# COMPUTATIONAL ECONOMICS

**Revised Edition**

**Ramsey Growth Model in Excel**

**David A. Kendrick**
**Ruben P. Mercado**
**Hans M. Amman**

# 1   Introduction

Most economists are familiar with the spreadsheet and even with the database capabilities of the Excel software, but fewer are aware that Excel also contains powerful procedures for solving both linear and nonlinear programming problems. As the Excel interface is such a familiar one and the specification of programming problems in Excel is relatively straightforward, there are times when it is the software of choice for solving certain types of optimization problems. In particular, when the models are small enough that the set-driven nature of GAMS does not give it an advantage over Excel, it may be advantageous to solve optimization problems in Excel. To illustrate this we use a one-sector growth model of the type that abounds in the economics literature—the famous Ramsey model of economic growth, Ramsey (1928). In particular, we follow the versions developed by Chakravarty (1962) and Taylor and Uhlig (1990). We employ a finite horizon version with a terminal capital stock constraint.

The model is first introduced in a mathematical form and then in a computational form.[1] The essential economics of the simple growth model used in this chapter is a trade-off between consumption and investment. More consumption in a time period means more utility in that time period but less investment and therefore less capital stock and less production in future time periods. Thus the key elements of the model are the production function with capital being used to produce output, the capital accumulation relationship with investment creating new capital, and the utility function with consumption resulting in utility.

# 2   Mathematical Form

The production side of the economy is specified in a stylized form by means of an aggregate production function

$$Y_t = \theta K_t^\alpha \tag{1}$$

where

---

[1]Most models used in this book cannot be solved analytically, so numerical methods are required. However, even when analytical solutions can be obtained, as shown later in Appendix *A*, page 22, it is still useful to get numerical solutions so that the code can be checked on simple models. Then the numerical methods can be used with more confidence when they are applied to more complex models that cannot be solved analytically.

$$
\begin{aligned}
Y_t &= \text{output in period } t \\
\theta &= \text{a technology parameter} \\
K_t &= \text{the capital stock in period } t \\
\alpha &= \text{exponent of capital in the production function}
\end{aligned}
$$

This is the widely used Cobb-Douglas form of a production function except that Cobb-Douglas usually includes both capital and labor inputs. However, for the sake of simplicity, the production function in this model includes only capital. Consider next the capital accumulation constraint

$$K_{t+1} = K_t + Y_t - C_t \tag{2}$$

where

$$C_t = \text{consumption in period } t$$

which says that the capital stock next period will be the same as this period plus the difference between output and consumption, which is saving or investment. For the moment, depreciation of the capital stock is ignored though you might want to add that to the model in an experiment. The production function (1) can be substituted into the capital accumulation equation (2) to obtain the equation

$$K_{t+1} = K_t + \theta K_t^\alpha - C_t \tag{3}$$

In addition, the model has an initial condition that specifies the size of the capital stock in the initial period,

$$K_0 \text{ given} \tag{4}$$

and includes a terminal condition that fixes a minimum amount of capital that must be left to the next generation after the time horizon covered by the model,

$$K_N \geqslant K^* \tag{5}$$

where

$$
\begin{aligned}
K^* &= \text{a lower bound on the amount of capital} \\
&\quad \text{required in the terminal period } N
\end{aligned}
$$

Finally, the model has a criterion function that is the discounted value of the utility that is obtained from consumption over all the covered periods, which is written in two steps. First the utility in each period is defined as

$$U\left(C_t\right) = \frac{1}{(1-\tau)} C_t^{(1-\tau)} \tag{6}$$

where

$$
\begin{aligned}
U\left(C_t\right) &= \text{the utility in period t as a function } C_t \\
\tau &= \text{a parameter in the utility function}
\end{aligned}
$$

This is a popular form of the utility function that is known as the *constant elasticity of intertemporal substitution* function. Roughly speaking, think of the elasticity of intertemporal substitution as measuring the degree of substitutability between consumption *today* and *tomorrow* or, in geometric terms, measuring the curvature of the indifference curves corresponding to consumption at any two points in time. For this function, the elasticity of substitution is constant and equal to $1/\tau$.

The sum of the discounted utilities is specified as

$$J = \sum_{t=0}^{N-1} \beta^t \, U\left(C_t\right) \tag{7}$$

where

$$
\begin{aligned}
J &= \text{the criterion value} \\
\beta &= \text{the discount factor}
\end{aligned}
$$

The discount factor is defined as

$$\beta = \frac{1}{1+\rho}$$

where $\rho$ is the discount rate. Substitution of equation (6) into equation (7) yields the criterion function

$$J = \sum_{t=0}^{N-1} \beta^t \frac{1}{(1-\tau)} C_t^{(1-\tau)} \tag{8}$$

In summary, the model consists of the criterion function (8), the capital accumulation equation (3), and the initial and terminal conditions (4) and (5), and can be stated as find $(C_0, C_1, \cdots, C_{N-1})$ to maximize

$$J = \sum_{t=0}^{N-1} \beta^t \frac{1}{(1-\tau)} C_t^{(1-\tau)}$$

subject to

$$K_{t+1} = K_t + \theta \, K_t^\alpha - C_t \tag{9}$$

$$K_0 \quad \text{given} \tag{10}$$

$$K_N \geqslant K^* \tag{11}$$

The essential problem, then, is to choose those levels of consumption over the time periods covered by the model that strike the right balance between consumption and investment. Lower consumption in any given period means less utility in that period but more savings and therefore larger capital stocks and more production in future years. This growth model is a nonlinear programming problem because of the nonlinearities in the criterion function (8) and the capital accumulation equation (3). It can be stated and solved rather nicely in Excel, as is discussed in the next section.

