Computational Physics

Topic 02 : Computational Problems involving Probability

Lecture 02 : Conditional Probability and Bayes Rule

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Outline

- Non-independent events
- Bayes Theorem

Outline

Conditional Probability

Probability values you calculate depend on the information you know (or are given) about related events.
i.e., Knowing that it was raining would change your estimate of the probability that the next person to enter this room would have a raincoat.
In more mathematical jargon ...

- Suppose we have an event A with a probability $\Pr(A)$.
- We obtain information that a related event B has occurred.
- We now want to use this information to update our estimate for the probability that A occurs.
  This new probability is written as

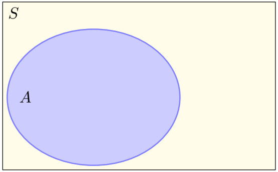
Probability Law: (Conditional Probability)

$$\Pr(A|B) = \frac{\Pr(A \text{ AND } B)}{\Pr(B)}$$

"Probability of A occurring given that B has occurred."

Justification for Conditional Probability Formula Using Sets

- The probability that event A occurs is given by
$$\Pr(A) = \frac{\#A}{\#S}$$



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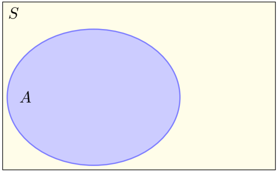
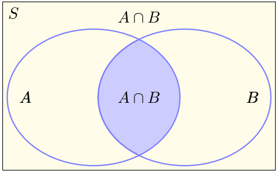


- If we know that event B has occurred then the only part of A that can then occur is in the set $A \cap B$.



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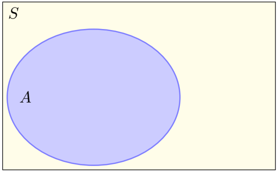

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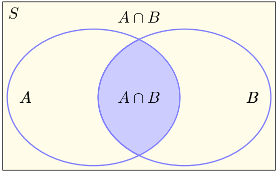
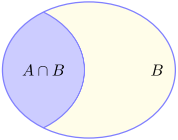


- Also since we know that B has occurred the sample space reduces from S to B so we have
$$\Pr(A|B) = \frac{\#(A \cap B)}{\#B} = \frac{\#(A \cap B)}{\#S} \cdot \frac{\#S}{\#B} = \frac{\Pr(A \text{ AND } B)}{\Pr(B)}$$



Example 1

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Out of 50 people surveyed in a study, 35 smoke in which there are 20 males. What is the probability that if the person surveyed is a smoker then he is a male?

EVENTS

- male = person selected is male
- smoker = person selected is a smoker $\Pr(\text{smoker}) = 35/50$
- $\Pr(\text{male AND smoker}) = 20/50$

Looking for the probability of the person being male given that they are a smoker ...

$$\Pr(\text{male}|\text{smoker}) = \frac{\Pr(\text{male AND smoker})}{\Pr(\text{smoker})} = \frac{20/50}{35/50} = \frac{20}{35} = 57.142\%$$

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Conditional Probability for Independent Events

Recall: If A and B are independent events then

$$\Pr(A \text{ AND } B) = \Pr(A)\Pr(B)$$

Substituting this formula into the conditional probability formula (which is true for all pair of events) we get

$$\Pr(A|B) = \frac{\Pr(A \text{ AND } B)}{\Pr(B)} = \frac{\Pr(A)\Pr(B)}{\Pr(B)} = \Pr(A)$$

Hence we have

Probability Law: (Conditional Probability for Independent Events)

If A and B are independent events then

$$\Pr(A|B) = \Pr(A) \tag{1}$$

In English "If A and B are independent then, knowing that event B has occurred does not change the probability of event A."

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Conditional Probability for Mutually Exclusive Events

Recall: If A and B are mutually exclusive events then

$$\Pr(A \text{ AND } B) = 0$$

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$$\Pr(A|B) = \frac{\Pr(A \text{ AND } B)}{\Pr(B)} = \frac{0}{\Pr(B)} = 0$$

Hence we have

Probability Law: (Conditional Probability for Mutually Exclusive)

If A and B are mutually exclusive then

$$\Pr(A|B) = 0 \tag{2}$$

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A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

EVENTS

- $\Pr(\text{red } 1^{\text{st}}) = 0.5$
- $\Pr(\text{red } 1^{\text{st}} \text{ AND blue } 2^{\text{nd}}) = 0.28$

We want probability $\Pr(\text{blue } 2^{\text{nd}})$, hence

$$\Pr(\text{blue } 2^{\text{nd}}) = \frac{\Pr(\text{red } 1^{\text{st}} \text{ AND blue } 2^{\text{nd}})}{\Pr(\text{red } 1^{\text{st}})} = \frac{0.28}{0.5} = 0.56$$

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Example 3

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What is the probability that the total of two dice will be greater than 9, given that the first die is a 5?

