PAPER_2

SECTION 1 (Maximum Marks : 12)

- **This section contains FOUR (04) questions.**
- **Each question has FOUR options (A), (B), (C) and (D). ONLY ONE of these four options is the correct answer.**
- **For each question, choose the option corresponding to the correct answer.**
- **Answer to each question will be evaluated according to the following marking scheme:**

Full Marks **: +3 If ONLY the correct option is chosen;**
Zero Marks **: 0 If none of the options is chosen (i.e. the question is unanswered);**
Negative Marks **: −1 In all other cases.**

1. Considering only the principal values of the inverse trigonometric functions, the value of $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)-2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ is

(A) $\frac{7}{24}$ (B) $\frac{-7}{24}$ (C) $\frac{-5}{24}$ (D) $\frac{5}{24}$

Answer (B)

Sol. $\tan\left(\sin^{-1}\left(\frac{3}{5}\right)-2\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$

Let $\sin^{-1}\frac{3}{5}=\alpha,\quad 2\cos^{-1}\frac{2}{\sqrt{5}}=\beta \quad \Rightarrow \cos\frac{\beta}{2}=\frac{2}{\sqrt{5}}$

$\because \sin\alpha=\frac{3}{5} \quad \Rightarrow \tan\alpha=\frac{3}{4} \quad \tan\beta=\frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{2\times\frac{1}{2}}{1-\frac{1}{4}}=\frac{4}{3}$

$\Rightarrow \tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\frac{\frac{3}{4}-\frac{4}{3}}{1+1}=-\frac{7}{24}$

2. Let $S=\left\{(x,y)\in R\times R : x\ge 0, y\ge 0, y^2\le 4x, y^2\le 12-2x \text{ and } 3y+\sqrt{8}x\le 5\sqrt{8}\right\}$. If the area of the region S is $\alpha\sqrt{2}$, then α is equal to

(A) $\frac{17}{2}$ (B) $\frac{17}{3}$ (C) $\frac{17}{4}$ (D) $\frac{17}{5}$

Answer (B)

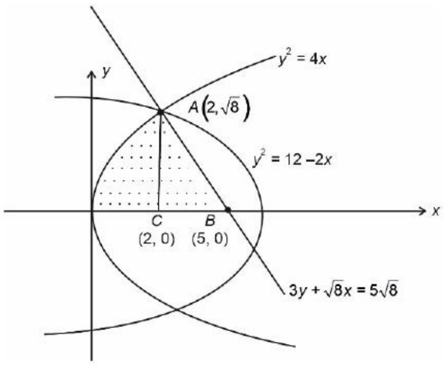
Sol.

$$y^2 = 4x, y^2 = 12 - 2x \Rightarrow x = 2, y = \sqrt{8}$$

A= $\int_0^2 2\sqrt{x}dx + \frac{1}{2} \times 3 \times \sqrt{8}$

$$= \left[2 \times \frac{2}{3} x^{\frac{3}{2}}\right]_0^2 + 3\sqrt{2} = \frac{4}{3} \times 2\sqrt{2} + 3\sqrt{2} = \frac{17}{3}\sqrt{2}$$

$$\because A = \alpha\sqrt{2} \Rightarrow \alpha = \frac{17}{3}$$



Option (B) is correct.

3. Let $k \in R$. If $\lim_{x \to 0+} (\sin(\sin kx) + \cos x + x)^{\frac{2}{x}} = e^6$, then the value of k is

(A) 1 (B) 2 (C) 3 (D) 4

Answer (B)

Sol.

$$I = \lim_{x \to 0^+} (\sin(\sin kx) + \cos x + x)^{\frac{2}{x}} = e^6$$

$$\Rightarrow \ln I = \lim_{x \to 0^+} \frac{2}{x} (\sin(\sin kx) + \cos x + x - 1)$$

$$\Rightarrow \ln I = \lim_{x \to 0^+} 2\left(\frac{\sin(\sin kx)}{\sin kx} \cdot \frac{\sin kx}{kx} \cdot \frac{kx}{x} + 1 - \frac{(1 - \cos x)}{x^2} \cdot x\right)$$

$$\Rightarrow \quad \ln I = 2(k+1) \quad \Rightarrow \quad I = e^{2(k+1)} = e^6$$

$$k + 1 = 3 \quad \Rightarrow k = 2$$

4. Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by

$$f(x)=\begin{cases} x^2\sin\left(\dfrac{\pi}{x^2}\right), & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

Then which of the following statements is TRUE?

