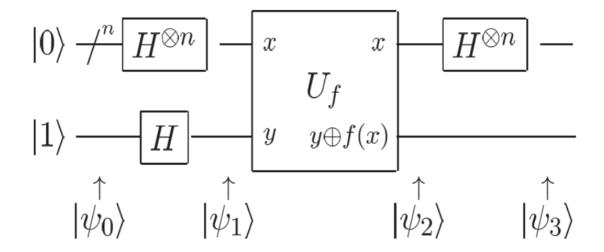
严格推导出Deutsch-Jozsa算法中每一步量子门作用后系统状态的变化过程



由公式

$$H: egin{cases} |0
angle
ightarrow rac{1}{\sqrt{2}}(|0
angle + |1
angle) \ |1
angle
ightarrow rac{1}{\sqrt{2}}(|0
angle - |1
angle) \end{cases}$$

$$U:|x
angle|y
angle
ightarrow|x
angle|y\oplus f(x)
angle$$

且已知

$$|\psi_0\rangle = \underbrace{|0\rangle \otimes |0\rangle \otimes \dots |0\rangle}_{n} \otimes |1\rangle$$

$$= |0\rangle^{\otimes n} |1\rangle$$

则易得

$$egin{aligned} |\psi_1
angle &= [rac{1}{\sqrt{2}}(|0
angle + |1
angle)]^{\otimes n}rac{1}{\sqrt{2}}(|0
angle - |1
angle) \ dots &: [rac{1}{\sqrt{2}}(|0
angle + |1
angle)]^{\otimes n} &= rac{1}{\sqrt{2^n}}\sum_{x_{n-1}=0}^1 \ldots \sum_{x_0=0}^1 |x_{n-1}, \ldots, x_0
angle \stackrel{x=x_{n-1}2^{n-1}+\cdots+x_02^0}{\longrightarrow} rac{1}{\sqrt{2^n}}\sum_{x=0}^{2^{n-1}} |x
angle \ dots &: |\psi_1
angle &= rac{1}{\sqrt{2^n}}\sum_{x=0}^{2^{n-1}} |x
angle rac{|0
angle - |1
angle}{\sqrt{2}} \end{aligned}$$

考虑

$$egin{aligned} |\psi_2
angle &= U |\psi_1
angle \ &= U rac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} |x
angle rac{|0
angle - |1
angle}{\sqrt{2}} \ &= rac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} |x
angle rac{U(|0
angle - |1
angle)}{\sqrt{2}} \end{aligned}$$

由于

$$egin{aligned} |0\oplus f(0)
angle - |1\oplus f(0)
angle \ &= egin{cases} |0\oplus 0
angle - |1\oplus 0
angle &= |0
angle - |1
angle &= (-1)^{f(0)}(|0
angle - |1
angle), f(0) &= 0 \ |0\oplus 1
angle - |1\oplus 1
angle &= |1
angle - |0
angle &= (-1)^{f(0)}(|0
angle - |1
angle), f(0) &= 1 \ &\therefore |0\oplus f(0)
angle - |1\oplus f(0)
angle &= (-1)^{f(0)}(|0
angle - |1
angle) \end{aligned}$$

同理可知

$$egin{aligned} |0\oplus f(1)
angle - |1\oplus f(1)
angle = (-1)^{f(1)}(|0
angle - |1
angle) \ dots & \forall f(x) \in \{0,1\}, U(|0
angle - |1
angle) = |0\oplus f(x)
angle - |1\oplus f(x)
angle = (-1)^{f(x)}(|0
angle - |1
angle) \end{aligned}$$

带入即可得

$$|\psi_2
angle=rac{1}{\sqrt{2^n}}\sum_{x=0}^{2^{n-1}}|x
angle(-1)^{f(x)}rac{|0
angle-|1
angle}{\sqrt{2}}$$

注意到

$$egin{aligned} dots H|0
angle &= rac{|0
angle + |1
angle}{\sqrt{2}} = rac{(-1)^{0.0}|0
angle + (-1)^{0.1}|1
angle}{\sqrt{2}} = rac{\sum_{z \in \{0,1\}} (-1)^{0.z}|z
angle}{\sqrt{2}} \ H|1
angle &= rac{|0
angle - |1
angle}{\sqrt{2}} = rac{(-1)^{1.0}|0
angle + (-1)^{1.1}|1
angle}{\sqrt{2}} = rac{\sum_{z \in \{0,1\}} (-1)^{1.z}|z
angle}{\sqrt{2}} \ dots H|x_i
angle &= rac{\sum_{z \in \{0,1\}} (-1)^{x_i \cdot z}|z
angle}{\sqrt{2}} \end{aligned}$$

那么

$$H^{\otimes n}|x_1,x_2,\ldots,x_n
angle = rac{\sum_{z_1,x_2,\ldots,z_n}(-1)^{x_1\cdot z_1+x_2\cdot z_2+\cdots+x_n\cdot z_n}|z_1,z_2,\ldots,z_n
angle}{\sqrt{2^n}}$$

若定义

$$x \cdot z = x_1 \cdot z_1 + x_2 \cdot z_2 + \dots + x_n \cdot z_n$$

则有

$$|H^{\otimes n}|x
angle = rac{1}{\sqrt{2^n}} \sum_{z=0}^{2^{n-1}} (-1)^{x\cdot z} |x
angle$$

因此

$$\begin{split} |\psi_3\rangle &= H^{\otimes n} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} |x\rangle (-1)^{f(x)} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} (-1)^{f(x)} H^{\otimes n} |x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} (-1)^{f(x)} \left[\frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^{n-1}} (-1)^{x \cdot z} |z\rangle \right] \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= \frac{1}{2^n} \sum_{x=0}^{2^{n-1}} (-1)^{f(x)} \left[\sum_{z=0}^{2^{n-1}} (-1)^{x \cdot z} |z\rangle \right] \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\ &= \frac{1}{2^n} \sum_{x=0}^{2^{n-1}} \left[\sum_{z=0}^{2^{n-1}} (-1)^{f(x) + x \cdot z} |z\rangle \right] \frac{|0\rangle - |1\rangle}{\sqrt{2}} \end{split}$$

练习1.5

$$\begin{aligned} |\psi_1\rangle &= |0001\rangle \\ |\psi_2\rangle &= |+++-\rangle \\ |\psi_3\rangle &= |+++-\rangle \\ |\psi_4\rangle &= |-++-\rangle \\ |\psi_5\rangle &= |--+-\rangle \\ |\psi_6\rangle &= |----\rangle \\ |\psi_7\rangle &= |111\rangle |-\rangle \end{aligned}$$

所以得到结果000的概率是0