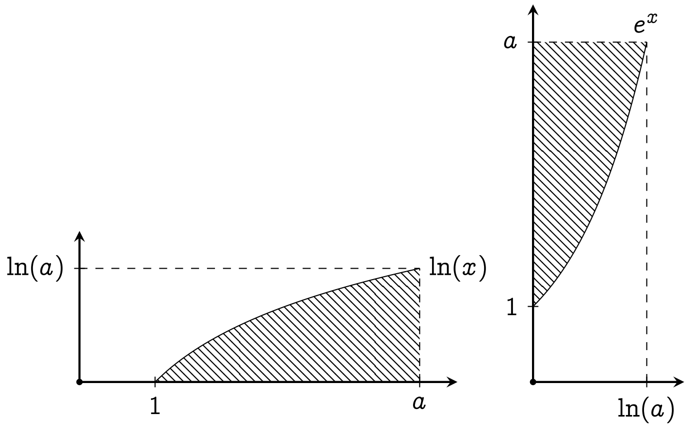


Figure 9.2: The integral of $\ln(x)$ and its physical interpretation

to the x axis. (Fig. 9.1) shows the graph of various functions with its inverse functions.

Consider $\int \ln(x)\,dx$. Let the answer of the integral be $A(x)$. $A(x)$ is the area under the graph of $\ln(x)$ from 0 to x. We still don't know if $\ln(x)$ has an integral or not. From the graph of $\ln(x)$ finding the area under the graph from 0 to a is not feasible because $\lim_{x\to 0}\ln(x) = -\infty$. Thus, we then try to analyze the area under the graph from 1 ($\ln(1) = 0$) to a instead.

(Fig. 9.2) Notice, the shaded areas in both graphs are the same. The shaded area in the left is equal to the rectangle $a \cdot \ln(x)$ minus the white area in the right side:

$$\int_1^a \ln(x)\,dx = a \cdot \ln(a) - \int_0^{\ln(a)} e^x\,dx$$

Then, we can easily evaluate this integral:

$$\begin{aligned}\int_1^a \ln(x)\,dx &= a \cdot \ln(a) - e^x\big|_0^{\ln(a)} \\ &= a \cdot \ln(a) - \left(e^{\ln(a)} - e^0\right) \\ &= a \cdot \ln(a) - a + 1\end{aligned}$$

From the physical interpretation, we can formulate the formula:

$$\int_a^b f(x)\,dx = (b-a)f(b) - \int_{f(a)}^{f(b)} f^{-1}(x)\,dx \tag{9.5}$$

One might be asking if this formula can output the indefinite integral of $\ln(x)$. By the fundamental theorem of calculus: if $\int f(x)\,dx = A(x)$, then $\int_0^a f(x)\,dx =$

$A(a)$. Substitute $a = x$, then you get the function $A(x)$.

$$\int_0^a f(x)\,dx = (a-0)\ln(a) - \int_{\ln(0)}^{\ln(a)} e^x\,dx = a\ln(a) - e^x\big|_{\ln(0)}^{\ln(a)} = a\ln(a) - a - e^{\ln(0)}$$

$\ln(0)$ is undefined, but we're in calculus; limits can help.

$$a(\ln(a)-1)\lim_{h\to 0} e^{\ln(x)} = a(\ln(a)-1)\cdot 1 \tag{9.6}$$

Then, substitute $a = x$ to get the indefinite integral of $\ln(x)$: $x(\ln(x)-1)$

Example 9.6.1: $\int_0^1 \arcsin(x)\,dx$

he graph of $\arcsin(x)$ *is shown in (Fig. 9.2). I'd like to point out that the integral* $\arcsin(x)$ *over the whole is zero; the left and right side cancel out. Also, since it's an odd function:*

$$\int_{-a}^{a} \arcsin(x)\,dx = 0 \tag{9.7}$$

We can safely use (Eq. 9.5)

$$\begin{aligned}\int_0^1 \arcsin(x)\,dx &= (1-0)\arcsin(1) - \int_{\arcsin(0)}^{\arcsin(1)} \sin(x)\,dx \\ &= \frac{\pi}{2} - \int_0^{\pi/2} \sin(x)\,dx \\ &= \frac{\pi}{2} - \cos(x)\big|_0^{\pi/2} \\ &= \frac{\pi}{2} + 1\end{aligned}$$

