Project Homework 1 for *Introduction to Adaptive Control*

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24th November 2020

Abstract

The problems in here will give you a head start for the final exam as you will need these results. You may document your results in any way (i.e. handwritten, Word, LaTeX, etc.). The result needs to be a *readable* .pdf and your simulation code. Please pack your results, including folder structure into a single compressed file and upload it.

1 MRAC of a First-Order System

In the following, you will design and implement a model-reference adaptive controller for a simple system.

1. Design a controller and implement it in Matlab/Simulink.
   Think of this as an *Industry Project*; that is: you are to write clean, readable and documented code. Separate initialization commands and the Simulink model. Do not use Simulink Callbacks. *Remember:* I need to be able to quickly see what you did.

2. Prove stability of the resulting closed-loop and prove convergence of the control error to zero. Show every step and be rigorous (i. e., mathematically exact)[1].

3. Include a Lyapunov-Like Function (and its derivative) in your simulation and show its decline over time.

4. In your written document, state your findings from the simulations. Investigate. Elaborate. Discuss. Only include plots that are relevant to your discussion.

[1] There will be no room for "hand waving". In class, I try to explain things intuitively. *You* need to be mathematically precise when it comes to the proof.

1.1 Linear Plant

Given. The plant G is

$$G: \quad \dot{x}_\mathrm{p} = a x_\mathrm{p} + b u;\ x_\mathrm{p}(0) = 0.$$

The real plant parameters are $a = 1$ and $b = 3$. These values are *unknown* to the adaptive controller.

The reference model M is

$$M: \quad \dot{x}_\mathrm{m} = -4 x_\mathrm{m} + 4r;\ x_\mathrm{m}(0) = 0.$$

Desired. Design an MRAC that produces the control input u s. t. the measurable plant output x_p follows the output x_m of the reference model for $t \to \infty$. In this homework[2], you are to use an adaptive gain $\gamma = 2$ for the adaptive law of all parameters.

Simulation run. Calculate the exact controller parameters (i. e., the parameters of the control law which would result in perfect tracking of the reference model without adaptation). In the following, compare the estimated parameters, to the exact parameters. Show the comparison in a Simulink plot and your written report. Discuss the findings.

Compare three different reference inputs:

- $r(t) = 4$ (i. e., a *single* step input of height 4).
- $r(t) = 4\sin(4t)$.
- $r(t)$ is a pulse-train of height 4 (i. e., it consists of repetitive up/down steps of height 4 with a period of your choosing.)

In your discussion, interpret the different signals of $r(t)$ as having different amounts of information. As information, count frequencies contained in $r(t)$. In loose[3] language, try to argue under which conditions the parameter values converge to the exact values.

1.2 Nonlinear Plant

This is a direct extension of the control problem above. The reference model is the same as in the previous section, so is the control problem as well as $\gamma = 2$.

[2] In real life, it would be part of your job to choose this, too.

[3] All you can do, at this point, is empirical investigations. Try to explain using reasoning.

Changes in the plant. Now, the plant G has an additional nonlinear term

$$G: \quad \dot{x}_{\mathrm{p}} = a x_{\mathrm{p}} + bu - bf(x_{\mathrm{p}}); \; x_{\mathrm{p}}(0) = 0.$$

The real plant parameters are $a = 1$ and $b = 3$. The nonlinear function $f(x)$ can be represented (exactly) by the linear combination of parameters θ_f and basis functions $\Phi_f(x)$:

$$f(x) = \theta_f^T \Phi_f(x), \text{ with the } \textit{known} \text{ basis functions}$$
$$\Phi_f = \begin{bmatrix} x^3 & \exp\left(-10(x+0.5)^2\right) & \exp\left(-10(x-0.5)^2\right) & \sin(2x) \end{bmatrix}^T.$$

The *unknown* parameters θ_f are

$$\theta_f = \begin{bmatrix} 0.01 & -1 & 1 & 0.5 \end{bmatrix}^T.$$

Draw the nonlinear function $f(x)$ with its parts.

Simulation run. The reference signal is $r(t) = \sin(3t) + \sin\left(\frac{3}{2}t\right) + \sin\left(\frac{3}{4}t\right) + \sin\left(\frac{3}{8}t\right)$. Is parameter estimation working well, in the sense that you can reproduce the nonlinear function $f(x)$? Is tracking of the reference model working well?

