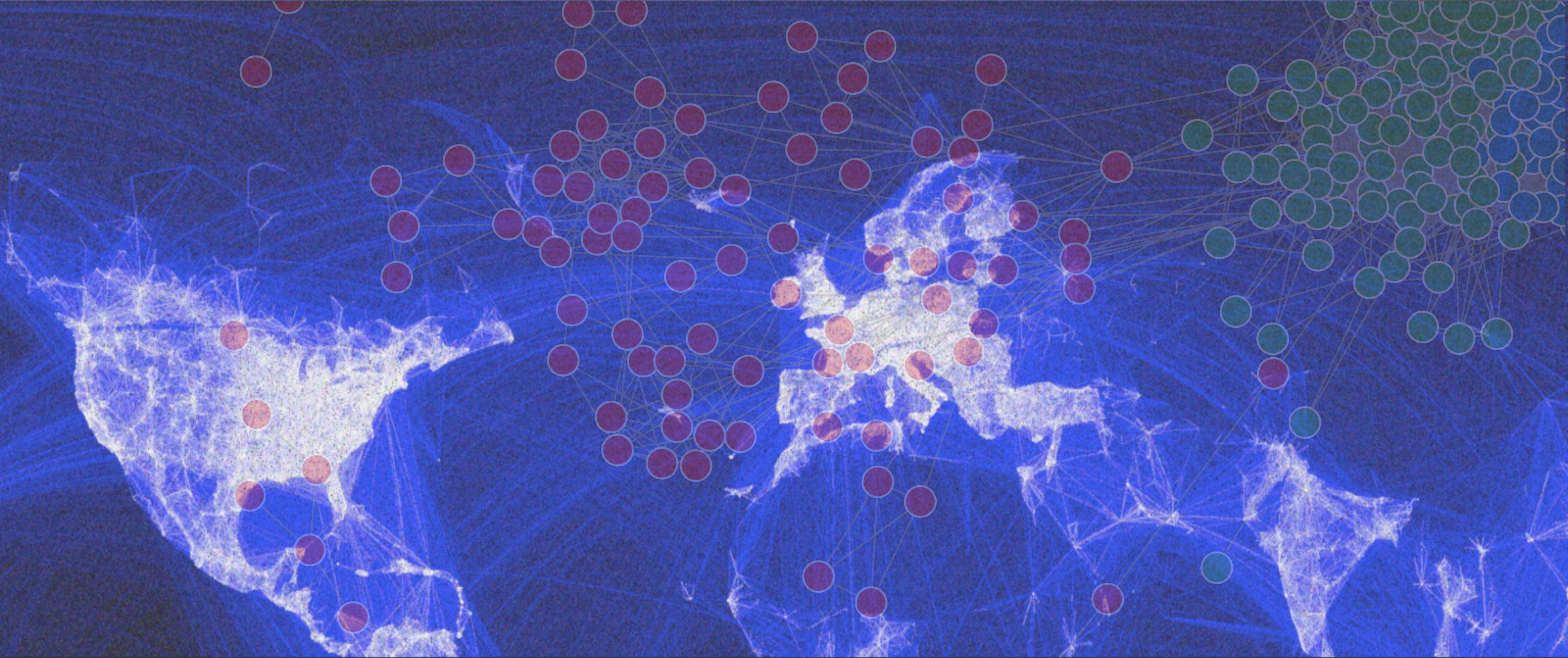


# Complex Networks


**Vincent Gauthier**

**Associate Professor**

**@vincentgauthier**

**http://complex.luxbulb.org**

**Telecom SudParis-Institut Mines-Telecom/CNRS SAMOVAR**

# Software

**R**

**Matlab**

**Python**

**Networkx [Python]**

**Graph-tool [Python+C++]**

**GraphLab [Python+C++]**

# Graph Editor

**Gephi**

**GraphViz**

**Pajek**

# Datasets

Mark Newman's network data set

Stanford Network Analysis Project

Carnegie Mellon CASOS data sets

NCEAS food web data sets

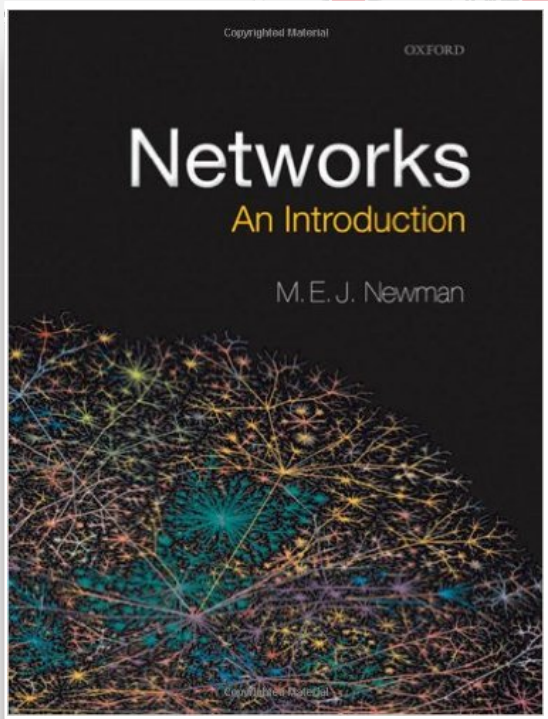
UCI NET data sets

Pajek data sets

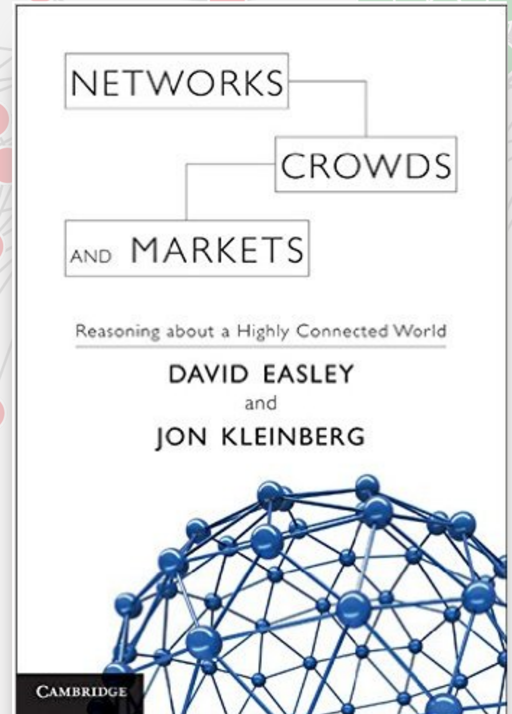
Linkgroup's list of network data sets

Barabasi lab data sets

Jake Hofman's online network data sets Alex Arenas's data sets




*Networks An Introduction*
Mark Newman, 2010.




*Networks, Crowds, and Markets*
*Reasoning About a Highly Connected World*
David Easley, Jon Kleinberg, 2010.

1. Definition of a network
2. Basics of graph theory
3. Describing a network
4. Network centrality
5. Eigen centrality, PageRank
6. Small world network (Watts-Strogatz)
7. Network Growth (Barabasi-Albert)
8. Social Network Analysis
9. Diffusion on network

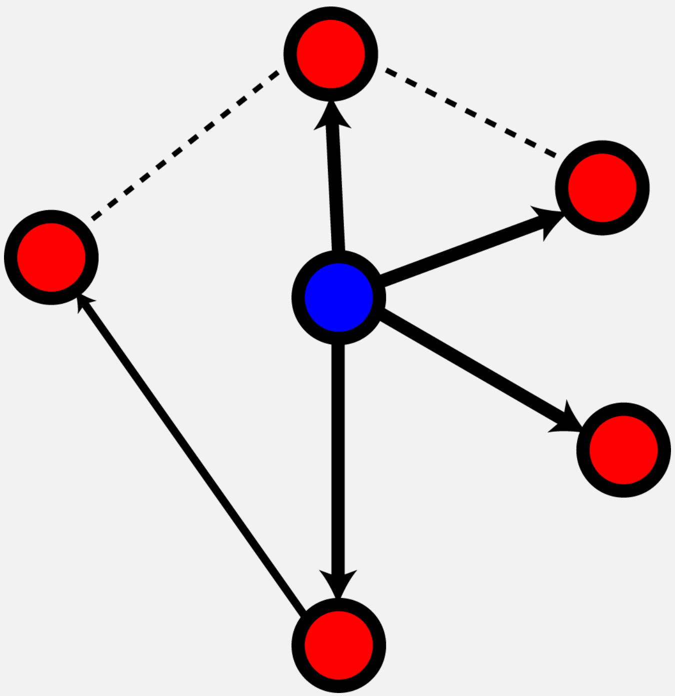
Lecture 1

Lecture 2

Lecture 3

# DESCRIBING NETWORK (Centrality Metrics)

# Describing Networks



## Position=Centrality

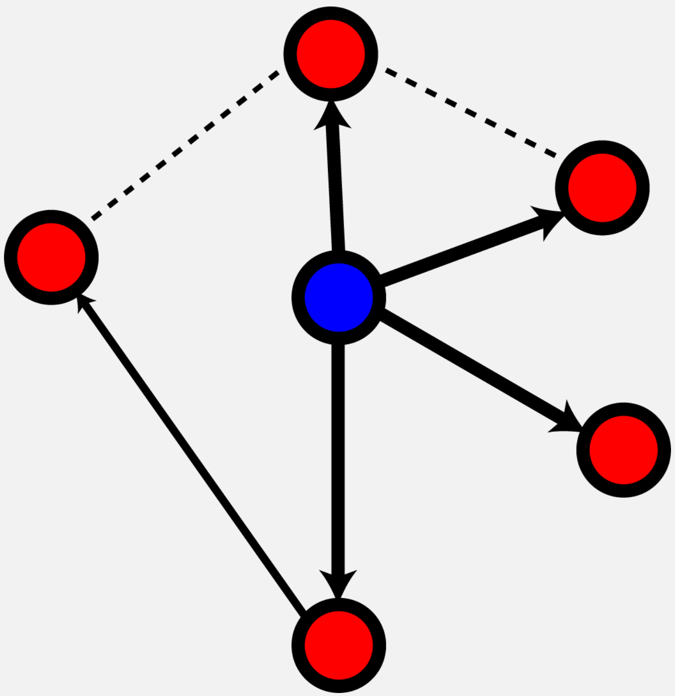
Measure of positional "importance"

| | |
|---|---|
| geometric | Harmonic centrality |
| | Closeness centrality |
| | Betweenness centrality |
| Connectivity | Degree centrality |
| | Eigenvector centrality |
| | PageRank |
| | Katz centrality |
| | Many many more |


Axioms for Centrality
Paolo Boldi & Sebastiano Vigna
Internet Mathematics Vol. 10, Iss. 3-4, 2014

Centrality and network flow
SP Borgatti
Social networks 27 (1), 55-71, 2005.

# Describing Networks



## Position=Centrality:

Harmonic, closeness centrality

Importance of being in the "center" of the network

Harmonic: $$c_i = \frac{1}{n-1}\sum_{j \neq i}\frac{1}{d_{ij}}$$

Length of shortest path

Distance: $$d_{ij} = \begin{cases} \boxed{l_{ij}} & \text{if } j \text{ reachable from } i \\ \infty & \text{otherwise} \end{cases}$$


Axioms for Centrality
Paolo Boldi & Sebastiano Vigna
Internet Mathematics Vol. 10, Iss. 3-4, 2014

Centrality and network flow
SP Borgatti
Social networks 27 (1), 55-71, 2005.

# Network position: example


## Robust Action and the Rise of the Medici

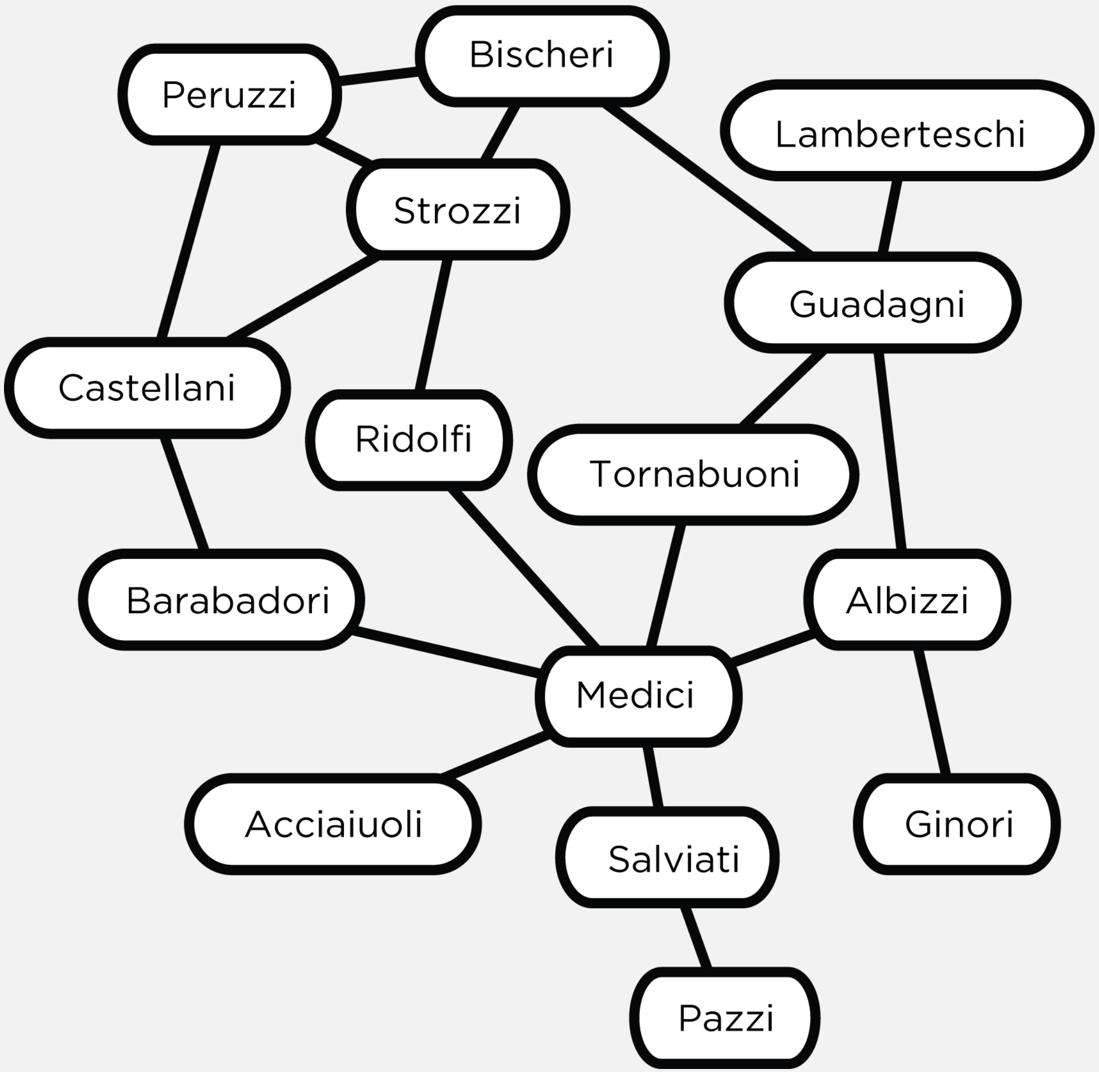
John F .Padgett and Christopher K. Ansell 1993

Duomo

Plazzo Medici


Giovanni de Medici

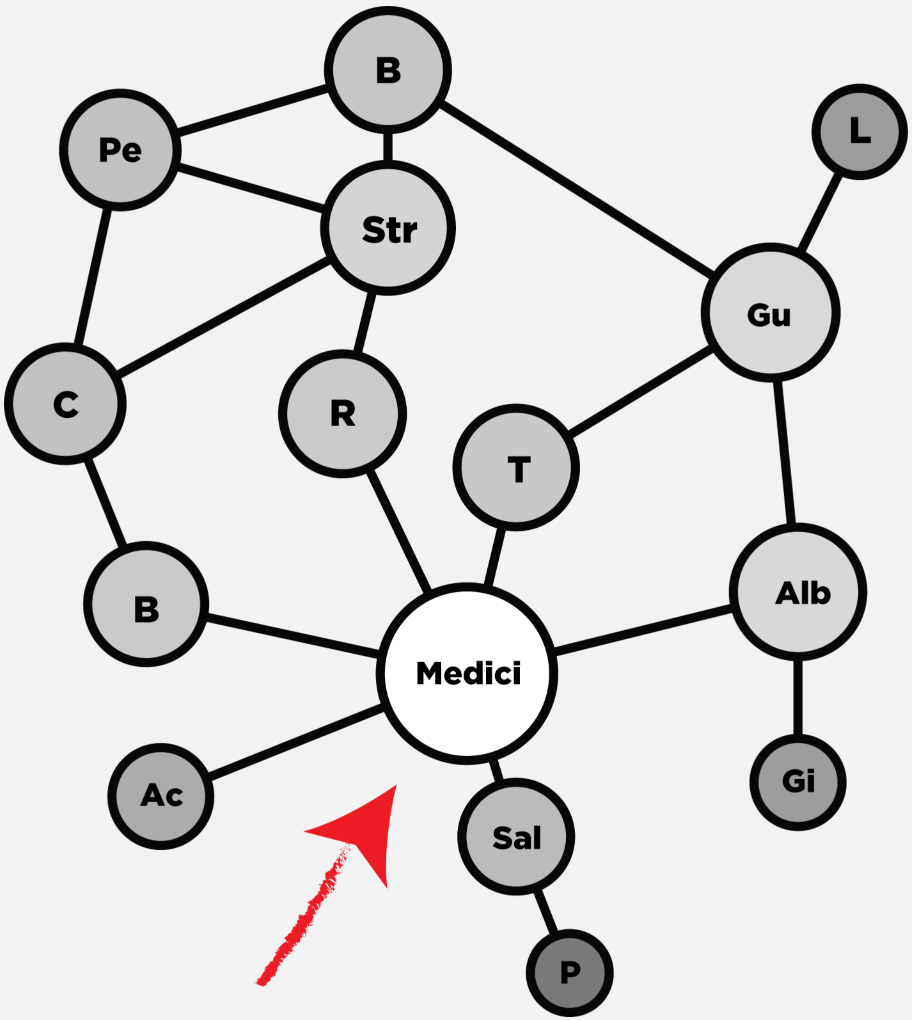
# Network position: closeness


**Node**: Florence families
**Edges**: inter-family marriages

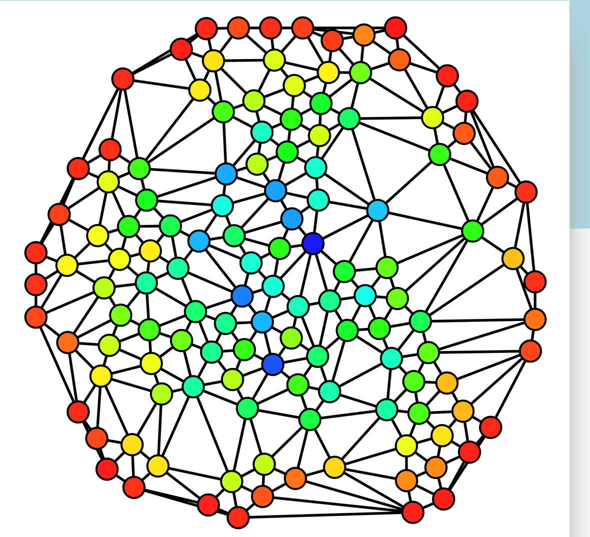
Which family is most central ?

# Network position: closeness




| | |
|---|---|
| Medici | 9.5 |
| Guadagni | 7.92 |
| Albizzi | 7.83 |
| Strozzi | 7.67 |
| Ridolfi | 7.25 |
| Bischeri | 7.2 |
| Tornabuoni | 7.17 |
| Barbadori | 7.08 |
| Peruzzi | 6.87 |
| Castellani | 6.87 |
| Salviati | 6.58 |
| Acciaiuoli | 5.92 |
| Ginori | 5.33 |
| Lamberteschi | 5.28 |
| Pazzi | 4.77 |

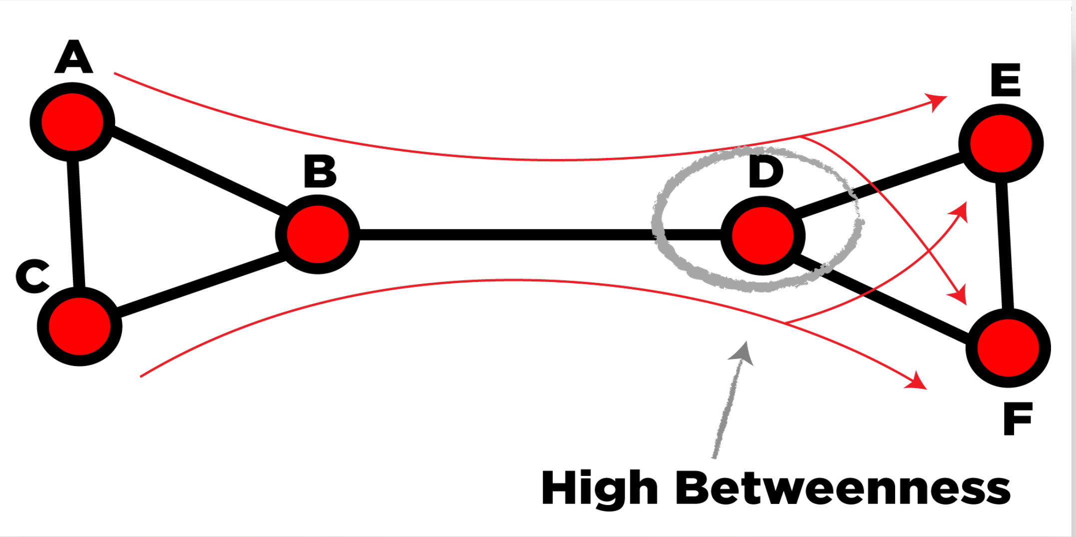
# Betweenness Centrality



$$g(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

Number of shortest path between node *s* and *t* passing through node *v*

Number of shortest path between node *s* and *t*



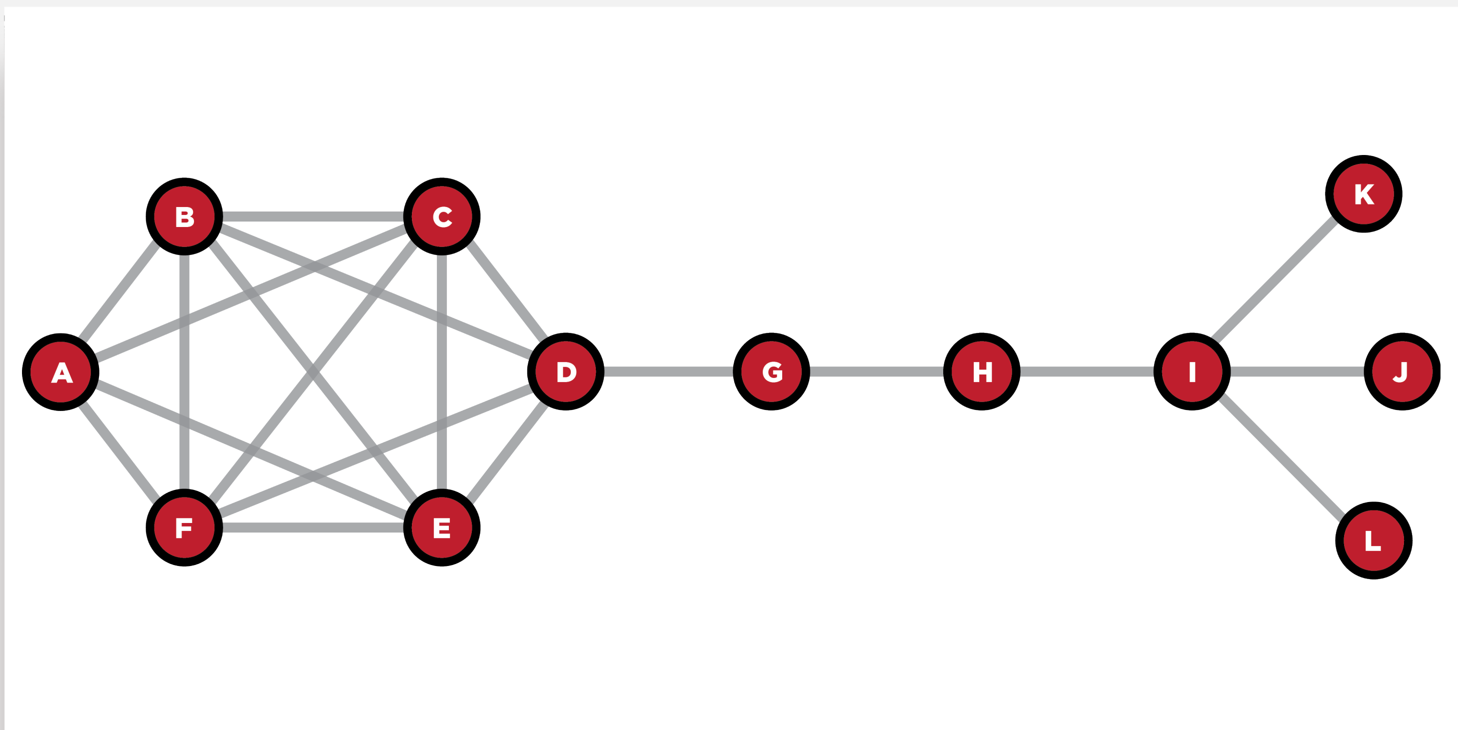

**Position=Centrality:**
Betweenness centrality

Importance of being a "bridge" in the network

**intuition:** how many pairs of individuals would have to go through you in order to reach one another in the minimum number of hops?

