COMPUTATIONAL ECONOMICS

Rational Expectations in Duali

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1 Introduction

In macroeconomics, the way in which expectations are modeled has a significant effect on model solution and simulation strategies. Some macroeconomic models include the assumption that economic agents form their expectations in a backward-looking adaptive way. That is, in order to form expectations in connection with the likely future value of a given macroeconomic variable, economic agents take into account the recent evolution of that variable and, perhaps, also that of other, closely related variables. For example, in Chapter **??**, we saw that the expected inflation rate was obtained as a weighted sum of the observed inflation rates in the previous two quarters. From a modeling point of view, that means that contemporary model expectational variables can be replaced by some combination of lagged variables.

In contrast, the assumption of rational expectations asserts that economic outcomes are not systematically different from economic agents' expectations about those outcomes. This implies that macroeconomic models should embed the notion that economic agents make use of all available information when forming their expectations. Thus, included in the agents' information set is the model of the economy that the modeler is using to capture their behavior. This assumption has a significant impact in terms of modeling and simulation, since under it, agents' expectations are a function of the whole macroeconomic model solution, while, at the same time, that solution is a function of agents' expectations. Moreover, model dynamics becomes more complex, since expectational variables are forward-looking variables that sometimes display a *jumping* behavior, instantaneously adjusting to changes in policy or exogenous variables. Finally, policy analysis is also more demanding, since policy makers have to take into account the agents' anticipatory behavior in response to their policy announcements and actions.

In this chapter, we perform simulations and policy experiments in the Duali software with John Taylor's rational expectations model. This is a prototype one-country model that is very useful as a training ground in the computational modeling of rational expectations. It is also a good introduction to Taylor's (1993) empirical multicountry models .

2 John Taylor's Closed Economy Model

John Taylor's closed economy model is a small prototype linear model with staggered contracts and rational expectations variables that generate an interesting pattern of dynamic behavior. It contains the following equations, variables, and parameters:

Equations:

$$x_t = \tfrac{\delta}{3} \sum_{i=0}^{2} \hat{w}_{t+i} + \tfrac{1-\delta}{3} \sum_{i=0}^{2} \hat{p}_{t+i} + \tfrac{\gamma}{3} \sum_{i=0}^{2} \hat{y}_{t+i} \tag{1}$$

$$w_t = \tfrac{1}{3} \sum_{i=0}^{2} x_{t-i} \tag{2}$$

$$p_t = \theta w_t \tag{3}$$

$$y_t = -d\, r_t + g_t \tag{4}$$

$$m_t - p_t = -b\, i_t + a y_t \tag{5}$$

$$r_t = i_t - \hat{p}_{t+1} + p_t \tag{6}$$

with the variables

x = contract wage
w = average wage
p = price level
y = output
i = nominal interest rate
r = real interest rate
m = money stock
g = government expenditure

In the original Taylor model, government expenditure appears implicitly as a shift factor in equation (4). Here, we make it an explicit variable in that equation and ˆ indicates expectation through period t.

Parameters:

$$\delta = 0.5 \quad \gamma = 1 \quad \theta = 1 \quad a = 1 \quad b = 4 \quad d = 1.2$$

The variables (all except i_t and r_t) are logarithms and are deviations from means or secular trends.

Equation (1) is a staggered-wage-setting equation. It is assumed that a wage decision lasts 3 years, with one-third of the wages being negotiated each year.

At any given time t, the contract wage depends on expectations of the values at times t, $t+1$, and $t+2$ of wages paid to other workers, the price level, and the real output. Equation (2) gives the average wage in the economy as the average of the contract wage in the current period and the two previous periods; equation (3) reflects markup pricing behavior by firms, that is, prices are set proportionally to the average wage; equation (4) defines a standard IS schedule; equation (5) is the money demand equation defining an LM schedule; and equation (6) gives the real interest rate as the nominal interest rate deflated by the rationally expected inflation rate, where the expected inflation rate is defined as

$$\hat{\pi}_t = \hat{p}_{t+1} - p_t$$

The model has six equations and six endogenous variables. It contains two policy variables, the money stock and government expenditure; is dynamic and linear; and has the *natural rate* property, in the sense that nominal shocks may affect real variables in the short run, but not in the long run.

3 Solving Optimal Control Rational Expectations Problems in Duali

As a rational expectations model, Taylor's model requires specific solution methods different from those applied to standard models. Many methods have been developed over the last two decades for solving rational expectations models. See, for example, Blanchard and Kahn (1980), Wallis (1980), Fair and Taylor (1983), Anderson and Moore (1985), Oudiz and Sachs (1985), Fisher, Holly, and Hughes-Hallett (1986), Pesaran (1987), Juillard (1996), Zadrozny and Chen (1999), Binder and Pesaran (2000), and Sims (2001). Some of those methods are analytical and they usually involve, for the case of linear models, the passage from the model structural form to a *pseudoreduced* form, in which the expectational variables are no longer present. Other methods are numerical. Moreover, as shown in Holly and Hughes-Hallett (1989, Chapter 7), the analysis of models' dynamic properties such as the computation of eigenvalues and the condition of dynamic controllability becomes more involved in rational expectation models.

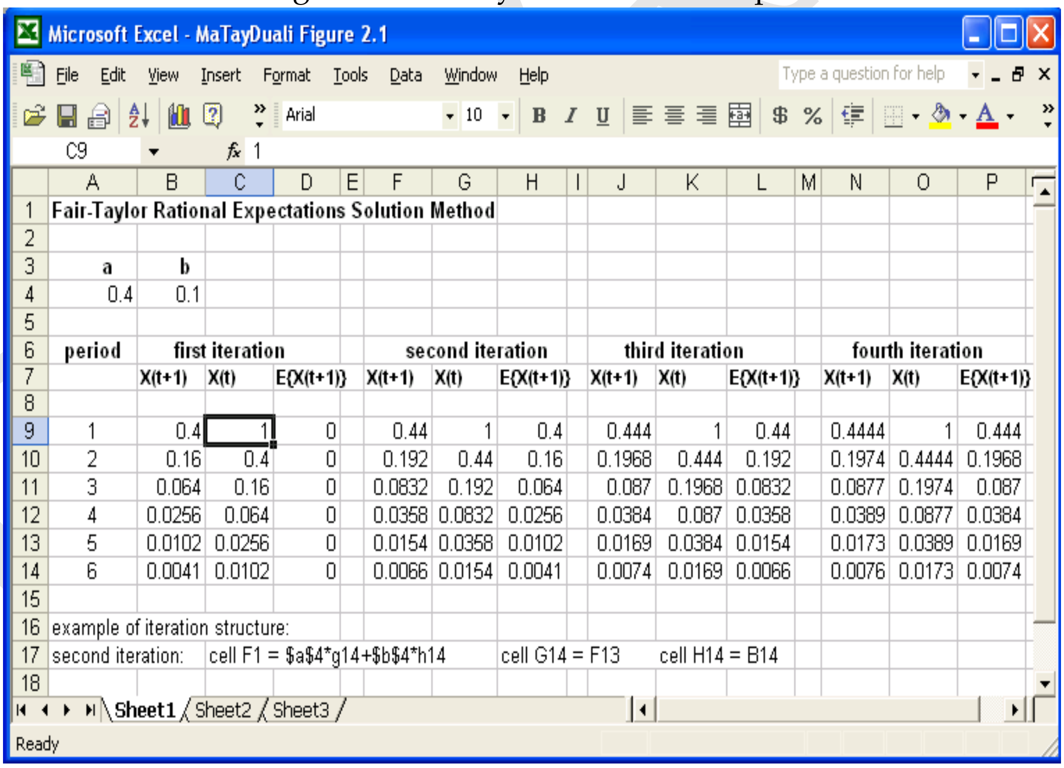
To solve optimal control problems containing rational expectations models, Duali uses a dynamic programming algorithm like the one presented in Chapter **??**, combined with the numerical method developed by Ray Fair and John Taylor to solve rational expectations models. The Fair and Taylor (1983) method is an iterative procedure that starts by solving the model for a set of arbitrary values —usually zeroes— for the path of each forward-looking variable. Then, after each iteration, the values of the forward-looking variables are updated with the solution values of the corresponding endogenous variable in the previous iteration. The process stops when convergence is obtained, that is, when the difference between the values of the forward variables in two successive iterations is smaller than a given tolerance value.

For example, suppose that we have a simple single-equation model like the one shown below, in which the future value of a variable (x_{t+1}) is a function of its current value (x_t) and also of its future expected value conditioned on the information available at time t ($x^e_{t+1|t}$):

$$x_{t+1} = ax_t + bx^e_{t+1\,|t}$$

Suppose also that the solution horizon covers only six periods, that $a = 0.4$, $b = 0.1$, and that the initial value for x_t is 1. The **EXCEL**spreadsheet in Figure 1 shows the results for the first four iterations, where we use E to denote expected value.

