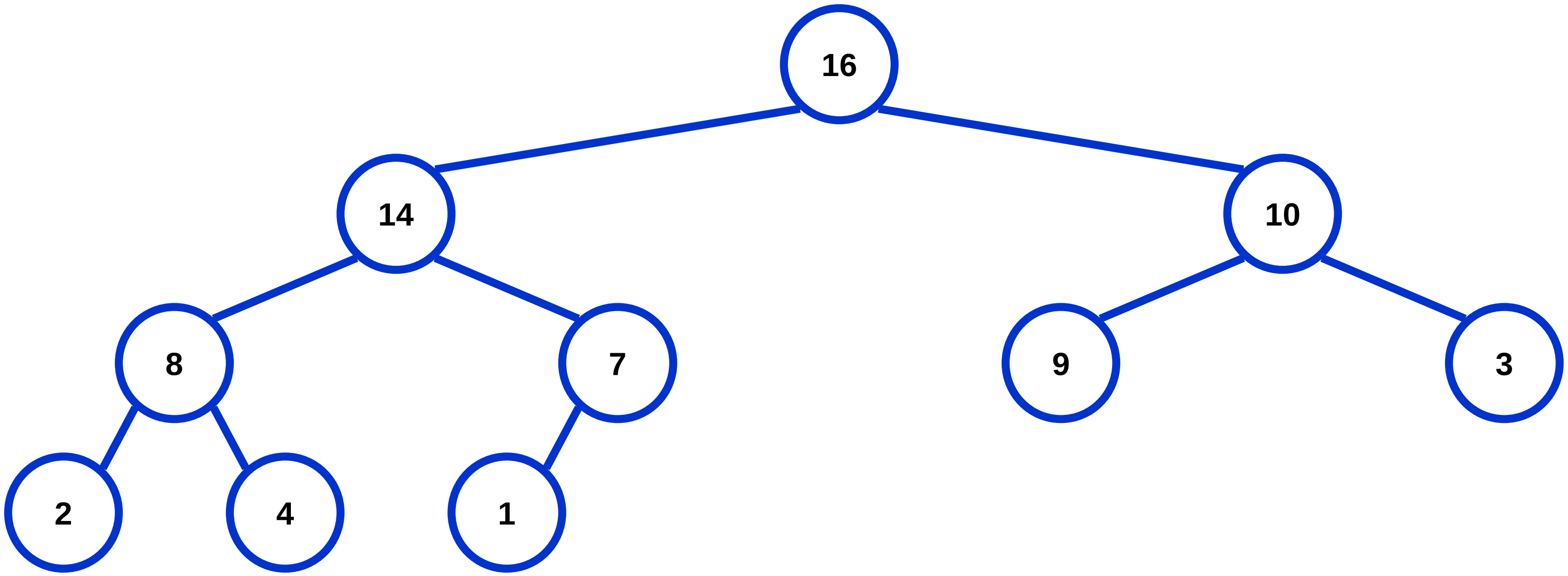
Heapsort

Heaps

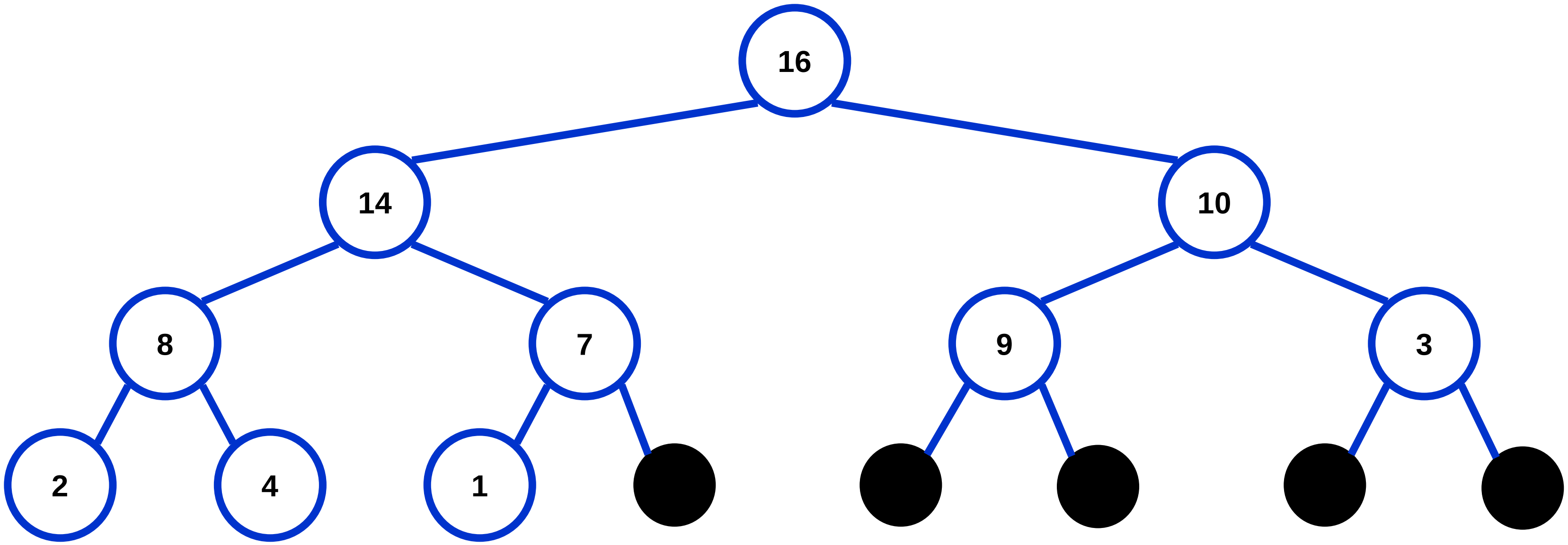
- A *heap* can be seen as a complete binary tree:



  - *What makes a binary tree complete?*
  - *Is the example above complete?*

Heaps

- A *heap* can be seen as a complete binary tree:



  - The book calls them "nearly complete" binary trees; can think of unfilled slots as null pointers

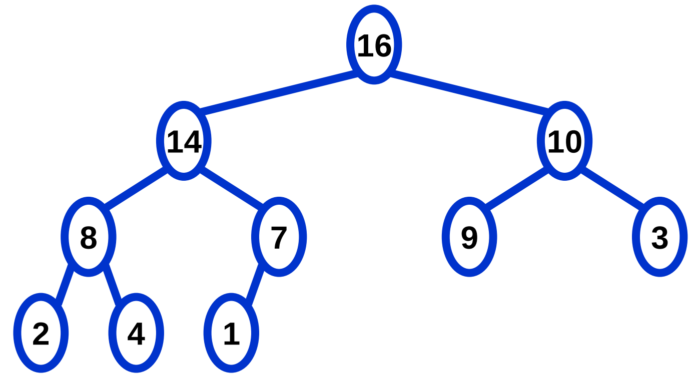
Heaps

- In practice, heaps are usually implemented as arrays:

A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

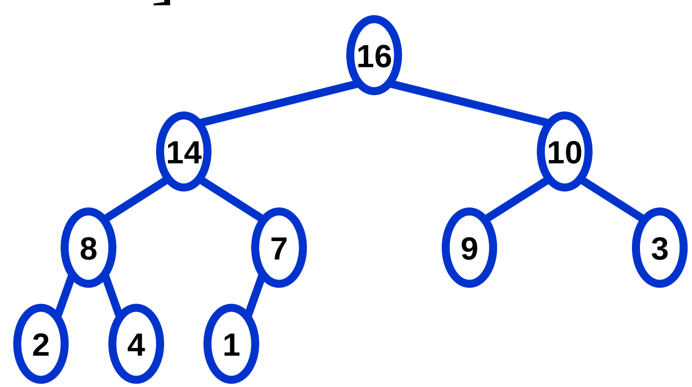
=



Heaps

- To represent a complete binary tree as an array:
  - The root node is A[1]
  - Node *i* is A[*i*]
  - The parent of node *i* is A[*i*/2] (note: integer divide)
  - The left child of node *i* is A[2*i*]
  - The right child of node *i* is A[2*i* + 1]

A =	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*	=



Referencing Heap Elements

- So…

```
Parent(A[i]) { return A[⌊i/2⌋]; }
Left(A[i]) { return A[2*i]; }
right(A[i]) { return A[2*i + 1]; }
```

The Heap Property

- Heaps also satisfy the *heap property*:

  A[*Parent*(A[i])] $\geq$ A[i]   for all nodes $i > 1$

  - In other words, the value of a node is at most the value of its parent
  - *Where is the largest element in a heap stored?*

Heap Height

- Definitions:
  - The *height* of a node in the tree = the number of edges on the longest downward path to a leaf
  - The height of a tree = the height of its root
- *What is the height of an n-element heap? Why?*
- This is nice: basic heap operations take at most time proportional to the height of the heap