## 3  Computational Form

Consider first a spreadsheet layout of the model as shown in Figure 1. The corresponding Excel file is in the book web page. Note first that the model horizon covers time periods numbered from zero through nine so that zero represents the initial period and nine the terminal period. The rows below the time periods display the consumption, $C_t$; production, $Y_t$; capital stock, $K_t$; and utility, $U(C_t)$ in each time period. All of these values are calculated when the model is solved and we show shortly how the calculations are structured. However, for now look only at the cell below the Total label, that is, cell L12, which is highlighted in the bottom right-hand corner of the spreadsheet. It contains the value *9.97*; however, we are not so much interested in that value as in how it is obtained. Look at the formula bar at the top of the spreadsheet, which contains the expression

```
SUM (B12:J12)
```

Figure 1: Dynamic Programming



This indicates that this cell contains the sum of the utility values for periods zero through eight that are contained in cells B12 through J12.

Actually, the value in each of the cells B12 through J12 is not, strictly speaking, the utility but rather the *discounted* utility for each period. This is illustrated in Figure 2. The cell D12 in the utility row is highlighted and the expression that is used to calculate the value in that cell is displayed in the formula bar as

```
= beta^D4*(1/(1-tau))*D5^(1-tau)
```

This is complicated so let us consider it one piece at a time. Begin with

```
beta^D4
```

which means that beta is raised to the power of the number in cell D4. This makes use of the *naming* capability for constants in Excel and is equivalent to B17ˆD4. The number in cell D4 is *2* so this term becomes $\beta^2$, which is the discount factor squared. $\beta$ is defined in line 17 of the spreadsheet as *0.98*. Moreover, since

Figure 2: Discounted utility



The formula bar shows: `D12` `=beta^D4*(1/(1-tau))*D5^(1-tau)`

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Growth Model | | | | | | | | | | | |
| 2 | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | |
| 4 | Time Period | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| 5 | Consumption | 0.347 | 0.351 | 0.355 | 0.358 | 0.361 | 0.364 | 0.366 | 0.368 | 0.370 | | |
| 6 | Production | 0.570 | 0.576 | 0.582 | 0.588 | 0.594 | 0.599 | 0.605 | 0.611 | 0.616 | | |
| 7 | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | Target |
| 9 | Capital | 7.000 | 7.223 | 7.448 | 7.676 | 7.906 | 8.138 | 8.373 | 8.612 | 8.854 | 9.100 | 9.100 |
| 10 | | | | | | | | | | | | |
| 11 | | | | | | | | | | | | Total |
| 12 | Utility | 1.178 | 1.161 | 1.144 | 1.126 | 1.108 | 1.090 | 1.072 | 1.054 | 1.035 | | 9.970 |
| 13 | | | | | | | | | | | | |
| 14 | | | | | | | | | | | | |
| 15 | | | | | | | | | | | | |
| 16 | tau | 0.5 | | | | | | | | | | |
| 17 | beta | 0.98 | | | | | | | | | | |
| 18 | alpha | 0.33 | | | | | | | | | | |
| 19 | theta | 0.3 | | | | | | | | | | |

$$\beta = \frac{1}{1+\rho}$$

we can infer that the discount rate $\rho$ is equal to about 0.02. Next consider the term

`(1/(1-tau))*D5^(1-tau)`

which can be rewritten as

$$\frac{1}{1-\tau}D5^{(1-\tau)}$$

Since the cell D5 contains consumption we can further rewrite this expression as

$$\frac{1}{1-\tau}C_t^{(1-\tau)}$$

8

which is the same as the utility function in equation (6). Thus, cell `D12` contains the mathematics

$$\beta^t \frac{1}{1-\tau} C_t^{(1-\tau)}$$

which is the discounted utility for period $t$. Also, the parameter $\tau$ of the utility function is defined in line 16 of the spreadsheet as being equal to *0.5*.

In summary, line 12 of the spreadsheet is used to calculate the discounted utility in each period and then to sum those values so as to obtain the total discounted utility in cell L12. Thus the criterion function for the model is contained in line 12.

Figure 3: Production expression

| | A | B | C | D | E | F | G | H | I | J | K | L | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Growth Model | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | |
| 4 | Time Period | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | | |
| 5 | Consumption | 0.347 | 0.351 | 0.355 | 0.358 | 0.361 | 0.364 | 0.366 | 0.368 | 0.370 | | | |
| 6 | Production | 0.570 | 0.576 | 0.582 | 0.588 | 0.594 | 0.599 | 0.605 | 0.611 | 0.616 | | | |
| 7 | | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | Target | |
| 9 | Capital | 7.000 | 7.223 | 7.448 | 7.676 | 7.906 | 8.138 | 8.373 | 8.612 | 8.854 | 9.100 | 9.100 | |
| 10 | | | | | | | | | | | | | |
| 11 | | | | | | | | | | | | Total | |
| 12 | Utility | 1.178 | 1.161 | 1.144 | 1.126 | 1.108 | 1.090 | 1.072 | 1.054 | 1.035 | | 9.970 | |
| 13 | | | | | | | | | | | | | |
| 14 | | | | | | | | | | | | | |
| 15 | | | | | | | | | | | | | |
| 16 | tau | 0.5 | | | | | | | | | | | |
| 17 | beta | 0.98 | | | | | | | | | | | |
| 18 | alpha | 0.33 | | | | | | | | | | | |
| 19 | theta | 0.3 | | | | | | | | | | | |

Next consider the constraints of the model. Begin with the expression for production, which is illustrated in Figure 4. In this figure cell `D6` is highlighted and the formula bar contains the expression

```
theta*D9^alpha
```

which is the same as equation. (1) for production, that is,

$$Y_t = \theta K_t^\alpha$$

since cell D9 contains the capital stock for period $t$; theta is defined near the bottom of the spreadsheet in line 19 as being equal to *0.3* and alpha is defined in line 18 as being equal to *0.33*.