Figure 1: Fair-Taylor method example


Microsoft Excel - MaTayDuali Figure 2.1

Fair-Taylor Rational Expectations Solution Method

a	b
0.4	0.1

period	first iteration X(t+1)	X(t)	E{X(t+1)}	second iteration X(t+1)	X(t)	E{X(t+1)}	third iteration X(t+1)	X(t)	E{X(t+1)}	fourth iteration X(t+1)	X(t)	E{X(t+1)}
1	0.4	1	0	0.44	1	0.4	0.444	1	0.44	0.4444	1	0.444
2	0.16	0.4	0	0.192	0.44	0.16	0.1968	0.444	0.192	0.1974	0.4444	0.1968
3	0.064	0.16	0	0.0832	0.192	0.064	0.087	0.1968	0.0832	0.0877	0.1974	0.087
4	0.0256	0.064	0	0.0358	0.0832	0.0256	0.0384	0.087	0.0358	0.0389	0.0877	0.0384
5	0.0102	0.0256	0	0.0154	0.0358	0.0102	0.0169	0.0384	0.0154	0.0173	0.0389	0.0169
6	0.0041	0.0102	0	0.0066	0.0154	0.0041	0.0074	0.0169	0.0066	0.0076	0.0173	0.0074

example of iteration structure:
second iteration: cell F1 = a4*g14+b4*h14 cell G14 = F13 cell H14 = B14

Sheet1 Sheet2 Sheet3

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Note that four iterations of the model solution are shown in the spreadsheet and that there are three columns of variables shown at each iteration, namely $x(t+1)$, $x(t)$, and $E[x(t+1)]$. The logic of the procedure is easy to follow. In the first iteration, column `D` is set to zero. Given those values and the initial value for x_t in cell `C9`, the model is solved for each of the remaining five time periods. The results in column B are then copied to column H in the second iteration and the model is solved again. The results are copied from column F to column L, and so on. Note how fast the results converge for this particular model and parameter values—the difference between columns P and N in the fourth iteration is quite small. What makes the Fair and Taylor method attractive is its simplicity and the fact that it can be applied to multiple-equation linear and nonlinear models.

Duali contains a method developed by Amman and Kendrick (1996) for solving optimal control problems with rational expectations. This procedure, which is described in what follows, uses the Fair and Taylor method as an intermediate step.

The problem is expressed as one of finding the controls $(u)_{t=0}^{N}$ to minimize a quadratic tracking criterion function J of the form

$$
\begin{aligned}
J \;=\; & \frac{1}{2}\left[x_N - \tilde{x}_N\right]' W_N \left[x_N - \tilde{x}_N\right] + \\
& \frac{1}{2}\sum_{t=0}^{N-1}\left(\left[x_t - \tilde{x}_t\right]' W_t \left[x_t - \tilde{x}_t\right] + \left[u_t - \tilde{u}_t\right]' \Lambda_t \left[u_t - \tilde{u}_t\right]\right)
\end{aligned}
\tag{7}
$$

subject, as a constraint, to the state-space representation of the economic model, also known as the regular form:

$$
x_{t+1} = Ax_t + Bu_t + Cz_t + D_1 x^e_{t+1\,|\,t} + D_2 x^e_{t+2\,|\,t}
\tag{8}
$$

where x, u, and z are state, control, and exogenous variables, respectively, $\tilde{x}$ and $\tilde{u}$ are desired paths for the state and control variables, and $x^e_{t+1|t}$ is a forward-looking variable equal to the expected value of the state variable at period $t+1$ conditioned on the information available at time t. Also A, B, C, D_1, and D_2 are matrices. In this example, the maximum lead for the forward-looking variables is two periods, but it could of course be larger.

A way of formalizing the rational expectations hypothesis is, for a determin-

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istic environment,

$$x^e_{t+1\mid t} = x_{t+1} \tag{9}$$

and, for a stochastic environment, where E is the mathematical expectation operator,

$$x^e_{t+1\mid t} = E_t x_{t+1} \tag{10}$$

Denote the expected value of the state variable at iteration v as $x^{ev}_{(t+1|t)}$. At the first iteration —iteration zero—the Amman and Kendrick procedure begins by setting $x^{e0}_{t+1|t} = x^0_{t+1} = 0$ for all t, and solving the resulting quadratic linear problem with a standard method, such as the one presented Chapter 16. The optimal state variables for the solution obtained—the *no lead* solution—are denoted as x^{NL}. Then, the expected values of the forward-looking variables are set equal to the solution for this first iteration, that is,

$$x^{e1}_{t+1\mid t} = x^{NL}_{t+1} \text{ and } x^{e1}_{t+2\mid t} = x^{NL}_{t+2} \quad \forall t \tag{11}$$

Thus, the system of equations corresponding to the first iteration is now

$$x^1_{t+1} = Ax^1_t + Bu^1_t + Cz_t + D_1 x^{e1}_{t+1\mid t} + D_2 x^{e1}_{t+2\mid t} \tag{12}$$

Note that the terms

$$Cz_t + D_1 x^{e1}_{t+1\mid t} + D_2 x^{e1}_{t+2\mid t} \tag{13}$$

are all known. This allows us to write the system of equations as

$$x^1_{t+1} = Ax^1_t + Bu^1_t + \tilde{C}\tilde{z}^1_t \tag{14}$$

where

$$\tilde{C} = \begin{bmatrix} C & D_1 & D_2 \end{bmatrix} \tag{15}$$

and

$$\tilde{z}^1_t = \begin{bmatrix} z_t \\ x^{e1}_{t+1\mid t} \\ x^{e1}_{t+2\mid t} \end{bmatrix} \tag{16}$$

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Again, we have a quadratic linear problem that can be solved with standard methods. When we do so, we have another set of solution values for the state variables, which are used as the values of the forward-looking variables in the next iteration, and so on. The procedure stops when convergence is obtained.

4 The Taylor Model in Duali

In the following we focus on the implementation of Taylor's model in Duali to perform simulations and optimal policy analyses. First we transform the model equations to make them more suitable for a matrix representation. As shown in Section 2, the model was

$$x_t = \frac{\delta}{3}\sum_{i=0}^{2}\hat{w}_{t+i} + \frac{1-\delta}{3}\sum_{i=0}^{2}\hat{p}_{t+i} + \frac{\gamma}{3}\sum_{i=0}^{2}\hat{y}_{t+i} \tag{17}$$

$$w_t = \frac{1}{3}\sum_{i=0}^{2}x_{t-i} \tag{18}$$

and from the previous section

$$p_t = \theta w_t \tag{19}$$

$$y_t = -d\, r_t + g_t \tag{20}$$

$$m_t - p_t = -b\, i_t + a y_t \tag{21}$$

$$r_t = i_t - \hat{p}_{t+1} + p_t \tag{22}$$

Expanding the summation signs, renaming some variables, and substituting the corresponding numerical values for the model parameters, we obtain the following model:

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$$
\begin{aligned}
x_t^{cw} &= 0.16\hat{w}_t + 0.16\hat{p}_t + 0.3\hat{y}_t + 0.16\hat{w}_{t+1} + 0.16\hat{p}_{t+1} + 0.3\hat{y}_{t+1} \\
&+ 0.16\hat{w}_{t+2} + 0.16\hat{p}_{t+2} + 0.3\hat{y}_{t+2} && (23) \\
w_t &= 0.3x_t^{cw} + 0.3xl_t^{cw} + 0.3xl_{t-1}^{cw} && (24) \\
p_t &= w_t && (25) \\
y_t &= -1.2r_t + g_{t-1} && (26) \\
i_t &= 0.25y_t + 0.25p_t - 0.25m_{t-1} && (27) \\
r_t &= i_t + p_t - hatp_{t+1} && (28) \\
xl_t^{cw} &= x_{t-1}^{cw} && (29)
\end{aligned}
$$

Note that x_t^{cw} is the contract wage in Taylor's model, which we relabeled here in equations (16) and (17) to avoid notational confusion with x_t , which is the vector of stacked variables of the model matrix representation. Also note that since in Taylor's model expectations are conditional on the information available at time t, we can write

$$w_{t|t}^e = w_t, \quad p_{t|t}^e = p_t, \quad y_{t|t}^e = y_t \tag{30}$$

which is why the variables in the first three right-hand-side terms in equation 30 are actual and not expected values, as is the case in the remaining terms.

In equation (24) there is a new variable xl_t^{cw}, defined in equation (29) as equal to lagged x_t^{cw}, that is, x_{t-1}^{cw}. Therefore, the variable xl_{t-1}^{cw} in equation (17) is equal to x_{t-2}^{cw}. In this way, using the same method we employed in Chapter **??**, we produce a one-lag-order reduction of equation (2). Since this is the only lagged equation in the model, we are left with a first-order model representation suitable for use in optimal control experiments.

In equation (27) we moved the interest rate i to the left-hand side to make its role as a state variable explicit. Finally, in Taylor's model, m and g appear as contemporaneous to the endogenous variables. By assuming that there is a one-period lag between a policy decision and its implementation, we can redefine these two control variables in equations (23) and (24) as m_{t-1} and g_{t-1}, since Duali, as well as the optimal control literature, works with one-lag policy variables.

