COMPUTATIONAL ECONOMICS

Stochastic Control

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1 introduction

This part of the book is different. *Parts I* and *II* covered a wide variety of topics in a way that provides an introduction to the computational methods used in the various fields. In contrast, in Part III we zero in on a narrow area of computational economic research that is of particular interest to us: the application of stochastic control theory methods to macroeconomic stabilization models. In the past we have done three kinds of work in this area: analytical [viz. Mercado (2004a)], computational with the **MATLAB** software [viz. Amman and Kendrick (2003)], and computational with the Duali software [viz Amman and Kendrick (1999b, 2008)]. Here we focus on the third of these since the Duali software provides a low-entry-cost way to begin work in this field. However, Duali represents a sharp contrast to the software systems used earlier in the book, which are all high-quality commercial programs. Duali, in contrast, is experimental software that is under development by two of us (Amman and Kendrick).

The Duali software is intended to provide a point and click interface for a stochastic control program that can be used to solve models with a quadratic tracking criterion function, linear systems equations, and stochastic specifications that may include additive and multiplicative noise terms, measurement errors, and uncertainty about initial conditions. It is not a commercial product and has not had the extensive testing that is a part of the history of such products. Rather, it is an academic piece of software for which there is no support staff or help line. Moreover, the software has not yet even reached the *beta* stage and is thus prone to crashes, so it must be used with care since it can cause not only the loss of the Duali input files but also of input files for other applications that are running concurrently.

On the other hand, this software has been used successfully by many undergraduate and graduate students in classes at the University of Texas and has provided an easy *on ramp* for many students into the field of stochastic control. In addition, it has been used by a number of graduate students in developing parts of their Ph.D. dissertation research. Therefore, we suggest that if you decide to move forward into this part of the book the gains may be substantial, but you should proceed with considerable caution.

If you choose to go forward, it is best to begin by making use of the *User's Guide for the Duali software*, which can be found by going to the book web site at

`https://github.com/hansamman/Computational_Economics_Lectures`

and proceeding to the Software section of the site. The User's Guide introduces the capabilities of Duali and take you through the steps to set up and solve a simple macroeconomic model. Once you have done that, the material in this third part of the book will follow logically.

2 Stochastic Control in Duali

In an earlier chapter we presented the Hall and Taylor macroeconomic model, a standard nonlinear dynamic model for an open economy with flexible exchange rates. There we represented and simulated the model in GAMS and introduced a basic form of optimal policy analysis. Working with the same model, in this chapter we take some steps forward in the realm of policy analysis, providing an introduction to the field of stochastic control.

A stochastic control problem can be posed as one in which a policy maker, manipulating a set of control variables, tries to influence the dynamics of an economic system in order to achieve various targets. For example, in a macroeconomic setting, the policy maker may use some controls—policy variables such as the money stock or government expenditure—to influence the behavior of the economy in order to maintain some target variables, such as unemployment and inflation, as close as possible to their desired paths. The economic model is usually represented in state-space form, that is, as a first-order system of dynamic equations. The policy maker has an objective function—usually a quadratic one—that specifies the target variables, the desired paths, and the relative weights put on the achievement of each target.

The solution of deterministic and stochastic control problems quickly becomes very involved. Thus, to make our task feasible, we have to rely on computational methods and specialized software. [1]Duali[2] software can receive the desired paths and corresponding weights for target and policy variables and the state-space representation of the economic model as inputs. It can then be used to generate simulation results and to compute the optimal policy rule and the implied solution paths for policy and target variables using the methods described in Chapter **??**. In what follows, we use Duali to perform first deterministic and then stochastic control experiments with the state-space representation of Hall and Taylor's

[1] This chapter draws extensively on both the verbal and mathematical development in Mercado and Kendrick (1999). Kluwer Academic Publishers have kindly granted us permission to reuse here substantial materials from the previously published work.

[2] See Amman and Kendrick (1999a). Special care should be taken when doing the experiments in this book that use Duali. If you have not already read about the Duali software in the introduction to this part of the book, please go back and do so.

model.

3 The Hall and Taylor Model in State-Space Form

Much undergraduate study of macroeconomics makes use of dynamic *nonlinear* models in levels; for example, the levels of government expenditure and the money supply are used to determine the levels of consumption, investment, output, interest rates, and net exports. In contrast, much empirical macroeconomic research centers on dynamic *linear* models in percentage deviations of variables from their steady-state values. In these empirical models one alters the percent deviation of government expenditures and the money supply from their steady-state levels and analyzes the resulting deviations of consumption, investment, output, interest rates, and net exports from their steady-state levels. However, the bridge between these two types of models is frequently not apparent. Therefore, for this chapter we begin with the dynamic nonlinear Hall and Taylor model in levels that we used with **GAMS** earlier in the book and transform it into a linearized model in percentage deviations of variables from their steady-state values in a similar fashion to the approach we used in the chapter that includes the *Johansen-type CGE model*. Moreover, we use here a four-equation linear version of the original Hall and Taylor twelve-equation nonlinear model that captures the essential behavior of the original model. Thus, our four-equation model's variables are as follows:

- Endogenous variables

$$\begin{aligned} Y^* &= \text{GDP} \\ R^* &= \text{Real interest rate} \\ plev^* &= \text{Domestic price level} \\ E^* &= \text{Nominal exchange rate} \end{aligned}$$

- Policy variables

$$\begin{aligned} M^* &= \text{Money stock} \\ G^* &= \text{Government expenditure} \end{aligned}$$

- Exogenous variables

$$\begin{aligned} plevw^* &= \text{Foreign price level} \\ YN^* &= \text{Potential GDP} \end{aligned}$$

The asterisks indicate percent deviations, for example, Y^* is the percent deviation of output from its steady-state value. This variable structure is one of the most common ways in which textbook macroeconomic models are presented. To transform the original twelve-equation nonlinear model we first collapsed it, by equation substitution, into a four-equation version. We then linearized the equations and represented the resulting model in matrix form. Next we solved the model for its reduced form representation, obtaining a third-order system of difference equations. Finally, we reduced that system to first order, that is, to its state-space form. Details on these transformations are provided in Appendix A.

The model's state-space representation is

$$x_{k+1} = Ax_k + Bu_k + Cz_k \tag{1}$$

where x is an augmented state vector defined as

$$x = \begin{bmatrix} X \\ XL \\ XLL \end{bmatrix}$$

with

$$X = \begin{bmatrix} Y^* \\ R^* \\ plev^* \\ E^* \end{bmatrix}$$

and XL and XLL are equal to the vector X lagged once and twice, respectively. We define the x vector in this way by augmenting the original state vector with lagged values in order to reduce the linearized model from a third-order representation to a first-order representation [see Kendrick (2002)]. The control vector and the exogenous variables vector are defined as

$$u = \begin{bmatrix} M^* \\ G^* \end{bmatrix} \quad z = \begin{bmatrix} YN^* \\ plevw^* \end{bmatrix}$$

.

The parameter matrices in the system equations are

$$A = \begin{bmatrix} -0.346 & 0 & -0.606 & 0 & 0 & 0 & 0.087 & 0 & 0 & 0 & 0.087 & 0 \\ 7.811 & 0 & 13.669 & 0 & 0 & 0 & -1.953 & 0 & 0 & 0 & -1.953 & 0 \\ 0.8 & 0 & 1.4 & 0 & 0 & 0 & -0.2 & 0 & 0 & 0 & -0.2 & 0 \\ 1.154 & 0 & 2.019 & 0 & 0 & 0 & -0.288 & 0 & 0 & 0 & -0.288 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.433 & 0.231 \\ -9.763 & 4.386 \\ 0 & 0 \\ -2.442 & 1.097 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0.346 & 0 \\ -7.811 & 0 \\ -0.800 & 0 \\ -1.154 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

where each of these matrix values is derived from the corresponding combination of parameter values in the original twelve-equation nonlinear Hall and Taylor model. See Appendix A for these derivations.

4 Introduction to Optimal Policy Analysis Methods with Duali

In an earlier chapter we used **GAMS** to study the responses of Hall and Taylor's model to changes in the policy variables. Optimal policy analysis employs a sort of *reverse* analysis. It begins by posing the question: how should policy variables be set in order for the target variables to follow prespecified paths?

The most popular way of stating this problem is as a quadratic linear problem

(QLP), and we introduced this type of problem in Chapters **??** and **??**. In formal terms, we express our problem here as one of finding the controls $(u)_{t=0}^{N}$ to minimize a quadratic tracking criterion function J of the form

$$
\begin{aligned}
J &= \frac{1}{2}\left[x_N - \tilde{x}_N\right]' W_N \left[x_N - \tilde{x}_N\right] \\
&+ \frac{1}{2}\sum_{k=0}^{N-1}\Big(\left[x_k - \tilde{x}_k\right]' W_k \left[x_k - \tilde{x}_k\right] + \left[u_k - \tilde{u}_k\right]' \Lambda_k \left[u_k - \tilde{u}_k\right]\Big) \qquad (2)
\end{aligned}
$$

subject to the state-state representation of the economic model given by equation (1), where $\tilde{x}$ and $\tilde{u}$ are desired paths for the state and control variables, respectively, and W and Λ are weighting matrices for states and controls, respectively.

