

Deep Reinforcement Learning and Control

Planning in Markov Decision Processes

Lecture 3, CMU 10703

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Markov Decision Process (MDP)

A **Markov Decision Process** is a tuple $(\mathcal{S}, \mathcal{A}, T, r, \gamma)$

- \mathcal{S} is a finite set of states
- \mathcal{A} is a finite set of actions
- T is a state transition probability function

$$T(s'|s, a) = \mathbb{P}[S_{t+1} = s' | S_t = s, A_t = a]$$

- r is a reward function

$$r(s, a) = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$$

- γ is a discount factor $\gamma \in [0, 1]$

Solving MDPs

- **Prediction:** Given an MDP $(\mathcal{S}, \mathcal{A}, T, r, \gamma)$ and a policy

$$\pi(a|s) = \mathbb{P}[A_t = a | S_t = s]$$

find the state and action value functions.

- **Optimal control:** given an MDP $(\mathcal{S}, \mathcal{A}, T, r, \gamma)$, find the optimal policy (aka the planning problem). Compare with the learning problem with missing information about rewards/dynamics.
- We still consider finite MDPs (finite \mathcal{S} and \mathcal{A}) with known dynamics!

Outline

- Policy iteration
- Value iteration
- Linear programming
- Asynchronous DP

Policy Evaluation

Policy evaluation: for a given policy π , compute the state value function

$$v_\pi(s) = \mathbb{E}_\pi [R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots | S_t = s]$$

where $v_\pi(s)$ is implicitly given by the **Bellman equation**

$$v_\pi(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) v_\pi(s') \right)$$

a system of $|\mathcal{S}|$ simultaneous equations.

MDPs to MRPs

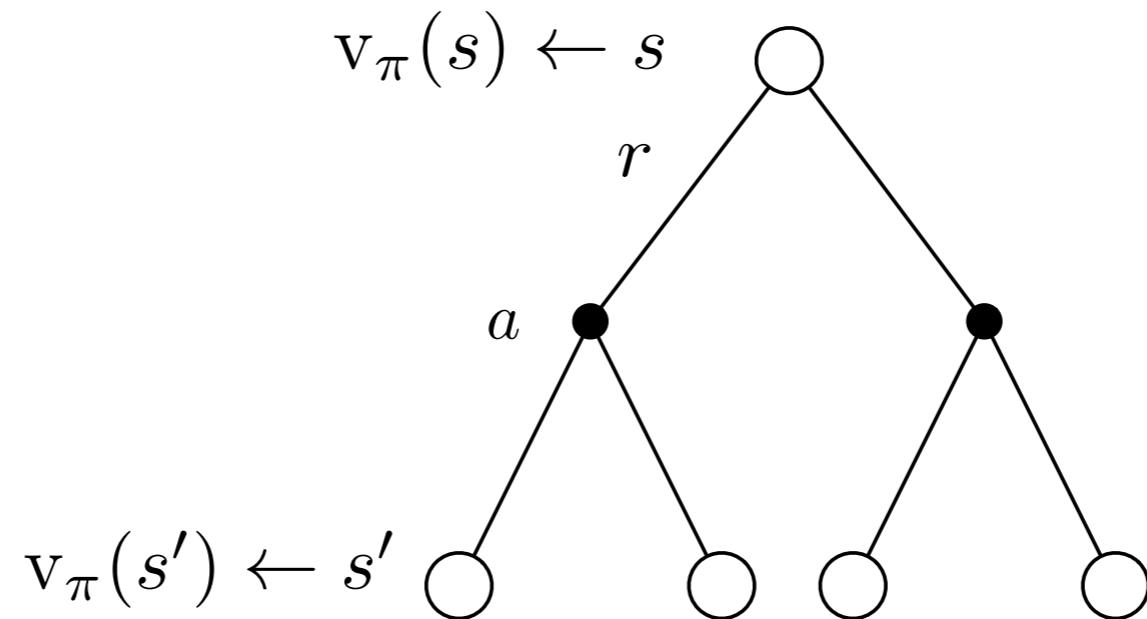
MDP under a fixed policy becomes **Markov Reward Process (MRP)**

$$\begin{aligned} v_\pi(s) &= \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) v_\pi(s') \right) \\ &= \sum_{a \in \mathcal{A}} \pi(a|s) r(s, a) + \gamma \sum_{a \in \mathcal{A}} \pi(a|s) \sum_{s' \in \mathcal{S}} T(s'|s, a) v_\pi(s') \\ &= r_s^\pi + \gamma \sum_{s' \in \mathcal{S}} T_{s's}^\pi v_\pi(s') \end{aligned}$$

where $r_s^\pi = \sum_{a \in \mathcal{A}} \pi(a|s) r(s, a)$ and $T_{s's}^\pi = \sum_{a \in \mathcal{A}} \pi(a|s) T(s'|s, a)$