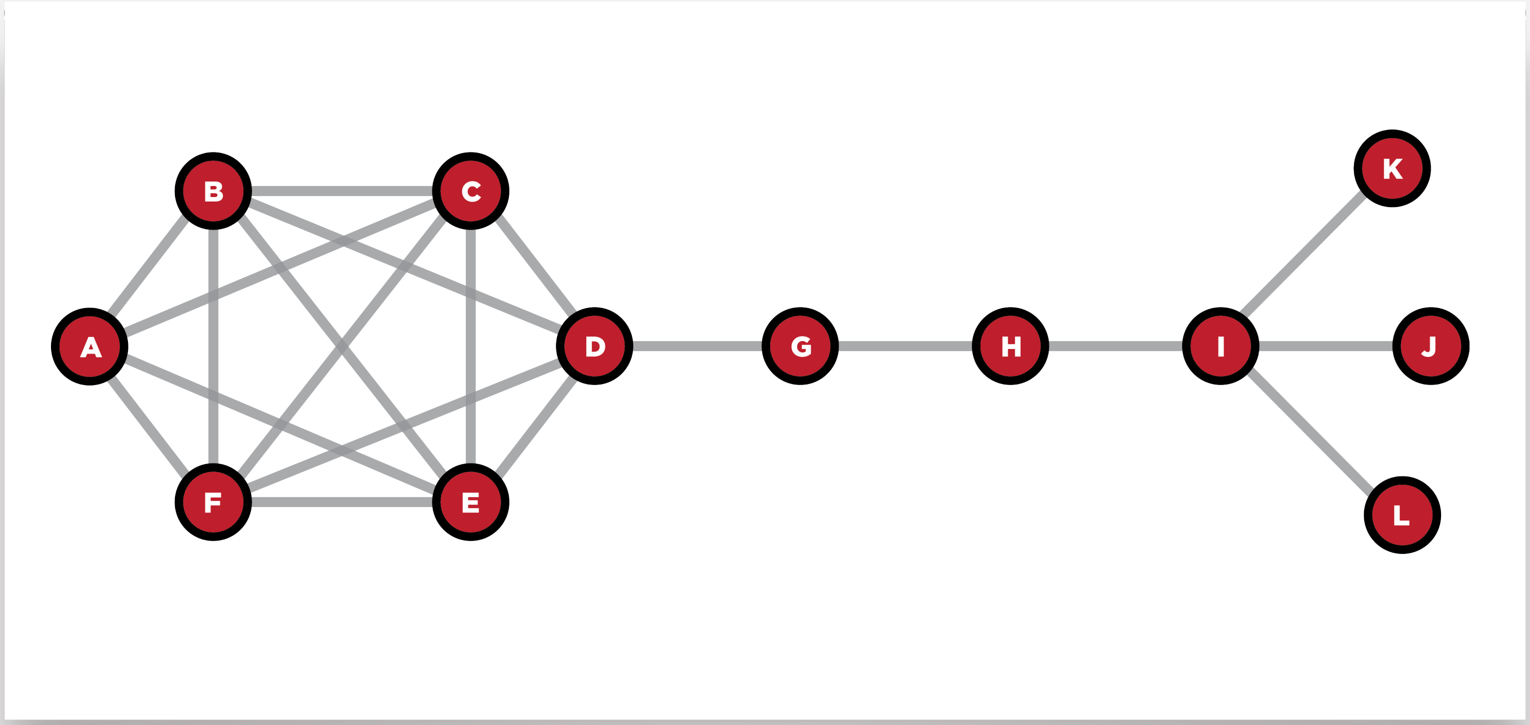
# Quiz:

## Find a node that has high betweenness but low degree

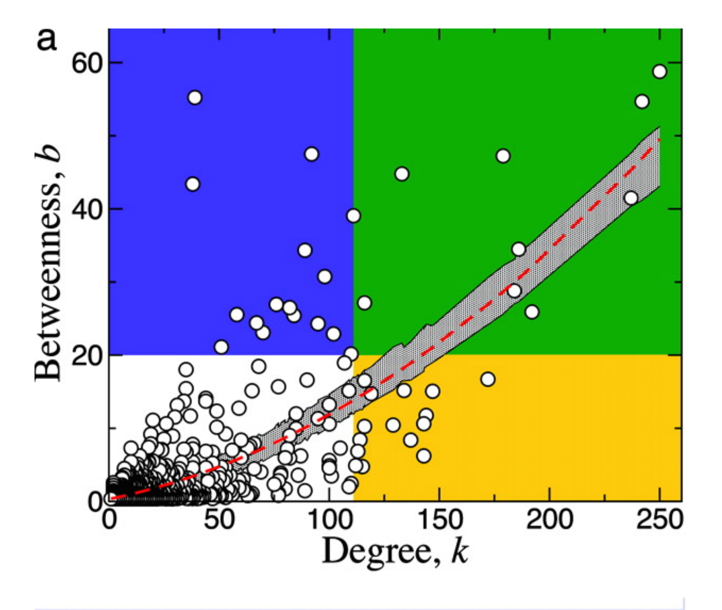


# Quiz:

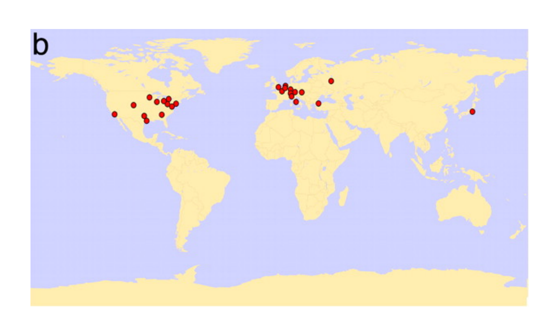
## Find a node that has low betweenness but high degree



# Centrality: Betweenness example




| Rank | City | b | $b/b_{ran}$ | Degree |
|---|---|---|---|---|
| 1 | Paris | 58.8 | 1.2 | 250 |
| 2 | Anchorage | 55.2 | 16.7 | 39 |
| 3 | London | 54.7 | 1.2 | 242 |
| 4 | Singapore | 47.5 | 4.3 | 92 |
| 5 | New York | 47.2 | 1.6 | 179 |
| 6 | Los Angeles | 44.8 | 2.3 | 133 |
| 7 | Port Moresby | 43.4 | 13.6 | 38 |





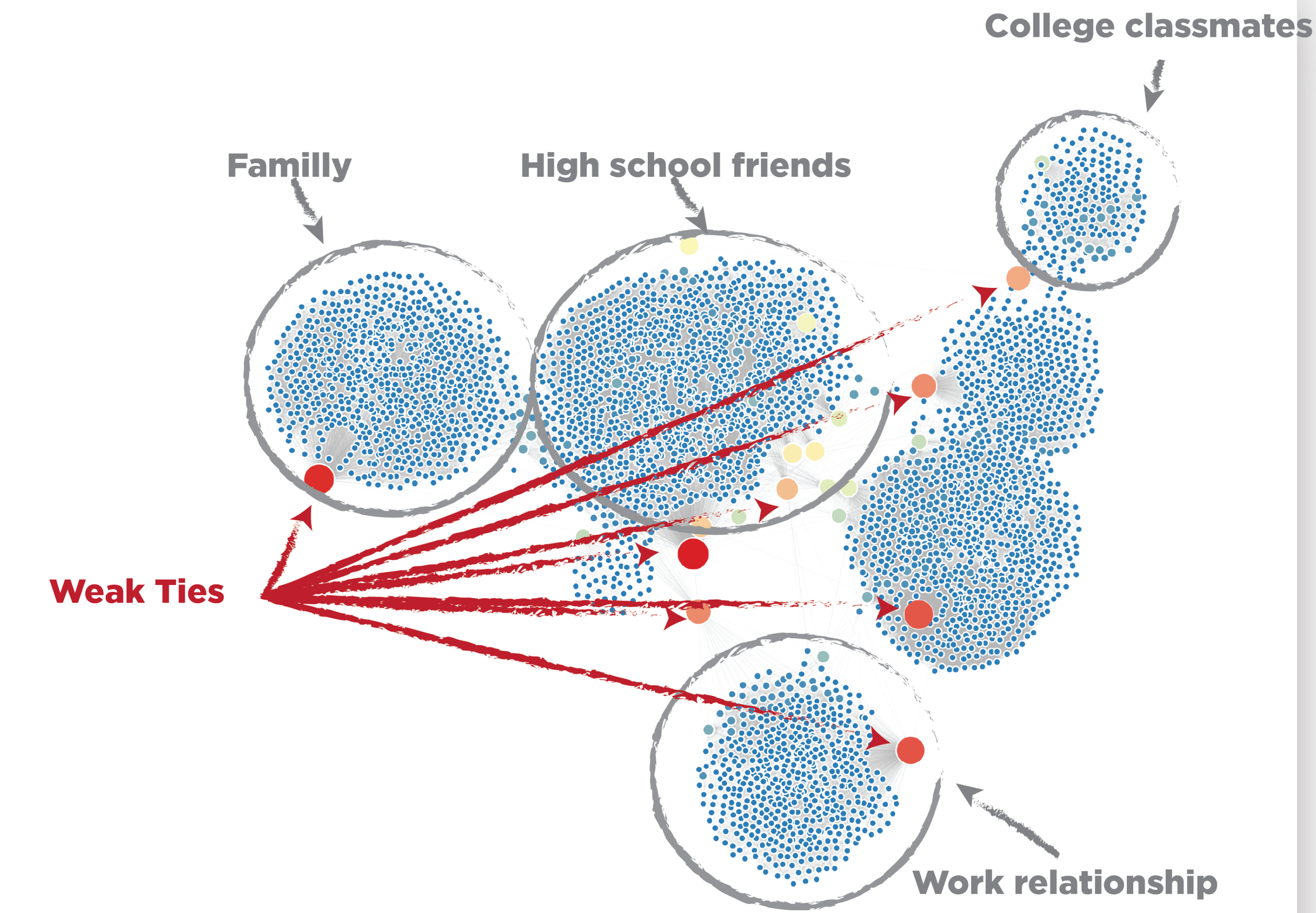

The worldwide air transportation network: Anomalous centrality, community structure, and cities' global
R. Guimerà, S. Mossa, A. Turtschi, and L. A. N. Amaral
PNAS, 2005.

# Betweenness centrality in social network

**Example: Facebook network**:

nodes are sized and colored by betweenness.

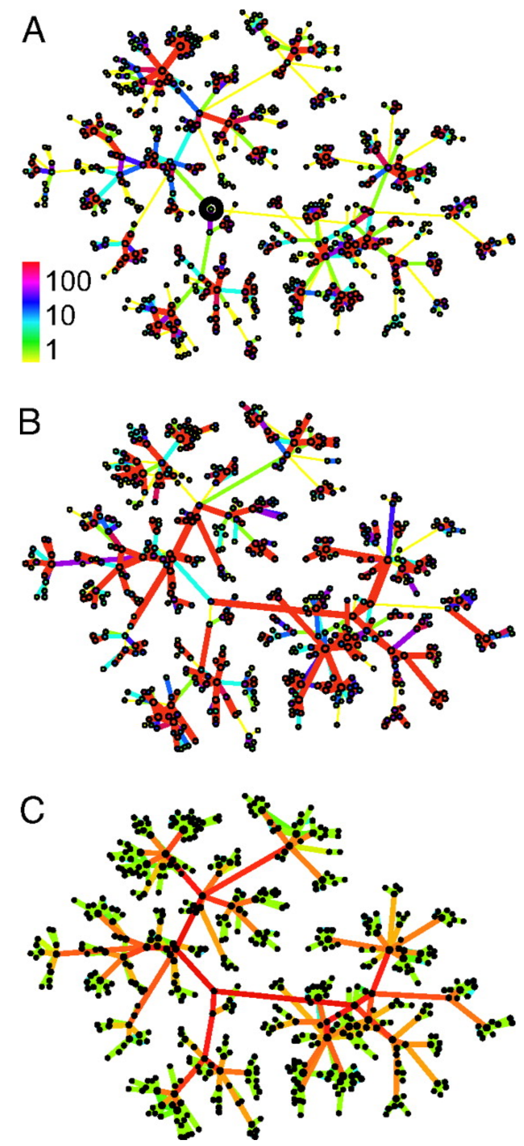
**Meaning:**
High betweeness is a sign of person is a bridge between different group of people



# Betweenness centrality Example: mobile network

**The structure of the Mobile Call Graph around a randomly chosen individual. Each link represents mutual calls between the two users, and all nodes are shown that are at distance less than six from the selected user, marked by a circle in the center.**

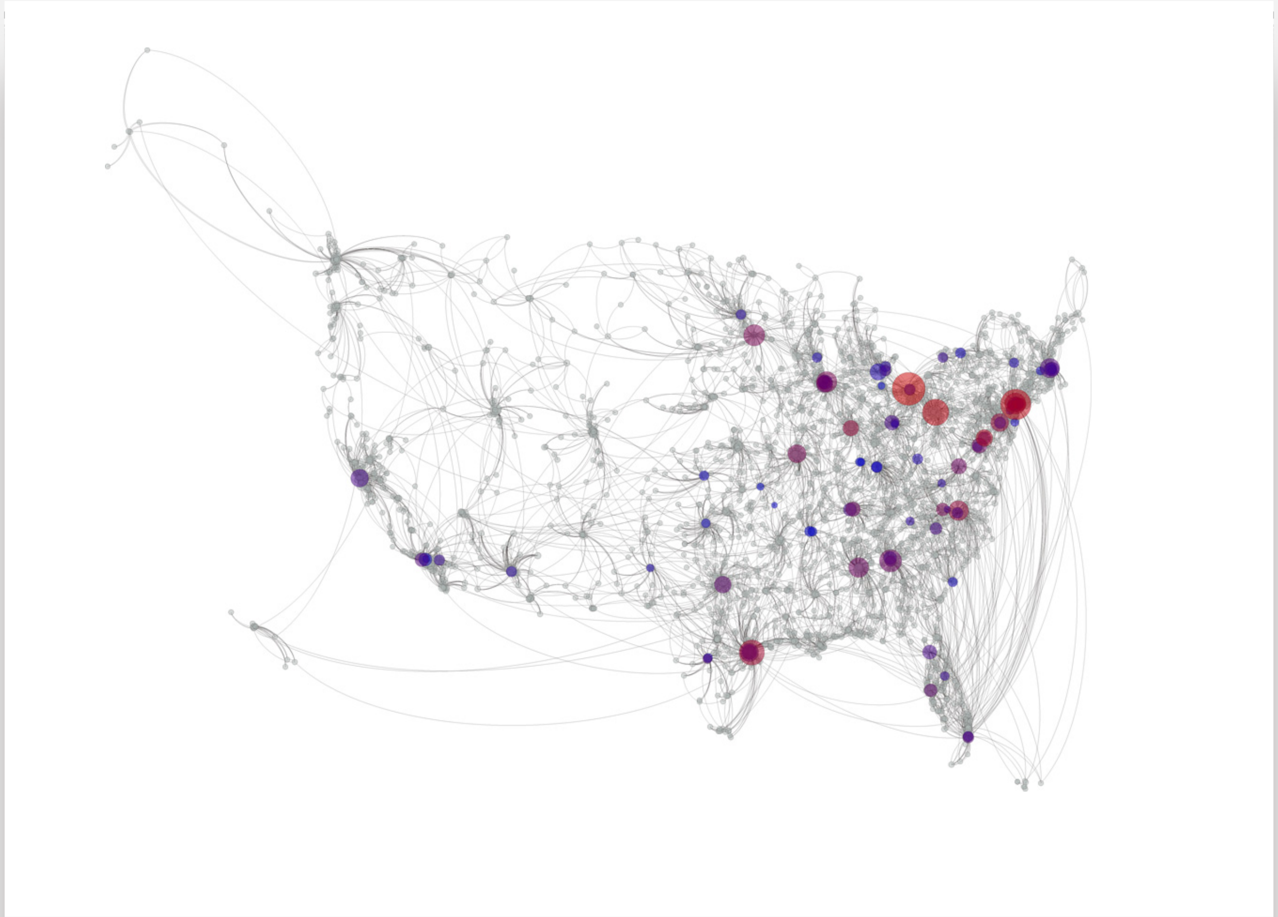
- (A) The real tie strengths, observed in the call logs, defined as the aggregate call duration in minutes (see color bar).
- (B) The dyadic hypothesis suggests that the tie strength depends only on the relationship between the two individuals. To illustrate the tie strength distribution in this case, we randomly permuted tie strengths for the sample in A.
- (C) The weight of the links assigned on the basis of their betweenness centrality $b_{ij}$ values for the sample in A as suggested by the global efficiency principle. In this case, the links connecting communities have high $b_{ij}$ values (red), whereas the links within the communities have low $b_{ij}$ values (green).





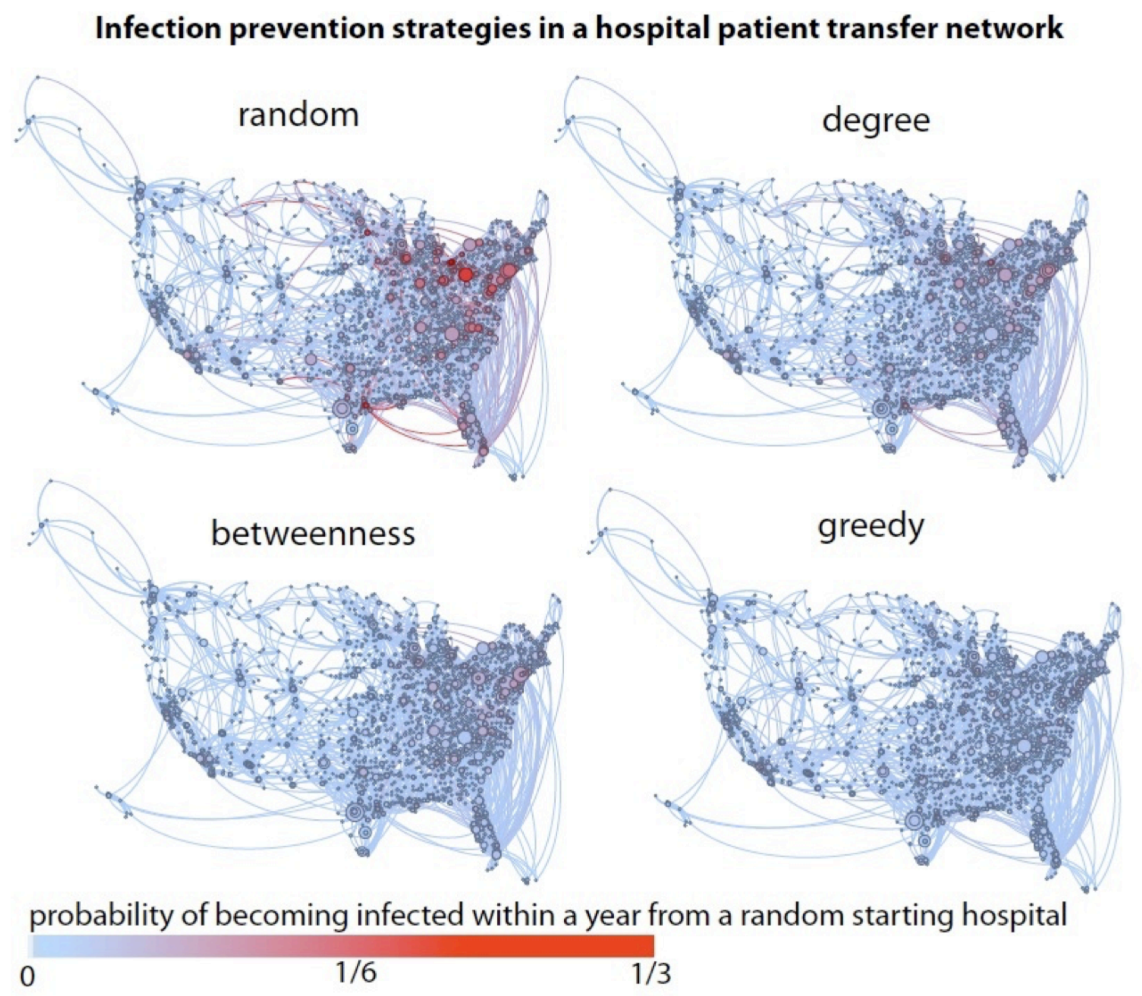
Structure and tie strengths in mobile communication networks
J.-P. Onnela, et al
PNAS, 2007.

# Hospital patient transfer



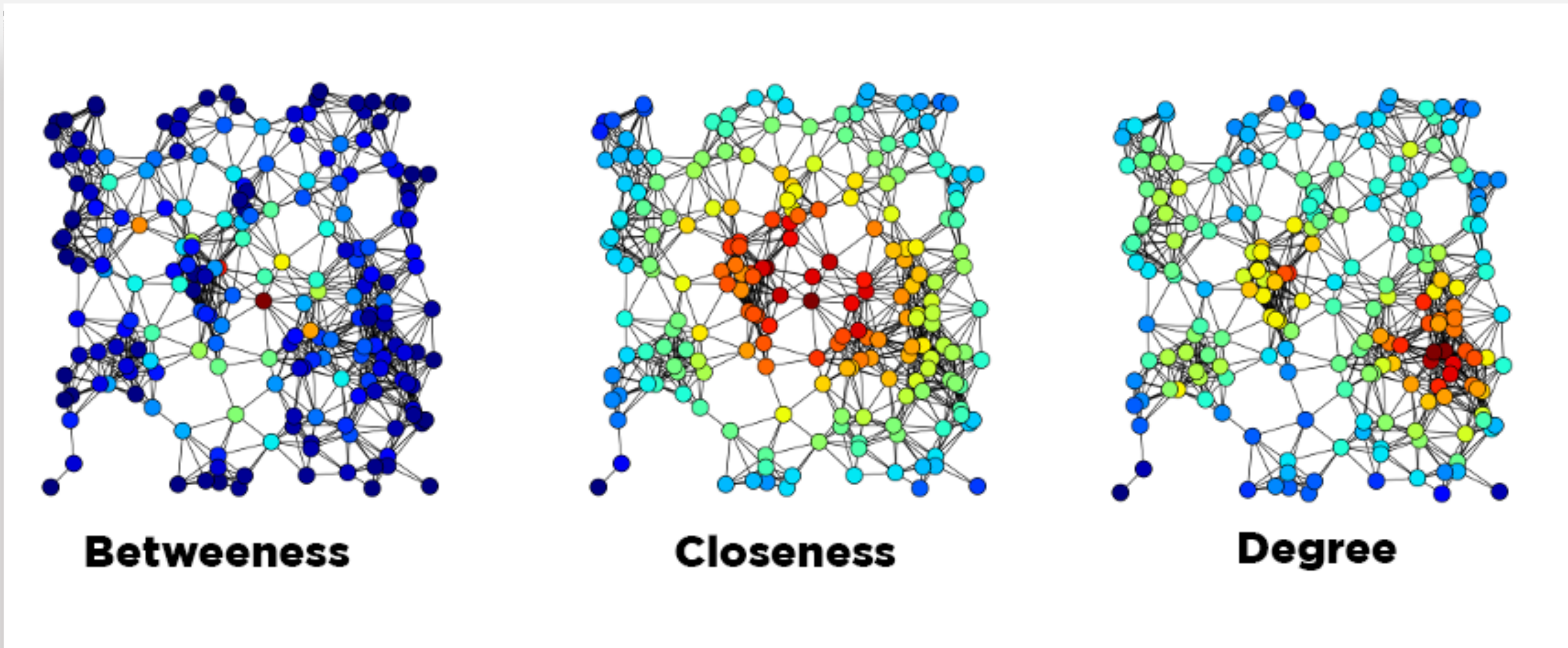
**Limiting the spread of highly resistant hospital-acquired microorganisms via critical care transfers: a simulation study**
Karkada UH, Adamic LA, Kahn JM, Iwashyna TJ
Intensive Care Med. 2011

# Hospital patient transfer



# Centrality continued

- Understanding whom are popular
- Understanding whom are at the center
- Understanding whom are receiving more in-flux

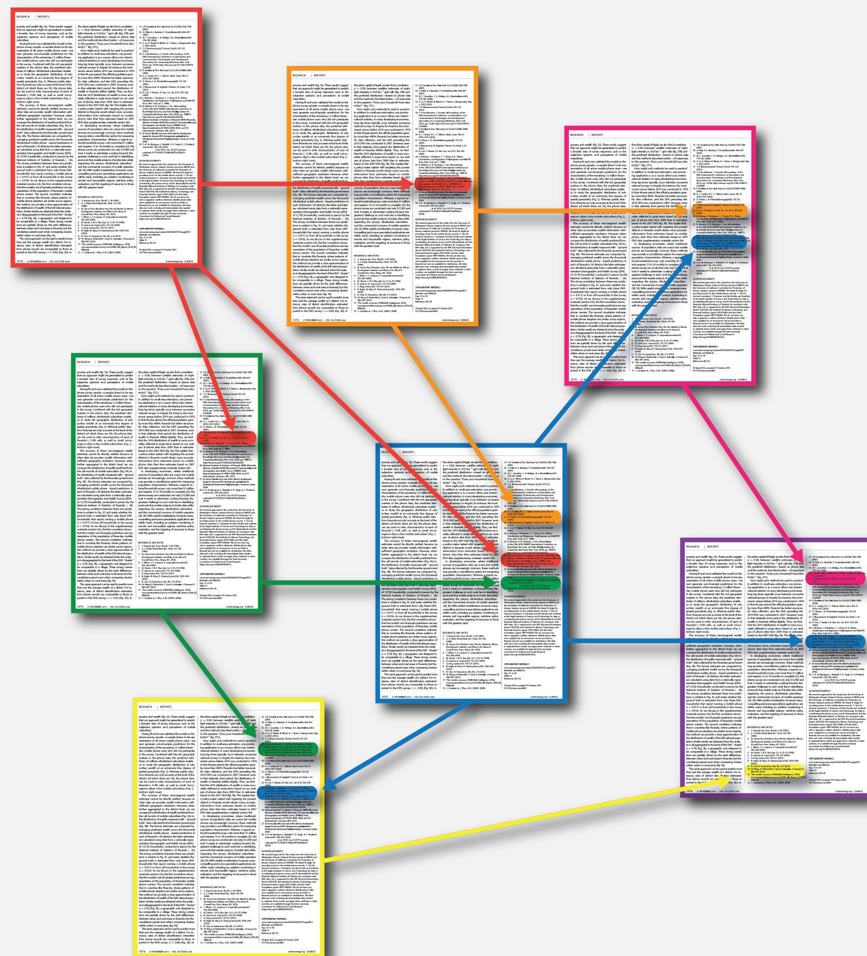


# DESCRIBING NETWORK (Eigen centrality, PageRank)

# Eigen centrality, PageRank

**The PageRank Citation Ranking: Bringing Order to the Web**

Larry Page and Sergey Brin 1999



## The problem:

How to provide a good algorithm to rank web page according to their importance in a given topics.

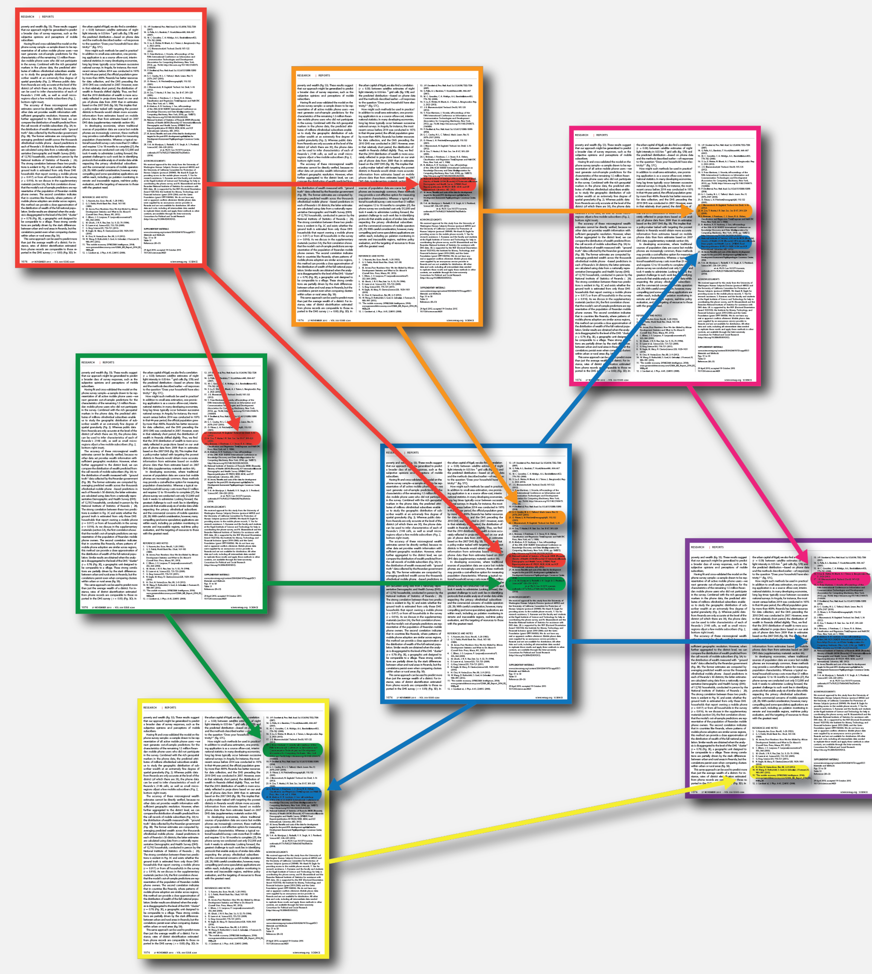
Same problem rank research paper according to their importance.