EVENTS

- $A =$ first die is 5 $\Pr(A) = 1/6$
- $B =$ total of two dice is greater than 9 Required
- $\Pr(A \text{ AND } B)$
  - Possible outcomes for A AND B: $(5,5)$, $(5,6)$
  - Hence $\Pr(A \text{ AND } B) = 2/36$

We want probability $\Pr(B)$, hence

$$\Pr(B) = \frac{\Pr(A \text{ AND } B)}{\Pr(A)} = \frac{2/36}{1/6} = \frac{1}{3} = 33.333\%$$

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Python Implementation

```
import numpy as np
import numpy.random as rnd

def run_trials(n):

    allowed, success = 0, 0

    for _ in range(n):
        dice = rnd.choice(range(1,7), size=2)

        if dice[0]==5:
            allowed += 1

            if sum(dice)>9:
                success += 1

    return success/allowed if allowed>0 else np.nan

print('%7s \t %s\n' % ('n', 'Pr(H)') + "="*25)
for n in [1, 10, 100, 1_000, 10_000, 100_000]:
    print('%7d \t %.5f' % (n, run_trials(n)))
```

```
      n      Pr(H)
=========================
      1      nan
     10      0.50000
    100      0.35294
   1000      0.32386
  10000      0.32445
 100000      0.33680
```

Extra Examples I

Example 4

A computer assembling company receives 24% of parts from supplier X, 36% of parts from supplier Y, and the remaining 40% of parts from supplier Z. Five percent of parts supplied by X, ten percent of parts supplied by Y, and six percent of parts supplied by Z are defective. If an assembled computer has a defective part in it, what is the probability that this part was received from supplier Z?

Example 5

At a plant, 20% of all the produced parts are subject to a special electronic inspection. It is known that any produced part which was inspected electronically has no defects with probability 0.95. For a part that was not inspected electronically this probability is only 0.7. A customer receives a part and find defects in it. What is the probability that this part went through an electronic inspection?

Extra Examples II

Example 6

A computer program consists of two blocks written independently by two different programmers. The first block has an error with probability 0.2. The second block has an error with probability 0.3. If the program returns an error, what is the probability that there is an error in both blocks?

Example 7

A computer maker receives parts from three suppliers, S1, S2, and S3. Fifty percent come from S1, twenty percent from S2, and thirty percent from S3. Among all the parts supplied by S1, 5% are defective. For S2 and S3, the portion of defective parts is 3% and 6%, respectively.

(a) What portion of all the parts is defective?

(b) A customer complains that a certain part in her recently purchased computer is defective. What is the probability that it was supplied by S1?

Example 8 I

Example 8

The following data represents the promotion record in a Police department of planet X.

	Female	Male	Total
Promoted	36	288	324
Not Promoted	204	672	876
Total	240	960	1200

i.e., 36 women have been promoted, in total, 324 people have been promoted, etc..

- After reviewing the record a claim could be made of discrimination (36 women to 288 men were promoted.)
- But this “low level” of female promotions could be due to the lower number of female staff in the force.
- We want to examine this using conditional probability.

Example 8 II

We want to examine this using conditional probability …

Events

F = event that an officer is female.

M = event that an officer is male. (So obviously $F^c = M$ and $M^c = F$)

P = event that an officer is promoted. (P^c is officer is not promoted.)

Probabilities

We want to convert our observed frequencies to probabilities …
Using the given table we have

$$\Pr(\text{officer is female AND is promoted}) = \Pr(F \text{ AND } P) = 36/1200 = 0.03$$

and so on for the rest of the table we have …

	Female	**Male**	**Total**
Promoted	36	288	324
Not Promoted	204	672	876
Total	240	960	1200

$\rightarrow$

	Female (F)	F^c	**Total**
Promoted (P)	0.03	0.24	0.27
P^c	0.17	0.56	0.73
Total	0.20	0.80	1.00

Example 8 III

	Female (F)	F^c	Total
Promoted (P)	0.03	0.24	0.27
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	Female (F)	F^c	Total
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- The probabilities in the centre of the table are called joint probabilities as they give the probability of two events happening together, i.e., jointly.