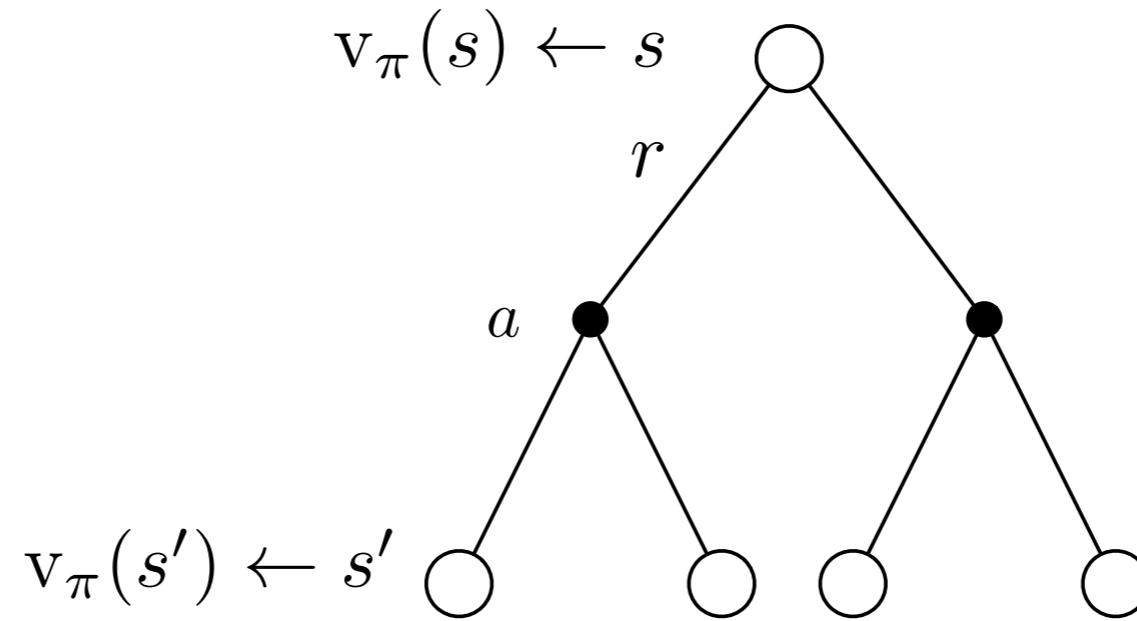
Back Up Diagram

MDP

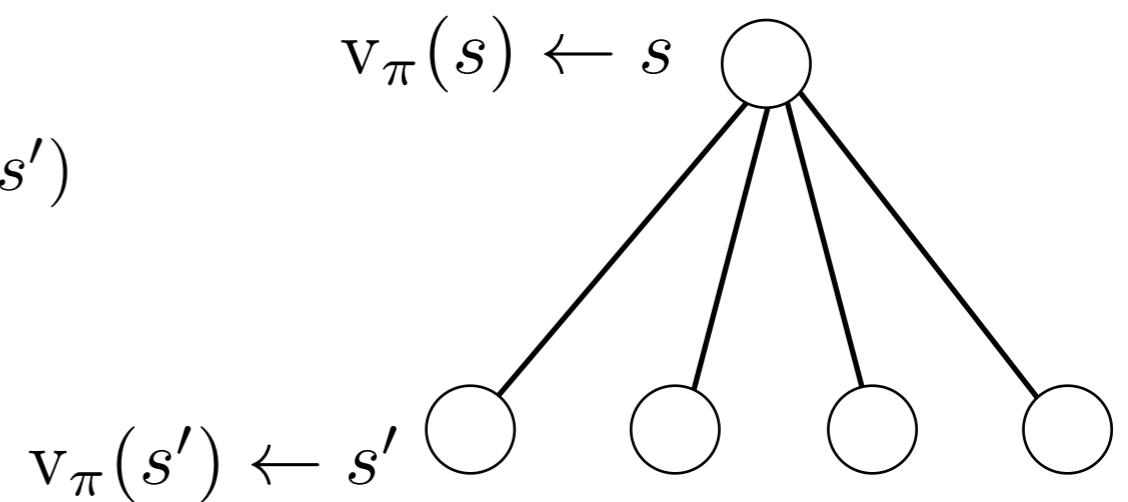


Back Up Diagram

MDP



MRP $v_\pi(s) = r_s^\pi + \gamma \sum_{s' \in \mathcal{S}} T_{s's}^\pi v_\pi(s')$



Matrix Form

The Bellman expectation equation can be written concisely using the induced MRP as

