Matlab script gauss.m: a few explanations

```
  function [x]  = gauss (A, b)
% function [x]  = gauss (A, b)
% solves A x  = b by Gaussian elimination
 n = size(A,1) ;
 A = [A,b];
 for k=1:n-1
    for i=k+1:n
      piv = A(i,k) / A(k,k) ;
      A(i,k+1:n+1)=A(i,k+1:n+1)-piv*A(k,k+1:n+1);
    end
 end
 x = backsolv(A,A(:,n+1));
```

Function

```
  function [x]  = gauss (A, b)
% function [x]  = gauss (A, b)
% solves A x  = b by Gaussian elimination
...
```

- The file containing the above script should be called `gauss.m`.
- The syntax for `function` is simple:

```
  function [Output-args] = func-name(Input-args)
% lines of comments
```

- Takes input arguments. Computes some values and returns them in the output arguments.

➤ The `gauss.m` script has 2 input arguments ($\boldsymbol{A}$ and $\boldsymbol{b}$) and one output argument ($\boldsymbol{x}$)

- % indicates a commented line. First few lines of comments after `function` header are echoed when you type

`>> help func-name`   For example `>> help gauss`

```
n = size(A,1) ;      % <-- n=Number of rows in matrix
A = [A,b];           % <-- Adds b as last column of A
                     % Now A contains augmented system.
                     % It has size n x  (n+1)
```

```
for k=1:n-1          % Main loop in GE -- for each k
   for i=k+1:n       % sweep rows  i=k+1 to i=n
      ...commands    % these commands will each combine
   end               % row i with a mulitple of row k
end
```

Step $\boldsymbol{k = 3}$ when $\boldsymbol{n = 6}$

```
for i=4:6
    piv=a(i,3)/a(3,3);
    row_i=row_i-piv*row_3;
end
```

$$\left[\begin{array}{cc|ccccc} * & * & * & * & * & * & * \\ \mathbf{0} & * & * & * & * & * & * \\ \hline \mathbf{0} & \mathbf{0} & * & * & * & * & * \\ \mathbf{0} & \mathbf{0} & * & * & * & * & * \\ \mathbf{0} & \mathbf{0} & * & * & * & * & * \\ \mathbf{0} & \mathbf{0} & * & * & * & * & * \end{array}\right]$$

```
piv = A(i,k) / A(k,k) ;
A(i,k+1:n+1)=A(i,k+1:n+1)-piv*A(k,k+1:n+1);
```

- The above: 1) computes the multiplier (pivot) to use in the elimination; 2) combines rows. Result $=$ a zero in position (i,k).

- When combining row i with row k no need to deal with zeros in columns 1 to $k-1$. Result will be zero.

- Also we know $A(i,k)$ will be zero – can be skipped.

- Result: need to combine rows from positions $k+1$ to $n+1$.

```
x = backsolv(A,A(:,n+1));
```

- The above invokes the back-solve script to solve the final system

THE ECHELON FORM [1.2]

The standard echelon form

A rectangular matrix is in echelon form (or row echelon form) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

- Each $\blacksquare$ is a nonzero (leading) entry.
- A $*$ can be a non-zero or a zero entry.

$$\begin{bmatrix} \blacksquare & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of these are / are not in Row Echelon Form? [$* =$ nonzero]

$$\begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- A sort of 'stretched' upper triangular form with added zero subcolumns

- Ignore the formal definition for a moment - use intuition from Gaussian Elimination. Example:

$$\begin{array}{ccccc} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{2} & \mathbf{-2} & \mathbf{3} & \mathbf{-1} & \mathbf{4} \\ \mathbf{3} & \mathbf{-3} & \mathbf{9} & \mathbf{-6} & \mathbf{6} \end{array}$$

- 1st step of Gaussian Elimination yields:

$$\begin{array}{ccccc} & \rightarrow & & & \\ \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \boxed{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{3} & \mathbf{-3} & \mathbf{0} \end{array}$$

$\leftarrow$ a_{22} is zero and pivoting does not help... Move to third position a_{23} which is -1:

- GE on rows 2 & 3:
- $row_3 := row_3 - (-3) * row_2$

Result: $\rightarrow$

$$\begin{array}{ccccc} \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}$$

In words: Row Echelon algorithm is a variant of Gaussia Elimination (with pivoting).

