THE LU FACTORIZATION [2.5]

LU factorization: Motivation

➤ Suppose we have to solve many linear systems

$$Ax = b^{(1)}, \quad Ax = b^{(2)}, \quad \cdots, \quad Ax = b^{(p)}$$

where matrix A is the same - but the right-hand sides are different

➤ Can solve each of them by Gaussian Elimination separately $\longrightarrow$ inefficient

✍ Cost?

➤ Can get the inverse A^{-1} then each solution is of the form $x^{(k)} = A^{-1}b^{(k)}$

✍ Cost? [Using method based on rref seen in Lec. Notes 8]

- Best option: Exploit the "LU factorization of A"
- Main result is this:

*Gaussian elimination algorithm can provide as a by-product a *factorization* of A into the product of a lower triangular matrix L with ones on the diagonal, and an upper triangular matrix U:*

$$\boxed{A = LU}$$

- In addition:

  This factorization is obtained at virtually no extra cost.

✍ How would you solve systems with multiple right-hand sides using this? What does this approach cost?

- Next: The LU factorization. Where does it come from and how to get it?

LU factorization – Revisiting GE

- We now ignore the right-hand side in GE

Recall: Gaussian elimination amounts to performing $n-1$ successive Gaussian transformations, i.e., multiplications (to the left) by elementary matrices that have ones on the diagonal and the negatives of the multipliers in the column being eliminated.

- Set $A_0 \equiv A$. Then – results of the $n-1$ steps:

$$\begin{aligned} A_1 &= E_1 A_0 \\ A_2 &= E_2 A_1 = E_2 E_1 A_0 \\ A_3 &= E_3 A_2 = E_3 E_2 E_1 A_0 \\ \cdots &= \cdots \\ A_{n-1} &= E_{n-1} E_{n-2} \cdots E_2 E_1 A_0 \end{aligned}$$

- $A_{n-1} \equiv U$ is an upper triangular matrix.
- We have $U = E_{n-1}E_{n-2}\cdots E_2E_1A$ or :

$$A \;=\; \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{n-1}^{-1}}_{L}U \equiv LU$$

- $E_1, E_2, \cdots, E_{n-1}$ are all lower triangular matrices with ones on the diagonal.
- Each E_j^{-1} is lower triangular with ones on the diagonal. (Why?)
- $L = E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{n-1}^{-1}$ is lower triangular (Why?)
- L has ones on the diagonal.

- In the end :

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} \quad \text{with:}$$
$$\boldsymbol{L} = \boldsymbol{E}_1^{-1}\boldsymbol{E}_2^{-1}\boldsymbol{E}_3^{-1}\cdots\boldsymbol{E}_{n-1}^{-1}$$
$$\boldsymbol{U} = \boldsymbol{A}_{n-1}$$

- Called the LU decomposition (or factorization) of $\boldsymbol{A}$.

Notes:

- $\boldsymbol{L}$ is Lower triangular, and has ones on the diagonal – We say that it is *unit lower triangular*

- $\boldsymbol{U}$ is the last matrix into which $\boldsymbol{A}$ is transformed from Gaussian elimination. It is *upper triangular.*

- We know how to get $\boldsymbol{U}$ [last matrix in GE]

- The main issue now is: How can we get $\boldsymbol{L}$?

How do we get L?

- Could we use: $L = E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{n-1}^{-1}$? Too complex!
- There is a simpler way:

Theorem. Assume that Gaussian elimination can terminate (no division by zero) and let U be the final triangular matrix obtained and L the lower triangular matrix with $l_{ii} = 1$, and, for $i > k$, $l_{ik} = piv_{ik}$, the multiplier used to eliminate row i in step k. Then: $A = LU$.

- $l_{kk} = 1$ and for $i \neq k$, $l_{ik} =$ multiplier a_{ik}/a_{kk} at k-th step of GE.

- The matrix A is the product of a unit lower triangular matrix L and an upper triangular matrix U.

LU factorization - an example

Example: Let $$A = \begin{pmatrix} 4 & -2 & 2 \\ -2 & 5 & 3 \\ 2 & 3 & 9 \end{pmatrix}$$

Step 1 of GE uses the multipliers $l_{21} = -1/2$, $l_{31} = 1/2$.

➤ Resulting matrix: $$A_1 = \begin{pmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \end{pmatrix}$$

Step 2 of Gaussian Elimination uses the multiplier $l_{32} = 1$.

➤ Resulting matrix $$A_2 = \begin{pmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{pmatrix} \equiv U$$

- Therefore : $\quad L = \begin{pmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{pmatrix}$

- Verify that $A = LU$

- LU factorization of the matrix $A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 5 & 9 \\ 1 & 0 & -12 \end{pmatrix}$

- For the same A compute the 3rd column of A^{-1}.

✍ How would you compute the inverse of a matrix given its LU factorization?

✍ Show how to use the LU factorization to solve linear systems with the same matrix $\boldsymbol{A}$ and different right-hand sides $\boldsymbol{b}$.

✍ True or false: "Computing the LU factorization of a matrix $\boldsymbol{A}$ involves more arithmetic operations than solving a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ by Gaussian elimination" ?