$$v_\pi = r^\pi + \gamma T^\pi v_\pi$$

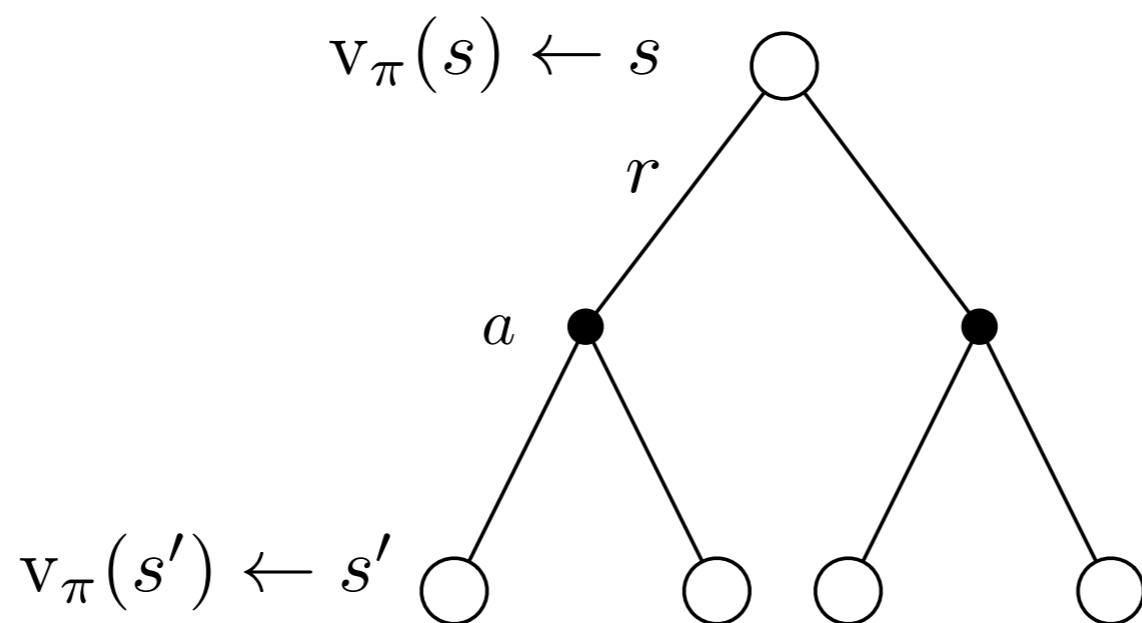
with direct solution

$$v_\pi = (I - \gamma T^\pi)^{-1} r^\pi$$

of complexity $O(N^3)$

Iterative Methods: Recall the Bellman Equation

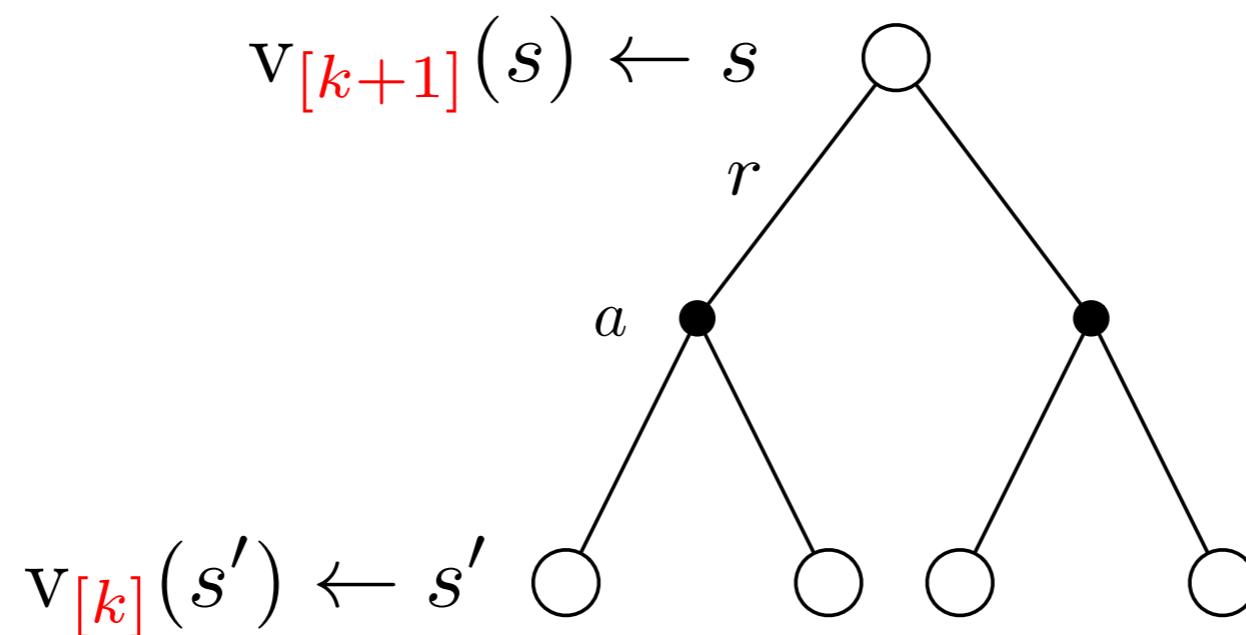
$$v_{\pi}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) v_{\pi}(s') \right)$$



Iterative Methods: Backup Operation

Given an expected value function at iteration k , we back up the expected value function at iteration $k+1$:

$$v_{[k+1]}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) v_{[k]}(s') \right)$$



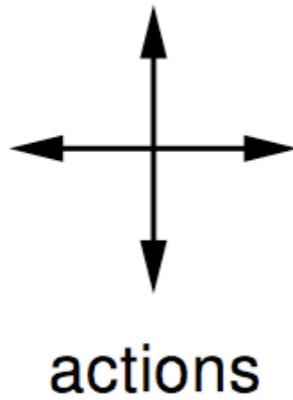
Iterative Methods: Sweep

A **sweep** consists of applying the backup operation $v \rightarrow v'$ for all the states in \mathcal{S}

Applying the back up operator iteratively

$$v[0] \rightarrow v[1] \rightarrow v[2] \rightarrow \dots v_\pi$$

A Small-Grid World



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

$\gamma = 1$

- An undiscounted episodic task
- Nonterminal states: 1, 2, ..., 14
- Terminal state: one, shown in shaded square
- Actions that would take the agent off the grid leave the state unchanged
- Reward is -1 until the terminal state is reached

Iterative Policy Evaluation

$V[k]$ for the random policy

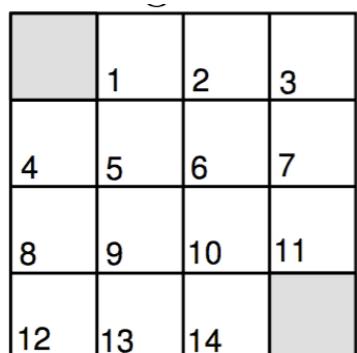
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 0$