* Step k now has two indices: pivot row k and pivot column l. (At the start $k = 1, l = 1$.)

* Step k : Try to eliminate entries $a_{k+1,l}$, $a_{k+2,l}$, ..., $a_{m,l}$.

* Do pivoting if necessary and try perform Gaussian Elimination.

* If the sub-column is all zero, set $l := l + 1$ and repeat.

$$\begin{array}{c|ccccc|} & & & l & & \\ \hline & * & * & * & * & * \\ k & 0 & 0 & 0 & * & * \\ & 0 & 0 & * & * & * \\ & 0 & 0 & * & * & * \\ \hline \end{array}$$

Case 1: swap row 2 and 3 (or 4). Then do GE step.

$$\begin{array}{c|ccccc|} & & & l & l_{new} & \\ \hline & * & * & * & * & * \\ k & 0 & 0 & 0 & * & * \\ & 0 & 0 & 0 & * & * \\ & 0 & 0 & 0 & * & * \\ \hline \end{array}$$

Case 2: Reset l to $l = 4$. Continue.

Terminology: Pivots, and pivot columns

PIVOTS

$$\begin{pmatrix} * & * & * & * & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

PIVOT COLUMNS

➤ Important in capturing the span of the columns of A (called the range of A - to be covered in detail later)

The reduced row echelon form

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form): $\boxed{\text{Matlab: } \texttt{rref}}$

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} & * & * & \mathbf{0} & * \\ & & \mathbf{1} & * & * & * & \mathbf{0} & * & * & \mathbf{0} & * \\ & & & & & & \mathbf{1} & * & * & \mathbf{0} & * \\ & & & & & & & & & \mathbf{1} & * \\ & & & & & & & & & & \end{bmatrix}$$

✍ How would you obtain the rref from the standard echelon form?

➤ Any nonzero matrix may be row reduced (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations.

➤ However, the reduced echelon form one obtains from a matrix is unique:

Each matrix is row equivalent to one and only one reduced echelon matrix.

➤ Remember that the permissible row operations are:

1) Interchange; 2) addition; 3) scaling.

Pivot position

➤ A pivot position in a matrix $\boldsymbol{A}$ is a location in $\boldsymbol{A}$ that corresponds to a leading 1 in the reduced echelon form of $\boldsymbol{A}$. A pivot column is a column of $\boldsymbol{A}$ that contains a pivot position.

$$\begin{bmatrix} \boxed{1} & * & \mathbf{0} & * & * & * & \mathbf{0} & * & * & \mathbf{0} & * \\ & & \boxed{1} & * & * & * & \mathbf{0} & * & * & \mathbf{0} & * \\ & & & & & & \boxed{1} & * & * & \mathbf{0} & * \\ & & & & & & & & & \boxed{1} & * \\ & & & & & & & & & & \end{bmatrix}$$

➤ In this example, the pivot columns are 1, 3, 7, and 10

✍ Find out how to get the pivot positions from matlab's `rref`

Example with standard echelon Form

Example: Row reduce the matrix $\boldsymbol{A}$ below to echelon form, and locate the pivot columns of $\boldsymbol{A}$.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

Solution: The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or pivot, must be placed in this position.

➤ Interchange rows 1 and 4 (note: in reality it is preferable to interchange rows 1 and 3. Why?)

$\downarrow$ Pivot

$$\begin{array}{ccccc} \boxed{1} & \mathbf{4} & \mathbf{5} & \mathbf{-9} & \mathbf{-7} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{-1} & \mathbf{3} & \mathbf{1} \\ \mathbf{-2} & \mathbf{-3} & \mathbf{0} & \mathbf{3} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-3} & \mathbf{-6} & \mathbf{4} & \mathbf{9} \end{array}$$

