OPTO6008 – Optical Fibre Technology I

Part 1: Fundamentals of Light Propagation in Optical Fibres

Lecture 1.8

Coupled Mode Theory

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Outline

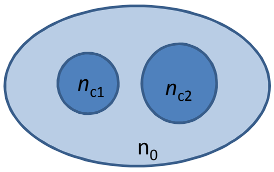
- Coupled mode theory
- Co-directional coupling
- Multimode interpretation
- Phase matching
- Bragg gratings
- Long period gratings

Coupled Mode Theory

In many photonic devices light is coupled between different modes / waveguides / fibres / directions

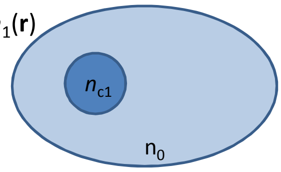
- **Example:** two fibre cores close together

Index profile $n(\mathbf{r})$

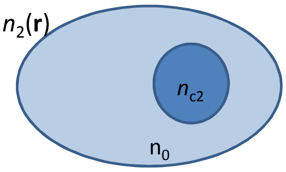


- We know how to calculate the **modes of each of the fibre cores**:

Index profile $n_1(\mathbf{r})$



Index profile $n_2(\mathbf{r})$



- Can we describe propagation of light in the full structure using this knowledge?

Coupled Mode Theory

- **Individual modes** fulfil $[\Delta + k^2 n_p^2(\mathbf{r})]E_p e^{i\beta_p z} = 0$
- **"Full" solution** $E = A(z)E_1 e^{i\beta_1 z} + B(z)E_2 e^{i\beta_2 z}$
  to fulfil $\Delta E + k^2 n^2(\mathbf{r})E = 0$
- Using the **slowly varying envelope approximation** (neglect terms d²*A*/d*z*², d²*B*/d*z*²):

$$2i\beta_1 \frac{dA}{dz} E_1 e^{i\beta_1 z} + Ak^2(n^2 - n_1^2)E_1 e^{i\beta_1 z} + 2i\beta_2 \frac{dB}{dz} E_2 e^{i\beta_2 z} + Bk^2(n^2 - n_2^2)E_2 e^{i\beta_2 z} = 0$$

- Next, apply $\int \ldots \cdot E_1^* \, dxdy$

small

$$2i\beta_1 \frac{dA}{dz} e^{i\beta_1 z} \int |E_1|^2 \, dxdy + Ae^{i\beta_1 z} k^2 \cancel{\int (n^2 - n_1^2)|E_1|^2}$$

$$+2i\beta_2 \frac{dB}{dz} e^{i\beta_2 z} \cancel{\int E_2 E_1^* \, dxdy} + Be^{i\beta_2 z} k^2 \int (n^2 - n_2^2) E_2 E_1^* \, dxdy = 0$$

small

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Coupled Mode Theory

- 2nd term small compared to 4th, 3rd term small compared to 1st:

$$\frac{dA}{dz} = i\frac{k^2\int(n^2-n_2^2)E_1^*E_2\,dxdy}{2\beta_1\int|E_1|^2\,dxdy}Be^{i(\beta_2-\beta_1)z}$$

$$\frac{dB}{dz} = i\frac{k^2\int(n^2-n_1^2)E_1E_2^*\,dxdy}{2\beta_2\int|E_2|^2\,dxdy}Ae^{i(\beta_1-\beta_2)z}$$

- Write as

$$\boxed{\begin{aligned}\frac{dA}{dz} &= i\kappa_{12}Be^{i2\Delta z}\\ \frac{dB}{dz} &= i\kappa_{21}Ae^{-i2\Delta z}\end{aligned}} \qquad \Delta = \frac{\beta_2-\beta_1}{2}$$