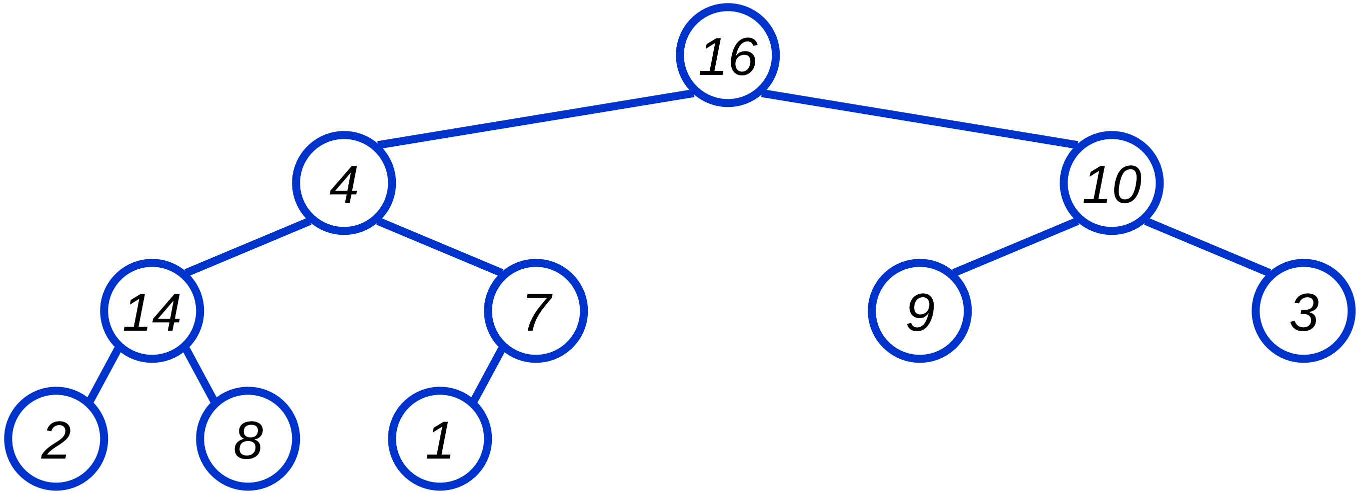
Heap Operations: Heapify()

- **`Heapify()`**: maintain the heap property
  - Given: a node *i* in the heap with children *l* and *r*
  - Given: two subtrees rooted at *l* and *r*, assumed to be heaps
  - Problem: The subtree rooted at *i* may violate the heap property (*How?*)
  - Action: let the value of the parent node "float down" so subtree at *i* satisfies the heap property
    - *What do you suppose will be the basic operation between i, l, and r?*

Heap Operations: Heapify()

```
Heapify(A, i)
{
  l = Left(i); r = Right(i);
  if (l <= heap_size(A) && A[l] > A[i])
      largest = l;
  else
      largest = i;
  if (r <= heap_size(A) && A[r] > A[largest])
      largest = r;
  if (largest != i)
      Swap(A, i, largest);
      Heapify(A, largest);
}
```

