1 Derivations from Maxwell Equations

$$\mathrm{B} = \nabla \times \mathrm{A}$$
$$\mathrm{E} = -\nabla \cdot \mathrm{v} - \frac{\partial \mathrm{A}}{\partial t}$$

1.1 Gauss' Law

$$\nabla \cdot \mathrm{E} = \frac{\rho}{\epsilon}$$
$$\nabla \cdot \left(-\nabla \cdot \mathrm{v} - \frac{\partial \mathrm{A}}{\partial t} \right) = \frac{\rho}{\epsilon}$$

The divergence is a linear operator :

$$-\nabla^2 \cdot \mathrm{v} - \nabla \cdot \frac{\partial \mathrm{A}}{\partial t} = \frac{\rho}{\epsilon}$$

We now multiply both terms by a test function X, and integrate it over a domain Ω:

$$-\int_\Omega \epsilon \cdot \left(\nabla^2 \cdot \mathrm{v}\right) \cdot \mathrm{X} \cdot \mathrm{d}\Omega - \int_\Omega \epsilon \nabla \cdot \frac{\partial \mathrm{A}}{\partial t} \cdot \mathrm{X} \cdot \mathrm{d}\Omega = \int_\Omega \rho \cdot \mathrm{X} \cdot \mathrm{d}\Omega$$
$$-\int_\Omega \epsilon \cdot \nabla \mathrm{v} \cdot \nabla \mathrm{X} \cdot \mathrm{d}\Omega - \int_\Omega \epsilon \nabla \cdot \frac{\partial \mathrm{A}}{\partial t} \cdot \mathrm{X} \cdot \mathrm{d}\Omega = \int_\Omega \rho \cdot \mathrm{X} \cdot \mathrm{d}\Omega$$

1.1.1 In a conductor

In a conductor, a common approximation is the current density being proportional to the electric field, $J = \sigma \cdot E$. σ being the conductivity of the conductor.

The law equivalent to Gauss' law in conductors becomes :

$$\nabla \cdot \mathrm{J} = 0$$
$$\nabla \cdot \sigma \mathrm{E} = 0$$
$$\sigma \nabla \cdot \left(-\nabla \cdot \mathrm{v} - \frac{\partial \mathrm{A}}{\partial t} \right) = 0$$

Thus,

$$-\int_\Omega \sigma \cdot \nabla \mathrm{v} \cdot \nabla \mathrm{X} \cdot \mathrm{d}\Omega - \int_\Omega \sigma \nabla \cdot \frac{\partial \mathrm{A}}{\partial t} \cdot \mathrm{X} \cdot \mathrm{d}\Omega = 0$$

1.1.2 In a conductor -DC

$$-\int_\Omega \sigma \cdot \nabla \mathrm{v} \cdot \nabla \mathrm{X} \cdot \mathrm{d}\Omega = 0$$

1.1.3 In a dielectric

There is no net charge in a dielectric. (That is, in a PCB)

$$-\int_{\Omega} \epsilon \cdot \nabla \mathrm{v} \cdot \nabla \mathrm{X} \cdot \mathrm{d}\Omega - \int_{\Omega} \epsilon \nabla \cdot \frac{\partial \mathrm{A}}{\partial t} \cdot \mathrm{X} \cdot \mathrm{d}\Omega = 0$$

1.1.4 In a dielectric - DC

$$\int_{\Omega} \epsilon \cdot \nabla \mathrm{v} \cdot \nabla \mathrm{X} \cdot \mathrm{d}\Omega = 0$$

1.2 Gauss' Law for Magnetism

$$\nabla \cdot \mathrm{B} = 0$$

$$\nabla \cdot \nabla \times \mathrm{A} = 0$$

The divergence of a curl is 0, hence :

$$0 = 0$$

Therefore, we cannot get anything useful from this law

1.3 Lenz-Faraday's Law

$$\nabla \times \mathrm{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

$$\nabla \times \left(-\nabla \cdot \mathrm{v} - \frac{\partial \mathrm{A}}{\partial t} \right) = -\frac{\partial \left(\nabla \times \mathrm{A} \right)}{\partial t}$$

$$\nabla \times \nabla \cdot \mathrm{v} - \frac{\partial \left(\nabla \times \mathrm{A} \right)}{\partial t} = -\frac{\partial \left(\nabla \times \mathrm{A} \right)}{\partial t}$$

