A MATHEMATICAL THEORY OF SAVING

I

THE first problem I propose to tackle is this: how much of its income should a nation save? To answer this a simple rule is obtained valid under conditions of surprising generality; the rule, which will be further elucidated later, runs as follows.

The rate of saving multiplied by the marginal utility of money should always be equal to the amount by which the total net rate of enjoyment of utility falls short of the maximum possible rate of enjoyment.

In order to justify this rule it is, of course, necessary to make various simplifying assumptions: we have to suppose that our community goes on for ever without changing either in numbers or in its capacity for enjoyment or in its aversion to labour; that enjoyments and sacrifices at different times can be calculated independently and added; and that no new inventions or improvements in organisation are introduced save such as can be regarded as conditioned solely by the accumulation of wealth.[1]

One point should perhaps be emphasised more particularly; it is assumed that we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination; we shall, however, in Section II include such a rate of discount in some of our investigations.

We also ignore altogether distributional considerations, assuming, in fact, that the way in which consumption and labour are distributed between the members of the community depends solely on their total amounts, so that total satisfaction is a function of these total amounts only.

Besides this, we neglect the differences between different kinds of goods and different kinds of labour, and suppose them to be expressed in terms of fixed standards, so that we can speak simply of quantities of capital, consumption and labour without discussing their particular forms.

Foreign trade, borrowing and lending need not be excluded, provided we assume that foreign nations are in a stable state, so

[1] *I.e.* they must be such as would not occur without a certain degree of accumulation, but could be foreseen given that degree.

that the possibilities of dealing with them can be included on the constant conditions of production. We do, however, reject the possibility of a state of progressive indebtedness to foreigners continuing for ever.

Lastly, we have to assume that the community will always be governed by the same motives as regards accumulation, so that there is no chance of our savings being selfishly consumed by a subsequent generation; and that no misfortunes will occur to sweep away accumulations at any point in the relevant future.

Let us then denote by $x(t)$ and $a(t)$ the total rates of consumption and labour of our community, and by $c(t)$ its capital at time t. Its income is taken to be a general function of the amounts of labour and capital, and will be called $f(a,c)$; we then have, since savings plus consumption must equal income,

$$\frac{dc}{dt} + x = f(a,c) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now let us denote by $U(x)$ the total rate of utility of a rate of consumption x, and by $V(a)$ the total rate of disutility of a rate of labour a; and the corresponding marginal rates we will call $u(x)$ and $v(a)$;

so that
$$u(x) = \frac{dU(x)}{dx}$$
$$v(a) = \frac{dV(a)}{da}.$$

We suppose, as usual, that $u(x)$ is never increasing and $v(a)$ never decreasing.

We have now to introduce a concept of great importance in our argument. Suppose we have a given capital c, and are going neither to increase nor decrease it. Then $U(x) - V(a)$ denotes our net enjoyment per unit of time, and we shall make this a maximum, subject to the condition that our expenditure x is equal to what we can produce with labour a and capital c. The resulting rate of enjoyment $U(x) - V(a)$ will be a function of c, and will, up to a point, increase as c increases, since with more capital we can obtain more enjoyment.

This increase of the rate of enjoyment with the amount of capital may, however, stop for either of two reasons. It might, in the first place, happen that a further increment of capital would not enable us to increase either our income or our leisure; or, secondly, we might have reached the maximum conceivable rate of enjoyment, and so have no use for more income or leisure. In either case a certain finite capital would give us the greatest rate of enjoyment economically *obtainable*, whether or not this was the greatest rate *conceivable*.

On the other hand, the rate of enjoyment may never stop increasing as capital increases. There are then two logical possibilities: either the rate of enjoyment will increase to infinity, or it will approach asymptotically to a certain finite limit. The first of these we shall dismiss on the ground that economic causes alone could never give us more than a certain finite rate of enjoyment (called above the maximum conceivable rate). There remains the second case, in which the rate of enjoyment approaches a finite limit, which may or may not be equal to the maximum conceivable rate. This limit we shall call the maximum *obtainable* rate of enjoyment, although it cannot, strictly speaking, be obtained, but only approached indefinitely.

