Calendar Descriptions
Computational Discovery on Jupyter

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January

A *doubly companion matrix* is a particular kind of rank-one update to a Frobenius companion matrix:

$$\begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 & -a_5 - b_5 \\ 1 & 0 & 0 & 0 & -b_4 \\ 0 & 1 & 0 & 0 & -b_3 \\ 0 & 0 & 1 & 0 & -b_2 \\ 0 & 0 & 0 & 1 & -b_1 \end{bmatrix}. \tag{1}$$

"Doubly companion" matrices were introduced in a 1999 paper by Butcher and Chartier in order to improve certain implicit Runge–Kutta methods, and later General Linear Methods, for numerically solving ordinary differential equations. See `https://link.springer.com/article/10.1023/A:1019176422613`. Doubly companion matrices have some interesting properties, and in particular one can choose the a_j in order to make the spectrum—given the b_j—have desirable properties, such as that all the eigenvalues coalesce to a single eigenvalue. But we chose to explore them as a Bohemian family. See Chapter 5 of the book.

This image is a density plot (hotter colors in the viridis palette correspond to higher density of eigenvalues) of all $43,046,721$ doubly companion matrices of dimension $m = 8$ with population $\{-1, 0, 1\}$. The plot window is a square $[-1.419, 1.419] \times [-1.419 \times 1.419]$ cut into a fine grid, 6000×6000. The eigenvalues for the image on the cover of the book were computed using standard eigenvalue software (`numpy.eig` in Python, `eig` in Matlab, and `Eigenvalues` in Maple™). In contrast, the eigenvalues for this image in the calendar were computed by first constructing all $2,184,139$ distinct characteristic polynomials (which all have integer coefficients), factoring them over the integers which detects multiplicity and in particular the multiplicity of the zero eigenvalue, and then solving the factors numerically via `fsolve` in Maple™. This removes the "rosette" of rounding errors that can be seen on the spine of the book cover. We deliberately left that rounding error rosette on the cover as an Easter egg teaching tool.

February

A density plot of all the eigenvalues of all dimension $m = 15$ upper Hessenberg Toeplitz matrices (with zero diagonal, and -1 on the subdiagonal) with population $\{-1, i, 1\}$ in the upper triangle. There are $4,782,969$ such matrices. The asymmetry across the real axis is notable. Near the left edge we see a "nonlinear Sierpinski" triangle beginning (fig 1). The plot window is $-3 \le x \le 3$, $-3 \le y \le 3$. For more on matrices like these, see "Bohemian Matrix Geometry" by Corless, Labahn, Piponi, and Rafiee Sevyeri at `https://doi.org/10.1145/3476446.3536177` (or at the free version on the arXiV).

March

A phase plot (also known as a "domain coloring") of the *lowest frequency continuous interpolant* for Narayana's cows sequence, which satisfies $a_0 = a_1 = a_2 = 1$ and $a_{n+1} = a_n + a_{n-2}$. Some of the (negative and complex) times at which there are zero cows are plainly visible in the figure. A floating-point version of this interpolant is shown in the code below.

The plot was produced in Julia using Evert Provoost's DomainColoring package at
`https://github.com/eprovst/DomainColoring.jl`. We are grateful to Toby Driscoll for both pointing out this package and helping us to use it to produce a high resolution image.

To make this version of the code fit on the page, we replaced the double-precision numbers, which arose from approximations to the roots of $\lambda^3 = \lambda^2 + 1$, with single-precision versions.

```
using CairoMakie, DomainColoring # hide
f = Figure(resolution=(3200,3200), fontsize=80)
ax = Axis(f[1,1], aspect = DataAspect())
domaincolor!(ax, z -> (0.61149199e0+0.0e0im) * 0.14655712e1 ^ z +
 0.38850801e0 * exp(-0.19112254e0 * z) * cos(0.18564785e1 * z)
  + 0.24509938e0 * exp(-0.19112254e0 * z) * sin(0.18564785e1 * z),
   [-5,10,-7.5,7.5], pixels=3000, abs=true
)
colsize!(f.layout, 1, Relative(0.95))
current_figure()
save("Narayana.png", current_figure()) # hide
nothing # hide
```

April

A "random" Sierpinski triangle produced in Python. See the report R4.1 on Julia set activity 1 in the book on p. 325. This one suggests either raindrops or ice pellets, depending on your imagination; in at least one of our locales, either might be seasonally appropriate.