The quadratic nature of the criterion function implies that deviations above and below target are penalized equally and that large deviations are more than proportionally penalized relative to small deviations. This particular form of the criterion function is not the only possible one, but is the most popular.[3]

For simplicity in the following we drop the asterisk from the variables. Thus we use Y, R, $plev$, and E instead of Y^*, R^*, $plev^*$, and E^* to indicate the state variables. However, we refer to the variables as percent deviations rather than as levels.

We assume that the policy goal is to stabilize Y, R, $plev$, and E around steady-state values (i.e., around zero). High and equal weights[4] are put on stabilizing Y and $plev$, lower and equal weights on R and E, and even lower and equal weights on the policy variables M and G. Neither the desired paths nor the following weighting matrices vary with time:

[3] For a discussion of the properties of different criterion functions, see Blanchard and Fischer (1989, Chapter 9)

[4] There is a conceptual difference between the weights used here and those that arise when the variables of interest are in levels rather than in percent deviations and are expressed in different units of measurement. For instance, if GDP is measured in dollars and prices are measured by an arbitrary price index, equal weights on these two variables would probably imply different policy priorities and vice versa. Since all variables in the state-space representation of Hall and Taylor's model are in percent deviations from steady state, weights and priorities can be considered as equivalent within certain limits. However, it should be clear that, for example, an interest rate 50 percent below steady-state values is something feasible, whereas a level of GDP 50 percent below steady state is not. In such a case, there is no analogy between weights and priorities. See Park (1997).

$$W = \begin{bmatrix} 100 &&&&&&&&&&& \\ & 50 &&&&&&&&&& \\ && 100 &&&&&&&&& \\ &&& 50 &&&&&&&& \\ &&&& 0 &&&&&&& \\ &&&&& 0 &&&&&& \\ &&&&&& 0 &&&&& \\ &&&&&&& 0 &&&& \\ &&&&&&&& 0 &&& \\ &&&&&&&&& 0 && \\ &&&&&&&&&& 0 & \\ &&&&&&&&&&& 0 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 25 & \\ & 25 \end{bmatrix}$$

Let us assume, for example, that the economy is going through a recession provoked by a temporary adverse shock to net exports that causes Y to be 4 percent below its steady-state value. Given the weight structure adopted previously, what are the optimal paths for government expenditure (G) and the money supply (M) in order to bring the economy back to its steady state? How do the optimal paths for the state variables compare with what would be the autonomous response of the system to that kind of shock? To answer these questions, we perform two experiments: one to obtain the optimal paths and a second to get the autonomous response of the economy. To run this simulation, use program ht-01.dui making the appropriate changes. (See the *Model Description* item in the Specification menu in Duali once the `ht-01.dui` file is opened in the application.)

To perform the first experiment in Duali, we have to set the problem as a deterministic one, set all the desired paths for states and controls equal to zero, impose the corresponding weights on states and controls, set an initial value for Y equal to –0.04, and solve the problem. Let us see in more detail how this is done.

Figure 1 shows the initial screen of the Duali software. The File and Edit menus are standard. The Specification and Data menus contain submenus related to the structure of the problem to be solved. The Solve menu presents options for different solution methods and the Results menu enables one to display the tables and graphs of the results. The Transformations menu contains several options to change the original structure of the problem, and the Preferences menu contains options related to the format of the display of results and to some specific types of experiments.

We begin by opening the `ht-01.dui` file using the File menu. Then select Specification:Stochastic Terms and note that the problem is set as deterministic,

Figure 1: Initial Screen

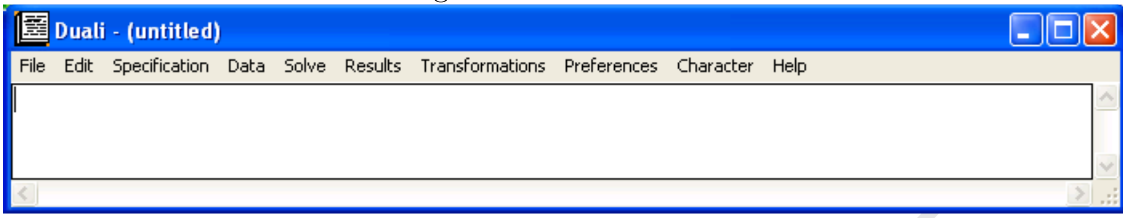
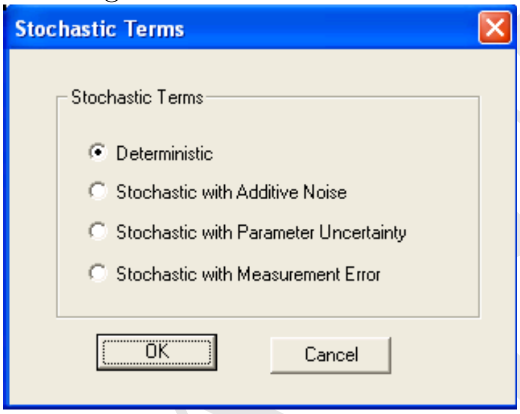


Figure 2: Stochastic Terms



as shown in Figure 2.

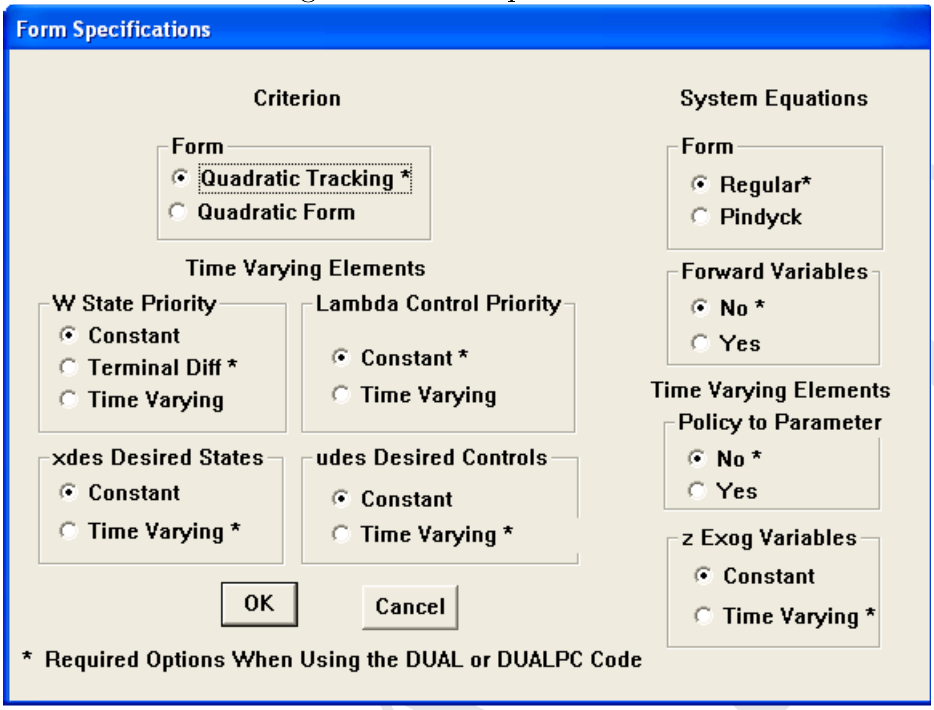
Then, from the Specification:Functional Form option we obtain the dialog box shown in Figure 3. Look at the Criterion side of this box and at the Form section on that side. There you can see that the problem is one in quadratic tracking. In fact, we try to minimize deviations of target variables from zero, since the model variables are already expressed in percent deviations from steady-state values. The W State Priority and Lambda Control Priority sections show that the weights on state and control variables are constant, that is, the same values for all periods. Desired state and control variables are also constant (all zeroes). The right-hand side of the dialog box shows the specification for the system equations. In particular, the Form section shows that the problem is written in:

1. Regular form, equation (1), that is, the standard state-space representation;

2. It does not contain forward variables (as is the case of models with rational expectations discussed later');

3. The policy variables do not affect the model parameters in this particular model. Finally, the exogenous variables remain constant over the time periods.