Figure 4: Capital; formation



| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Growth Model | | | | | | | | | | | |
| 2 | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | |
| 4 | Time Period | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| 5 | Consumption | 0.347 | 0.351 | 0.355 | 0.358 | 0.361 | 0.364 | 0.366 | 0.368 | 0.370 | | |
| 6 | Production | 0.570 | 0.576 | 0.582 | 0.588 | 0.594 | 0.599 | 0.605 | 0.611 | 0.616 | | |
| 7 | | | | | | | | | | | | |
| 8 | | | | | | | | | | | | Target |
| 9 | Capital | 7.000 | 7.223 | 7.448 | 7.676 | 7.906 | 8.138 | 8.373 | 8.612 | 8.854 | 9.100 | 9.100 |
| 10 | | | | | | | | | | | | |
| 11 | | | | | | | | | | | | Total |
| 12 | Utility | 1.178 | 1.161 | 1.144 | 1.126 | 1.108 | 1.090 | 1.072 | 1.054 | 1.035 | | 9.970 |
| 13 | | | | | | | | | | | | |
| 14 | | | | | | | | | | | | |
| 15 | | | | | | | | | | | | |
| 16 | tau | 0.5 | | | | | | | | | | |
| 17 | beta | 0.98 | | | | | | | | | | |
| 18 | alpha | 0.33 | | | | | | | | | | |
| 19 | theta | 0.3 | | | | | | | | | | |

Next consider the expression for the capital accumulation constraint, shown in Figure 5, where cell D9 is highlighted. The expression in the formula bar this time,

```
C9 + theta*C9^alpha - C5
```

contains at its core the expression for production that we just developed, that is,

```
theta*C9^alpha
```

10

Thus we can translate the entire expression as

$$K_{t-1} + Y_t - C_{t-1}$$

since row 9 contains the capital stock figures and row 5 contains the consumption figures. As you can see by comparing the expression above to the capital accumulation constraint in equation (2) with the time periods each decreased by one period, that is,

$$K_t = K_{t-1} + Y_{t-1} - C_{t-1}$$

the timing in the spreadsheet calculations is slightly off for production, but as that slight timing error may make the spread slightly easier to specify we leave the error for the time being.

In addition, note that at the end of row 9 in the spreadsheet there is a target capital stock. We discuss this in detail when we describe how the model is actually solved in Excel. However, before we do that it is necessary to indicate how the initial condition for capital stock is specified. This is shown in Figure (6), where cell B9 is highlighted. When cell B9 is highlighted the fomula bar does not show a mathematical expression like those in the other cells in line 9, but rather just the number 7. This is the initial capital stock that was specified in the mathematical statement of the models in equation (4) as

$$K_0 \qquad \text{given}$$

So the initial capital stock is given and it has been specified as equal to 7 in this version of the model.

The tour of the model in Excel we have just given is slightly confusing because it discusses both the data elements that the user must provide and the variables that are calcuated when the spreadsheet is solved. Now let us separate the two by looking again at Figure 6. The user must supply the time period numbers in row 4, the initial capital stock in cell B9, and the parameter values $\tau$, $\beta$, $\alpha$, and $\theta$ in cells B16 through B19, as well as the terminal capital stock target in cell L9. Excel computes all the rest. Then why are all those other numbers shown in Figure 6? Those other numbers were all computed the last time the model was solved and are updated if you alter one of the inputs just mentioned and then solve the model again.

Let us consider next how the model is solved. This is accomplished by selecting the *Tools menu* and the *Solver option* from that menu. When you do

Figure 5: Capital Constraint expression



Figure 6: Solver window

this the dialog box in Figure 7 appears.[2] Consider first the top line in this box in the section called Set Target Cell. The Edit box to the right of this caption indicates that cell L12 has been chosen, and this corresponds to the total discounted utility on the right-hand side of the utility line in the spreadsheet. Just beneath this the user can specify whether the value in the cell is to be maximized or minimized. In the growth model at hand we are seeking to maximize the total discounted utility so Max is selected.

The next line is used to specify cells to be changed while seaching for the solution to the model. In the growth model we are solving for the values of consumption in each period that provide the best trade-off between utility in that period and saving that becomes future capital stocks and permits more production later. Therefore, we specify here that the variables to be used in searching for the optimum are those in cells B5 to J5, which are the consumption values.

Next consider the box labeled Subject to the Constraints, in which appears the constraint

    K9 > = L9

As cell K9 contains the capital stock for period 9 and cell L9 contains the target capital stock, this constraint requires that the terminal period capital stock computed by the model be greater than or equal to the user-specified target, which in this case is set to *9.1*, that is, *30* percent higher than the intitial capital stock. This corresponds to the mathematical constraint in equation (5, that is,

$$K_N \geqslant K^*$$

where $K_N$ is the capital stock in the terminal period and $K^*$ is the target capital stock.

