COMPUTATIONAL ECONOMICS

Option Modeling

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1 Introduction

In the last few decades the derivative markets have become a dominant part of the financial markets in the world. A derivative is a financial instrument which derives its value from the value of some other financial instrument or variable. Derivatives come in many shapes, but stock options are the most dominant.

Options are financial instruments that convey the right, but not the obligation, to engage in a future transaction on some underlying security, or in a futures contract. In other words, the holder does not have to exercise this right, unlike a forward or future. There are options that give you the right a buy stock at a predetermined price. These are called *call* options. Options that give you the right to sell at a predetermined price are called *put* options.

Each of these option come in all sorts of flavors like European options, American options, Asian options, Exotic options, Basket options and many more. In this chapter we will concentrate on the most regular types options and provide means to evaluate less regular types of options.

2 European Options

The theoretical value of an option can be determined by a variety of models. These models, can also predict how the value of the option will change in the face of changing conditions. The most simpel types of options are the European call, C_E en put, P_E option. European options are options that can only be exercised, at the time they mature, T.

The Nobel laureates Black and Scholes, derive in the 1973 (1973) their well know formula for the pricing of European Options. In many respects is, the so-called, still the working horse for the pricing of options. The derivation of the Black and Scholes pricing formula is rather complicated, however, the resulting equations are relatively easy to compute.

The starting point for the pricing of a European options is the following equation

$$C_E(S,t) = SN(d_1) - Xe^{-r(T-t)}N(d_2) \tag{1}$$

In this equation $C_E(S,t)$ is the price of the European call option as a function of the stock price S given the point in time t and T the time period at which the option may be exercised. X is the exercise price at which the underlying stock can be bought. The derivation of what comes is rather complicated and beyond the scope of this chapter. Good sources are Black and Scholes (1973), Wilmott et al. (1993) and Hull (1993).

However, it possible to give intuitive insight on how the Black and Scholes formula works. For instance, when $t = T$ the option has matured and the value of the options is

$$C_E(S,T) = max[S - X, 0] \tag{2}$$

Similar if $S = 0$ then

$$C_E(0,t) = 0 \tag{3}$$

which is the (theoretical) lower bound of the call option for any t.

The function $N(.)$ is the so called *normalized form of the cumulative normal distribution function*, which is used to measure the level of uncertainty. The cumulative normal distribution function distribution is defined as

$$N(d_i|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\infty}^{d_i} e^{\frac{-(t-\mu)^2}{2\sigma}} dt \tag{4}$$

In the above equation (1), the normalized in the sense that $\mu = 0$ and $\sigma = 1$ in that case equation (3) reduces to

$$N(d_i) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{d_i} e^{-\frac{1}{2}t^2} dt \tag{5}$$

which has the shape
and is the cumulative chance. As you can see from the plot

$$N(-\infty) = 0 \quad \text{and} \quad N(0) = \frac{1}{2} \quad \text{and} \quad N(\infty) = 1 \tag{6}$$

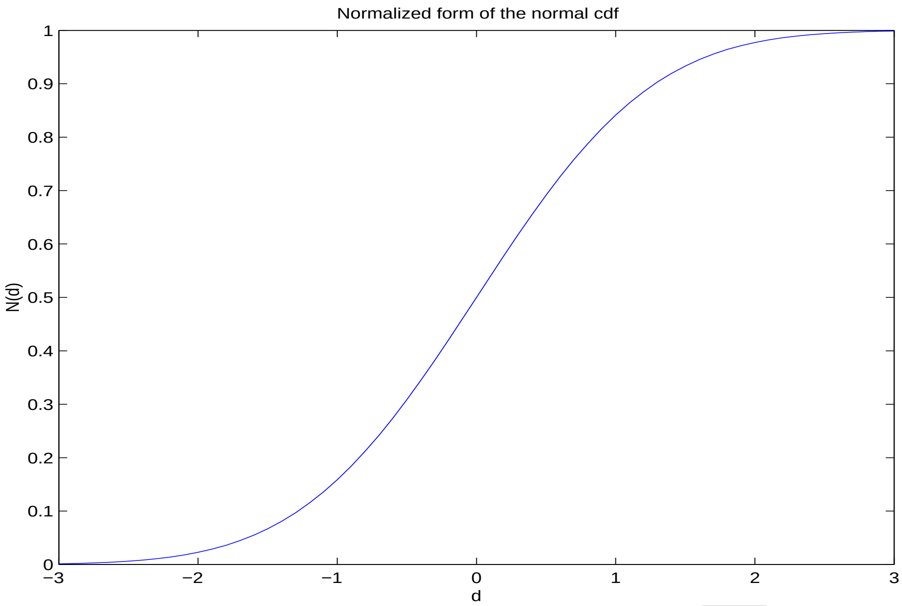
The above plot allows to interpret equation (1). For instance, the first element on the right hand side is

