پایه‌ای برای زیرفضای W که شامل تمامی بردارهایی در R^4 می‌شود که بر ستون‌های ماتریس A عمود هستند را پیدا کنید.

$$A = \begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$$

پاسخ)

Let us write $A = [A_1\, A_2\, A_3\, A_4]$, where A_i is the i-th column vector of A for $i = 1, 2, 3, 4$.
First we claim that a vector $\mathbf{x} \in \mathbb{R}^4$ is perpendicular to all column vectors A_i if and only if $\mathbf{x} \in \mathcal{N}(A^{\mathrm{T}})$.
To see this, we compute

$$A^{\mathrm{T}}\mathbf{x} = \begin{bmatrix} A_1^{\mathrm{T}} \\ A_2^{\mathrm{T}} \\ A_3^{\mathrm{T}} \\ A_4^{\mathrm{T}} \end{bmatrix} \mathbf{x} = \begin{bmatrix} A_1^{\mathrm{T}}\mathbf{x} \\ A_2^{\mathrm{T}}\mathbf{x} \\ A_3^{\mathrm{T}}\mathbf{x} \\ A_4^{\mathrm{T}}\mathbf{x} \end{bmatrix}.$$

From this equality the claim follows immediately.

So we proved that $\mathcal{N}(A^{\mathrm{T}}) = W$. From this, we see that W is actually a subspace of $\mathbb{R}^4$.

Thus, we need to find a basis for the null space of the transpose A^{T}.

We apply elementary row operations to A^{T} and obtain a reduced row echelon form

$$A^{\mathrm{T}} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The last two columns correspond to two free variables. Let s and t be free variable.

Then $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathcal{N}(A^{\mathrm{T}})$ if and only if $\mathbf{x}$ satisfies

$$\begin{aligned} x_1 &= s + 2t \\ x_2 &= -2s - 3t \\ x_3 &= s \\ x_4 &= t, \end{aligned}$$

equivalently

$$\mathbf{x} = s \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore a basis of $W = \mathcal{N}(A^{\mathrm{T}})$ is

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$