(A) $f(x)=0$ has infinitely many solutions in the interval $\left[\dfrac{1}{10^{10}},\infty\right)$.

(B) $f(x)=0$ has no solutions in the interval $\left[\dfrac{1}{\pi},\infty\right)$.

(C) The set of solutions of $f(x)=0$ in the interval $\left(0,\dfrac{1}{10^{10}}\right)$ is finite.

(D) $f(x)=0$ has more than 25 solutions in the interval $\left(\dfrac{1}{\pi^2},\dfrac{1}{\pi}\right)$.

Answer (D)

Sol.

$$f(x)\ =\begin{cases} x^2\sin\left(\dfrac{\pi}{x^2}\right), & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

$$f(x)=0\Rightarrow \sin\left(\dfrac{\pi}{x^2}\right)=0$$

$$\Rightarrow \frac{\pi}{x^2}=n\pi$$

$$\Rightarrow x^2=\frac{1}{n}$$

$$\Rightarrow x=\frac{1}{\sqrt{n}}$$

$\text{If } x\in\left[\dfrac{1}{10^{10}},\infty\right)$	$\text{If } x\in\left[\dfrac{1}{\pi},\infty\right)$	$\text{If } x\in\left(0,\dfrac{1}{10^{10}}\right)$
$\dfrac{1}{\sqrt{n}}\in\left[\dfrac{1}{10^{10}},\infty\right)$	$\dfrac{1}{\sqrt{n}}\in\left[\dfrac{1}{\pi},\infty\right)$	$\sqrt{n}\in\left(10^{10},\infty\right)$

$\sqrt{n} \in \left(0, 10^{10}\right]$	$\sqrt{n} \in (0, \pi]$	n infinite
$n \in \left(0, \left(10^{10}\right)^2\right]$	$n \in \left(0, \pi^2\right]$	If $x \in \left(\frac{1}{\pi^2}, \frac{1}{\pi}\right)$
Finite values of n	$n = 1, 2, 3 \ldots 9$	$\sqrt{n} \in \left(\pi, \pi^2\right)$
		$n \in \left(\pi^2, \pi^4\right)$
		$n \in (9.8, 97.2\ldots)$
		More than 25 solutions

SECTION 2 (Maximum Marks : 12)

- This section contains THREE (03) questions.
- Each question has FOUR options (A), (B), (C) and (D). ONE OR MORE THAN ONE of these four option(s) is(are)

correct answer(s).

- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 ONLY if (all) the correct option(s) is(are) chosen;

Partial Marks : +3 If all the four options are correct but ONLY three options are chosen;

Partial Marks : + 2 If three or more options are correct but ONLY two options are chosen, both of which are correct;

Partial Marks : +1 If two or more options are correct but ONLY one option is chosen and it is a correct option;

Zero Marks : 0 If unanswered;

Negative Marks : –2 In all other cases.

5. Let S be the set of all $(\alpha, \beta) \in R \times R$ such that

$$\lim_{x \to \infty} \frac{\sin\left(x^2\right)(\log_e x)^{\alpha} \sin\left(\frac{1}{x^2}\right)}{x^{\alpha\beta}\left(\log_e(1+x)\right)^{\beta}} = 0$$

Then which of the following is (are) correct?

(A) $(-1, 3) \in S$

(B) $(-1, 1) \in S$

(C) $(1, -1) \in S$

(D) $(1, -2) \in S$

Answer (B,C)

Sol.

$$\lim_{x\to\infty}\frac{\sin\left(x^2\right)\sin\left(\frac{1}{x^2}\right)(\ln x)^{\alpha}}{x^{\alpha\beta}(\ln(1+x))^{\beta}}=0$$

$$=\lim_{x\to\infty}\frac{\left(\sin x^2\right)\sin\left(\frac{1}{x^2}\right)\frac{1}{x^2}}{\left(\frac{1}{x^2}\right)x^{\alpha\beta}(\ln(1+x))^{\beta}}=0$$

It is possible if $\alpha\beta+2>0 \;\; \alpha\beta>-2$

(A) $\alpha\beta=-3$ (B) $\alpha\beta=-1$ (C) $\alpha\beta=-1$ (D) $\alpha\beta=-2$

6. A straight line drawn from the point $P(1,3,2)$, parallel to the line $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z-6}{1}$, intersects the plane $L_1: x-y+3z=6$ at the point Q. Another straight line which passes through Q and is perpendicular to the plane L_1 intersects the plane $L_2: 2x-y+z=-4$ at the point R. Then which of the following statements is(are) TRUE?