Note that it is not necessary in the Solver dialog box to specify all of the capital accumulation constraints in line 9 of the spreadsheet as constraints. Rather they are effectively linked together by the mathematical expressions so it is necessary to include only cell K9 when specifying the constraints.

To solve the model one selects the Solve button in the Solver dialog box in Figure 7. What happens behind the scenes in Excel next is the solution of the nonlinear programming model that is represented by the growth model. A *Newton method* or a *conjugate gradient* method can be used in the program to solve the model. A

---

[2]In case the dialog box does not appear, see *Appendix B*, page 25.

brief discussion of nonlinear optimization methods is provided in *Appendix C*, page 26, at the end of the book.

Figure 7: Solver options



Clicking on the `Options button` in the *Solver dialog box* displays the Solver `Options dialog` box shown in Figure 8. In this box you are able to change different parameters—maximum time, number of iterations, precision, tolerance, and convergence—that allow you to control the performance of the nonlinear optimization method used by Excel. Note that the Assume Non-Negative option has been selected to constrain the solution values of the model to nonnegative values.

# 4   Results

When solving the growth model with the Excel Solver it is useful to remember essential trade-off in the model. More consumption today means more utility today. However, less consumption today means more saving and more investment today, which means more capital stock in the future and thus more output and more consumption possibilities in the future.

Therefore, the problem is to find just the right level of consumption in each time period given the parameters of the model. The key parameters are

$$
\begin{aligned}
\beta &= 0.98 & \text{beta discount factor} \\
K^* &= 9.10 & \text{target capital stock} \\
\theta &= 0.30 & \text{theta production function parameter} \\
\alpha &= 0.33 & \text{alpha production function exponent} \\
K_0 &= 7.00 & \text{initial capital stock} \\
\tau &= 0.50 & \text{tau utility function parameter}
\end{aligned}
$$

The discount factor is the most intuitive of these parameters. Recall that it is equal to

$$
\beta = \frac{1}{1+\rho} \tag{12}
$$

Solving equation (12) for the discount rate $\rho$ yields

$$
\rho = \frac{1}{\beta} - 1 \tag{13}
$$

Thus when $\beta = 0.98$

$$
\rho = \frac{1}{0.98} - 1 = 1.02 - 1.00 = .02 \tag{14}
$$

and when $\beta = 0.95$

$$
\rho = \frac{1}{0.95} - 1 = 1.052 - 1.00 = .052 \tag{15}
$$

So to a reasonable approximation in the range of interest

$$
\beta \approx 1.00 - \rho \tag{16}
$$

Thus a discount rate of 6 percent or 0.06 implies a discount factor of *0.94*. Next assume that the criterion function in equation (8) includes the discount factor, $\beta$, raised to the power $t$, that is,

$$
J = \sum_{t=0}^{N} \beta^t \frac{1}{(1-\tau)} C_t^{(1-\tau)}
$$

and consider how $\beta^t$ varies with $\beta$ and $t$ as shown in Table 4.

You can see that when the discount rate is *5* percent the term $\beta^t$ becomes smaller much faster as the time period increases than it does when the discount rate is *2* percent. Thus when the discount rate is higher, future utility

Table 1: Values of Discount Term

| $\rho$ | $\beta$ | Values of $\beta^t$ corresponding to each time period | | | | | |
|--------|---------|------------------|------|------|------|------|------|
| | | Time periods | | | | | |
| | | 0 | 1 | 2 | 3 | 4 | 5 |
| 0.02 | 0.98 | 1.00 | 0.98 | 0.96 | 0.94 | 0.92 | 0.90 |
| 0.05 | 0.95 | 1.00 | 0.95 | 0.90 | 0.86 | 0.81 | 0.77 |

is *discounted* more heavily, that is, given less weight in the criterion function, so if your discount rate is 2 percent you have relatively more interest in your consumption in future years than if your discount rate is 5 percent. Therefore, altering $\beta$ is one of the interesting experiments to do with this model. As you increase the discount rate (and thus decrease the discount factor $\beta$) you should expect to see more consumption early in the time horizon covered by the model. An illustration of this result is shown in Figure 9, which contains plot lines for three experiments corresponding to three different values of $\beta$.

Figure 8: Solution for various $\beta$ values
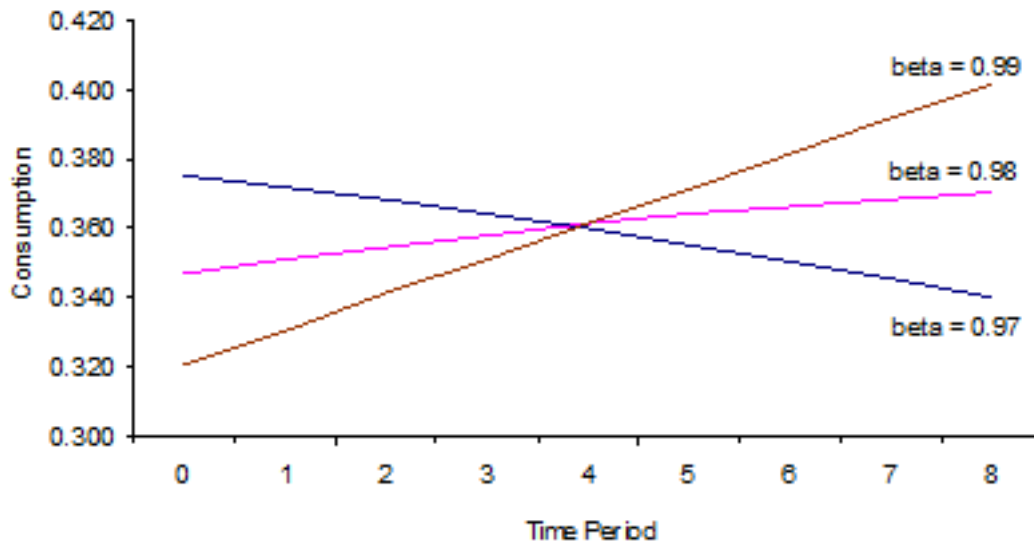