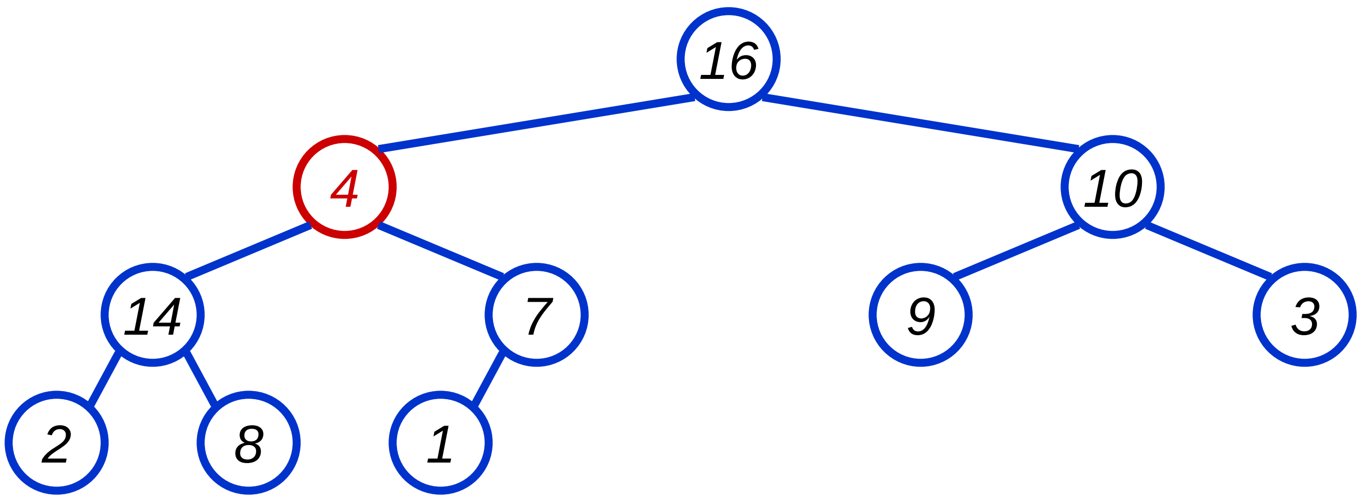
Heapify() Example



A =

16	4	10	14	7	9	3	2	8	1

Heapify() Example



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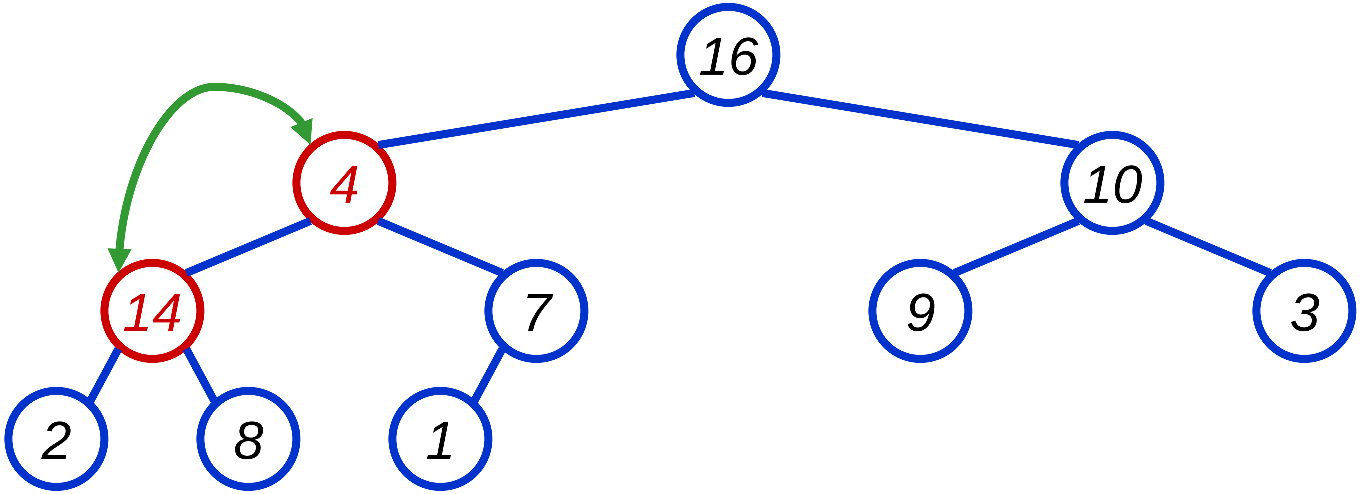

