Quick reference of Fourier transform for Fourier domain optical coherence tomography

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Ver. 1.1.1 released on 2025-03-24

Contents

1 Preface

This short memorandum provides a quick reference for Fourier analysis. Among various textbooks on the subject, this note is distinguished by its focus on Fourier-domain optical coherence tomography (FD-OCT). FD-OCT is typically formulated using the Fourier pair (z, k), where z represents depth position and k denotes angular wavenumber. In contrast, most Fourier analysis textbooks use the Fourier pair (t, ν), where t represents time and ν is frequency. Since the frequency ν for t corresponds to the oscillation wavenumber κ for z, the standard textbook notation cannot be directly applicable to FD-OCT. Although converting between the (z, κ) and (z, k) formulations is conceptually straightforward, it can sometimes be intricate and prone to inconsistencies. This reference provides a set of consistent formulas based on the (z, k) Fourier pair, and hence, can serve as a toolbox for learning and developing FD-OCT theory. Additionally, the same formulas can be applied to (t, ω) Fourier analysis by substituting z with t and k with ω, where t is time and ω is angular frequency. Therefore, this reference can also be used for constructing theories based on the (t, ω) formulation.

2 Notations

In this memorandum, the following notations are used:

λ denotes wavelength, typically measured in meters (m) in the SI system.

k is the angular wavenumber, defined as $k = 2\pi/\lambda$. In FD-OCT formulations, the spectral domain variable is typically k.

κ is the oscillation wavenumber given by $\kappa = k/\pi = 1/\lambda$. It is rarely used in FD-OCT formulations.

z represents the double-path axial optical depth and is the Fourier pair of k. The single-path physical depth is given by $z/2n$, where n is the refractive index.

$\mathcal{F}[\quad]$ and $\mathcal{F}^{-1}[\quad]$ denote the Fourier transform and inverse Fourier transform, respectively.

$f(z)$ and $\tilde{f}(k)$ represent a function of z and its Fourier spectrum in the k-domain, respectively. In general, any function with a tilde ($\tilde{}$) represents the Fourier spectrum of the corresponding function without the tilde.

The operators $*$ and $\otimes$ represent convolution and correlation, respectively.

The superscript $*$ denotes the complex conjugate.

3 Definition of Fourier transform

3.1 Fourier transform

The Fourier transform that transforms a function of z to its k spectrum is defines as follows.

$$\tilde{f}(k) = \mathcal{F}\left[f(z)\right] \equiv \int_{-\infty}^{+\infty} f(z)e^{-ikz}dz, \tag{1}$$

where $\mathcal{F}[\quad]$ represents Fourier transform.

3.2 Inverse Fourier transform

The inverse Fourier transform from the k domain to the z domain is defined as

$$f(z) = \mathcal{F}^{-1}\left[\tilde{f}(k)\right] \equiv \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikz}dk, \tag{2}$$

where $\mathcal{F}[\quad]$ represents inverse Fourier transform. $1/2\pi$ is a constant to make the Fourier transform and the inverse Fourier transform as a reversible pair.

3.3 Euler's formula and cos-sin-form of Fourier transform

Euler's formula can decompose exponential function into cosine and sine functions as

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{3}$$

and

$$e^{-i\theta} = \cos\theta - i\sin\theta, \tag{4}$$

where θ is a phase value or function.

Using Euler's formula, the Fourier and the inverse Fourier transforms can be written in the cos-and-sin form as

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} f(z)e^{-ikz}dz = \int_{-\infty}^{+\infty} f(z)\cos(kz)\ dz - i\int_{-\infty}^{+\infty} f(z)\sin(kz)\ dz, \tag{5}$$

and

$$f(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{f}(k)e^{ikz}dk = \frac{1}{2\pi}\left[\int_{-\infty}^{+\infty} \tilde{f}(k)\cos(kz)dk + i\int_{-\infty}^{+\infty} \tilde{f}(k)\sin(kz)dk\right]. \tag{6}$$

4 Convolution and convolution theorem

4.1 Convolution

The convolution of two functions $f(z)$ and $g(z)$ is defines as

$$f(z) * g(z) \equiv \int_{-\infty}^{+\infty} f(z')\, g(z - z')\, dz', \tag{7}$$

where $*$ denotes convolution.