# Eigen centrality, PageRank

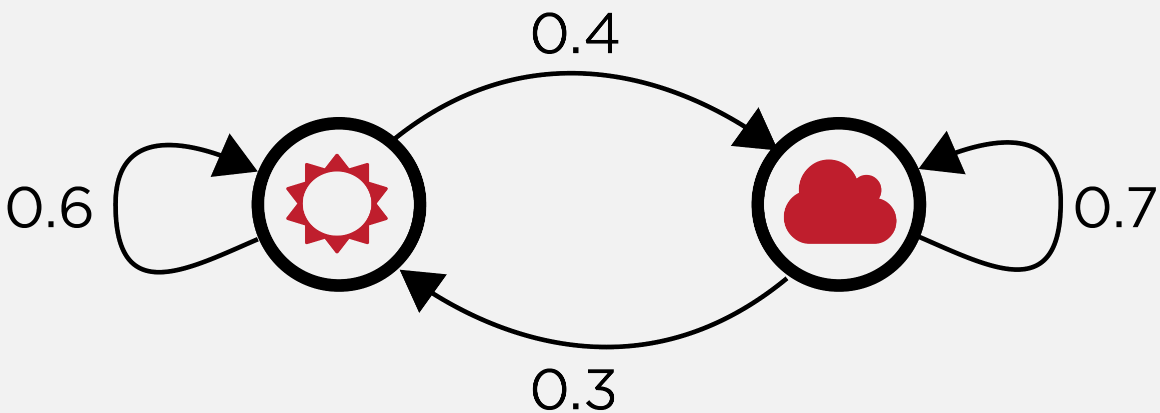
**The Idea behind Eigen Centrality:**

- Define $\pi_i$ as the importance of the node *i*
- $\pi_i$ depend on $\pi_j$ if *j* is a neighbor of *i*
- The importance is transmitted through the network

$$\pi_i \propto \sum_j a_{ji} \pi_j$$



# Remember the Markov Chains




Weather Tomorrow

Weather Prediction

Weather Today

Tomorrow

Today

$$\begin{bmatrix}0.6\\0.4\end{bmatrix} = \begin{bmatrix}0.6 & 0.4\\ \boxed{0.3} & 0.7\end{bmatrix}^T \times \begin{bmatrix}1\\0\end{bmatrix}$$

Probability of a sunny weather tomorrow given that today it is cloudy

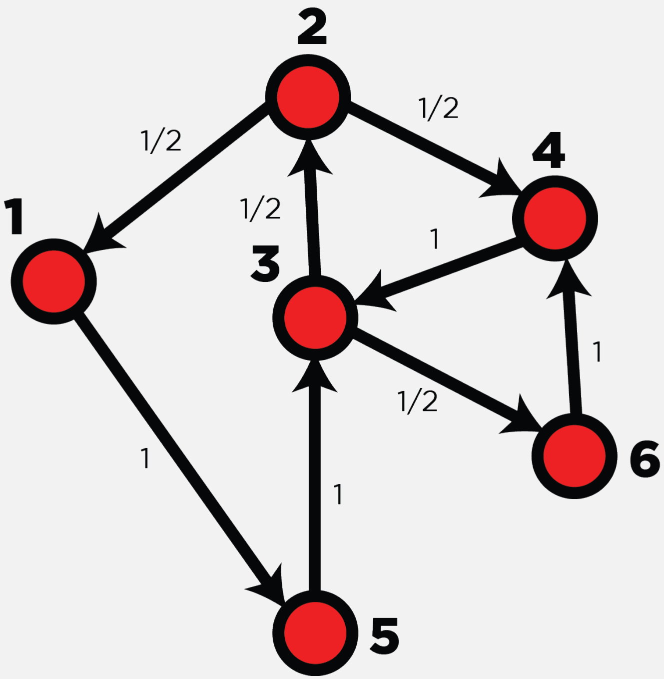
$$\begin{bmatrix}0.48\\0.52\end{bmatrix} = \left(\begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\end{bmatrix}^T\right)^{2} \times \begin{bmatrix}1\\0\end{bmatrix}$$

Weather of the day after Tomorrow

Weather Prediction

Weather Today

# Eigen centrality




$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad \pi_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Row stochastic matrix

$$\begin{aligned} \pi_1 &= \mathbf{A}^T \pi_0 \\ \pi_2 &= \mathbf{A}^T \pi_1 \\ &\vdots \\ \pi_k &= \mathbf{A}^T \pi_{k-1} \\ \pi_k &= \left(\mathbf{A}^T\right)^k \pi_0 \end{aligned}$$

Steady state

Eigen centrality

# Eigen centrality

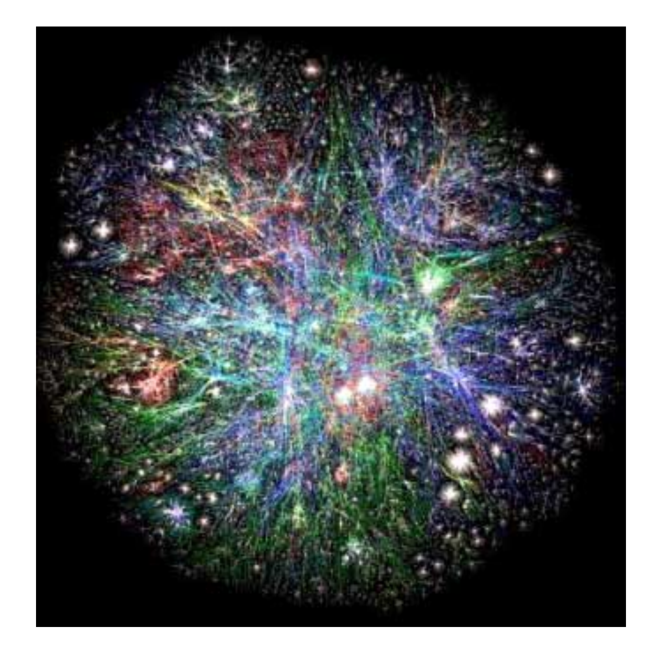
## continued

$$\pi_{k+1} = A^T \pi_k$$

$$\underbrace{\pi = A^T \pi}_{\text{Stationary state}} \; when \; k \rightarrow \infty$$

$$\lambda \pi = A^T \pi$$

Eigen value = 1      Eigen vector

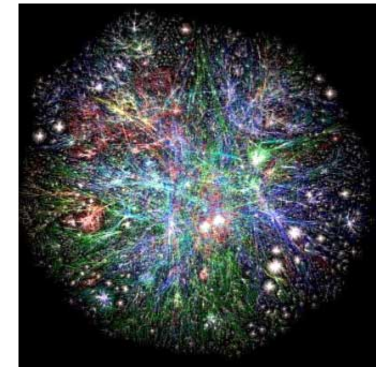


Perron–Frobenius theorem

- If A is a positive matrix and primitive
  - ✔A has a real eigenvalue
  - ✔There exist a eigenvalue $\lambda_{max} = 1$ (if A is stochastic row vector )

# Eigen centrality

**continued**

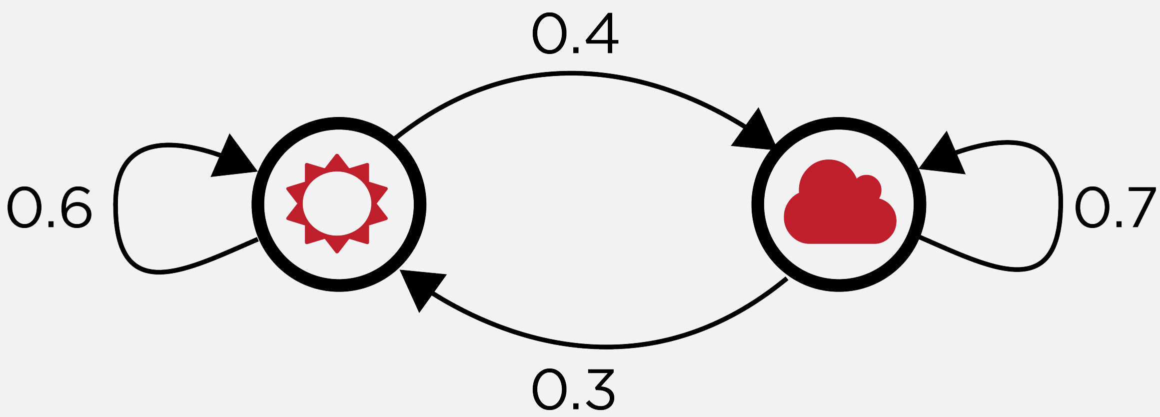


How to understand Perron-Frobenius theorem:

- The network is irreducible: meaning there is only one component: “all the node are reachable”
- The network is aperiodic: meaning there is no sink state in the network

# The Eigen value problem

## finding the eigen value




$$\begin{aligned} \mathbf{A}^T \pi &= \lambda \pi \\ \mathbf{A}^T \pi - \lambda \pi &= \vec{0} \\ \left[\mathbf{A}^T - \lambda \mathbf{I}\right] \pi &= \vec{0} \end{aligned}$$

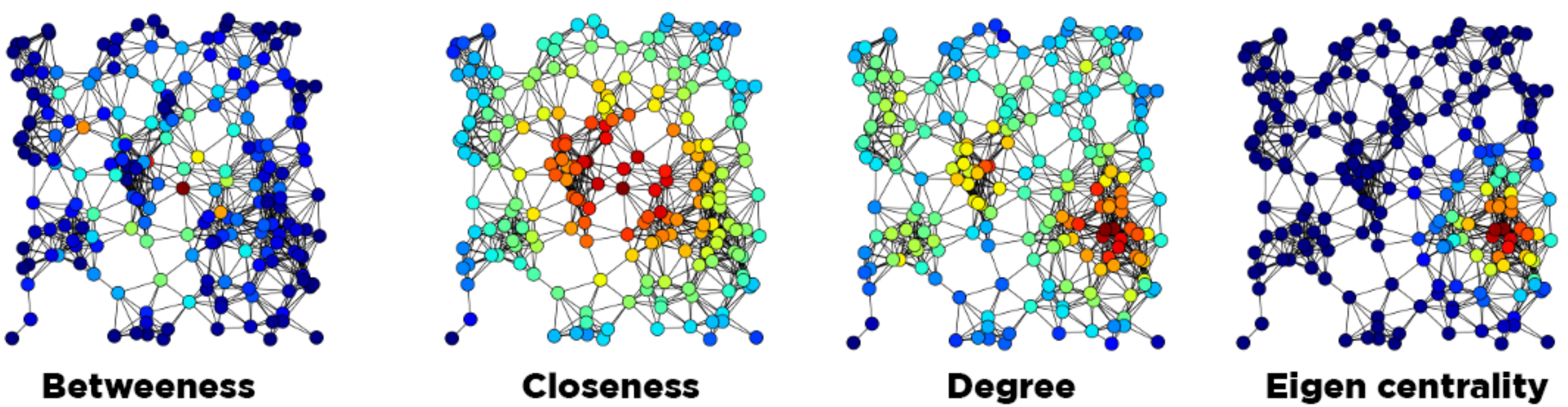
$$\boxed{\det\left[\mathbf{A}^T - \lambda \mathbf{I}\right] = 0}$$

**Find the eigenvalue $\lambda$ by solving the equation**

$$\text{solve} \quad \begin{vmatrix} 0.6 - \lambda & 0.3 \\ 0.4 & 0.7\lambda \end{vmatrix} = 0$$

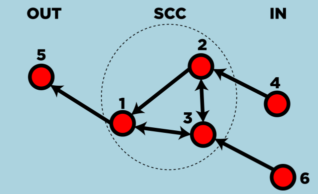
# Centrality continued

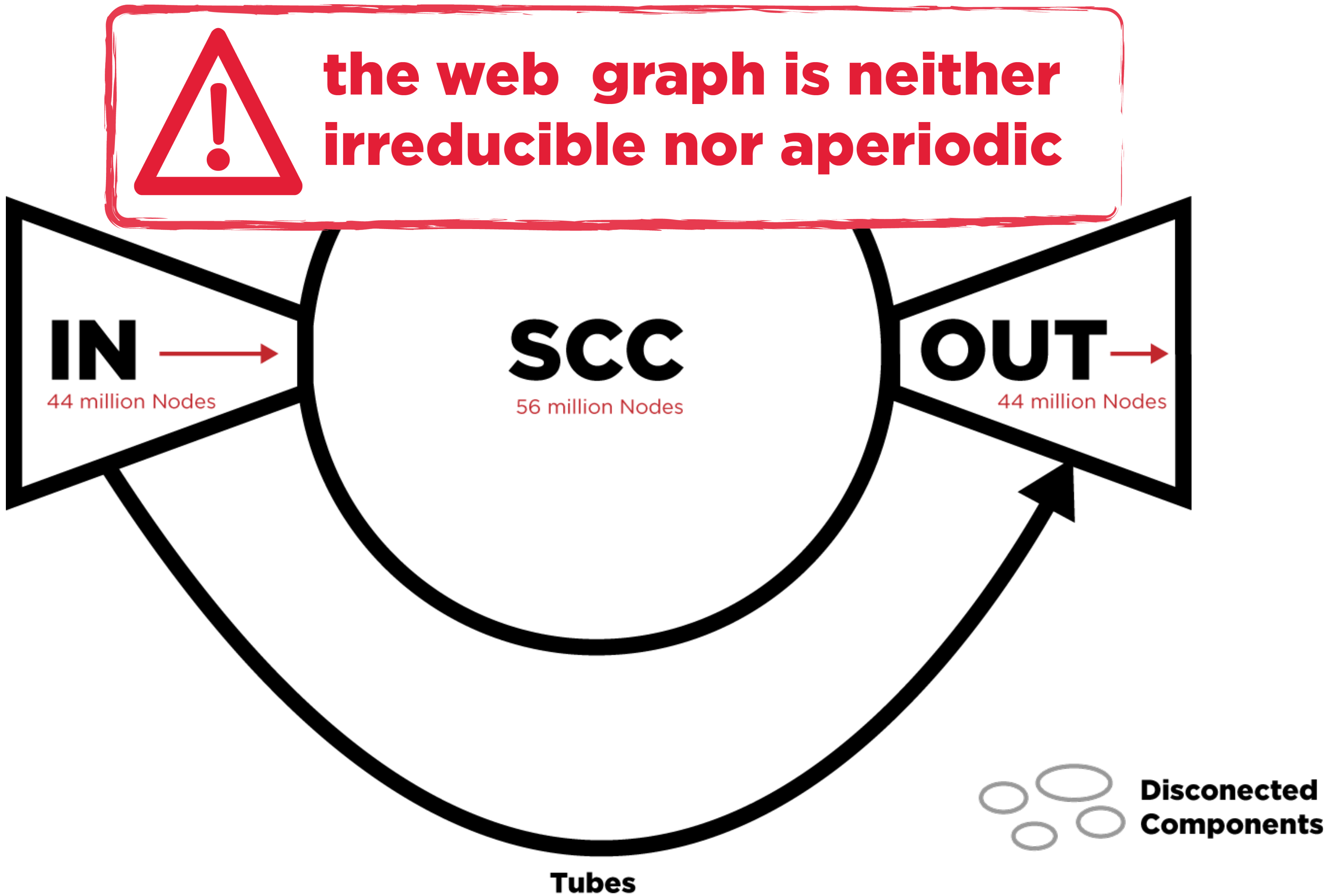
- Understanging whom are popular
- Understanging whom are at the center
- Understanginh whom are receving more in-flux
- Receiving more Trust



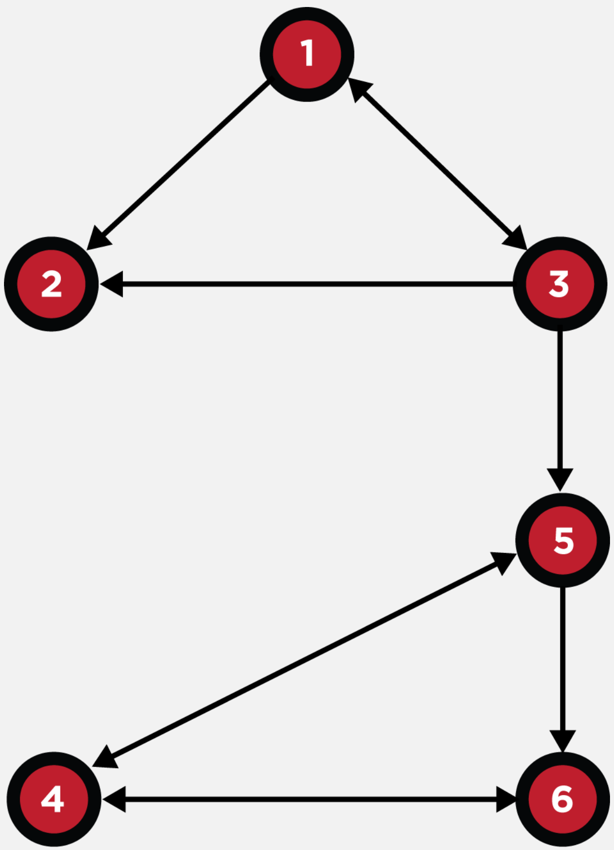
# Wait ... the SCC

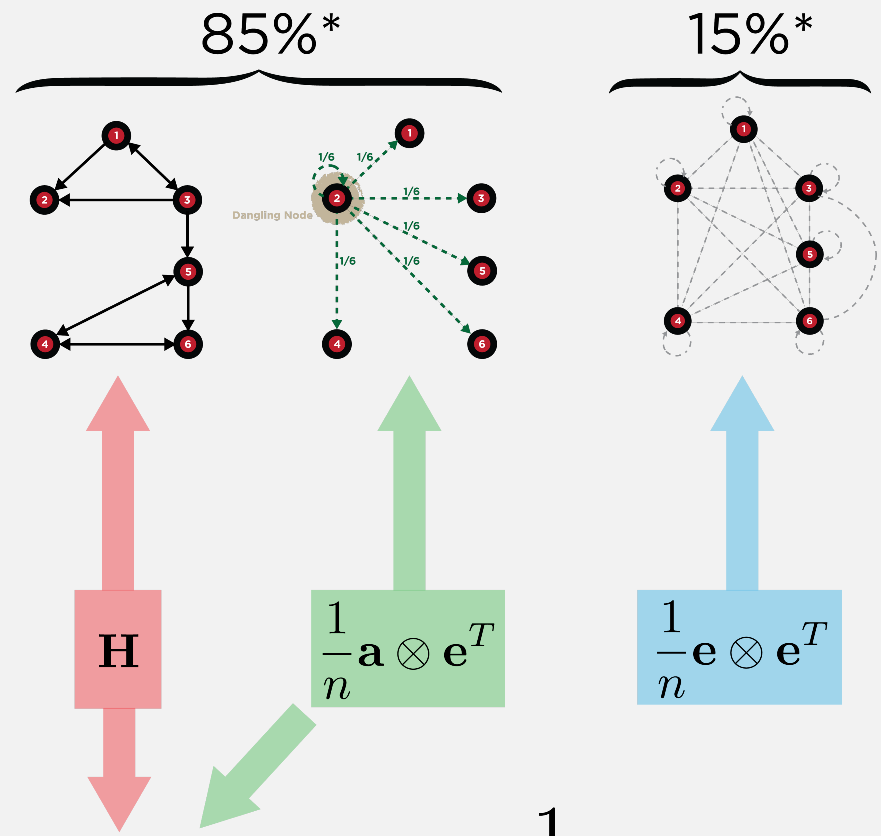
## of the Web Graph



# PageRank

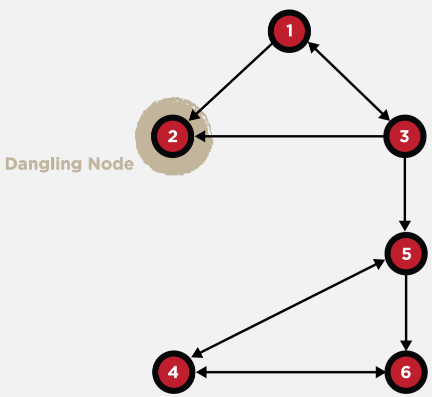



$$\mathbf{G} = \alpha \mathbf{S} + (1-\alpha)\frac{1}{n}\mathbf{e} \otimes \mathbf{e}^T$$

*Pour $\alpha = 0.85$

# Build the Google matrix




$$\mathbf{a} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{G} = \alpha S + (1-\alpha)\frac{1}{n}\mathbf{e} \otimes \mathbf{e}^T$$
$$\mathbf{S} = \mathbf{H} + \frac{1}{n}\mathbf{a} \otimes \mathbf{e}^T$$

$$\frac{1}{n}\mathbf{e} \otimes \mathbf{e}^T = \begin{bmatrix} 1/n & 1/n & 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n & 1/n & 1/n & 1/n \end{bmatrix}$$

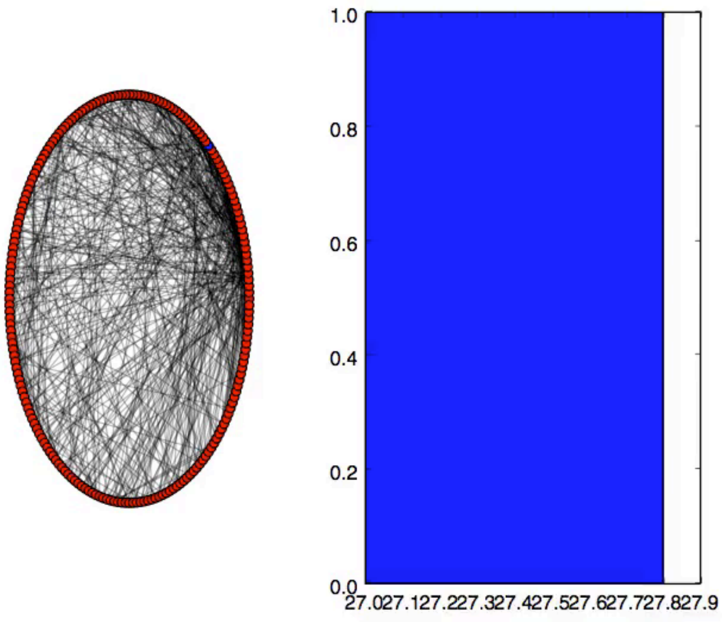
$$\frac{1}{n}\mathbf{a} \otimes \mathbf{e}^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1/n & 1/n & 1/n & 1/n & 1/n & 1/n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

## Power iteration to solve the PR

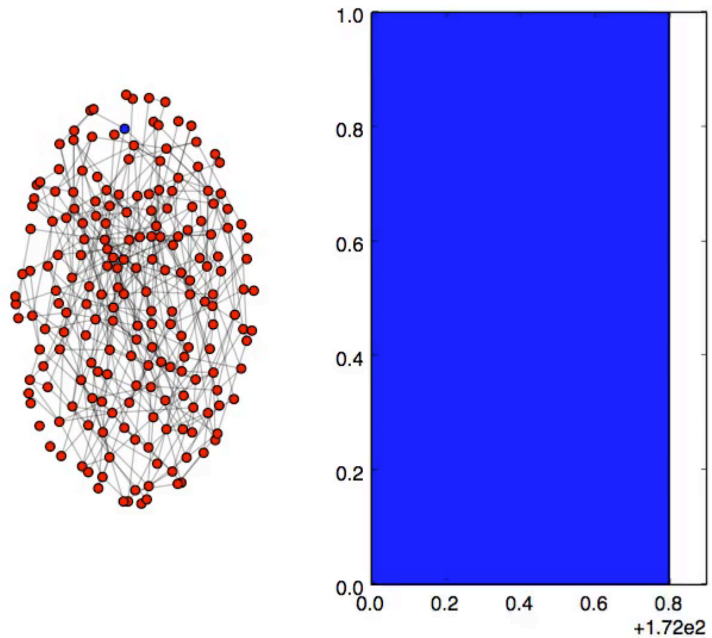
**Data:** Google Matrice $\mathbf{G}$, Nombre d'itération *MaxIter*, Tolérance $\epsilon$
**Result:** Vecteur $\mathbf{x}$
$\mathbf{x} := 1/n$
**xold** $:= 0$
**for** *i in range(MaxIter)* **do**
  $\mathbf{xold}^T := \mathbf{x}^T$
  $\mathbf{x}^T := \mathbf{x}^T\mathbf{G}$
  **if** $\epsilon > n\|\mathbf{x} - \mathbf{xold}\|$ **then**
    **return** $\mathbf{x}^T$
  **end**
**end**

# Simulation of a random walk

Non uniform degree distribution




Uniform degree distribution



# Google search engine

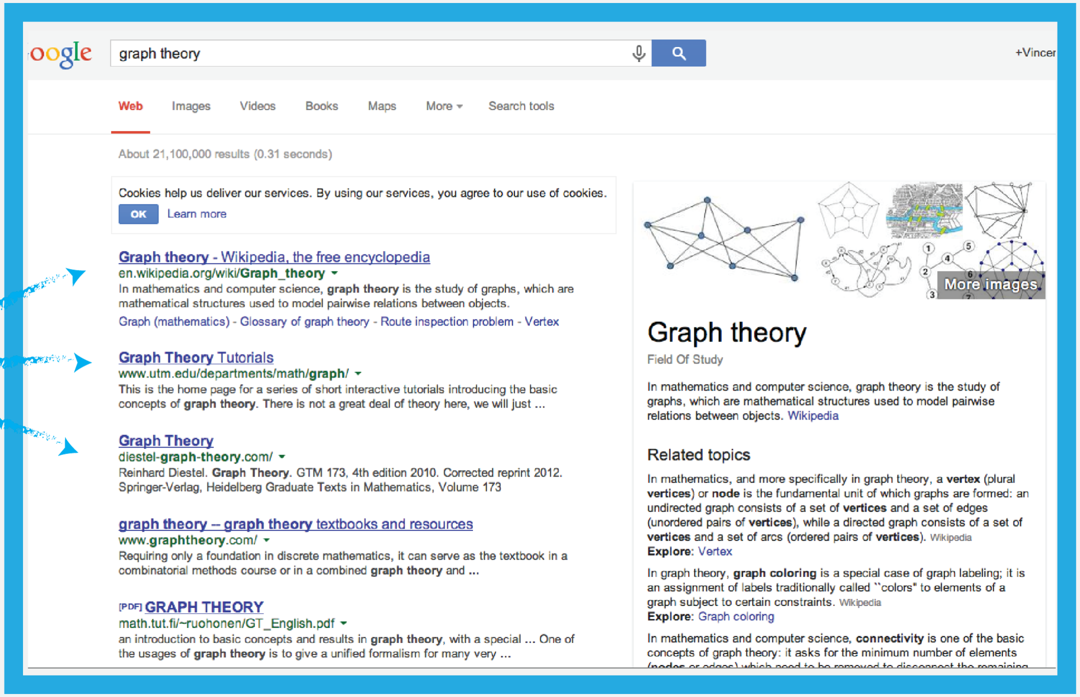
Google

## Google general overview:

1. Compute the PageRank for the web graph
2. For a request filter by keywords
3. Display result as function of their PageRank

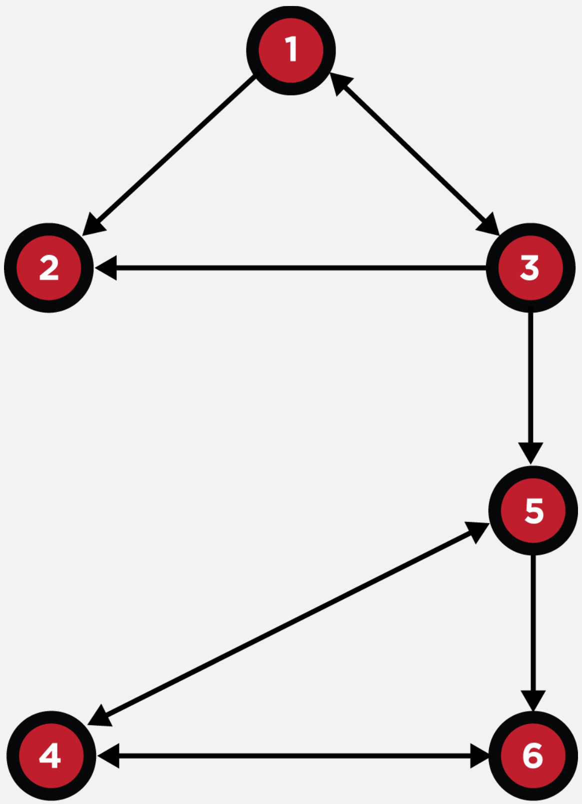
**Filter by beywords**

**Display result as function of their PR**



# Google search engine

Google




**Search for: Algorithms**

| Page Id | PR | Keyworks |
|---|---|---|
| $x_1$ | 0.5 | Flower, Garden, Pool |
| $x_2$ | 0.2 | News, Computer Science |
| $x_3$ | 0.1 | Computer Science, Algorithms |
| $x_4$ | 0.1 | Algorithms, Graph, Dijkstra |
| $x_5$ | 0.05 | Algorithms, Graph, Dijkstra, C++ |
| $x_6$ | 0.05 | Network Science, Graphs, Algorithms |