Example 8 III

$\Pr(F \text{ AND } P)$

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$\Pr(F \text{ AND } P)$ — 0.03

$\Pr(F \text{ AND } P^C)$ — 0.17

$\Pr(F^C \text{ AND } P)$ — 0.24

$\Pr(F^C \text{ AND } P^C)$ — 0.56

- The probabilities in the centre of the table are called joint probabilities as they give the probability of two events happening together, i.e., jointly.
- The probability along the margins (that were calculated using the total column and total row) are called marginal probabilities.

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$\Pr(F)$

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$\Pr(P)$ $\Pr(P^c)$

$\Pr(F)$ $\Pr(F^c)$

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- The marginal probabilities tell us that …
  20% of the work force is remale, 80% is female.
  27% of the work force received promotion, etc..

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$\Pr(P)$
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- The marginal probabilities tell us that …
  20% of the work force is remale, 80% is female.
  27% of the work force received promotion, etc..
- We could also get the marginal probabilities by summing the joint probabilities.

Example 8 IV

Now calculating the probability of promotion given office is female

$$\Pr(P|F) = \frac{\Pr(P \text{ AND } F)}{\Pr(F)} = \frac{0.03}{0.20} = 0.1$$

And the probability of promotion given office is male

$$\Pr(P|\overline{F}) = \frac{\Pr(P \text{ AND } \overline{F})}{\Pr(\overline{F})} = \frac{0.24}{0.80} = 0.3$$

Hence males are three times more likely to be promoted than females.

Outline

Motivational Example I

Example 9

Assume that you are working for a company that has a mandatory drug testing policy. It is estimated that 2% of the employees use a certain drug, and the company is giving a test that is 99% accurate in identifying users of this drug. What is the probability that if an employee is identified by this test as a drug user, the person is innocent?

EVENTS

- drug = "person is a drug user" $\Pr(\text{drug}) = 0.02$
- positive = "person tests positive for the drug"
- Test is 99% accurate in identifying users of this drug

$$\Pr(\text{positive}|\text{drug}) = \Pr(\overline{\text{positive}}|\overline{\text{drug}}) = 0.99$$

We want probability of not a drug user given that they tested positive test, i.e.,

$$\Pr(\overline{\text{drug}}|\text{positive}) = \frac{\Pr(\overline{\text{drug}} \text{ AND positive})}{\Pr(\text{positive})}$$

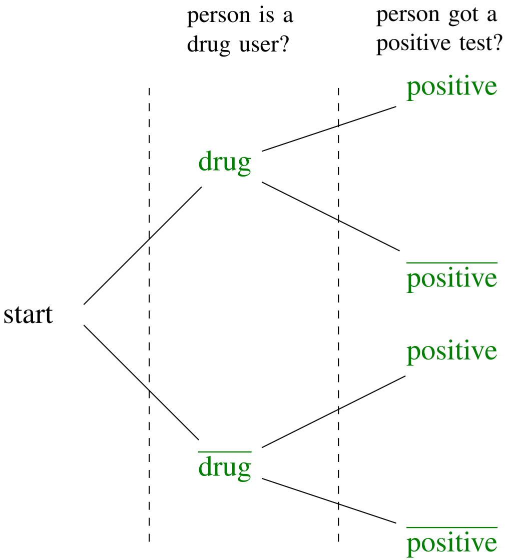
Motivational Example — Tree Diagram II

person is a drug user?

person got a positive test?

start

Motivational Example — Tree Diagram II



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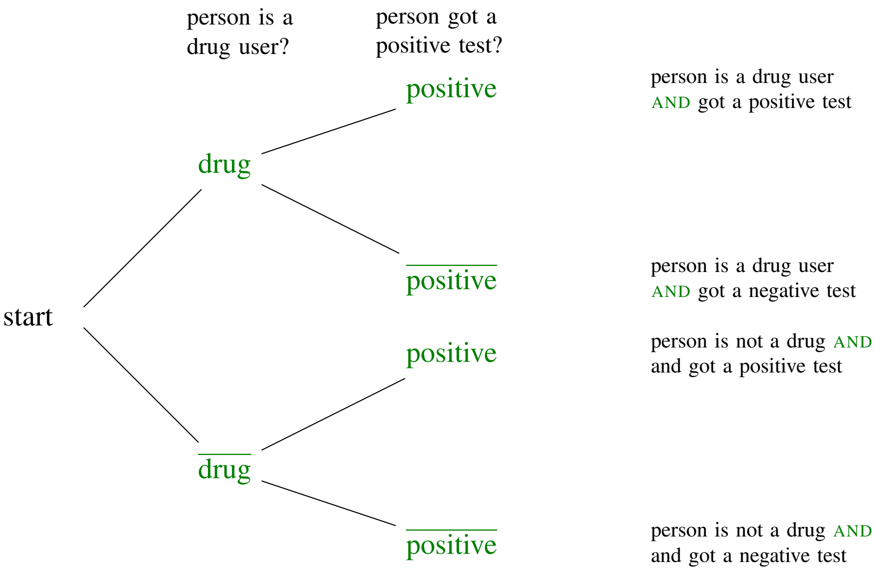

