LINEAR MAPPINGS [1.8]

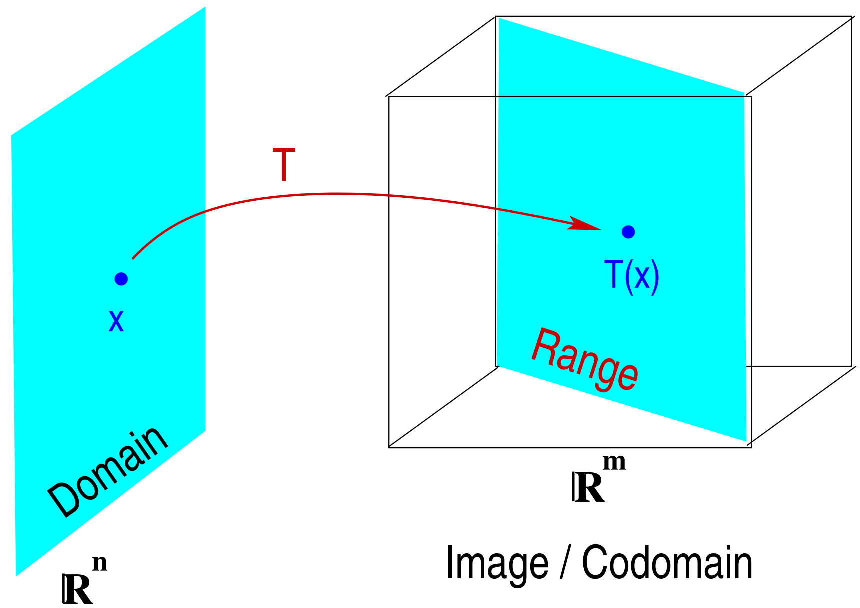
Introduction to linear mappings [1.8]

➤ A transformation or function or mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to every $\boldsymbol{x}$ in $\mathbb{R}^n$ a vector $\boldsymbol{T}(\boldsymbol{x})$ in $\mathbb{R}^m$.

➤ $\mathbb{R}^n$ is called the domain space of $\boldsymbol{T}$ and $\mathbb{R}^m$ the image space or co-domain of $\boldsymbol{T}$.

➤ Notation:

$$\boldsymbol{T}: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$



➤ $\boldsymbol{T}(\boldsymbol{x})$ is the image of $\boldsymbol{x}$ under $\boldsymbol{T}$

Example: Take the mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T: \quad \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{R}^3$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$$

Example: Another mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T: \quad \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{R}^3$$
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + 5x_2 \end{pmatrix}$$

✍ What is the main difference between these 2 examples?

Definition A mapping T is linear if:

(i) $T(u+v) = T(u) + T(v)$ for u, v in the domain of T

(ii) $T(\alpha u) = \alpha T(u)$ for all $\alpha \in \mathbb{R}$, all u in the domain of T

➤ The mapping of the second example given above is linear - but not for the first one.

➤ If a mapping is linear then $T(0) = 0$. (Why?)

Observation: A mapping is linear if and only if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

for all scalars α, β and all u, v in the domain of T.

✍ Prove this

➤ Given an $m \times n$ matrix A, consider the special mapping:

$$\begin{aligned} T: \quad \mathbb{R}^n & \longrightarrow \mathbb{R}^m \\ x & \longrightarrow y = Ax \end{aligned}$$

✍ Domain == ??; Image space == ??

➤ From what we saw earlier ['Properties of the matrix-vector product'] such mappings are linear

➤ As it turns out:

If T is linear, there exists a matrix A such that $T(x) = Ax$ for all x in $\mathbb{R}^n$

➤ In plain English: 'A linear mapping can be represented by a matvec'

➤ A is the representation of T.

✍ Let $\boldsymbol{A}$ be a square matrix. Is the mapping $\boldsymbol{x} \to \boldsymbol{x} + \boldsymbol{A}\boldsymbol{x}$ linear? If so find the matrix associated with it.

✍ Same questions for the mapping $\boldsymbol{x} \to \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\alpha}\boldsymbol{x}$ - where $\boldsymbol{\alpha}$ is a scalar.

✍ Express the following mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ in matrix/vector form:

$$\begin{aligned} y_1 &= 2x_1 - x_2 + 1 \\ y_2 &= x_2 - x_3 - 2 \end{aligned}$$

➤ Is this a linear mapping?

✍ Read Section 1.9 and explore the notions of onto mappings ('surjective') and one-to-one mappings ('injective') in the text. You must at least know the definitions.

✍ A mapping is onto if and only if ....

✍ A mapping is one-to-one if and only if ....

MATRIX OPERATIONS [2.1]

Matrix operations

➤ If A is an $m \times n$ matrix (m rows and n columns) –then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A.

$$\begin{array}{cc} & \text{Column } j \\ & \downarrow \\ \text{Row } i \rightarrow & \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} = A \\ & \begin{array}{ccccc} \uparrow & \qquad & \uparrow & \qquad & \uparrow \\ a_1 & & a_j & & a_n \end{array} \end{array}$$

- The number a_{ij} is the ith entry (from the top) of the jth column
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$ called a column vector
- The columns are denoted by $a_1, ..., a_n$, and the matrix A is written as $A = [a_1, a_2, \cdots, a_n]$

➤ The diagonal entries in an $m \times n$ matrix A are a_{11}, a_{22}, a_{33}, ..., and they form the main diagonal of A.