A second key parameter of the model that also plays a role in this matter of time preference of consumption and may affect your results in the experiments

16

just described. is the target capital stock, $K^*$. The relevant constraint of the model is equation (5), that is,

$$K_N \geqslant K^*$$

which requires that the capital stock in the terminal period exceed the target. This can be thought of as a constraint that represents the interest of the next generation. Without such a constraint, the optimal solution to the growth model is to invest little or nothing in the last years covered by the model and to make consumption very high in those periods. Thus a constraint of this sort is normally added to numerical growth models.

There can be an interplay between the choice of discount rate and the choice of the target capital stock. If you choose a high-target capital stock, then changes in the discount rate may not have much effect on the pattern of consumption over time since consumption must in any event be very low in order to ensure that there is enough investment to meet the target capital stock in the terminal period.

One of the most straightforward experiments with the model is to increase the initial capital stock. This has the effect of permitting more consumption with less investment, and one would expect to see higher levels of both output and consumption in the model solution.

If you alter the $\theta$ parameter in the production function in equation (1), that is,

$$Y_t = \theta K_t^\alpha$$

you are modifying the efficiency of the production process. For example, if you increase $\theta$, more output can be produced with the same capital stock and you should find higher levels of both output and consumption in the model solution. Similarly altering the $\alpha$ parameter affects the efficiency of the production process.

The last parameter that can be modified is $\tau$, a parameter in the utility function. Intuition here is a little hard to come by, but as $\tau$ approaches zero the utility function becomes linear and as $\tau$ approaches one it becomes logarithmic, so it may be useful to think of $\tau$ as a parameter that affects the curvature of the utility function or the degree of diminishing marginal utility.

Note that when you perform these experiments, if the changes you make in the parameter values are relatively small, the Excel Solver easily converges to a

new solution. This may not be the case for significant changes. Thus in those model runs you may have to *guess* and provide new values for the sequence of consumption values to be used by the Solver as new starting values, or you may have to play with different Solver options to control the Solver

In contrast to numerical growth models, theoretical growth models are usually solved for infinite horizons and do not have a terminal capital stock target. As an approximation to this, some numerical growth models are solved for much longer time horizons than the period of interest and the solution is used only for a shorter period. Thus if one is interested in a 20-year period the model might be solved for 40 or 60 years so that the end conditions do not have much effect on the solution paths for the first 20 years. When extending the time horizon, make sure that as you insert more columns in the Excel spreadsheet the equations of the model are copied in a proper manner, that the cell containing the sum of utilities is properly updated to cover the new range, and that the specifications of the target cell, the changing cell, and the constraint are properly updated in the Solver dialog box.

An interesting experiment is to impose a terminal capital stock equal to the initial capital stock and solve the model for different time horizons. The optimal capital stock path for an experiment like this is shown in Figure 9.

We can see that optimal values for the capital stock first increase then decrease. If we keep extending the time horizon, we generate a sequence of even higher arches whose top parts become flatter as they get closer to an upper limit value of about 10.5. This behavior is known at the *turnpike property*. To understand this, we have to point out that a model like the one presented in this chapter has a steady-state solution—a solution that, given enough time, the consumption and capital stock levels would converge to and stay at forever. It can be shown (see *Appendix A*, page 22) that for this model the steady-state capital stock is

$$K_{ss} = \left(\frac{1-\beta}{\beta\alpha\theta}\right)^{\frac{1}{\alpha-1}} \tag{17}$$

Substituting the corresponding parameter values we obtain $K_{ss} = 10.559$[3]. Thus, any finite optimal path tends to reach the steady-state value, stay there or close to it as long as possible, and then leave it to go back to the target capital stock.

---

[3]To confirm that this is indeed a steady-state solution, you may want to impose this value as the initial and target capital stock values and solve the model with Excel.