Figure 3: Form Specification



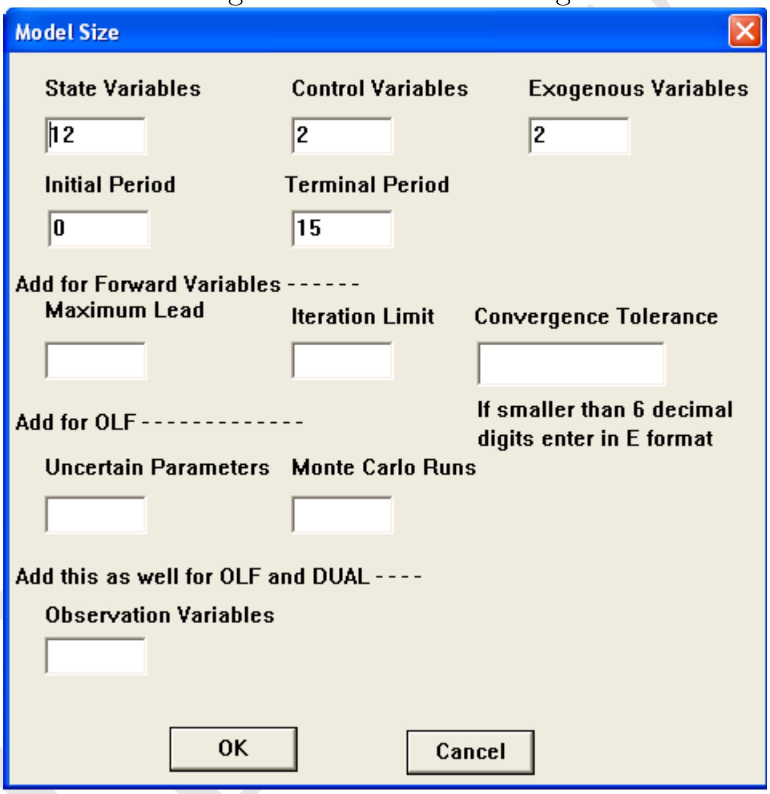
From the `Data:Size` menu we obtain the dialog box shown in Figure 4. The model is specified as containing twelve state variables (actually, four contemporaneous and eight lagged), two control variables, and two exogenous variables; the simulation covers sixteen periods.

The `Data:Acronyms` menu option contains the assignment of labels to the model variables and to the time periods. The `Data:Equations` section contains the numerical values for matrices A, B, C and for the initial-state variable values, and the `Data:Criterion` section contains the values for the W and Λ weighting matrices and the desired path values for state and control variables.

Choosing the menu option `Solve:QLP`, we solve the problem as a QLP using the solution procedure described in Chapter ??. The numerical results are then displayed automatically. The Results menu options allow us to define different display, plotting, and printing options.

The results of this experiment to obtain the optimal states are shown in Figures 5-5, and the graphs in that figure show the autonomous state and control paths. In order to obtain the autonomous path of the system, we impose zero

Figure 4: Model size dialog



weights on the state variables, very high and equal weights on the controls, and, as in the first experiment described earlier, set an initial value for Y equal to –0.04. This has the effect of leaving the state variables free to take on any values while restricting the policy variables from deviating from their steady-state values.

In Figures 5-8 the vertical axes show the percent deviations from steady-state values and the horizontal axes show the time periods. In these plots a value of 0.02 means *2 percent above steady state*. It does not mean *2 percent increase with respect to the previous period*. Thus, a 10 percent permanent increase in M means that the money stock is increased by 0.1 at the initial period and kept constant at the new level from then on. Since all variables (endogenous, policy, and exogenous) are in percent deviations, their steady-state values are all zeroes.

Figure 5: Gross Domestic Product

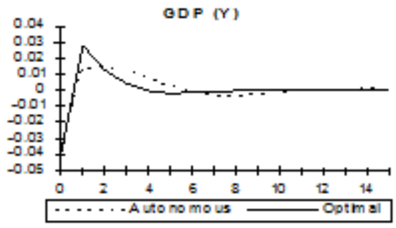


Figure 6: Real Interest Rate

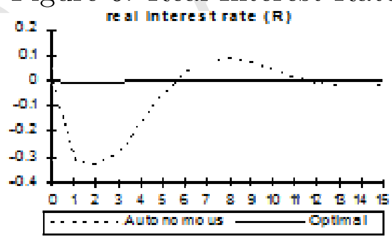


Figure 7: Price Level

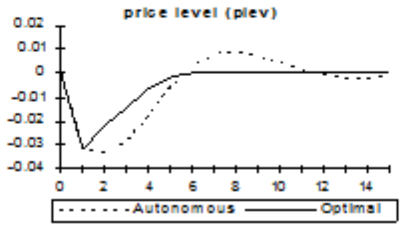
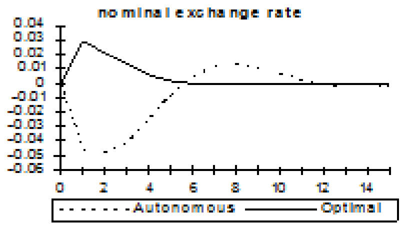


Figure 8: Nominal Exchange Rate



The optimal solution paths for the states outperform the autonomous responses of the system for all four target variables. This comes as no surprise, though it may not always be the case. Indeed, remember that the optimal solutions are obtained from the minimization of an overall loss function. On some occasions, depending on the weight structure, it may be better to not do as well as the autonomous response for some targets in order to obtain more valuable gains from others.

Why does the autonomous path of the economy display the observed behavior? Here is how Hall and Taylor explain it:[5].

> With real GDP below potential GDP after the drop in net exports, the price level will begin to fall. Firms will have found that the demand for their products has fallen off and they will start to cut their prices (...). The lower price level causes the interest rate to fall.[6] With a lower interest rate, investment spending and net exports will increase.[7] The increase in investment and net exports will tend to offset the original decline in net exports. This process of gradual price adjustment will continue as long as real GDP is below potential GDP.

What explains the observed optimal path of the four variables of interest? We can see in Figures 5-8 that Y is brought up very quickly, going from 4 percent below steady state to 3 percent above steady state, and then decays slowly to its steady-state value. This performance could be attributed to the more than 6 percent increase in G that can be observed in the optimal policy variables' paths (Figure 9).

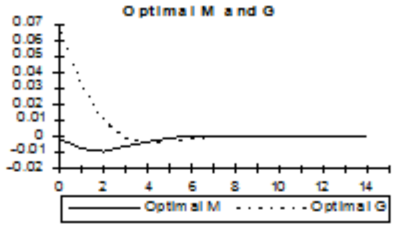
Meanwhile in Figure 5-8, R experiences almost no variation when compared to the big drop of almost 35 percent implied by the autonomous behavior of the

[5]Hall and Taylor (1997, Page 232).

[6]Since less money is demanded by people for transaction purposes.

[7]Since the price level falls much less than the real interest rate during the first periods of the adjustment, the nominal exchange rate has to fall too, as can be derived from equation (9) in the original Hall and Taylor's model. This implies that the real exchange rate will fall, thus causing net exports (see their equation (10)) to rise.

Figure 9: The Optimal Controls



system. Once again, the increase in G exerts upward pressure on the interest rate, thus keeping it from falling. Finally, the nominal exchange rate has to go up to compensate for the fall in prices, given that the real interest rate does not change much.

We can also see in Figure 9 that monetary policy plays a minor role when compared to fiscal policy.[8] Even though we put the same weights on both variables, government expenditure appears to be more effective in bringing the economy out of its recession given the weight structure we put on the target variables.

It is interesting to analyze the different combinations of behavior of variables that the policy maker can achieve given a model and a criterion function. The curve showing those combinations is known as the policy frontier.[9] For instance, we may want to depict the trade-off between the standard deviations of Y and *plev* in Hall and Taylor's model when, as earlier, Y is shocked by a negative 4 percent in period zero. To obtain the corresponding policy frontier, we have to vary the relative weights on Y and *plev*, perform one simulation for each weight combination, and compute the corresponding standard deviations. The results of six such experiments, keeping the same weights on the remaining states and controls as in the foregoing simulation, are shown in Table 1 and Figure 10.

The policy frontier for Y and *plev* is clearly shown in the Figure 10, where each diamond represents the result of an experiment. The higher the weight on Y relative to that of *plev*, the lower its standard deviation, and vice versa. The flatness of the curve indicates that it is easier to achieve a reduction in the percent deviation from target for *plev* than for Y. Of course, the shape and location of this particular policy frontier are conditional on the weight structure imposed on

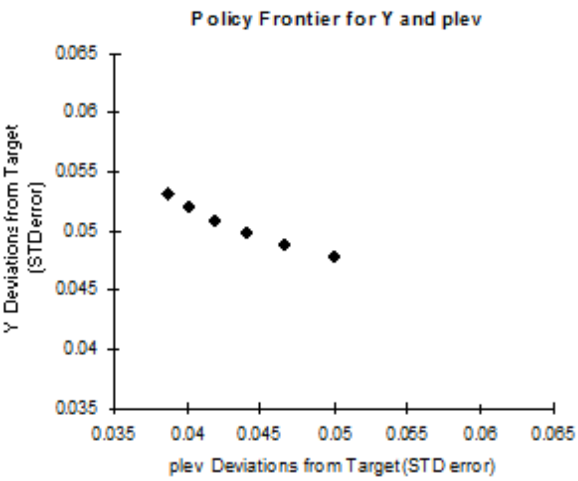
[8]Note that the optimal values for the policy variables are computed for periods zero to fourteen only. Given that we are working with a state-space representation of the model, policy variables can only influence the next period state variables. That is, the controls at period zero are chosen, with a feedback rule, as a function of period-zero states, but they determine period one states, and so on. See Kendrick (1981).

[9]See Hall and Taylor (1997, Chapter 17).