**Return Results**

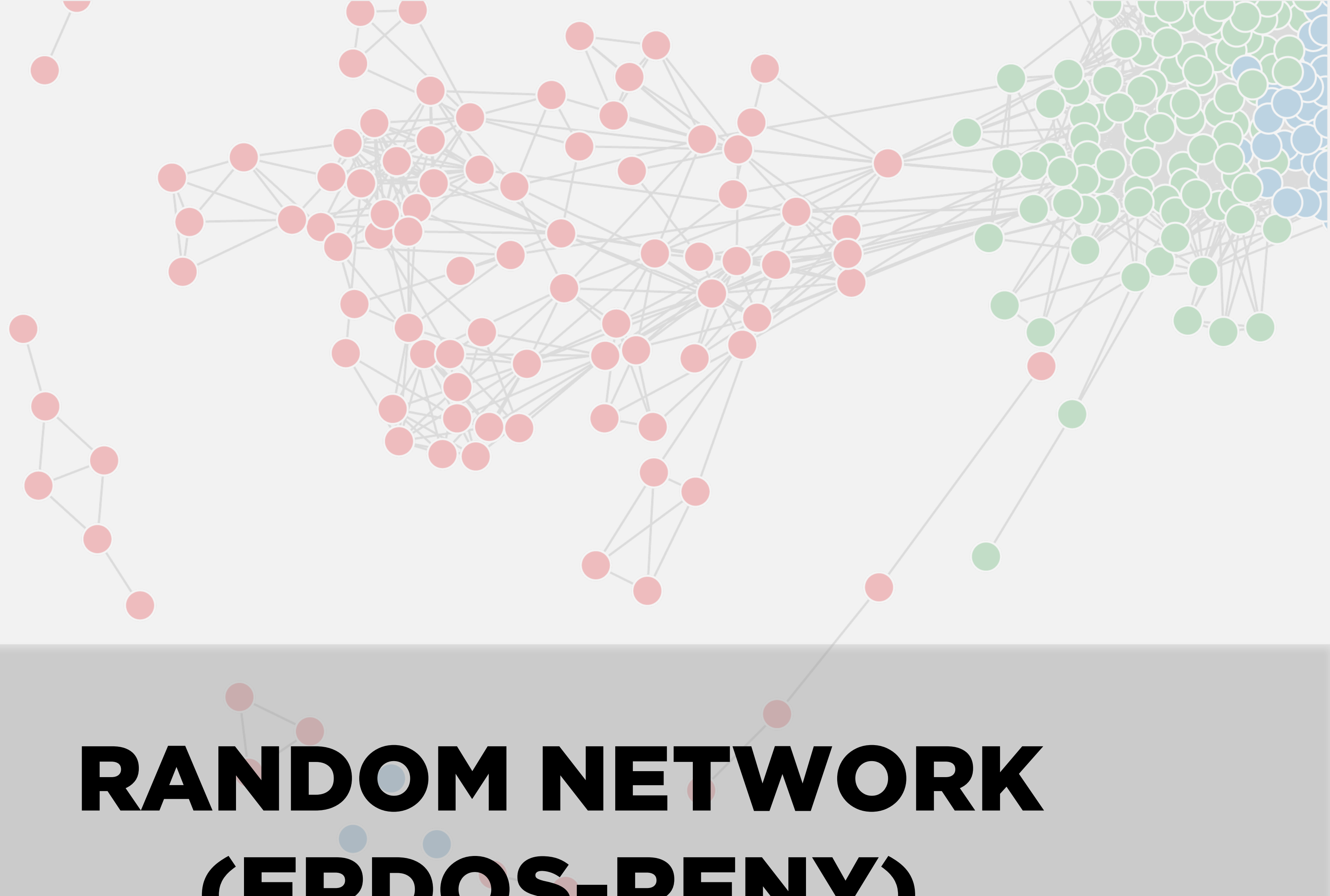
**Ordered by their PR**

# Many other applications

## Ranking anything

- Language
- Social Network (klout)
- Bitcoin
- College Football

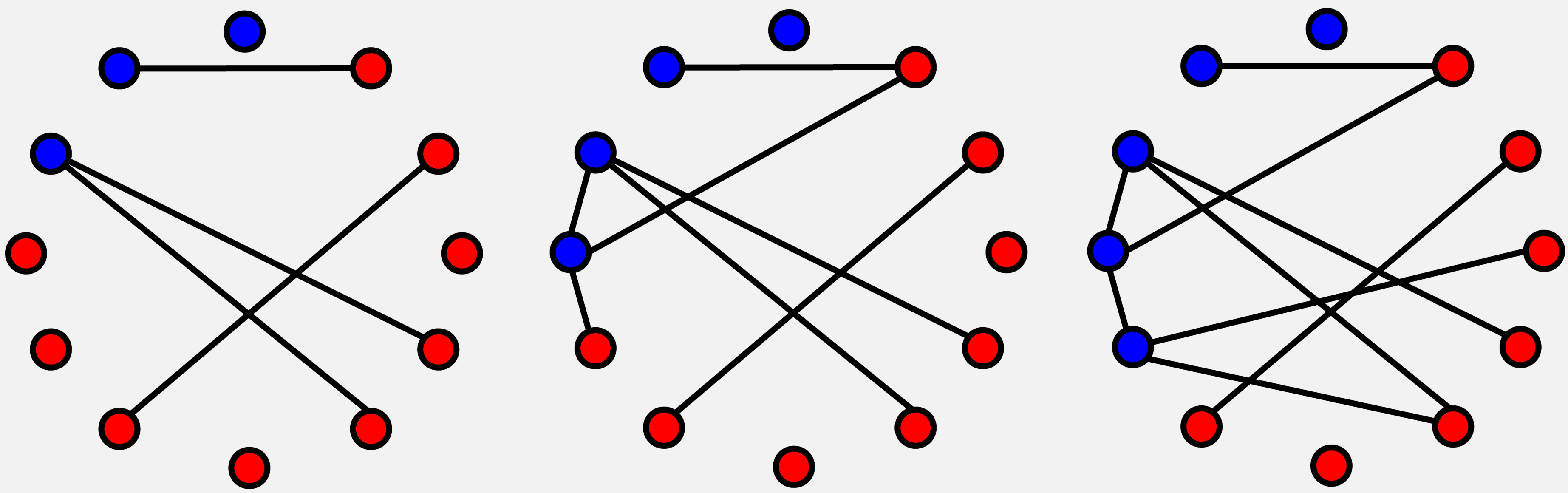
network-based ranking system for US college football
J. Park, M. J. Newman
J. Stat. Mech. 2005



# RANDOM NETWORK (ERDOS-RENY)

# Random Network Models

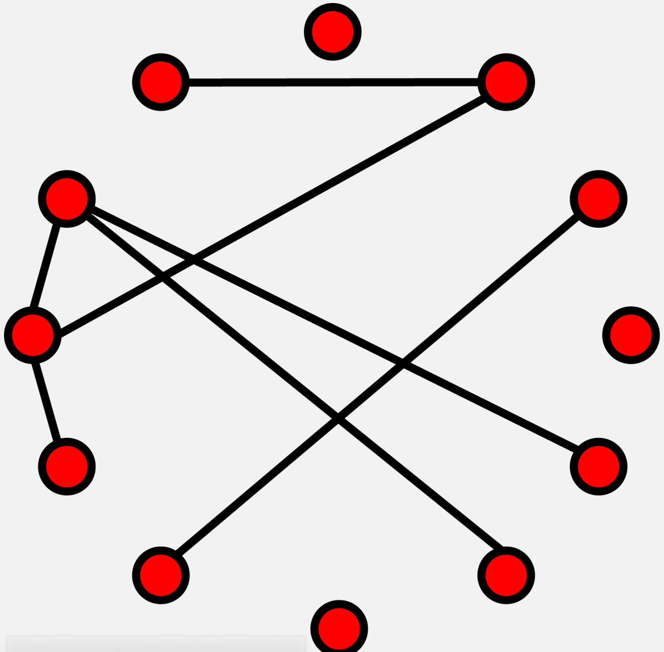
## Erdos-Reny Model



## The NULL model

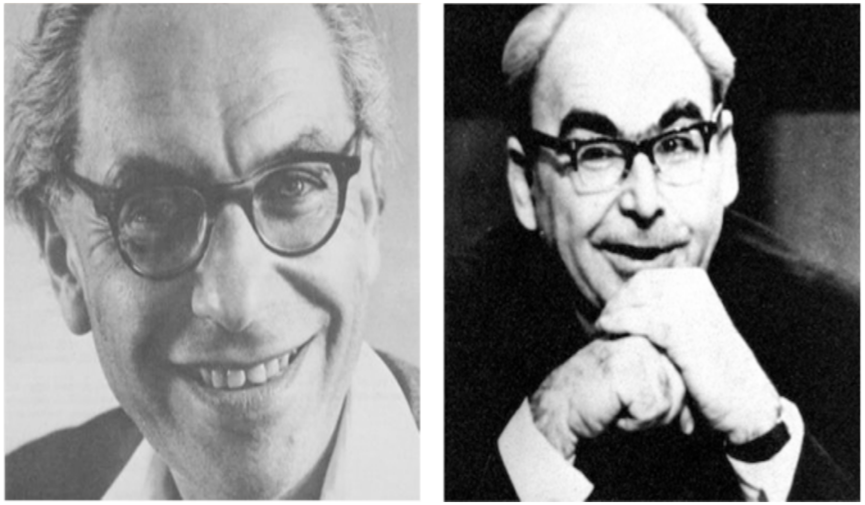
# Erdos-Reny Model (1960)

**Connect with probability p**



$N = 12, p = 1/10$

$<k> = 1.1$



*Pál Erdös* (1913-1996) *Alfréd Rényi* (1921-1970)

**Definition:**

A **random graph** is a graph of *N* labeled nodes where each pair of nodes is connected by a preset probability **p**.

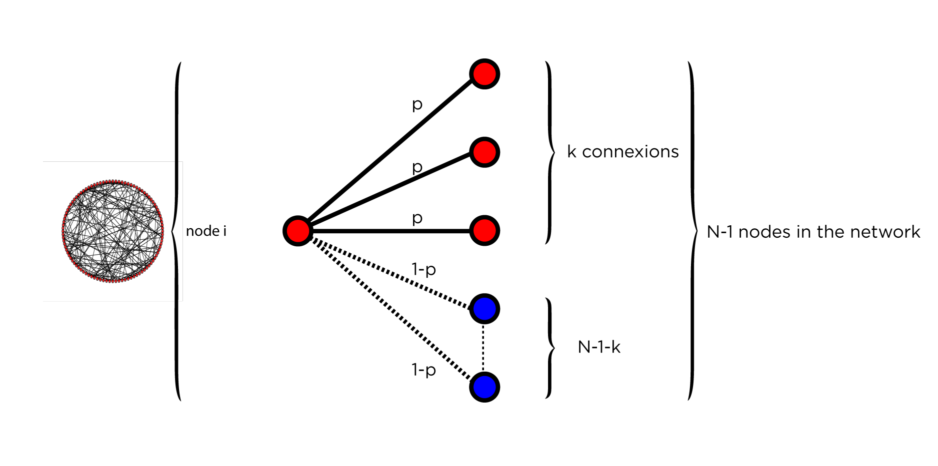
We will call is G(N, p).

# Degree distribution

Each node has *(N – 1)* tries to get edges

Each try is a success with probability ***p***

The binomial distribution gives us the probability that a node has degree ***k***




**Probability of having exactly k edges:**

$$P_k = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

**How many edges per node?**

$$\langle k \rangle = (N-1)p$$

$$\sigma_k \approx \frac{1}{(N-1)^{1/2}}$$

As the network size increases, the distribution becomes increasingly narrow—we are increasingly confident that the degree of a node is in the vicinity of <k>.

**G(N,p)-model:**

For each potential edge we flip a biased coin:

with probability ***p*** we add the edge
with probability *(1-p)* we don't

# Reminder on Binomial Distribution

$$\langle k\rangle = E(K) = \sum_{k=0}^{N-1} kP_k = \sum_{k=0}^{N-1} k\begin{pmatrix} N-1 \\ k \end{pmatrix} p^k(1-p)^{N-1-k}$$

$$= \sum_{k=1}^{N-1} k\frac{(N-1)!}{k!(N-1-k)!} p^k(1-p)^{N-1-k}$$

$$= (N-1)p\sum_{k=1}^{N-1} \frac{(N-2)!}{(k-1)!(N-1-k)!} \quad p^{k-1}(1-p)^{N-1-k}, \quad a = k-1,\ b = N-2, N-1 = b+1, k = a+1$$

$$= (N-1)p\sum_{a=0}^{b} \frac{b!}{a!(b-a)!} p^a(1-p)^{b-a}$$

$$= (N-1)p\sum_{a=0}^{b} \begin{pmatrix} b \\ a \end{pmatrix} p^a(1-p)^{b-a}$$

**Average Degree of G(N,p)**

$$\langle k\rangle = (N-1)p$$

# Approximations

$$p_k = \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

Binomial

$\downarrow$ limit $p$ small

$$p_k = \frac{z^k e^{-z}}{k!}$$

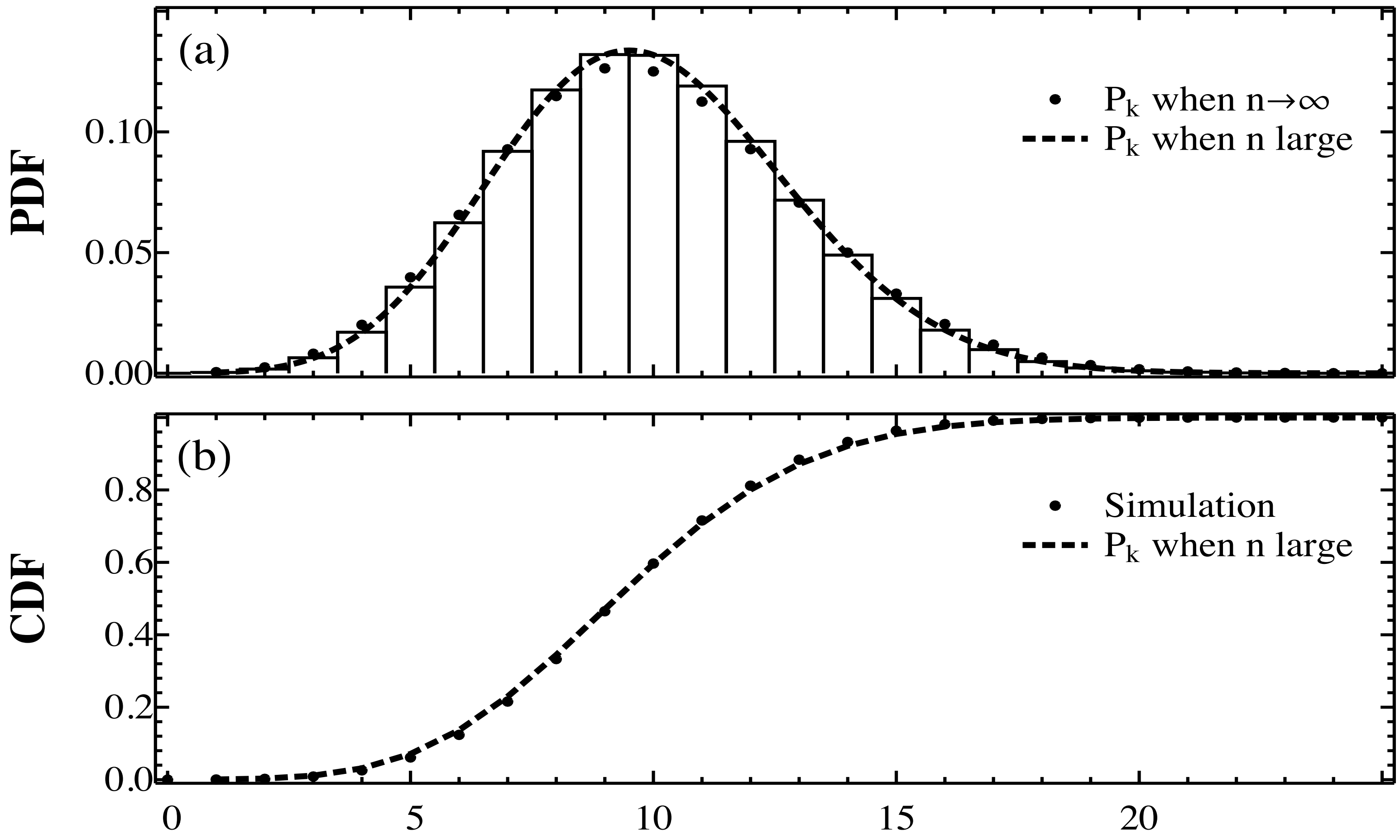
Poisson

$\downarrow$ limit large $n$

$$p_k = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(k-z)^2}{2\sigma^2}}$$

Normal

# Degree Distribution



# Quiz Q1

**The maximum degree of a node in a simple (no multiple edges between the same two nodes) N node graph is:**

a) N

b) N - 1

c) N / 2

# Quiz Q2

What is the probability that a specific node in a N nodes graph as exactly 4 edges (when N=100 and p=0.1):

$$P_4 = \frac{99!}{4!95!} 0.1^4 (1-0.1)^{95}$$

# Quiz Q3

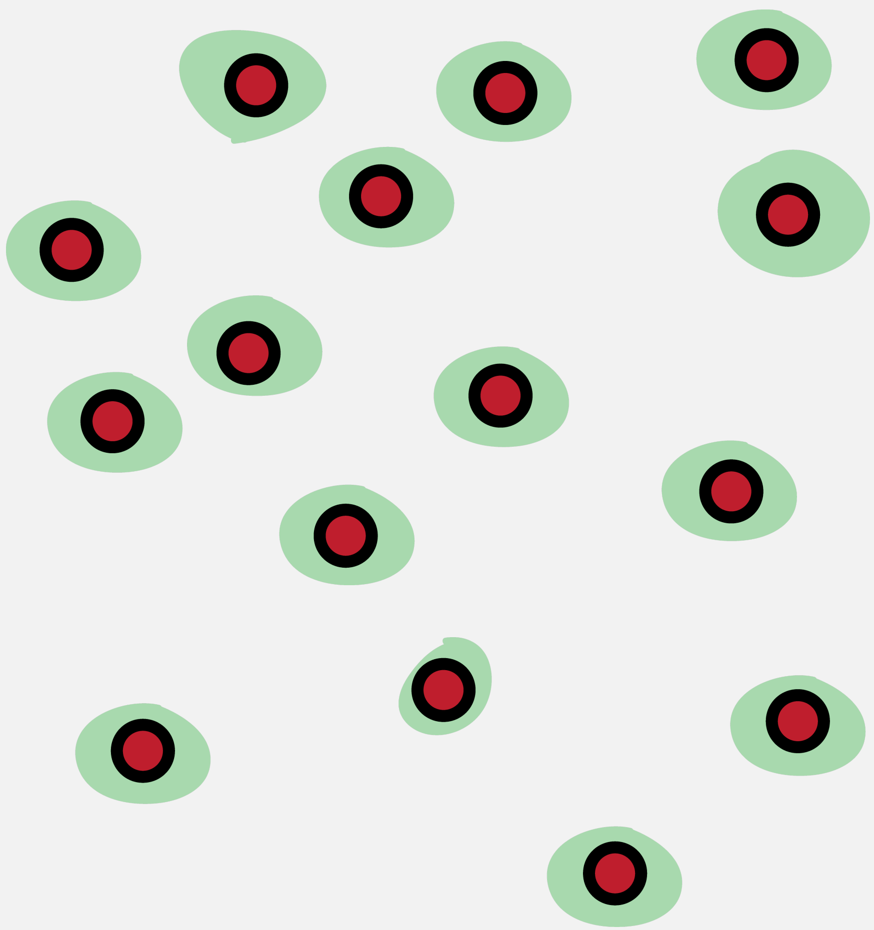
What is the average degree a node in a network G(N,p) where N=20, p=0.1:

a) 1.0

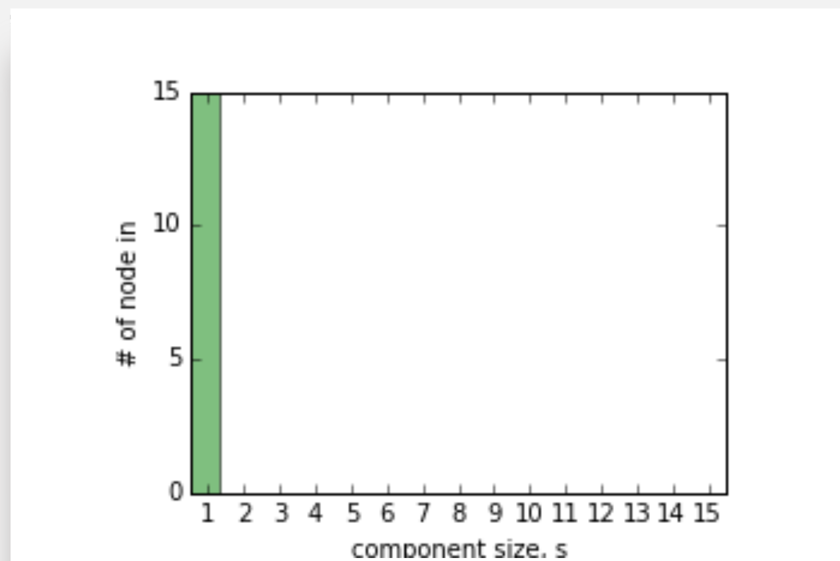
b) 1.9

c) 5.4

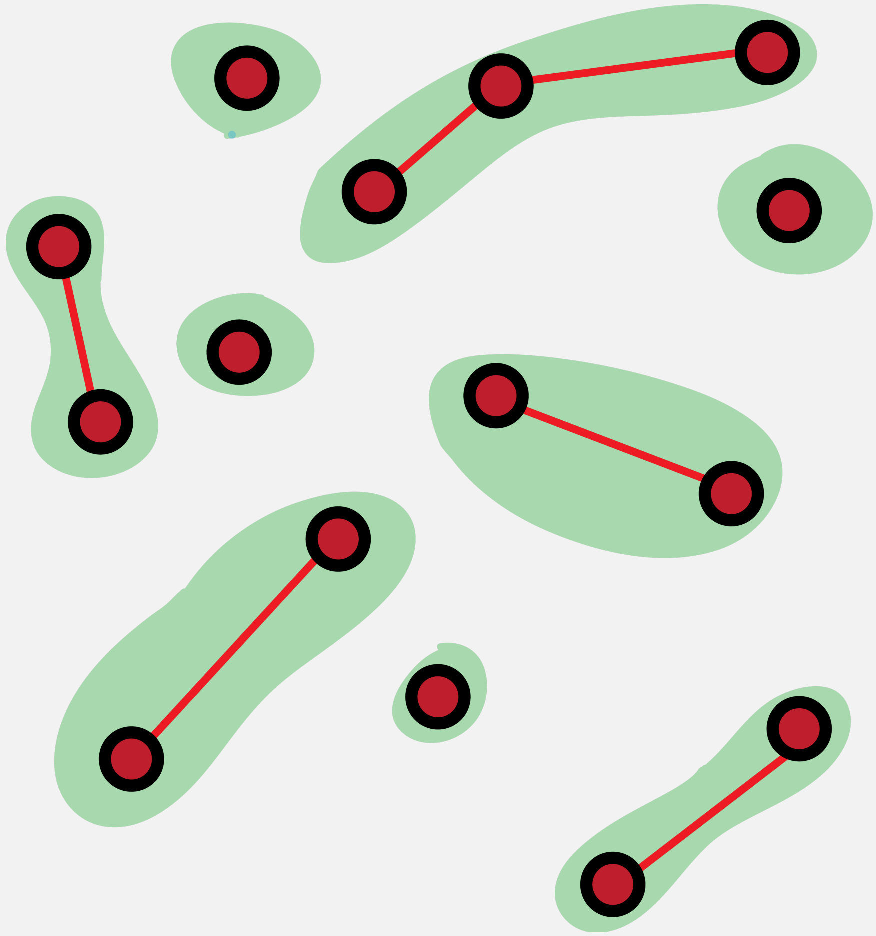
# Percolation



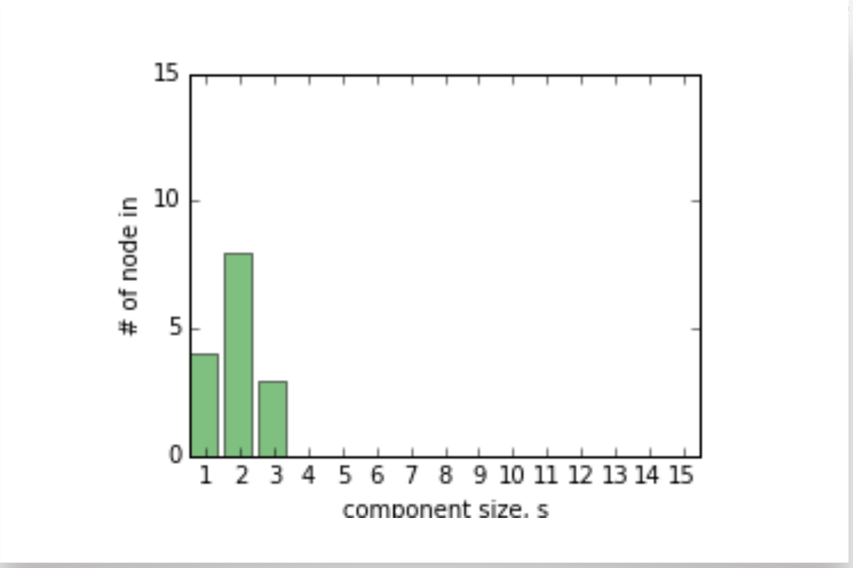
**The percolation game:**
choose *X* random pairs
connect them repeat (count #
of components)



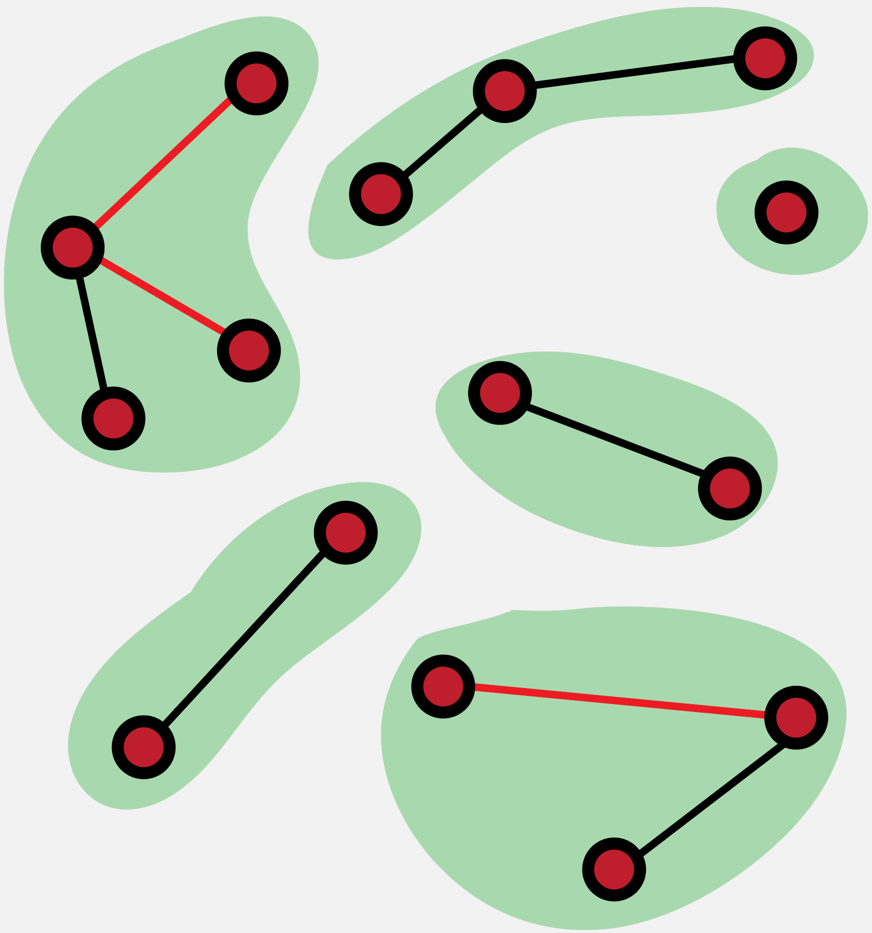
# Percolation



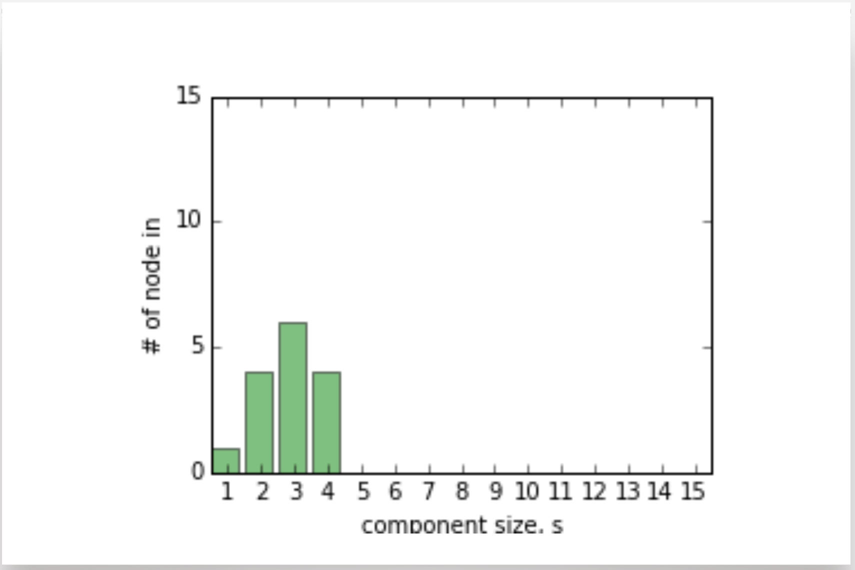
**The percolation game:**
choose *X* random pairs connect them repeat (count # of components)



