Proyecto Compilando Conocimiento

Ecuaciones Diferenciales

La Transformada de Laplace

Introducción

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Capítulo 1

La Transformada de Laplace

1.1. Definición

Dada una función $f(t)$ definida para toda $t \geq 0$ la tranformada de Laplace de $f(t)$ es la función $F(s)$ definida de la Siguiente manera:

$$\mathscr{L}\left\{f(t)\right\} = F(s) = \int_0^\infty e^{-st} f(t) dt \tag{1.1}$$

Recuerda que una Integral Impropia:

$$\int_0^\infty g(t)dt = \lim_{b\to\infty} \int_a^b g(t)dt$$

1.2. Tabla de Transformación

$f(t)$	$\mathscr{L}\left\{f(t)\right\} = F(s)$
k	$\frac{k}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$sen(kt)$	$\frac{k}{s^2+k^2}$
$cos(kt)$	$\frac{s}{s^2+k^2}$
$senh(kt)$	$\frac{k}{s^2-k^2}$
$cosh(kt)$	$\frac{s}{s^2-k^2}$
$y'(t)$	$s[\mathscr{L}\left\{y(t)\right\}] - y(0)$
$y''(t)$	$s^2[\mathscr{L}\left\{y(t)\right\}] - sy(0) - y'(0)$
$e^{at} \cdot t^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{at} \cdot cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$

$f(t)$	$\mathscr{L}\left\{f(t)\right\} = F(s)$
$e^{at} \cdot sen(kt)$	$\dfrac{k}{(s-a)^2 + k^2}$

1.3. Demostraciones de la Tabla

1.3.1. $f(t) = k$

Calcule la Tranformada de Laplace cuando $f(t) = k$

Recuerda que podemos hacer que: $\mathscr{L}\{k\} = k \cdot \mathscr{L}\{1\}$

$$\begin{aligned}
\mathscr{L}\{1\} = F(s) &= \int_0^\infty e^{-st} 1 dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st} dt \\
&= \lim_{b\to\infty} \frac{e^{-st}}{-s}\Big|_0^b \\
&= \lim_{b\to\infty} \left[\frac{1}{s} - \frac{e^{-sb}}{s}\right] \\
&= \frac{1}{s}
\end{aligned}$$

Por lo tanto:

$$\mathscr{L}\{k\} = \frac{k}{s} \tag{1.2}$$

1.3.2. $f(t) = e^{at}$

Calcule la Tranformada de Laplace cuando $f(t) = e^{at}$

$$\begin{aligned}
\mathscr{L}\{e^{at}\} = F(s) &= \int_0^\infty e^{-st} \cdot e^{at} dt \\
&= \lim_{b\to\infty} \int_0^b e^{-st+at} dt \\
&= \lim_{b\to\infty} \int_0^b e^{-(s-a)t} dt \\
&= \lim_{b\to\infty} \frac{e^{-(s-a)t}}{-(s-a)}\Big|_0^b \\
&= \lim_{b\to\infty} \left[\frac{e^{-(s-a)b}}{-(s-a)} - \frac{e^{-(s-a)0}}{-(s-a)}\right] \\
&= \frac{1}{s-a}
\end{aligned}$$

1.3.3. $f(t) = sen(kt)$

Calcule la Tranformada de Laplace cuando $f(t) = sen(kt)$

$$\begin{aligned}
\mathscr{L}\{sen(kt)\} = F(s) &= \mathscr{L}\left\{\frac{e^{kti} - e^{-kti}}{2i}\right\} \\
&= \frac{1}{2i}\left(\mathscr{L}\{e^{kti}\} - \mathscr{L}\{e^{-kti}\}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s-ki} - \frac{1}{s+ki}\right) \\
&= \frac{1}{2i}\left(\frac{s+ki-(s-ki)}{s^2+k^2}\right) \\
&= \frac{1}{2i}\left(\frac{2ki}{s^2+k^2}\right) \\
&= \frac{k}{s^2+k^2}
\end{aligned}$$

1.3.4. $f(t) = cos(kt)$

Calcule la Tranformada de Laplace cuando $f(t) = cos(kt)$

$$\begin{aligned}
\mathscr{L}\{cos(kt)\} = F(s) &= \mathscr{L}\left\{\frac{e^{kti} + e^{-kti}}{2i}\right\} \\
&= \frac{1}{2i}\left(\mathscr{L}\{e^{kti}\} + \mathscr{L}\{e^{-kti}\}\right) \\
&= \frac{1}{2i}\left(\frac{1}{s-ki} + \frac{1}{s+ki}\right) \\
&= \frac{1}{2i}\left(\frac{s-ki+s+ki}{s^2+k^2}\right) \\
&= \frac{1}{2i}\left(\frac{2si}{s^2+k^2}\right) \\
&= \frac{s}{s^2+k^2}
\end{aligned}$$

1.3.5. $f(t) = senh(kt)$

Calcule la Tranformada de Laplace cuando $f(t) = senh(kt)$

$$
\begin{aligned}
\mathscr{L}\{senh(kt)\} = F(s) &= \mathscr{L}\left\{\frac{e^{kt} - e^{-kt}}{2i}\right\} \\
&= \frac{1}{2}\left(\mathscr{L}\{e^{kt}\} - \mathscr{L}\{e^{-kt}\}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-k} - \frac{1}{s+k}\right) \\
&= \frac{1}{2}\left(\frac{s+k-(s-k)}{s^2-k^2}\right) \\
&= \frac{1}{2}\left(\frac{2k}{s^2-k^2}\right) \\
&= \frac{k}{s^2-k^2}
\end{aligned}
$$

