Climbing Powers Solution Analysis

Question:

Climbing Powers

Age 16 to 18 ★★

We can define 2^{3^4} either as $(2^3)^4$ or as $2^{(3^4)}$. Does it make any difference?

Now calculate $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ and $\sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)}$ and answer the following question for the natural extension of both definitions.

Which number is the biggest

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}}$$

where the powers of root 2 go on for ever, or

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}?$$

**WARNING: BAD HANDWRITING IS PRESENT IN THIS PRESENTATION

From exponent laws, we know that the second expression, namely

$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$

Now, we are looking at the first expression, namely

$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$ **to infinity**

There are two ways to interpret $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$, as shown below:

Well done Paul Jefferys, you got close to a complete solution here. We have to consider two different values of these climbing powers depending on the order of operations which can be shown by putting in brackets. We can define 2^{3^4} either as $(2^3)^4 = 2^{12}$ or as $2^{(3^4)} = 2^{81}$. In the same way there are two interpretations of $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ The first of these is $f(f(\sqrt{2}))$ where $f(x) = x^{\sqrt{2}}$ which gives:

$$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\times\sqrt{2}} = \sqrt{2}^2 = 2$$

$f(x) = x^{\sqrt{2}}$

$f(\sqrt{2}) = \sqrt{2}^{\sqrt{2}}$

$f(\sqrt{2}^{\sqrt{2}}) = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = 2$

In the second case we get $g(g(\sqrt{2}))$ where $g(x) = (\sqrt{2})^x$, and using a calculator to get an approximate value gives:

$$\sqrt{2}^{(\sqrt{2}^{\sqrt{2}})} = \sqrt{2}^{1.63...} = 1.76 \text{ to } 2 \text{ decimal places.}$$

$g(x) = (\sqrt{2})^x$

$g(\sqrt{2}) = (\sqrt{2})^{\sqrt{2}}$

$g((\sqrt{2})^{\sqrt{2}}) = (\sqrt{2})^{(\sqrt{2})^{\sqrt{2}}}$

So

$$\sqrt{2}^{(\sqrt{2}^{\sqrt{2}})} < (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}.$$

Now consider

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot}}}}}}}$$

where the powers of $\sqrt{2}$ go on forever. ~~We have seen that we have two possibilities.~~

Now that we know that we can interpret $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ in these two different ways, we can apply this knowledge when figuring out

$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$ **to infinity.**

where the powers of $\sqrt{2}$ go on forever. We have seen that we have two possibilities, namely

$$X_1 = \lim x_n \text{ where } x_1 = \sqrt{2},\ x_{n+1} = x_n^{\sqrt{2}} \text{ or}$$
$$X_2 = \lim x_n \text{ where } x_1 = \sqrt{2},\ x_{n+1} = (\sqrt{2})^{x_n}.$$

$\sqrt{2},\ (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}},\ ((\sqrt{2}^{\sqrt{2}})^{\sqrt{2}})^{\sqrt{2}}$

$\sqrt{2},\ \sqrt{2}^{\sqrt{2}},\ \sqrt{2}^{(\sqrt{2}^{\sqrt{2}})}, \ldots$

N.B. Both iterations can be done on a calculator or computer: $X_1 = \lim x_n$ is equivalent to iterating $f(x) = x^{\sqrt{2}}$ and $X_2 = \lim x_n$ is equivalent to iterating $g(x) = (\sqrt{2})^x$. If you do this experimentally, in each case starting with $x_1 = \sqrt{2}$, you will find that the first iteration appears to converge to infinity and the second appears to converge to 2. We claim $X_1 = +\infty$.

The two different ways to interpret it can be calculated using a computer, using the IDLE shell (python) and Mathematica. I coded it into both of the programs, but I'm going to put Mathematica's here, since I'm using Mathematica to write this.

This code runs X1; the different iterations it calculates is illustrated with the red drawings above.

```
In[]:= v = Sqrt[2];
For[i = 0, i < 19, i++, v = v^Sqrt[2]; Print[N[v]]]
```

```
1.63253
2.
2.66514
4.
7.10299
16.
50.4525
256.
2545.46
65536.
```

6.47935×10^{6}

4.29497×10^{9}

4.19819×10^{13}

1.84467×10^{19}

1.76248×10^{27}

3.40282×10^{38}

3.10635×10^{54}

1.15792×10^{77}

9.64939×10^{108}

From the output, we can clearly see that it diverges to positive infinity, just like it says in the image. Now, let's try coding X2 and seeing what results it outputs.

In[∘]:=
```
n = 1;
For[i = 0, i < 19, i++, n = Sqrt[2]^n; Print[N[n]]]
```

```
1.41421
1.63253
1.76084
1.84091
1.89271
1.927
1.95003
1.96566
1.97634
1.98367
1.98871
1.99219
1.99459
1.99626
1.99741
1.9982
1.99876
1.99914
1.9994
```

From the output, we can clearly see that it converges to 2 (with more iterations it will be even clearer), just like it says in the image.

Therefore, $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ = X2, but $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ ≠ X1. This isn't a solid proof, though, so the next segment covers the proof.

Proof

Let's first prove that X1 diverges to infinity, without coding it up.