$$SN(d_1) \tag{7}$$

which is the expected value of the stock at time t, give the time to maturity $T - t$. Similar the element

$$Xe^{-r(T-t)}N(d_2) \tag{8}$$

Figure 1: Plot of the Normalized form (4)



is the discounted value of the X exercise price at time t. Hence, the function $N(d_i)$ is used to compute the likelihood of the two elements in equation (1). Hence, the (expected) value of the call options is the difference between the expected stock value and the present value of the X exercise price.

The missing element of equation (1) are the d_1 and d_2. These are defined as

$$d_1 = \frac{\log(S/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{9}$$

and

$$d_2 = \frac{\log(S/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \tag{10}$$

with equation (9) and (10) we have all the parts in in order to compute the call option value in equation (1).

The above equations can be very efficiently be implemented in **MATLAB**. The corresponding **MATLAB** code is

```
function [price] = blackscholes_call(s, x, r, sigma, t)
d1 = (log(s / x) + (r + 0.5 * sigma * sigma) * t) / (sigma * (t ^ 0.5));
d2 = (log(s / x) + (r - 0.5 * sigma * sigma) * t) / (sigma * (t ^ 0.5));
price = s * normcdf(d1) - x * exp(-r * t) * normcdf(d2);
```

The function `blackscholes_call` returns the value `price`. The parameters s, the stock price, x, the exercise price, r, the default free short term interest rate, σ, the volatility, and the time to maturity, t are parsed through the function call.

The pricing formula for a *put* option, using Black and Scholes, is

$$P_E(S,t) = Xe^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{11}$$

where d_1 and d_2 are defined as in (9) and (10) respectively. Like (2)

$$P_E(S,T) = max[X - S, 0] \tag{12}$$

and

$$P_E(0,t) = Xe^{-r(T-t)} \tag{13}$$

which is the upper bound condition for the put option. The corresponding **MATLAB** code is

```
function [price] = blackscholes_put(s, x, r, sigma, t)
d1 = (log(s / x) + (r + 0.5 * sigma * sigma) * t) / (sigma * (t ^ 0.5));
d2 = (log(s / x) + (r - 0.5 * sigma * sigma) * t) / (sigma * (t ^ 0.5));
price =  x * exp(-r * t) *normcdf(-d2) - s *normcdf(-d1);
```

which completes this section.

3 American Options

As already mentioned, European options are options that can only be exercised, at the time they expire at the expiration date T. With American option this is different. American options may be exercised on any moment during the contract and not only at the time the contract expires. For this reason, American type options have a higher value than European options. By the way, most options that are traded world wide at international option exchanges are American options.

This

```
function [price] = binomial_call(s, x, r, sigma, t, nstep)
% Matlab function to compute the binomial tree for American options

f=zeros(nstep+1,nstep+1);

dt = t / nstep;
```

```
u = exp(sigma * (dt ^ 0.5));
d = 1/u;
a = exp(r * dt);
p = (a - d) / (u - d);
uloop=0;
u2=0;

u2=u*u;
uloop=u^(-nstep);
for j=0:nstep
    sum = s*uloop - x;
    f(nstep+1,j+1)=max(sum,0);
    uloop=uloop*u2;
end

for i=nstep-1:-1:0
      uloop=u^(-i);
      u2=u*u;
      for j=0:i
          sum = s*uloop - x;
          sum1= exp(-r*dt)*(p*f(i+2,j+2)+(1.0-p)*f(i+2,j+1));
          f(i+1,j+1)=max(sum,sum1);
          uloop=uloop*u2;
      end
end

price = f(1,1);
```