May

A phase plot (also known as a "domain coloring") of the Mandelbrot polynomial $p_4(z) = z^7 + 4z^6 + 6z^5 + 6z^4 + 5z^3 + 2z^2 + z + 1$ arising from the iteration $p_0 = 0$ and $p_{n+1} = zp_n^2 + 1$. All seven of the zeros are visible as locations where all colours meet. The contour that intersects itself happens to be where $|p_4(z)| = 1$. As in the image for the month of July in this calendar, the plot window is $-2.25 \leq x \leq 0.75$, $-1.5 \leq y \leq 1.5$. Plot produced in Julia using Evert Provoost's DomainColoring package at `https://github.com/eprovst/DomainColoring.jl`. Again, thanks go to Toby Driscoll for pointing us at this package.

June

Adventure is, by definition, an alluring and captivating pursuit due to the unknown and unexpected elements it encompasses. Mathematical exploration, too, is an adventure, akin to embarking on a mountain expedition to unveil enchanting landscapes, unreachable peaks, or hidden lakes. I came across this image during a computational exploration of the eigenvalues of random matrices generated by randomly sampling their elements from the set $\{0, i, -i, 0.3\}$. In each matrix, two random elements are substituted with two real parameters, t_1 and t_2.

The specific matrix employed to generate the image is as follows.

$$\mathbf{M} = \begin{pmatrix} -i & -i & -i & 0.3 & -i & -i & 0 & i \\ 0.3 & i & -i & i & 0 & i & 0.3 & 0 \\ -i & 0.3 & i & 0 & 0 & i & -i & i \\ 0 & i & -i & -i & 0.3 & 0.3 & 0 & 0 \\ 0 & 0.3 & i & 0 & 0.3 & 0 & t_1 & i \\ -i & i & 0.3 & 0 & 0 & 0.3 & i & t_2 \\ -i & i & 0.3 & i & i & -i & 0.3 & 0.3 \\ 0 & 0.3 & -i & 0 & i & i & 0.3 & 0.3 \end{pmatrix}$$

The image illustrates eigenvalues of M in the $\mathbb{C}$ plane, computed for 2,000,000 randomly sampled values of t_1 and t_2 within the interval $[-20, 20]$.

 Generated using the code from `https://github.com/profConradi/eigenfish/`.

July

Newton's method applied to z times the Mandelbrot polynomial $p_4(z) = z^7 + 4z^6 + 6z^5 + 6z^4 + 5z^3 + 2z^2 + z + 1$ arising from the iteration $p_0 = 0$ and $p_{n+1} = zp_n^2 + 1$. All seven of the nonzero roots are visible. Different colors show where different initial approximations go to each root. Compare with the figure from the month of May. As there, the plot window is $-2.25 \leq x \leq 0.75$, $-1.5 \leq y \leq 1.5$.

August

Independence Day of Ukraine is celebrated on August 24, in commemoration of the Declaration of Independence in 1991. Russia's unprovoked invasion of Ukraine, starting in 2014 and erupting into open war in February 2021, is a horror and tragedy of the world. The colours and aspect ratio of this image, which match that of the flag of Ukraine, were chosen in support of Ukraine against this invasion.

The mathematical aspects of the image include that the fractal boundary separates the region of attraction of the root at $z = -i$ from that of the root at $z = i$ under third-order Schröder iteration,

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f''(z_n)f(z_n)^2}{2f'(z_n)^3}$$

where $f(z) = z^2 + 1$. See Corless and Li, "Revisiting Gilbert Strang's *A Chaotic Search for i*", `https://doi.org/10.1145/3363520.3363521`

September

Persymmetric doubly companion matrices are like those of the January image, except all $a_j = b_j$.