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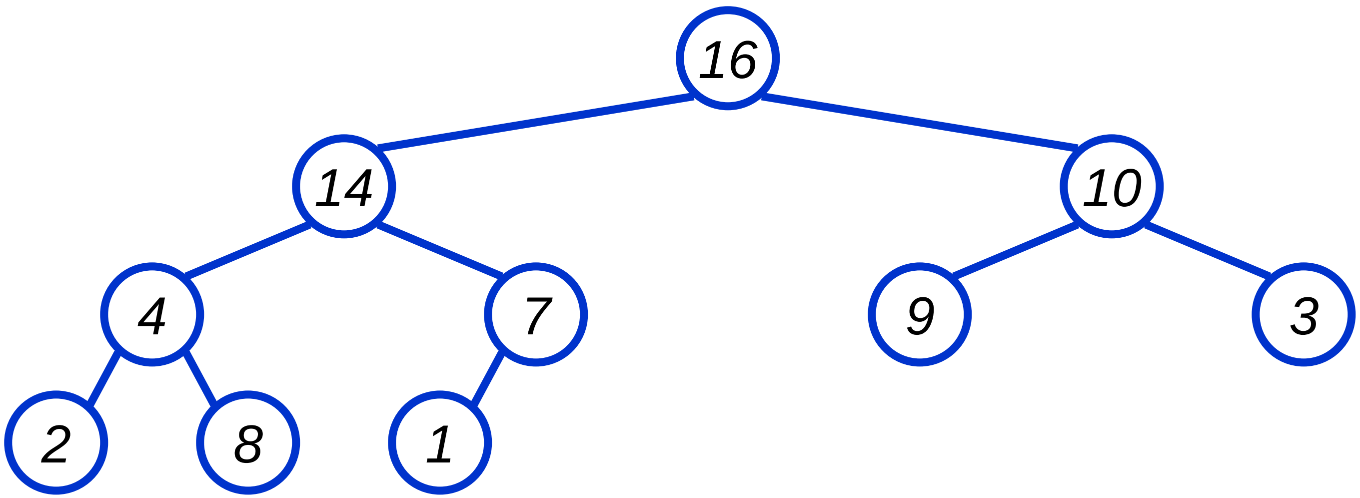
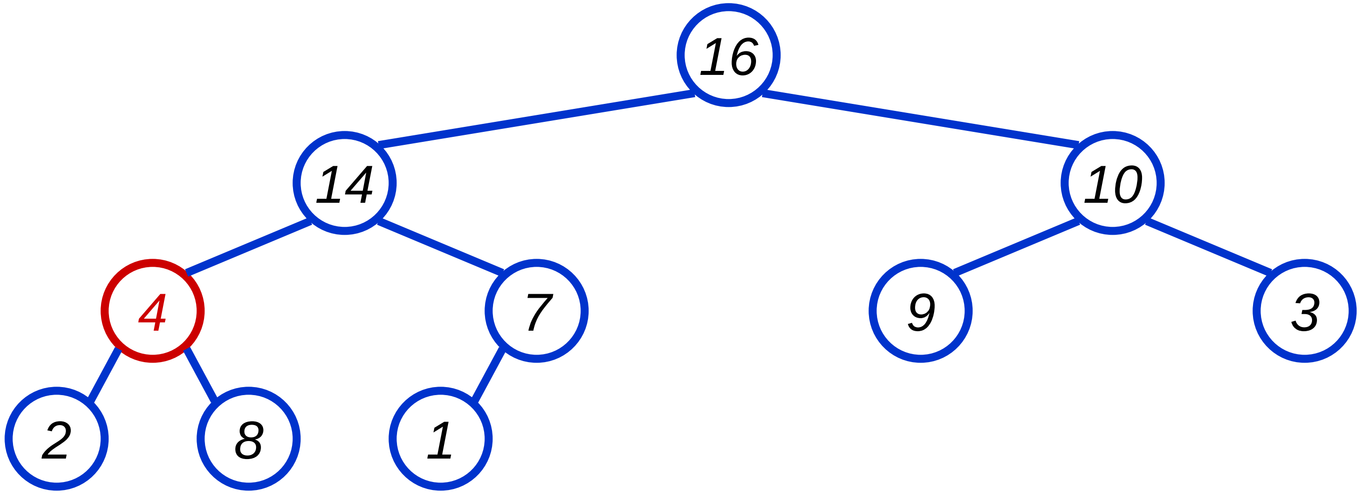
Heapify() Example