- Energy conservation implies $|A(z)|^2+|B(z)|^2 = \text{const.}$

$$\Rightarrow \quad \kappa_{12} = \kappa_{21}^*$$

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Co-directional Coupler

- **Co-directional coupler** β_1, β_2>0: input $A(0)$, $B(0)$

  Solution of coupled mode equation is:

$$A(z) = \left\{\left[\cos(qz) - i\frac{\Delta}{q}\sin(qz)\right]A(0) + i\frac{\kappa}{q}\sin(qz)B(0)\right\}e^{i\Delta z}$$

$$B(z) = \left\{\left[\cos(qz) + i\frac{\Delta}{q}\sin(qz)\right]B(0) + i\frac{\kappa}{q}\sin(qz)A(0)\right\}e^{-i\Delta z}$$

$$q = \sqrt{\kappa^2+\Delta^2}$$

- Light launched into one core, $A(0)=A_0$, $B(0)=0$, gives **power flow**:

$$P_a(z) = \frac{|A(z)|^2}{|A_0|^2} = 1 - F\sin^2(qz)$$

$$P_b(z) = \frac{|B(z)|^2}{|A_0|^2} = F\sin^2(qz)$$

$$F = (\kappa/q)^2 = \frac{1}{1+(\Delta/\kappa)^2}$$

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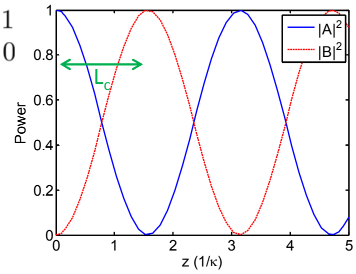
Co-directional Coupler - Examples

$A(0) = 1$
$B(0) = 0$
$\Delta = 0$

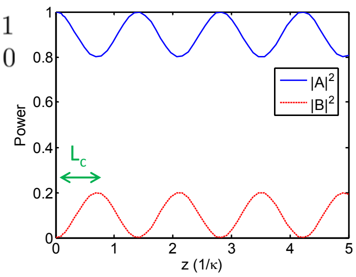


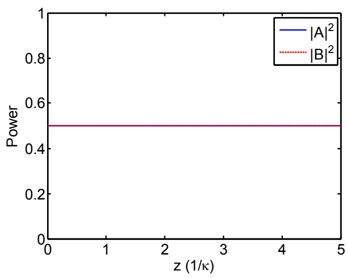
$$L_c = \frac{\pi}{2\sqrt{\kappa^2 + \Delta^2}}$$ coupling length

Coupling length L_c scales with $1/\kappa$:

- large κ, strong coupling, fast oscillations
- small κ, weak coupling, slow oscillations

$A(0) = 1$
$B(0) = 0$
$\Delta = 2\kappa$





$A(0) = \sqrt{2}$
$B(0) = \sqrt{2}$
$\Delta = 0$

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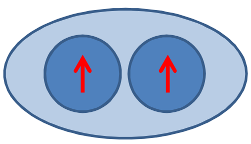
Multimode Interpretation

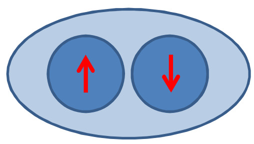
- For simplicity, assume Δ=0 here ($\beta_1=\beta_2=\beta$): $$\frac{dA}{dz} = i\kappa B$$ $$\frac{dB}{dz} = i\kappa A$$
- Look for solutions:

$$\begin{pmatrix} A(z) \\ B(z) \end{pmatrix} = e^{i\lambda z} \begin{pmatrix} A(0) \\ B(0) \end{pmatrix} \Longrightarrow \begin{cases} \lambda = \kappa,\ B(0) = A(0) \\ \lambda = -\kappa,\ B(0) = -A(0) \end{cases}$$

- Corresponding electric mode fields:

$$E_+ = (E_1 + E_2)e^{i(\beta+\kappa)z} \qquad E_- = (E_1 - E_2)e^{i(\beta-\kappa)z}$$





Symmetric superposition:

- Larger propagation constant
- Larger effective index

Antisymmetric superposition:

- Lower propagation constant
- Lower effective index

Multimode Interpretation

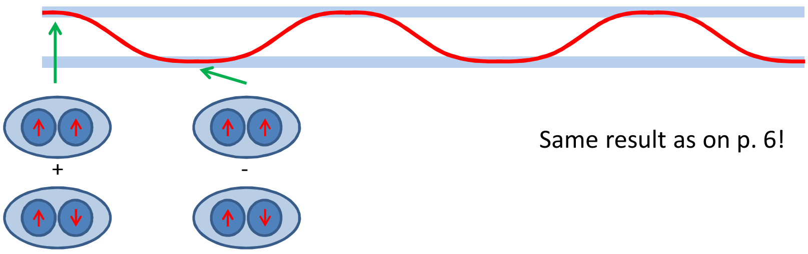
- The oscillation of power between the two fibres is then explained as **interference** of the two modes of the coupler

Example: assume light launched into core 1 at z=0: excites symmetric and antisymmetric mode with same amplitude

$$E(0) = A_0 E_1 = A_0 \tfrac{1}{2}[E_+(0) + E_-(0)]$$

- After propagation

$$E(z) = A_0 \tfrac{1}{2}[E_+(z) + E_-(z)] = A_0 e^{i\beta z}[E_1 \cos(\kappa z) + iE_2 \sin(\kappa z)]$$

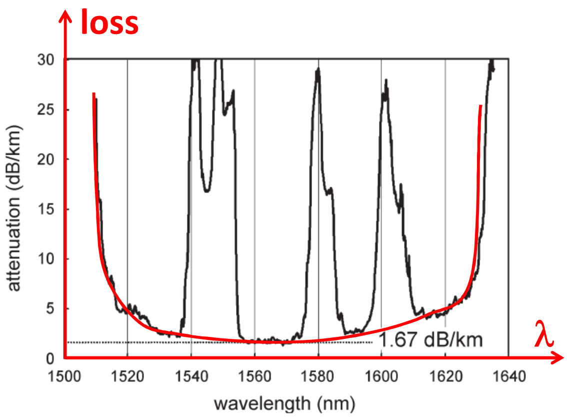


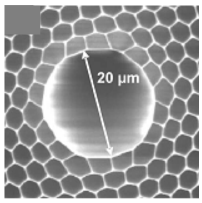
Same result as on p. 6!

Consequences of Mode Coupling (1)

Photonic Bandgap Fibres (see Lecture 1.7):

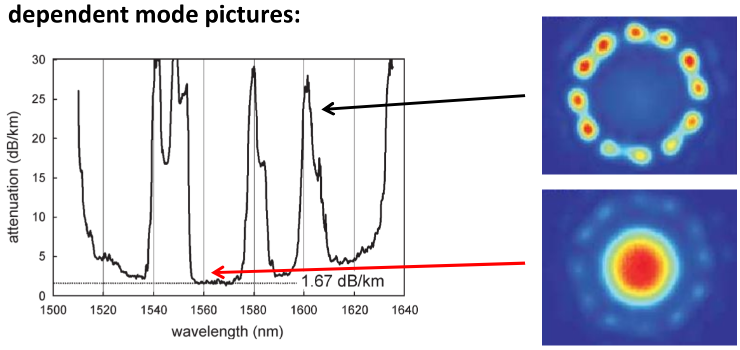
- Within the bandgap, low losses expected
- **But:** Experiments show **strong absorption bands**





Consequences of Mode Coupling (2)

Wavelength dependent mode pictures:

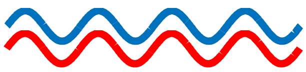


- Hollow core supports a **core mode** by bandgap guidance
- Glass structures in the cladding support index-guided modes ("**surface modes**")
- At certain wavelengths, core mode and surface modes have same β