We now represent the model in what is known as the Pindyck or $I - A$ form, which is an equivalent representation to the form shown in equation (8). The

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Pindyck form of Taylor's model can be written as[1]

$$x_t = A_0 x_t + A_1 x_{t-1} + B_1 u_{t-1} + C_1 z_{t-1} + \hat{D}_1 x^e_{t/t} + \hat{D}_2 x^e_{t+1/t} + \hat{D}_3 x^e_{t+2/t} \quad (31)$$

where

$$x_t = \begin{bmatrix} x^{cw}_t \\ w_t \\ p_t \\ y_t \\ i_t \\ r_t \\ xl^{cw}_t \end{bmatrix} \quad (32)$$

$$u_{t-1} = \begin{bmatrix} m_{t-1} \\ g_{t-1} \end{bmatrix} \quad (33)$$

$$A_0 = \begin{bmatrix} 0 & 0.1666 & 0.1666 & 0.3333 & 0 & 0 & 0 \\ 0.3333 & 0 & 0 & 0 & 0 & 0 & 0.3333 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.2 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (34)$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3333 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ -0.25 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[1] In Taylor's model, expectations are conditioned on the information available at t. In Duali, when a model is written in the Pindyck form, expectations are conditioned at $t-1$. This change in the timing of the information does not appear as problematic for the Taylor model, since Duali replicates the results obtained by the original Taylor simulations. However, different assumptions concerning the information set timing may be relevant for other models.

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$$\hat{D}_2 = \begin{bmatrix} 0 & 0.1666 & 0.1666 & 0.3333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{D}_3 = \begin{bmatrix} 0 & 0.1666 & 0.1666 & 0.3333 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In equation (28), z_{t-1} is a vector of exogenous variables and C_1 is a matrix. They are both equal to zero, since the model does not contain exogenous variables. Note also that $\hat{D}_1$ is set equal to zero, since the model does not contain contemporaneous expected variables. Finally, we set equal to four the maximum number of decimals for parameter values.

5 Dynamic Simulation

As a way of getting acquainted with some dynamic properties of the Taylor model, we analyze its dynamic evolution for given changes in its policy variables. The general problem to be solved in Duali is the one of finding the controls $(u)_{t=0}^{N-1}$ to minimize a quadratic tracking criterion function J of the form

$$J = \left\{ \frac{1}{2} [x_N - \tilde{x}_N]' W_N [x_N - \tilde{x}_N] + \frac{1}{2} \sum_{t=0}^{N-1} \left([x_t - \tilde{x}_t]' W_t [x_t - \tilde{x}_t] + [u_t - \tilde{u}_t]' \Lambda_t [u_t - \tilde{u}_t] \right) \right\} \tag{35}$$

subject to

$$x_t = A_0 x_t + A_1 x_{t-1} + B_1 u_{t-1} + C_1 z_{t-1} + \hat{D}_1 x^e_{t/t} + \hat{D}_2 x^e_{t+1/t} + \hat{D}_3 x^e_{t+2/t} \tag{36}$$

where the variables and parameters were defined in previous sections. Though the Duali software is oriented toward solving optimization problems like the one just presented, it can also handle standard simulations such as the

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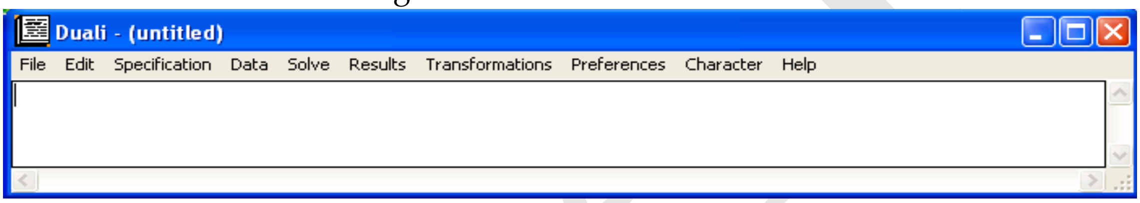
experiments in this section, where we change the values of the policy variables to see their dynamic impacts on the endogenous variables of the model. To do so, the weights on the controls in the Λ matrix are set to relatively high values, while the weights on the states in the W matrix are set to relatively small ones. We then define the desired paths for the controls as equal to the policy change to be introduced. In this way we force the system to respond to the prespecified changes in the policy variables. In fact, what we are doing is ignoring the optimization part of the solution method presented in the previous section and using the Fair-Taylor method only to simulate the rational expectations model.

We begin from the main menu shown in the Duali main window in Figure 2. From the File option we open the file `tay-sim.dui`. In the `Specification:Stochastic Terms` menu option, we see that the problem is set as deterministic, as shown in Figure 3. We then select the `Specification:Functional Form` option and obtain the dialog box in Figure 4.

Figure 2: Dual main window



On the Criterion side of the dialog box we see that the problem is a Quadratic-Tracking problem with constant state and control priorities and that the desired states and controls are constant. They are all set equal to zero, since in the optimal control experiment in the next section we seek to minimize deviations of target variables from means or secular trends. Also recall that the model variables are already expressed in deviation form.

On the System Equations side of the dialog box in Figure 4 we see that the Pindyck form is selected, the option Yes is selected for Forward Variables, there are no policy to parameter effects, and the exogenous variables are constant.

From the `Data:Size` menu we obtain the dialog box shown in Figure 5. The model is specified as containing seven state variables (in fact, six contemporaneous and one lagged), two control variables, and one exogenous variable, and the simulation covers eleven periods. The Maximum Lead for forward variables is set equal to three. This is telling Duali that the model contains three $\hat{D}$ matrices, as seen in equation (28). The Iteration Limit is set to 50 and the

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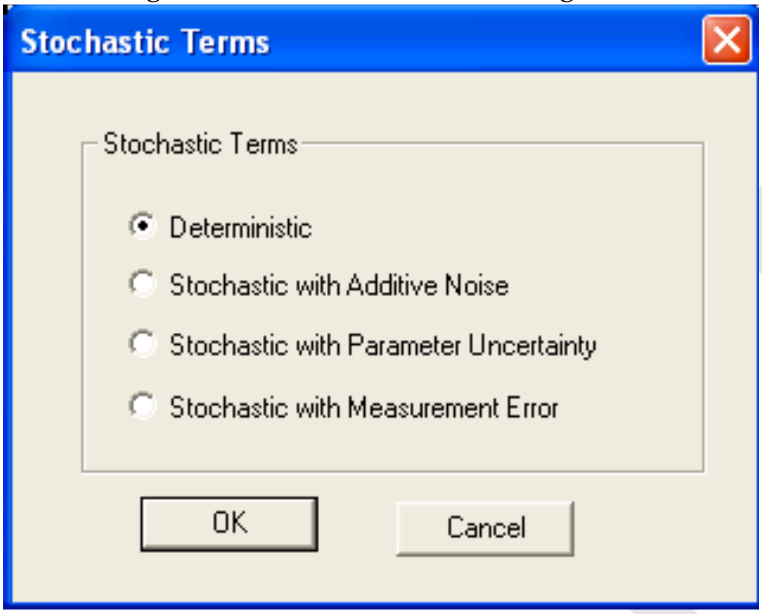
Figure 3: Stochastic terms dialog box.



Convergence Tolerance to $1.6E-12$, that is, a very small number in exponential notation. Thus, if the sum of the squared differences among all the control variables in all time periods in one iteration and the previous iteration is less than the convergence tolerance number, then the iterations are halted and convergence is declared. If convergence is not achieved once the iteration limit is reached, an error message is displayed.

Such a small convergence tolerance number is necessary to perform simulation experiments in which we force the controls to follow given paths, thus allowing them to experience only very minor changes from period to period. Therefore, since Duali computes convergence over changes in the controls, and given that we allow only minor changes in them, we need to impose a very small convergence tolerance number to be able to run simulation experiments. Such a small number is not be mandatory in the optimal policy experiments to be introduced later in this chapter.

The Data:Acronyms menu option contains the assignment of labels to the model variables and time periods. The Data:System Equations section contains the numerical values for matrices A_0, A_1, B_1, C_1, $\hat{D}_1$, $\hat{D}_2$, and $\hat{D}_3$. The Data:Criterion section contains the values for the W and Λ weighting matrices and the desired path values for state and control variables. We see that the weights on the controls in the Λ matrix are set to 99, a relatively large value, while the weights on

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Figure 4: Form specifications dialog box.