Table 1: Optimal Policy Frontier

Experiment	Weight on Y	Weigh on $plev$	STD Y	STD $plev$
1	100	0	0.0479	0.0500
2	80	20	0.0489	0.0466
3	60	40	0.0499	0.0440
4	40	60	0.0509	0.0419
5	20	80	0.0520	0.0401
6	0	100	0.0531	0.0386

Figure 10: Optimal Policy Frontier



the model's other variables. For example, if we increase the weight on the policy variables, the policy frontier shifts up and to the right, farther away from the origin [the (0,0) point of zero deviations for Y and *plev*]. This is due to the more restricted possibilities for actively using the policy variables to reach the targets for Y and *plev*.

5 Stochastic Control

We now begin to take uncertainty into account. Indeed, macroeconomic models are only empirical approximations to reality. Thus, we want to consider that there are random shocks hitting the economy every time period (additive uncertainty), that the model parameters are just estimated values with associated variances and covariances (multiplicative uncertainty), and that the actual values of the model's variables and initial conditions are never known with certainty (measurement error).[10]

Stochastic control methods artificially generate a dynamic stochastic environment through random shock generation. They use specific procedures for choosing the optimal values for each period's policy variables: certainty equivalence (CE) when there is additive uncertainty only, open-loop feedback (OLF) when there is parameter uncertainty, and DUAL (adaptive control) when there is active learning. Moreover, there are specific mechanisms for updating parameter estimates, so that these methods allow us to perform sophisticated simulations.

In this section, we perform experiments incorporating some forms of additive and multiplicative uncertainty into Hall and Taylor's model. We proceed in three steps. First, we analyze the differences in qualitative behavior of the policy variables when different procedures for choosing their optimal values are used (specifically, CE versus OLF w/o update). Second, we compare the quantitative performances of the CE and OLF procedures within artificially generated stochastic environments, including passive learning mechanisms. Finally, we compute an optimal policy frontier.

Years ago William Brainard (1967) showed that, for a static model, the existence of parameter uncertainty causes the optimal policy variable to be used in a more conservative way as compared to the case of no parameter uncertainty. However, this finding cannot be translated to the case of dynamic models. The existence of dynamics makes the situation much more complex and opens new possibilities for policy management. One of the earliest applications of an OLF procedure in

[10]See Kendrick (1981).

a dynamic setting was by Tinsley, Craine, and Havenner (1974). Some analytical results have been provided by Chow (1973), Turnovsky (1975), Shupp (1976a, 1976b), and Craine (1979) and more recently by Mercado and Kendrick (2000) and Mercado (2004b) in connection with the qualitative behavior of the policy variables when the OLF procedure is used in a model with one state and one or two controls. There are no straightforward theoretical results for the case of models with several states and controls.

As shown in Chapter **??**, the procedure for choosing the controls in the presence of parameter uncertainty (OLF) differs from the standard deterministic QLP procedure or its certainty equivalent (CE) in that in the first case the variances and covariances associated with the model parameters have to be taken into account.

To illustrate some possible outcomes and show a first contrast between patterns of behavior generated by QLP and OLF w/o update procedures,[11] we perform an experiment with Hall and Taylor's model. As in the previous section, we assume that Y is 4 percent below its steady-state value at time zero and we keep the same weight structure and desired paths. We also assume that there is uncertainty in connection with six out of the eight control parameters in the B matrix, and that the standard deviation of each of these parameters is equal to 20 percent.

To carry out the experiment, we use the program `ht-02.dui`, which is basically the same as `ht-01.dui`, with some changes that we discuss later. From the `Specification:Stochastic Terms` option (Figure 11) we see that the problem is set as stochastic with parameter uncertainty. Then, from the Data:Size option, we see that we defined six uncertain parameters and use one Monte Carlo run, as shown in Figure 11. From the `Specification:Source` of `Random Terms` main menu option, we select the Read In option, as shown in Figure 12. However, we set all those random terms equal to zero. To do so, we go to the `Data:Additive Noise Terms` main menu option and, as shown in Figure 13, select the `XSIS` option. When we do so, a dialog box containing the matrix of additive noise terms is displayed, and we see that all its element are set to zero.

The information related to the uncertain parameter is provided in Duali by means of one vector and two matrices. The theta vector of the initial values of uncertain parameters (`TH0`) contains the uncertain parameters values. The matrix that indicates which parameters in the model are treated as uncertain (`ITHN`) provides a mapping from the position in the `TH0` vector to a position in the system equations matrices. The first column indicates the matrix (0 for the A matrix, 1

[11] For a detailed discussion of the OLF without update procedure see Amman and Kendrick (1999a, Chapter 5).

Figure 11: Stochastic Terms

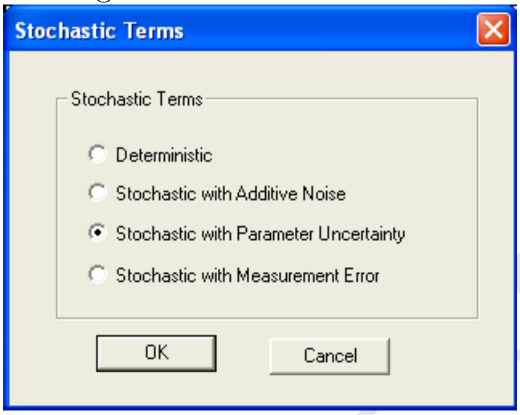


Figure 12: Model Size

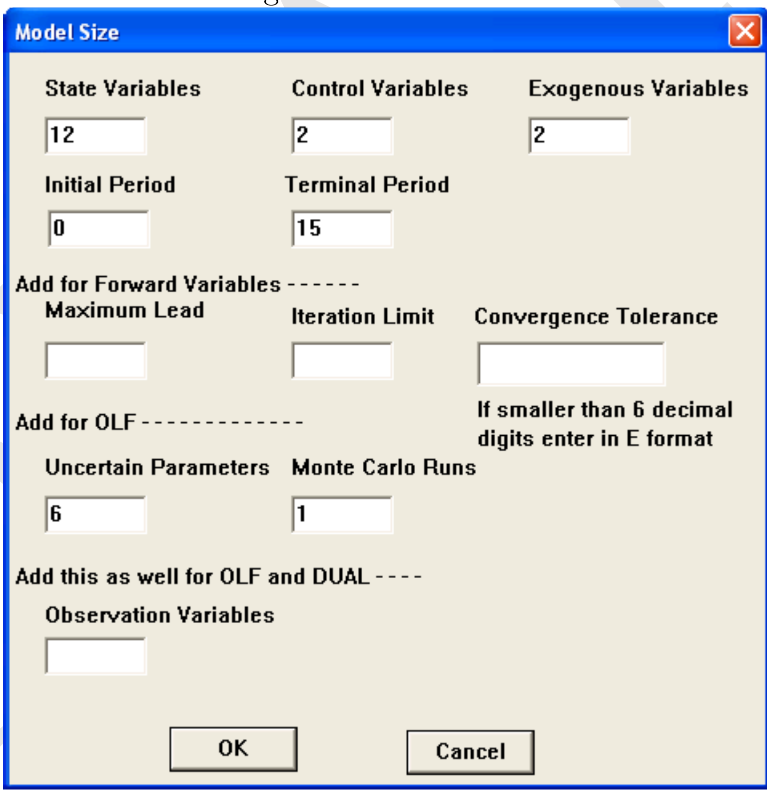


Figure 13: Source Random Terms

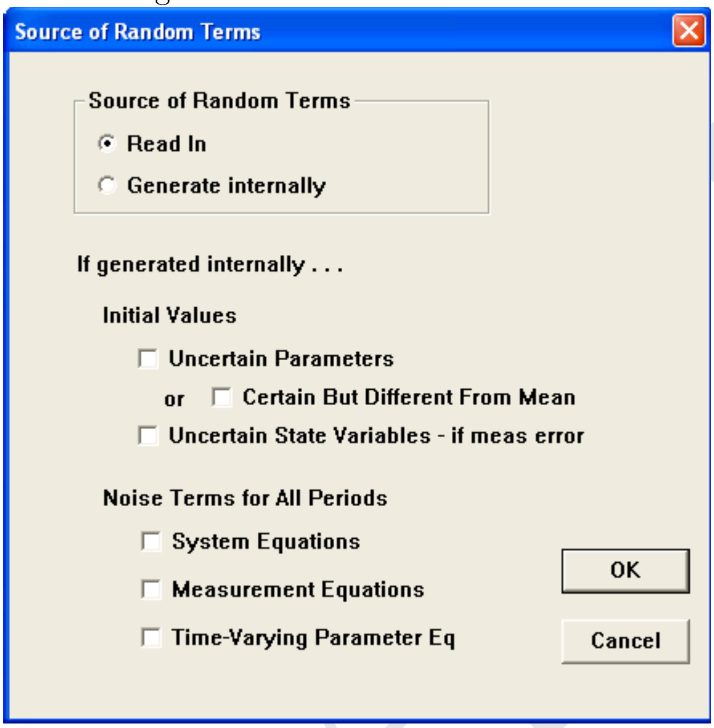
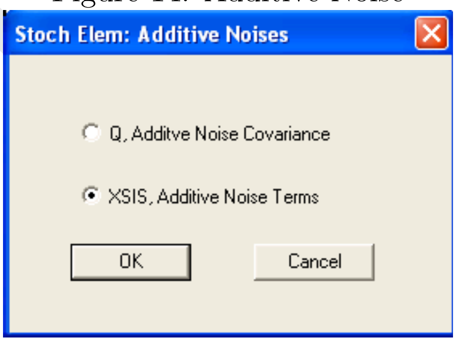