  ⇨ **strong coupling**, leading to loss of power from core into cladding

Phase Matching

- So far we found that light can be coupled between modes efficiently when they have the same propagation constant, this is called **phase matching**

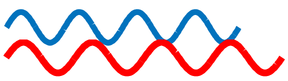


$$\beta_1 = \beta_2$$

- Phase matching is related to **momentum conservation** for exchange of photons

$$\hbar\beta_1 = \hbar\beta_2$$

- **No coupling** when phase matching is not fulfilled



$$\beta_1 \neq \beta_2$$

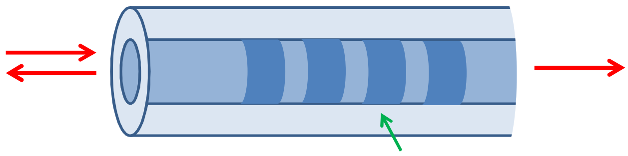
- Phase matching can still be achieved when **a third wave** is added

$$\beta_1 = \beta_2 + K$$

- This third wave can be anything that couples to light: **light, sound, a spatial modulation of geometry, of refractive index, etc.**

Fibre Bragg Gratings

- Another example of coupled modes:



Spatially modulated refractive index

- Refractive index modulation

$$n(x,y,z) = n_1(x,y) + \frac{\delta n}{2}(e^{iKz} + e^{-iKz}) = n_1(x,y) + \delta n \cos(Kz)$$

$K = \dfrac{2\pi}{\Lambda}$ $\quad\Lambda$ ... grating period

- Light can be reflected: coupling of forward and backward propagating modes

Grating Theory

- Fibre mode: $E_1 e^{\pm i\beta z}$ $\qquad (\nabla^2 + k^2 n_1^2) E_1 e^{\pm i\beta z} = 0$
- with grating: $E = A(z)E_1 e^{i\beta z} + B(z) E_1 e^{-i\beta z}$ $\qquad (\nabla^2 + k^2 n^2) E = 0$

- Using the **slowly varying approximation** (neglect terms *d²A/dz²*, *d²B/dz²*):

$$2i\beta \frac{dA}{dz} E_1 + Ak^2 n_1 \delta n (e^{\cancel{i}Kz} + e^{-\cancel{i}Kz}) E_1$$
$$-2i\beta \frac{dB}{dz} E_1 e^{-\cancel{i}2\beta z} + Bk^2 n_1 \delta n (e^{iKz} + e^{-\cancel{i}Kz}) E_1 e^{-i2\beta z} = 0$$

- **Neglect fast oscillating terms** (*K*, *-K*, *-2β*, *-K-2β*)
- Term *K-2β* can be made 0 (or close to 0) by choice of Λ: **Phase matching**

$$\frac{dA}{dz} = iB \frac{k^2 \int n_1 \delta n |E_1|^2 \, dxdy}{2\beta \int |E_1|^2 \, dxdy} e^{i(K-2\beta)z} \approx i \underbrace{\frac{\pi \delta n}{\lambda}}_{=\kappa} B e^{i(K-2\beta)z}$$

$$\frac{dB}{dz} \approx -i \frac{\pi \delta n}{\lambda} A e^{-i(K-2\beta)z}$$

Grating Theory Results

Bragg wavelength: $\lambda_B = 2n_{\text{eff}}\Lambda$

Coupling coefficient: $\kappa = \dfrac{\pi\delta n}{\lambda_B}$

Detuning: $\Delta = \beta - \dfrac{\pi}{\Lambda}$

Forward and backward propagating power: $P_f(z) = \dfrac{|A(z)|^2}{|A_0|^2}$, $P_b(z) = \dfrac{|B(z)|^2}{|A_0|^2}$

For $|\Delta| > \kappa$:

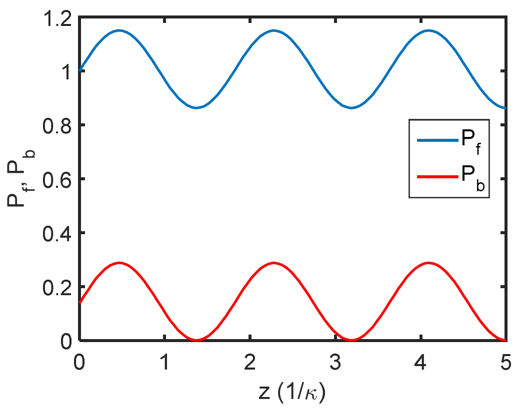
$$\rho = \sqrt{\Delta^2 - \kappa^2}$$
$$P_f(z) = \frac{\rho^2 + \kappa^2 \sin^2[\rho(z-L)]}{\rho^2 + \kappa^2 \sin^2(\rho L)}$$
$$P_b(z) = \frac{\kappa^2 \sin^2[\rho(z-L)]}{\rho^2 + \kappa^2 \sin^2(\rho L)}$$

For $|\Delta| < \kappa$:

$$\alpha = \sqrt{\kappa^2 - \Delta^2}$$
$$P_f(z) = \frac{\alpha^2 + \kappa^2 \sinh^2[\alpha(z-L)]}{\alpha^2 + \kappa^2 \sinh^2(\alpha L)}$$
$$P_b(z) = \frac{\kappa^2 \sinh^2[\alpha(z-L)]}{\alpha^2 + \kappa^2 \sinh^2(\alpha L)}$$

Grating Theory Results

Δ=2κ

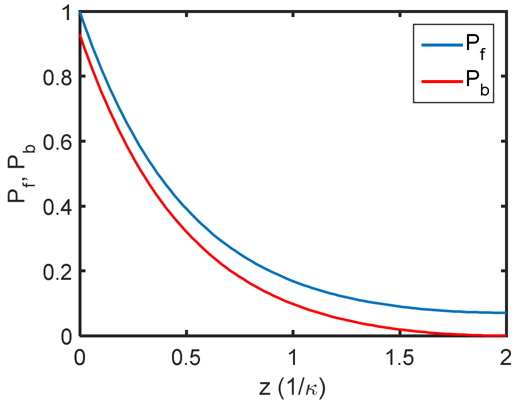


Note:

- Power conservation for grating length L: $P_f(L)+P_b(0) = P_f(0)$
- Locally inside the grating the total power $P_f(z)+P_b(z)$ can be > $P_f(0)$

Grating Theory Results

Δ=0



- **On resonance** the forward propagating power decreases on length scale $1/\alpha$
- In this case reflection is phase matched $\lambda_B = 2n_{\text{eff}}\Lambda \iff \beta = K - \beta$

Bragg Grating Reflectance

Reflectance: $R = \frac{|B(0)|^2}{|A(0)|^2}$

Transmittance: $T = 1 - R = \frac{|A(L)|^2}{|A(0)|^2}$

For $|\Delta| > \kappa$: $R = \frac{\kappa^2 \sin^2(\rho L)}{\rho^2 + \kappa^2 \sin^2(\rho L)}$ $\rho = \sqrt{\Delta^2 - \kappa^2}$

For $|\Delta| < \kappa$: $R = \frac{\kappa^2 \sinh^2(\alpha L)}{\alpha^2 + \kappa^2 \sinh^2(\alpha L)}$ $\alpha = \sqrt{\kappa^2 - \Delta^2}$

Maximum reflectance: $\Delta = 0,\ \lambda = \lambda_B,\ \alpha = \kappa$

$$R = \tanh^2(\kappa L)$$

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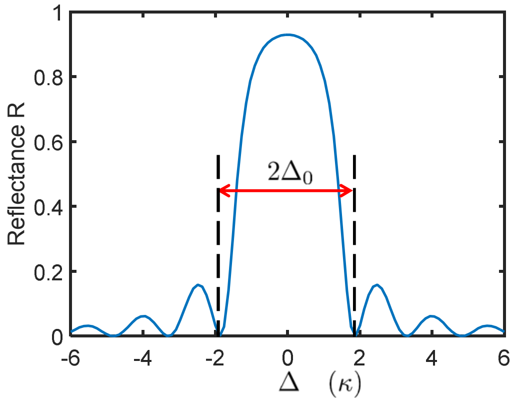
Bragg Grating Reflectance