What we have in the several cases called the maximum obtainable rate of enjoyment or utility we shall call for short *Bliss* or *B*. And in all cases we can see that the community must save enough either to reach *Bliss* after a finite time, or at least to approximate to it indefinitely. For in this way alone is it possible to make the amount by which enjoyment falls short of bliss summed throughout time a finite quantity; so that if it should be possible to reach bliss or approach it indefinitely, this will be infinitely more desirable than any other course of action. And it is bound to be possible, since by setting aside a small sum each year we can in time increase our capital to any desired extent.[1]

Enough must therefore be saved to reach or approach bliss some time, but this does not mean that our whole income should be saved. The more we save the sooner we shall reach bliss, but the less enjoyment we shall have now, and we have to set the one against the other. Mr. Keynes has shown me that the rule governing the amount to be saved can be determined at once from these considerations. But before explaining his argument it will be best to develop equations which can be used in the more general problems which we shall consider later.

[1] As it stands this argument is incomplete, since in the last case considered above bliss was the limiting value, as capital tends to infinity, of the enjoyment obtainable by spending our *whole income*, and so making no provision for increasing capital further. The lacuna can easily be filled by remarking that to save £$\frac{1}{n}$ in the nth year would be sufficient to increase capital to infinity (since $\Sigma\frac{1}{n}$ is divergent), and that the loss of income $\left(£\frac{1}{n}\right)$ would then decrease to zero, so that the limiting values of income and expenditure would be the same.

The first of these comes from equating the marginal disutility of labour at any time to the product of the marginal efficiency of labour by the marginal utility of consumption at that time,

i.e.
$$v(a) = \frac{\partial f}{\partial a} u(x) \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The second equates the advantage derived from an increment Δx of consumption at time t, to that derived by postponing it for an infinitesimal period Δt, which will increase its amount to $\Delta x\left(1 + \frac{\partial f}{\partial c}\Delta t\right)$, since $\frac{\partial f}{\partial c}$ gives the rate of interest earned by waiting. This gives

$$u\{x(t)\} = \left\{1 + \frac{\partial f}{\partial c}\Delta t\right\} u\{x(t + \Delta t)\}$$

or in the limit

$$\frac{d}{dt}u(x(t)) = -\frac{\partial f}{\partial c} \, . \, u(x(t)) \quad . \quad . \quad . \quad . \quad (3)$$

This equation means that $u(x)$, the marginal utility of consumption, falls at a proportionate rate given by the rate of interest. Consequently x continually increases unless and until either $\frac{\partial f}{\partial c}$ or $u(x)$ vanishes, in which case it is easy to see that bliss must have been attained.

Equations (1), (2) and (3) are sufficient to solve our problem provided we know c_0, the given capital with which the nation starts at $t = 0$, the other "initial condition" being supplied by considerations as to the behaviour of the function as $t \to \infty$.

To solve the equations we proceed as follows: noticing that x, a and c are all functions of one independent variable, the time, we have

$$\begin{aligned}\frac{d}{dx}\{u(x) \, . \, f(a,c)\} &= \frac{du}{dx} \, . \, f(a,c) + u(x)\frac{\partial f}{\partial a}\frac{da}{dx} + u(x)\,\frac{\partial f}{\partial c}\frac{dc}{dt}\frac{dt}{dx} \\ &= \frac{du}{dx}f(a,c) + v(a)\frac{da}{dx} - \frac{du(x)}{dt}\{f(a,c) - x\}\frac{dt}{dx} \\ &= x\frac{du}{dx} + v(a)\frac{da}{dx}. \quad \text{(Using (2), (3) and (1).)}\end{aligned}$$

Consequently, integrating by parts

$$u(x) \, . \, f(a,c) = xu(x) - U(x) + V(a) + \text{a constant } K,$$

or
$$\frac{dc}{dt} = f(a,c) - x = \frac{K - \{U(x) - V(a)\}}{u(x)}. \quad . \quad . \quad . \quad (4)$$

We have now to identify K with what we called B, or bliss. This is most easily done by starting in a different way.

$\int_0^\infty (B - U(x) + V(a))dt$ represents the amount by which enjoyment falls short of bliss integrated throughout time; this is (or can be made) finite, and our problem is to minimise it. If we apply the calculus of variations straight away, using equation (1), we get equations (2) and (3) again; but if, instead of this, we first change the independent variable to c, we get a great simplification. Our integral becomes

$$\int_{c_0}^\infty \frac{B - U(x) + V(a)}{dc/dt} dc\ [1]$$

or

$$\int_{c_0}^\infty \frac{B - U(x) + V(a)}{f(a,c) - x} dc. \quad \text{Using (1).}$$

Now in this x and a are entirely arbitrary functions of c, and to minimise the integral we have simply to minimise the integrand by equating to zero its partial derivatives. Taking the derivative with respect to x we obtain:

$$\frac{-u(x)}{f(a,c) - x} + \frac{B - U(x) + V(a)}{\{f(a,c) - x\}^2} = 0;$$

consequently
$$\frac{dc}{dt} = f(a,c) - x = \frac{B - (U(x) - V(a))}{u(x)} \quad . \quad . \ (5)$$

or, as we stated at the beginning,

rate of saving multiplied by marginal utility of consumption should always equal bliss minus actual rate of utility enjoyed.