4.2 z-domain convolution theory

The Fourier transform of convolution of functions in z becomes a multiplication of the Fourier transformations of these functions.

$$\mathcal{F}\left[f(z) * g(z)\right] = \tilde{f}(k)\tilde{g}(k). \tag{8}$$

Similarly,

$$\mathcal{F}^{-1}\left[\tilde{f}(k)\tilde{g}(k)\right] = f(z) * g(z). \tag{9}$$

This relationship is known as convolution theory.

4.3 k-domain convolution theory

The convolution theory for the convolution of k-domain functions is

$$\mathcal{F}^{-1}\left[\tilde{f}(k) * \tilde{g}(k)\right] = 2\pi f(z)g(z). \tag{10}$$

Similarly,

$$\mathcal{F}\left[f(z)g(z)\right] = \frac{1}{2\pi}\tilde{f}(k) * \tilde{g}(k). \tag{11}$$

4.4 Multiple-convolution and associative law of convolution

By sequentially applying convolution theory, it can be extended into three or more functions as

$$\mathcal{F}\left[\{f(z) * g(z)\} * h(z)\right] = \mathcal{F}\left[f(z) * g(z)\right]\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)\tilde{h}(k). \tag{12}$$

It implies the associative rule of the convolution

$$\left[f(z) * g(z)\right] * h(z) = f(z) * \left[g(z) * h(z)\right]. \tag{13}$$

Similarly, the k-domain convolution theory of three functions becomes

$$\mathcal{F}^{-1}\left[\left\{\tilde{k}(k) * \tilde{g}(k)\right\} * \tilde{h}(k)\right] = \frac{1}{2\pi}\mathcal{F}^{-1}\left[\tilde{f}(k) * \tilde{g}(k)\right]h(z) = \frac{1}{(2\pi)^2}f(z)g(z)h(z). \tag{14}$$

Note that, for multiple convolution of n functions, the coefficient in the right hand side of the equation becomes $1/(2\pi)^{(n-1)}$.

The associative law of convolution of k-domain function is the same with that of z-domain function as

$$\left[\tilde{f}(k) * \tilde{g}(k)\right] * \tilde{h}(k) = \tilde{f}(k) * \left[\tilde{g}(k) * \tilde{h}(k)\right]. \tag{15}$$

4.5 Commutative law of convolution

The commutative law stands for convolution as

$$f(z) * g(z) = g(z) * f(z), \tag{16}$$

and

$$\tilde{f}(k) * \tilde{g}(k) = \tilde{g}(k) * \tilde{f}(k). \tag{17}$$

5 Delta-function and shift law

5.1 Delta function and its Fourier transform

The Dirac delta function, or "delta function" in short, is defined as

$$\delta(x) = \begin{cases} 0 & \text{for} \quad x \neq 0 \\ +\infty & \text{for} \quad x = 0 \end{cases}, \tag{18}$$

where x is a general variable, and

$$\int_{-\infty}^{+\infty} \delta(x)\ dx = 1. \tag{19}$$

The Fourier transform of a delta-function is a complex harmonic function whose frequency is defined by the position of the delta function as

$$\mathcal{F}\left[\delta\left(z - \mathrm{z}_1\right)\right] = e^{-i\mathrm{z}_1 k} \qquad \Leftrightarrow \qquad \mathcal{F}^{-1}\left[e^{-i\mathrm{z}_1 k}\right] = \delta\left(z - \mathrm{z}_1\right), \tag{20}$$

where z_1 is a constant representing the position of the z-domain delta function.

Similarly, for k-domain delta function

$$\mathcal{F}^{-1}\left[\delta\left(k - \mathrm{k}_1\right)\right] = \frac{1}{2\pi} e^{i\mathrm{k}_1 z} \qquad \Leftrightarrow \qquad \mathcal{F}\left[e^{i\mathrm{k}_1 z}\right] = 2\pi\delta\left(k - \mathrm{k}_1\right), \tag{21}$$

where k_1 is a constant representing the position of the k-domain delta function.