1.3.6. $f(t) = cosh(kt)$

Calcule la Tranformada de Laplace cuando $f(t) = cosh(kt)$

$$
\begin{aligned}
\mathscr{L}\{cosh(kt)\} = F(s) &= \mathscr{L}\left\{\frac{e^{kt} + e^{-kt}}{2i}\right\} \\
&= \frac{1}{2}\left(\mathscr{L}\{e^{kt}\} - \mathscr{L}\{e^{-kt}\}\right) \\
&= \frac{1}{2}\left(\frac{1}{s-k} + \frac{1}{s+k}\right) \\
&= \frac{1}{2}\left(\frac{s+k+s-k}{s^2-k^2}\right) \\
&= \frac{1}{2}\left(\frac{2s}{s^2-k^2}\right) \\
&= \frac{s}{s^2-k^2}
\end{aligned}
$$

1.3.7. $f(t) = g'(t)$

Calcule la Tranformada de Laplace cuando $f(t) = cosh(kt)$

Recuerda que vamos a usar este cambio de variable:

- $u = e^{-st} \quad \rightarrow \quad du = -se^{-st}dt$
- $dv = g'(t) \quad \rightarrow \quad v = g(t)$

$$
\begin{aligned}
\mathscr{L}\{g'(t)\} = F(s) &= \int_0^\infty e^{-st} g'(t) dt \\
&= \rightarrow_{AplicandoCambios} \\
&= e^{-st} f(t)|_0^\infty - \int_0^\infty -se^{-st} g(t) dt \\
&= e^{-st} f(t)|_0^\infty + s \int_0^\infty e^{-st} g(t) dt \\
&= e^{-st} f(t)|_0^\infty + s\mathscr{L}\{g(t)\} \\
&= (e^{-s\infty} f(\infty)) - (e^{-s0} f(0)) + s\mathscr{L}\{g(t)\} \\
&= -f(0) + s\mathscr{L}\{g(t)\} \\
&= s\mathscr{L}\{g(t)\} - f(0)
\end{aligned}
$$

1.4. Ejemplos Útiles

$f(t) = e^{t+7}$

Calcule la Tranformada de Laplace cuando $f(t) = e^{t+7}$

$$
\begin{aligned}
\mathscr{L}\{e^{t+7}\} = & \\
&= \mathscr{L}\{e^t \cdot e^7\} \\
&= e^7 \cdot \mathscr{L}\{e^t\} \\
&= e^7 \frac{1}{s-1} \\
&= \frac{e^7}{s-1}
\end{aligned}
$$

Capítulo 2

Ecuaciones con Laplace

2.1. Ecuaciones de 2do Grado

$y' + ay = 0$

Usando la Transformada de Laplace:

$$
\begin{aligned}
\mathscr{L}\{y' - ay\} &= L\{0\} \\
\mathscr{L}\{y'\} - a\mathscr{L}\{y\} &= L\{0\} \\
s\mathscr{L}\{y\} - y(0) - a\mathscr{L}\{y\} &= 0 \\
(s-a)\mathscr{L}\{y\} &= y(0) \\
\mathscr{L}\{y\} &= \frac{y(0)}{s-a} \\
\mathscr{L}\{y\} &= y(0)\mathscr{L}\{e^{ax}\} \\
\mathscr{L}\{y\} &= \mathscr{L}\{y(0) \cdot e^{ax}\} \\
y &= y(0) \cdot e^{ax}
\end{aligned}
$$

$y'' + 4y' + 4y = t^2$

Donde $y(0) = y'(0) = 0$ Usando la Transformada de Laplace:

$$
\begin{aligned}
\mathscr{L}\{y'' + 4y' + 4y\} &= L\{t^2\} \\
\mathscr{L}\{y''\} + 4\mathscr{L}\{y'\} + 4\mathscr{L}\{y\} &= \frac{2!}{s^3} = \frac{2}{s^3} \\
s^2[\mathscr{L}\{y(t)\}] - sy(0) - y'(0) + 4s[\mathscr{L}\{y(t)\}] - 4y(0) + 4\mathscr{L}\{y\} &= \frac{2}{s^3} \\
s^2[\mathscr{L}\{y(t)\}] + 4s[\mathscr{L}\{y(t)\}] + 4\mathscr{L}\{y\} &= \frac{2}{s^3} \\
\mathscr{L}\{y(t)\}(s^2 + 4s + 4) &= \frac{2}{s^3} \\
\mathscr{L}\{y(t)\} &= \frac{\frac{2}{s^3}}{(s^2 + 4s + 4)} \\
\mathscr{L}\{y(t)\} &= \frac{2}{s^3 \cdot (s^2 + 4s + 4)} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s^3 \cdot (s^2 + 4s + 4)}\right\} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s^3 \cdot (s^2 + 4s + 4)}\right\} \\
y(t) &= \mathscr{L}^{-1}\left\{\frac{2}{s^3 (s + 2)^2}\right\}
\end{aligned}
$$

Ahora hagamos un paréntesis, tenemos ya solo que resolver esto por fracciones parciales:

Bibliografía

[1] ProbRob
`Youtube.com`