$L = 2/\kappa$



Bandwidth of grating = distance between first zeros of reflectance

$$2\Delta_0 = \frac{2}{L}\sqrt{\kappa^2 L^2 + \pi^2}$$

$$\delta\lambda \simeq \frac{\lambda_B^2}{\pi n_{\text{eff}} L}\sqrt{\kappa^2 L^2 + \pi^2}$$

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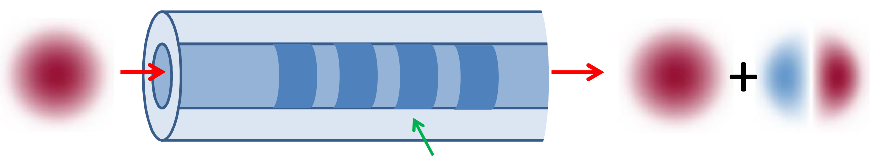
Long Period Gratings

In multimode fibres, Bragg gratings can be used to **couple light between modes**



Spatially modulated refractive index

Grating must fulfil:

- **Nonuniform transverse profile**: coupling requires nonzero overlap

$$\int \delta n(x,y) E_1(x,y) E_2^*(x,y)\, dx\, dy \neq 0$$

- **Phase matching**: $K = \beta_1 - \beta_2$

  The periodicity Λ is much larger (K much smaller) here than for reflection gratings (previous pages): **long period gratings**

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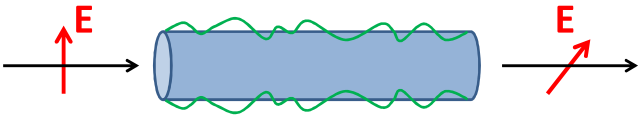
Polarisation Mode Coupling

From Lecture 1.4: in circularly symmetric fibres, the two polarisation modes of the LP_{01} mode are degenerate ($\beta_x = \beta_y$):

- Modes are **phase matched** ⇨ strong polarisation coupling
- No phase matching **in birefringent fibres** ⇨ polarisation maintaining



Can understand this in terms of long period gratings:

- Consider fluctuations of fibre properties along the fibre length, for example, fluctuations of fibre radius ***r(z)***
- Write radius fluctuations as Fourier integral: $r(z) = \mathrm{Re}\left\{\int c(K) e^{iKz}\, dK\right\}$

Surface fluctuations are a superposition of long-period gratings!

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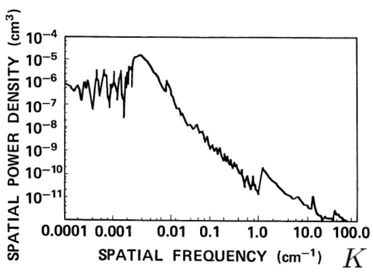
Polarisation Mode Coupling

$r(z) = \mathrm{Re}\left\{\int c(K) e^{iKz}\, dK\right\}$

- Polarisations are coupled by component *K* with: $K = \beta_x - \beta_y$
- Coupling strength depends on the amplitude of fluctuations: $|c(K)|^2$

Measurement of |*c*(*K*)|² on an actual fibre:

$|c(K)|^2$



c(*K*) decreases for large *K*

⇨ Polarisations not coupled for large β_x-β_y

⇨ Polarisation-maintaining (PM) fibres

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References

- K. Okamoto, "Fundamentals of Optical Waveguides", 2nd edition, Academic Press, 2006
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- P. St. J. Russell, "Photonic-Crystal Fibers", J. Lightwave Technol. **24**, 4729 (2006)