[redacted].

$\log x_n^{\sqrt{2}} = \sqrt{2} \log x_n$

(properties of logs)

x_1

Proof

We have $\boxed{x_1 = \sqrt{2}}$ and $x_{n+1} = x_n^{\sqrt{2}}$, thus

$$\log x_{n+1} = \sqrt{2} \log x_n, \quad \boxed{\log x_1 = \log \sqrt{2}.}$$

Thus

take the log of both sides

If we try different values of *n*, we get:

$$\log x_2 = \boxed{\sqrt{2}^{\,1} \log \sqrt{2}}$$

$$\log x_3 = \sqrt{2}\left(\sqrt{2}\log\sqrt{2}\right) = \boxed{\left(\sqrt{2}\right)^2 \log\sqrt{2}} = 2\log 2$$

$$\log x_4 = \sqrt{2}\left(\left(\sqrt{2}\right)^2 \log\sqrt{2}\right) = \boxed{\left(\sqrt{2}\right)^3 \log\sqrt{2}}$$

and so on

We can notice that the answers we get are all in the form

$$\log x_{n+1} = \left(\sqrt{2}\right)^n \log\sqrt{2},$$

As $n \to \infty$, $\log(x_n) \to +\infty$. If $\log(x_n) \to +\infty$, it must be true that $x_n \to \infty$.

Now that we've proven X1 diverges to +infinity, let's prove that X2 converges 2, by induction.

$$X_2 = \lim x_n \text{ where } x_1 = \sqrt{2},\ x_{n+1} = (\sqrt{2})^{x_n}.$$

$\sqrt{2},\ \sqrt{2}^{\sqrt{2}},\ \sqrt{2}^{\left(\sqrt{2}^{\sqrt{2}}\right)},\ \ldots$

To prove that X2 converges to 2, we need to prove that the sequence increases (i.e. $x_n < x_{n+1}$) and the upper limit is 2 (it doesn't go beyond 2) $x_n < 2$.

PROOF BY INDUCTION: $x_n < 2$

If we prove the statement "If $x_n < 2$, then $x_{n+1} < 2$" then it is proven that $x_n < 2$. This is because it shows that as the sequence progresses, it is always less than 2.

Statement 1: We assume that $x_n < 2$.
Statement 2: When n is 1, the first term is $x_1 = \sqrt{2}$. $\sqrt{2}$ is clearly less than 2, so we know that $x_1 < 2$.
Statement 3: We know that $x_{n+1} == \left(\sqrt{2}\right)^{x_n}$ by definition.
Statement 4: From statement 1, we know that $\left(\sqrt{2}\right)^{x_n} < \left(\sqrt{2}\right)^2$.
From statements 3 and 4, $x_{n+1} < 2$.

Therefore, if $x_n < 2$, then $x_{n+1} < 2$.
So, by induction, $x_n < 2$.

PROOF BY INDUCTION: $x_n < x_{n+1}$

Statement 1: Now, we assume that $\mathbf{x_{n-1} < x_n}$ (similar to the statement we are proving.)

Statement 2: When n is 2, $x_{2-1} = \sqrt{2} < x_2 = \left(\sqrt{2}\right)^{\sqrt{2}}$. This is because $1 < \sqrt{2}$.

Statement 3: We know that $x_{n+1} == \left(\sqrt{2}\right)^{x_n}$ by definition.

Statement 4: From statement 3, $\frac{x_{n+1}}{x_{(n+1)-1}} = \frac{\left(\sqrt{2}\right)^{x_n}}{\left(\sqrt{2}\right)^{x_{n-1}}} = \left(\sqrt{2}\right)^{x_n - x_{n-1}}$

Statement 5: From statement 1, $x_n - x_{n-1} > 0$.

Statement 6: From statement 5, $\left(\sqrt{2}\right)^{x_n - x_{n-1}} > 1$.

Statement 7: From statements 4 and 6, $\frac{x_{n+1}}{x_n} > 1$.

Statement 8: From multiplying both sides by $x_n$in statement 7, we get $x_{n+1} > x_n$.

Therefore, $x_{n+1} > x_n$ by induction.

Now, we know that the sequence is increasing and is less than 2.

To prove that X2 converges to 2, we assign $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}$ to a variable x.

$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}} = x$

Since the exponent is the same thing, we can simplify it to

$\sqrt{2}^{x} = x$

There are two solutions of x that satisfy this:

x = 2, 4

However, since the sequence is less than 2, it must converge to a number less than or equal to 2. (X2 ≤ 2). Since 2 is the only one that fits, and 4 is not less than or equal 2, X2 must converge to 2.

Conclusion

If the definitions of X2 and X1 are defined by this image below:

where the powers of $\sqrt{2}$ go on forever. We have seen that we have two possibilities, namely

$$X_1 = \lim x_n \text{ where } x_1 = \sqrt{2},\ x_{n+1} = x_n^{\sqrt{2}} \text{ or}$$
$$X_2 = \lim x_n \text{ where } x_1 = \sqrt{2},\ x_{n+1} = (\sqrt{2})^{x_n}.$$

$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = X2$

$(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \neq X1$

If you read the whole thing, you are a legend :)