Figure 9: Optimal capital formation
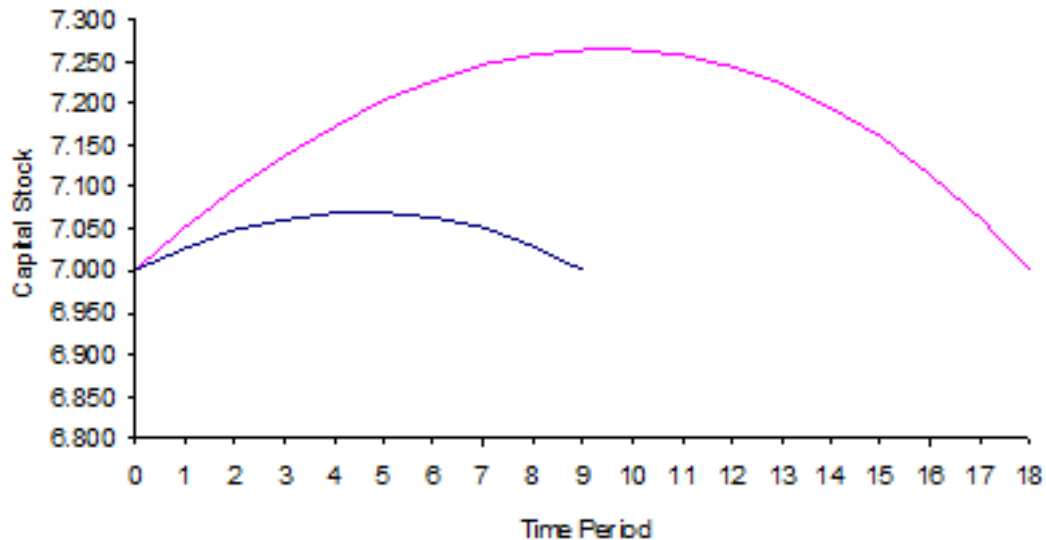


# 5   Experiments

Computational economics is not a subject that is easy to learn with the traditional lecture and exam style of teaching. Rather the crucial learning process is to first solve the models that other scholars have used, then to repeatedly make minor modifications and solve the model again in order to gain a clear understanding of how the model works and its strengths and weaknesses. At a later stage substantial structural changes can be made to the model so that it is more applicable to an economic situation of interest to the student.

Perform a series of experiments by modifying one of the parameters discussed earlier and observing the effects on the paths for capital stocks, output, and consumption. Though it might be interesting to change several parameters at one time it is usually better when you are first studying a model to only change one parameter at a time. Save your results from one run to the next so that you can use Excel to plot the results across runs as in Figures 8 and 9.

A more challenging experiment that you may want to undertake (or may not want to undertake at this stage) is to treat the technology parameter, $\theta$, as stochas-

tic. For example, you can define it as having a uniform distribution. To do so, you can use the Excel function RAND, which generates random numbers uniformly distributed between zero and one. Be aware that you should generate a random number for each time period.

# 6 Further Reading

Jones (1998) provides a systematic introduction to growth models. Azariadis (1993) and Barro and Sala-i-Martin (1995), at a more advanced level, present a variety of optimal growth Ramsey-type models similar to the one developed in this chapter. Aghion and Howitt (1997) present a systematic treatment of endogenous growth models. Ros (2001) develops a presentation of growth models for developing countries. Mercado, et al. (2003) present a GAMS version of a single-sector growth model like the one used in this chapter and a multisector optimal growth model in GAMS that is an extension of the Kendrick and Taylor (1970) model. See Judd (1998, Chapter 13) for perturbation methods for solving growth models.

# References

Aghion, P. and Howitt, P.: 1997, *Endogenous Growth Theory*, MIT Press, Cambridge, Massachussets, USA.

Azariadis, C.: 1993, *Intertemporal macroeconomics*, Blackwell, Oxford, United Kingdom.

Barro, R. and Sala-i-Martin, X.: 1995, *Economic Growth*, MIT Press, Cambridge, Massachussets, USA.

Chakravarty, S.: 1962, Optimal savings with a finite planning horizon, *International Economic Review* **3**, 338–355.

Judd, K. L.: 1998, *Numerical methods in economics*, MIT Press, Cambridge, Massachussets.

Kendrick, D. A. and Taylor, L.: 1970, Numerical solutions of nonlinear planning models, *Econometrica* **38**, 453–467.

Mercado, P. R., Lin, L. and Kendrick, D. A.: 2003, Modeling economic growth with gams, *in* A. K. Dutt and J. Ros (eds), *Development Economics and Structuralist Macroeconomics: Essays in Honor of Lance Taylor*, Edward Elgar, Cheltenham, United Kingdom, chapter 2, pp. 31–51.

Miranda, M. J. and Fackler, P. L.: 2002, *Applied Computational Economics and Finance*, MIT Press, Cambridge, Massachussets.

Ramsey, F. P.: 1928, A mathematical theory of saving, *The Economic Journal* **38**(152), 543–559.
   **URL:** *http://www.jstor.org/stable/2224098*

Ros, J.: 2001, *Development Theory and the Economics of Growth*, University of Michigan Press, Ann Arbor, Michigan, USA.

Taylor, J. B. and Uhlig, H.: 1990, Solving nonlinear stochastic growth models: A comparison of alternative solution methods, *Journal of Business and Economic Statistics* **8**, 1–17.

# Appendices

## A   Obtaining the Steady State of the Growth Model

Here we are interested in deriving the steady-steady solution of the model presented in Chapter 1. More detailed derivation steps can be found in Azariadis (1993).

We begin by defining the utility function

$$u(C_t) = \frac{1}{1-\tau}C_t^{1-\tau} \tag{A-1}$$

and the production function

$$f(K_t) = \theta K_t^{\alpha} \tag{A-2}$$

Thus, the model we want to solve can be stated as: find $(C_0, C_1, \cdots, C_{N-1})$ to maximize

$$J = \sum_{t=0}^{\infty} \beta^t u(C_t) \tag{A-3}$$

subject to

$$K_{t+1} = K_t + f(K_t) - C_t \tag{A-4}$$

$$K_0 \text{ given} \tag{A-5}$$

$$\lim_{t \to \infty} \left[ \beta^t u'(C_t) K_t \right] = 0 \tag{A-6}$$

where (A-3) is the criterion function, (A-4) is the capital accumulation equation, and (A-5) is the initial condition. Since we are now interested in deriving the steady-state solution of the model, we consider an infinite horizon problem. Thus, instead of a fixed terminal condition, we now impose the transversality condition (A-6), where $u'(C_t)$ is the derivative of the utility function. This condition states that the discounted lifetime utility is maximal when the capital stock is zero or, in other terms, that at time $t$ the present value of capital tends to zero as time goes to infinity.