➤ A diagonal matrix is a matrix whose nondiagonal entries are zero

➤ An important example is the $n \times n$ identity matrix, I_n (each diagonal entry equals one) - Example:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

➤ Another important matrix is the zero matrix (all entries are 0). It is denoted by O.

Equality of two matrices: Two matrices A and B are equal if they have the same size (they are both $m \times n$) and if their entries are all the same.

$$a_{ij} = b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Sum of two matrices: If A and B are $m \times n$ matrices, then their sum $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries in A and B.

- If we call C this sum we can write:

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

✍

$$\begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix} = ??; \qquad \begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix} = ??$$

scalar multiple of a matrix If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose entries are r times the corresponding entries in A.

$$(\alpha A)_{ij} = \alpha a_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Theorem Let A, B, and C be matrices of the same size, and let α and β be scalars. Then

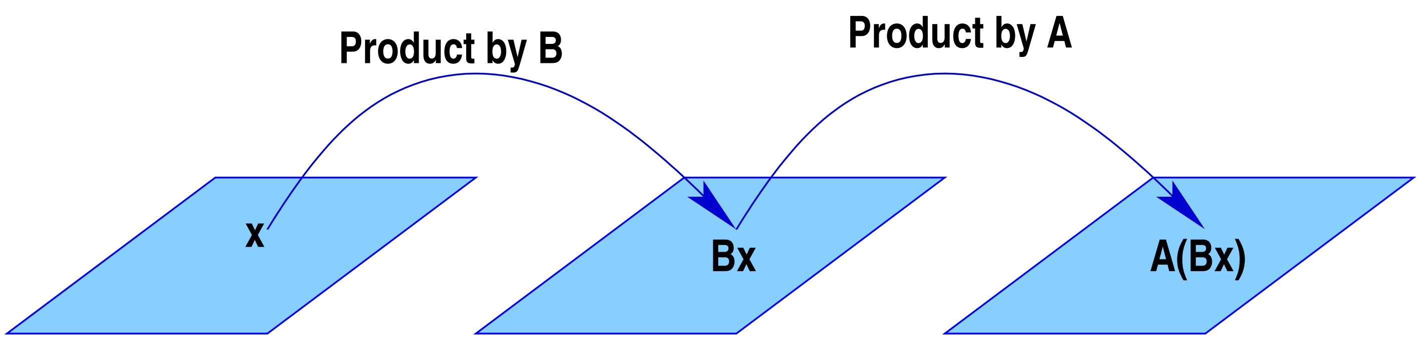
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $\alpha(A + B) = \alpha A + \alpha B$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $\alpha(\beta A) = (\alpha\beta)A$

✍ Prove all of the above equalities

Matrix Multiplication

- When a matrix $\boldsymbol{B}$ multiplies a vector $\boldsymbol{x}$, it transforms $\boldsymbol{x}$ into the vector $\boldsymbol{B}\boldsymbol{x}$.

- If this vector is then multiplied in turn by a matrix $\boldsymbol{A}$, the resulting vector is $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$.



- Thus $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$ is produced from $\boldsymbol{x}$ by a composition of mappings– the linear transformations induced by $\boldsymbol{B}$ and $\boldsymbol{A}$.

Goal: to represent this composite mapping as a multiplication by a single matrix, call it C for now, so that

$$A(Bx) = Cx$$

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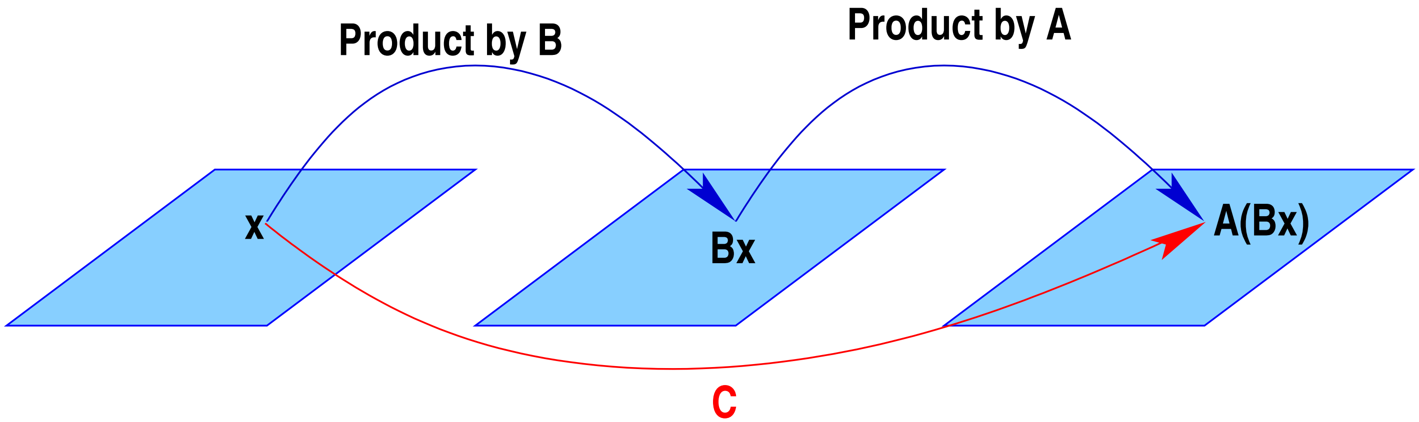