Example 9.6.2: $\int \arcsin(x)\,dx$

et $\int \arcsin(x)\,dx = A(x)$*, then* $A(a) = \int_0^a \arcsin(x)\,dx$

$$\begin{aligned}\int_0^a \arcsin(x)\,dx &= (a-0)\arcsin(a) - \int_{\arcsin(0)}^{\arcsin(a)} \sin(x)\,dx \\ &= a\arcsin(a) - (-\cos(x))\big|_0^{\arcsin(a)} \\ &= a\arcsin(a) + \cos(\arcsin(a))\end{aligned}$$

From here, simplify $\cos(\arcsin(a))$ *by using trigonometric manipulation.*

Then, we finally get the indefinite integral of inverse sine:

$$A(a) = a\arcsin(a) + \frac{1}{\sqrt{1-a^2}}$$
$$A(x) = x\arcsin(x) + \frac{1}{\sqrt{1-x^2}}$$

The same could be done for all inverses of trigonometric functions. Answers of integrals of inverse trigonometric functions are at the end of the chapter.

9.7. Trigonometric substitution

If we'd like to find the area under a circle radius R, we must find a function that describes a circle, then integrate it. It's well known that $x^2 + y^2 = r^2$ plots a circle. We must rearrange it into $y = f(x)$: $f(x) = \sqrt{r^2 - x^2}$

$$A = \int_{-r}^{r} f(x)\,dx$$
$$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$$

9.8. Integration using taylor series

9.9. Partial fractions

9.10. Feynman's integration techniques

9.11. Cauchy's repeated integration formula

Formulas in Chapter 9

Integrals of inverse trigonometric functions

1. $\displaystyle\int \arcsin(x)\,dx = x\arcsin(x) + \sqrt{1-x^2} + \mathrm{C}$
2. $\displaystyle\int \arccos(x)\,dx = x\arccos(x) - \sqrt{1-x^2} + \mathrm{C}$
3. $\displaystyle\int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln\left(1+x^2\right) + \mathrm{C}$
4. $\displaystyle\int \operatorname{arccsc}(x)\,dx = x\operatorname{arccsc}(x) + \ln\left(x + \sqrt{x^2 = 1}\right) + \mathrm{C}$
5. $\displaystyle\int \operatorname{arcsec}(x)\,dx = x\operatorname{arcsec} x - \ln\left(x + \sqrt{x^2-1}\right) + \mathrm{C}$
6. $\displaystyle\int \operatorname{arccot}(x)\,dx = x\operatorname{arccot}(x) + \frac{1}{2}\ln\left(1+x^2\right) + \mathrm{C}$

Chapter 10

Series

Part II

The applications

Chapter 11

Multiple integrals

Chapter 12

Introduction to differential equations

Abstract

This chapter, as said, is an introduction to differential equations. It does not dives into the solving, but dives into the understanding of differential equations. Solving different forms of differential equations are in (Ch. 14). A lot of examples using various forms of mechanics are used here. Also, this chapter contains numerical methods with the Fortran language.

Abbreviations:

1. DE: Differential equation
2. ODE: Oridinary differential equation
3. PDE: Partial differential equation

DEs are a family of equations that relate one or more unknown functions and their derivatives. It arises when one tries to describe a system that's changing over a variable, e.g., population size, physical systems, heat flow. One thing to keep in mind while learning differential equations is: differential equations are extremely hard to solve. There are forms to these differential equations. Actually, most of the DEs don't have an analytical solution. As said back in the preface, everything in nature is written in the language of DEs. Thus, it is necessary to understand how these equations formed together to describe a system.