5.2 Shift law of Fourier transform

By using the convolution theorem and the Fourier transform of delta-function, the Fourier transform of a shifted function can be obtained as

$$\mathcal{F}\left[f(z) * \delta\left(z - \mathrm{z}_1\right)\right] = \tilde{f}(k) e^{-i\mathrm{z}_1 k} \tag{22}$$

Its k-domain version is

$$\mathcal{F}^{-1}\left[\tilde{f}(k) * \delta\left(k - \mathrm{k}_1\right)\right] = f(z) e^{i\mathrm{k}_1 z}. \tag{23}$$

Shift law indicates that if the signal is shifted, a linear phase slope is added to its spectrum. Similarly, if the spectrum of a function is shifted, a linear phase slope is added to the function.

6 Comb function and its distributional formulation

6.1 Comb-function and Fourier transform

A comb-function is defined as a train of delta-functions as

$$\operatorname{comb}\left(\frac{z}{\Delta z}\right) \equiv \sum_{n=-\infty}^{+\infty} \delta\left(z - n\Delta\right), \tag{24}$$

where Δz is a period of the delta-functions train, i.e., the separation of two adjacent delta-functions.

The Fourier transform of the comb function becomes

$$\mathcal{F}\left[\mathrm{comb}\left(\frac{z}{\Delta z}\right)\right] = \frac{2\pi}{\Delta z}\mathrm{comb}\left(k\frac{\Delta z}{2\pi}\right). \tag{25}$$

Namely, the Fourier transform of a comb function with a period of Δz is a comb function with a period of $2\pi/\Delta z$. It is noteworthy that shorter the period in the z domain results in the longer the period in the k domain and *vice versa*.

The inverse Fourier transform of a k-domain comb function is

$$\mathcal{F}^{-1}\left[\mathrm{comb}\left(\frac{k}{\Delta k}\right)\right] = \frac{1}{\Delta k}\mathrm{comb}\left(z\frac{\Delta k}{2\pi}\right). \tag{26}$$

Namely, the inverse Fourier transform of a comb function with a period of Δk is a comb function with a period of $2\pi/\Delta k$. Similar to the former case, shorter the period in the k domain results in the longer the period in the z domain and *vice versa*.

6.2 Distributional form of comb-function

By using Poisson summation formula, the comb-function can be expressed in its distribution form as

$$\mathrm{comb}\left(\frac{z}{\Delta z}\right) = \sum_{n=-\infty}^{+\infty} \delta\left(z - n\Delta z\right) = \frac{1}{\Delta z}\sum_{m=-\infty}^{+\infty} \exp\left(i2\pi\frac{m}{\Delta z}z\right) \tag{27}$$

The Fourier transform of a comb-function [Eq. (25)] can be derived by using this distributional form and the formula of the Fourier transform of delta-function [Eq. (21)].

7 Auto-correlation and Wiener-Khintchine's theorem

NOTE!: This section may be inconsistent for complex functions, and may consist some errors. Don't blindly trust this section. The material will be corrected soon.

7.1 Cross- and auto-correlation functions

The cross-correlation function of two functions $f(z)$ and $g(z)$ is defines as

$$f(z) \otimes g(z) \equiv \int_{-\infty}^{+\infty} f(z')g(z'-z)\,dz', \tag{28}$$

where $\otimes$ represents correlation.

The correlation function between a same function is called as auto-correlation, and is written as

$$f(z) \otimes f(z) = \int_{-\infty}^{+\infty} f(z')f(z'-z)\,dz'. \tag{29}$$

The correlation and convolution are related as

$$f(z) \otimes g(z) = f(z) * g(-z). \tag{30}$$

This can be proven using $g'(z) \equiv g(-z)$ as

$$f(z) \otimes g(z) = \int_{-\infty}^{+\infty} f(z')g(z'-z)\,dz' = \int_{-\infty}^{+\infty} f(z')g'(z-z')\,dz' = f(z) * g(-z). \tag{31}$$

7.2 Wiener-Khintchine's theorem

The inverse Fourier transform of the power spectrum of a function is the auto-correlation of the function;

$$\mathcal{F}^{-1}\left[\left|\tilde{f}(k)\right|^{2}\right] = f(z) \otimes f(z). \tag{32}$$

This relation between the power spectrum and the auto-correlation is known as Wiener-Khintchine's theorem.