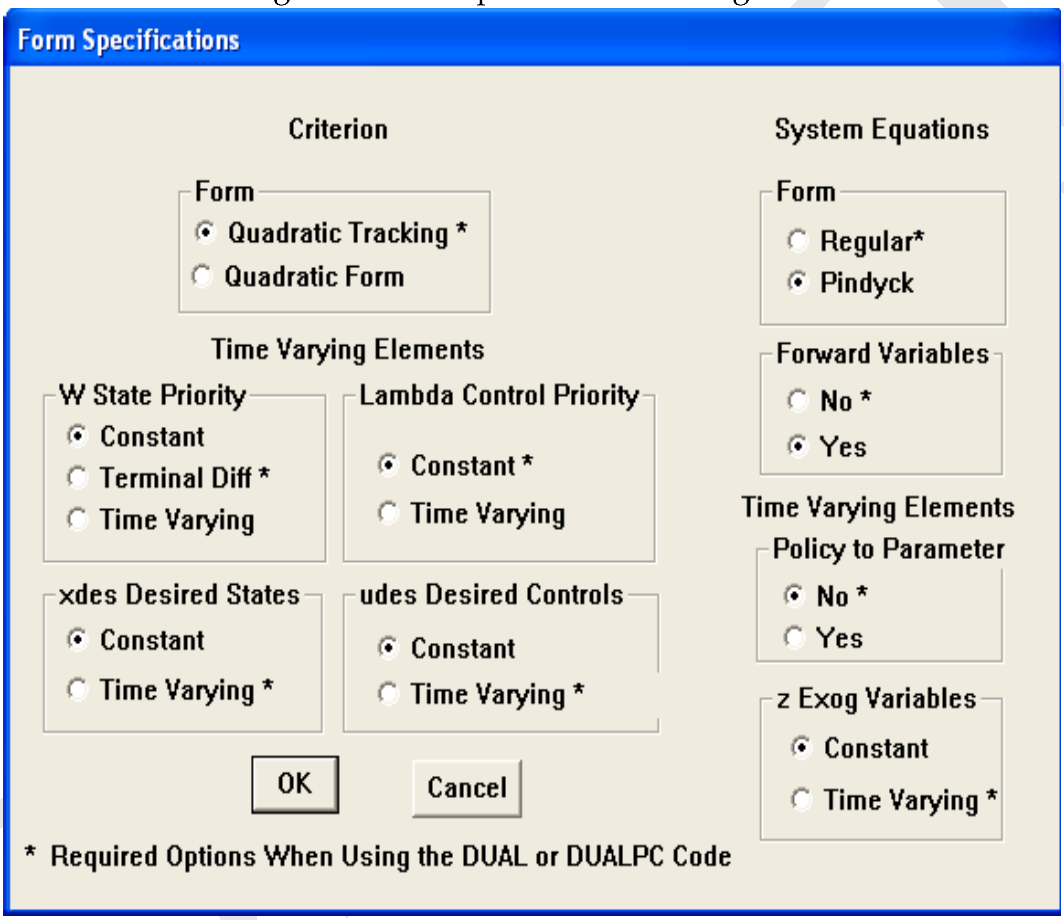

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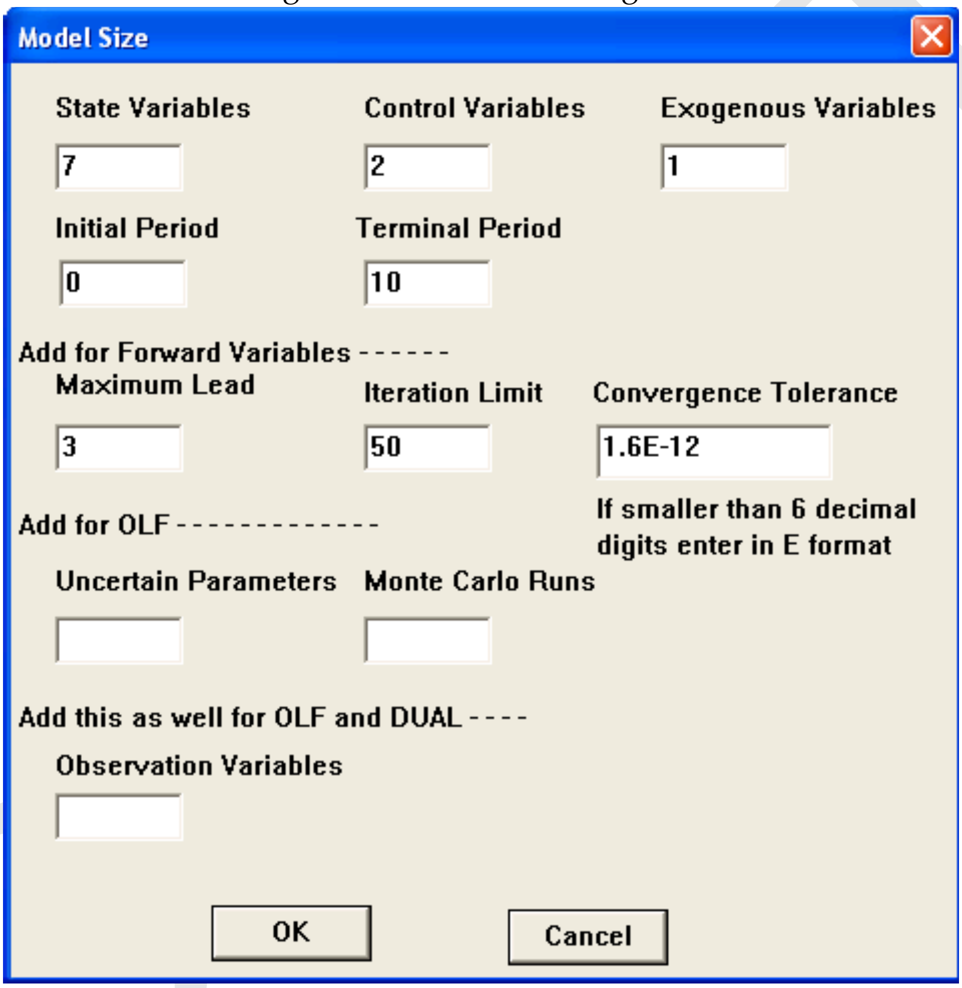
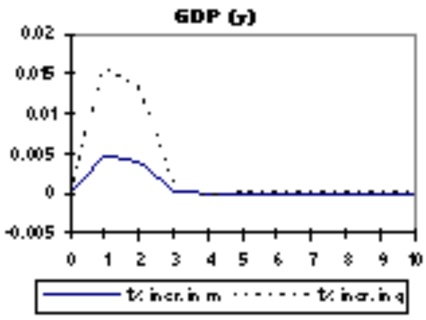
Figure 5: Model size dialog box.



the states in the W matrix are set to 1, a relatively small number, and the desired paths for the controls are set equal to the policy change to be introduced, that is, to 0.01 for the experiment to follow. In this way we force the system to respond to the prespecified changes in the policy variables. Finally, choosing the menu option `Solve:QLP` the problem is solved and the numerical results are displayed automatically. The menu option Results allows us to define different display, plotting, and printing options. Moreover, for this and the other experiments in this chapter, it is convenient to set the display of results to four decimals. This can be done in the `Preferences:Results` menu option, choosing the corresponding value in the Format section.

Figure 6 shows the results of two experiments: a 1 percent unanticipated permanent increase in the money supply (m) and a 1 percent unanticipated permanent increase in government expenditure (g). That is, m and g increase by 0.01 in the first period of each of the two experiments and are kept at the new values from the second period onward. The horizontal axes show the time periods, and the vertical axes correspond to percent deviations from steady-state values for y and p and show percent points for i and r.[2] Thus a value of 0.01 in the GDP graph means that GDP goes from \$600 to \$606 billion, while a value of 0.01 in the nominal interest rate graph means that that rate changes from 5 to 6 percent.[3]

Figure 6: GDP (y).



Here is how John Taylor explains the observed behavior of the model for the two experiments:

[2]Remember that in Taylor's model, y and p are in logs, which is equivalent to percent deviations from steady state, whereas i and r are not.

[3]Taylor (1993, Chapter 3), presents graphs conveying the same information as the ones we show here. However, he presents the results in levels.

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Figure 7: Price level (p).

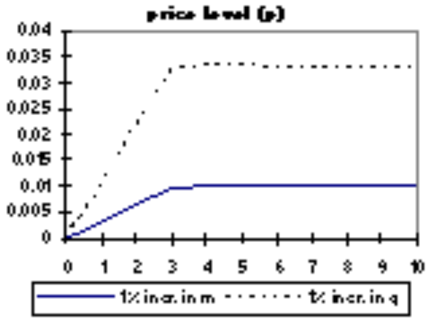


Figure 8: Nominal interest rate (i).

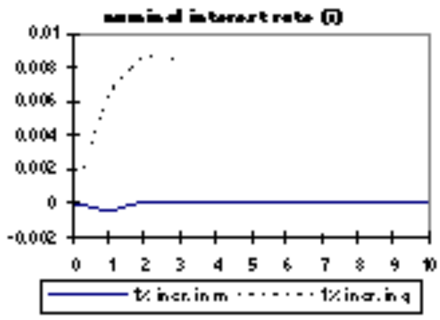
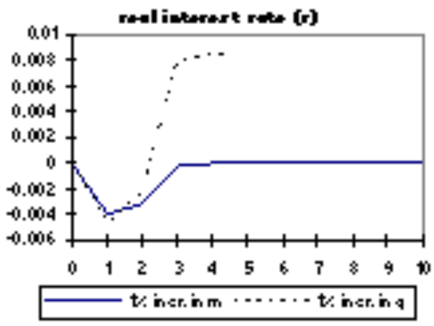


Figure 9: Real interest rate (r).