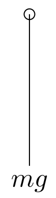


Figure 12.1: Free body diagram of projectile motion

There are two types of DE: ODE and PDE. ODEs are DE with just normal derivatives which are generally easier to solve than PDEs but still extremely difficult to solve. PDEs on the other hand, contains partial derivatives. Most of the time, PDEs do not have a beautiful analytical solution. Usually, we don't solve these DEs even, but we use numerical methods in order to approximate how the final shape looks like; then, analyze the behavior of the equation after. We jump to the answer before gaining equations.

12.1. Newtonian mechanics, gravity, and second order ODEs

Newtonian mechanics is centralized on force. Force changes acceleration, acceleration changes velocity, velocity changes the object's positon, and vice versa. It's written as in (Eq. 12.1)

$$F = ma = m\frac{dv}{dt} = \frac{d}{dt}\left(\frac{d}{dx}\right) \tag{12.1}$$

12.1.1. Projectile motion

Projectile motion, at least on the surface of Earth, assumes universal gravity. There are gravitational forces acting on the object. The free-body diagram is as (Fig. 12.1)

12.1.2. Swinging pendulum

12.1.3. Time of collision

Chapter 13

Numerical Analysis

Chapter 14

General forms of differential equations

Chapter 15

Vector calculus I:

Chapter 16

Signal analysis

Part III

The extensions

Chapter 17

Discrete calculus

Chapter 18

Fractional calculus

Chapter 19

Quantum calculus

Part IV

The foundations, reimagined: real analysis

Chapter 20

The reals

Part V

Beyond imagination: complex analysis

Chapter 21

Treatment of the Complex numbers

Abstract

The inspiration for the creation of complex numbers can be found in (Ch. 9.3). This chapter focuses on the rigorous treatment of complex numbers, mainly, algebra of complex numbers. This chapter also explores stereographic mapping.

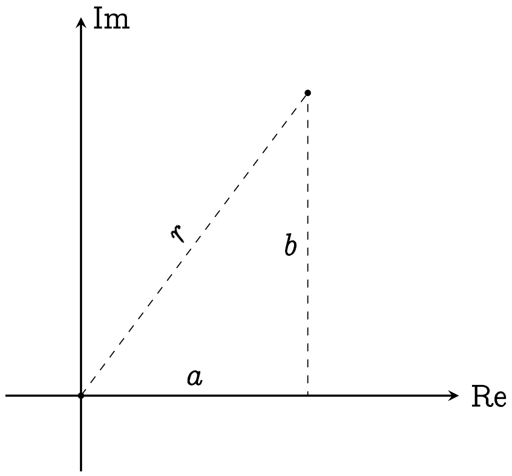
21.1. Complex numbers: a two-dimensional number

A "number" is understood as existing in a one-dimensional line. Complex numbers on the other hand is a number, in its own right, but it exists in a two dimensional plane we call the **complex plane**. A complex number can be specified in two forms: rectangular ($a + b\imath$) and polar ($re^{\imath\theta}$). These two forms are connected by (Eq. 9.2), the Euler's formula:

$$re^{\imath\theta} = r\cos(\theta) + \imath r\sin(\theta).$$

To convert one form to the other, we match coefficient. The method to convert from polar into rectangular is already given by the Euler's formula itself. To convert from rectangular into polar, we turn to the physical interpretation of the polar form.

This two numbers shouldn't be thought as a vector, even though sometimes it acts like one. Rather, it should be thought as an individual, single number.