4 The Greeks

It is important to know, to what measure the option price is sensitive to changes in the parameters. For this purpose we can derive what is is called, the *Greeks*. The most common Greeks are the (call) option

$$\Delta = \frac{\partial C}{\partial S} \quad \text{effect of a change in the stock price} \tag{14}$$

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \quad \text{rate of change of } \Delta \tag{15}$$

$$\Theta = \frac{\partial C}{\partial t} \quad \text{effect of a change in the duration} \tag{16}$$

$$\Lambda = \frac{\partial C}{\partial \sigma} \quad \text{effect of a change in volatility} \tag{17}$$

$$\tag{18}$$

5 Computing Implied Volatility

In the previous section we have assumed that the parameters of the option model are know. However, that is not always the case. This holds especially for the volatility parameter σ, which is in practice is hard to determine. Therefore, sometimes, it is interesting to compute the *implied volatility*. The implied volatility is the $\sigma = \sigma_i$ as implied in the actual stock price S and the quoted option price in the option market.

Mathematically this means that equation (1) can be written as solving the equation

$$f(\sigma_i) = C_E - SN(d_1(\sigma_i)) + Xe^{-r(T-t)}N(d_2(\sigma_i)) = 0 \tag{19}$$

using σ_i as a variable, treating the market value of the call option , C_E, as a constant. The Newton–Raphson method, see Press et al. (1986), is a well known approach of a finding the roots of an equation. This method works like this. Take

$$f(\sigma_i) = 0 \tag{20}$$

and make a the first order expansion around the point $\bar{\sigma}_i$

$$f(\sigma_i) \approx f(\bar{\sigma}_i) + f'(\bar{\sigma}_i)(\sigma_i - \bar{\sigma}_i) \tag{21}$$

if we have a good estimate for $\bar{\sigma}_i$ then

$$f(\bar{\sigma}_i) + f'(\bar{\sigma}_i)(\sigma_i - \bar{\sigma}_i) \approx 0 \tag{22}$$

which means that

$$\sigma_i \approx \bar{\sigma}_i - \frac{f'(\bar{\sigma}_i)}{f(\bar{\sigma}_i)} \tag{23}$$

Starting with the (fair) estimate for $\bar{\sigma}_i = \sigma_i^0$, we may iterate with $j = 0, 1, 2...$ to approximate the solution of (20)

$$\sigma_i^{j+1} \approx \sigma_i^j - \frac{f'(\sigma_i^j)}{f(\sigma_i^j)} \tag{24}$$

Usually the stopping criterium for the iteration scheme is the norm

$$\left|\left|\frac{f'(\sigma_i^j)}{f(\sigma_i^j)}\right|\right| < \epsilon \tag{25}$$

The derivative

$$f'(\sigma) \tag{26}$$

can be numerically approximated through central differencing

$$\frac{f(\sigma_i + \delta) - f(\sigma_i - \delta)}{2\delta} \tag{27}$$

δ being an arbitrary small number.

The iteration scheme of (24) can be used to compute the implied volatility from equation (19). We have implemented the above iteration scheme in the **MATLAB** code

```
function [sigma1]=implied_blackscholes_call(s, x, r, sigma0, t, price)
    eps = 1E+16;
    while eps > 0.01
        mylc = (blackscholes_call(s, x, r, sigma0 + 0.001, t) - ...
               blackscholes_call(s, x, r, sigma0 - 0.001, t)) / 0.002;
        sigma1 = sigma0 + (price - blackscholes_call(s, x, r, sigma0, t)) ...
                / mylc;
        eps = (sigma1 - sigma0) ^ 2 / sigma0 ^ 2;
        sigma0 = sigma1;
    end
```

The function has an additional parameter `price` which is the market quote of the call option. `sigma0` is the initial value of σ to start the iterative proicedure to solve the implied volatility as implied in `price`.

In this specific case of the Black and Scholes model is is also an analytic for the derivative $f'(\sigma_i)$.

6 Suggested Reading

As already mentioned, good sources are Wilmott et al. (1993) and Hull (1993). There are also a number of reference that concentrate on the computational aspects of financial modeling. The ones we would like to mention are Brandimarte (2001) and Jäckel (2002).

References

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