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Monetary policy has an expected positive effect on output that dies out as prices rise and real-money balances fall back to where they were at the start. Note that the real interest rate drops more than the nominal rate because of the increase in expected inflation that occurs at the time of the monetary stimulus. For this set of parameters the nominal interest hardly drops at all; all the effect of monetary policy shows up in the real interest rate. Fiscal policy creates a similar dynamic pattern for real output and for the price level. Note, however, that there is a surprising *crowding-in* effect of fiscal policy in the short run as the increase in the expectation of inflation causes a drop in the real interest rate. Eventually the expected rate of inflation declines and the real interest rate rises; in the long run, private spending is completely crowded out by government spending.[4]

6 Optimal Policy Analysis

We now apply optimal control techniques to Taylor's model. The problem is to find the optimal paths for the policy variables given desired paths for the target variables, and it can be stated in the same form as was done at the beginning of Section **??**. We assume that the policy goal is to stabilize y, p, i, and r around steady-state values (i.e., around zero). We put high and equal weights on stabilizing y and p, lower and equal weights on i and r, and even lower weights on the policy variables m and g. The corresponding weighting matrices, which follow, remain constant through time:

$$W = \begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 100 & & & & \\ & & & 100 & & & \\ & & & & 50 & & \\ & & & & & 50 & \\ & & & & & & 0 \end{bmatrix} \tag{37}$$

$$\Lambda = \begin{bmatrix} 25 & \\ & 25 \end{bmatrix} \tag{38}$$

To perform a deterministic experiment, we assume that the economy is going through a recession engendered by a temporary adverse shock to y that brings it to 4 percent below its steady-state value. What would be, in this situation, the optimal paths for m and g? What would be the optimal paths for the

[4]Taylor (1993, Page 25).

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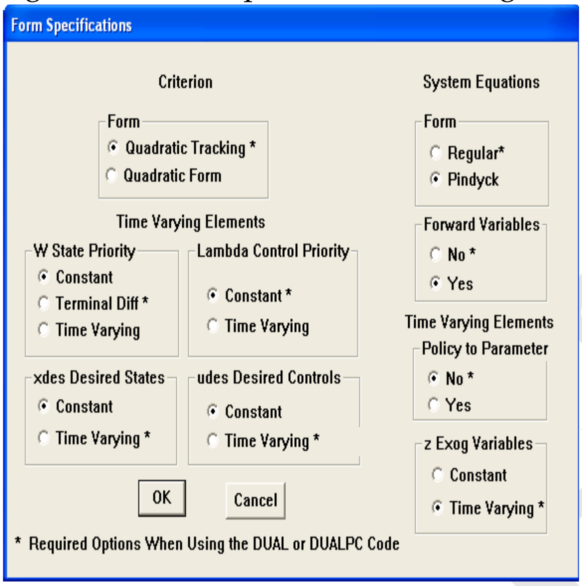
Figure 10: Form specifications dialog box.



state variables as compared with the autonomous response of the system?

To perform this experiment in Duali, we use the file tay-qlp.dui, which is essentially the same as the one used in the previous section, with some modifications. In the `Data:Criterion` section we see that the values of the W and Λ weighting matrices are now set as in equations (37)-(38), while the desired paths for the controls are set to zero.

In order to implement the shock to y in the first simulation period we have to introduce an artificial time-varying exogenous variable. That is, the shock is defined as a first-period change in an arbitrary exogenous variable affecting only the state variable y. To do this, in the `Specifications:Functional Forms` option, in the `z Exog Variables` section the option `Time Varying` is now selected, as shown in the dialog box in Figure 10. Then, in the `Data:System` Equations option we set the fourth element of the matrixC_1 equal to 1 and set the first element of the exogenous variable z equal to –0.04, while all the remaining elements are set to zero, as shown in Figures 10-12.

Note that this procedure is different from the one we used to implement an analogous shock in Chapter **??**. There, we applied the shock to the initial value of the shocked variable, that is, we defined the shock in the Duali option `Data:System Equations:x0`. We cannot do that here, since the variable of interest

Figure 11: C_1 matric input window.

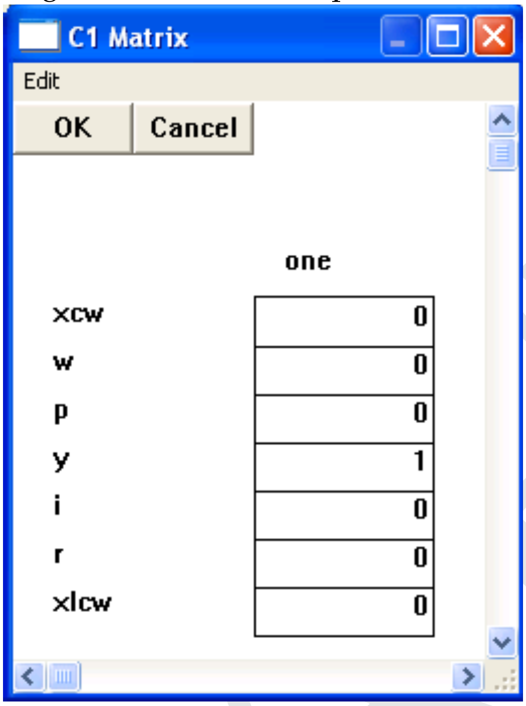


Figure 12: z_t elements input window.

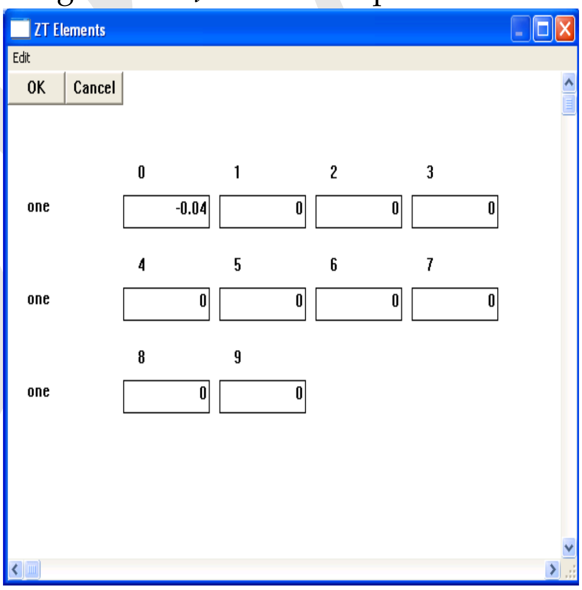


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Figure 13: Gross Domestic Product

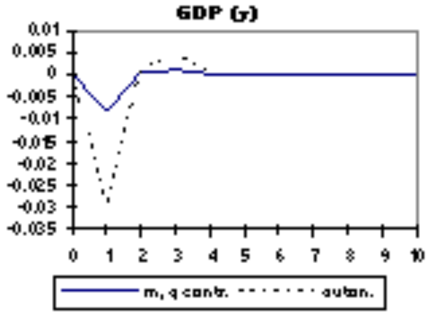
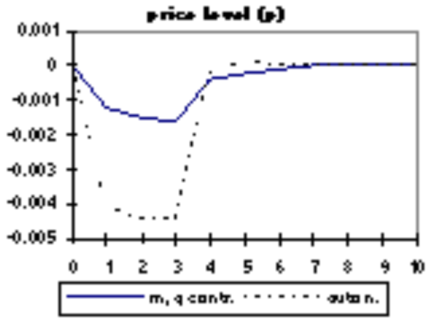


Figure 14: Price Level



(y) does not appear with lagged values in Taylor's model.[5]

Finally, we solve the problem choosing the menu option Solve:QLP. The graphs in Figures 13-17 show the autonomous response of the system to a –0.04 unanticipated transitory shock to y, and the behavior obtained when applying deterministic optimal control (QLP) to face the same shock, that is, when actively using m and g as controls.

We can observe how the behavior of the state variables under the optimal control solution outperforms substantially the autonomous response of the system, reducing the cost of getting the economy out of the recession. In order to generate that behavior, as can be seen in the policy variables graph, the optimal policy mix relies on a 2.5 percent transitory expansion in government expenditure during the first period, and at the same time it also requires a small 0.5 percent transitory increase of the money supply.

[5]We could use the option `System Equations-x0` if, instead of shocking the variable y, we decide to shock the contract wage, since the contract wage is the only variable with lagged values in Taylor's model.

Figure 15: Nominal Interest Rate

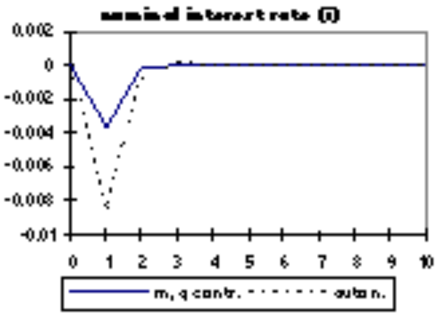


Figure 16: Real Interest Rate

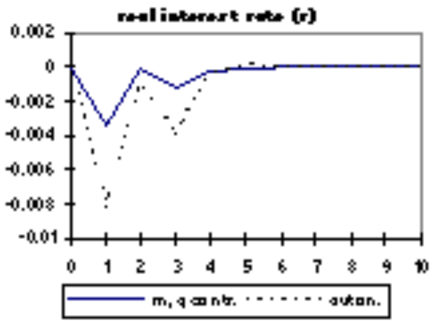
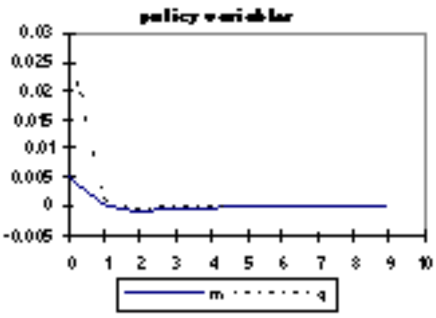