$$\nabla \times \nabla \cdot \mathrm{v} = 0$$

The curl of a gradient is 0, hence :

$$0 = 0$$

Therefore, we cannot get anything useful from this law

1.4 Ampere's Law

$$\nabla \times \mathrm{B} = \mu \left(\mathrm{J} + \epsilon \frac{\partial \mathrm{E}}{\partial t} \right)$$

$$\nabla \times \left(\nabla \times \mathrm{A} \right) = \mu \left(\mathrm{J} + \epsilon \frac{\partial \left(-\nabla \cdot \mathrm{v} - \frac{\partial \mathrm{A}}{\partial t} \right)}{\partial t} \right)$$

$$\frac{1}{\mu} \nabla \times \left(\nabla \times \mathrm{A} \right) = \mathrm{J} - \epsilon \frac{\partial \left(\frac{\partial \mathrm{A}}{\partial t} \right)}{\partial t} - \epsilon \frac{\partial \left(\nabla \cdot \mathrm{v} \right)}{\partial t}$$

$$\frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A}) = \mathrm{J} - \epsilon\frac{\partial^2 A}{\partial^2 t} - \epsilon\nabla\cdot\frac{\partial\,(\mathrm{v})}{\partial t}$$

We now multiply both terms by a test function X, and integrate it over a domain Ω:

$$\int_\Omega \frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A})\cdot X\cdot d\Omega = \int_\Omega J\cdot X\cdot d\Omega - \int_\Omega \epsilon\frac{\partial^2 A}{\partial^2 t}\cdot X\cdot d\Omega - \int_\Omega \epsilon\nabla\cdot\frac{\partial\,(\mathrm{v})}{\partial t}\cdot X\cdot d\Omega$$

$$\int_\Omega \frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A})\cdot X\cdot d\Omega = \int_\Omega J\cdot X\cdot d\Omega - \int_\Omega \epsilon\frac{\partial^2 A}{\partial^2 t}\cdot X\cdot d\Omega - \int_\Omega \epsilon\nabla\cdot\frac{\partial\,(\mathrm{v})}{\partial t}\cdot X\cdot d\Omega$$

1.4.1 In a conductor - General case

In a conductor, a common approximation is the current density being proportional to the electric field, $J = \sigma\cdot E$. σ being the conductivity of the conductor.

$$\int_\Omega \frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A})\cdot X\cdot d\Omega = -\int_\Omega \sigma\nabla v\cdot X\cdot d\Omega - \int_\Omega \sigma\frac{\partial \mathrm{A}}{\partial t}\cdot X\cdot d\Omega - \int_\Omega \epsilon\frac{\partial^2 A}{\partial^2 t}\cdot X\cdot d\Omega - \int_\Omega \epsilon\nabla\cdot\frac{\partial\,(\mathrm{v})}{\partial t}\cdot X\cdot d\Omega$$

We could even add that in a good conductor, there is little to no displacement current:

$$\int_\Omega \frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A})\cdot X\cdot d\Omega = -\int_\Omega \sigma\nabla v\cdot X\cdot d\Omega - \int_\Omega \sigma\frac{\partial \mathrm{A}}{\partial t}\cdot X\cdot d\Omega$$

1.4.2 In a conductor - DC

$$\int_\Omega \frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A})\cdot X\cdot d\Omega = -\int_\Omega \sigma\nabla v\cdot X\cdot d\Omega$$

1.4.3 In a conductor - DC - Ignoring magnetic field

$$\int_\Omega \sigma\nabla v\cdot X\cdot d\Omega = 0$$

1.4.4 In a dielectric

There cannot be any current flowing in a good dielectric, hence J = 0

$$\int_\Omega \frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A})\cdot X\cdot d\Omega = \epsilon\frac{\partial^2 A}{\partial^2 t}\cdot X\cdot d\Omega - \int_\Omega \epsilon\nabla\cdot\frac{\partial\,(\mathrm{v})}{\partial t}\cdot X\cdot d\Omega$$

1.4.5 In a dielectric - DC

$$\int_\Omega \frac{1}{\mu}\nabla\times(\nabla\times \mathrm{A})\cdot X\cdot d\Omega = 0$$