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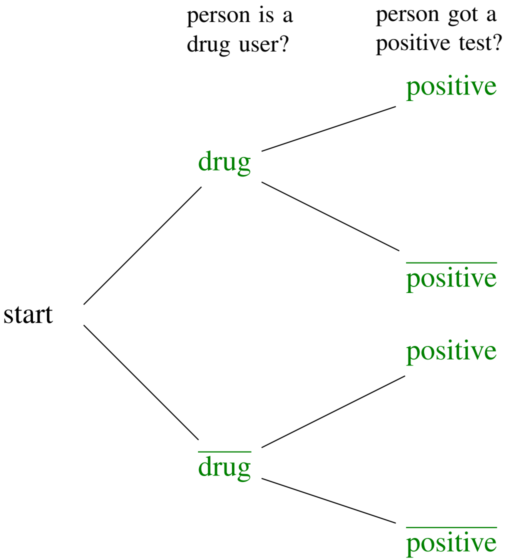

person is a drug user AND got a positive test ✔

(False negative)

person is a drug user AND got a negative test ✘

person is not a drug AND and got a positive test ✘

(False positive)

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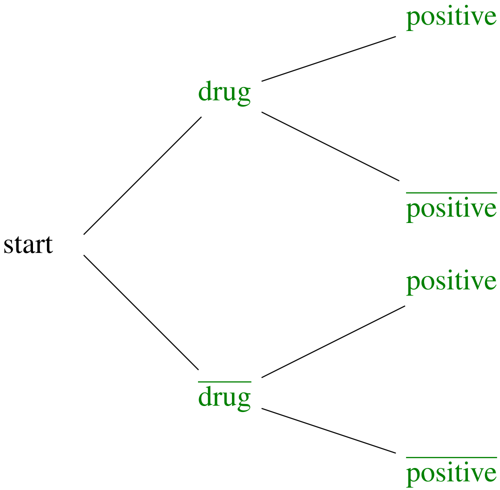
Motivational Example — Tree Diagram II

person is a drug user? | person got a positive test?



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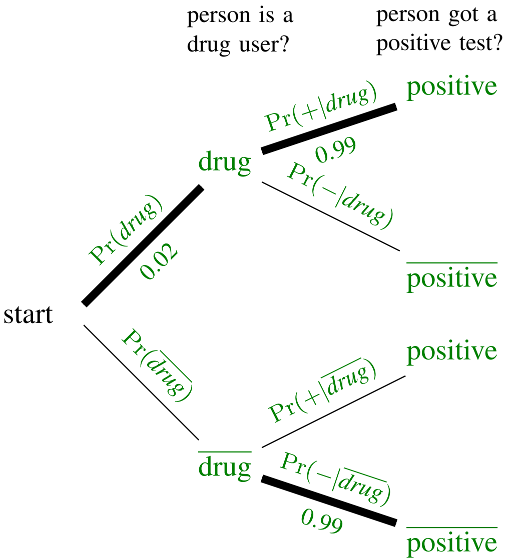

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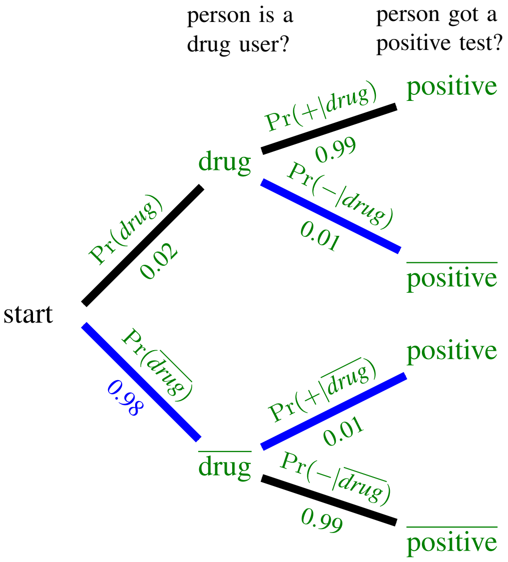

