Quick Reference of Fourier Transform for Fourier Domain Optical Coherence Tomography

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Note: This version is a pre-view version and may contains several errors. Your error report is quite welcome. Please send your error report to Yasuno (yoshiaki.yasuno@gmail.com).

Contents

Preface

This short note aims at providing a quick reference of Fourier analysis. Among several such textbooks, this short note is characterized by its focus toward Fourier domain optical coherence tomography (FD-OCT). FD-OCT is commonly formulated by using a Fourier pair of z (depth position) and k (angular wavenumber), while the majority of the textbooks of Fourier analysis use a Fourier pair of t and a frequency ν. ν for t corresponds to κ (oscillation wavenumber) for z, and hence, formulas provided by these textbooks are not directly applicable for FD-OCT. Although the conversion between z-κ formulation and z-k formulation is essentially not difficult, it can sometimes be elaborated and causes error of inconsistency. This short reference provides a set of consistent formula which is based on z-k Fourier pair. So it can be utilized as a tool box to make your own theory for FD-OCT. In addtion, the same formula can be utilized for t-ω Fourier analysis by substituting z by t and k by ω. And hence, this quick reference also can be utilized for building a theory based on t-ω foumulation.

1 Notations

In this short note, the following notations are utilized.

λ denotes wavelength and its unit is typically m in SI system.

k is an *angular* wavenumber. $k = 2\pi/\lambda$. In the formulation of FD-OCT, a variable of spectral domain is typically k.

κ is an *oscillation* wavenumber; $\kappa = k/2\pi = 1/\lambda$. κ is rarely used in formulation of FD-OCT.

z is a *double-path* axial optical depth, and is the Fourier pair of k. The single path physical depth is then represented as $z/2n$ with n as a refractive index.

$\mathfrak{F}[\quad]$ and $\mathfrak{F}^{-1}[\quad]$ respectively represent Fourier transform and inverse Fourier transform.

$f(z)$ and $\tilde{f}(k)$ respectively represent a function of z and its Fourier spectrum in k-domain. In general, any function with a tilde ($\tilde{}$) is the Fourier spectrum of the function of the same notation but without the tilde.

A operator $*$ and $\otimes$ represents convolution and correlation, respectively.

The superscript of $*$ represents complex conjugate.

2 Definition of Fourier transform

Fourier transform

The Fourier transform that transforms a function of z to its k spectrum is defines as follows.

$$\tilde{f}(k) = \mathfrak{F}\left[f(z)\right] = \int_{-\infty}^{+\infty} f(z)e^{-ikz}dz, \tag{1}$$

where $\mathfrak{F}[\quad]$ represents Fourier transform.

Inverse Fourier transform

The inverse Fourier transform (k to z) is defined as

$$f(z) = \mathfrak{F}^{-1}\left[\tilde{f}(k)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikz}dk, \tag{2}$$

where $\mathfrak{F}[\quad]$ represents inverse Fourier transform. $1/2\pi$ is a constant to make the Fourier transform and the inverse Fourier transform as a reversible pair.

3 Convolution and convolution theorem

Convolution

The convolution of two functions $f(z)$ and $g(z)$ is defines as

$$f(z) * g(z) \equiv \int_{-\infty}^{+\infty} f(z')g(z-z')dz', \tag{3}$$

where $*$ denotes convolution.

z-domain convolution theory

The Fourier transform of convolution of functions in z becomes a multiplication of the Fourier transformations of these functions.

$$\mathfrak{F}\left[f(z) * g(z)\right] = \tilde{f}(k)\tilde{g}(k). \tag{4}$$

Similarly,

$$\mathfrak{F}^{-1}\left[\tilde{f}(k)\tilde{g}(k)\right] = f(z) * g(z). \tag{5}$$

This relationship is known as convolution theory.

k-domain convolution theory

The convolution theory of the convolution of k domain functions is

$$\mathfrak{F}^{-1}\left[\tilde{f}(k) * \tilde{g}(k)\right] = 2\pi f(z)g(z). \tag{6}$$

Similarly,

$$\mathfrak{F}\left[f(z)g(z)\right] = \frac{1}{2\pi}\tilde{f}(k) * \tilde{g}(k). \tag{7}$$