Policy π , an equiprobable random action

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 1$



$k = 2$

- An undiscounted episodic task
- Nonterminal states: 1, 2, ..., 14
- Terminal state: one, shown in shaded square
- Actions that would take the agent off the grid leave the state unchanged
- Reward is -1 until the terminal state is reached

$k = 3$

$k = 10$

$k = \infty$

Iterative Policy Evaluation

$V[k]$ for the random policy

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 0$

Policy π , an equiprobable random action

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 1$

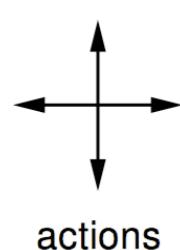
0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 2$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 3$

$k = \infty$



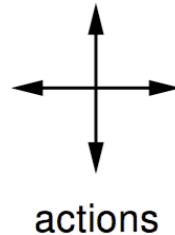
- An undiscounted episodic task
- Nonterminal states: 1, 2, ..., 14
- Terminal state: one, shown in shaded square
- Actions that would take the agent off the grid leave the state unchanged
- Reward is -1 until the terminal state is reached

	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$k = 10$

Iterative Policy Evaluation

Policy π , an equiprobable random action



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

- An undiscounted episodic task
- Nonterminal states: 1, 2, ..., 14
- Terminal state: one, shown in shaded square
- Actions that would take the agent off the grid leave the state unchanged
- Reward is -1 until the terminal state is reached

$V[k]$ for the random policy

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 0$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 1$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 2$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 3$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = 10$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

$k = \infty$

Contraction Mapping Theorem

An operator F on a normed vector space \mathcal{X} is a **γ -contraction**, for $0 < \gamma < 1$, provided for all $x, y \in \mathcal{X}$

$$\|T(x) - T(y)\| \leq \gamma \|x - y\|$$

Theorem (Contraction mapping)

For a γ -contraction F in a complete normed vector space \mathcal{X}

- F converges to a unique fixed point in \mathcal{X}
- at a linear convergence rate γ

Remark. In general we only need metric (vs normed) space

Value Function Space

- Consider the vector space V over value functions
- There are $|\mathcal{S}|$ dimensions
- Each point in this space fully specifies a value function $v(s)$
- Bellman backup brings value functions closer in this space?
- And therefore the backup must converge to a unique solution

Value Function ∞ -Norm

- We will measure distance between state-value functions u and v by the ∞ -norm
- i.e. the largest difference between state values,

$$\|u - v\|_\infty = \max_{s \in \mathcal{S}} |u(s) - v(s)|$$

Bellman Expectation Backup is a Contraction

- Define the Bellman expectation backup operator

$$F^\pi(v) = r^\pi + \gamma T^\pi v$$

- This operator is a γ -contraction, i.e. it makes value functions closer by at least γ ,

$$\begin{aligned} \|F^\pi(u) - F^\pi(v)\|_\infty &= \|(r^\pi + \gamma T^\pi u) - (r^\pi + \gamma T^\pi v)\|_\infty \\ &= \|\gamma T^\pi(u - v)\|_\infty \\ &\leq \|\gamma T^\pi\| \|u - v\|_\infty \\ &\leq \gamma \|u - v\|_\infty \end{aligned}$$

Convergence of Iter. Policy Evaluation and Policy Iteration

- The Bellman expectation operator F^π has a unique fixed point
- v_π is a fixed point of F^π (by Bellman expectation equation)
- By contraction mapping theorem
- Iterative policy evaluation converges on v_π

Policy Improvement

- Suppose we have computed v_π for a deterministic policy π
- For a given state s , would it be better to do an action $a \neq \pi(s)$?

- It is better to switch to action a for state s if and only if

$$q_\pi(s, a) > v_\pi(s)$$

- And we can compute $q_\pi(s, a)$ from v_π by:

$$\begin{aligned} q_\pi(s, a) &= \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s, A_t = a] \\ &= r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) v_\pi(s') \end{aligned}$$

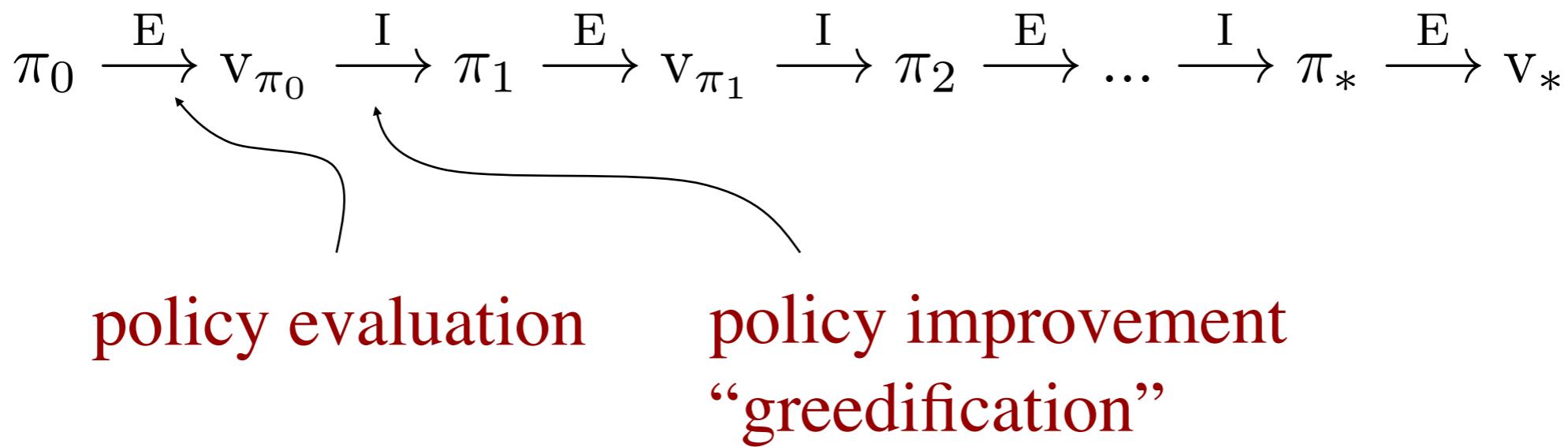
Policy Improvement Cont.