Figure 14: Additive Noise



for the B matrix, and 2 for the c vector) and the second and third columns indicate the row and column number of the parameter in the matrix. Finally, `SITT0` is the variance-covariance matrix corresponding to the uncertain parameters:

$$\texttt{TH0} = \begin{bmatrix} b_{11} = 0.433 \\ b_{12} = 0.231 \\ b_{21} = -9.763 \\ b_{22} = 4.386 \\ b_{41} = -2.442 \\ b_{42} = 1.097 \end{bmatrix}$$

$$\texttt{ITHN} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 4 & 1 \\ 1 & 4 & 2 \end{bmatrix}$$

$$\texttt{SITT0} = \begin{bmatrix} 0.00749 & & & & & \\ & 0.00213 & & & & \\ & & 3.81264 & & & \\ & & & 0.76947 & & \\ & & & & 0.23853 & \\ & & & & & 0.04813 \end{bmatrix}$$

All three matrices remain constant during the simulation. The elements in `SITT0` are computed by taking 20 percent of the corresponding element in TH0 and then squaring the result. Thus, for the b_{11} coefficient this is

$$[(0.2)\ (0.433)]^2 = 0.00749$$

From the `Data:Parameter` Uncertainty menu option we obtain the dialog box shown in Figure 14. When selecting each of the first three options, the corresponding vector or matrix is displayed.

The graphs in Figure 16 show the results obtained for government expenditure and for the money supply when selecting the main menu option `Solve: OLF (w/o update)`. They also contrast these results with those corresponding to the deterministic (QLP) solution as obtained in Section using the program `ht-01.dui`.

As can be seen in the graphs in Figures 16-17, the use of government expenditure is slightly more *cautious* with the OLF w/o update procedure in the first few periods. This is in line with the Brainard result mentioned before. However, the reverse is true for the case of the money supply, which is used *more aggressively*

Figure 15: Uncertain Parameters

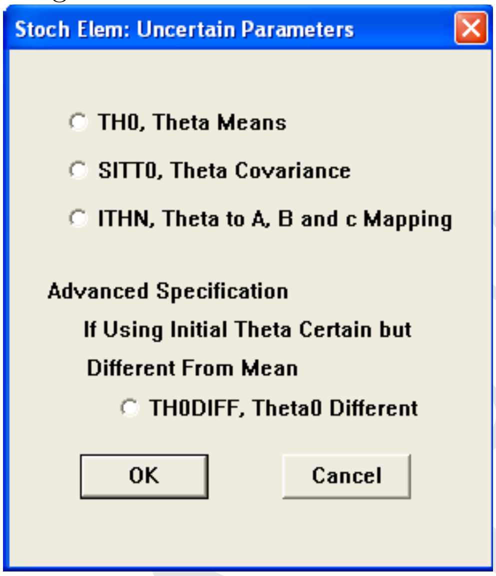


Figure 16: Government Spending

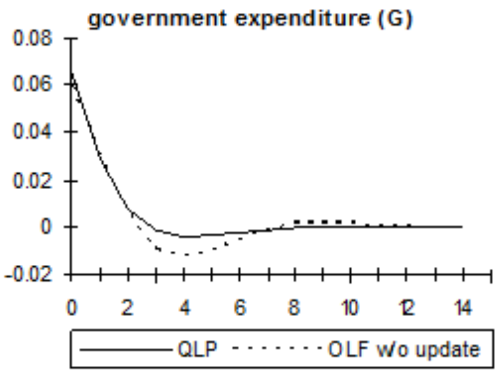
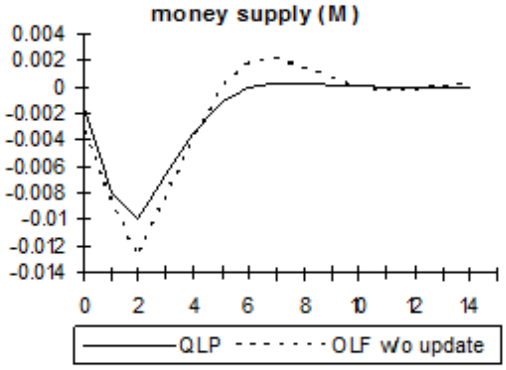


Figure 17: Money Supply



with OLF w/o update. Thus, we can see how going from a univariate to a multivariate setting may have important consequences, as is also the case of a change from static to dynamic models.

It is interesting to explore the consequences of increasing the level of uncertainty of the model parameter's corresponding to one of the policy variables. For example, let us assume that we now double the standard deviation of the parameters corresponding to government expenditure (parameters b_{12} from 0.00213 to 0.00853, b_{22} from 0.76947 to 3.07791, and b_{42} from 0.04813 to 0.19254, while leaving the other elements of SITT0 unchanged), that is, increasing the variance of these three parameters, which are associated with government expenditures, from 20 to 40 percent. Then the `SITT0` matrix becomes

$$SITT0 = \begin{bmatrix} 0.00749 & & & & & \\ & 0.00853 & & & & \\ & & 3.81264 & & & \\ & & & 3.07791 & & \\ & & & & 0.23853 & \\ & & & & & 0.19254 \end{bmatrix} \quad (3)$$

The graphs in Figure 18 contrast the behavior of the policy variables for this experiment (named OLF w/o update-B) against the behavior shown by the same variables in the experiment analyzed earlier (named, as previously, OLF w/o update). To run this experiment, use file `ht-02.dui`, introducing the corresponding changes in the `SITT0` matrix.

As one might expect, the increase in the relative uncertainty of government expenditure parameters induces a more cautious use of that policy variable, at

Figure 18: Government Expenditure

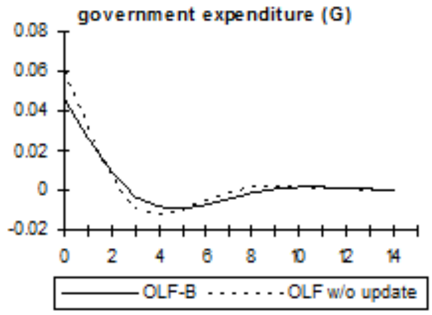
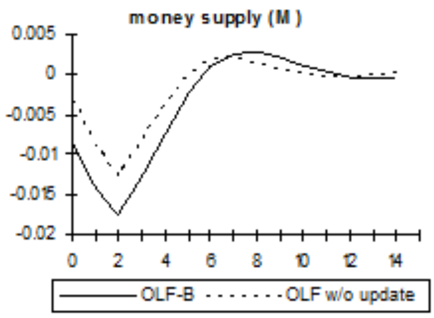


Figure 19: Money Supply



least during the first periods. At the same time the money supply, now with a relatively lower associated uncertainty, is also used more actively during the first periods. Though these findings seem plausible, they do not reflect any theoretical result, since such results are not yet available. As with the previous experiment, we might find different results with a different model.

6 Stochastic Control with Parameter Updating

We now move toward a more complex stochastic environment. As in the previous section, we assume that some of the model parameters are uncertain, but now we also assume that the model is constantly shocked by additive noise, that the true model is not known to the policy maker, and that a passive-learning process takes place. We perform several Monte Carlo runs to contrast the performance of two procedures: CE and OLF.

The general structure of each Monte Carlo run is as follows. At time zero, a vector of model parameters is drawn from a normal distribution whose mean and variances are those of matrices `TH0` and `SITT0`. Then, at each time t, we have:

1. Random generation of a vector of an additive shocks.
2. Computation of the optimal controls for periods k to N(terminal period).
3. Propagation of the system one period forward (from period k to period $k+1$) applying the vector of controls (for period k only) computed in step 2.
4. Updating of the next period parameter estimates (both means and variance-covariance elements).

For choosing the optimal control at each period (step 2) we use either a CE procedure or, alternatively, an OLF procedure. For the projection-updating mechanism (step 4) we use a Kalman filter.

Thus, each Monte Carlo run begins with a vector of parameter estimates that is different from their *true* value. Using this parameter vector, the policy maker computes (with a CE or an OLF procedure) the optimal values of the controls and then applies those values corresponding to time k only. However, the response of the economic system (its forward movement from time k to time $k+1$ is generated by the computer using the *true* parameter values, which are unknown to the policy maker. Then, at period $k+1$ a new observation is made of the state vector, which is used to compute updated parameter estimates with a Kalman filter. After a number of time periods, the sequence of updated estimates should begin to converge to their *true* value.