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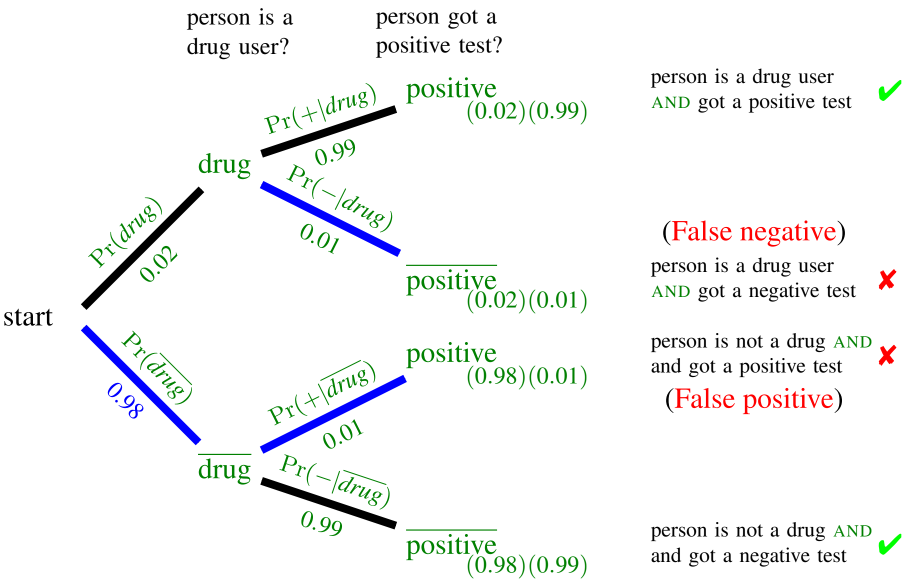

Motivational Example III

$$\Pr(\text{positive}) = \underbrace{(0.02)(0.99)}_{\Pr(\text{positive AND drug})} + \underbrace{(0.98)(0.01)}_{\Pr(\text{positive AND } \overline{\text{drug}})} = 0.0296$$

Using the conditional probability formula

$$\Pr(\overline{\text{drug}}|\text{positive}) = \frac{\Pr(\overline{\text{drug}} \text{ AND positive})}{\Pr(\text{positive})} = \frac{(0.98)(0.01)}{0.0296} = 0.331$$

In other words, if a person tests positive, there is a roughy 33% chance that the person is innocent!

Bayes' Theorem

The conditional probability law states

$$\Pr(A|B) = \frac{\Pr(A \text{ AND } B)}{\Pr(B)} \implies \Pr(A|B)\Pr(B) = \Pr(A \text{ AND } B)$$

and

$$\Pr(B|A) = \frac{\Pr(A \text{ AND } B)}{\Pr(A)} \implies \Pr(B|A)\Pr(A) = \Pr(A \text{ AND } B)$$

Combing the two results we have

$$\Pr(B|A)\Pr(A) = \Pr(A|B)\Pr(B)$$

Hence we have

Probability Law: (Bayes' Theorem)

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)}$$

In English "Bayes Theorem allows us to switch the order of terms in a conditional probability"

Bayes' Theorem with Total Law of Probability

We often use the Total Law of Probability to expand the denominator in Bayes' Theorem into known quantities:

$$\begin{aligned}\Pr(A) &= \Pr(A \text{ AND } B) + \Pr(A \text{ AND } \overline{B}) \\ &= \Pr(A|B)\Pr(B) + \Pr(A|\overline{B})\Pr(\overline{B})\end{aligned}$$

Hence Bayes' Theorem can be written as

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)} = \frac{\Pr(A|B)\Pr(B)}{\Pr(A|B)\Pr(B) + \Pr(A|\overline{B})\Pr(\overline{B})}$$

Similarly, if $B_1, B_2, \ldots, B_n$ are mutually exclusive and exhaustive events we have

$$\begin{aligned}\Pr(A) &= \Pr(A \text{ AND } B_1) + \Pr(A \text{ AND } B_2) + \cdots + \Pr(A \text{ AND } B_n) \\ &= \Pr(A|B_1)\Pr(B_1) + \Pr(A|B_2)\Pr(B_2) + \cdots + \Pr(A|B_n)\Pr(B_n)\end{aligned}$$

and the corresponding Bayes' Theorem formula.