Figure 17: Policy variable



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It may be surprising to find such a positive active policy role in the presence of rational expectations since that specification is sometimes identified with the idea of policy ineffectiveness. However, we have to remember that Taylor's model contains a built-in rigidity—a staggered contracts mechanism—that breaks down the ineffectiveness of policy in the short run.[6]

More generally, rational expectations tend to increase the degree of controllability of an economic system, unless the particular structure and/or parameter values of the model imply a complete neutralization of the effects of policy variables.[7] Indeed, not only can the policy maker influence the economy through past and current controls, but he can also affect the economic system through the preannouncement of future control values. However, for these announcements to have a positive effect on economic performance, they have to be credible, that is, the policy maker has to be committed to carrying them out.[8] These issues have led some researchers to focus their policy analyses on the evaluation of alternative rules that policy makers are presumed to follow. Two of the most influential researchers engaged in this type of work are John Taylor and Michael Woodford.[9]

For example, using the Taylor model, we may be interested in evaluating the performance of a monetary policy rule in which the monetary authority, having as an implicit target the stabilization of the price level, changes the money stock in an inverse proportion to the changes in the price level. In formal terms, a simple rule of that type can be written as

$$m_t = a_r p_t \tag{39}$$

where

$$\begin{aligned} m &= \text{money stock} \\ p &= \text{price level} \\ a_r &= \text{constant coefficient} \end{aligned}$$

[6] To learn about the role of nominal and real *rigidities* in macroeconomic models, see Blanchard and Fischer (1989).

[7] See Holly and Hughes-Hallett (1989, Chapter 7)

[8] Lack of credibility may lead to problems of *time inconsistency*. See Holly and Hughes-Hallett (1989, Chapter 8); and Blanchard and Fischer (1989, Chapter 11). For an appraisal of the practical importance of this issue, see Blinder (1997).

[9] See Taylor (1999) and Woodford (2003).

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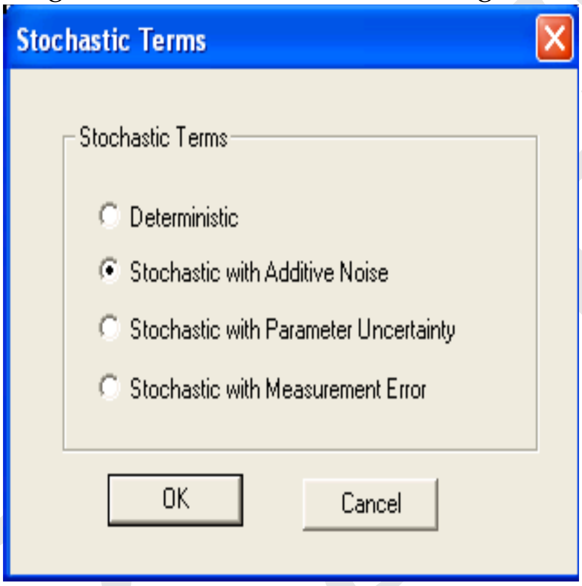
$a_r < 0$ is called the feedback gain coefficient in control theory. Our goal is to evaluate how the variance of the price level changes as the absolute value of the a_r coefficient increases (i.e., as the monetary authority responds more strongly to changes in the price level) when the model is shocked by an additive noise.

To perform these experiments in Duali we use the file `tay-hcfr.dui`. In the `Specification:Stochastic Terms` option, the problem is defined as stochastic with additive noise, as shown in Figure 18.

Figure 18: Stochastic terms dialog box.



In a similar fashion as in the previous QLP, in the `Specifications:Functional Forms` option, in the `z Exog Variables` section, the option Time Varying is selected. But here we have to do so in order to be able to define the source of random terms. Then, in the Specification:Source of Random Terms option, the Generate Internally option is selected as shown in Figure 19, indicating that Duali's random numbers generator is used to generate the shocks. Moreover, in the Noise Terms for All Periods section, the System Equations option is selected, indicating that the shocks are applied only to the system equations.

In the `Specification:Options Monte Carlo` option, we can select the starting period for the calculation of the variance of state and control variables over time. As shown in the dialog box in Figure 20, we selected period zero as the starting period. In the `Data:Size` option, we now have to specify the number of Monte Carlo runs. As shown in Figure 21, we chose 1000.

The next step is to define the variance of the shocks to be applied to the model during the Monte Carlo runs. We perform experiments in which the

Figure 19: Sources of random terms dialog box.

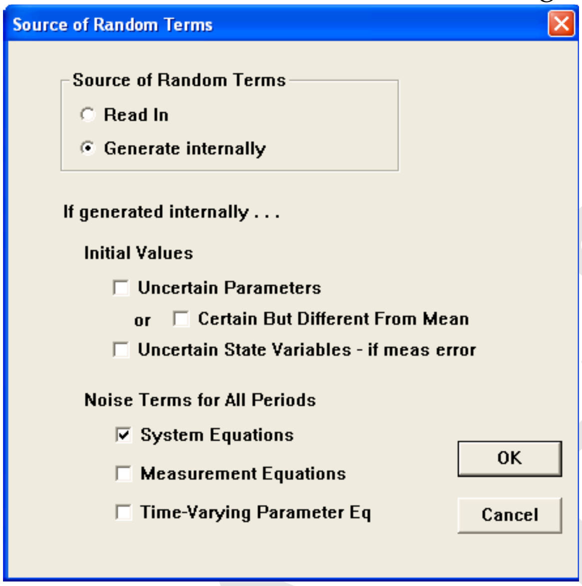


Figure 20: Monte Carlo options dialog box.

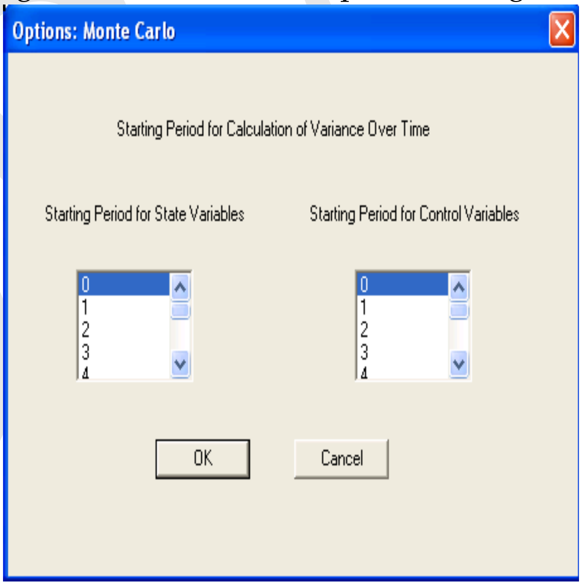
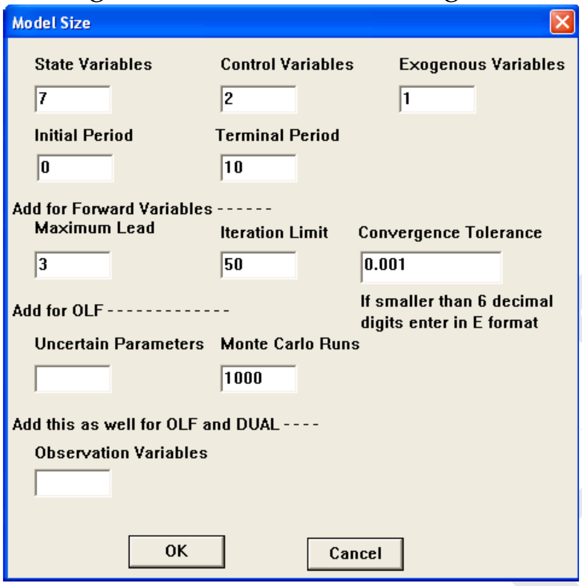


Figure 21: Model size dialog box



shocks are applied only to the contract wage equation. To do so, as shown in the dialog boxes in Figure 22, we first select, in the Data:Additive Noise Terms option, the Q, `Additive Noise Covariance` option. This selection triggers the display of the Q covariance matrix of the additive noise terms. There we assign the value 0.1 to the diagonal element corresponding to the contract wage variable x^{cw}. All the other values should be zeroes. However, having zeroes in the diagonal of the Q matrix causes difficulties when Duali tries to find its inverse during the solution of the problem. Thus, we assign very small values (0.00001) to the remaining diagonal elements.