- Do this for all states to get a new policy $\pi' \geq \pi$ that is greedy with respect to v_π :

$$\begin{aligned}\pi'(s) &= \arg \max_a q_\pi(s, a) \\ &= \arg \max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(s') | S_t = s, A_t = a] \\ &= \arg \max r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a)v_\pi(s')\end{aligned}$$

- What if the policy is unchanged by this?
 - Then the policy must be optimal!

Policy Iteration



Policy Iteration

1. Initialization

$V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation

Repeat

$$\Delta \leftarrow 0$$

For each $s \in \mathcal{S}$:

$$v \leftarrow V(s)$$

$$V(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) (r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) V(s'))$$

$$\Delta \leftarrow \max(\Delta, |v - V(s)|)$$

until $\Delta < \theta$ (a small positive number)

3. Policy Improvement

policy-stable $\leftarrow true$

For each $s \in \mathcal{S}$:

$$a \leftarrow \pi(s)$$

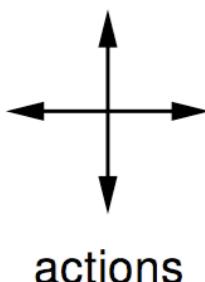
$$\pi(s) \leftarrow \arg \max r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) v_\pi(s')$$

If $a \neq \pi(s)$, then *policy-stable* $\leftarrow false$

If *policy-stable*, then stop and return V and π ; else go to 2

Iterative Policy Eval for the Small Gridworld

Policy π , an equiprobable random action



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

$\gamma = 1$

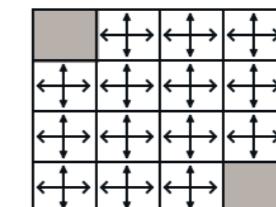
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V_k for the
Random Policy

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 0$

Greedy Policy
w.r.t. V_k



random
policy

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 1$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 2$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 3$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

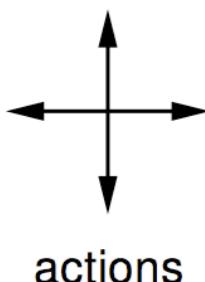
$k = 10$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

$k = \infty$

Iterative Policy Eval for the Small Gridworld

Policy π , an equiprobable random action



	1	2	3
4	5	6	7
8	9	10	11
12	13	14	

$R = -1$
on all transitions

$\gamma = 1$

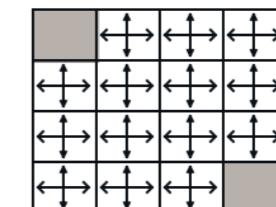
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- Reward is -1 until the terminal state is reached

V_k for the
Random Policy

0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0

$k = 0$

Greedy Policy
w.r.t. V_k



0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0

$k = 1$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0

$k = 2$

0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 3$

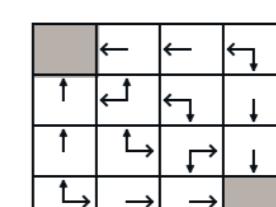
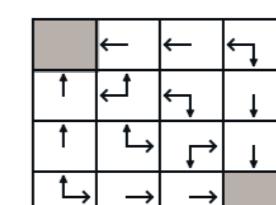
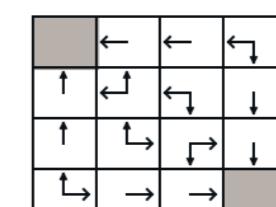
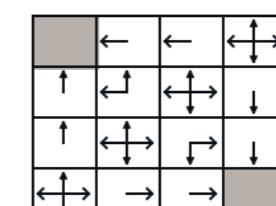
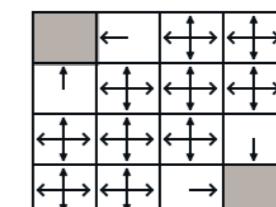
0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = 10$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0