21.2. Algebra of complex numbers

Raising a complex number to a real exponent is like

Addition, subtraction, and multiplication works like the reals:

$$(a + b\imath) + (c + d\imath) = (a + c) + (b + d)\imath,$$
$$(a + b\imath) - (c + d\imath) = (a - c) + (b - d)\imath,$$
$$(a + b\imath)(c + d\imath) = ac + bc\imath + ad\imath + bd\imath^2 = (ac - bd) + (ad + bc)\imath.$$

21.3. Stereographic mapping of complex numbers

Appendix A

Infinity and the Cantor's paradox

Appendix B

Fundamental of physics

Abstract

This is the essence of fundamental physics and kinematics. Most of the contents here are from Newton's Principia, edited by the writer, and does not include the symbolic part of calculus. Reading the Newton Principia on its own is highly recommended; it offers extensive insight to the physical world. One concept should be kept in mind; not everything is definable. No one can exactly define what a primitive quantity is, otherwise, it wouldn't be primitive. As far as definition goes, we can tell what a primitive quantity describe and interact with its environment, but not what it is.

Albert's Einstein relativistic effect aren't considered in this appendix

I. Physical quantities

A physical quantity is a physical property of a body or system that can be quantified by measurement. A physical quantity is a *value* that is the result of an algebraic number and a unit. E.g. "5 meters" is formed from "5", an algebraic number and "meters", the unit.

There are two types of physical quantities

1. **Scalar**: A quantity that has magnitude, but no direction e.g. masses, temperature, distance.[1] Its name came from the word "scaling"

2. **Vector**: A quantity that has both magnitude and direction e.g. position, velocity, force. You have to tell the direction, otherwise it will lose its meaning. "Your velocity is $5\,\mathrm{m\,s^{-1}}$" is meaningless because you don't know the direction of it.

i. Disambiguing vectors

That definition before is a kind of non-rigorous definiton. I do not want to cause further confusion of what a vector is later on, because this textbook also has vector calculus infused in it.

II. Explicit properties

Even though these properties are trivial; still, they should be addressed and explained before progressing further. Differences between absolute, relative, and apparent are also addressed.

Property B.1

> ***Time*** *is an explicit property of nature; it doesn't reference anything external. Time is a measure that flows and increases uniformly everywhere in the universe.*

There are two type of time. Absolute time flows since the start of the universe and has no unit. Apparent time has unit, e.g. day, month, and year.

The definition of time stated earlier was from the 1600's. Later, Albert Einstein disproved this. In his book *On the electrodynamics of moving bodies*, he stated that time does not flow uniformly for every observer and an observer moving close to the speed of light will experience time contraction. These relativistic effects are not considered in this textbook.

[1] Not to be confused with position

Property B.2

__Space__ is an explicit property of nature; it doesn't reference anything external.

Absolute space is homogeneous and immovable. Relative space is a movable; such a relative dimension is determined by the senses of the observer. A moving body experiences relative space. That body is the center of this relative space, but it does not change absolute space. Sometimes, relative space is called the reference frame.

Absolute space and relative space has the same magnitude, same unit of reference. Numerically, the position in this space can change depending on where the origin is placed. E.g. $1\,\mathrm{cm}$ of length is the same everywhere but its position might not be.

Property B.3

__Place__ is the part of space that a body occupies, not the position of that body. In some sense, place is a quantity which depends on the space, both absolute or relative.

Property B.4

__Motion__ is the change of position of a body from one place to another through time.

Absolute motion is the change of position of a body from one point in space to another through absolute time and vice versa for relative motion. A resting body on Earth does not have relative motion but has absolute motion because the Earth revolves around the sun. Motion can be quantified. Quantified motion are called the quantity of motion or the momentum (Def. B.2).

III. On coordinates

Dimension refers to the number used to specify a quality of a body that's in a space e.g., mass, position, velocity, or time.

Coordinate system is a system that assigns every point in a space to a different number. A **coordinate** is a number used for specifying the position of a point in a space.

To say "Object 'A' and 'B' is two apart" is meaningless because the direction and the magnitude is not specified. "A" could be two to the right of "B", or 2 to the left. "two" of what and to what direction. Hence, a coordinate system uses a **unit** that acts like a reference; it defines what "one" is in that coordinate.