As in the previous section, we assume that there is uncertainty in connection with six of the control parameters in the B matrix and that the standard deviation of each of these parameters is equal to 20 percent. Then matrices `TH0`, `SITT0`, and `ITHN` are the same as in equation (3). We also assume that GDP (Y) and the price level (*plev*) are hit by additive shocks with 2 percent standard deviation, while the real interest rate (R) and the nominal exchange rate (E) experience shocks with 5 percent standard deviation. Thus, the variance-covariance matrix of additive noises (Q) is as follows[12]:

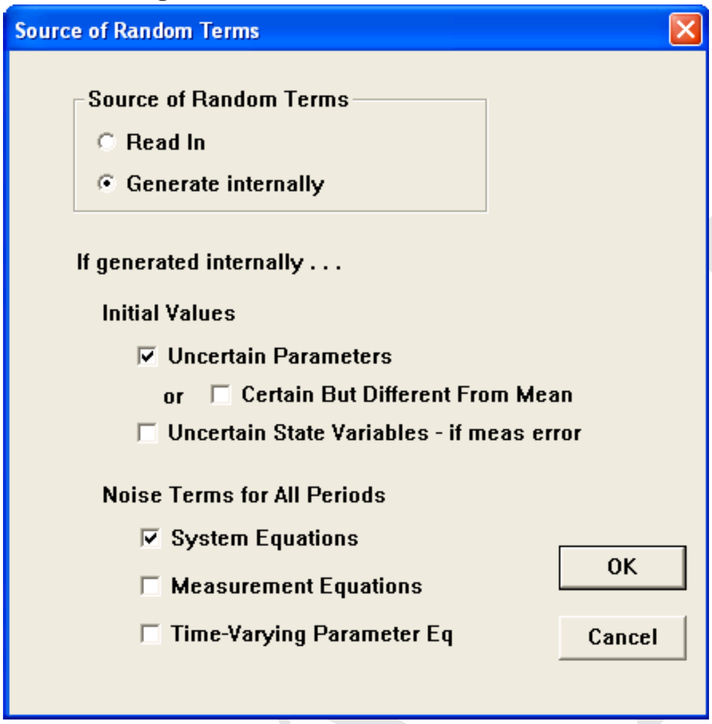
$$Q = \begin{bmatrix} 0.0004 & & & & & & & & & & & \\ & 0.0025 & & & & & & & & & & \\ & & 0.0004 & & & & & & & & & \\ & & & 0.0025 & & & & & & & & \\ & & & & 0.1^{(-9)} & & & & & & & \\ & & & & & 0.1^{(-9)} & & & & & & \\ & & & & & & 0.1^{(-9)} & & & & & \\ & & & & & & & 0.1^{(-9)} & & & & \\ & & & & & & & & 0.1^{(-9)} & & & \\ & & & & & & & & & 0.1^{(-9)} & & \\ & & & & & & & & & & 0.1^{(-9)} & \\ & & & & & & & & & & & 0.1^{(-9)} \end{bmatrix}$$

We perform 100 Monte Carlo runs to compare the performance of the CE procedure against that of OLF. To do so, we use the file ht-03.dui, which is similar to the `ht-02.dui` file used in the previous sections, but has some modifications. If we select the `Data:Additive Noise Terms:Additive Noise Covariance menu` options we see the Q matrix shown above. In the `Specification:Data:Size` menu option we have to specify the number of Monte Carlo runs. That option is set to three. It may be better to make a first run like this with a small number of Monte Carlo runs to gain familiarity with the procedure. However, to perform a more serious experiment, we set it equal to 100. Be aware that this may take some minutes to run, depending on the computer. Then, in the `Specification:Source of Random Terms` option, we check the options:

1. Generate Internally;
2. Uncertain Parameters;
3. System Equations, as shown in Figure 20.

[12] We want the shocks to affect only contemporaneous variables and not their lagged values. However, if we set to zero the elements of the Q matrix corresponding to lagged variables, Duali gives us an error message. That is why we set those elements equal to the minimum possible value (0.000000001).

Figure 20: Source Random Terms



We then choose the `Solve:Compare Print option`, obtaining a dialog box like the one shown in Figure 21, where we see that the options CE and OLF have been selected. When we click OK, we are asked to provide a debug file name. After we do so, a dialog box like the one shown in Figure 22 is displayed.

In the box shown in Figure 22 we have many options to build a very detailed solution report with summary, intermediate, and final results, among other things. We just check the *Only results summary* option, leaving all the others blank, then click OK, and Duali starts solving the problem. In the meantime, two dialog boxes named Method Count and Average Criterion Values are displayed. We click OK for each of them. Finally, once the run is completed, the results are stored in the file we specified as the debug print file. It is best to exit from Duali before examining the results file in an editor. When doing Monte Carlo runs in Duali it is important to look for the results in the debug print file and not in the Display results on line since the Display numbers are only for the last Monte Carlo run and not for the averages across all the runs.

The results in the debug print file corresponding to our 100 Monte Carlo runs are shown in Table 2. The OLF procedure does slightly better than the CE, not only in connection with the average criterion value, but also in terms of the number of Monte Carlo runs with the lowest criterion. As can be appreciated in

Figure 21: Method

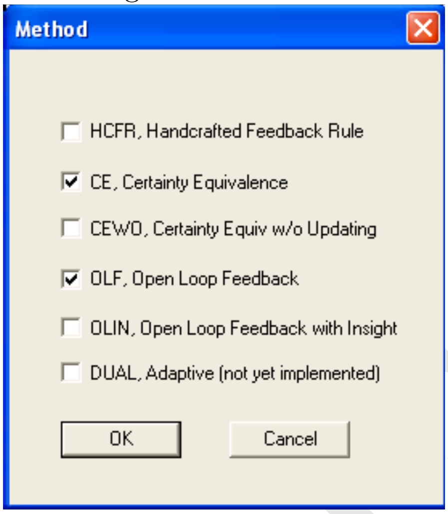


Figure 22: Debug Print

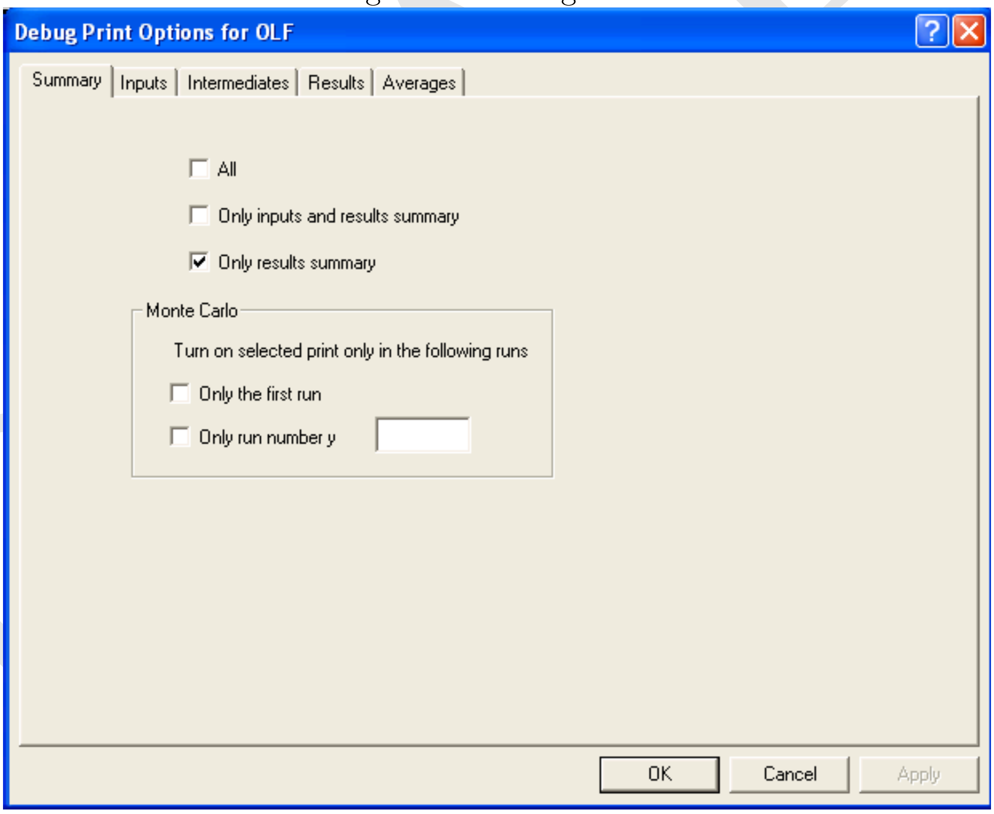
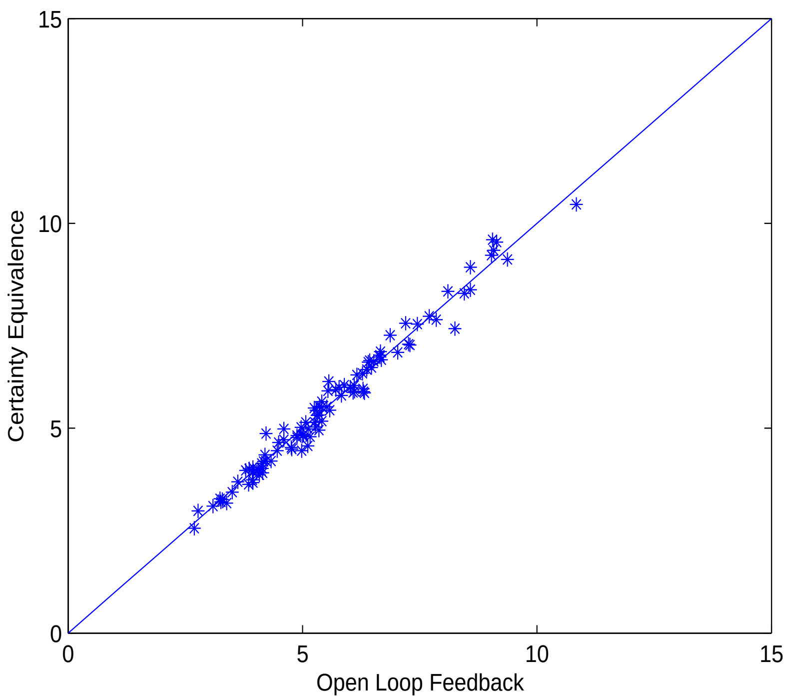


Table 2: Monte Carlo Results

	CE	OLF
Average criterion value	5.60	5.59
Runs with lowest criterion	47	53

the graph shown in Figure 23, where each diamond represents the value of the criterion function for one Monte Carlo run, most of the diamonds are close to the 45° line, indicating a similar performance for both procedures. There are no significant outliers that could be introducing a bias in the computed average criterion values.