A =

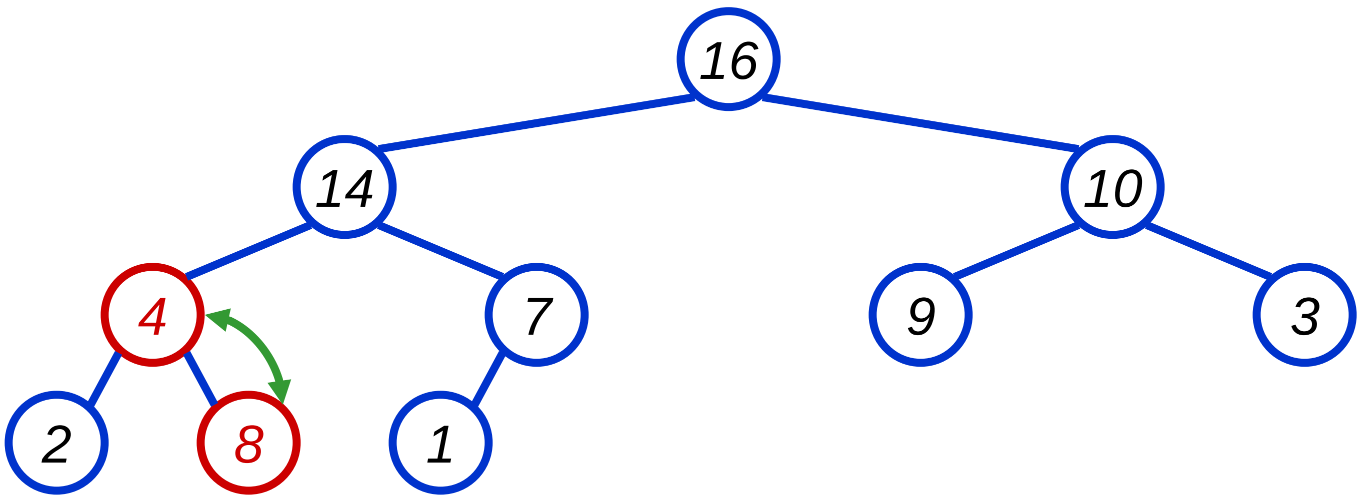
16	14	10	4	7	9	3	2	8	1

Heapify() Example



$A =$										
	16	14	10	4	7	9	3	2	8	1

Heapify() Example



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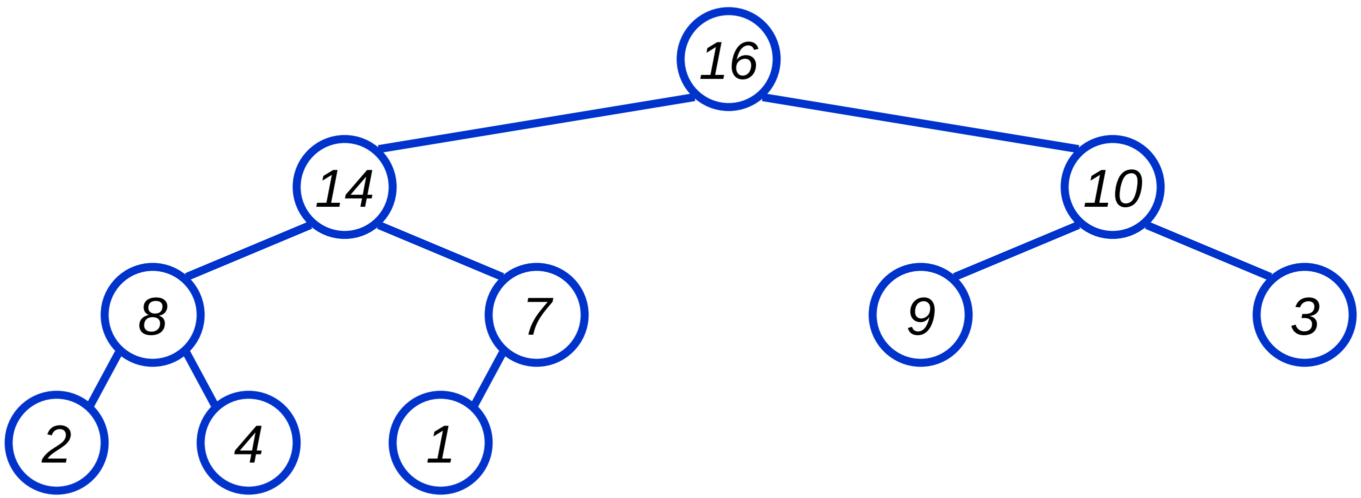
Heapify() Example