Mr. Keynes, to whom I am indebted for several other suggestions, has shown me that this result can also be obtained by the following simple reasoning.

Suppose that in a year we ought to spend £x and save £z. Then the advantage to be gained from an extra £1 spent is $u(x)$, the marginal utility of money, and this must be equated to the sacrifice imposed by saving £1 less.

Saving £1 less in the year will mean that we shall only save £z in $1 + \frac{1}{z}$ years, not, as before, in one year. Consequently, we shall be in $1 + \frac{1}{z}$ year's time exactly where we should have been in one year's time, and the whole course of our approach

[1] The upper limit will not be ∞, but the least capital with which bliss can be obtained, if this is finite. c steadily increases with t, at any rate until the integrand vanishes, so that the transformation is permissible.

to bliss will be postponed by $\frac{1}{z}$ of a year, so that we shall enjoy $\frac{1}{z}$ of a year less bliss and $\frac{1}{z}$ of a year more at our present rate. The sacrifice is, therefore,

$$\frac{1}{z}\{B - (U(x) - V(a)\}.$$

Equating this to $u(x)$, we get equation (5) again, if we replace z by $\frac{dc}{dt}$, its limiting value.

Unfortunately this simple reasoning cannot be applied when we take account of time-discounting, and I have therefore retained my equations (1)–(4), which can easily be extended to deal with more difficult problems.

The most remarkable feature of the rule is that it is altogether independent of the production function $f(a, c)$, except in so far as this determines bliss, the maximum rate of utility obtainable. In particular the amount we should save out of a given income is entirely independent of the present rate of interest, unless this is actually zero. The paradoxical nature of this result will to some extent be mitigated later, when we find that if the future is discounted at a constant rate ρ and the rate of interest is constant and equal to r, the proportion of income to be saved is a function of the ratio ρ/r. If $\rho = 0$ this ratio is 0 (unless r be 0 also) and the proportion to be saved is consequently independent of r.

The rate of saving which the rule requires is greatly in excess of that which anyone would normally suggest, as can be seen from the following table, which is put forward merely as an illustration.

Family income per annum.	Total utility.
£150	2
£200	3
£300	4
£500	5
£1000	6
£2000	7
£5000	8 = Bliss.

If we neglect variations in the amount of labour, the amount that should be saved out of a family income of £500 would be about £300. For then bliss minus actual rate of utility = 8 — = 5. Savings = £300 and marginal utility of consumption at

£200 = about $\frac{\cdot 1}{£60}$. (From £150 to £300 $U(x) = \frac{13x}{300} - 3 - \frac{x^2}{15,000}$, approximating by fitting a parabola, so that $u(x) = \frac{13}{300} - \frac{x}{7,500} = \frac{1}{60}$ if $x = 200$.)

It is worth pausing for a moment to consider how far our conclusions are affected by considerations which our simplifying assumptions have forced us to neglect. The probable increase of population constitutes a reason for saving even more, and so does the possibility that future inventions will put the bliss level higher than at present appears. On the other hand, the probability that future inventions and improvements in organisation are likely to make income obtainable with less sacrifice than at present is a reason for saving less. The influence of inventions thus works in two opposite ways: they give us new needs which we can better satisfy if we have saved up beforehand, but they also increase our productive capacity and make preliminary saving less urgent.

The most serious factor neglected is the possibility of future wars and earthquakes destroying our accumulations. These cannot be adequately accounted for by taking a very low rate of interest over long periods, since they may make the rate of interest actually negative, destroying as they do not only interest, but principal as well.

II

I propose now to assume that returns to capital and labour are constant and independent,[1] so that

$$f(a,c) = pa + rc,$$

where p, the rate of wages, and r, the rate of interest, are constants.