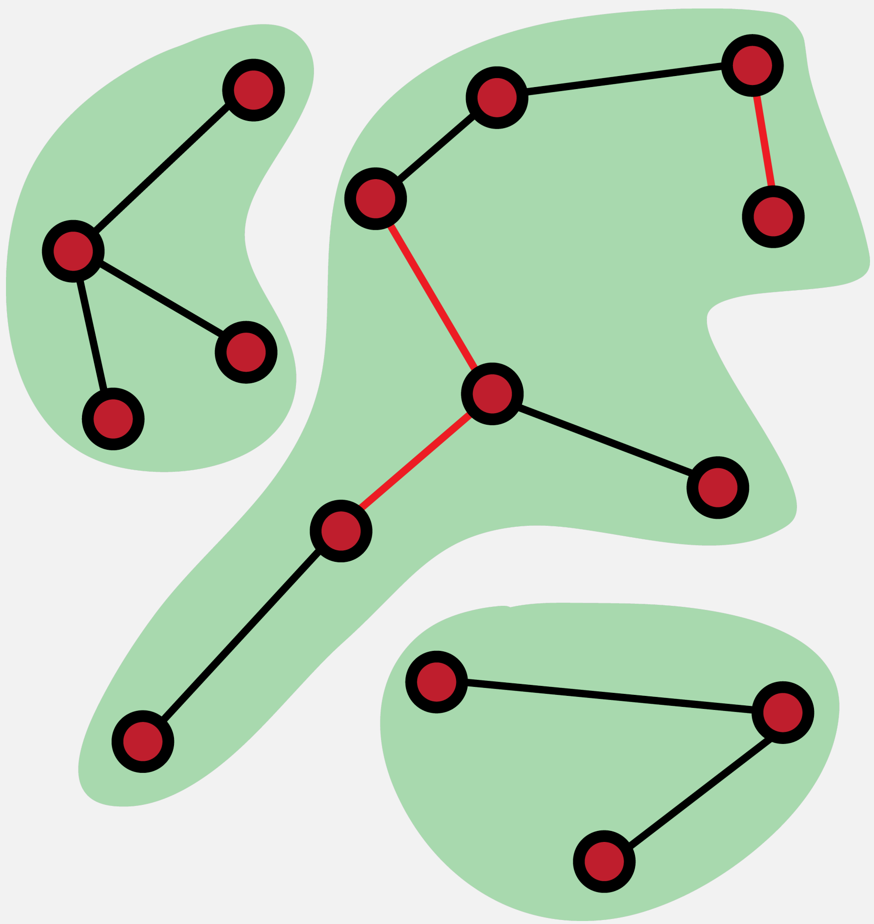
# Percolation



**The percolation game:**
choose *X* random pairs
connect them repeat (count #
of components)



# Percolation



**The percolation game:**
choose *X* random pairs
connect them repeat (count #
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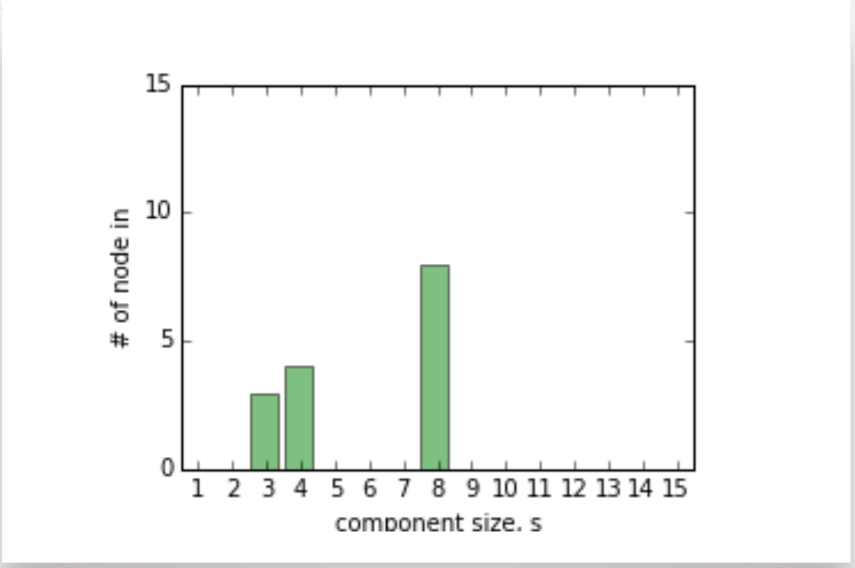
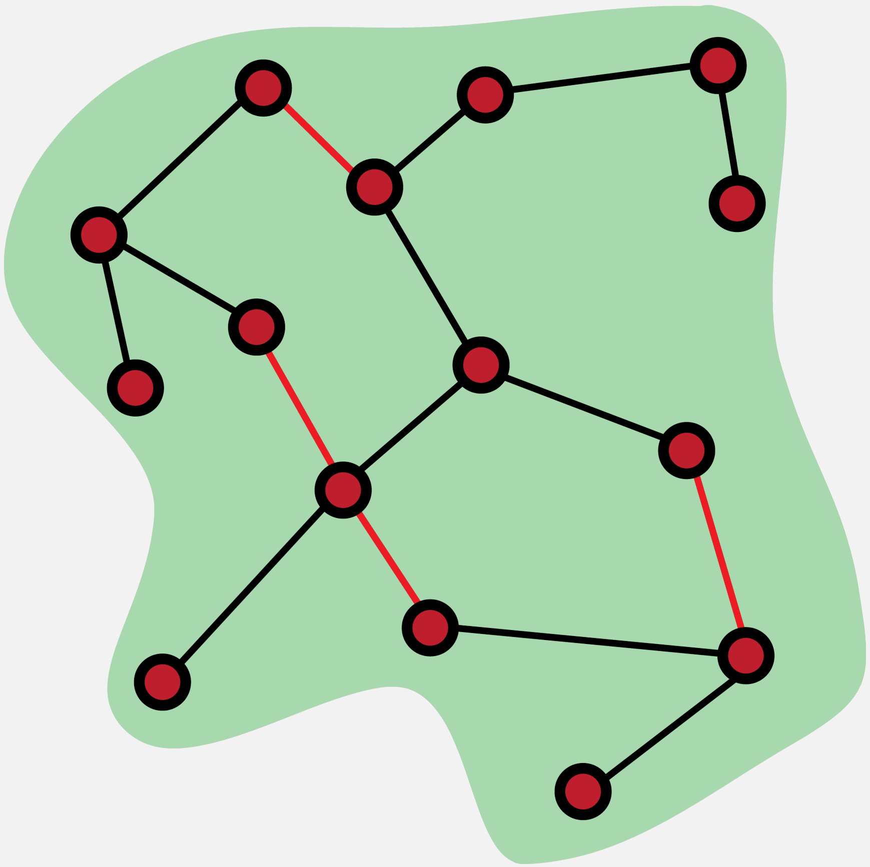
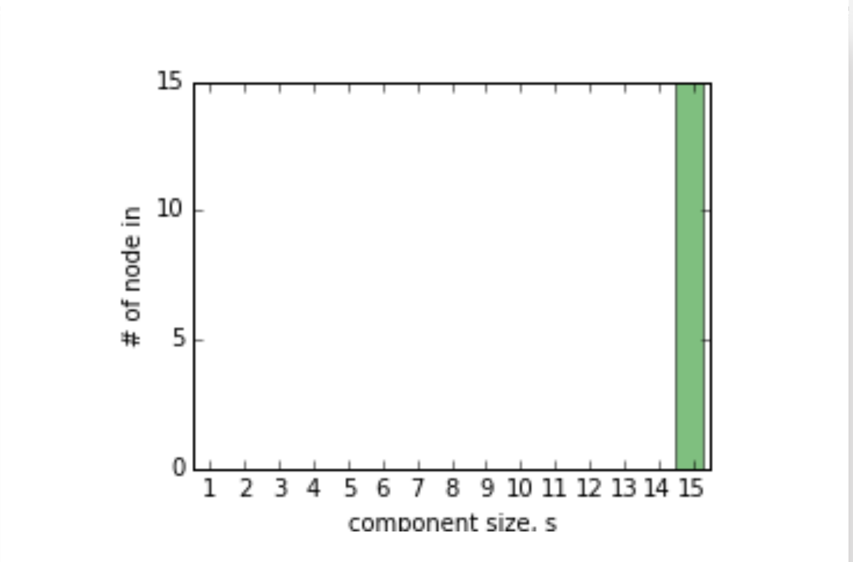


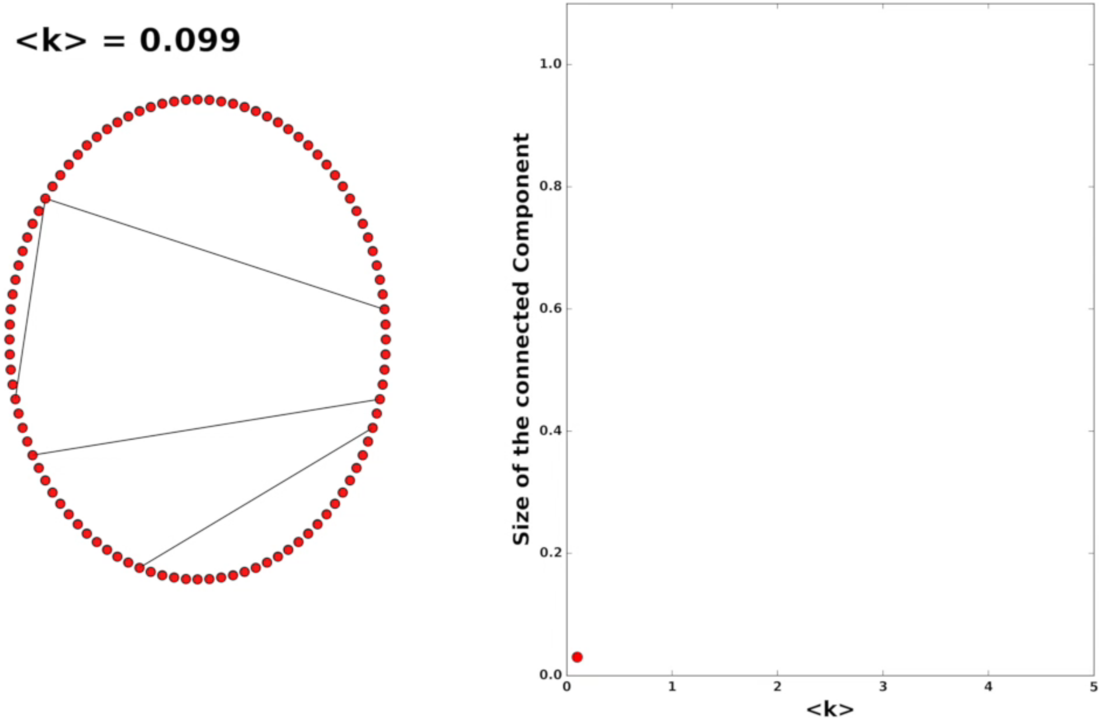
# Percolation



**The percolation game:**
choose *X* random pairs
connect them repeat (count #
of components)



# Percolation in action




connected

disconnected

**Probability p increases**

low

high

# Quiz Q5:

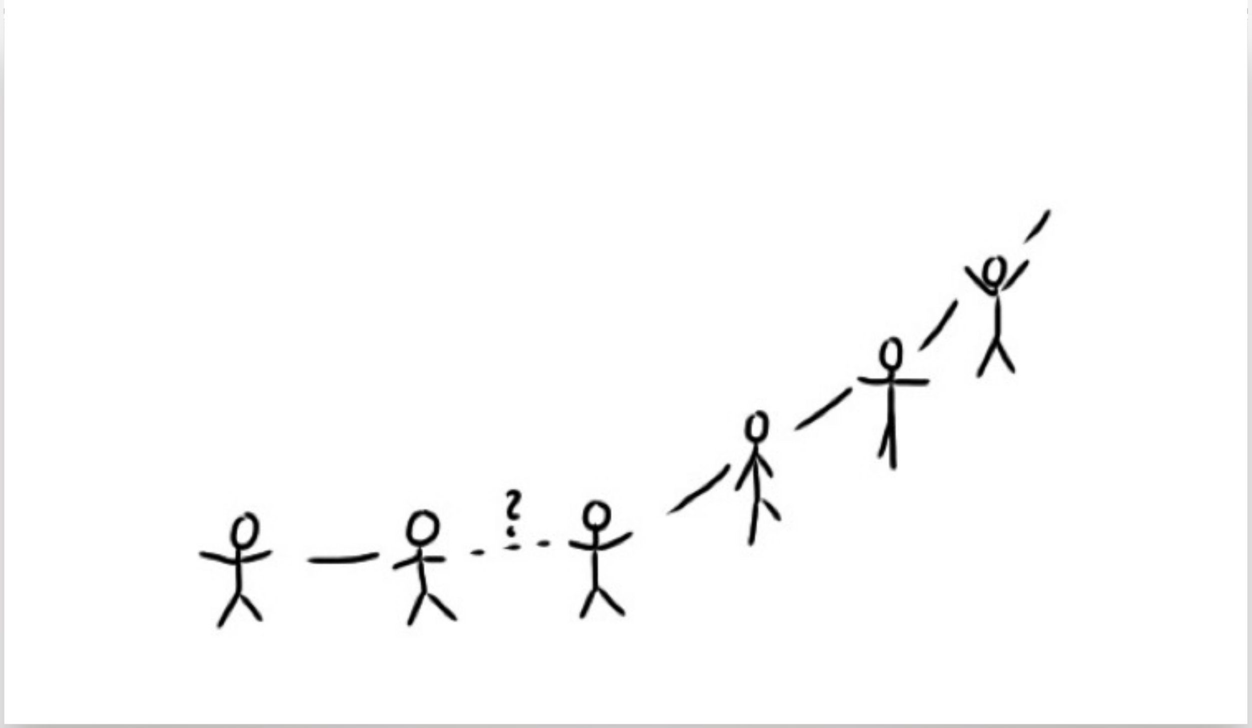
What is the average degree <k> at which the giant component start to emerge:

a) 5

b) 3

c) 1

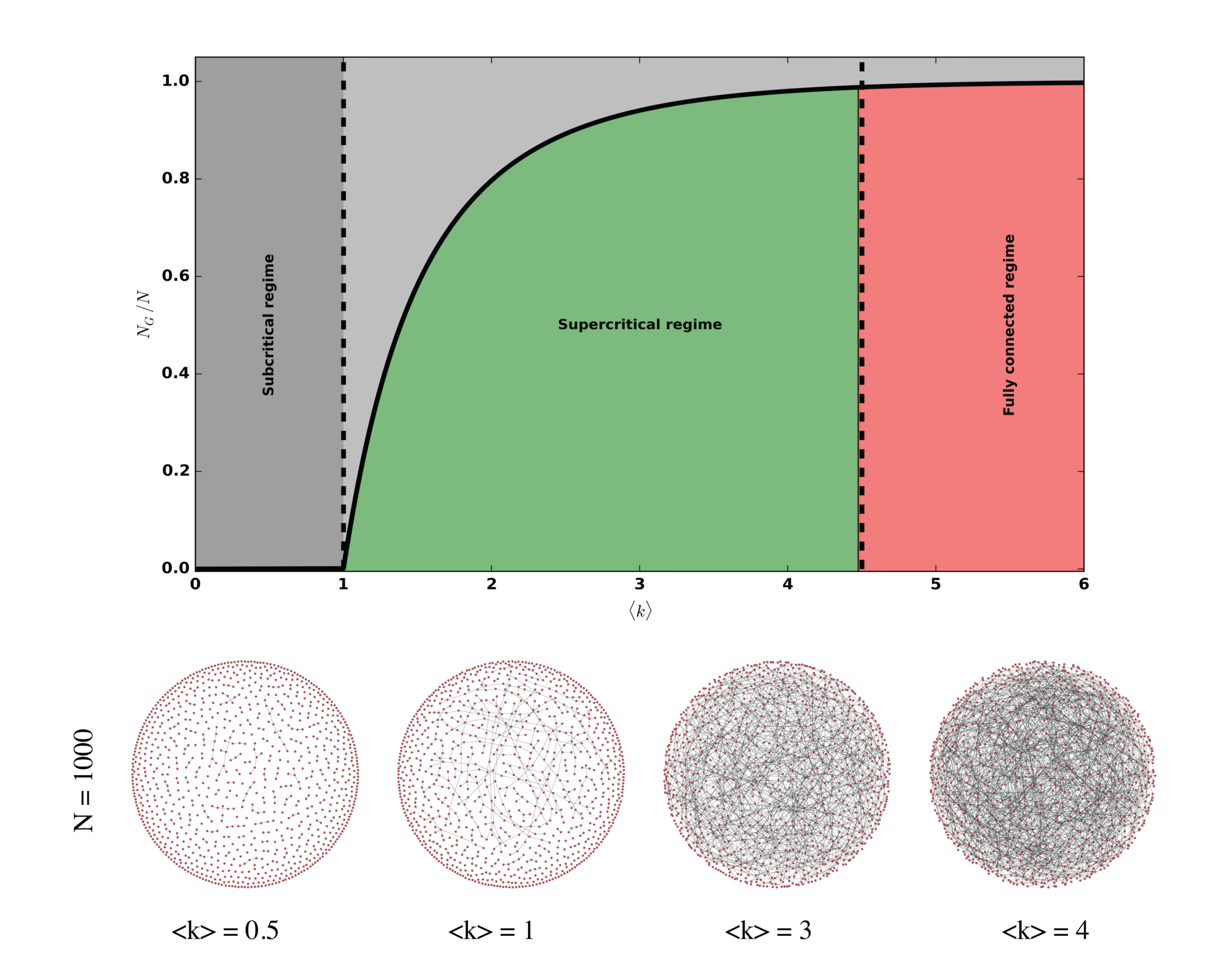
# Giant component – another angle



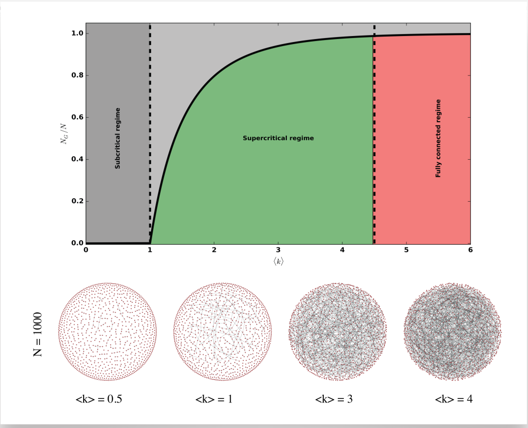
**By property of degree distribution:**

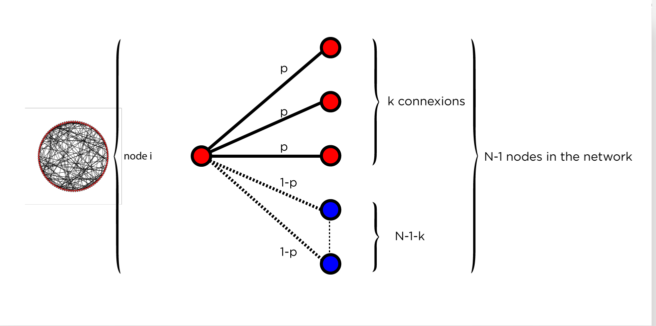
the average degree of your friends, you excluded, is <k> so at <k> = 1, each of your friends is expected to have another friend, who in turn have another friend, etc. the giant component emerges

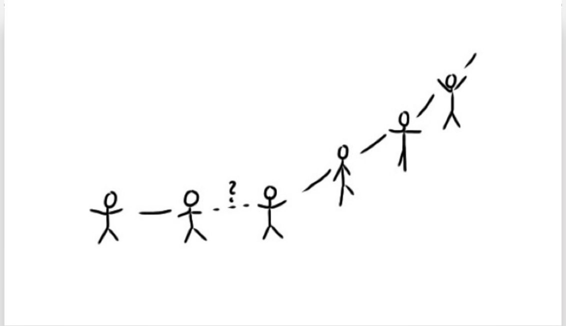
# Critical Regimes



# Critical regimes

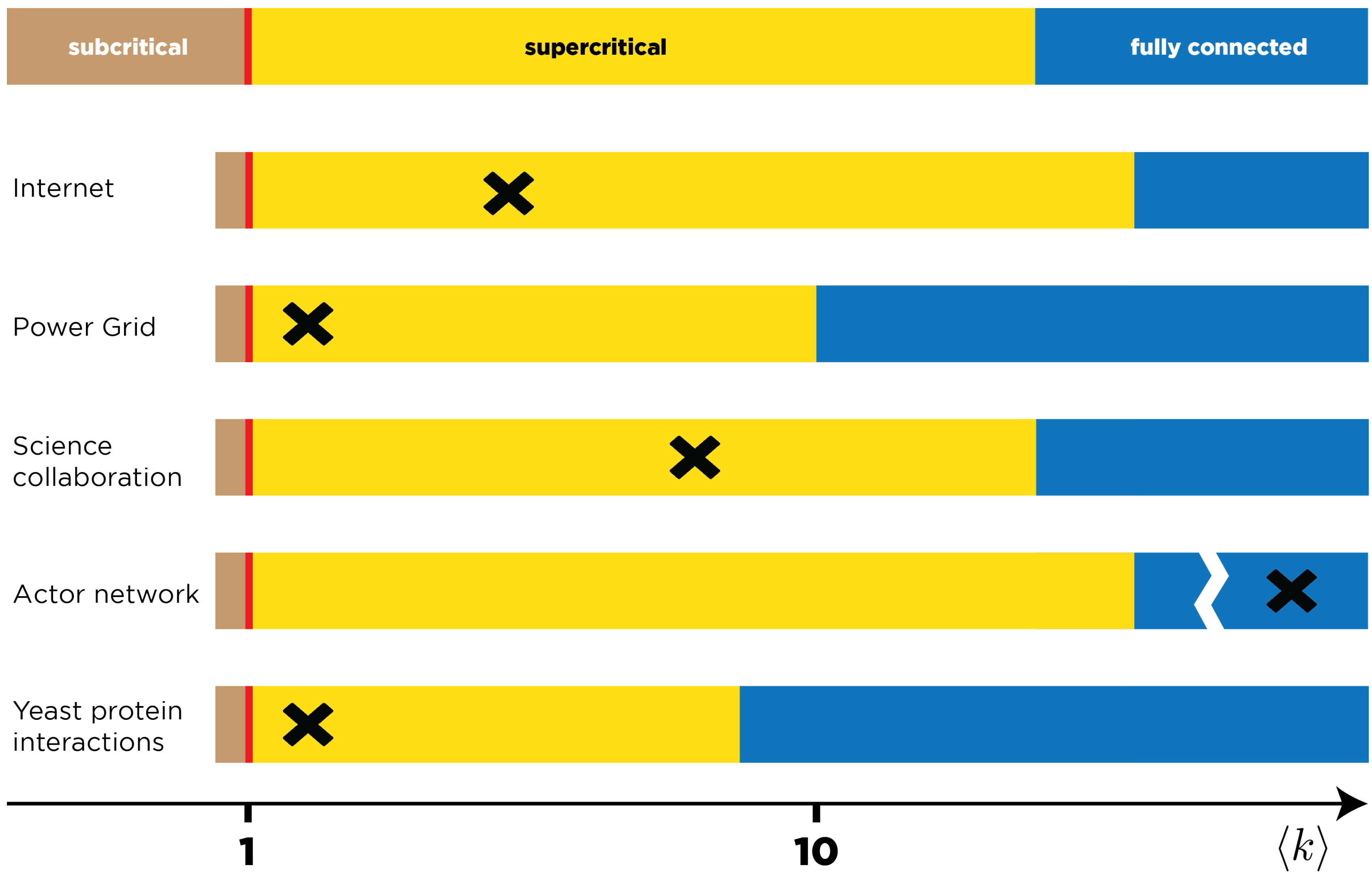


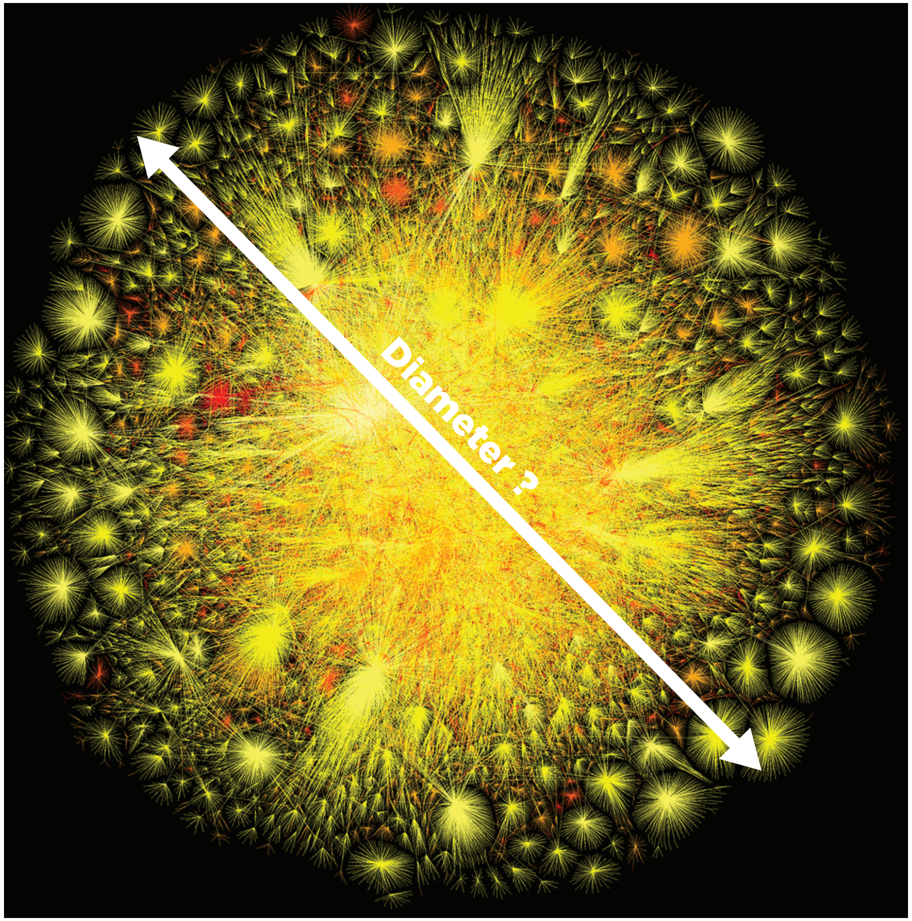



1) Once the average degree exceeds <k>= 1, a giant component should emerge that contains a finite fraction of all nodes. Hence only for <k> > 1 the nodes organize themselves into a recognizable network.
2) For <k> > ln(N) all components are absorbed by the giant component, resulting in a single fully connected network.