Multiple-convolution and associative law of convolution

By sequentially applying convolution theory, it can be extended into three or more functions as

$$\mathfrak{F}\left[\{f(z) * g(z)\} * h(z)\right] = \mathfrak{F}\left[f(z) * g(z)\right]\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)\tilde{h}(k). \tag{8}$$

It implies the associative rule of the convolution

$$[f(z) * g(z)] * h(z) = f(z) * [g(z) * h(z)] \tag{9}$$

Similarly, the k-domain convolution theory of three functions becomes

$$\mathfrak{F}^{-1}\left[\left\{\tilde{f}(k) * \tilde{g}(k)\right\} * \tilde{h}(k)\right] = \frac{1}{2\pi}\mathfrak{F}^{-1}\left[\tilde{f}(k) * \tilde{g}(k)\right] h(z) = \frac{1}{(2\pi)^2} f(z)g(z)h(z). \tag{10}$$

Note that, for multiple convolution of n functions, the coefficient in the right hand side of the equation becomes $1/(2\pi)^{(n-1)}$.

The associative law of convolution of k-domain function is the same with that of z-domain function as

$$\left[\tilde{f}(k) * \tilde{g}(k)\right] * \tilde{h}(k) = \tilde{f}(k) * \left[\tilde{g}(k) * \tilde{h}(k)\right]. \tag{11}$$

Commutative law of convolution

The commutative law stands for convolution as

$$f(z) * g(z) = g(z) * f(z), \tag{12}$$

and

$$\tilde{f}(k) * \tilde{g}(k) = \tilde{g}(k) * \tilde{f}(k). \tag{13}$$

4 Delta-function and shift law

Delta-function and its Fourier transform

The Fourier transform of a delta-function is a complex harmonic function whose frequency is defined by the position of the delta function;

$$\mathfrak{F}\left[\delta(z - z_0)\right] = e^{-ikz_0} \quad \Leftrightarrow \quad \mathfrak{F}^{-1}\left[e^{-ikz_0}\right] = \delta(z - z_0). \tag{14}$$

Similarly, for k-domain delta function

$$\mathfrak{F}^{-1}\left[\delta(k-k_0)\right] = \frac{1}{2\pi}e^{ik_0z} \quad \Leftrightarrow \quad \mathfrak{F}\left[e^{ik_0z}\right] = 2\pi\delta(k-k_0). \tag{15}$$

Shift law of Fourier transform

By using the convolution theorem and the Fourier transform of delta-function, the Fourier transform of a shifted function can be obtained as

$$\mathfrak{F}\left[f(z) * \delta(z-z_0)\right] = \tilde{f}(k)e^{-ikz_0} \tag{16}$$

Its k-domain version is

$$\mathfrak{F}^{-1}\left[\tilde{f}(k) * \delta(k-k_0)\right] = f(z)e^{ik_0z} \tag{17}$$

Shift law indicates that if the signal is shifted, a linear phase slope is added to its spectrum. Similarly, if the spectrum of a function is shifted, a linear phase slope is added to the function.

5 Comb function and its distributional formulation

Comb-function and Fourier transform

A comb-function is defined as a train of delta-functions as

$$\mathrm{comb}\left(\frac{z}{\Delta z}\right) = \sum_{n=-\infty}^{\infty} \delta(z - n\Delta z), \tag{18}$$

where Δz is a period of (separation between) the delta-functions.

The Fourier transform of the comb function becomes

$$\mathfrak{F}\left[\mathrm{comb}\left(\frac{z}{\Delta z}\right)\right] = \frac{2\pi}{\Delta z}\mathrm{comb}\left(k\frac{\Delta z}{2\pi}\right). \tag{19}$$

Namely, the Fourier tranform of a comb function with a period of Δz is a comb function with a period of $2\pi/\Delta z$.

The inverse Fourier tranform of a k-domain comb function is

$$\mathfrak{F}^{-1}\left[\mathrm{comb}\left(\frac{k}{\Delta k}\right)\right] = \frac{1}{\Delta k}\mathrm{comb}\left(z\frac{\Delta k}{2\pi}\right). \tag{20}$$

Namely, the inverse Fourier transform of a comb function with a period of Δk is a comb function with a period of $2\pi/\Delta k$.