This assumption will enable us

(*a*) To represent our former solution by a simple diagram;
(*b*) To extend it to the case of an individual who only lives a finite time;
(*c*) To extend it to include the problem in which future utilities and disutilities are discounted at a constant rate.

[1] It is worth noting that in most of (*a*) we only require independence of returns, and not constancy, and that nowhere do we really require *wages* to be constant, but these assumptions are made throughout to simplify the statement. They are less absurd if the state is one among others which are only advancing slowly, so that the rates of interest and wages are largely independent of what our particular state saves and earns.

On our new hypothesis the income of the community falls into two clearly defined parts, pa and rc, which it will be convenient to call its *earned* and *unearned* income respectively.

(*a*) Equation (2), which now reads

$$v(a) = pu(x),$$

determines a as a function of x only, and we can conveniently put

$$\begin{aligned} y &= x - pa = \text{consumption} - \text{earned income} \\ w(y) &= u(x) = v(a)/p \\ W(y) &= \int w(y)dy = \int(u(x)dx - v(a)da) = U(x) - V(a). \end{aligned}$$

$W(y)$ may be called the total and $w(y)$ the marginal utility of unearned income, since they are the total and marginal utilities arising from the possession of an unearned income y available for consumption.

Equation (5) now gives

$$rc - y = f(a,c) - x = \frac{B - W(y)}{w(y)} \quad . \quad . \quad . \quad . \quad (6)$$

or

$$B - W(y) = \frac{dW}{dy}(rc - y),$$

which means that the point (rc, B) lies on the tangent at y to the curve $z = W(y)$.

Figure (1) shows the curve $z = W(y)$, which either attains the value B at a finite value y_1 (the case shown in the figure) or else approaches it asymptotically as $y \to \infty$.

In order to determine how much of a given unearned income rc should be saved, we take the point P, (rc, B), on the line $z = B$, and from it draw a tangent to the curve (not $z = B$, which will always be one tangent, but the other one). If the abscissa of Q, the point of contact, is y, an amount y of the unearned income should be consumed, and the remainder, $rc - y$, should be saved. Of course y may be negative, which would mean that not only would the whole unearned income be saved, but part of the earned income also.

It is easy to see that there must always be such a tangent, because the curve $z = W(y)$ will have a tangent or asymptote $y = -\eta$, where η is the greatest excess of earnings over consumption compatible with continued existence.

This rule determines how much of a given income should be spent, but it does not tell us what our income will amount to after a given lapse of time. This is obtained from equation (3), which now gives us

$$\frac{d}{dt}w(y) = -rw(y)$$

or

$$w(y) = Ae^{-rt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Here $A = w(y_0)$, where y_0 is the value of y for $t = o$ determined as the abscissa of Q, where P is (rc_0, B).

Supposing, then, we want to find the time taken in accumulating a capital c from an initial capital c_0, we take P to be the

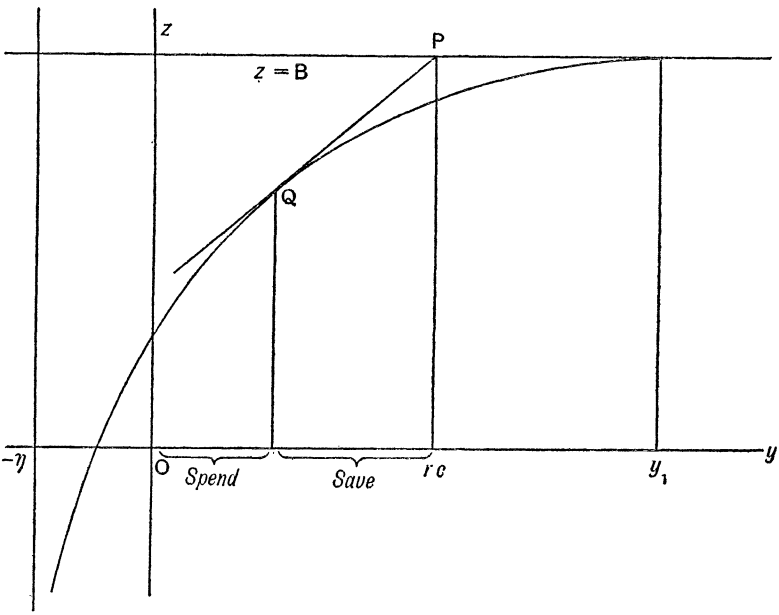

Fig. 1.