Figure 23: Scatter diagram OLF versus Certainty Equivalence



These results are contrary to what one would expect intuitively, since in the presence of parameter uncertainty OLF might be expected to do not only slightly but significantly better than CE. However, we have to mention that no theoretical results have as yet been developed in connection with the relative performance of CE versus OLF. The experimental results are conditioned on the model structure, its parameter mean, and variance values, and may well change (in any direction) in a different context. For example, working with a different model Amman and Kendrick (1999b) find OLF results that are substantially better than the CE results, and Lee (1998) obtains similar results from a substantially larger model.

7 Experiments

In Section 2 we analyzed the autonomous and optimal policy responses to a negative shock in net exports. You may want to analyze other shocks implying different initial conditions for the model's endogenous variables. Or you may analyze the effects of changes in the exogenous variables and the corresponding optimal policy responses. You may also want to put a very high weight (priority) on the money supply or government expenditure so that in fact only one policy variable is used to control the system. Then, you might contrast these cases against the analysis performed in this chapter in which the two controls were assigned equal weights. Finally, you may want to assign different sets of equal values to the weights on the control variables for the experiment presented earlier to observe the displacement effects that these changes have on the optimal policy frontier.

In Section 4 we analyzed the optimal response of the policy variables when parameter uncertainty was taken into account. In particular, we increased the relative uncertainty of government expenditure parameters and found that this induced a more cautious use of that policy variable during the first periods. You may want to continue increasing the level of uncertainty of those parameters and see the pattern of responses in the policy variables. Or you might increase the relative uncertainty of the money supply parameters.

8 Further Reading

For one of the first applications of control theory methods to macroeconomics models see Pindyck (1973). Chow (1975) provides an introduction to the analysis and control of dynamic economic systems. Kendrick (1981) presents a systematic treatment of stochastic control for economic models, with particular focus on passive and active learning methods. Holly and Hughes-Hallett (1989) also present a systematic treatment of optimal control methods, with special treatment of expectations and uncertainty. Sengupta and Fanchon (1997) describe methods and a wide range of applications of control theory in economics. Chiarella and Flaschel (2000) provide a nonlinear dynamics approach to macroeconomics. For a related global dynamics approach to analyzing overlapping generation models see Gomis and Haro (2003).

Kendrick (2005) reviews the historical development and likely future paths in the field of stochastic control in economics.

For a most interesting visual approach to the use of control theory methods in economics that uses the Simulink system with **MATLAB** see Herbert and Bell (1997). For an observer approach to control methods in economics see Herbert (1998).

Amman and Kendrick (1999a) provide a users' guide to Duali, with a variety of tutorial level chapters dealing with different control methods and models.

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Appendix

A Linearization of Hall and Taylor's Model

Introduction

The linearization method that we use is known as Johansen's method [see Johansen (1960)]. It involves transforming all the variables in the model into percentage changes with respect to a base case. We introduced this method in Chapter 8, where we learned that there are some rules, analogous to differentiation, that simplify the task of linearizing a model, and we apply those rules here.

Remember that since the Hall and Taylor model is a dynamic one, all its variables have an explicit or implicit time subscript. It is important to understand that the percentage changes in each variable are changes with respect to a baseline case (the point of linearization) and not with respect to *the previous period*. If our baseline case is the steady state and, say, X^*_{t+4} takes the value 0.01, this means that the variable X, at time $t+4$, is 1 percent higher than its steady-state value. It does not mean that X^*_{t+4} is 1 percent higher than X^*_{t+3}.

Linearization of the model

The steady-state solution for Hall and Taylor's original nonlinear model in levels is: $Y = 6000$, $R = 0.05$, $plev = 1$, and $E = 1$. These steady-state values correspond to the following values for policy and exogenous variables: $M = 900$, $G = 1200$, $YN = 6000$, and $plevw = 1$. We pick the steady-state solution as our baseline or point of linearization. Thus, the expression in the sum rule in Chapter 8 for

$$X = Y + Z \tag{A-1}$$

becomes

$$X^* = s_y Y^* + s_z Z^* \tag{A-2}$$

where X^*, Y^*, and Z^* are percentage deviations of the corresponding level variables and s_y and s_z are the shares

$$s_y = \frac{Y_{ss}}{Y_{ss} + Z_{ss}} \tag{A-3}$$

and

$$s_z = \frac{Z_{ss}}{Y_{ss} + Z_{ss}} \tag{A-4}$$

where the subscript ss means steady-state value.
The original twelve-equation model contains the following equations:

IS-LM

GDP identity:

$$Y = C + I + G + X \tag{A-5}$$

Disposable income:

$$Y^d = (1 - t)\, Y \tag{A-6}$$

Consumption:

$$C = a + bY^d \tag{A-7}$$

Investment:

$$I = e - dR \tag{A-8}$$

Money demand:

$$M/P = kY - hR \tag{A-9}$$

Expectations Augmented Phillips Curve
Expected inflation:

$$\pi^e = \alpha\pi_{-1} + \beta\pi_{-2} \tag{A-10}$$

Inflation rate:

$$\pi = \pi^e + f\left\{\left(Y_{-1} - Y_N\right)/Y_N\right\} \tag{A-11}$$

Price level:

$$P = P_{-1}\left(1 + \pi\right) \tag{A-12}$$

Foreign Sector
Real exchange rate:

$$EP/P_W = q + vR \tag{A-13}$$

Net exports:

$$X = g - mY - nEP/P_W \tag{A-14}$$

IS-LM
Government deficit:

$$G_d = G - tY \tag{A-15}$$

Unemployment Rate:

$$U = U_N - \mu\,\{(Y - Y_N)/Y_N\} \tag{A-16}$$

To obtain the equation for Y^* (i.e., GDP percent deviation from steady state), we substitute equations (A-6), (A-7), (A-8), and (A-14) into equation (A-5). Linearizing, we obtain

$$Y^* = -sa_{12}R^* - sa_{13}plev^* - sa_{14}E^* + sb_{12}G^* + sc_{12}plevw^* \tag{A-17}$$

where[13]

$$aux \;=\; \{1 - [b(1-t) - n]\} \tag{A-18}$$

$$sa_{12} \;=\; (dR_{ss})/(auxY_{ss}) \tag{A-19}$$

$$\begin{aligned} sa_{13} &= (n\ E_{ss}\ plevw_{ss}\ plev_{ss})/(aux\ plevw_{ss}^2\,Y_{ss}) \\ sa_{14} &= (n\ plev_{ss}\ plevw_{ss}E_{ss})/(aux\ plevw_{ss}2\,Y_{ss}) \end{aligned} \tag{A-20}$$

$$\begin{aligned} sb_{12} &= G_{ss}/(aux\ Y_{ss}) \\ sc_{12} &= (n\ E_{ss}\ plev_{ss}\ plevw_{ss})/(aux\ plevw_{ss}2\ Y_{ss}) \end{aligned} \tag{A-21}$$

To derive the equation for R^* (real interest rate), linearizing and rearranging equation (refeq5) we obtain

$$R^* = -sa_{21}\ Y^* - \ sa_{23}\ plev^* + \ sb_{21}\ M^* \tag{A-22}$$

where

$$sa_{21} = -(k\ Y_{ss})/(h\ R_{ss});\ \ sa_{23} = -M_{ss}\ plev_{ss}/(h\ plev_{ss}2\ R_{ss});\ \ sb_{21} = -M_{ss}/(h\ plev_{ss}\ R_{ss}). \tag{A-23}$$

To obtain the equation for $plev^*$ (domestic price level), substitute equation (refeq6)

$$\pi^e = \alpha\pi_{-1} + \beta\pi_{-2} \tag{A-24}$$

[13] The reason we define the coefficients as sa_{12}, and so on., becomes clear later, when we write the model in matrix notation.