Change in exact parameter values. Now, do the same again, but with the parameter values

$$\theta_f = \begin{bmatrix} 0 & -1 & 1 & 0 \end{bmatrix}^T.$$

Assume the zeros to be known (i. e., use the same model but the parameters given as zero are known to the controller) What change is introduced? Discuss the resulting parameter estimation.

2 Parameter Convergence to the Exact Values

Investigate under which condition the parameters converge to their true values. Let me give you some hints.

We know that the error dynamics are

$$e(t) = \frac{1}{k^*} M(s) \left[\tilde{\theta}^T(t) \phi(t)\right]$$

with the parameter errors $\tilde{\theta}(t)$ and the regressor $\phi(t)$ containing external functions. We *know* that $e(t)$ goes to zero, always (even if it takes a really long time). Discuss why.

Assume, for this task, that $e \approx 0$, i. e., the error is negligibly small. Set it to zero. This simplifies your derivation in the following. Note that this is an unrealistic assumption. But it helps you in the following; the real derivation would be too difficult. Under this special assumption we now *know* that the parameters $\theta(t)$ will have *some* constant values. Why?

The question is, under which conditions on $\phi(t)$ does $\tilde{\theta}(t)$ go to zero? In other words: What do we require of the regressor such that the parameters converge to their true values?

Answer the question above by following these steps:

Consider $M(s)$ in the error dynamics as a stable linear system/filter with a nonzero DC-gain. If we assume that e is zero (from some time on to infinity), then what can you conclude about the input signal $\left[\tilde{\theta}^T(t)\phi(t)\right]$?

Finally, using the answer of the last question: Find conditions on $\phi(t)$ such that the only solution is $\tilde{\theta}(t) = 0$ when the error is converged to zero and θ is constant.

3 Investigate the High-Frequency Gain

I am asking you something we have not discussed, yet. In the following, k is the so-called *high-frequency gain*. Investigate what it means.

Let

$$G(s) = \frac{b_1 s + b_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

be some system. Represent this system as

$$G(s) = k\frac{Z(s)}{R(s)},$$

with $Z(s)$, $R(s)$ being monic[4] polynomials. What is the relative degree n^*? Interpret the relative degree (explain it with loose language). What is the DC-gain of G? What is the gain of G for very high frequencies (i. e., $|G(j\omega)|_{\omega\to\infty}$).

Now that you have looked at this system, it's time to investigate the high-frequency gain k. Take the n^*-th derivative[5] of $G(s)$ and call it $W(s)$.

[4] Look up what "monic" means.

[5] Hint: I would stay in s-Domain for this.

Consider a step input (i. e., $u = \frac{1}{s}$) to $W(s)$. Show that the immediate gain[6] from u to the output is k. Interpret your results.

The following is geared for you to understand why a known sign of the high-frequency gain is helpful for control (generally, not just adaptive control). Let's investigate the meaning of the sign of the high-frequency gain for control. Let

$$G_{\mathrm{a}}(s) = \frac{\lambda}{s+\lambda}$$
$$G_{\mathrm{b}}(s) = \frac{1}{s+\lambda}$$

be two systems, each with input u and output y. And assume that the pole λ may lie between $\lambda \in [-1, 1]$. In what ranges lie the DC-gains and high-frequency gains of G_{a} and G_{b}?

Using a traditional (i. e., non-adaptive) pure proportional controller $u = ky$: Can you find a fixed (constant) value for k that stabilizes G_{a} for all admissible λ? Can you find a (possibly different) fixed value for k that stabilizes G_{b} for all admissible λ? To show this, use any tool that you desire[7]. Would something change if the feedback controller can be of arbitrary order (but fixed)?

[6] Direct feedthrough gain, or properly stated, the output at $t = 0+$, resulting from a unit step at $t = 0$ at the input. (remember the inital value theorem? Use it.)

[7] e. g., Nyquist criterion, Root-locus, or computation of the closed-loop pole. Actually, the simplest is computing the closed-loop pole. But the most insight results from using the Nyquist Theorem, in my opinion, if you feel comfortable with it (it is more work).