point (rc, B) and P_0 to be (rc_0, B). $w(y)$ is then the slope of the tangent from P, and $w(y_0)$ the slope of the tangent from P_0, so that the time in question

$$= \frac{1}{r}\log_e \frac{w(y_0)}{w(y)} = \frac{1}{r}\log_e \frac{\text{slope of tangent from } P_0}{\text{slope of tangent from } P}.$$

(*b*) Suppose now that we are concerned with an individual who lives only for a definite time, say T years, instead of with a community which lives for ever. We still have equation (4)

$$f(a,c) - x = \frac{K - (Ux - V(a))}{u(x)}$$

or

$$rc - y = \frac{K - W(y)}{w(y)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

but K is no longer equal to B, and has still to be determined. In order to find it we must know how much capital our man feels it necessary to leave his heirs; let us call this c_3.

Equation (8) means, as before, that y can be found as the abscissa of the point of contact Q of a tangent drawn from (rc, K) or P to the curve. P always lies on $z = K$, and its abscissa begins by being rc_0 and ends by being rc_3. K we can take as being less than B, since a man who lives only a finite time will save less than one who lives an infinite time, and the greater K is, the greater will be the rate of saving. Consequently $z = K$ will meet the curve, say at P_4.

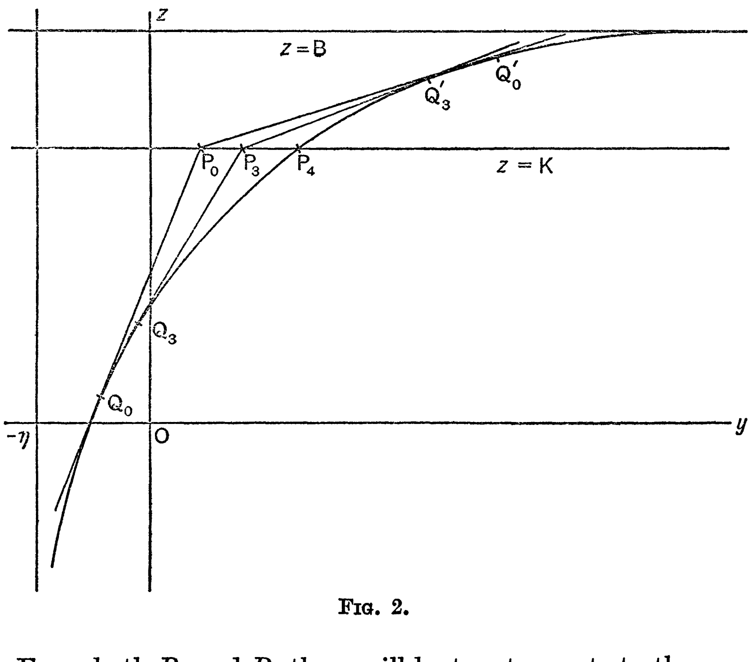

FIG. 2.

From both P_0 and P_3 there will be two tangents to the curve, of which either the upper or the lower can, for all we know, be taken as determining y_0 and y_3. If, however, $c_3 > c_0$ as in Fig. 2, we can only take the lower tangent from P_0, since the upper tangent gives a value of y_0 greater than either of the values of y_3, which is impossible, as y continually increases. Taking, then, Q_0 as the point of contact of the lower tangent from P_0, there are two possible cases, according as we take as giving y_3 either Q_3, the lower, or Q_3', the upper value. If we take Q_3, P_0 moves straight to P_3, and there is saving all the time; this happens when T is small. But if T is large, Q_0 moves right along to Q_3', and P_0 goes first up to P_4, and then back to P_3;

to begin with there is saving, and subsequently splashing. Similarly, if $c_0 > c_3$, there are two possible cases, and in this case it is the lower tangent from P_3 that cannot be taken.

In order to determine which tangents to take and also the value of K we must use the condition derived from equation (7)

$$\frac{\text{slope of tangent taken from } P_0}{\text{slope of tangent taken from } P_3} = \frac{w(y_0)}{w(y_3)} = e^{rT}.$$

This, together with the fact that the abscissae of P_0 and P_3 are c_0, c_3, and that they have the same ordinate K, suffices to fix both K and the tangents to be taken.

(*c*) We have now to see how our results must be modified when we no longer reckon future utilities and disutilities as equal to present ones, but discount them at a constant rate ρ.