Having defined the stochastic structure of the simulations, we now have to define and assign values to the feedback rule to be evaluated. The mathematical form of this rule was given earlier in equation (**??**) of Chapter **??** as

$$u_k = G_k x_k + g_k$$

Thus to modify the feedback rule we have to change the elements in either the feedback gain matrix G or the feedback gain vector g. To do so, we select the Data:Handcrafted Feedback Rule option. As shown in the dialog box in Figure 24, we select the capital G option, which is the feedback gain matrix of the rule to be applied in the experiments. We leave the small g option blank, since it corresponds to a vector of constant terms that is absent from the specific rule we evaluate as defined in equation (31), that is,

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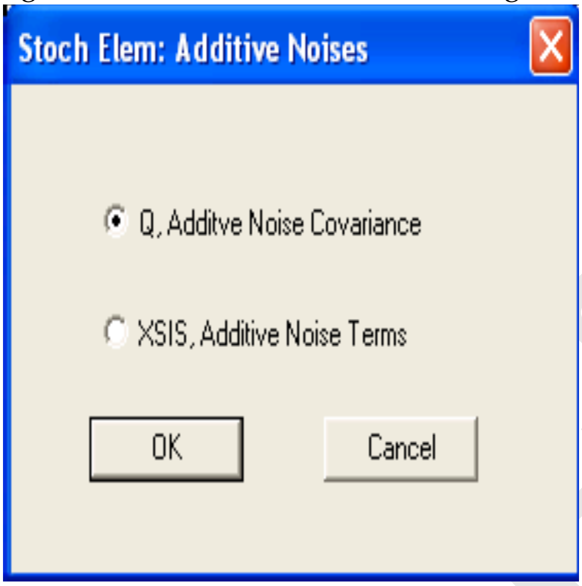
Figure 22: Additive noise term dialog box.



$$m_t = a_r p_t \tag{40}$$

When making the selection of G, the corresponding window is displayed as shown in Figure 24. We see that the value –0.1 is the only one assigned, and it corresponds to the value of the a_r coefficient in equation (31). We also see that more complex rules could be easily defined by assigning values to other cells in the matrix.

Having defined the stochastic structure of the experiments to be performed, and the rule to be evaluated, we are now ready to move on to the selection of the solution method and the storage and display of results. We first select the `Solve:Compare Print` option and obtain a dialog box that displays several solution methods. We could select some or all of them in case we want to perform experiments comparing their relative performance. Since that is not our goal here, we just select the `HFCR, Handcrafted Feedback Rule` option as shown in Figure 26.

When doing so, we are asked to provide a debug file name; we could use, for example, the name `tay-hcfr.dbg`. This file contains the simulation results. After providing the file name, a dialog box containing many options related to the generation of results is displayed, as shown in the dialog box in Figure 27. Given the nature of our experiment, we keep all the options blank except two. In the Averages section, we select the Average Average over Monte Carlo Runs and the Average Variance over Monte Carlo Runs options.

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Figure 23: Q matrix input window.

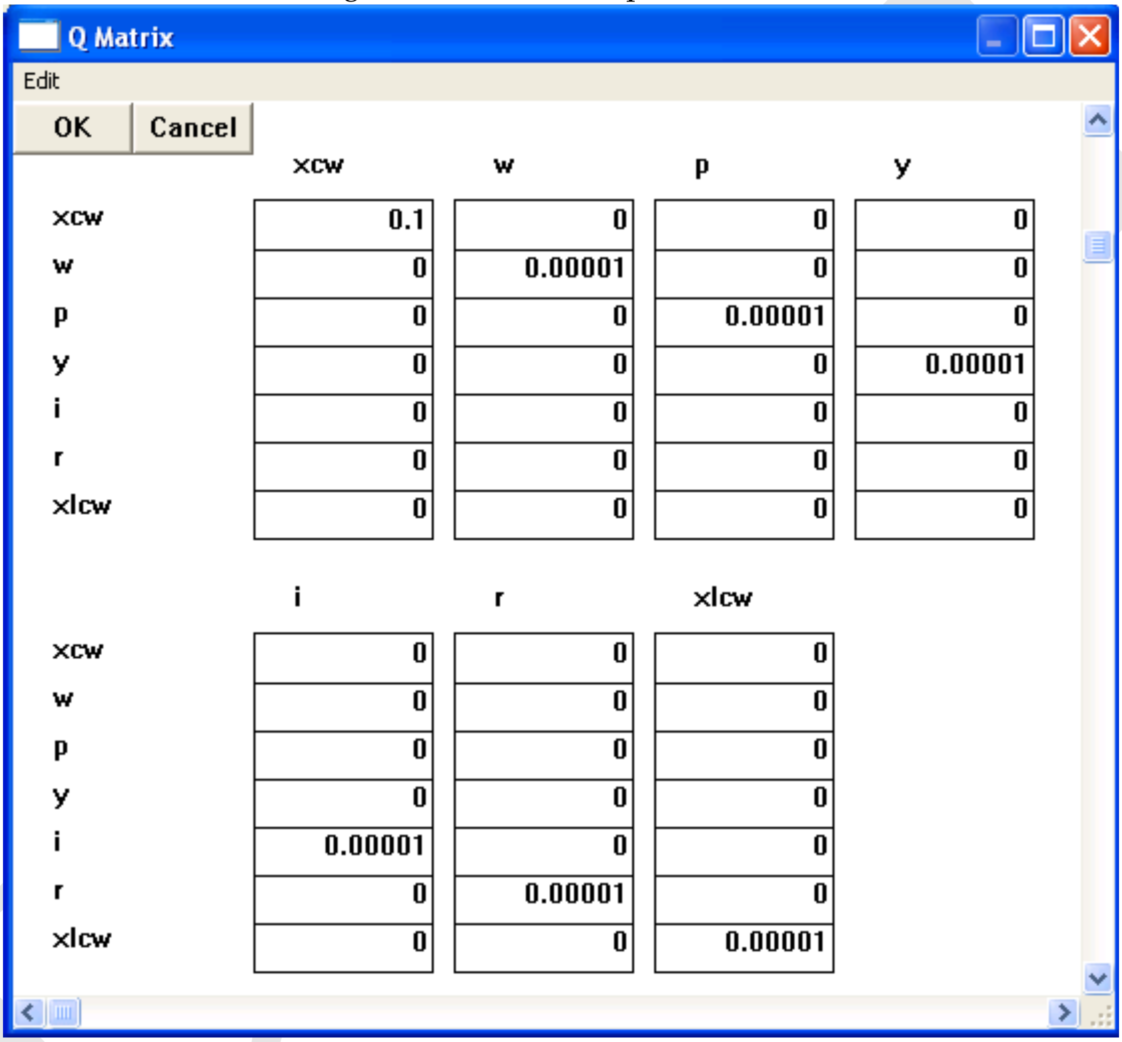


Figure 24: G and g dialog box.

Figure 25: Feedback gain matrix G input window.

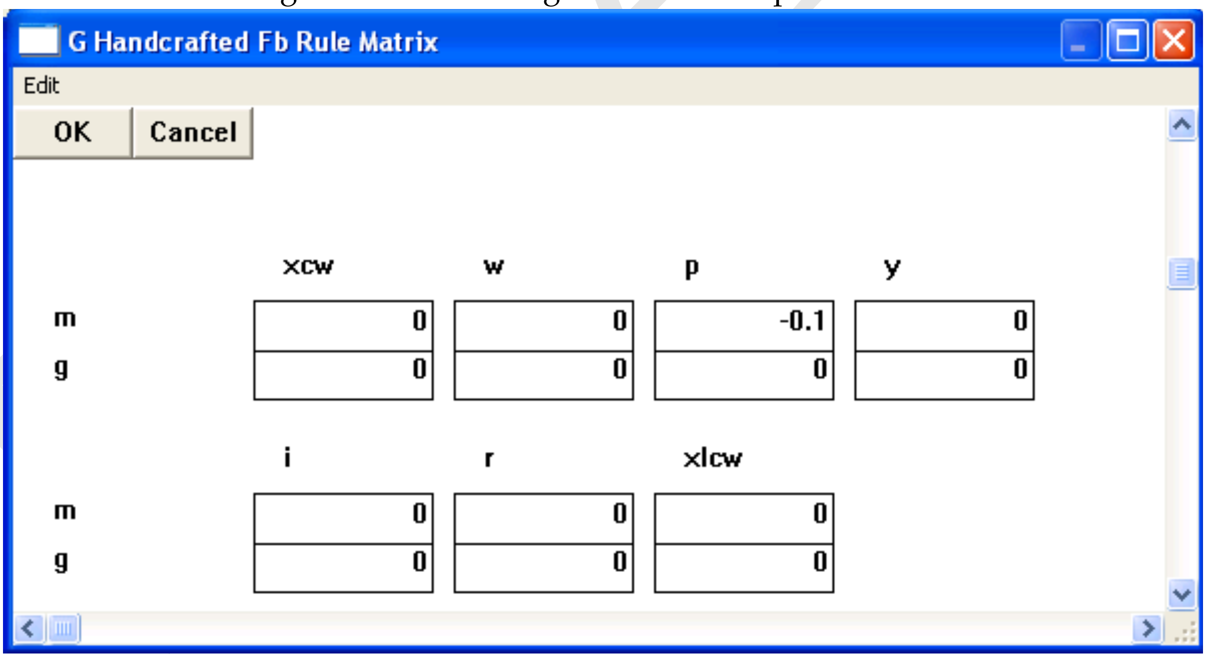
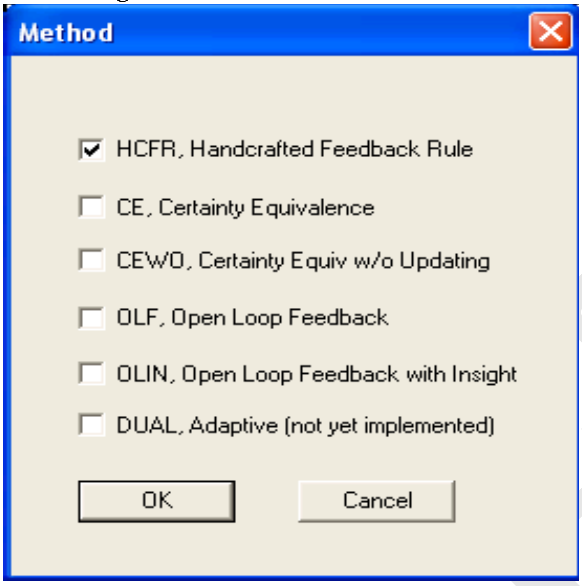


Figure 26: Method dialo box.