Rearranging (A-4) and substituting for $C_t$ in (A-3), we obtain

$$J = \sum_{t=0}^{\infty} \beta^t v(K_t, K_{t+1}) \tag{A-7}$$

where

$$v(K_t, K_{t+1}) = u\left[f(K_t) + K_t - K_{t+1}\right] \tag{A-8}$$

Differentiating (A-7) with respect to $K_{t+1}$, we get, for each time period $t$, the first-order condition

$$\beta^t v_2(K_t, K_{t+1}) + \beta^{t+1} v_1(K_{t+1}, K_{t+2}) = 0 \tag{A-9}$$

where $v_1$ and $v_2$ are the partial derivatives of the function $v$ and

$$v_1 = \left[1 + f'(K_{t+1})\right] u'(C_{t+1}) \tag{A-10}$$

and

$$v_2 = -u'(C_{t+1}) \tag{A-11}$$

We now divide (A-9) by $\beta^t$ to obtain

$$\beta v_1 + v_2 = 0 \tag{A-12}$$

or, substituting (A-10) and (A-11) into (A-12)

$$\beta\left[1 + f'(K_{t+1})\right] u'(C_{t+1}) = u'(C_t) \tag{A-13}$$

Equation (A-13) together with equation (A-14),

$$K_{t+1} = K_t + f(K_t) - C_t \tag{A-14}$$

form a dynamical system that describes the evolution of the time paths for consumption and the capital stock. Given the initial condition (A-5), this system has a solution for each terminal value of the capital stock. The transversality condition (A-6) ensures that, out of the many possible solutions, we pick the optimum one.

To compute the steady state, we eliminate the time subscripts from (A-13) and (A-4) to obtain, respectively,

$$f'(K) = \frac{1 - \beta}{\beta} \tag{A-15}$$

23

and

$$C = f(K) \tag{A-16}$$

Finally, to obtain the steady state for the capital stock as in equation (17), page 18, in Chapter 1, we substitute equation (A-2) into (A-14) and solve for $K$, thus obtaining

$$K_{ss} = \left( \frac{1 - \beta}{\beta \alpha \theta} \right)^{\frac{1}{\alpha - 1}} \tag{A-17}$$

# B   Running the Solver in Excel

Download the file for the growth model or for the neural net from the book web site. Your version of Excel may not have the Solver option available by default. To check this look for the Solver option on the Tools menu. If you do not find it, click on `Add-Ins`, check Solver Add-in, and click OK. Then look in the Tools menu again.

# C Nonlinear Optimization Solvers

Solving nonlinear optimization problems usually requires the use of numerical methods. In general, those methods consist of a *smart* trial-and-error algorithm that is a finite sequence of computational steps designed to look for convergence to a solution. There is a variety of algorithms to solve nonlinear problems. Some of them are global methods, in the sense that they perform a parallel exploration of many regions of the optimization space, for example, the genetic algorithm. Other are local, as they tend to focus on the exploration of a particular region of the optimization space. We introduce here two of the most popular local methods—the gradient and the Newton methods—used by the solvers in Excel, **GAMS**, and **MATLAB**, but before introducing them, we give with a simple example.

Suppose that we are trying to find the maximum of a nonlinear function

$$y = f(x) \tag{C-1}$$

such as the one represented in Figure 10.


A simple and very rudimentary algorithm to find the solution might be as follows: We choose an arbitrary initial value for $x$, such as $x_0$ in Figure 10, and compute the corresponding $y_0 = f(x_0)$. We then increase that value by a constant magnitude $h$ (we name this magnitude the *search step*) that we also choose in an arbitrary way. For the new value of $x$, that is $x_1$, we compute the corresponding value of

$$y_1 = f(x_1) = f(x_0 + h) \tag{C-2}$$

and compare this value to the one obtained in the previous step. We continue to do this as long as the differences between two successive values of $y$ are positive (negative for a minimization problem). As soon as we compute a difference with a negative sign (in Figure F.1 this would correspond to $x_2$), we reverse the direction of the search. We begin to move in the opposite direction along $x$ (i.e., subtracting $h$ from $x$) and use a smaller value for $h$ than the one we were using while we moved in the opposite direction. We continue like this until we again find a difference between two successive values of $y$ that is negative, at which point we again reverse the direction of the search and further reduce the size of $h$, and so on. We stop when the difference between two successive values of $y$ falls below a preestablished tolerance limit.

The gradient and the Newton methods are iterative like the one just presented.