The direction in a coordinate system is given by the **unit vector** which depends on the user. The imaginary line that stretches along the unit vector to infinity is called the **axis**.

IV. Definitions

Definition B.1

*Quantity of matter, aka. **mass**, is a measure of matter that arises from its density*[2] *and volume conjointly.*

Mass is a scalar physical quantity. If the density of matter is doubled in the same space, the mass is also doubled. If the size is doubled but the density stays constant, the mass doubles also. The relation between mass (m), density (ρ), and volume (V) can be written as in (Eq. B.1)

$$m = \rho \cdot V \tag{B.1}$$

[2] In older translation, the density is called the bulk

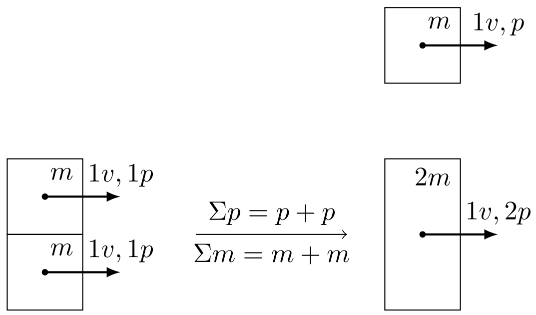


Figure B.1: A body twice the mass moving at the same velocity has twice the momentum

Definition B.2

Quantity of motion, aka. ***momentum*** *is a measure of motion that arises from the velocity and the mass conjointly.*

Since velocity is a vector quantity, momentum is also a vector quantity. Momentum represents the *amount* of movement. The momentum of the whole is the sum of the momentum of all its parts. Hence, a body twice the mass moving at the same velocity has twice the momentum, illustrated in (Fig. B.1).

The relationship between mass (m), velocity ($\vec{v}$), and momentum ($\vec{p}$) can be written as in (Eq. B.2)

$$\vec{p} = m\vec{v} \tag{B.2}$$

Definition B.3

Force *is a primitive influence that can change the movement quantity (momentum) of an object. The*

The understanding of force as an action of pushing and pulling is a surprisingly good starting point. Humans are completely aware of the force that's exerted on them. If I'm pushed, I'm aware that there are forces exerted on me. If I push something, I'm exerting force on that thing, and I'm aware of it. When I carry a book, I can feel the weight of the book, that means there are forces that the book exerts on me. If a force is exerted on a body, the velocity of that body will change; hence, changing the momentum of the body. However, not all forces can change the momentum of a body. What forces can change the momentum will be discussed later in (Sec. V)

V. Axioms, or laws of motion

i. Newton's first and second law

Laws of Motion B.1

Every body preserves in its state of being at rest or of moving uniformly straight forward, except if it's compelled to change its state by forces ***impressed***

Newton's first law give definition to what *zero* force is; constant momentum.

Laws of Motion B.2

A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is ***impressed***

Newton's second law can be restated as: if a force $\vec{F}$ is *impressed* on an object, that force will change the momentum of an object depending on the direction that the force is impressed. It's written as (Eq. B.3). Moreover, a change in momentum also means that the velocity is changed; hence, a force impressed on a body must also accelerates the body (Eq. B.4). The dot on the symbol is a common notation in physics meaning that that variable is changing through time.

$$\vec{F} = \dot{\vec{p}} \tag{B.3}$$
$$\vec{F} = m\dot{\vec{v}}$$
$$\vec{F} = m\vec{a} \tag{B.4}$$

On the surface, newton's second law includes the first law. It seems like if the force is zero

ii. Newton's third law

Laws of Motion B.3

To any action there is always an opposite and equal reaction. In other words, the actions of two bodies upon each other are always equal and always opposite in direction.

Appendix C

Lebesque integral

Appendix D

Binomial theorem

Appendix E

Integral of the reciprocal

Appendix F

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Appendix G

The Jacobian

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