This rate of discounting future *utilities* must, of course, be distinguished from the rate of discounting future sums of money. If I can borrow or lend at a rate r I must necessarily be equally pleased with an extra £1 now and an extra £$(1 + r)$ in a year's time, since I could always exchange the one for the other. My marginal rate of discount for money is, therefore, necessarily r, but my rate of discount for utility may be quite different, since the marginal utility of money to me may be varying by my increasing or decreasing my expenditure as time goes on.

In assuming the rate of discount constant, I do not mean that it is the same for all individuals, since we are at present only concerned with one individual or community, but that the present value of an enjoyment at any future date is to be obtained by discounting it at the rate ρ. Thus, taking it to be about $\frac{3}{4}$ per cent., utility at any time would be regarded as twice as desirable as that a hundred years later, four times as valuable as that two hundred years later and so on at a compound rate. This is the only assumption we can make, without contradicting our fundamental hypothesis that successive generations are actuated by the same system of preferences. For if we had a varying rate of discount—say a higher one for the first fifty years—our preference for enjoyments in 2000 A.D. over those in 2050 A.D. would be calculated at the lower rate, but that of the people alive in 2000 A.D. would be at the higher.

Let us suppose first that the rate of discount for utility ρ is less than the rate of interest r.

Then equations (1) and (2) are unchanged, but equation (3) becomes

$$\frac{d}{dt}u(x) = -u(x)\left\{\frac{\partial f}{\partial c} - \rho\right\}$$
$$= -u(x)(r - \rho) \quad . \quad . \quad . \quad . \quad . \quad (9)$$

as we are now assuming $\frac{\partial f}{\partial c}$ constant and equal to r;

consequently $$w(y) = u(x) = Ae^{-(r-\rho)t} \quad . \quad . \quad . \quad . \quad (9a)$$

and $$rc - y = \frac{dc}{dt} = \frac{dc}{dw} \cdot \frac{dw}{dt} = -(r - \rho)w\frac{dc}{dw}$$

so $$\frac{dc}{dw} + \frac{rc}{(r-\rho)w} = \frac{y}{(r-\rho)w},$$

where $$cw^{r/(r-\rho)} = \int \frac{yw^{\rho/(r-\rho)}}{r-\rho}dw + \frac{K}{r}$$
$$= \frac{1}{r}yw^{r/(r-\rho)} - \frac{1}{r}\int_b^y w^{r/(r-\rho)}(y)dy + \frac{K}{r}$$

(K, b constants.)

and $$\frac{dc}{dt} = rc - y = \frac{K - \int_b^y w^{r/(r-\rho)}(y)dy}{w^{r/(r-\rho)}(y)} \quad . \quad . \quad . \quad (10)$$

This equation is the same as (8) except that instead of $w(y)$ and $W(y)$, which is $\int w(y)dy$, we have $w^{r/(r-\rho)}(y)$ and $\int w^{r/(r-\rho)}(y)dy$. The method of solution both for a community and for an individual is therefore the same as before, except that instead of the real utility of unearned income we have to consider what we can call its modified utility, obtained by integrating the marginal utility to the power $r/(r - \rho)$. This has the effect of accelerating the decrease of marginal utility and lessening the relative importance of high incomes. We can in this way translate our discounting of the future into a discounting of high incomes. The rate at which this is done is governed solely by the ratio of ρ to r, so that if ρ is 0 it is independent of the value of r, provided this is not also 0. The main conclusion of section I is thus confirmed.

There is, however, a slight difficulty, because we have not really shown yet that if we are considering an infinite time, the constant K is to be interpreted as what might be called "modified bliss," *i.e.* the maximum value of $\int_b^y w^{r/(r-\rho)}(y)dy$. This modified bliss would require the same income as bliss does, the modification being solely in the value set on it. This result can, however, be deduced at once from equation (9*a*), which shows that y increases until bliss is reached, so that $\frac{dc}{dt}$ can never become negative and K cannot be less than modified bliss. On the other hand, provided this condition is fulfilled, 9(a) shows that the larger y is initially, the smaller will be A, and the larger will be y throughout future time. Hence K must be as small as possible (provided it is not so small as to make $\frac{dc}{dt}$ ultimately negative); so that K cannot be greater than modified bliss. Hence as it is neither less nor greater it must be equal.

As in (b), we can adapt our solution to the case of an individual with only a finite time to live, in this case drawing tangents to the modified utility curve.