This image is a density plot (hotter colors in the viridis palette correspond to higher density of eigenvalues) of all 3^{12} persymmetric doubly companion matrices of dimension $m = 12$ with population $\{-1, 0, 1\}$. The plot window is a square $[-2.25, 2.25] \times [-2.25 \times 2.25]$ cut into a fine grid, 6000×6000. Compare with the January image and with the cover of the book.

October

On September 30, Miller Christen, a student at Clemson University, sent us an email pointing out a typo on page 137, where we had said $P = [1, 1 - i]$ when we had meant $P = [1 + i, 1 - i]$. She then gave us an image of what the unit upper Hessenberg Bohemian eigenvalue density plot *should* have been, if we had coded the example the way we had said we had. So here is a density plot of the eigenvalues of all $2^{21} = 2,097,152$ matrices of dimension $m = 6$ when the population is $P = [1, 1 - i]$. They are plotted on $-1.25 \leq \Re(\lambda) \leq 3.75$, $-1.8 \leq \Im(\lambda) < 1.2$. The similarities to the image in the book are striking: there are still spirals present, for instance. The differences are also striking: the image is clearly disconnected, for instance. None of these features have been explained. We are particularly intrigued by the spirals in both images, and by the pentagonal symmetry in one hole in this image.

November

Circulant matrices can be diagonalized quickly using the FFT. A particular kind of circulant matrix, useful in the study of magic squares, was populated with $P = \{-1, i, 1\}$. The number of free entries was 14 so the number of matrices constructed (actually their first rows, with the first entry and hence the diagonal being zero) was $3^{14} = 4,782,969$, but the dimension of each matrix was 29, resulting in $138,706,101$ eigenvalues being computed. The density plot on $-19 \leq x \leq 19$ and $-19 \leq y \leq 19$ used a 3200 by 3200 grid, with the viridis colour palette and a custom frequency function so that "hotter" colours indicate a greater density of eigenvalues. Computations and imaging done in Maple™. We thank Matthew Lettington (Cardiff University) for teaching us about magic squares and inspiring this image.

December

This is an eigenvalue density plot for skew-symmetric tridiagonal matrices with population $(2, 5i, 2 + i)$, on $-10 \leq \Re(\lambda) \leq 10$, $-5 \leq \Im(\lambda) \leq 5$ (axes not identically scaled). Eigenvalues of all $3^{13} = 1,594,323$ dimension 14 matrices were computed and imaged in Maple™. What can you say about that "vortex" in the middle? We don't understand it, and we'd love you to explain it for us!

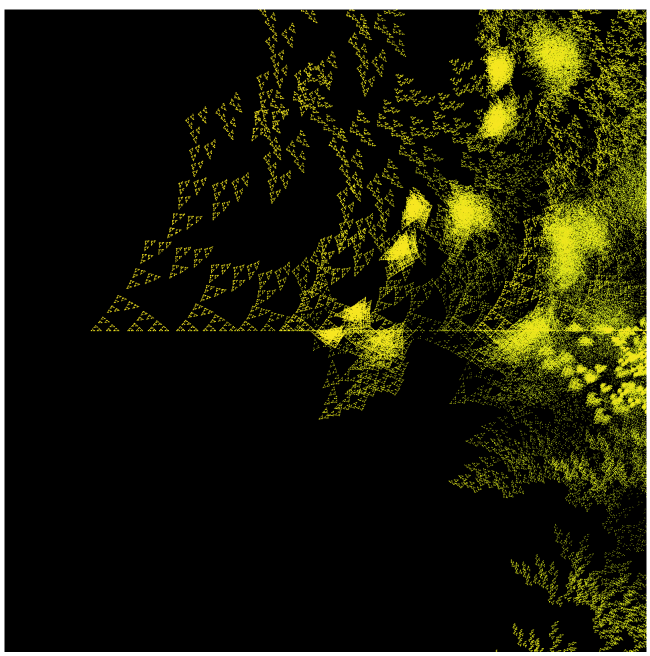

Figure 1: One last image; a zoom in on the February image on the left.