$k = \infty$

optimal
policy

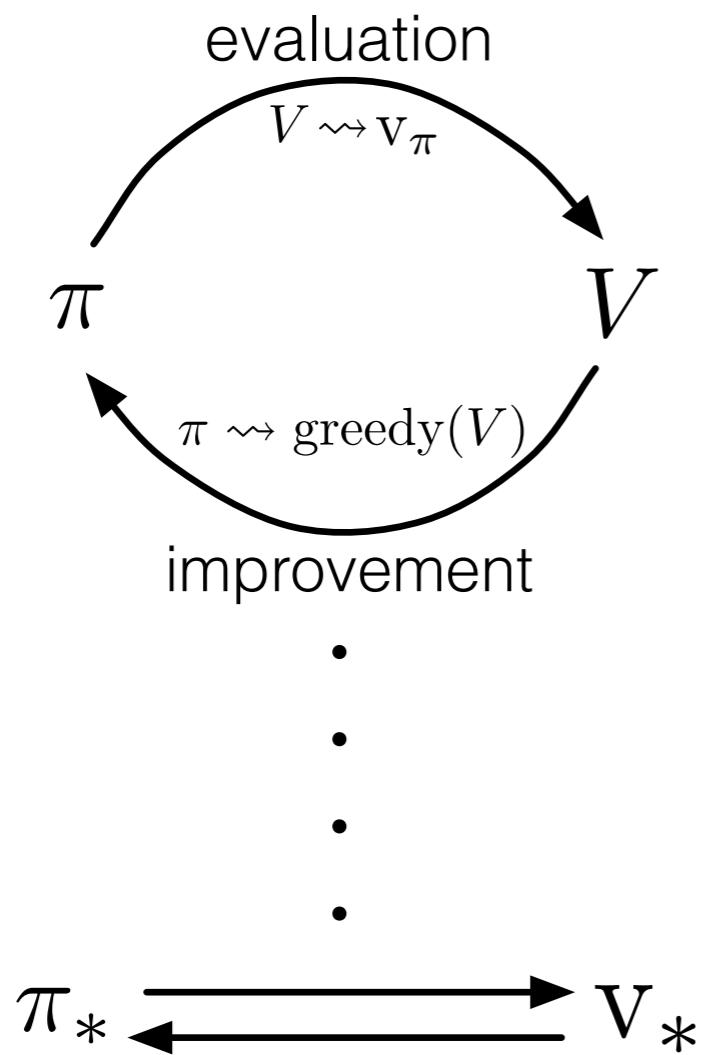


Generalized Policy Iteration

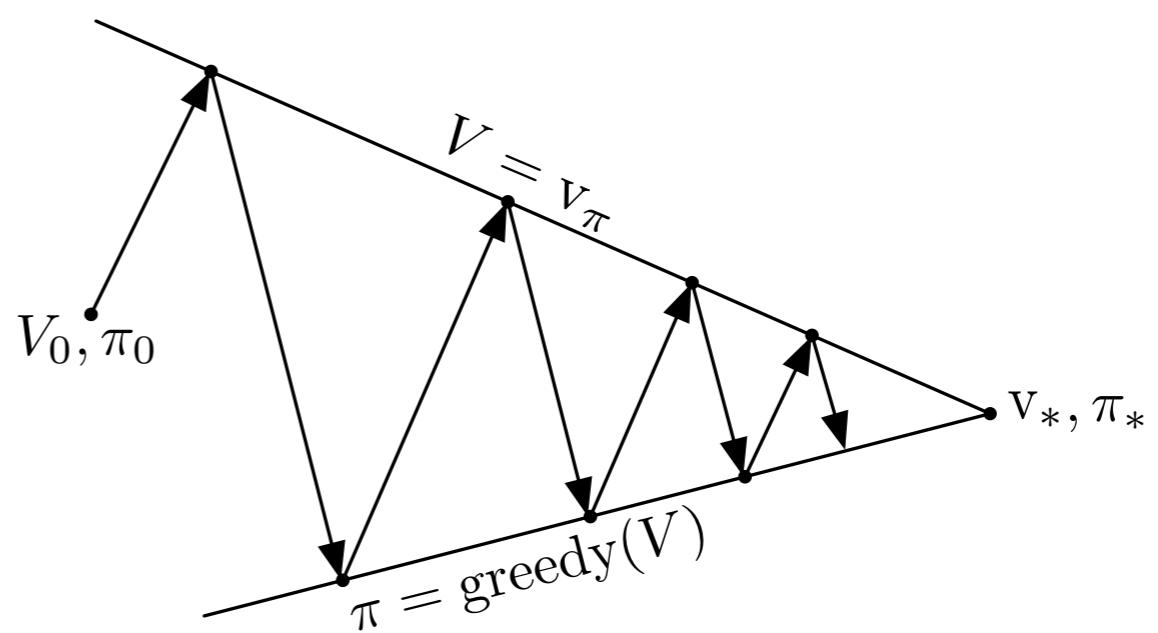
- Does policy evaluation need to converge to v_π ?
- Or should we introduce a stopping condition
 - e.g. ϵ -convergence of value function
- Or simply stop after k iterations of iterative policy evaluation?
- For example, in the small grid world $k = 3$ was sufficient to achieve optimal policy
- Why not update policy every iteration? i.e. stop after $k = 1$
 - This is equivalent to value iteration (next section)

Generalized Policy Iteration

Generalized Policy Iteration (GPI): any interleaving of policy evaluation and policy improvement, independent of their granularity.