We are then ready to perform our experiment. Once we click the OK button, the Monte Carlo runs begin. Since we are performing 1000 runs, it may take awhile before results are displayed. Two dialog boxes like the ones shown in Figure 28 appear while Duali is running (one after the other). We just dismiss them by clicking the `OK` button, since they display results corresponding to experiments with cross comparison of methods, something we are not interested in here.

We exit from Duali and then open the debug file, `tay-hcfr.dbg`, as we named it earlier, with an editor. Since we have performed a large number of Monte Carlo runs, the output is quite large as it displays some basic results corresponding to each run. Moving down to the end of the output, our results of interest are just the following:

```
AvgVarXsTimeHcfr
0.0924 0.0168 0.0168 0.0129 0.0003

AvgVarXsTimeHcfr
0.0089 0.0825

...

AvgVarUsTimeHcfr
0.0002 0.0000
```

`AvgVarXsTimeHcfr` means the average variance of the state variables across time for the handcrafted feedback rule solution method. We see that there are

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Figure 27: Debug print dialog box.

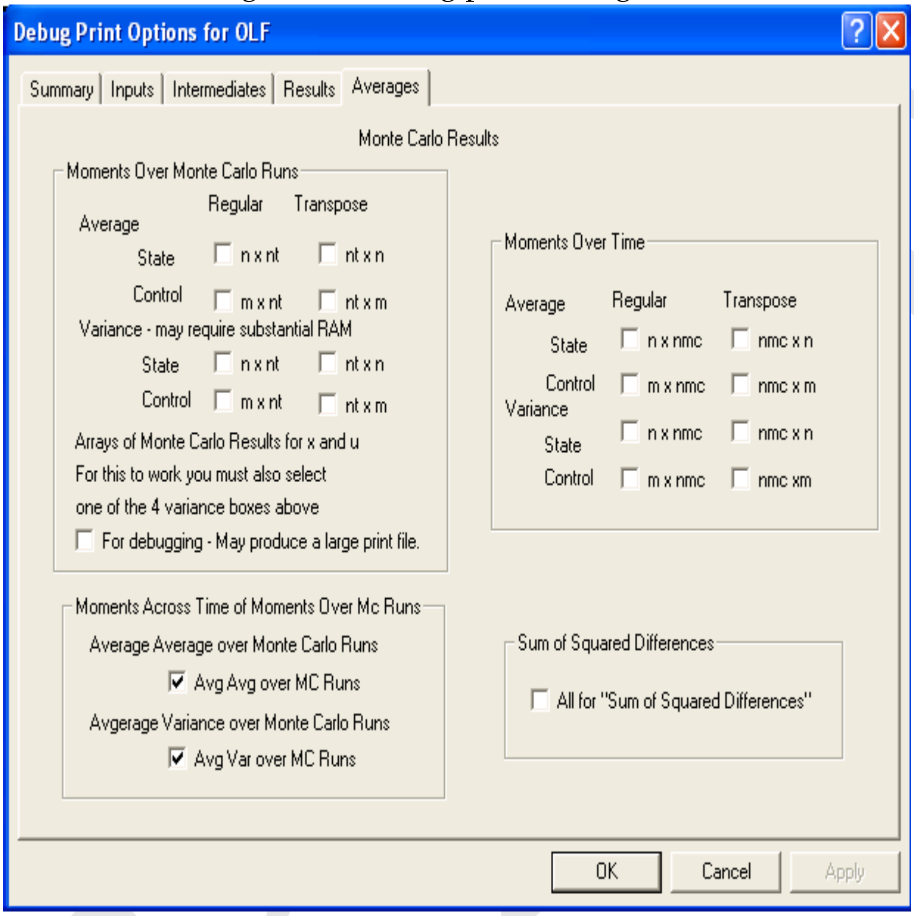


Figure 28: Method count window.

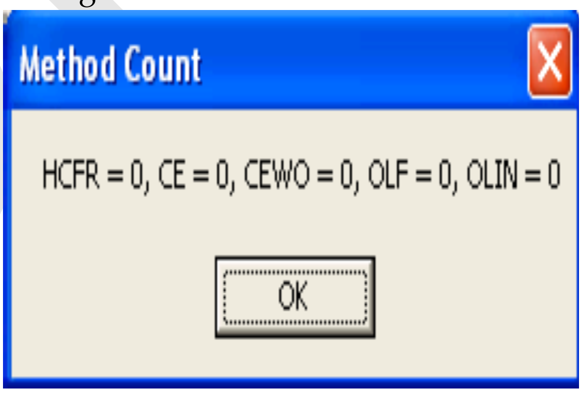
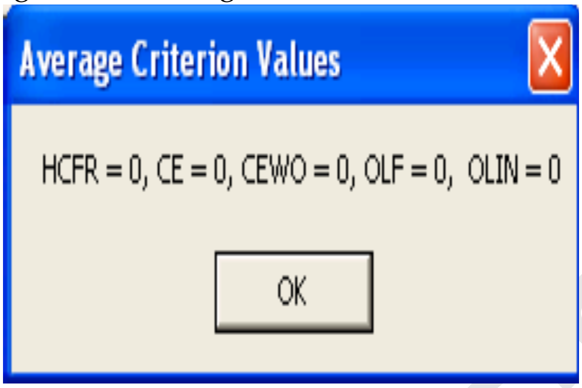


Figure 29: Average criterion value window.



seven results, each one corresponding to an element of the transpose of the state variable vector as defined earlier in equation (8), that is,

$$x'_t = [\ x_t^{cw} \quad w_t \quad p_t \quad y_t \quad i_t \quad r_t \quad xl_t^{cw}] \tag{41}$$

Thus, the result we are interested in is the third one to the right in the first row (0.0168) since it corresponds to the average variance of the price level variable. AvgVarUsTimeHcfr contains the results corresponding to the control variables, so the first one (0.0002), the one corresponding to the variance of the lagged money supply stock, is the one of interest to us.

These results considered by themselves are not very informative. However, we can repeat the experiment for different values of the a_r coefficient to obtain a comparative performance. Table 6 shows the results of ten experiments.

We can see that as the absolute value of the feedback gain coefficient a_r increases, the variance of the price level tends to decrease, whereas the variance of the money stock increases. That is, a stronger response of the monetary authority to changes in the price level reduces the variance of that variable but at the cost of an increased variance of the policy tool. A natural question to be asked is what would be the optimal rule in this case, that is, the optimal level of the feedback gain coefficient. If the only concern is the variance of the price level, the response is easy: it is the highest possible absolute value. However, if the variance of the money stock is also a concern, relative priorities should be explicit.

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Table 1: Comparative Rules Experiments

a_r	Variance of p	Variance of m
–0.1	0.0168	0.0002
–0.2	0.0156	0.0006
–0.3	0.0158	0.0014
–0.4	0.0157	0.0024
–0.5	0.0155	0.0038
–0.6	0.0150	0.0053
–0.7	0.0154	0.0072
–0.8	0.0149	0.0093
–0.9	0.0145	0.0113
–1	0.0143	0.0138

7 Experiments

As a first and relatively simple experiment, you can perform optimal policy experiments like the one shown in Figures 13-17, changing the priorities on state and control variables. Then, you can also change the nature of the initial shock.

Alternatively, you can specify different handcrafted feedback rules to perform experiments like the one illustrated in Figure ??. For example, you might want to specify a rule in which the money supply is a function of output instead of the price level or design more complex rules, with money supply and government spending as controls and one or more state variables as target variables.

Finally, you might also define a rule in which the real interest rate—instead of the money supply (as was the case in the experiment presented here)—is used to respond to changes in prices. This type of rule is typically used by many researchers—for example, Taylor (1999) and Woodford (2003)—to discuss monetary policy rules in the United States. To do so, note that you have to redefine the interest rate as a control variable and the money supply as a state variable, because when the interest rate is used as a control, the money supply becomes an endogenous variable. Since this is a substantial change in the model structure, it may require you to start from scratch in order to input the new model in Duali.

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8 Further Reading

The prototype Taylor model presented in this chapter, together with its U.S. and multicountry extended econometric versions are developed in Taylor (1993). Holly and Hughes-Hallett (1989, Chapter 7) provide an introduction to the application of optimal control techniques to rational expectations models. Amman and Kendrick (1996, 1999, 2000, 2003) develop optimal control techniques and applications for a variety of rational expectations models. Taylor (1999) and Woodford (2003) provide a broad treatment of the application of policy rules to rational expectations models. For a useful starting point for coming abreast of recent work on the variety of optimizing trend-deviation macroeconomic models, see Kozicki and Tinsley (Kozicki and Tinsley (2002)).

For discussion of the robust control approach to stochastic control, see Deissenberg (1987), Rustem (1992), Hansen and Sargent ((2007), and Rustem and Howe (2002)

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