Distributional formulation of comb-function

By using Poisson summation formula, the comb-function can be expressed in its distribution form as

$$\mathrm{comb}\left(\frac{z}{\Delta z}\right) = \sum_{n=-\infty}^{\infty} \delta(z - n\Delta z) = \frac{1}{\Delta z} \sum_{m=-\infty}^{\infty} \exp\left(i2\pi \frac{m}{\Delta z} z\right). \tag{21}$$

The Fourier transform of a comb-function (Eq. (19)) can be derived by using this distributional form and the formula of the Fourier transform of delta-function (Eq. (15)).

6 Auto-correlation and Wiener-Khintchine's theorem

Cross- and auto-correlation functions

The cross-correlation function of two functions $f(z)$ and $g(z)$ is defines as

$$f(z) \otimes g(z) \equiv \int_{-\infty}^{+\infty} f(z')g(z' - z)dz', \tag{22}$$

where $\otimes$ represents correlation.

The correlation function between a same function is called as auto-correlation, and is defined as

$$f(z) \otimes f(z) \equiv \int_{-\infty}^{+\infty} f(z')f(z' - z)dz'. \tag{23}$$

Wiener-Khintchine's theorem

The inverse Fourier transform of the power spectrum of a function is the auto-correlation of the function;

$$\mathfrak{F}^{-1}\left[\left|\tilde{f}(k)\right|^2\right] = f(z) \otimes f(z). \tag{24}$$

This relation between the power spectrum and the auto-correlation is known as Wiener-Khintchine's theorem.

7 Scaling and mirroring of a signal

Scaling property of Fourier transform

The Fourier transform of a signal scaled in z is

$$\mathfrak{F}\left[f(az)\right] = \frac{1}{|a|}\tilde{f}\left(\frac{k}{a}\right). \tag{25}$$

Similarly,

$$\mathfrak{F}^{-1}\left[\tilde{f}(bk)\right] = \frac{1}{|b|} f\left(\frac{z}{b}\right), \tag{26}$$

where a and b are scaling factors.

Fourier transform of a mirrored signal

The Fourier transform of a mirrored signal is obtained as

$$\mathfrak{F}\left[f(-z)\right] = \tilde{f}(-k) \quad \Leftrightarrow \quad \mathfrak{F}^{-1}\left[\tilde{f}(-k)\right] = f(-z). \tag{27}$$

This is a special case of the scaling property with a scaling factor of -1.

8 Fourier transform of real function and complex conjugate

Fourier transform of real function

The Fourier transform spectrum of a real function $r(z)$ is a mirror-conjugate-symmetric function;

$$\tilde{r}(k) = \tilde{r^*}(-k) \tag{28}$$

where $\tilde{r}(k) = \mathfrak{F}\left[r(z)\right]$ and $r(z)$ is a real function.

Similarly, for a real spectrum $\tilde{s}(k)$,

$$s(z) = s^*(-z) \tag{29}$$

where $s(k) = \mathfrak{F}^{-1}\left[\tilde{s}(k)\right]$ and $s(k)$ is a real function.

This relation is exemplified by FD-OCT. A spectral interferogram is a real function, and hence its Fourier transform in z-domain is mirror-conjugate-symmetric. (Note that it is not true if additional phase is applied to the spectral interferogram, as it is frequently done to correct dispersion.)

Fourier transform of complex conjugate

The Fourier transform of the complex conjugate of a function becomes a mirrored complex conjugate of the original spectrum;

$$\mathfrak{F}\left[f^*(z)\right] = \tilde{f}^*(-k), \tag{30}$$

where superscript of $*$ denotes complex conjugate. Similarly,

$$\mathfrak{F}^{-1}\left[\tilde{f}^*(k)\right] = f^*(-z). \tag{31}$$

This relationship can be derived by resolving $f(z)$ into two real functions as $f_r(z)+ if_i'(z)$, Fourier transform it, and then applying the rule for the Fourier transform of a real function.

9 Parseval's theorem

Parseval's theorem is the law of energy conservation during Fourier transform.

$$\int_{-\infty}^{+\infty} \left|f(z)\right|^2 dz = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\tilde{f}(k)\right|^2 dk \tag{32}$$

Outlooks

The following topics are going to be added soon.

- Window functions and is Fourier transform
  - Rectangle function
  - Gauss function
  - Hanning function
  - Hamming function
- Hilbert transform as Fourier filter

Release log

2014-08-21 (Thu) [Ver1.0] The first pre-view release by Yasuno.