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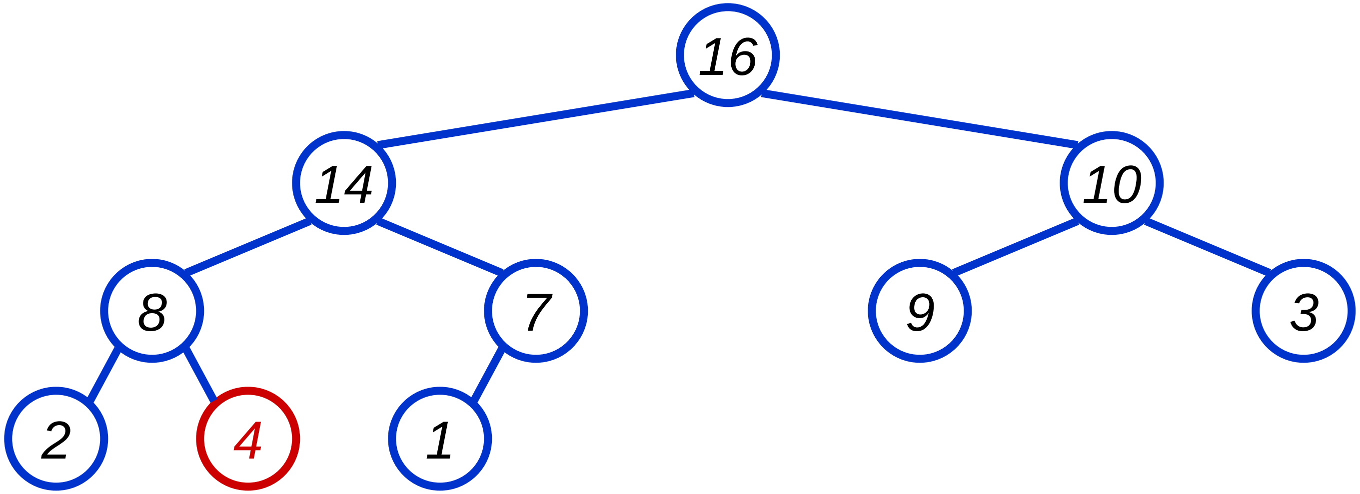
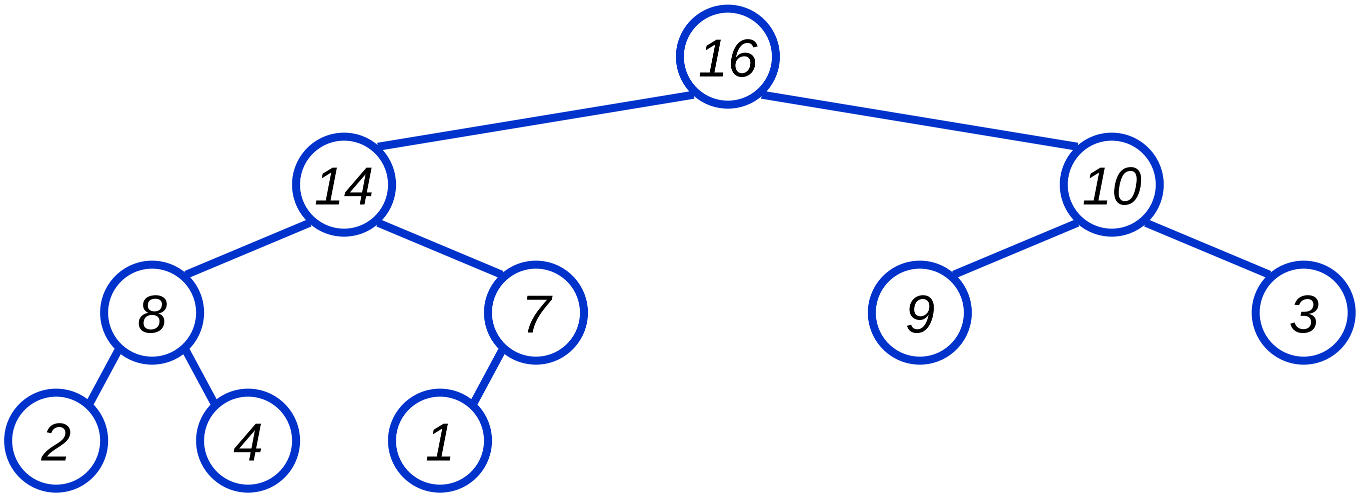
Heapify() Example



A =

16	14	10	8	7	9	3	2	4	1

Heapify() Example



A =									
16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

Analyzing Heapify(): Informal

- *Aside from the recursive call, what is the running time of* **`Heapify()`***?*
- *How many times can* **`Heapify()`** *recursively call itself?*
- *What is the worst-case running time of* **`Heapify()`** *on a heap of size n?*

Analyzing Heapify(): Formal

- Fixing up relationships between *i*, *l*, and *r* takes $\Theta(1)$ time
- *If the heap at i has n elements, how many elements can the subtrees at l or r have?*
  - Draw it
- Answer: $2n/3$ (worst case: bottom row 1/2 full)
- So time taken by **`Heapify()`** is given by

$$T(n) \le T(2n/3) + \Theta(1)$$

Analyzing Heapify(): Formal

- So we have

  $T(n) \leq T(2n/3) + \Theta(1)$

- By case 2 of the Master Theorem,

  $T(n) = O(\lg n)$

- Thus, **`Heapify()`** takes logarithmic time

Heap Operations: BuildHeap()

