1. 作业1

1.1. 题目

First order condition for a convex function Assume that f is differentiable. dom f is convex. Then f is convex iff

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle \quad \text{for } \forall \boldsymbol{x}, y \in \operatorname{dom} f$$

1.2. 解

$$\begin{aligned}
& f(ty + (1-t)x) \leq tf(y) + (1-t)f(x) \\
& \Longrightarrow f(x + t(y-x)) \leq f(x) + t(f(y) - f(x)) \\
& \Longrightarrow f(x + t(y-x)) - f(x) \leq t(f(y) - f(x)) \\
& \Longrightarrow \frac{f(x + t(y-x)) - f(x)}{t} \leq f(y) - f(x) \\
& \Longrightarrow f(y) \geq f(x) + \frac{f(x + t(y-x)) - f(x)}{t}
\end{aligned}$$

let $g(t) = f(x + t(y-x))$

then $f(y) \geq f(x) + \frac{g(t)-g(0)}{t}$

when $t \to 0$, $\frac{g(t)-g(0)}{t} = g'(t) \leq f(y) - f(x) = [\text{介值定理}] f'(a)(y-x)$ where a is between x and y

when $t \to 0$, we have $lim f'(a) = f(x)$

So $f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \langle \nabla f(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle$

2. 作业2

2.1. 题目

根据PPT中shrinkage,绘制二维的结果

Shrinkage Operator

a) For given $\boldsymbol{b} \in \mathbb{R}^n$ and $\lambda > 0$, the solution to

$$\min_{\boldsymbol{u} \in \mathbb{R}^n} \lambda \|\boldsymbol{u}\|_2 + \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{b}\|^2$$

is

$$\boldsymbol{u}^* = \max\{\|\boldsymbol{b}\|_2 - \lambda, 0\}\frac{\boldsymbol{b}}{\|\boldsymbol{b}\|_2} := \text{shrink}\,(\boldsymbol{b}, \lambda)$$

Proof.

i) For $\boldsymbol{u} \neq \boldsymbol{0}$, 可导, 得$\lambda\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_2} + \boldsymbol{u} - \boldsymbol{b} = \boldsymbol{0}$, 整理得

$$(\frac{\lambda}{\|\boldsymbol{u}\|_2} + 1)\boldsymbol{u} = \boldsymbol{b} \qquad (\boldsymbol{u} \parallel \boldsymbol{b})$$

两边取norm, 得 $\lambda + \|\boldsymbol{u}\|_2 = \|\boldsymbol{b}\|_2$, i.e., $\|\boldsymbol{u}\|_2 = \|\boldsymbol{b}\|_2 - \lambda$, i.e. $\boldsymbol{u} = (\|\boldsymbol{b}\|_2 - \lambda)\frac{\boldsymbol{b}}{\|\boldsymbol{b}\|_2}$.

ii) $u = 0$, 原式分成2部分, 重点是第一部分

$$\lambda \boldsymbol{g} + \boldsymbol{u} - \boldsymbol{b} = \boldsymbol{0} \qquad \text{for } \|\boldsymbol{g}\| \leq 1 \quad \text{and } \boldsymbol{u} = 0$$

i.e. $\boldsymbol{b} = \lambda \boldsymbol{g}$, thus $\|\boldsymbol{b}\|_2 \leq \lambda$. 所以 $\boldsymbol{u} = \max\{\|\boldsymbol{b}\|_2 - \lambda, 0\}\frac{\boldsymbol{b}}{\|\boldsymbol{b}\|_2}$ □

Shrinkage Operator

b) the solution to

$$\min_{\boldsymbol{u}\in\mathbb{R}^n} \lambda\|\boldsymbol{u}\|_1 + \frac{1}{2}\|\boldsymbol{u}-\boldsymbol{b}\|^2$$

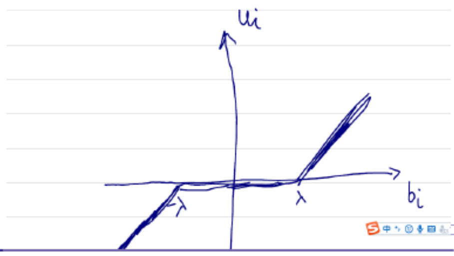
is

$$\boldsymbol{u}^* = \max\{|b_i|-\lambda, 0\}\text{sign}\,(b_i)$$

Proof.

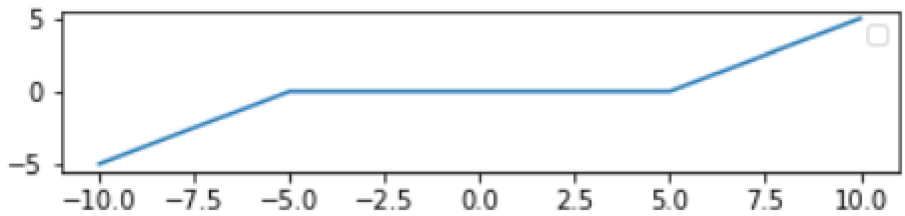
原式化为 $\min_{u_1,\cdots,u_n}\sum_i(|u_i|+\frac{1}{2}|u_i-b_i|^2)$

$$u_i = \max\{|b_i|-\lambda, 0\}\text{sign}\, b_i$$



绘制u关于b的图像

2.2. 解



https://www.kaggle.com/tianyilt/shrinkage-operator-in-2d

参考王佳镐同学,感觉coding很规范,改成了标量版本