A geometric metaphor for convergence of GPI:



Principle of Optimality

- Any optimal policy can be subdivided into two components:
 - An optimal first action \mathcal{A}_*
 - Followed by an optimal policy from successor state \mathcal{S}'
- Theorem (Principle of Optimality)
 - A policy $\pi(a|s)$ achieves the optimal value from state s , $v_\pi(s) = v_*(s)$, if and only if
 - For any state s' reachable from s , π achieves the optimal value from state s' , $v_\pi(s') = v_*(s')$

Value Iteration

- If we know the solution to subproblems $v_*(s')$
- Then solution $v_*(s')$ can be found by one-step lookahead

$$v_*(s) \leftarrow \max_{a \in \mathcal{A}} r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a)v_*(s')$$

- The idea of value iteration is to apply these updates iteratively
- Intuition: start with final rewards and work backwards
- Still works with loopy, stochastic MDPs

Example: Shortest Path

g			

Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

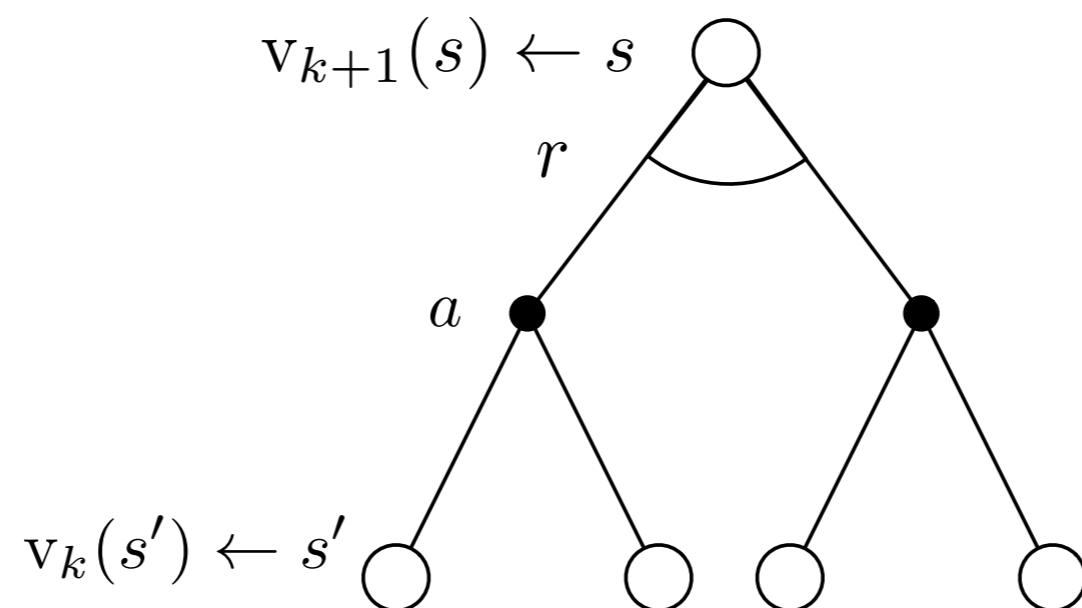
0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

Value Iteration

- Problem: find optimal policy π
- Solution: iterative application of Bellman optimality backup
- $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_*$
- Using synchronous backups
 - At each iteration $k + 1$
 - For all states $s \in \mathcal{S}$
 - Update $v_{k+1}(s)$ from $v_k(s')$

Value Iteration (2)



$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a)v_k(s') \right)$$

$$v_{k+1} = \max_{a \in \mathcal{A}} r(a) + \gamma T(a)v_k$$

Bellman Optimality Backup is a Contraction

- Define the Bellman optimality backup operator F^* ,

$$F^*(v) = \max_{a \in \mathcal{A}} r(a) + \gamma T(a)v$$

- This operator is a γ -contraction, i.e. it makes value functions closer by at least γ (similar to previous proof)

$$\|F^*(u) - F^*(v)\|_\infty \leq \gamma \|u - v\|_\infty$$

Convergence of Value Iteration

- The Bellman optimality operator F^* has a unique fixed point
- v_* is a fixed point of F^* (by Bellman optimality equation)
- By contraction mapping theorem
- Value iteration converges on v_*

Synchronous Dynamic Programming Algorithms

Problem	Bellman Equation	Algorithm
Prediction	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman Optimality Equation	Value Iteration

- Algorithms are based on state-value function $v_\pi(s)$ or $v_*(s)$
- Complexity $O(mn^2)$ per iteration, for m actions and n states
- Could also apply to action-value function $q_\pi(s, a)$ or $q_*(s, a)$
- Complexity $O(m^2n^2)$ per iteration

Efficiency of DP

- To find an optimal policy is polynomial in the number of states...
- BUT, the number of states is often astronomical, e.g., often growing exponentially with the number of state variables (what Bellman called “the curse of dimensionality”).
- In practice, classical DP can be applied to problems with a few millions of states.

Asynchronous DP

- All the DP methods described so far require exhaustive sweeps of the entire state set.
- Asynchronous DP does not use sweeps. Instead it works like this:
 - Repeat until convergence criterion is met:
 - Pick a state at random and apply the appropriate backup
 - Still need lots of computation, but does not get locked into hopelessly long sweeps
 - Guaranteed to converge if all states continue to be selected
 - Can you select states to backup intelligently? YES: an agent's experience can act as a guide.

