Introduction to Quantum Computing (Fall, 2022)
Exam Sheet

1. (10 points) Show that unitaries cannot "delete" information: there is no 1-qubit unitary U that maps $|\psi\rangle \mapsto |0\rangle$ for every 1-qubit state $|\psi\rangle$.

2. (10 points) Suppose we have the following Pauli matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .$$

   (a) Show that the set of matrices $\{I, X, Y, Z\}$ forms a basis for any 2×2 Hermitian matrix, i.e. any 2×2 Hermitian matrix is a linear combination of the Pauli matrices.

   (b) Show that the set of matrices $\{\sigma \otimes \sigma' \mid \sigma, \sigma' \in \{I, X, Y, Z\}\}$ form a basis for any 4×4 Hermitian matrix.

3. (20 points) Suppose $\{L_l\}$ and $\{M_m\}$ are two sets of measurement operators. Show that a measurement defined by the measurement operators $\{L_l\}$ followed by a measurement defined by the measurement operators $\{M_m\}$ is physically equivalent to a single measurement defined by measurement operators $\{N_{l,m}\}$ with the representation $N_{l,m} = M_m L_l$.

4. (20 points) Alice and Bob share an EPR-pair, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

   (a) Let C be a 2×2 matrix. Show that $\mathrm{tr}((C \otimes I)|\psi\rangle\langle\psi|) = \frac{1}{2}\mathrm{tr}(C)$.

   (b) Alice could apply one of the four Pauli matrices I, X, Y, Z to her qubit. Use part (a) to show that the four resulting 2-qubit states form an orthonormal set.

   (c) Suppose Alice applies one of the four Pauli matrices to her qubit and then sends that qubit to Bob. Give the four projectors of a 4-outcome projective measurement that Bob could do on his 2 qubits to find out which Pauli matrix Alice actually applied.

5. (20 points) Let $\theta \in [0, 2\pi)$, $U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $|\psi\rangle = U_\theta|0\rangle$ and $|\psi^\perp\rangle = U_\theta|1\rangle$.

   (a) Show that $ZX|\psi^\perp\rangle = |\psi\rangle$.

   (b) Show that an EPR-pair, $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, can also be written as $\frac{1}{\sqrt{2}}(|\psi\rangle|\psi\rangle + |\psi^\perp\rangle|\psi^\perp\rangle)$.

   (c) Suppose Alice and Bob start with an EPR-pair. Alice applies U_θ^{-1} to her qubit and then measures it in the computational basis. What pure state does Bob have if her outcome was 0, and what pure state does he have if her outcome was 1?

   (d) Suppose Alice knows the number θ but Bob does not. Give a protocol that uses one EPR-pair and one classical bit of communication where Bob ends up with the qubit $|\psi\rangle$ (in contrast to general teleportation of an unknown qubit, which uses one EPR-pair and two bits of communication).

6. (20 points) Give a circuit for implementing the inverse quantum Fourier transform.

7. (extra 20 points) Let ρ be a density matrix. A *minimal ensemble* for ρ is an ensemble $\{p_i, |\psi_i\rangle\}$, i.e. $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, containing a number of elements equal to the rank of ρ. Let $|\psi\rangle$ be any state in the support of ρ. (The *support* of a Hermitian matrix A is the vector space spanned by the eigenvectors of A with non-zero eigenvalues.) Show that there is a minimal ensemble for ρ that contains $|\psi\rangle$, and moreover that in any such ensemble $|\psi\rangle$ must appear with probability

$$p_0 = \frac{1}{\langle\psi_i|\,\rho^{-1}\,|\psi_i\rangle},$$

where ρ^{-1} is defined to be the inverse of ρ, when ρ is considered as a matrix acting only on the support of ρ. (This definition removes the problem that ρ may not have an inverse.)