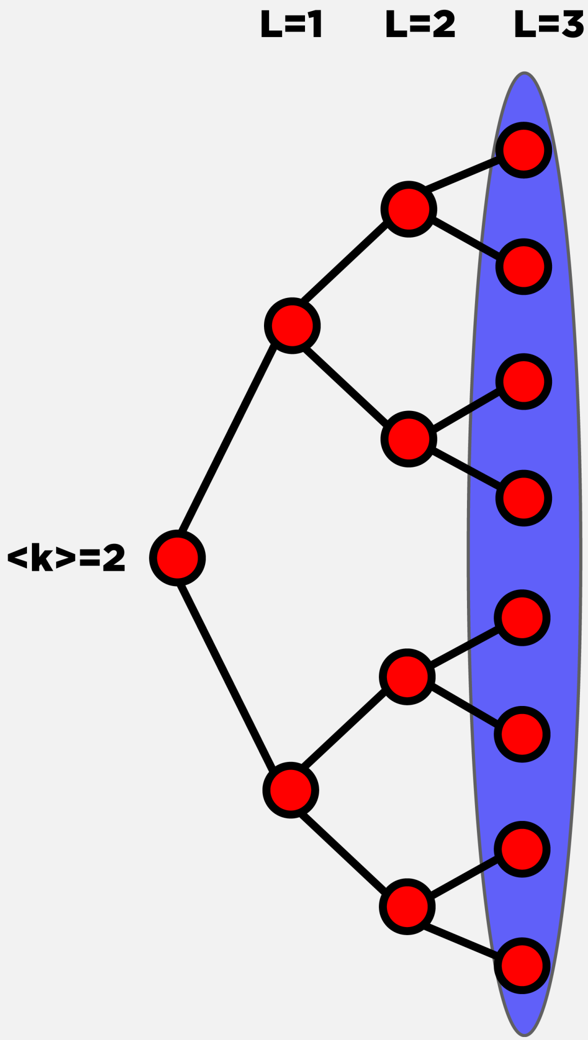
# Most real networks are supercritical



# Average shortest path



# Average shortest path



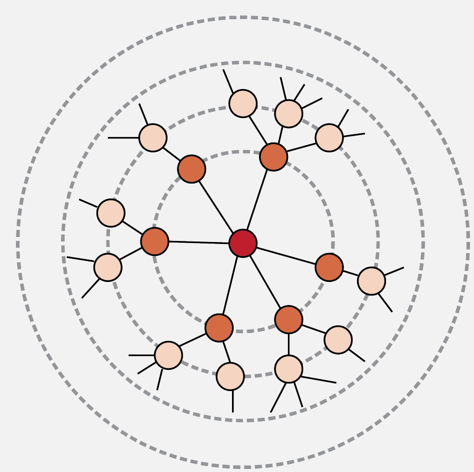

$n_l = \langle k \rangle^l$

Scaling of the average shortest path:

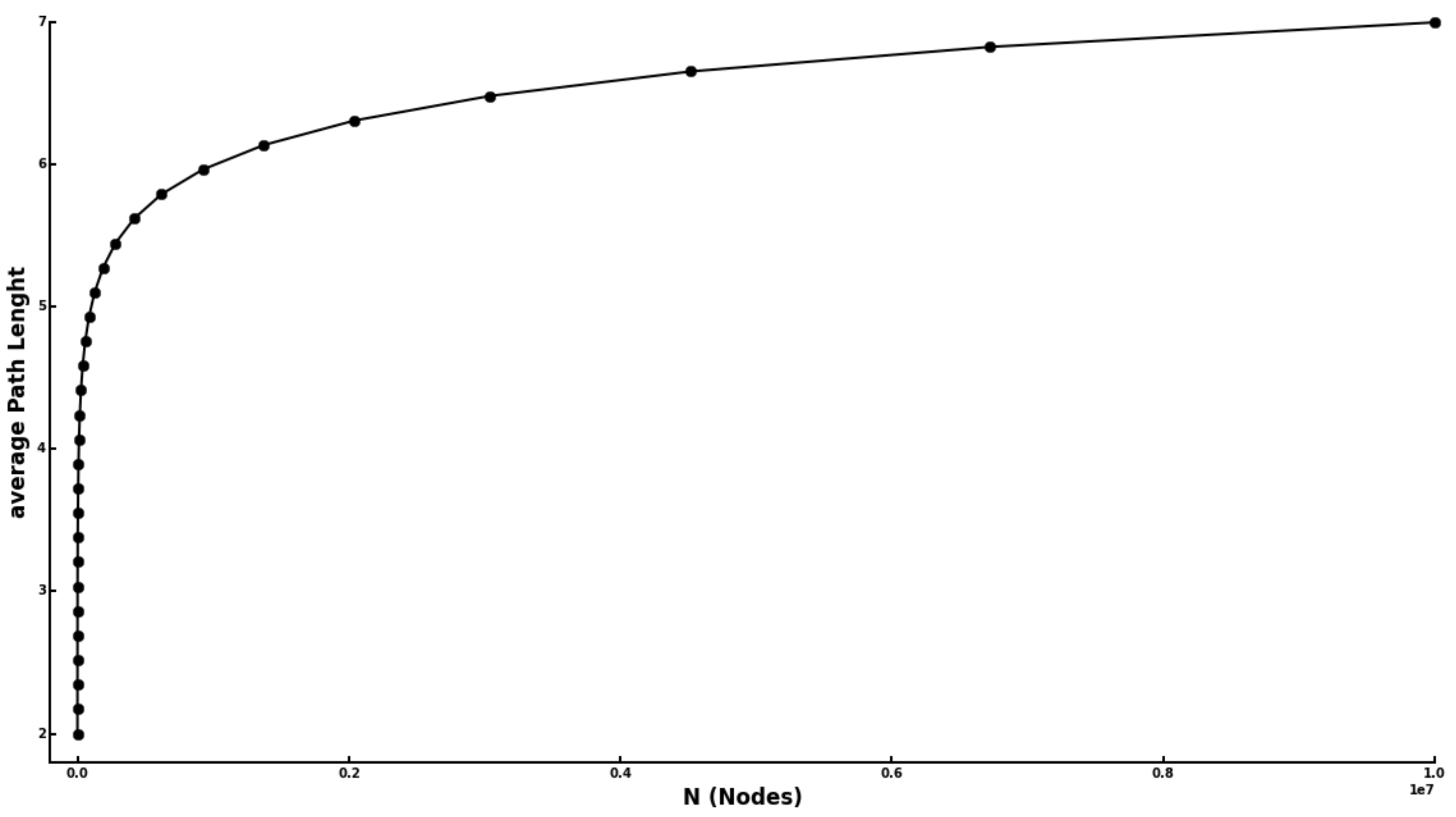
$$l_{avg} \sim \frac{\log N}{\log \langle k \rangle} \quad \text{Eq. (1)}$$

a rough argument:

- ER graph is locally tree-like (no loops; low clustering coefficient)



# Average Path Length



# Real Network

| Network | N | L | <k> | <d> | $d_{max}$ | lnN/ln<k> |
|---|---|---|---|---|---|---|
| *Internet* | 192,244 | 609,066 | 6.34 | 6.98 | 26 | 6.58 |
| *WWW* | 325,729 | 1,497,134 | 4.60 | 11.27 | 93 | 8.31 |
| *Power Grid* | 4,941 | 6,594 | 2.67 | 18.99 | 46 | 8.66 |
| *Mobile Phone Calls* | 36,595 | 91,826 | 2.51 | 11.72 | 39 | 11.42 |
| *Email* | 57,194 | 103,731 | 1.81 | 5.88 | 18 | 18.4 |
| *Science Collaboration* | 23,133 | 93,439 | 8.08 | 5.35 | 15 | 4.81 |
| *Actor Network* | 702,388 | 29,397,908 | 83.71 | 3.91 | 14 | 3.04 |
| *Citation Network* | 449,673 | 4,707,958 | 10.43 | 11,21 | 42 | 5.55 |
| *E. Coli Metabolism* | 1,039 | 5,802 | 5.58 | 2.98 | 8 | 4.04 |
| *Protein Interactions* | 2,018 | 2,930 | 2.90 | 5.61 | 14 | 7.14 |

**Table**

The average distance and the maximum distance dmax for the ten reference networks. The last column provides predicted by (eq.1 slide 33), indicating that it offers a reasonable approximation to the measured . Yet, the agreement is not perfect.

# Quiz Q5:

Ignore for the time being the fact that many of your friends' friends are your friends as well. If everyone has 500 friends, the average person would have how many friends of friends?

# SMALL WORLD NETWORK (WATTS-STROGATZ MODEL)

# Milgram experiment


**Instructions:**

Given a target individual (stockbroker in Boston), pass the message to a person you correspond with who is "closest" to the target.

**Outcome:**

20% of initiated chains reached target
average chain length = 6.5

**"Six degrees of separation"**

# Milgram's experiment repeated

**Email experiment:**

- 18 targets
- 13 different countries
- 60,000+ participants
- 24,163 message chains
- 384 reached their targets
- average path length 4.0

Is 6 is a surprising number?
In the 1960s? Today? Why?

Pool and Kochen in (1978 established that the average person has between 500 and 1500 acquaintances)

Source: NASA, U.S. Government; http://visibleearth.nasa.gov/view_rec.php?id=2429

An Experimental Study of Search in Global Social Networks
Peter Sheridan Dodds, Roby Muhamad, Duncan J. Watts
Science, 2003: 301 (5634), 827-829.

# Interpreting Milgram's experiment

- **Is 6 is a surprising number?**
  In the 1960s? Today? Why?

- **If social networks were random... ? Pool and Kochen (1978) ~500-1500 acquaintances / person:**
  500 choices 1st link
  $500^2$ = 250,000 potential 2nd degree neighbors
  $500^3$ = 125,000,000 potential 3rd degree neighbors

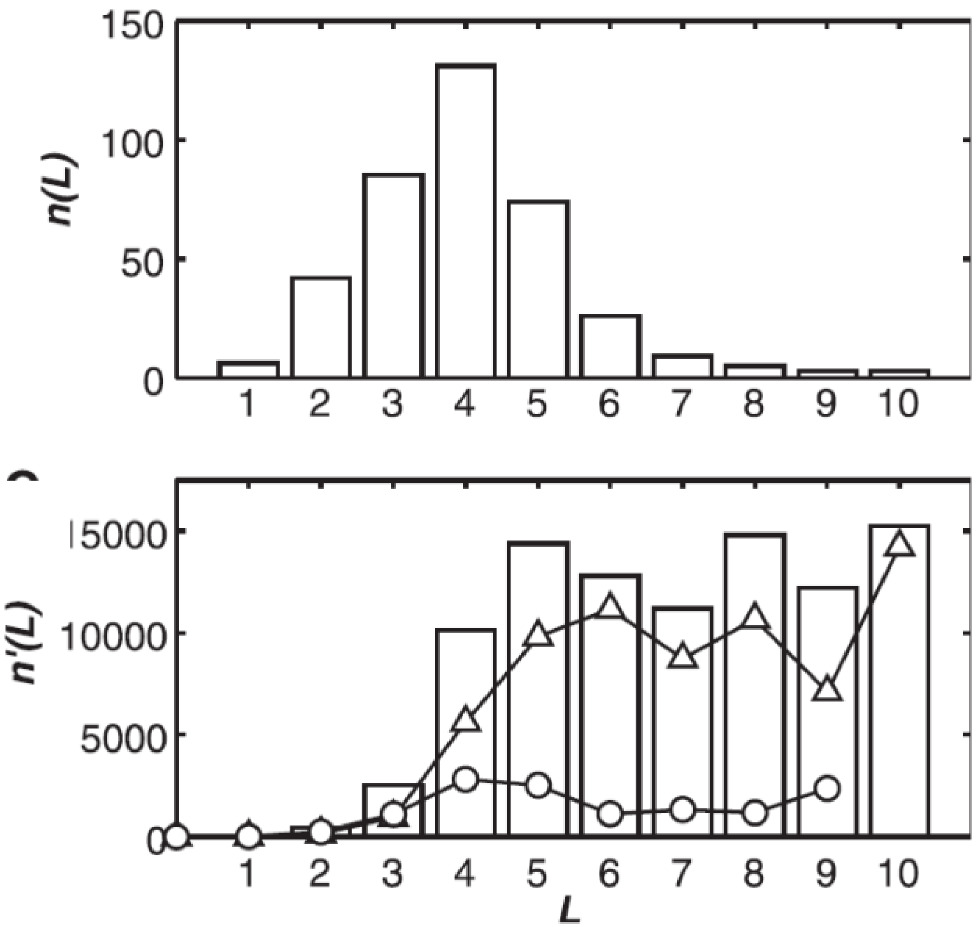
- **If networks are completely cliquish?**
  all my friends' friends are my friends
  what would happen?

# Quiz:

**If the network were completely cliquish, that is all of your friends of friends were also directly your friends, what would be true:**

a) I am in an isolated group
b) Your shortest path length to your friend's friends would be 2
c) None of your friendship edges would be a part of a triangle (closed triad)

# Estimating the true distance




observed chain lengths

recovered histogram of path length

○ **inter-country**
△ **intra-country**

**An Experimental Study of Search in Global Social Networks**
Peter Sheridan Dodds, Roby Muhamad, and Duncan J. Watts
Science 301 (5634), 827, 20013.

# Navigability and search strategy: Small world experiment @ Columbia

**Successful chains disproportionately used**

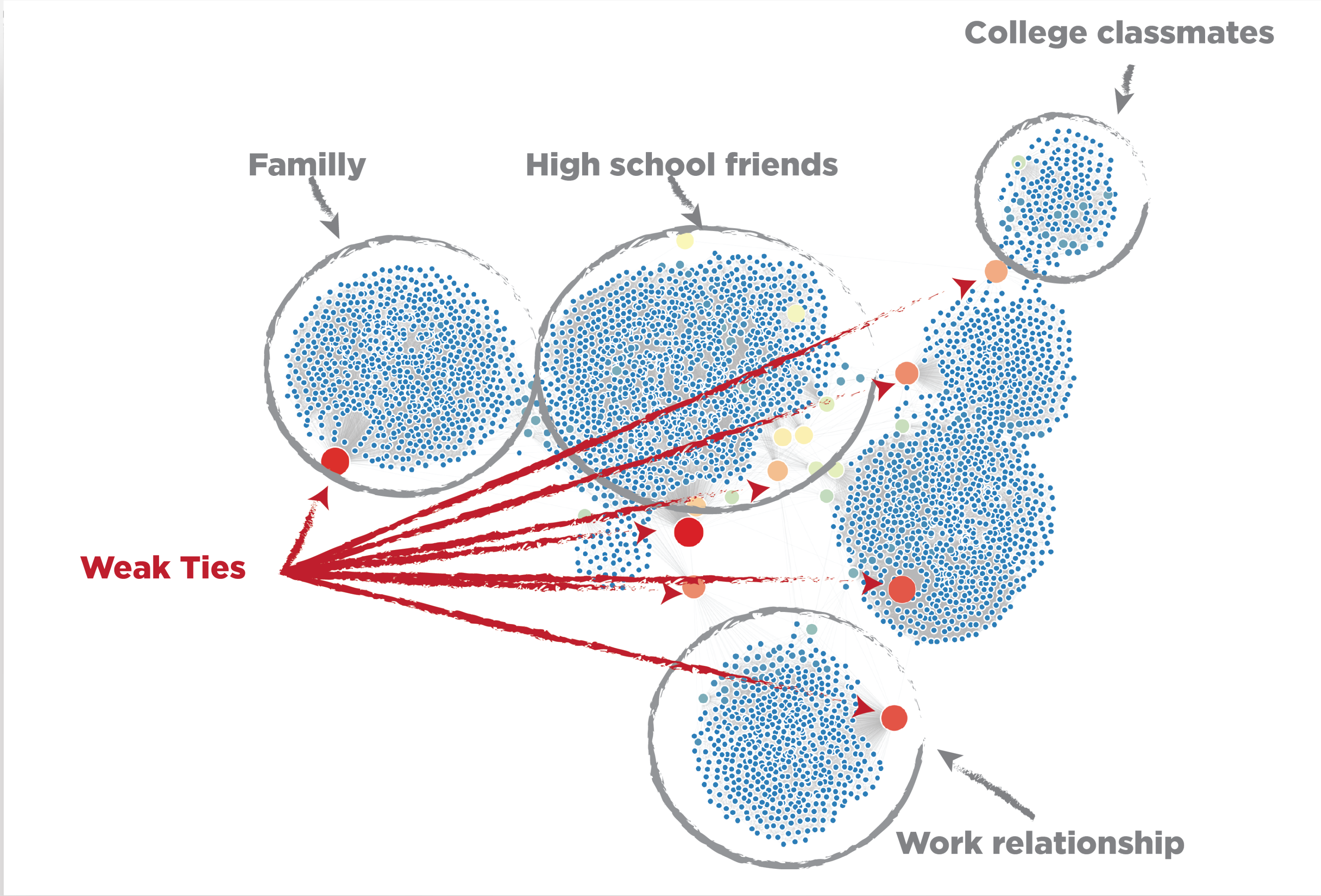
- weak ties (Granovetter)
- professional ties (34% vs. 13%)
- ties originating at work/college
- target's work (65% vs. 40%)

**. . . and disproportionately avoided**

- hubs (8% vs. 1%) (+ no evidence of funnels)
- family/friendship ties (60% vs. 83%)

**Strategy: Geography ➡ Work**

# Strenght of weak ties




*The Strengh of weak ties*
Mark S. Granovetter
American journal of sociology, 1973

# Remember the Clustering

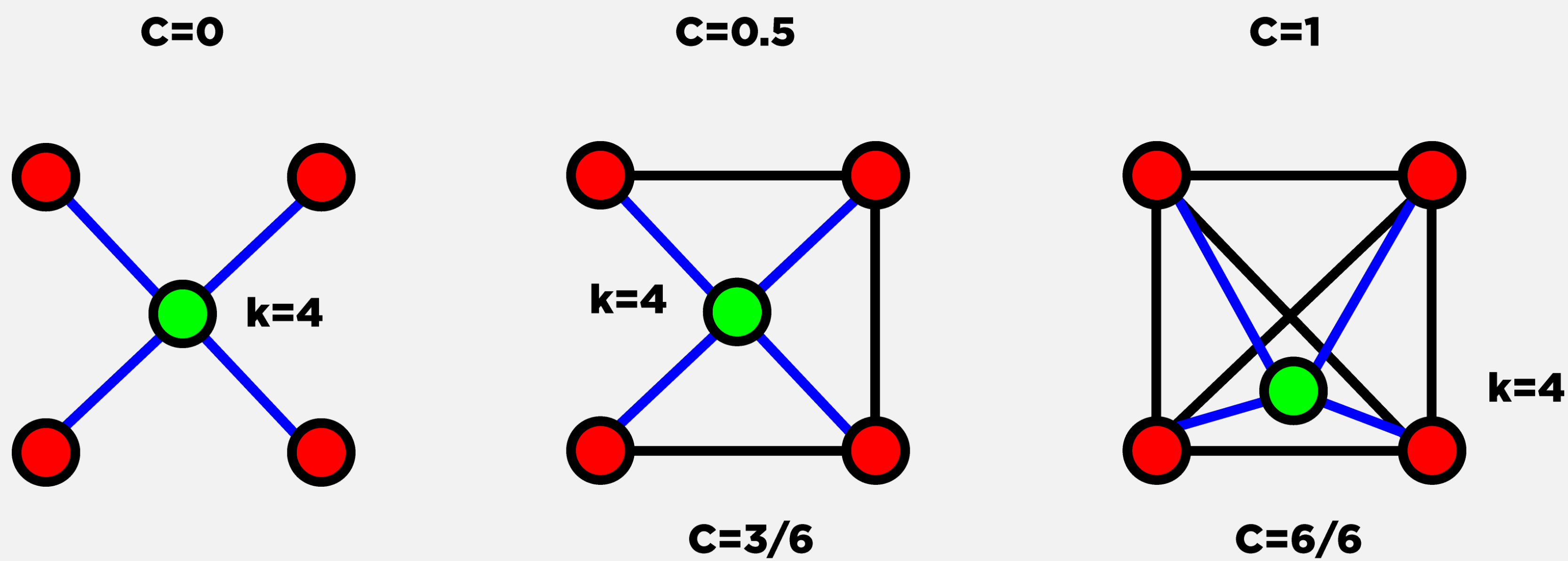
**How many of my friends are also firends ?**

Local Clustering Coeficient

$$C(i) = \frac{\#\ of\ links\ between\ neighbors}{\frac{k(k-1)}{2}}$$

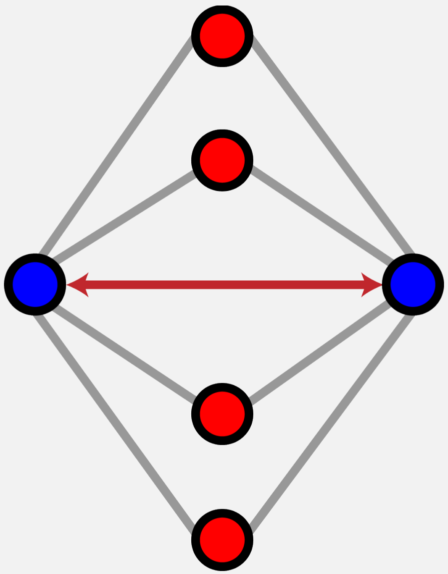
Average Clustering Coeficient

$$\bar{C} = \frac{1}{N}\sum_{i=1}^{N} C(i)$$



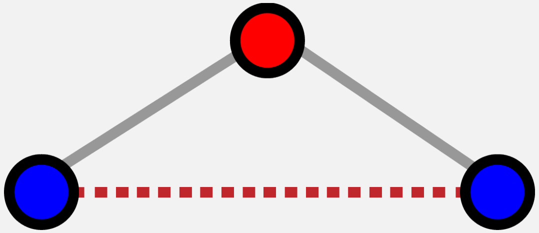
# Strong tie and embeded structure

embeddeness: number of common neighbors the two endpoints have

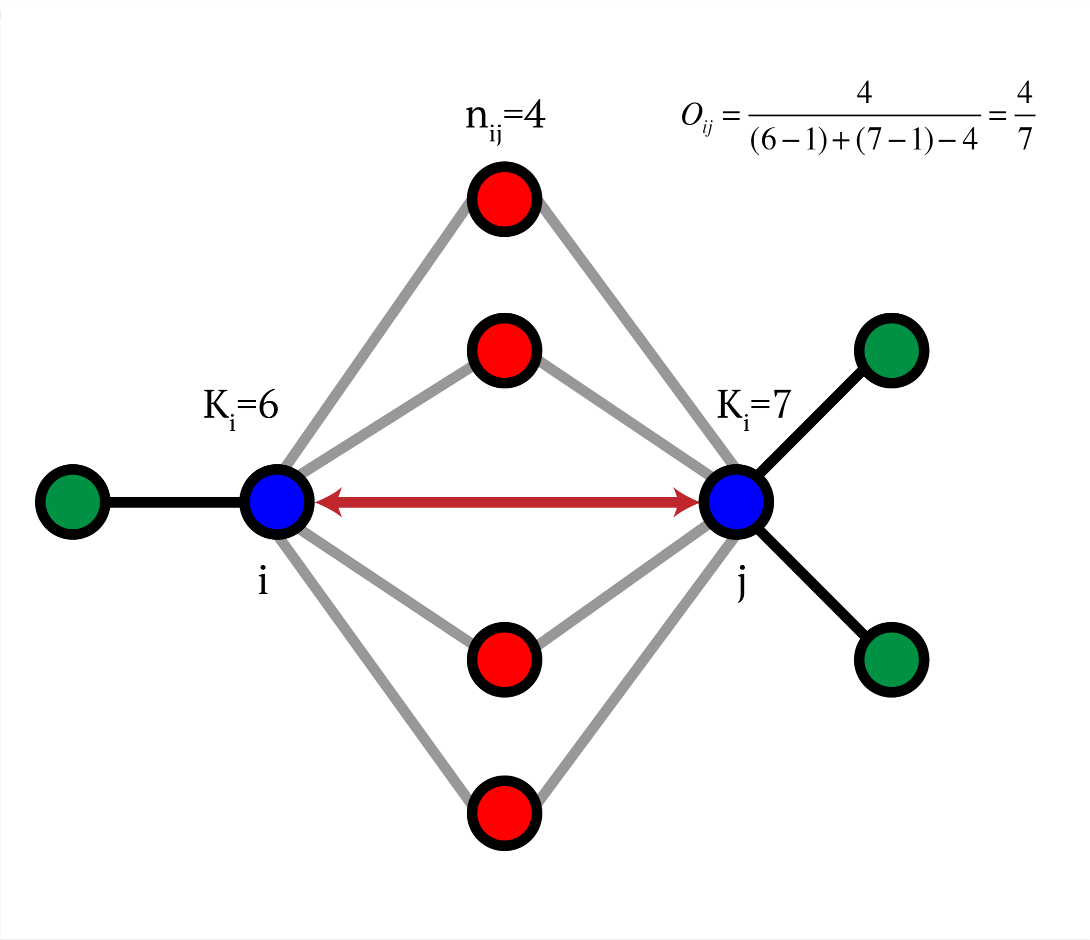


- **A strong tie:**
  frequent contact
  affinity
  many mutual contacts

- **Forbidden triad:**
  strong ties are likely to “close”



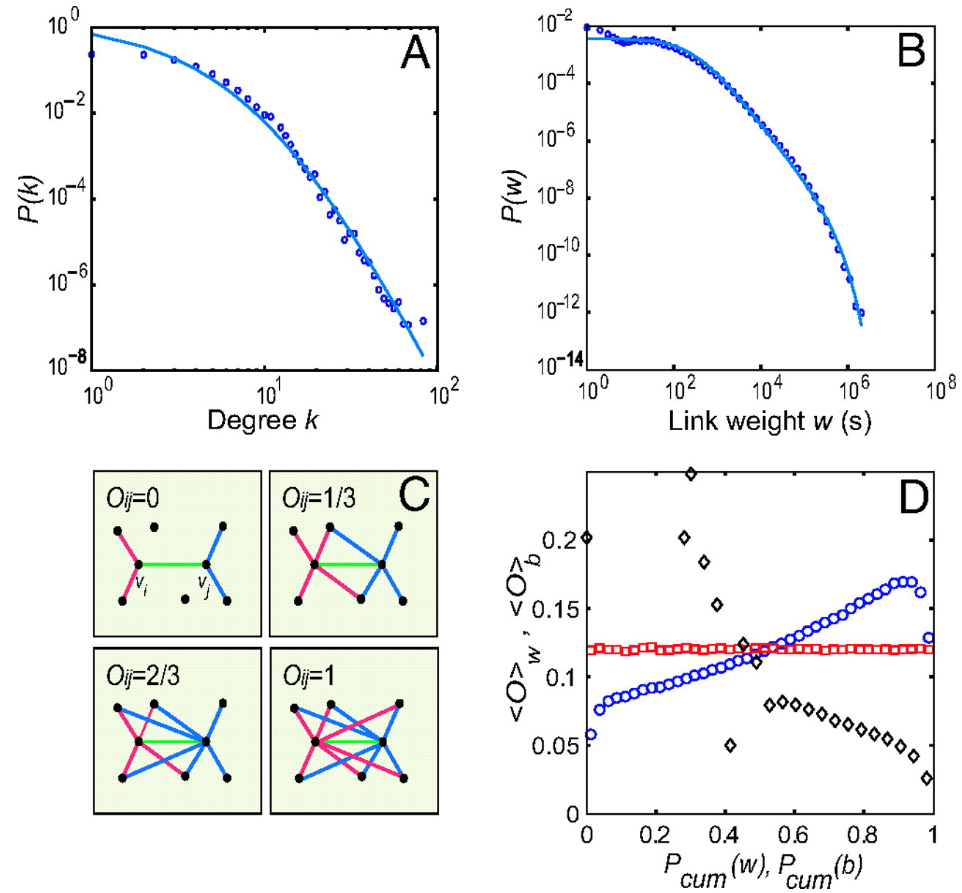
# Embeddeness




How many common neighbors a pair of node has

$$O_{ij} = \frac{n_{ij}}{(k_i - 1) + (k_j - 1) - n_{ij}}$$

# Characterizing the large-scale structure and the tie strengths of the mobile call graph

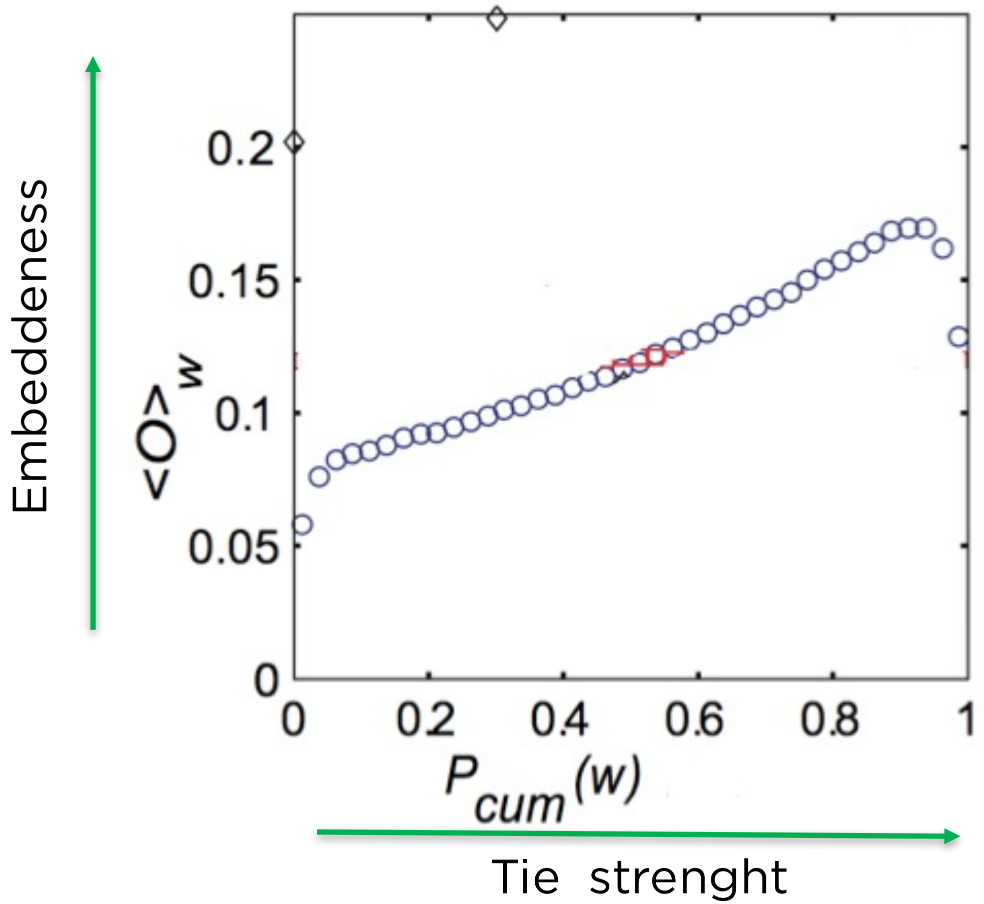



📖 Structure and tie strengths in mobile communication networks

J.-P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, D. Lazer, K. Kaski, J. Kertész, and A.-L. Barabási