An interesting special case is that of a community for which

$$w(y) = Dy^{-\alpha} \qquad (\alpha > 1)$$

we shall have $w^{r/(r-\rho)}(y) = Ey^{-\beta}$, $\beta = \frac{r\alpha}{r-\rho}$, $E = D^{r/(r-\rho)}$

$$\text{savings} = \frac{K - \int w^{r/(r-\rho)}(y)dy}{w^{r/(r-\rho)}(y)dy} = \frac{K - K_1 + \frac{Ey^{1-\beta}}{\beta - 1}}{Ey^{-\beta}}.$$

It is clear that corresponding to $K = B$ in the case when $\rho = 0$

we have here $\qquad K = K_1$

and savings $\qquad = \frac{y}{\beta - 1}$

i.e., a constant proportion $\frac{r-\rho}{r(\alpha - 1) + \rho}$ of unearned income should be saved, which if $\rho = 0$ is $\frac{1}{\alpha - 1}$, and independent of r.

If the rate of interest is less than the rate of discounting utility, we shall have similar equations, leading to a very different result. The marginal utility of consumption will rise at a rate $\rho - r$, and consumption will fall towards the barest subsistence level at which its marginal utility may be taken as infinite, if we disregard the possibility of suicide. During this process all capital will be exhausted and debts incurred to the extent to which credit can be obtained, the simplest assumption on this point being that it will be possible to borrow a sum such that it is just possible to keep alive after paying the interest on it.

III

Let us next consider the problem of the determination of the rate of interest.

(α) In the first place we will suppose that everyone discounts future utility for himself or his heirs, at the same rate ρ.

Then in a state of *equilibrium* there will be no saving and

$$\frac{dx}{dt} = \frac{dc}{dt} = 0,$$

so that we have

$$x = f(a,c)$$

$$v(a) = \frac{\partial f}{\partial a} u(x)$$

$$\frac{\partial f}{\partial c} = \rho\,;$$

three equations to determine x, a and c.

The last equation tells us that the rate of interest as determined by the marginal productivity of capital, $\frac{\partial f}{\partial c}$, must be equal to the rate of discounting ρ.[1]

But suppose that at a given time, say the present, $\frac{\partial f}{\partial c} > \rho$. Then there will not be equilibrium, but saving, and since a great deal cannot be saved in a short time, it may be centuries before equilibrium is reached, or it may never be reached, but only approached asymptotically; and the question arises as to how, in the meantime, the rate of interest is determined, since it cannot be by the ordinary equilibrium equation of supply and demand.

The difficulty is that the rate of interest functions as a demand price for a whole quantity of capital, but as a supply price, not for a quantity of capital, but for a rate of saving. The resulting state of affairs is represented in Fig. 3, in which, however, variations in the amount of labour are neglected. This shows the demand curve for capital $r = \frac{\partial f}{\partial c}$, the ultimate supply curve $r = \rho$ and the temporary supply curve $c = c_0$. It is clear that the rate of interest is determined directly by the intersection of the demand curve with the temporary supply curve $c = c_0$. The ultimate supply curve $r = \rho$ only comes in as governing the rate at which c_0 approaches its ultimate value OM, a rate which depends roughly on the ratio of PM to QN. We see, therefore, that the rate of interest is governed primarily by the demand price, and may greatly exceed the reward ultimately necessary to induce abstinence.

[1] Equilibrium could, however, also be obtained either at bliss with $\rho < \frac{\partial f}{\partial c}$, or at the subsistence level with $\rho > \frac{\partial f}{\partial c}$. Cf. ($\gamma$) below.

Similarly, in the accounting of a Socialist State the function of the rate of interest would be to ensure the wisest use of existing capital, not to serve in any direct way as a guide to the proportion of income which should be saved.

(β) We must now try to take some account of the fact that different people discount future utility at different rates, and, quite apart from the time factor, are not so interested in their heirs as in themselves.