(A) The length of the line segment PQ is $\sqrt{6}$

(B) The coordinates of R are $(1,6,3)$

(C) The centroid of the triangle PQR is $\left(\frac{4}{3},\frac{14}{3},\frac{5}{3}\right)$

(D) The perimeter of the triangle PQR is $\sqrt{2}+\sqrt{6}+\sqrt{11}$

Answer (A,C)

Sol. Equation of line parallel to $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z-6}{1}$ through $P(1,3,2)$ is

$$\frac{x-1}{1}=\frac{y-3}{2}=\frac{z-2}{1}=\lambda \text{ (let)}$$

Now, putting any point $(\lambda+1,2\lambda+3,\lambda+2)$ in L_1

$\boxed{\lambda=1}$

$\Rightarrow$ Point $Q(2,5,3)$

Equation of line through $Q(2,5,3)$ perpendicular to L_1 is

$$\frac{x-2}{1}=\frac{y-5}{-1}=\frac{z-3}{3}=\mu \text{ (Let)}$$

Putting any point $(\mu+2,-\mu+5,3\mu+3)$ in L_2

$\mu = -1$

$\Rightarrow$ Point $R(1,6,0)$

(A) $PQ = \sqrt{1+4+1} = \sqrt{6}$

(B) $R(1,6,0)$

(C) Centroid $\left(\frac{4}{3}, \frac{14}{3}, \frac{5}{3}\right)$

(D) $PQ + QR + PR = \sqrt{6} + \sqrt{11} + \sqrt{13}$

7. Let A_1, B_1, C_1 be three points in the xy-plane. Suppose that the lines A_1C_1 and B_1C_1 are tangents to the curve $y^2 = 8x$ at A_1 and B_1, respectively. If $O = (0,0)$ and $C_1 = (-4,0)$, then which of the following statements is (are) TRUE?

(A) The length of the line segment OA_1 is $4\sqrt{3}$

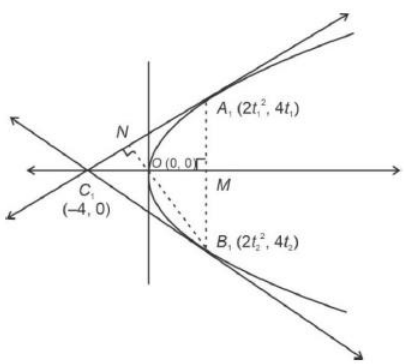
(B) The length of the line segment A_1B_1 is 16

(C) The orthocentre of the triangle $A_1B_1C_1$ is $(0,0)$

(D) The orthocentre of the triangle $A_1B_1C_1$ is $(1,0)$

Answer (A, C)

Sol.



Let $A_1 = \left(2t_1^2, 4t_1\right)$ and $B_1 = \left(2t_2^2, 4t_2\right)$

$C \equiv (-4,0) \equiv \left(2t_1t_2, 2\left(t_1+t_2\right)\right)$

$\Rightarrow t_2 = -t_1$ and $t_1\left(-t_1\right) = -2$

$t_1 = \sqrt{2}, t_2 = -\sqrt{2}$

$A_1 \equiv (4, 4\sqrt{2}), B_1 \equiv (4, -4\sqrt{2})$

$\therefore \quad OA_1 = \sqrt{4^2 + (4\sqrt{2})^2} = 4\sqrt{3}$

$A_1B_1 = 8\sqrt{2}$

Altitude $C_1M : y = 0$

Altitude $B_1N : \sqrt{2}x + y = 0$

$\therefore$ Orthocentre $\equiv (0, 0)$

SECTION 3 (Maximum Marks : 24)

• This section contains SIX (06) questions.