into equation (refeq7)

$$\pi = \pi^e + f\left\{\left(Y_{-1} - Y_N\right)/Y_N\right\} \tag{A-25}$$

to get

$$\pi = \ \alpha\pi_{-1} + \ \beta\pi_{-2} + \ f(Y_{-1} \ - \ YN)/YN \tag{A-26}$$

This expression combines variables in levels and variables in rates of growth. To avoid the confusion that may arise from working with percentage changes of rates of growth, we proceed as follows. Taking into account that the percent deviation of a variable for small deviations is approximately equal to its corresponding log difference, we can write

$$(Y_{-1} - YN)/YN \approx \ln Y_{-1} - \ln YN \tag{A-27}$$

Now, we can rewrite equation (refeq8), that is,

$$\begin{array}{l} P = P_{-1}\left(1+\pi\right) \\ 1+\pi = \frac{P}{P_{-1}} \\ \pi = \frac{P-P_{-1}}{P} \end{array} \tag{A-28}$$

as

$$\pi = (plevplev_{-1})/plev_{-1} \tag{A-29}$$

and applying the same property as above, we can write

$$\pi \approx \ln plev - \ln plev_{-1} \tag{A-30}$$

and then

$$\pi_{\text{-}1} \ \approx \ \ln \ plev_{-1} - \ \ln \ plev_{-2} \tag{A-31}$$

$$\pi_{-2} \approx \ln plev_{-2} - \ln plev_{-3} \tag{A-32}$$

Now, substituting (e.4) and (e.6) –(e.8) into (e.3) and linearizing, we obtain

$$plev^* = sa1_{31}Y^*_{-1} + \ sa1_{33}\ plev^*_{-1} + \ sa2_{33}\ plev^*_{-2} + \ sa3_{33}\ plev^*_{-3} + \ sc_{31}\ YN^* \tag{A-33}$$

where

$$sa1_{31} = f\ ;\ sa1_{33} = \ 1+\alpha\ ;\ sa2_{33} = \ \beta - \alpha\ ;\ sa3_{33} = -\beta\ ;\ sc_{31} = -f. \tag{A-34}$$

Finally, to derive the equation for E^* (nominal exchange rate), linearizing equation (refeq9) we obtain

$$E^* \;=\; -\, sa_{42}\; R^* - \; sa_{43}\; plev^* + \; sc_{42}\; plevw^* \tag{A-35}$$

where

$$sa_{42} = -\; v\; plevw_{ss}\; R_{ss}/(plev_{ss}\; E_{ss});\;\; sa_{43} = 1;\;\; sc_{42} = 1. \tag{A-36}$$

Since variables G_d (government deficit) and U (unemployment rate) do not have any *feedback* with the other equations in the model, we can ignore equations (A-15) and (A-16). In summary, the four equations of our model are (e.1), (e.2), (e.9), and (e.10):

$$Y^* \;= -sa_{12}R^* - \; sa_{13}plev^* - \; sa_{14}E^* + \; sb_{12}G^* + \; sc_{12}plevw^* \tag{A-37}$$

$$R^* = -sa_{21}\; Y^* - \; sa_{23}\; plev^* + \; sb_{21}\; M^* \tag{A-38}$$

$$plev^* = sa1_{31}Y^*_{-1} + \; sa1_{33}\; plev^*_{-1} + \; sa2_{33}\; plev^*_{-2} + \; sa3_{33}\; plev^*_{-3} + \; sc_{31}\; YN^* \tag{A-39}$$

$$E^* \;=\; -\, sa_{42}\; R^* - \; sa_{43}\; plev^* + \; sc_{42}\; plevw^* \tag{A-40}$$

Note that since in this linearized representation all variables are in percent deviations, their steady-state values are all zeroes.

Writing our structural model in matrix notation, we obtain

$$SA\; X \;=\; SA1\; X_{-1} \;+\; SA2\; X_{-2} \;+\; SA3\; X_{-3} \;+\; SB\; U \;+\; SC\; V \tag{A-41}$$

where:

$$X = \begin{bmatrix} Y^* \\ R^* \\ plev^* \\ E^* \end{bmatrix} \quad U = \begin{bmatrix} M^* \\ G^* \end{bmatrix} \quad V = \begin{bmatrix} YN^* \\ plevw^* \end{bmatrix} \tag{A-42}$$

and

$$SA = \begin{bmatrix} 1 & sa_{12} & sa_{13} & sa_{14} \\ sa_{21} & 1 & sa_{23} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & sa_{42} & sa_{43} & 1 \end{bmatrix} \quad SA1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ sa1_{31} & 0 & sa1_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{A-43}$$

$$\begin{gathered} SA2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & sa2_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad SA2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & sa3_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ SB = \begin{bmatrix} 0 & sb_{12} \\ sb_{21} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad SC = \begin{bmatrix} 0 & sc_{21} \\ 0 & 0 \\ sc_{31} & 0 \\ 0 & sc_{42} \end{bmatrix} \end{gathered} \tag{A-44}$$

We have obtained above a structural model that is, of course, a simultaneous system of equations. To find its reduced form, we have to get rid of this simultaneity and express each endogenous variable as only a function of policy, exogenous, and predetermined variables. This can be done easily.

From Eequation (e.11) the reduced form can be obtained as

$$X \;=\; RA1\; X_{-1} + \; RA2\; X_{-2} \;+\; RA3\; X_{-3} \;+\; RB\; U \;+\; RC\; V \tag{A-45}$$

where

$$\begin{aligned} RA1 &= SA^{-1}SA1 \\ RA2 &= SA^{-1}SA2 \\ RA3 &= SA^{-1}SA3 \\ RB &= SA^{-1}SB \\ RC &= SA^{-1}SC \end{aligned}$$

Equation (e.12) is a third-order system difference equation (the maximum lag is equal to 3). It is necessary to reduce it to a first-order system, which is called the *state-space* representation.[14] For instance, to analyze some dynamic properties of the linearized model we have to know its characteristic roots, and these are equal to the eigenvalues of the matrix of the first-order version (matrix

[14]The concept of state-space goes beyond this, but we do not deal with it here.

A below).[15] Moreover, to determine the model's controllability or to perform policy experiments with Duali, the input model has to be in state-space form. To make this transformation, we augment the state variable by taking the following steps:We define the new vectors XL_{-1} and XLL_{-1} as

$$XL_{-1} = \begin{bmatrix} xlY^*_{-1} \\ xlR^*_{-1} \\ xlplev^*_{-1} \\ xlE^*_{-1} \end{bmatrix} = X_{-2} = \begin{bmatrix} Y^*_{-2} \\ R^*_{-2} \\ plev^*_{-2} \\ E^*_{-2} \end{bmatrix} \tag{A-46}$$

$$XLL_{-1} = \begin{bmatrix} xllY^*_{-1} \\ xllR^*_{-1} \\ xllplev^*_{-1} \\ xllE^*_{-1} \end{bmatrix} = XL_{-2} = \begin{bmatrix} xlY^*_{-2} \\ xlR^*_{-2} \\ xlplev^*_{-2} \\ xlE^*_{-2} \end{bmatrix} = X_{-3} = \begin{bmatrix} Y^*_{-3} \\ R^*_{-3} \\ plev^*_{-3} \\ E^*_{-3} \end{bmatrix} \tag{A-47}$$

We then rewrite (e.12) as

$$X = RA1X_{-1} + RA2\,XL_{-1} + RA3\,XLL_{-1} + RBU + RCV \tag{A-48}$$

define the augmented state vector x

$$x = \begin{bmatrix} X \\ XL \\ XLL \end{bmatrix} \tag{A-49}$$

rewrite (e.13) and (e.14) as

$$XL \;=\; X_{-1} \tag{A-50}$$

$$XLL \;=\; XL_{-1} \;=\; X_{-2} \tag{A-51}$$

and finally transform (e.15) into its state-space representation as

$$x \;=\; A\,x_{-1} \;+\; B\,U \;+\; C\,V \tag{A-52}$$

where U and V are the same as before, and

$$A = \begin{bmatrix} RA1 & RA2 & RA3 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \; B = \begin{bmatrix} RB \\ 0 \\ 0 \end{bmatrix} \; C = \begin{bmatrix} RC \\ 0 \\ 0 \end{bmatrix} \tag{A-53}$$

[15]See Mercado and Kendrick (1999).

and I is a (4×4) identity matrix and the 0's are (4×4) and (4×2) matrices of zeros as appropriate.

In Hall and Taylor's model, the policy variables contemporaneously affect the model's endogenous variables, and this is also true for its state-space representation. In order to obtain a proper state-state representation, that is, one in which the control variables also appear with one lag, we have to assume that there is a one-period delay between a policy decision and its implementation. Then, we can substitute M^*_{-1} for M^*, and G^*_{-1} for G^*. We also assume that the exogenous variables YN^* and $plevw^*$ affect the system with one lag instead of contemporaneously. Expressing the model in this way, we can make use of many results from the optimal control literature, which works with models with one-lag controls. The Duali software also works in this way.

Thus, in matrix notation, with numerical parameter values derived from the corresponding original model parameter values, where all the variables are percent deviations from the steady state, the state-space representation of Hall and Taylor's model can then be written as in equation (A-5) in Chapter 17.