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and the corresponding Bayes' Theorem formula.

Example 10

Example 10

Suppose we have two bowls, both containing red and blue coloured balls: one has 4 red balls and 6 green balls, the other has 6 red and 4 green. We toss a fair coin, if heads, pick a random ball from the first bowl, if tails from the second.

What is the probability of getting a red ball?

EVENTS

- $R =$ red ball
- $H =$ getting a head in the coin toss $\Pr(H) = 1/2$
- $T =$ getting a tail in the coin toss $\Pr(T) = 1/2$

  Note H and T are mutually exclusive and exhaustive, since $T = \overline{H}$

$$\begin{aligned}\Pr(R) &= \Pr(R \text{ AND } H) + \Pr(R \text{ AND } T)\\ &= \Pr(R|H)\Pr(H) + \Pr(R|T)\Pr(T) = \frac{4}{10}\frac{1}{2} + \frac{6}{10}\frac{1}{2} = \frac{10}{20} = \frac{1}{2}\end{aligned}$$

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Example 11 I

Example 11

A factory uses three machines X, Y and Z. Suppose

- Machine X produces 50% of the items, of which 3% are defective
- Machine Y produces 30% of the items, of which 4% are defective
- Machine Z produces 20% of the items, of which 5% are defective

(a) Determine the probability that a randomly selected item is defective.

(b) Determine the probability that the item came from each of the machines.

EVENTS

- D = item selected is defective
- X = item selected came from machine X

$$\Pr(X) = 0.5$$
$$\Pr(D|X) = 0.03$$

- Y and Z defined similarly to X

  Note X, Y, and Z are mutually exclusive and exhaustive events.

Example 11 II

1. *Determine the probability that a randomly selected item is defective:*

$$\begin{aligned}\Pr(D) &= \Pr(D \text{ AND } X) + \Pr(D \text{ AND } Y) + \Pr(D \text{ AND } Z)\\ &= \Pr(D|X)\Pr(X) + \Pr(D|Y)\Pr(Y) + \Pr(D|Z)\Pr(Z)\\ &= (0.03)(0.5) + (0.04)(0.3) + (0.05)(0.2)\\ &= 3.7\%\end{aligned}$$

2. *Determine the probability that the item came from each of the machines:*

$$\Pr(X|D) = \frac{\Pr(D|X)\Pr(X)}{\Pr(D)} = \frac{(.03)(.5)}{.037} = 40.5\%$$

$$\Pr(Y|D) = \frac{\Pr(D|Y)\Pr(Y)}{\Pr(D)} = \frac{(0.04)(0.3)}{0.037} = 32.5\%$$

$$\Pr(Z|D) = \frac{\Pr(D|Z)\Pr(Z)}{\Pr(D)} = \frac{(0.05)(0.2)}{0.037} = 27\%$$

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Example 12 I

Example 12

Suppose we have a blood test for a relatively rare disease, and

- The probability a person tests positive given they have the disease is 99%.
- The probability they test positive given they do not have the disease is 1%.
- The disease is prevalent in 1% of the population.

What is the probability that someone who tests positive actually has the disease?

EVENTS

- $D =$ person has desease $\Pr(D) = 0.01$
- $P =$ person tested positive $\Pr(P) = ?$
- $\Pr(P|D) = 0.99$
- $\Pr(P|\overline{D}) = 0.01$

We want $\Pr(D|P)$

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We want $\Pr(D|P)$ but we have $\Pr(P|D)$ ... use Bayes' rule

$$\overbrace{\Pr(D|P)}^{\text{✘}} = \frac{\overbrace{\Pr(D \text{ AND } P)}^{?}}{\underbrace{\Pr(P)}_{?}} = \frac{\overbrace{\Pr(P \text{ AND } D)}^{?}}{\underbrace{\Pr(P)}_{?}} = \frac{\overbrace{\Pr(P|D)}^{\text{✔}}\overbrace{\Pr(D)}^{\text{✔}}}{\underbrace{\Pr(P)}_{?}}$$

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$$\Pr(D|P) = \frac{(0.99)(0.01)}{0.0198} = 0.5$$

So half of those who test positive do not have the disease!
Think of what this means in terms of human cost, or system cost (treating a patient for a condition they do not have), etc.

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