• The answer to each question is a NON-NEGATIVE INTEGER.

• For each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

• Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +4 If ONLY the correct integer is entered;

Zero Marks : 0 In all other cases.

8. Let $f : R \to R$ be a function such that $f(x+y) = f(x) + f(y)$ for all $x, y \in R$, and $g : R \to (0, \infty)$ be a function such that $g(x+y) = g(x)g(y)$ for all $x, y \in R$. If $f\left(\frac{-3}{5}\right) = 12$ and $g\left(\frac{-1}{3}\right) = 2$, then the value of $\left(f\left(\frac{1}{4}\right) + g(-2) - 8\right)g(0)$ is ....

Answer (51)

Sol. $f(x+y) = f(x) + f(y)$

$\Rightarrow f(x) = kx$

$f\left(\frac{-3}{5}\right) = 12 \Rightarrow k = -20$

$\therefore \quad f(x) = -20x$

$g(x+y) = g(x)g(y) \Rightarrow g(x) = a^x$

$g\left(\frac{-1}{3}\right) = 2 \Rightarrow a = \frac{1}{8}$

$\therefore \quad g(x) = \left(\frac{1}{8}\right)^x$

$$\left(f\left(\frac{1}{4}\right)+g(-2)-8\right)g(0)=(-5+64-8)\times 1=51$$

9. A bag contains N balls out of which 3 balls are white, 6 balls are green, and the remaining balls are blue. Assume that the balls are identical otherwise. Three balls are drawn randomly one after the other without replacement. For i=1,2,3, let W_i, G_i, and B_i denote the events that the ball drawn in the i^{th} draw is a white ball, green ball, and blue ball, respectively, If the probability $P\left(W_1 \cap G_2 \cap B_3\right)=\frac{2}{5N}$ and the conditional probability $P\left(B_3 \mid W_1 \cap G_2\right)=\frac{2}{9}$, then N equals..........

Answer (11)

Sol. N Balls $=3W+6G+(N-9)B$

$$P\left(W_1 \cap G_2 \cap B_3\right)=\frac{2}{5N}$$

$$\Rightarrow \frac{3}{N}\times\frac{6}{N-1}\times\frac{N-9}{N-2}=\frac{2}{5N}$$

$$\Rightarrow N^2-48N+407=0$$

N = 11 or 37

$$P\left(B_3 \mid W_1 \cap G_2\right)=\frac{2}{9}$$

$$\Rightarrow \frac{P\left(W_1 \cap G_2 \cap B_3\right)}{P\left(W_1 \cap G_2\right)}=\frac{2}{9}$$

$$\Rightarrow \frac{\frac{2}{5N}}{\frac{3}{N}\times\frac{6}{N-1}}=\frac{2}{9}$$

$$\Rightarrow \frac{N-1}{45}=\frac{2}{9}$$

$$\Rightarrow N=11$$

10. Let the function $f:\mathbb{R}\rightarrow\mathbb{R}$ be defined by

$$f(x)=\frac{\sin x}{e^{\pi x}}\frac{\left(x^{2023}+2024x+2025\right)}{\left(x^2-x+3\right)}+\frac{2}{e^{\pi x}}\frac{\left(x^{2023}+2024x+2025\right)}{\left(x^2-x+3\right)}$$

Then the number of solutions of $f(x)=0$ in R is

Answer (01)

Sol. $f(x)=0$

$$\Rightarrow \frac{x^{2023}+2024x+2025}{\left(x^2-x+3\right)}\left[\frac{\sin x+2}{e^{\pi x}}\right]=0$$

$$\Rightarrow x^{2023}+2024x+2025=0$$

Let $g(x)=x^{2023}+2024x+2025$

$g'(x)=2023x^{2022}+2024>0\forall x\in R$

$\therefore$ f(x)=0 has only one solution

11. Let $\vec{p}=2\hat{i}+\hat{j}+3\hat{k}$ and $\vec{q}=\hat{i}-\hat{j}+\hat{k}$. If for some real numbers α,β and γ, we have $15\hat{i}+10\hat{j}+6\hat{k}=\alpha(2\vec{p}+\vec{q})+\beta(\vec{p}-2\vec{q})+\gamma(\vec{p}\times\vec{q})$, then the value of γ is

Answer (2)