PNAS 2007

# Edge neighborhood overlap as a function of tie strength



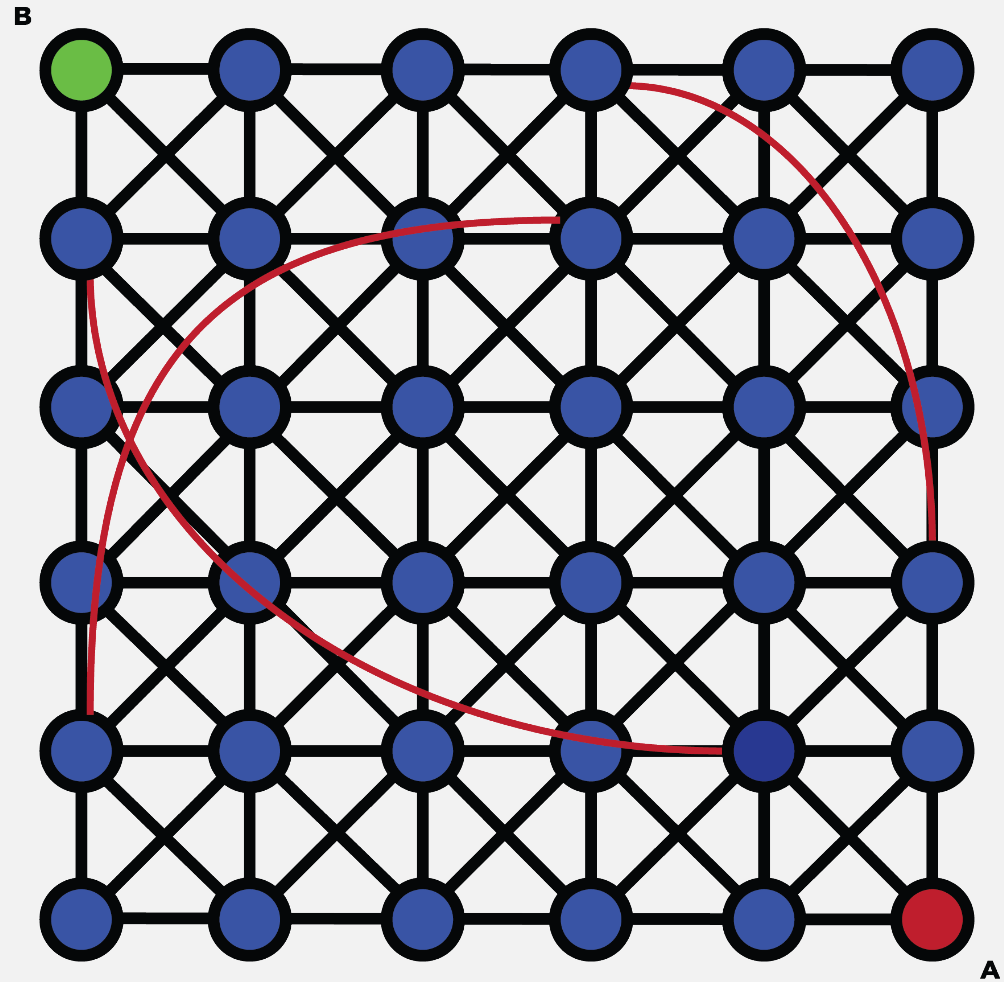

Structure and tie strengths in mobile communication networks
J.-P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, D. Lazer, K. Kaski, J. Kertész, and A.-L. Barabási
PNAS 2007

# Randomness + regularity

- Need "clustering" (your friends are likely to know each other)
- Regularities bring clustering
- Connected random networks have short average path lengths

$$d_{ab} = 6$$

$$d_{ab} = 3$$




With the red edge the overall $\langle d \rangle$ decrease

# Watts-Strogatz model

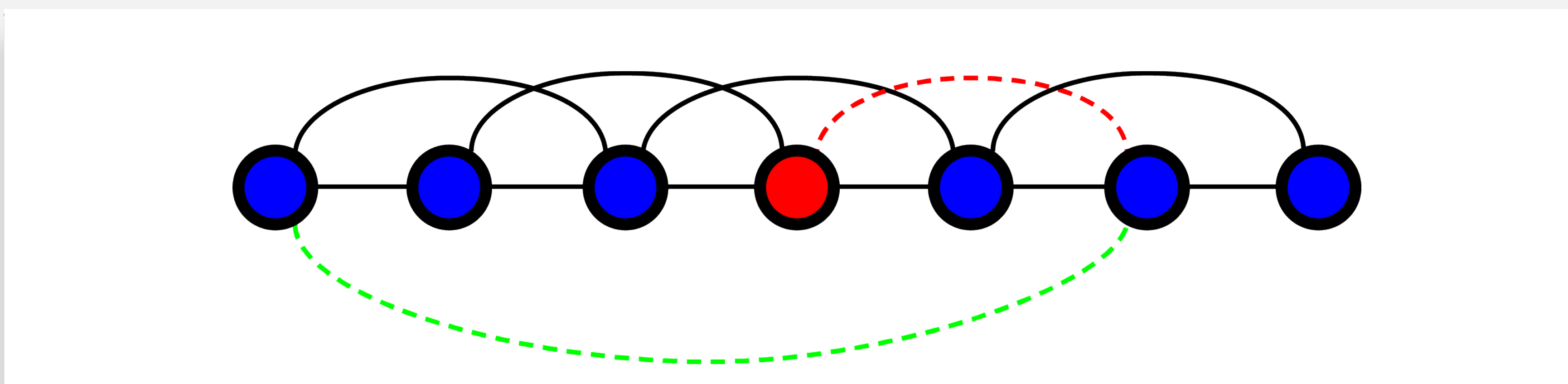
## Generating small world graphs

- Each node has K>=4 nearest neighbors
- tunable: vary the probability p of rewiring any given edge
- small p: regular lattice
- large p: classical random graph

**Collective dynamics of "small-world" networks**

Duncan J. Watts & Steven H. Strogatz

Nature 1998



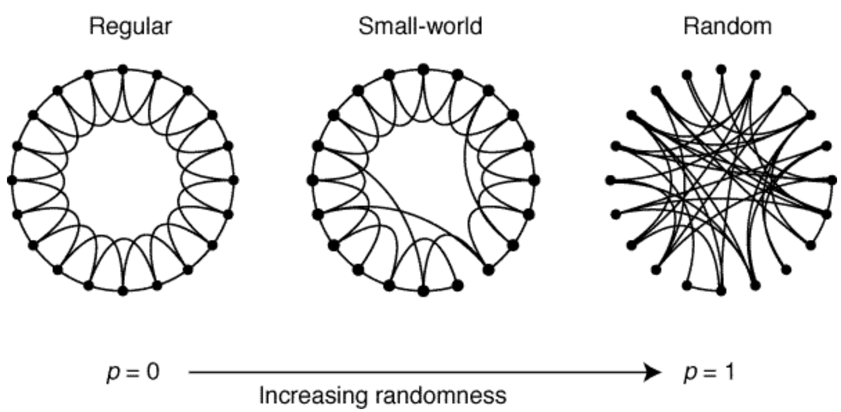
# It is a small world after all !

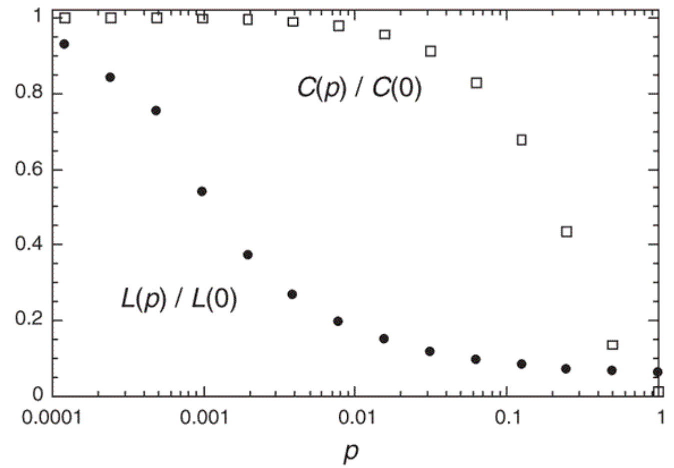
| Network | $L_{actual}$ | $L_{random}$ | $C_{actual}$ | $C_{random}$ |
|---|---|---|---|---|
| Film actors | 3.65 | 2.99 | 0.79 | 0.00027 |
| Power grid | 18.7 | 12.40 | 0.08 | 0.005 |
| C. elegans | 2.65 | 2.25 | 0.28 | 0.05 |

**Small-world networks were found everywhere:**

- neural network of C. elegans,
- semantic networks of languages,
- actor collaboration graph,
- food webs,
- social networks of comic book characters.

Bret Victor's Scientific Communication As Sequential Art




Collective dynamics of 'small-world' networks
Duncan J. Watts and Steven H. Strogatz
Nature 393, 440-442, 1998.

# Small World (Watts-Strogatz model)

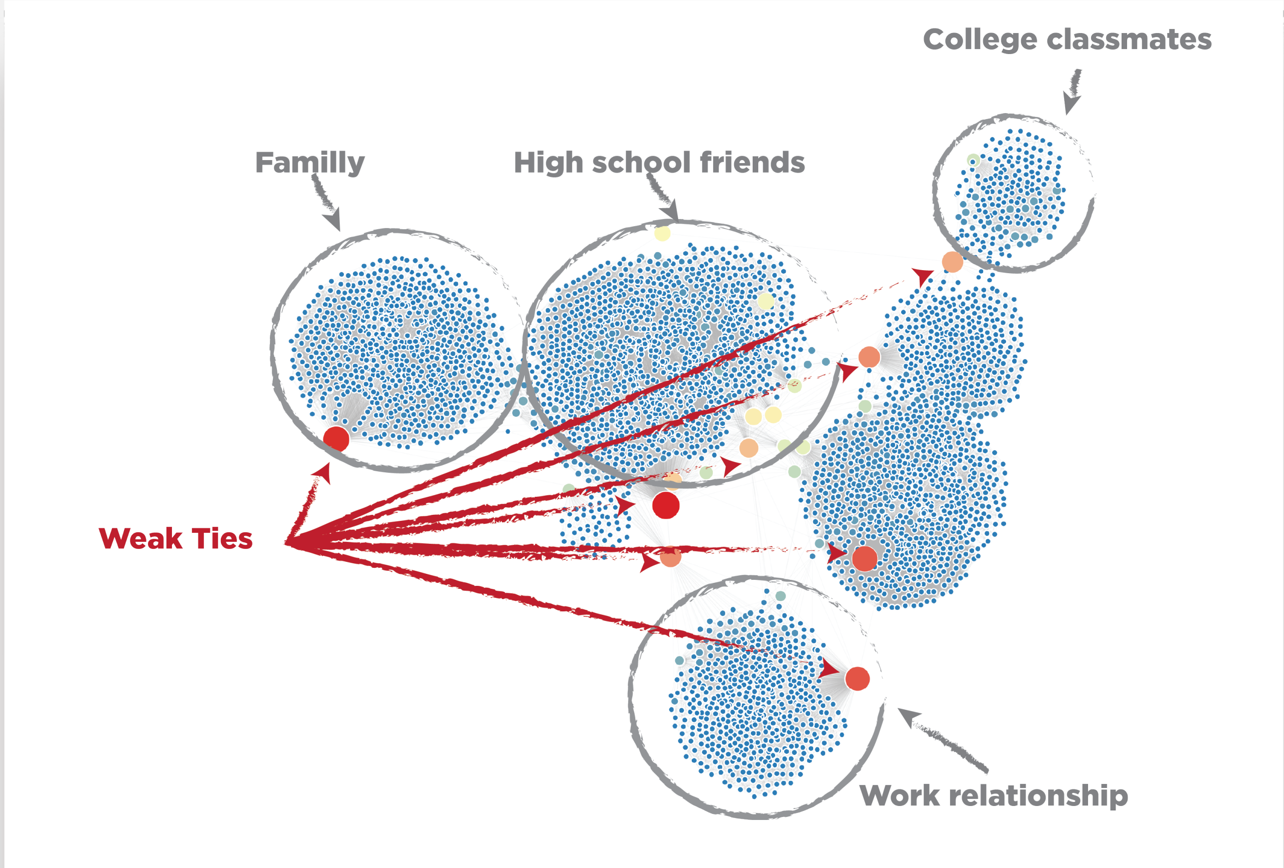
- **Small World Property**

  In real networks the average distance between two nodes depends logarithmically on N, rather than following a polynomial expected for regular lattices

- **High Clustering**

  The average clustering coefficient of real networks is much higher than expected for a random network of similar number of node and diameter (or average path length)

# Social network




- Caracteristics of social Network
  - High Clustering
  - Few Bridge Node

# Ties and geography

“ The geographic movement of the [message] from Wyoming to Massachusetts is striking. There is a progressive closing in on the target area as each new person is added to the chain ”

S. Milgram “The small world problem”

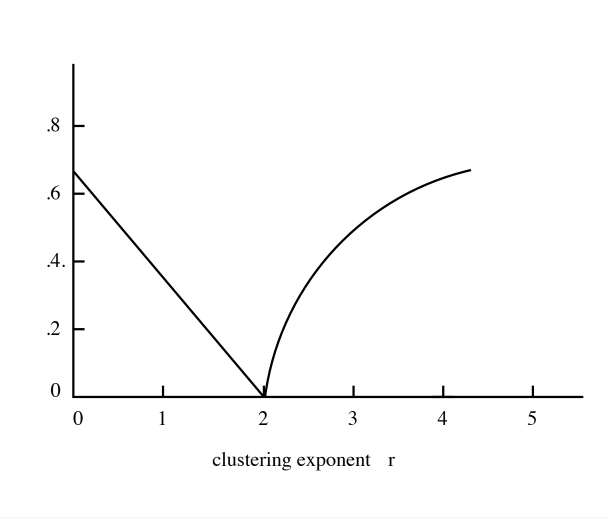
Psychology Today 1, 61, 1967

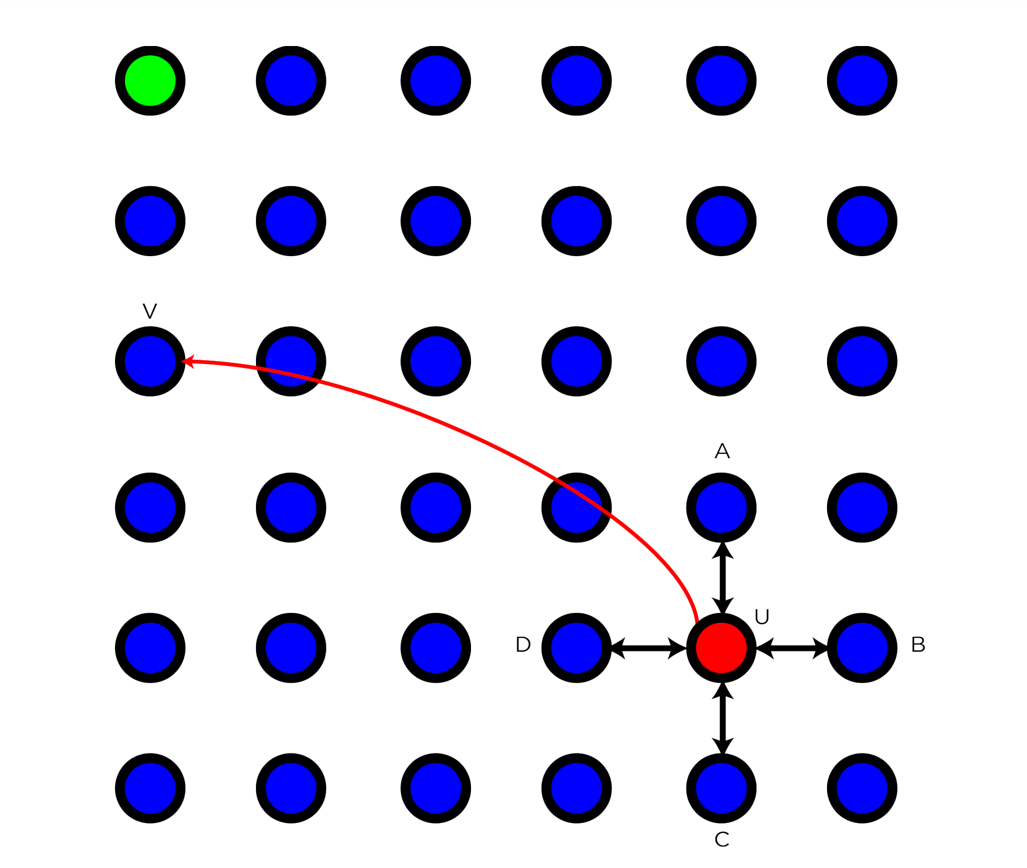
# Kleinberg's geographical small world model

nodes are placed on a lattice and connect to nearest neighbors

additional links placed with:

$$P_{link}(u,v) = d(u,v)^r$$

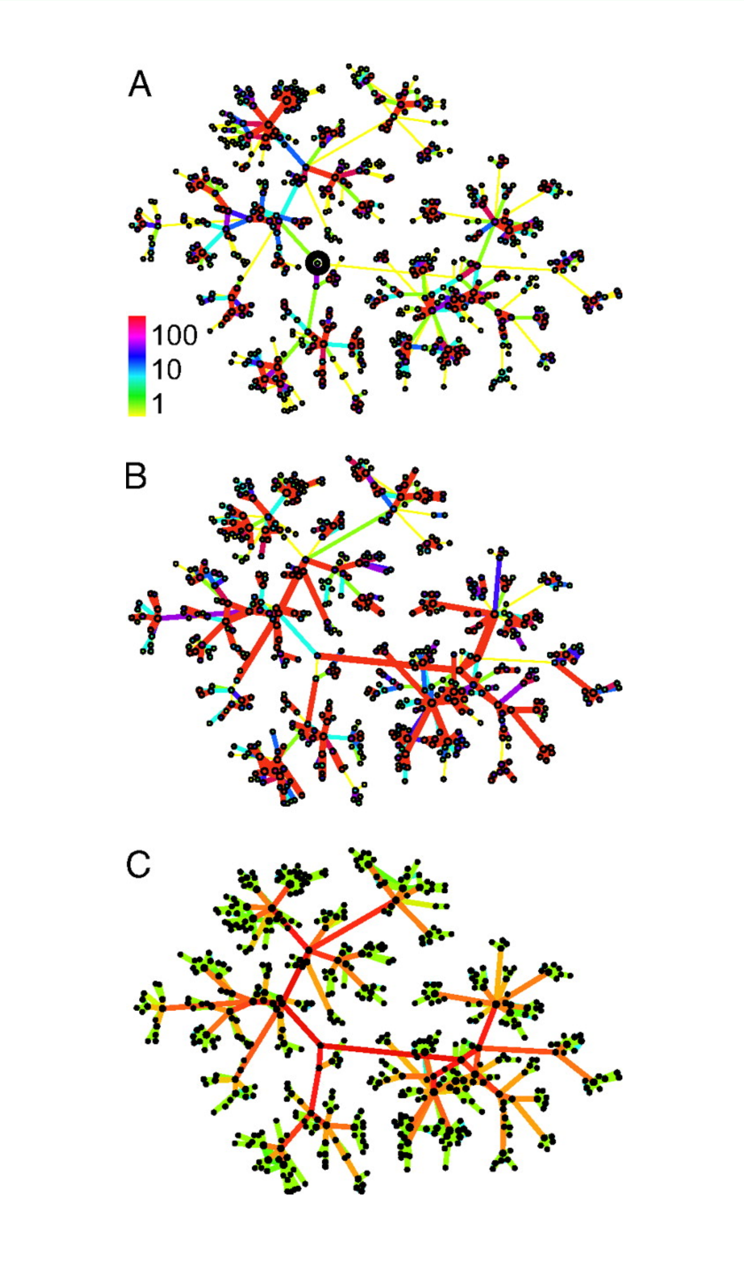



The Small World Phenomenon, An Algorithmic Perspective
Jon Kleinberg
Nature 2000

# Conclusion

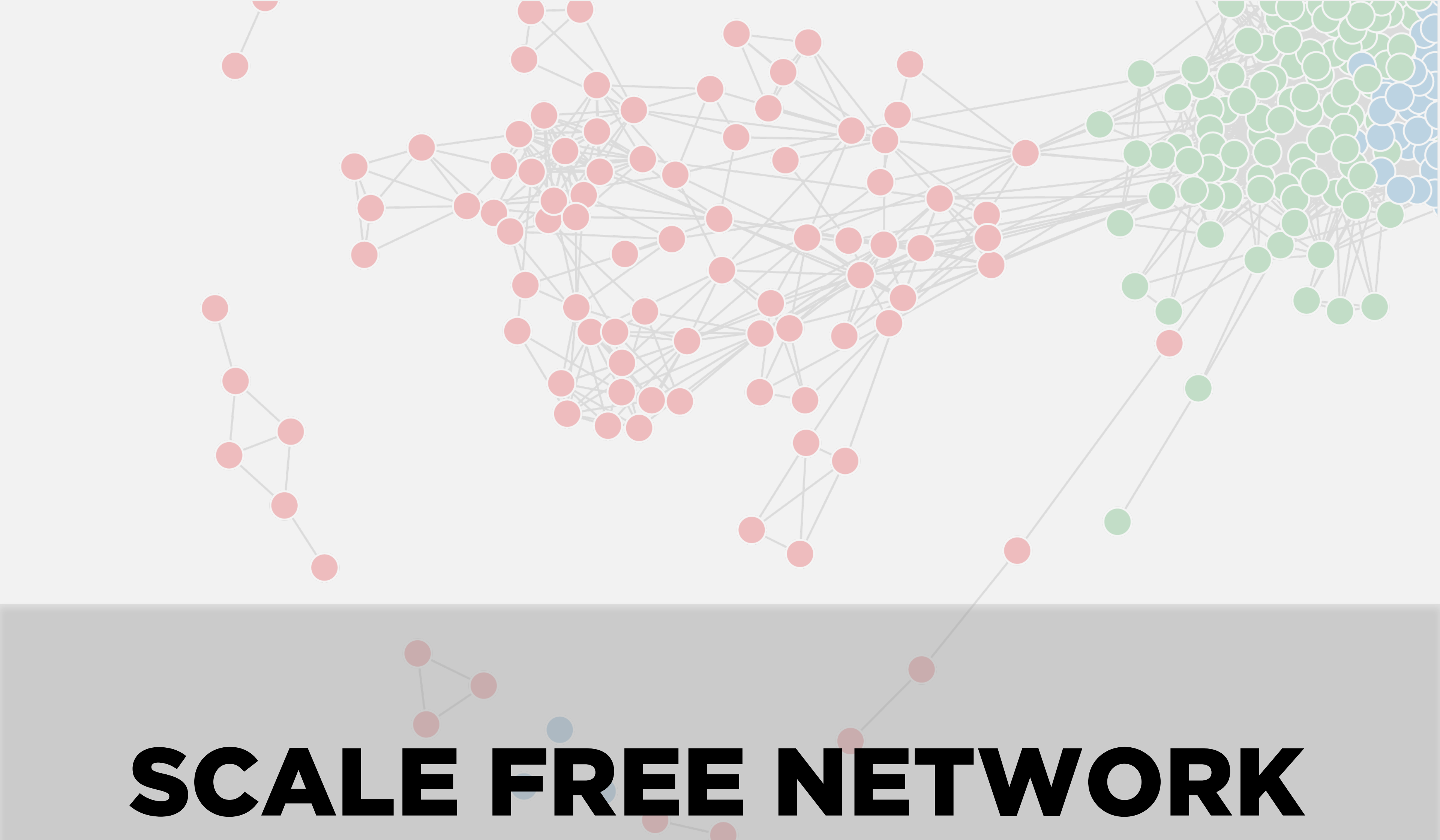
- clustering and embeddeness play a important role especially in social network
- there is an interplay between network diameter and the clustering coefficient
- in social network node tend to form group (or community)
- strong tie and weak tie both play an important role




**We need a better understanding of the link between clustering coefficient and the network diameter**

# What we learned

1. Most of graph in nature are supercritical
2. Once the average degree exceeds <k>= 1, a giant component should emerge that contains a finite fraction of all nodes
3. Clustering shape the way network a designed
4. In real networks the average distance between two nodes depends logarithmically on N



# SCALE FREE NETWORK

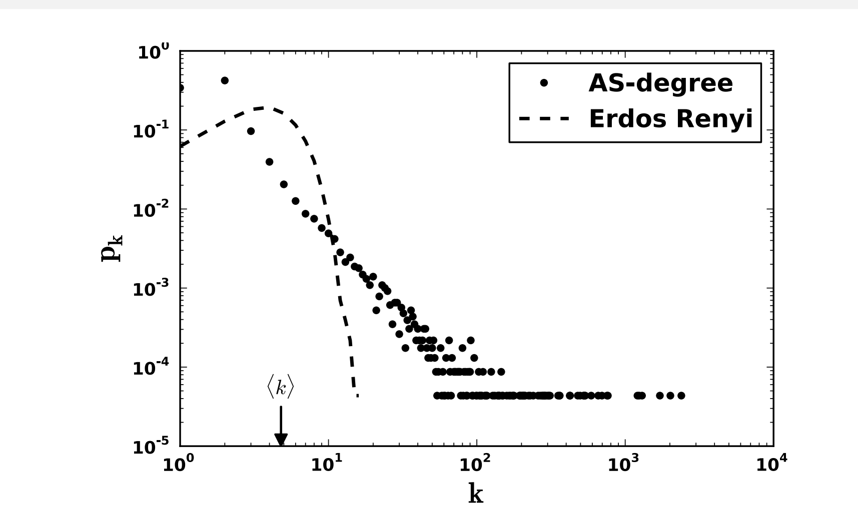
# How real networks behave ?