Asynchronous Dynamic Programming

- Three simple ideas for asynchronous dynamic programming:
 - In-place dynamic programming
 - Prioritized sweeping
 - Real-time dynamic programming

In-Place Dynamic Programming

- Synchronous value iteration stores two copies of value function

- for all s in \mathcal{S}

$$\mathbf{v}_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) \mathbf{v}_{old}(s') \right)$$

$$\mathbf{v}_{old} \leftarrow \mathbf{v}_{new}$$

- In-place value iteration only stores one copy of value function

- for all s in \mathcal{S}

$$\mathbf{v}(s) \leftarrow \max_{a \in \mathcal{A}} \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) \mathbf{v}(s') \right)$$

Prioritized Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a)v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman poor of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

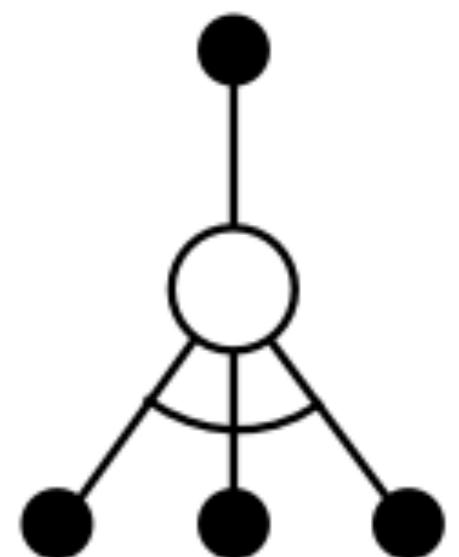
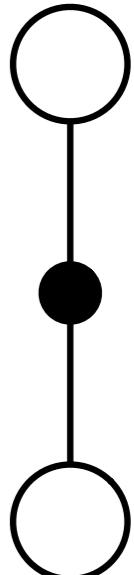
Real-time Dynamic Programming

- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step $\mathcal{S}_t, \mathcal{A}_t, r_{t+1}$
- Backup the state \mathcal{S}_t

$$v(\mathcal{S}_t) \leftarrow \max_{a \in \mathcal{A}} \left(r(\mathcal{S}_t, a) + \gamma \sum_{s' \in \mathcal{S}} T(s' | \mathcal{S}_t, a) v(s') \right)$$

Sample Backups

- In subsequent lectures we will consider sample backups
- Using sample rewards and sample transitions $(\mathcal{S}, \mathcal{A}, r, \mathcal{S}')$
- Instead of reward function and transition dynamics
- Advantages:
 - Model-free: no advance knowledge of MDP required
 - Breaks the curse of dimensionality through sampling
 - Cost of backup is constant, independent of $n = |\mathcal{S}|$



Approximate Dynamic Programming

- Approximate the value function
- Using a function approximate $\hat{v}(s, w)$
- Apply dynamic programming to $\hat{v}(\cdot, w)$
- e.g. Fitted Value Iteration repeats at each iteration k ,
 - Sample states $\tilde{\mathcal{S}} \subseteq \mathcal{S}$
 - For each state $s \in \tilde{\mathcal{S}}$, estimate target value using Bellman optimality equation,
$$\tilde{v}_k(s) = \max_{a \in \mathcal{A}} \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) \hat{v}(s', w_k) \right)$$
 - Train next value function $\hat{v}(\cdot, w_{k+1})$ using targets $\{\langle s, \tilde{v}_k(s) \rangle\}$

Linear Programming (LP)

- Recall, at value iteration convergence we have

$$\forall s \in \mathcal{S} : V^*(s) = \max_{a \in \mathcal{A}} \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) V^*(s') \right)$$

- LP formulation to find

$$\min_V \quad \sum_{\mathcal{S}} \mu_0(s) V(s)$$

s.t. $\forall s \in \mathcal{S}, \forall a \in \mathcal{A}$:

$$V(s) \geq r(s, a) + \gamma \sum_{s' \in \mathcal{S}} T(s'|s, a) V^*(s')$$

μ_0 is a probability distribution over \mathcal{S} , with $\mu_0(s) > 0$ for all s in \mathcal{S} .

Theorem. V^* is the solution to the above LP.

LP con't

- Let F be the Bellman operator, i.e., $V_{i+1}^* = F(V_i)$. Then the LP can be written as:

$$\begin{aligned}\min_V \quad & \mu_0^\top V \\ \text{s.t.} \quad & V \geq F(V)\end{aligned}$$

Monotonicity Property: If $U \geq V$ then $F(U) \geq F(V)$.

Hence, if $V \geq F(V)$ then $F(V) \geq F(F(V))$, and by repeated application, $V \geq F(V) \geq F^2V \geq F^3V \geq \dots \geq F^\infty V = V^*$

Any feasible solution to the LP must satisfy $V \geq F(V)$, and hence must satisfy $V \geq V^*$. Hence, assuming all entries in μ_0 are positive, V^* is the optimal solution to the LP.

LP: The Dual $\rightarrow q^*$

$$\min_{\lambda} \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}} \sum_{s' \in \mathcal{S}} \lambda(s, a) T(s, a, s') r(s, a)$$

$$\text{s.t. } \forall s \in \mathcal{S} : \sum_{a' \in \mathcal{A}} \lambda(s', a') = \mu_0(s) + \gamma \sum_{s \in \mathcal{S}} \sum_{a \in \mathcal{A}} \lambda(s, a) T(s, a, s')$$

- Interpretation
 - $\lambda(s, a) = \sum_{t=0}^{\infty} \gamma^t P(s_t = s, a_t = a)$
 - Equation 2: ensures that λ has the above meaning
 - Equation 1: maximize expected discounted sum of rewards
- Optimal policy: $\pi^*(s) = \arg \max_a \lambda(s, a)$