Sol. $2\vec{p}+\vec{q}=5\hat{i}+\hat{j}+7\hat{k}$

$\vec{p}-2\vec{q}=0\hat{i}+3\hat{j}+\hat{k}$

$$\vec{p}\times\vec{q}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 3\\ 1 & -1 & 1\end{vmatrix}$$

$=\hat{i}(4)-\hat{j}(-1)+\hat{k}(-3)$

$=4\hat{i}+\hat{j}-3\hat{k}$

$15\hat{i}+10\hat{j}+6\hat{k}=\alpha(5\hat{i}+\hat{j}+7\hat{k})+\beta(3\hat{j}+\hat{k})+\gamma(4\hat{i}+\hat{j}-3\hat{k})$

$\therefore\ 15=5\alpha+4\gamma$

$10=\alpha+3\beta+\gamma$

$6=7\alpha+\beta-3\gamma$

$\therefore\ \alpha=\frac{7}{5},\beta=\frac{11}{5},\gamma=2$

$\therefore\ \gamma=2$

12. A normal with slope $\frac{1}{\sqrt{6}}$ is drawn from the point $(0,-\alpha)$ to the parabola $x^2=-4ay$, where $a>0$. Let L be the line passing through $(0,-\alpha)$ and parallel to the directrix of the parabola. Suppose that L

intersects the parabola at two points A and B. Let r denote the length of the latus rectum and s denote the square of the length of the line segment AB. If $r : s = 1 : 16$, then the value of 24a is____

Answer (12)

Sol. $x^2 = -4ay$

Equation of normal

$$y = mx - 2a - \frac{a}{m^2}$$

$$-\alpha = -2a - \frac{a}{\frac{1}{6}} = -8a$$

$$\Rightarrow \alpha = 8a$$

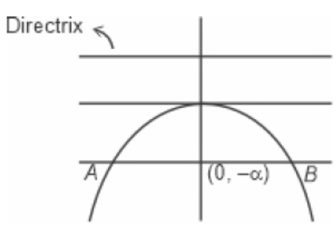
Equation of required line

$$y = -\alpha$$

$$\Rightarrow y = -8a, \text{ solving with } x^2 = -4ay$$

$$\Rightarrow x^2 = 32a^2$$

$$\Rightarrow x = \pm 4\sqrt{2}a$$



$$= \pm \frac{\alpha}{\sqrt{2}}$$

$$A\left(\frac{\alpha}{\sqrt{2}}, -\alpha\right), B\left(\frac{-\alpha}{\sqrt{2}}, -\alpha\right) \Rightarrow AB = \sqrt{2}\alpha$$

$$\Rightarrow \frac{r}{s} = \frac{4a}{2\alpha^2} = \frac{1}{16} \Rightarrow \frac{4a}{2 \times 64a^2} = \frac{1}{16}$$

$$\Rightarrow a = \frac{1}{2}$$

$$\Rightarrow \boxed{24a = 12}$$

13. Let the function $f : [1, \infty) \to R$ be defined by

$$f(t)=\begin{cases}(-1)^{n+1}2, & \text{if } t=2n-1, n\in N,\\ \frac{(2n+1-t)}{2}f(2n-1)+\frac{(t-(2n-1))}{2}f(2n+1), & \text{if } 2n-1<t<2n+1, n\in N.\end{cases}$$

Define $g(x)=\int_1^x f(t)dt, x\in(1,\infty)$. Let α denote the number of solutions of the equation $g(x)=0$ in the interval

$(1,8]$ and $\beta=\lim_{x\to 1+}\frac{g(x)}{x-1}$. Then the value of $\alpha+\beta$ is equal to

Answer (5)

Sol. $f(t)=\left(\frac{(2n+1)-t}{2}\right)(-1)^{n+1}2+\left(\frac{t-(2n-1)}{2}\right)(-1)^{n+2}2, t\in(2n-1,2n+1)$