**Emergence of scaling in random networks**

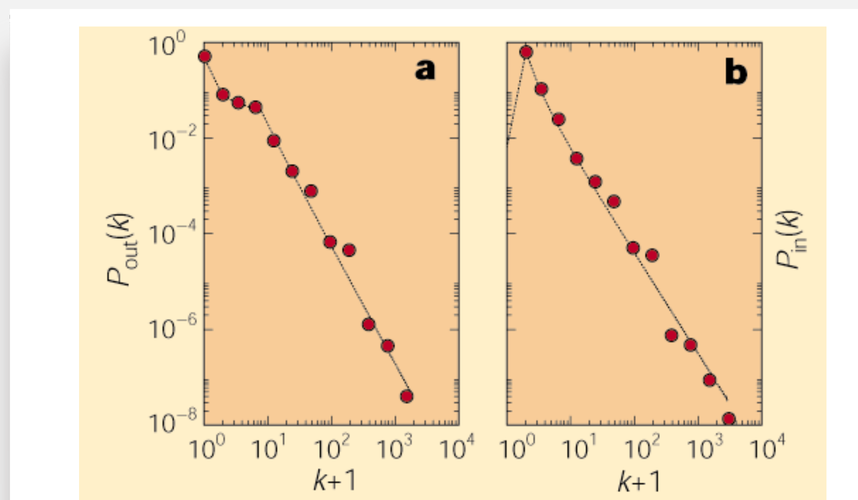
1999

AL Barabási, R Albert

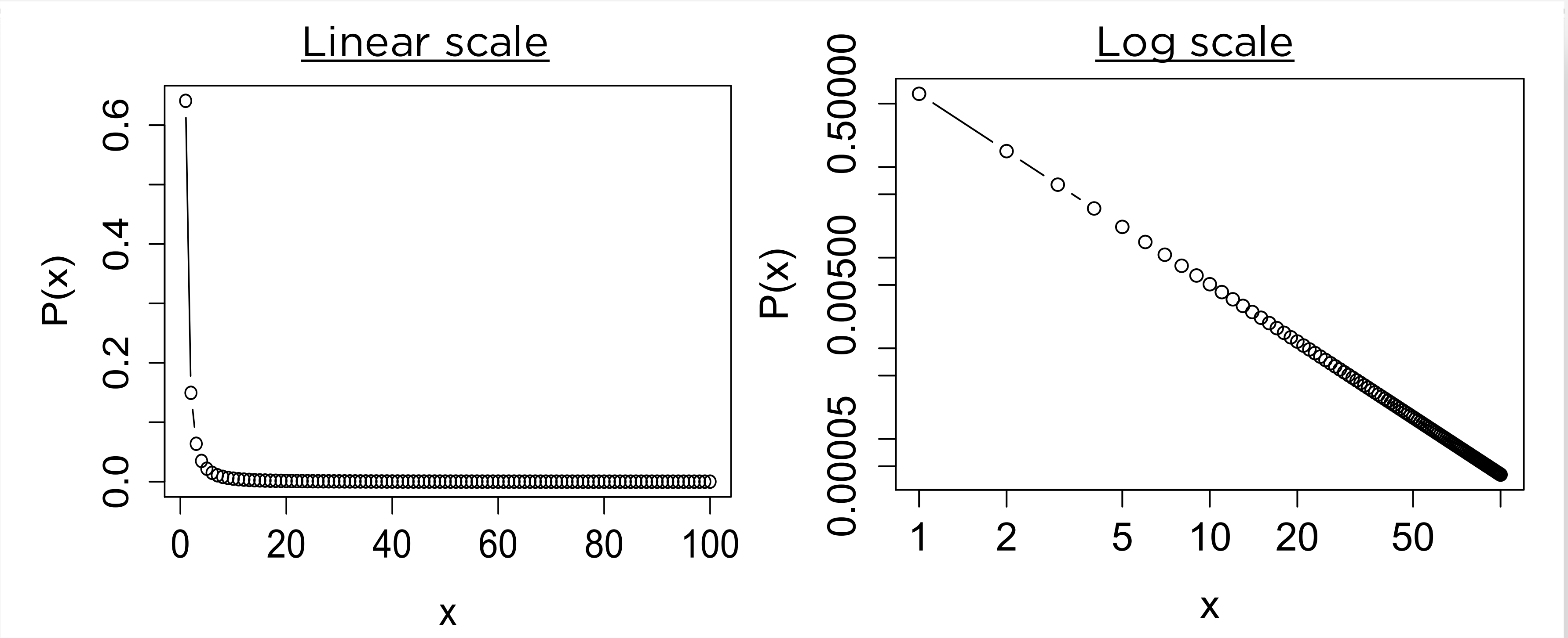
## Internet AS




## WWW

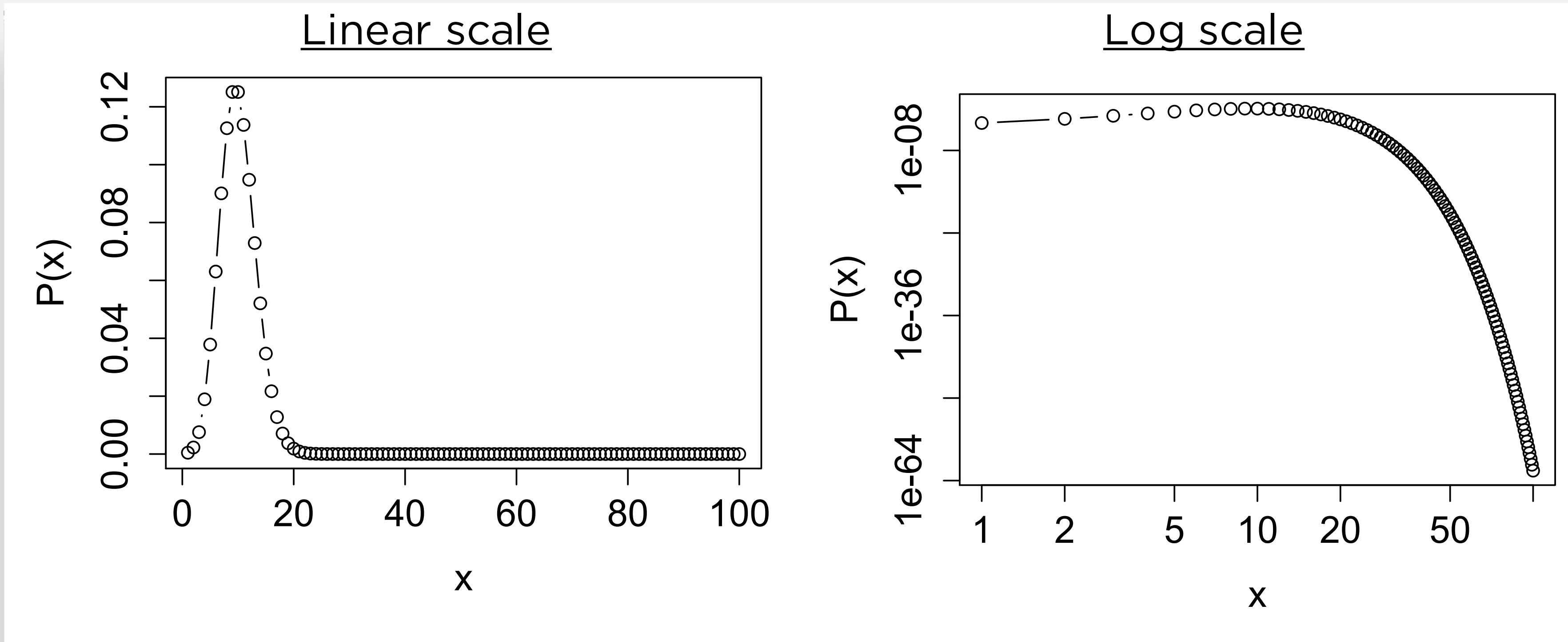


# Power-law distribution




- high skew (asymmetry)
- straight line on a log-log plot

# Poisson distribution




- little skew (asymmetry)
- curved on a log-log plot

# Power law distribution

Straight line on a log-log plot

$$\ln(p(k)) = c - \alpha \ln(k)$$

Exponentiate both sides to get that p(k), the probability of observing an node of degree k is given by:

$$p(k) = Ck^{-\alpha}$$

power law exponent

normalization
constant (probabilities over
all *k* must sum to 1)

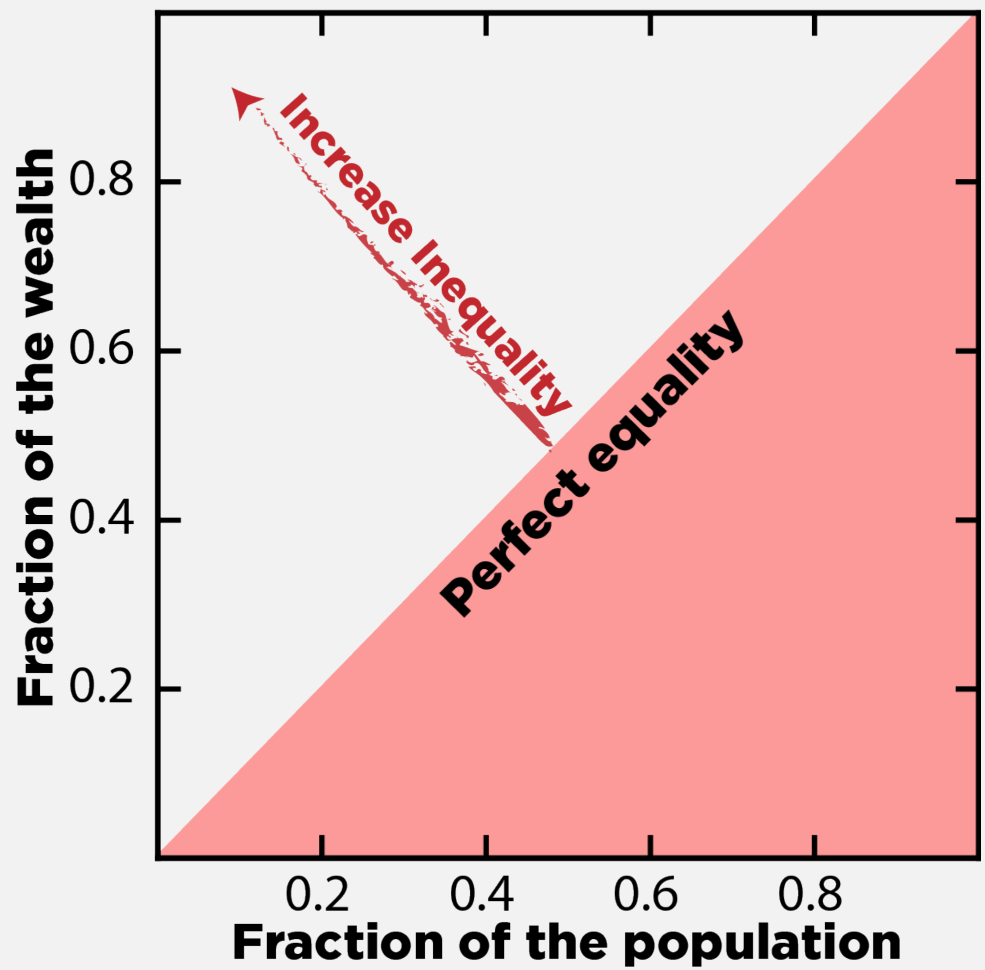
# Degree distribution

**power law distribution what does it mean ?**

## Degree
## “wealth”

What fraction of total wealth W is owned by richest fraction P

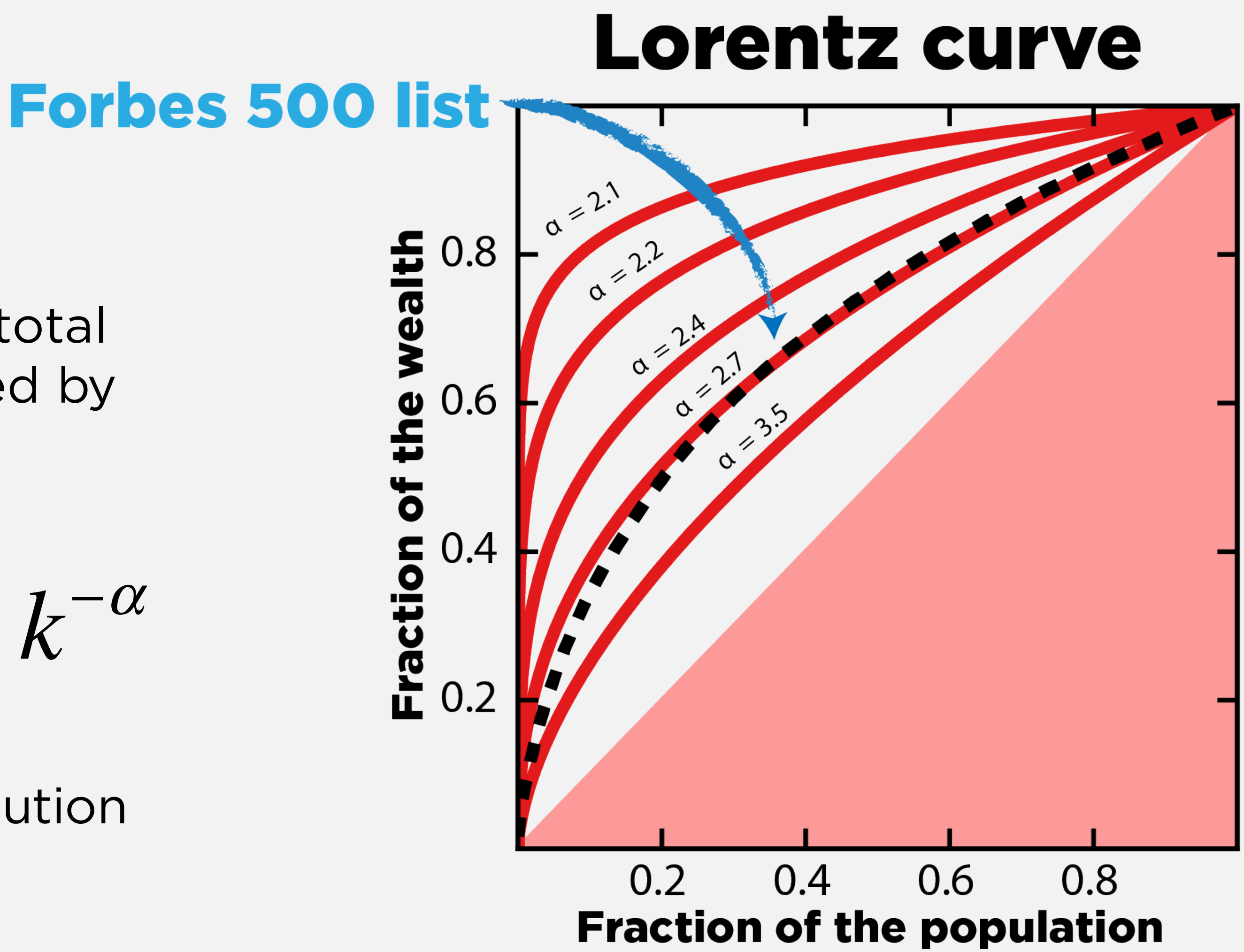
## Lorentz curve



# Degree distribution

**power law probability distribution what does it mean ?**

## Degree “wealth”



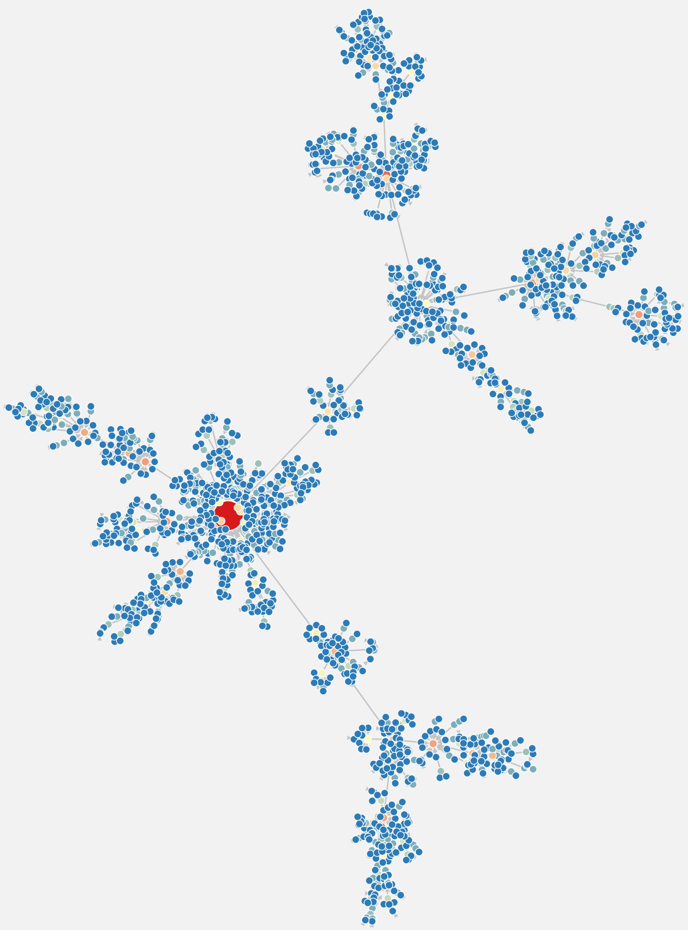

What fraction of total wealth W is owned by richest fraction P

$$\Pr(k) \propto k^{-\alpha}$$

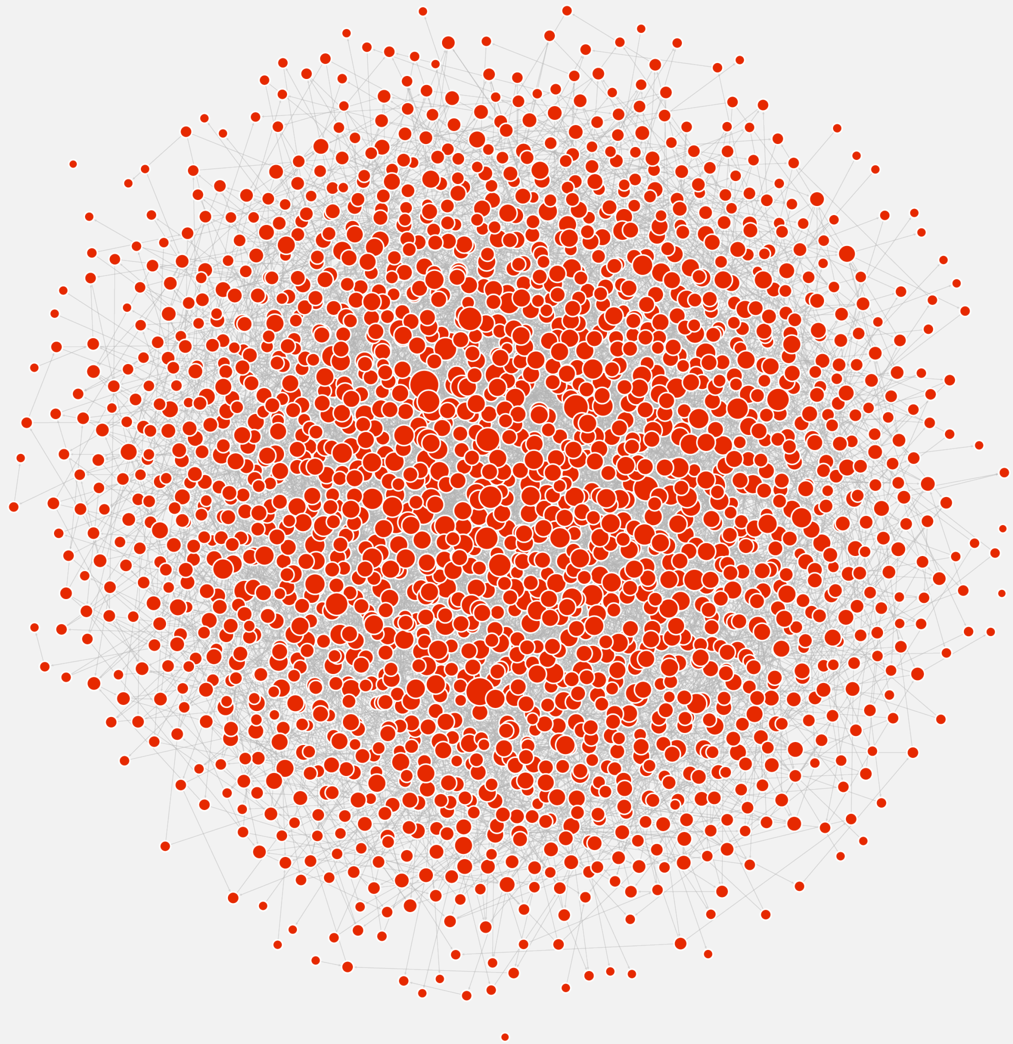
power-law distribution
**80/20 rule**

# Scale free compare to Erdos-Reny

Scale free network



Random network



# Quiz:

**As the exponent $\alpha$ increases, the downward slope of the line on a log-log plot:**

a) stays the same
b) becomes milder
c) becomes steeper

# Quiz:

**As the exponent $\alpha$[a] decreases, the difference between lowest degree node and the highest degree node :**

a) stays the same

b) becomes milder

c) becomes steeper

a: for $\alpha > 1$

# 2nd ingredient: preferential attachment

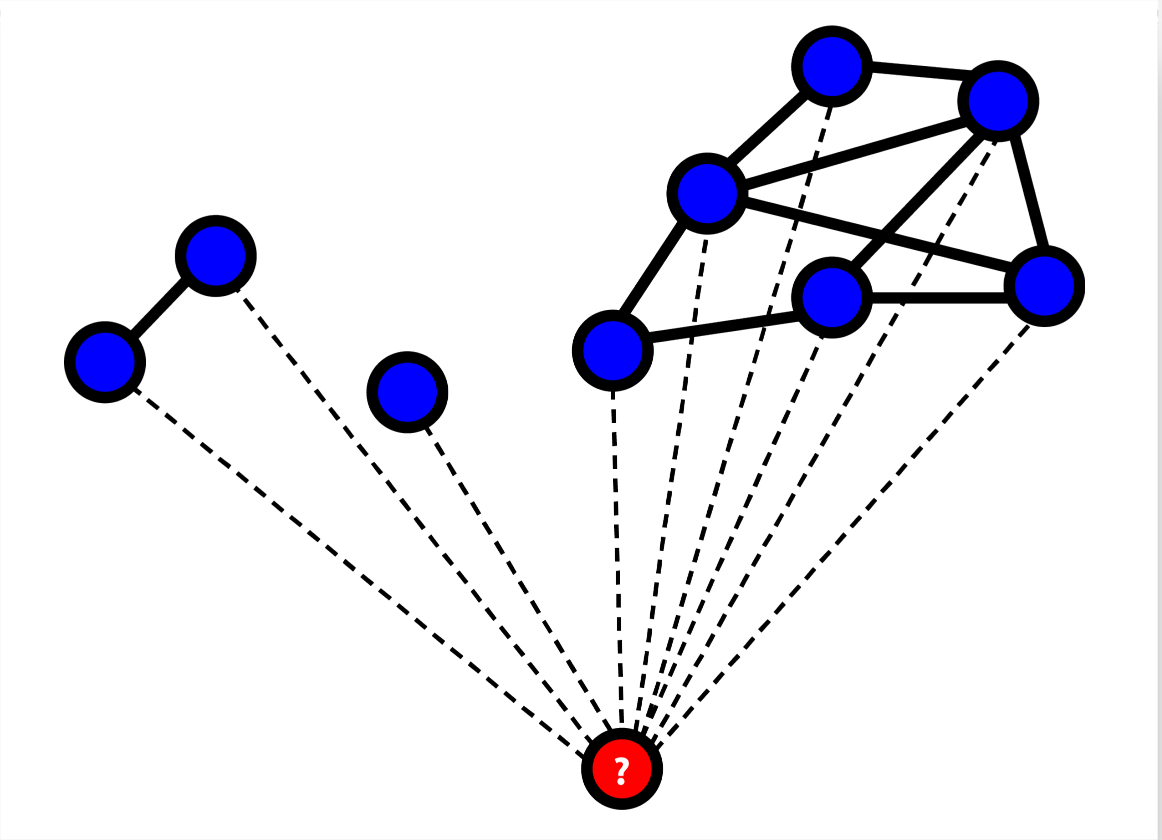
**Preferential attachment:**

- new nodes prefer to attach to well-connected nodes over less-well connected nodes

**Process also known as**

- cumulative advantage
- rich-get-richer
- Matthew effect

# Generating power-law networks




Nodes prefer to attach to nodes with many connections
(preferential attachment, cumulative advantage)

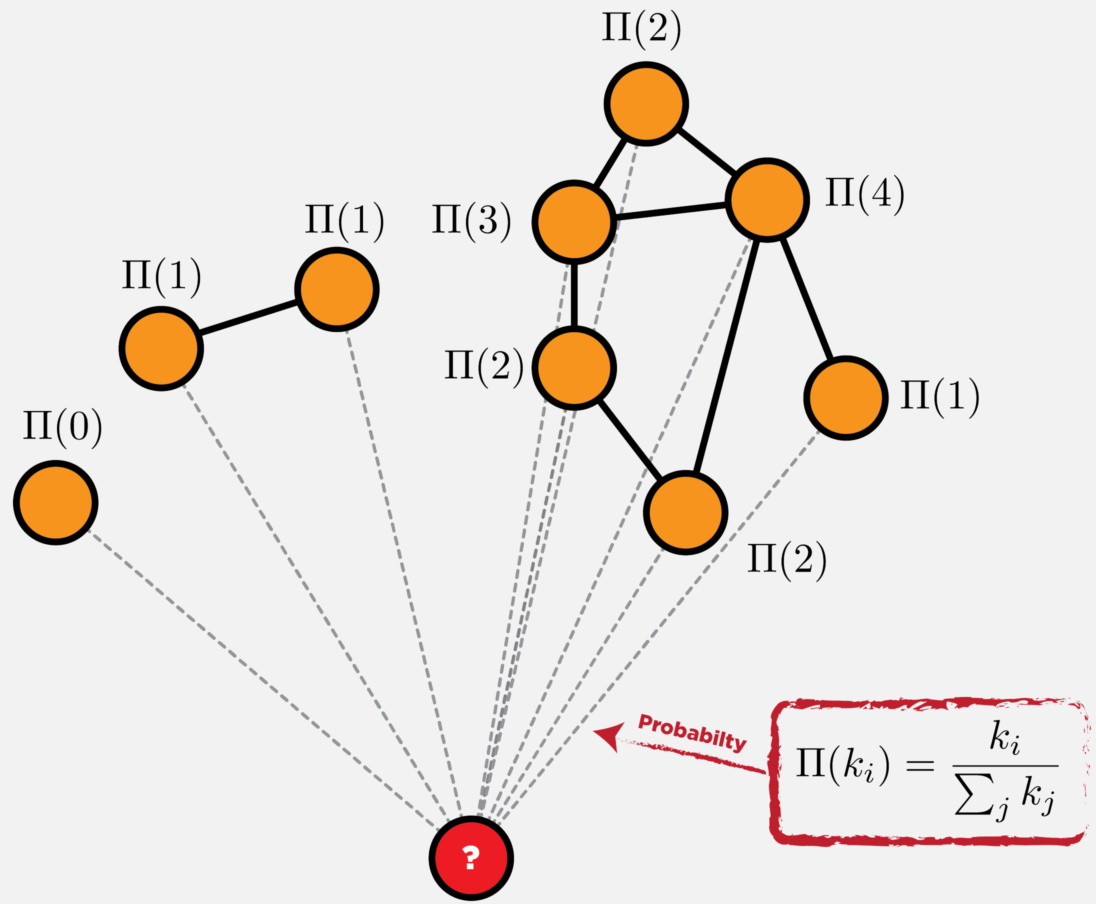
# Barabasi-Albert model

**First used to describe skewed degree distribution of the World Wide Web.**

**Growth + Preferential attachment**

**Each node connects to other nodes with probability proportional to their degree**

- Step 1: Start with $m_0$ disconnected nodes
- Step 2:
  1. Growth a node appears at each time step
  2. Each node make m connections to nodes already present
  3. Preferential attachment probability of connecting the ith node is $\propto k_i$
- Results in power-law with exponent α = 3



# Quiz Q:

**How could one make the growth model more realistic for social networks?**

a) old nodes die
b) some nodes are more sociable
c) friendships vane over time
d) all of the above

# Quiz Q:

**The degree distribution for a growth model where new nodes attach to old nodes at random will be**

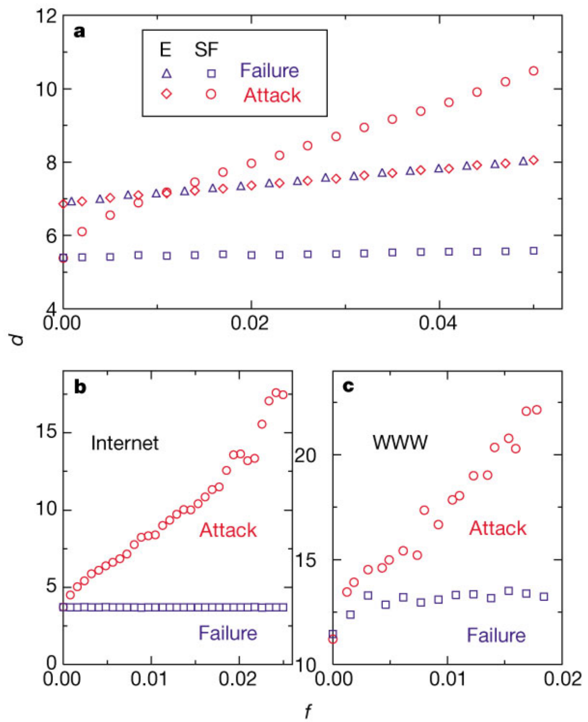
a) a curved line on a log-log plot

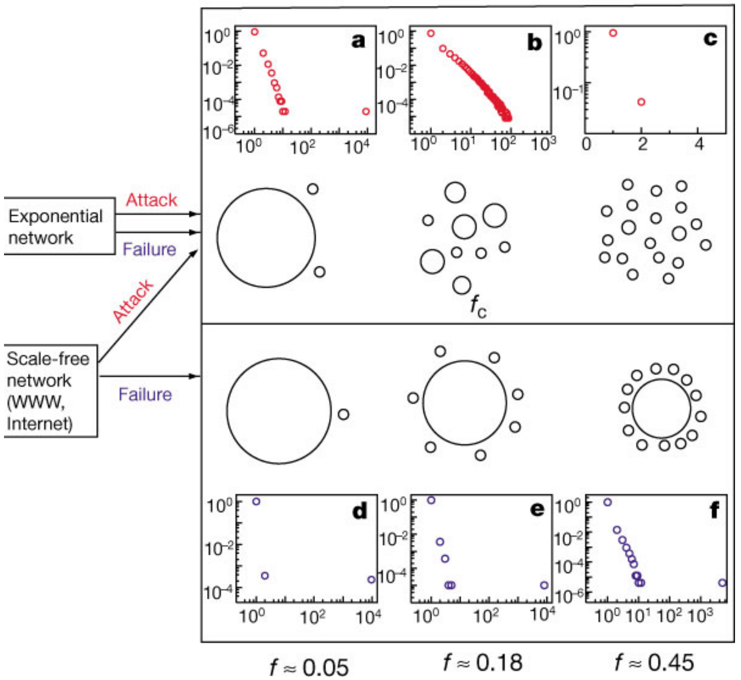
b) a straight line on a log-log plot

# Robust yet fragile

- Plots of network diameter as a function of fraction of nodes removed
- Erdős-Rényi versus scale-free networks

- blue symbols = random removal
- red symbols = targeted removal (most connected first)




Error and attack tolerance of complex networks
Albert et al.
Nature, 2000

# Robust yet fragile

**Scale-free networks are thus robust to random failures yet fragile to targeted ones**

- All very reasonable: Hubs are a big deal.
- Representing all webpages as the same size node is obviously a stretch (e.g., google vs. a random person's webpage)
- Most connected nodes are either:
  1. Physically larger nodes that may be harder to 'target'
  2. or subnetworks of smaller, normal-sized nodes.
- Need to explore cost of various targeting schemes.

# Quiz:

**Which network is more diffusive:**

a) Scale Free Network

b) Erdos-Reny Network

# End of lecture 2

# CREDITS

- Complex Network  Peter Dodds
- Five Lectures on Networks, Aaron Clauset
- Network Science course, Albert-László Barabási


- Networks An Introduction  Mark Newman, 2010.
- Networks, Crowds, and Markets, D. Easley, J. Kleinberg, 2010.
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