Let us suppose that they are not concerned with their heirs

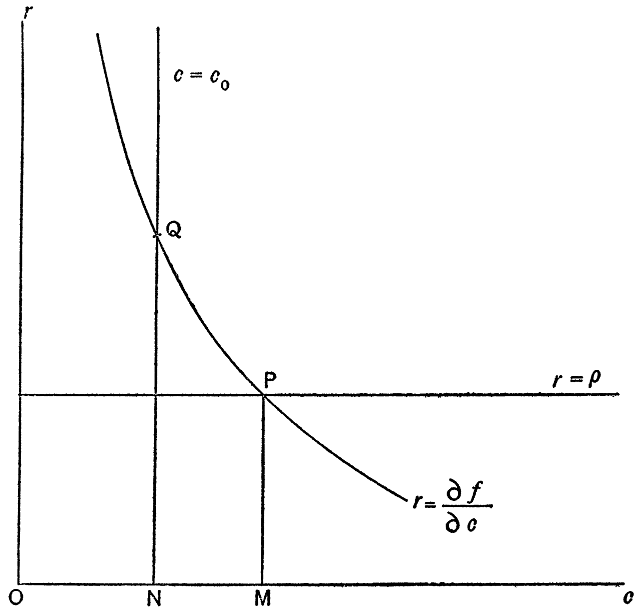

FIG. 3.

at all; that each man is charged with a share of the maintenance of such children as are necessary to maintain the population, but starts his working life without any capital and ends it without any, having spent his savings on an annuity; that within his own lifetime he has a constant utility schedule for consumption and discounts future utility at a constant rate, but that this rate may be supposed different for different people.

When such a community is in equilibrium, the rate of interest must, of course, equal the demand price of capital $\frac{\partial f}{\partial c}$. And it will also equal the "supply price," which arises in the following way. Suppose that the rate of interest is constant and equal to

r, and that the rate of discount for a given individual is ρ. Then if $r > \rho$, he will save when he is young, not only to provide for loss of earning power in old age, but also because he can get more pounds to spend at a later date for those he forgoes spending now. If we neglect variations in his earning power, his action can be calculated by modifying the equations of II*c* to apply to a finite life as in II*b*. He will for a time accumulate capital, and then spend it before he dies. Besides this man, we must suppose there to be in our community other men, exactly like him except for being born at different times. The total capital possessed by n men of this sort whose birthdays are spread evenly through the period of a lifetime will be n times the *average* capital possessed by each in the course of his life. The class of men of this sort will, therefore, possess a constant capital depending on the rate of interest, and this will be the amount of capital supplied by them at that price. (If $\rho > r$, it may be negative, as they may borrow when young and pay back when old.) We can then obtain the total supply curve of capital by adding together the supplies provided at a given price by each class of individual.

If, then, we neglect men's interest in their heirs, we see that capital has a definite supply price to be equated to its demand price. This supply price depends on people's rates of discount for utility, and it can be equated to the rate of discount of the "marginal saver" in the sense that someone whose rate of discount is equal to the rate of interest will neither save nor borrow (except to provide for old age).

But the situation is different from the ordinary supply problem, in that those beyond this "margin" do not simply provide nothing, but provide a negative supply by borrowing when young against their future earnings, and so being on the average in debt.

(γ) Let us now go back to case (α) by supposing men, or rather families, to live for ever, and discount future utility at a constant rate, but let us try this time to take account of variations in the rate of discount from family to family.

For simplicity let us suppose that the amount of labour is constant, so that the total income of the country can be regarded as a function $f(c)$ of the capital only. The rate of interest will then be $f'(c)$. Let us also suppose that every individual could attain the maximum *conceivable* utility with a finite income x_1, and that no one could support life on less than x_2.

Now suppose equilibrium [1] is obtained with capital c, income $f(c)$ and rate of interest $f'(c)$ or r. Then those families, say $m(r)$ in number, whose rate of discount is less than r must have attained bliss or they would still be increasing their expenditure according to equation (9a). Consequently they have between them an income $m(r) \,.\, x_1$. The other families, $n - m(r)$ in number (where n is the total number of families), must be down to the subsistence level, or they would still be decreasing their expenditure. Consequently they have between them a total income $\{n - m(r)\}x_2$,

whence
$$\begin{aligned} f(c) &= m(r)x_1 + \{n - m(r)\}x_2 \\ &= n \,.\, x_2 + m(r)\{x_1 - x_2\}, \end{aligned}$$

which, together with $r = f'(c)$, determines r and c. $m(r)$ being an increasing function of r, it is easy to see, by drawing graphs of r against $f(c)$, that the two equations have in general a unique solution.[2]

In such a case, therefore, equilibrium would be attained by a division of society into two classes, the thrifty enjoying bliss and the improvident at the subsistence level.

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[1] We suppose each family in equilibrium, which is the only way in which that state could be maintained, since otherwise, although the savings of some might at any moment balance the borrowings of others, they would not continue to do so except by an extraordinary accident.

[2] We have neglected in this the negligible number of families for which ρ is exactly equal to r.