$\Rightarrow f(t)=2(-1)^{n+1}(2n-t), t\in(2n-1,2n+1)$

$\Rightarrow g(x)=\int_1^x f(t)dt, x\in(1,8]$

$$=\begin{cases}\int_1^x 2(2-t)dt, 1<x\le 3, n=1\\ 3\\ \int_1^3 2(2-t)dt+\int_3^x(2t-8)dt, 3<x\le 5, n=2\\ 3\\ \int_1^3 2(2-t)dt+\int_3^5(2t-8)dt+\int_5^x 2(6-t)dt, 5<x\le 7, n=3\\ 3\\ \int_1^3 2(2-t)dt+\int_3^5(2t-8)dt+\int_5^7 2(6-t)dt+\int_7^x(2t-16)dt, x\in(7,8], n=4\end{cases}$$

$$=\begin{cases}-x^2+4x-3, 1<x\le 3,\\ x^2-8x+15, 3<x\le 5\\ -x^2+12x-35, 5<x\le 7\\ x^2-16x+63, 7<x\le 8\end{cases}=\begin{cases}-(x-1)(x-3), 1<x\le 3\\ (x-3)(x-5), 3<x\le 5\\ -(x-5)(x-7), 5<x\le 7\\ (x-7)(x-9), 7<x\le 8\end{cases}$$

$\Rightarrow g(x)=0\Rightarrow x=3,5,7\Rightarrow\alpha=3$

$\beta=\lim_{x\to 1^+}\left(\frac{g(x)}{x-1}\right)=\lim_{x\to 1^+}-\frac{(x-1)(x-3)}{x-1}=2$

$\Rightarrow\alpha+\beta=5$

SECTION 4 (Maximum Marks : 12)

• This section contains TWO (02) paragraphs.

• Based on each paragraph, there are TWO (02) questions.

• The answer to each question is a NUMERICAL VALUE.

• For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.

• If the numerical value has more than two decimal places, truncate/round-off the value to TWO decimal places.

• Answer to each question will be evaluated according to the following marking scheme:

Full Marks : +3 If ONLY the correct numerical value is entered in the designated place;

Zero Marks : 0 In all other cases.

PARAGRAPH I

Let $S=\{1,2,3,4,5,6\}$ and X be the set of all relations R from S to S that satisfy both the following properties:

i. R has exactly 6 elements.

ii. For each $(a,b)\in R$, we have $|a-b|\geq 2$.

Let $Y=\{R\in X$: The range of R has exactly one element $\}$ and $Z=\{R\in X : R$ is a function from S to $S\}$.

Let $n(A)$ denote the number of elements in a set A.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

14. If $n(X)={}^{m}C_6$, then the value of m is

Answer (20)

Sol. $S=\{1,2,3,4,5,6\}\quad R:S\to S$

Number of elements in $R=6$ and for each $(a,b)\in R; |a-b|\geq 2$

$X\to$ set of all relation $R:S\to S$

If

$$\left.\begin{array}{ll}
a=1,\ b=3,4,5,6 & \to 4\\
a=2,\ b=4,5,6 & \to 3\\
a=3,\ b=1,5,6 & \to 3\\
a=4,\ b=1,2,6 & \to 3\\
a=5,\ b=1,2,3 & \to 3\\
a=6,\ b=1,2,3,4 & \to 4
\end{array}\right]\text{— Total number of ordered pairs } (a,b) \text{ s.t. } |a-b|\geq 2 \;\; = 20$$

$\therefore\quad n(X)=$ number of elements in X

$= {}^{20}C_6 \quad \therefore m = 20$

PARAGRAPH I

Let $S = \{1, 2, 3, 4, 5, 6\}$ and X be the set of all relations R from S to S that satisfy both the following properties:

i. R has exactly 6 elements.

ii. For each $(a, b) \in R$, we have $|a - b| \geq 2$.

Let $Y = \{R \in X$: The range of R has exactly one element $\}$ and $Z = \{R \in X : R$ is a function from S to $S\}$.

Let $n(A)$ denote the number of elements in a set A.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

15. If the value of $n(Y) + n(Z)$ is k^2, then $|k|$ is _____

Ans (36)

Sol. $S = \{1, 2, 3, 4, 5, 6\}$ $\qquad R : S \to S$

Number of elements in R = 6 and for each $(a, b) \in R; |a - b| \geq 2$

$X \to$ set of all relation $R : S \to S$

If

$a = 1$	$b = 3, 4, 5, 6$	$\to$	4
$a = 2$	$b = 4, 5, 6$	$\to$	3
$a = 3$	$b = 1, 5, 6$	$\to$	3
$a = 4$	$b = 1, 2, 6$	$\to$	3
$a = 5$	$b = 1, 2, 3$	$\to$	3
$a = 6$	$b = 1, 2, 3, 4$	$\to$	4

Total number of ordered pairs (a, b) s. t. $|a - b| \geq 2 = 20$

$\therefore \quad n(X) =$ number of elements in X

$= {}^{20}C_6 \quad \therefore m = 20$

$Y=\{R\in X$: The range of R has exactly one element}

From above, if range of R has exactly one element, then maximum number of elements in R will be 4 .