➤ Assume A is $m \times n$, B is $n \times p$, and x is in $\mathbb{R}^p$

➤ Denote the columns of B by $b_1, \cdots, b_p$ and the entries in x by $x_1, \cdots, x_p$. Then:

$$Bx = x_1 b_1 + \cdots + x_p b_p$$

➤ By the linearity of multiplication by $\boldsymbol{A}$:

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{B}\boldsymbol{x}) &= \boldsymbol{A}(\boldsymbol{x}_1\boldsymbol{b}_1) + \cdots + \boldsymbol{A}(\boldsymbol{x}_p\boldsymbol{b}_p) \\ &= \boldsymbol{x}_1\boldsymbol{A}\boldsymbol{b}_1 + \cdots + \boldsymbol{x}_p\boldsymbol{A}\boldsymbol{b}_p \end{aligned}$$

➤ The vector $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$ is a linear combination of $\boldsymbol{A}\boldsymbol{b}_1, \cdots, \boldsymbol{A}\boldsymbol{b}_p$, using the entries in $\boldsymbol{x}$ as weights.

➤ In matrix notation, this linear combination is written as

$$\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x}) = [\boldsymbol{A}\boldsymbol{b}_1, \boldsymbol{A}\boldsymbol{b}_2, \cdots \boldsymbol{A}\boldsymbol{b}_p].\boldsymbol{x}$$

➤ Thus, multiplication by $[\boldsymbol{A}\boldsymbol{b}_1, \boldsymbol{A}\boldsymbol{b}_2, \cdots, \boldsymbol{A}\boldsymbol{b}_p]$ transforms $\boldsymbol{x}$ into $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$.

➤ Therefore the desired matrix $\boldsymbol{C}$ is the matrix

$$\boldsymbol{C} = [\boldsymbol{A}\boldsymbol{b}_1, \boldsymbol{A}\boldsymbol{b}_2, \cdots, \boldsymbol{A}\boldsymbol{b}_p]$$

➤ Denoted by $\boldsymbol{A}\boldsymbol{B}$

Definition: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product AB is the matrix whose p columns are $Ab_1, \cdots, Ab_p$. That is:

$$AB = A[b_1, b_2, \cdots, b_p] = [Ab_1, Ab_2, \cdots, Ab_p]$$

➤ Important to remember that :

Multiplication of matrices corresponds to composition of linear transformations.

✍ Operation count: How many operations are required to perform product AB?

✍ Compute AB when

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

✍ Compute AB when

$$A = \begin{bmatrix} 2 & -1 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 3 & -2 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -2 & 2 \\ 2 & 1 & -2 \\ -1 & 3 & 2 \end{bmatrix}$$

✍ Can you compute AB when

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ -1 & 4 \end{bmatrix}?$$

Row-wise matrix product

➤ Recall what we did with matrix-vector product to compute a single entry of the vector Ax

➤ Can we do the same thing here? i.e., How can we compute the entry c_{ij} of the product AB without computing entire columns?

✍ Do this to compute entry $(2, 2)$ in the first example above.

✍ Operation counts: Is more or less expensive to perform the matrix-vector product row-wise instead of column-wise?

Properties of matrix multiplication

Theorem Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$ (associative law of multiplication)
- $A(B + C) = AB + AC$ (left distributive law)
- $(B + C)A = BA + CA$ (right distributive law)
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$ for any scalar α
- $I_m A = AI_n = A$ (product with identity)

If $AB = AC$ then $B = C$ ('simplification') : True-False?

If $AB = 0$ then either $A = 0$ or $B = 0$: True or False?

$AB = BA$: True or false??

Square matrices. Matrix powers

- Important particular case when $n = m$ - so matrix is $n \times n$
- In this case if x is in $\mathbb{R}^n$ then $y = Ax$ is also in $\mathbb{R}^n$
- AA is also a square $n \times n$ matrix and will be denoted by A^2
- More generally, the matrix A^k is the matrix which is the product of k copies of A:

$$A^1 = A; \quad A^2 = AA; \quad \cdots \quad A^k = \underbrace{A \cdots A}_{k \text{ times}}$$

- For consistency define A^0 to be the identity: $A^0 = I_n$,

✍ $A^l \times A^k = A^{l+k}$ – Also true when k or l is zero.

Transpose of a matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem : Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(\alpha A)^T = \alpha A^T$ for any scalar α
- $(AB)^T = B^T A^T$