LINEAR INDEPENDENCE AND BASES[4.3]

Recall: Linear independence

➤ The set $\{v_1, ..., v_p\}$ is said to be linearly dependent if there exist weights $\alpha_1, ..., \alpha_p$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_p v_p = 0 \qquad (1)$$

➤ It is linearly independent otherwise

➤ The above equation is called linear dependence relation among the vectors $v_1, \cdots, v_p$

➤ The set $v_1, v_2, \cdots, v_p$ is linearly dependent if and only if equation (1) has a nontrivial solution, i.e., if there are some weights, $\alpha_1, ..., \alpha_p$, not all zero, such that (1) holds.

➤ In such a case, (1) is called a linear dependence relation among $v_1, ..., v_p$.

Theorem: An indexed set $\{v_1, ..., v_p\}$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent if and only if some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, ..., v_{j-1}$.

✍ As an exercise prove formally this theorem

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{b_1, ..., b_p\}$ in V is a basis for H if:

1. $\mathcal{B}$ is a linearly independent set, and
2. The subspace spanned by $\mathcal{B}$ coincides with H; that is, $H = \mathrm{span}\{b_1, ..., b_p\}$

- The definition of a basis applies to the case when $\boldsymbol{H} = \boldsymbol{V}$, (any vector space is a subspace of itself)

- A basis of $\boldsymbol{V}$ is a linearly independent set that spans $\boldsymbol{V}$.

- Note that condition (2) implies that each of the vectors $\boldsymbol{b_1, ..., b_p}$ must belong to $\boldsymbol{H}$, because $\mathbf{span}\{\boldsymbol{b}_1, ..., \boldsymbol{b}_p\}$ contains $\boldsymbol{b_1, ..., b_p}$.

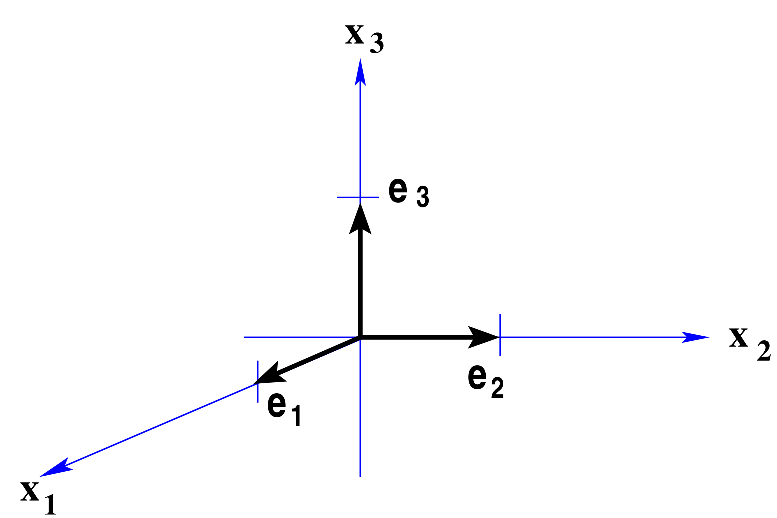
Standard basis of $\mathbb{R}^n$

Let $e_1, ..., e_n$ be the columns of the $n \times n$ matrix, I_n.

That is,

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \; e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}; \; \cdots \; ; e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix};$$

- The set $\{e_1, \cdots, e_n\}$ is called the standard basis for $\mathbb{R}^n$.
- Sometimes the term canonical basis is used



Spanning set theorem

Theorem: Let $S = \{v_1, ..., v_p\}$ be a set in V, and let $H = \mathbf{span}\{v_1, ..., v_p\}$.

1. If one of the vectors in S–say, v_k–is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.
2. If $H \neq \{0\}$, some subset of S is a basis for H.

Proof: 1. By rearranging the list of vectors in S, if necessary, we may assume that v_k is the last vector of the list, i.e., v_p, so:

$$v_p = a_1 v_1 + ... + a_{p-1} v_{p-1} \tag{1}$$

➤ Given any x in H, we may write

$$x = \alpha_1 v_1 + ... + \alpha_{p-1} v_{p-1} + \alpha_p v_p \tag{2}$$

for suitable scalars $\alpha_1, ..., \alpha_p$.

➤ Substituting the expression for v_p from (1) into (2) it is easy to see that x is a linear combination of $v_1, ... v_{p-1}$.

➤ Vector x was arbitrary – Thus $\{v_1, ..., v_{p-1}\}$ spans H -

2. If the original spanning set S is linearly independent, then it is already a basis for H.

➤ Otherwise, one of the vectors in S depends on the others and can be deleted, by part (1).

➤ Repeat this process until the spanning set is linearly independent and hence is a basis for H. (If the spanning set is eventually reduced to one vector, that vector will be nonzero because $H \neq \{0\}$) ■

✍ Let $\boldsymbol{H = \mathrm{span}\{v_1, v_2, v_3\}}$ with

$$\boldsymbol{v_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix};}$$

Show that $\boldsymbol{v_3}$ is a linear combination of the first 2 vectors and then find a basis of $\boldsymbol{H}$.

Basis of Col(A)

Theorem:
The pivot columns of a matrix A form a basis for Col(A).

Proof: Let B be the reduced echelon form of A. The set of pivot columns of B is linearly independent (no vector in the set is a linear combination of the vectors that precede it).

$$B = \begin{bmatrix} 1 & * & 0 & * & * & * & 0 & * & * & 0 & * \\ & & 1 & * & * & * & 0 & * & * & 0 & * \\ & & & & & & 1 & * & * & 0 & * \\ & & & & & & & & & 1 & * \\ & & & & & & & & & & \end{bmatrix}$$

- Since A is row equivalent to B, the pivot columns of A are linearly independent as well
- Every nonpivot column of A is a linear combination of the pivot columns of A.