Figure 10: A nonlinear function.
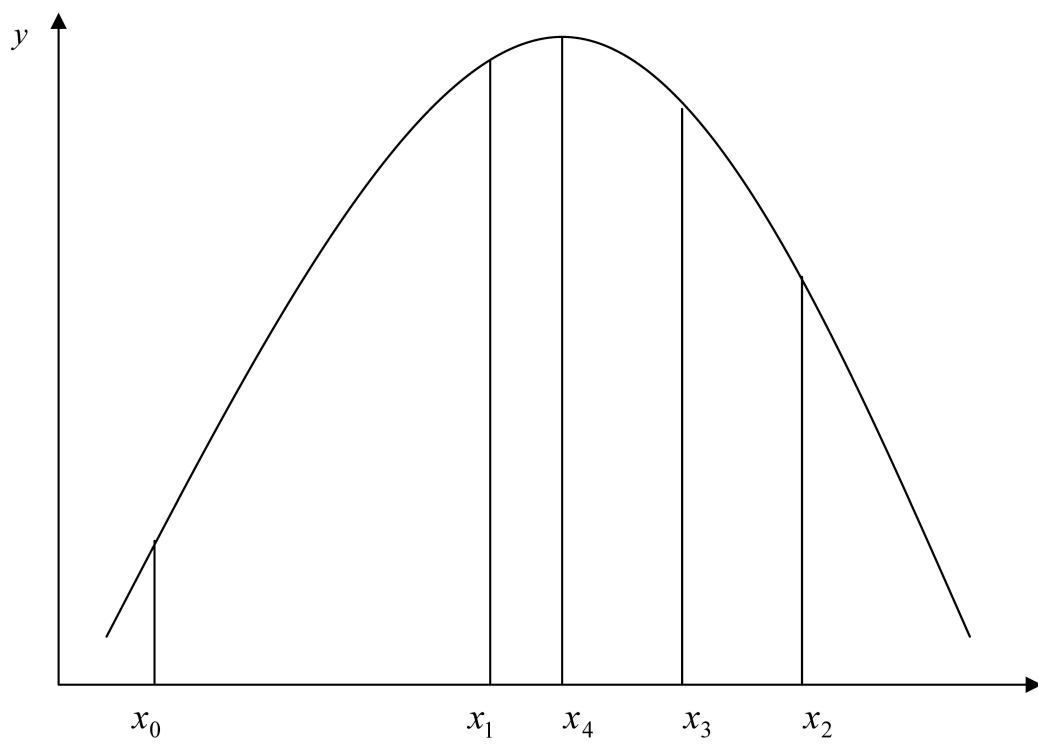
However, they exploit local information about the form of the function; that is, they use the function's derivatives. To illustrate this we change to a multivariate example. In this case we use the following equation to obtain each new value of the vector $x$:

$$x_{n+1} = x_n + h\Delta x \qquad \text{(C-3)}$$

where $h$ is the search step—now always a positive value—and $\Delta x$ is the direction of change, which, as we will see, is determined by the function's derivatives.

The gradient method uses the first derivative or gradient, which gives us information about how the function changes in the neighborhood of a given point. Its basic framework is the well-known first-order Taylor approximation,

$$f(x_{n+1}) \cong f(x_n) + h\nabla f(x_n)\Delta x \qquad \text{(C-4)}$$

where $\nabla f(x_n)$ is the gradient vector. Note that since $h$ is supposed to be positive, the best direction of motion is

$$\Delta x = \nabla f(x_n) \qquad \text{(C-5)}$$

for a maximization problem, since

$$f(x_{n+1}) \cong f(x_n) + h\nabla(f(x_n))^2 > f(x_n) \qquad \text{(C-6)}$$

In addition, for a minimization problem

$$\Delta x = -\nabla f(x_0) \qquad \text{(C-7)}$$

since

$$f(x_{n+1}) \cong f(x_n) - h\nabla(f(x_n))^2 < f(x_n) \qquad \text{(C-8)}$$

The basic framework of the Newton method is the second-order Taylor approximation

$$f(x_{n+1}) \cong f(x_n) + h\nabla f(x_n)\Delta x + \frac{h}{2}\Delta x' H(x_n)\Delta x \qquad \text{(C-9)}$$

where $H(x_0)$ is the second-order derivative or Hessian, which tells us how the slope of the function changes in a neighborhood of a given point.

Assuming the Taylor expansion of a function $f$ is a good global approximation of that function, we approximate the optimum value of $f$ by optimizing its

Taylor expansion. In our case, this is equivalent to saying that to determine the best direction of motion $\Delta x$ we have to optimize the expression (f.9). Differentiating (f.9) with respect to $\Delta x$, setting the result equal to zero, and solving for $\Delta x$, we obtain

$$\Delta x = -\frac{\nabla f(x_n)}{H(x_n)} \tag{C-10}$$

which is the best direction of motion for the Newton method.

Sometimes iterative methods like the ones presented here do not converge to a solution after a finite number of iterations. This problem can be overcome by changing the maximum number of iterations, or the size of the search step, or the tolerance limit, or the initial value of the search. Most solvers allow you to change these parameters.

Note also, as is the general case for numerical methods dealing with nonlinear optimization problems, that if there is more than one local optimum we will find only one of them. Thus, we never know for sure if the optimum we reached was a local or a global one. A rough way of dealing with this difficulty is to solve the problem providing the algorithm with alternative initial values of the search.

In this appendix we presented three numerical methods of increasing complexity. Of course, the more complex ones make use of more information, thus reducing, in general, the number of steps needed to achieve convergence. However, those steps become more complex, since they require the computation of a gradient or a Hessian. Then there are trade-offs to be evaluated when choosing a solution method.

There are additional methods for solving nonlinear problems numerically—for example, the conjugate gradient method, the penalty function method, and the sequential quadratic programming—a number of which extend, combine, or mimic the ones introduced here. For a comprehensive presentation refer to Judd (1998) and Miranda and Fackler (2002). The Excel solver uses a conjugate gradient method or a Newton method. **GAMS** uses a variety of methods, depending on what you choose or have set up as the default nonlinear solver. The **MATLAB** solver used in Chapter 7 and invoked by the fmincon function uses a sequential quadratic programming method. For details on the specific methods used by **Excel**, **GAMS**, and **MATLAB** refer to their corresponding user's and solver's manuals.