$\therefore \quad n(Y)=0$

$Z=\{R\in X : R \text{ is a function from } S \text{ to } S\}$

$n(Z)={}^4C_1\times{}^3C_1\times{}^3C_1\times{}^3C_1\times{}^3C_1\times{}^4C_1$

$=(36)^2$

$n(y)+n(z)=0+(36)^2=k^2$

$\Rightarrow \quad |k|=36$

PARAGRAPH II

Let $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ be the function defined by $f(x)=\sin^2 x$ and let $g:\left[0,\frac{\pi}{2}\right]\to[0,\infty)$ be the function defined by $g(x)=\sqrt{\frac{\pi x}{2}-x^2}$.

(There are two questions based on PARAGRAPH "II", the question given below is one of them)

16. The value of $2\int_0^{\frac{\pi}{2}} f(x)g(x)dx-\int_0^{\frac{\pi}{2}} g(x)dx$ is

Answer (0)

Sol. $f(x)=\sin^2 x, g(x)=\sqrt{\frac{\pi}{2}x-x^2}$

Here $f\left(\frac{\pi}{2}-x\right)=\cos^2 x, g\left(\frac{\pi}{2}-x\right)=g(x)$

Let $I_1=2\int_0^{\frac{\pi}{2}} f(x)g(x)=2\int_0^{\frac{\pi}{2}}\sin^2 x\cdot g(x)dx$ ...(1)

as $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$\Rightarrow I_1=2\int_0^{\frac{\pi}{2}}\cos^2 xg(x)dx$ ...(2)

$(1)+(2)$

$\Rightarrow 2I_1=2\int_0^{\frac{\pi}{2}} g(x)dx$

$$\Rightarrow I_1 = \int_0^{\frac{\pi}{2}} g(x)dx$$

PARAGRAPH II

Let $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ be the function defined by $f(x)=\sin^2 x$ and let $g:\left[0,\frac{\pi}{2}\right]\to[0,\infty)$ be the function defined by $g(x)=\sqrt{\frac{\pi x}{2}-x^2}$.

(There are two questions based on PARAGRAPH "II", the question given below is one of them)

17. The value of $\frac{16}{\pi^3} \int_0^{\frac{\pi}{2}} f(x) g(x) d x$ is $\qquad$

Answer (0.25)

Sol. According to Q .16

$$2\int_0^{\frac{\pi}{2}} f(x)g(x)dx=\int_0^{\frac{\pi}{2}} g(x)dx=l_1 \text{ (let)}$$

Now, $l_1=\int_0^{\frac{\pi}{2}} g(x)dx=\int_0^{\frac{\pi}{2}}\sqrt{\frac{\pi}{2}x-x^2}\,dx$

$$I_1=\int_0^{\frac{\pi}{2}}\sqrt{\left(\frac{\pi}{4}\right)^2-\left(\frac{\pi}{4}-x\right)^2}$$

Put $\frac{\pi}{4}-x=t$

$$\Rightarrow dx=-dt$$

$$I_1=-\int_{\frac{\pi}{4}}^{\frac{-\pi}{4}}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2}\,dt$$

$$l_1=\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2}\,dt$$

$$l_1=2\int_0^{\frac{\pi}{4}}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2}\,dt=2\left[\frac{t}{2}\sqrt{\left(\frac{\pi}{4}\right)^2-t^2}+\frac{\pi^2}{32}\sin^{-1}\left(\frac{4t}{\pi}\right)\right]_0^{\frac{\pi}{4}}$$

$$l_1=\frac{\pi^3}{32}$$

Now, $I=\frac{8}{\pi^3}l_1$

$$I=\frac{1}{4}=0.25$$