8 Scaling and mirroring of function

8.1 Scaling property of Fourier transform

The Fourier transform of a function scaled in z is

$$\mathcal{F}\left[f(\mathrm{a}z)\right] = \frac{1}{|\mathrm{a}|}\tilde{f}\left(\frac{k}{\mathrm{a}}\right), \tag{33}$$

where a is a constant scaling the function along z. Namely, $f(\mathrm{a}z)$ is $1/\mathrm{a}$-times narrower than $f(z)$.

Similarly,

$$\mathcal{F}^{-1}\left[\tilde{f}(\mathrm{b}k)\right] = \frac{1}{|\mathrm{b}|}\tilde{f}\left(\frac{z}{\mathrm{b}}\right), \tag{34}$$

where b is another scaling factor.

8.2 Fourier transform of a mirrored signal

The Fourier transform of a mirrored signal is obtained as

$$\mathcal{F}\left[f(-z)\right] = \tilde{f}(-k) \qquad \Leftrightarrow \qquad \mathcal{F}^{-1}\left[\tilde{f}(-k)\right] = f(-z), \tag{35}$$

where $f(-z)$ is the mirrored signal of $f(z)$.

Note that this is the special case of the Fourier transform of the scaled function [Eqs. (33) and (34)] with the scaling factor of -1.

9 Fourier transform of real function and complex conjugate

9.1 Fourier transform of real function

The Fourier transform spectrum of a real function $r(z)$ is a mirror-conjugate-symmetric function, namely,

$$\tilde{r}(k) = \tilde{r^{*}}(-k), \tag{36}$$

where $\tilde{r}(k)$ is $\mathcal{F}\left[r(z)\right]$, and the superscript of $*$ denotes complex conjugate. Note that $\tilde{r}(k)$ is not necessarily a real function. Namely, the Fourier transform of a real function can be a complex function.

Similarly, for a real k-domain function, i.e., a spectrum, $\tilde{s}(k)$,

$$s(z) = s^{*}(-z), \tag{37}$$

where $s(k)$ is $\mathcal{F}^{-1}[\tilde{s}(k)]$, and $s(k)$ is a real functions.

This relation is exemplified by FD-OCT. A spectral interferogram is a real function, and hence its Fourier transform in z-domain is mirror-conjugate-symmetric. (Note that it is not true if additional phase is applied to the spectral interferogram, as it is frequently done to correct dispersion.)

9.2 Fourier transform of complex conjugate

The Fourier transform of the complex conjugate of a function becomes a mirrored complex conjugate of the original Fourier tansform [i.e., spectrum $\tilde{f}(k)$] as

$$\mathcal{F}[f^*(z)] = \tilde{f}^*(-k), \tag{38}$$

where superscript of $*$ denotes complex conjugate.

Similarly,

$$\mathcal{F}^{-1}\left[\tilde{f}^*(k)\right] = f^*(-z). \tag{39}$$

This relationship can be derived by resolving $f(z)$ into two real functions as $f(z) = f_r(z) + i f_i(z)$, where both $f_r(z)$ and $f_i(z)$ are real functions, Fourier transform it, and then applying the rule for the Fourier transform of a real function.

10 Parseval's theorem

Parseval's theorem is the law of energy conservation during Fourier transform, and is expressed as

$$\int_{-\infty}^{+\infty} |f(z)|^2 \, dz = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\tilde{f}(k)\right|^2 \, dk. \tag{40}$$

Outlooks

The following topics are going to be added.

- Following Window functions and is Fourier transform
  - Rectangle function
  - Gauss function
  - Hanning function
  - Hamming function
- The relationship between convolution and correlation functions.
- Hilbert transform as Fourier filter

Release log

- 2025-04-15 (Tue): [ver1.3pre01] Dirac delta function and sin-and-cos form of Fourier transform are added by Yasuno.
- 2025-03-27 (Thu): [Ver1.2pre01] Reformatted and several interpretations of equations added by Yasuno.

- 2015-06-02 (Tue): [Ver1.1] "Comb function" was added by Yasuno.
- 2014-08-21 (Thu): [Ver1.0] The first pre-view release by Yasuno.