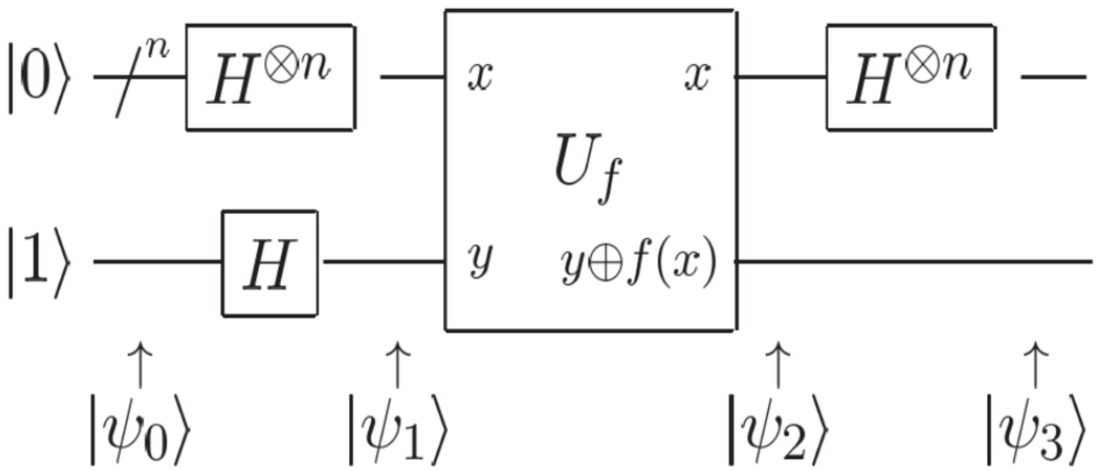
严格推导出Deutsch-Jozsa算法中每一步量子门作用后系统状态的变化过程



由公式

$$H:\begin{cases}|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \\ |1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\end{cases}$$

$$U:|x\rangle|y\rangle \rightarrow |x\rangle|y\oplus f(x)\rangle$$

且已知

$$\begin{aligned}|\psi_0\rangle &= \underbrace{|0\rangle\otimes|0\rangle\otimes\ldots|0\rangle}_{n}\otimes|1\rangle \\ &= |0\rangle^{\otimes n}|1\rangle\end{aligned}$$

则易得

$$|\psi_1\rangle = [\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)]^{\otimes n}\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

$$\because [\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)]^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x_{n-1}=0}^{1}\ldots\sum_{x_0=0}^{1}|x_{n-1},\ldots,x_0\rangle \xrightarrow{x=x_{n-1}2^{n-1}+\cdots+x_0 2^0} \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^{n-1}}|x\rangle$$

$$\therefore |\psi_1\rangle = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^{n-1}}|x\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}}$$

考虑

$$\begin{aligned}|\psi_2\rangle &= U|\psi_1\rangle \\ &= U\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^{n-1}}|x\rangle\frac{|0\rangle-|1\rangle}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^{n-1}}|x\rangle\frac{U(|0\rangle-|1\rangle)}{\sqrt{2}}\end{aligned}$$

由于

$$
\begin{aligned}
&|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle \\
&= \begin{cases} |0 \oplus 0\rangle - |1 \oplus 0\rangle = |0\rangle - |1\rangle = (-1)^{f(0)}(|0\rangle - |1\rangle), f(0) = 0 \\ |0 \oplus 1\rangle - |1 \oplus 1\rangle = |1\rangle - |0\rangle = (-1)^{f(0)}(|0\rangle - |1\rangle), f(0) = 1 \end{cases} \\
&\therefore |0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle = (-1)^{f(0)}(|0\rangle - |1\rangle)
\end{aligned}
$$

同理可知

$$
\begin{aligned}
&|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle = (-1)^{f(1)}(|0\rangle - |1\rangle) \\
&\therefore \forall f(x) \in \{0,1\}, U(|0\rangle - |1\rangle) = |0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle = (-1)^{f(x)}(|0\rangle - |1\rangle)
\end{aligned}
$$

带入即可得

$$
|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} |x\rangle (-1)^{f(x)} \frac{|0\rangle - |1\rangle}{\sqrt{2}}
$$

注意到

$$
\begin{aligned}
\because H|0\rangle &= \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{(-1)^{0\cdot 0}|0\rangle + (-1)^{0\cdot 1}|1\rangle}{\sqrt{2}} = \frac{\sum_{z\in\{0,1\}}(-1)^{0\cdot z}|z\rangle}{\sqrt{2}} \\
H|1\rangle &= \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \frac{(-1)^{1\cdot 0}|0\rangle + (-1)^{1\cdot 1}|1\rangle}{\sqrt{2}} = \frac{\sum_{z\in\{0,1\}}(-1)^{1\cdot z}|z\rangle}{\sqrt{2}} \\
\therefore H|x_i\rangle &= \frac{\sum_{z\in\{0,1\}}(-1)^{x_i\cdot z}|z\rangle}{\sqrt{2}}
\end{aligned}
$$

那么

$$
H^{\otimes n}|x_1, x_2, \ldots, x_n\rangle = \frac{\sum_{z_1, x_2, \ldots, z_n}(-1)^{x_1\cdot z_1 + x_2\cdot z_2 + \cdots + x_n\cdot z_n}|z_1, z_2, \ldots, z_n\rangle}{\sqrt{2^n}}
$$

若定义

$$
x \cdot z = x_1 \cdot z_1 + x_2 \cdot z_2 + \cdots + x_n \cdot z_n
$$

则有

$$
H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^{n-1}} (-1)^{x\cdot z}|x\rangle
$$

因此

$$
\begin{aligned}
|\psi_3\rangle &= H^{\otimes n} \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} |x\rangle (-1)^{f(x)} \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} (-1)^{f(x)} H^{\otimes n}|x\rangle \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^{n-1}} (-1)^{f(x)} \left[\frac{1}{\sqrt{2^n}} \sum_{z=0}^{2^{n-1}} (-1)^{x\cdot z}|z\rangle\right] \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{1}{2^n} \sum_{x=0}^{2^{n-1}} (-1)^{f(x)} \left[\sum_{z=0}^{2^{n-1}} (-1)^{x\cdot z}|z\rangle\right] \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \frac{1}{2^n} \sum_{x=0}^{2^{n-1}} \left[\sum_{z=0}^{2^{n-1}} (-1)^{f(x) + x\cdot z}|z\rangle\right] \frac{|0\rangle - |1\rangle}{\sqrt{2}}
\end{aligned}
$$

练习1.5

$$
\begin{aligned}
|\psi_1\rangle &= |0001\rangle \\
|\psi_2\rangle &= |+++-\rangle \\
|\psi_3\rangle &= |+++-\rangle \\
|\psi_4\rangle &= |-++-\rangle \\
|\psi_5\rangle &= |--+-\rangle \\
|\psi_6\rangle &= |----\rangle \\
|\psi_7\rangle &= |111\rangle|-\rangle
\end{aligned}
$$

所以得到结果000的概率是0