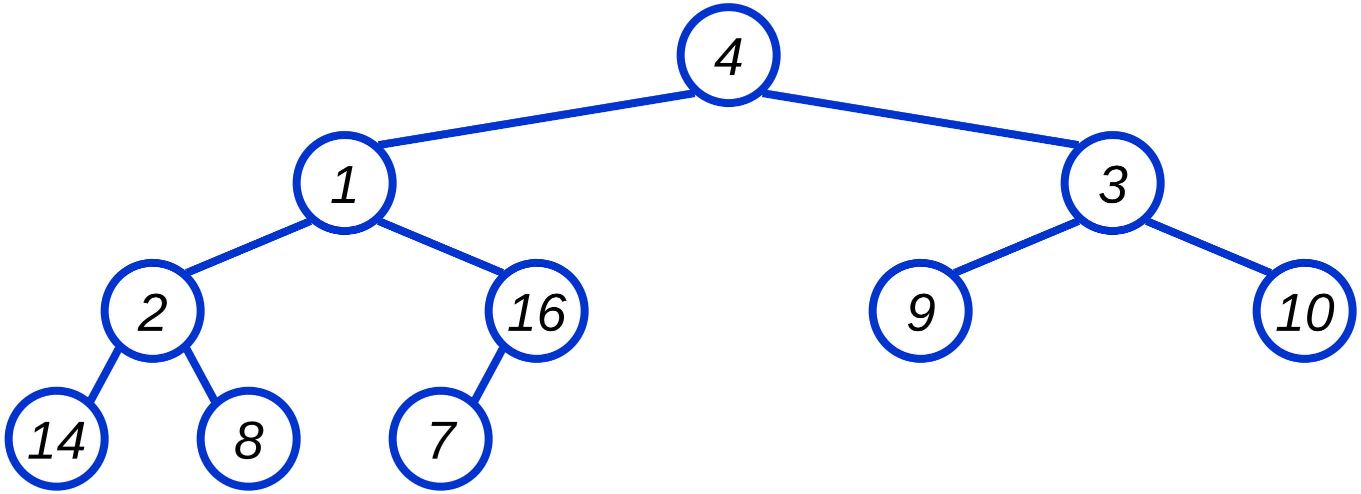
- We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays
  - Fact: for array of length *n*, all elements in range $A[\lfloor n/2 \rfloor + 1 .. n]$ are heaps (*Why?*)
  - So:
    - Walk backwards through the array from n/2 to 1, calling **`Heapify()`** on each node.
    - Order of processing guarantees that the children of node *i* are heaps when *i* is processed

BuildHeap()

```
// given an unsorted array A, make A a heap
BuildHeap(A)
{
  heap_size(A) = length(A);
  for (i = ⌊length[A]/2⌋ downto 1)
      Heapify(A, i);
}
```

BuildHeap() Example

- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



Analyzing BuildHeap()

- Each call to **`Heapify()`** takes O(lg *n*) time
- There are O(*n*) such calls (specifically, $\lfloor n/2 \rfloor$)
- Thus the running time is O(*n* lg *n*)
  - *Is this a correct asymptotic upper bound?*
  - *Is this an asymptotically tight bound?*
- A tighter bound is O(*n*)
  - *How can this be? Is there a flaw in the above reasoning?*

Analyzing BuildHeap(): Tight

- To **`Heapify()`** a subtree takes O(h) time where h is the height of the subtree
  - h = O(lg m), m = # nodes in subtree
  - The height of most subtrees is small
- Fact: an n-element heap has at most $\lceil n/2^{h+1} \rceil$ nodes of height h

Heapsort

- Given **`BuildHeap()`**, an in-place sorting algorithm is easily constructed:
  - Maximum element is at A[1]
  - Discard by swapping with element at A[n]
    - Decrement heap_size[A]
    - A[n] now contains correct value
  - Restore heap property at A[1] by calling **`Heapify()`**
  - Repeat, always swapping A[1] for A[heap_size(A)]

Heapsort

```
Heapsort(A)
{
    BuildHeap(A);
    for (i = length(A) downto 2)
    {
        Swap(A[1], A[i]);
        heap_size(A) -= 1;
        Heapify(A, 1);
    }
}
```

Analyzing Heapsort

- The call to **`BuildHeap()`** takes $O(n)$ time
- Each of the $n - 1$ calls to **`Heapify()`** takes $O(\lg n)$ time
- Thus the total time taken by **`HeapSort()`**
  $= O(n) + (n - 1)\, O(\lg n)$
  $= O(n) + O(n \lg n)$
  $= O(n \lg n)$

Priority Queues

- Heapsort is a nice algorithm, but in practice Quicksort (coming up) usually wins
- But the heap data structure is incredibly useful for implementing *priority queues*
  - A data structure for maintaining a set *S* of elements, each with an associated value or *key*
  - Supports the operations **`Insert()`**, **`Maximum()`**, and **`ExtractMax()`**
  - ***What might a priority queue be useful for?***

Implementing Priority Queues with Heaps

The heap data structure with the `Heapify-down` and `Heapify-up` operations can efficiently implement a priority queue that is constrained to hold at most N elements at any point in time. Here we summarize the operations we will use.

- `StartHeap`(N) returns an empty heap H that is set up to store at most N elements. This operation takes $O(N)$ time, as it involves initializing the array that will hold the heap.
- `Insert`(H, v) inserts the item v into heap H. If the heap currently has n elements, this takes $O(\log n)$ time.
- `FindMin`(H) identifies the minimum element in the heap H but does not remove it. This takes $O(1)$ time.
- `Delete`(H, i) deletes the element in heap position i. This is implemented in $O(\log n)$ time for heaps that have n elements.
- `ExtractMin`(H) identifies and deletes an element with minimum key value from a heap. This is a combination of the preceding two operations, and so it takes $O(\log n)$ time.