➤ Thus the nonpivot columns of a may be discarded from the spanning set for Col($\boldsymbol{A}$), by the Spanning Set Theorem.

➤ This leaves the pivot columns of $\boldsymbol{A}$ as a basis for Col($\boldsymbol{A}$). ■

Note: The pivot columns of a matrix $\boldsymbol{A}$ are evident when $\boldsymbol{A}$ has been reduced to an echelon form $\boldsymbol{B}$ (standard or reduced). However be sure to use the pivot columns of $\boldsymbol{A}$ itself for the basis of Col($\boldsymbol{A}$), not those of $\boldsymbol{B}$

Two Views of a Basis:

- When the Spanning Set Theorem is used, the deletion of vectors from a spanning set must stop when the set becomes linearly independent.

- If an additional vector is deleted, it will not be a linear combination of the remaining vectors, and hence the smaller set will no longer span V.

- Thus a basis is a spanning set that is as small as possible.

- A basis is also a linearly independent set that is as large as possible.

- If S is a basis for V, and if S is enlarged by one vector –say, w–from V, then the new set loses linear independence [Explain why]

Dimension and rank

➤ It can be shown that the number of vectors in a basis of a subspace H is always the same –

✍ Show the result

Hence the definition:

Definition: The **dimension** of a subspace H is the number of vectors in any basis for H. When $H = \{0\}$ its dimension is defined to be zero.

➤ Notation $\mathbf{dim}(H)$

- Related (and important) definition

Definition: The rank of a matrix $\boldsymbol{A}$ is the dimension of its column space.

- Notation: rank$(\boldsymbol{A})$.
- Note: rank$(\boldsymbol{A})$ = number of pivot columns in $\boldsymbol{A}$.
- Recall from an earlier example that we could find a spanning set of Nul$(\boldsymbol{A})$ which has as many vectors as there are free variables.
- Therefore $\mathbf{dim}(\text{Nul}(\boldsymbol{A}))$ = number of free variables. Hence the important result

$$\text{rank}(\boldsymbol{A}) + \mathbf{dim}(\text{Nul}(\boldsymbol{A})) = \boldsymbol{n}$$

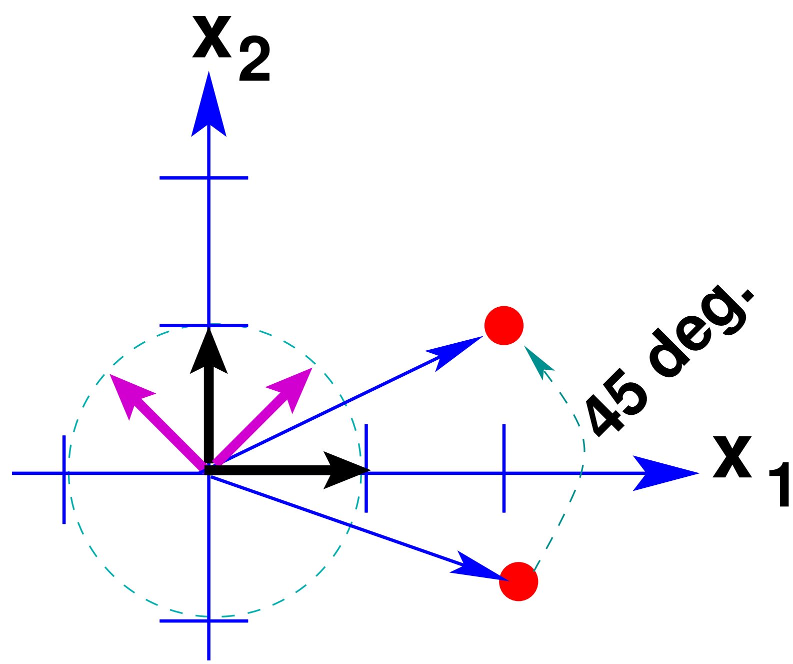
- Known as the Rank+Nullity theorem
- rank$(\boldsymbol{A})$ = rank$(\boldsymbol{A}^T)$ [row-rank=column rank]

APPLICATION: ROTATION AND TRANSLATIONS [2.7]

Application: Rotations and translations in $\mathbb{R}^2$

➤ In the form of exercises. Try to answer all questions before class [see textbook if needed]

✍ Consider the mapping that sends any point $\boldsymbol{x}$ in $\mathbb{R}^2$ into a point $\boldsymbol{y}$ in $\mathbb{R}^2$ that is rotated from $\boldsymbol{x}$ by an angle $\boldsymbol{\theta}$. Is the mapping linear?



✍ Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed]

Rotations and translations in $\mathbb{R}^2$

- We will now deal with Translations or shifts
- Another very important operation..
- Recall: Not a linear mapping – but called affine mapping..
- This will require a little artifice..

✍ How can you now represent a translation via a matrix-vector product? [Hint: add an artificial compoment of 1 at the end of vector x]

- Called Homogeneous coordinates
- Try this in matlab

Rotations and translations in $\mathbb{R}^2$

✍ The most important mapping in real life is a combination of Rotation and Translation. How do you represent these?

➤ We will use the Homogeneous coordinates introduced above

➤ Need to combine two mappings: rotation and then translation

✍ Does the order matter? Reason from the geometry and then from the derivation of your matrix

✍ Find the combined mapping

➤ Try this in matlab