$\uparrow$ Pivot Column

➤ Create zeros below the pivot, 1, by adding multiples of the first row to the rows below $\longrightarrow$ Next matrix:

$$\begin{array}{ccccc} \boxed{1} & \mathbf{4} & \mathbf{5} & \mathbf{-9} & \mathbf{-7} \\ \mathbf{0} & \boxed{2} & \mathbf{4} & \mathbf{-6} & \mathbf{-6} \\ \mathbf{0} & \mathbf{5} & \mathbf{10} & \mathbf{-15} & \mathbf{-13} \\ \mathbf{0} & \mathbf{-3} & \mathbf{-6} & \mathbf{4} & \mathbf{9} \end{array}$$

$\uparrow$

Next pivot column

- Next pivot column: Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

  Result: $\longrightarrow$

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- Can't create a leading entry in column 3 $\longrightarrow$ Move to col. 4. Swap rows 3 & 4. Done.
- Pivot columns: 1, 2, and 4.

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Recall: Algorithm to get standard row echelon form is a form of Gaussian elimination with pivoting
- Algorithm to get reduced row echelon form is a form of Gauss-Jordan elimination with pivoting
- Next: same example done with reduced form

Same example with reduced echelon Form

- Initial matrix below. First step same: swap rows 1 & 4

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- Next: Create zeros below the pivot, 1, by adding multiples of the first row to the rows below it.

$\longrightarrow$ Next matrix:

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- Scale 2nd row:

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

- Next step: create zeros in column 2 [except position (2,2)]

$$\begin{bmatrix} \boxed{1} & 0 & -3 & 3 & 5 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

- Move l to column 4; Swap rows 4 , 5;

$$\begin{bmatrix} \boxed{1} & 0 & -3 & 3 & 5 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Scale row 3

$$\begin{bmatrix} \boxed{1} & 0 & -3 & 3 & 5 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Finally: create zeros in column 4 except position (3,4).

$$\begin{bmatrix} \boxed{1} & 0 & -3 & 0 & 5 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

➤ Pivot positions: 1, 2, 4. They are (always) identical with those obtained from Standard Row Echelon form.

APPLICATIONS OF THE ECHELON FORM [1.3]

Solving a general linear system

Question: What are *all* the solutions of a linear system $[A, b]$

➤ Recall that we have 3 scenarios: 1) 0 solution; 2) infinitely many sols.; 3) exactly one solution.

➤ Set is called "general solution" or "complete solution"

➤ Answer provided by echelon form [reduced or standard]

Step 1 Form the augmented system $[A, b]$

$$\left[\begin{array}{cccccccccc|c} * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \\ * & * & * & * & * & * & * & * & * & * & * \end{array}\right]$$

Step 2 Obtain the reduced echelon form.

➤ Result is something like:

$$\left[\begin{array}{cccccccccc|c} \mathbf{1} & * & \mathbf{0} & * & * & * & \mathbf{0} & * & * & \mathbf{0} & * \\ & & \mathbf{1} & * & * & * & \mathbf{0} & * & * & \mathbf{0} & * \\ & & & & & & \mathbf{1} & * & * & \mathbf{0} & * \\ & & & & & & & & & \mathbf{1} & * \\ & & & & & & & & & & \end{array}\right]$$

Important: Solutions to this system same as those of $[\boldsymbol{A}, \boldsymbol{b}]$. So w'll find the solutions from this reduced system

✍ What can you say if the last column (RHS) happens to be a pivot column?

➤ Unknowns associated with pivots are called basic

➤ Others are called free

➤ In above example: 1, 3, 7, 10 are basic, 2, 4, 5, 6, 8, 9, are free, and column 11 is the RHS (not a variable).

Step 3 Write solutions: solutions depend on parameters which are the free variables.

- Express basic variables in terms of the free variables
- For any values given to the free variables you will get a solution
- For example for the above picture: $x_{10} = b_4$;

$x_7 = b_3 -$ scalar.$x_8 -$ scalar.x_9 etc..

✍ Find general solution when augmented matrix is:

$$\left[\begin{array}{ccccccc|c} 1 & 2 & 0 & 0 & 1 & 0 & 0 & -2 \\ 1 & 2 & -1 & -2 & 2 & 2 & 3 & -3 \\ 1 & 2 & 2 & 4 & -3 & -4 & -4 & -6 \\ 0 & 0 & 1 & 2 & -3 & 0 & -5 & -3 \\ 1 & 2 & 0 & 0 & 2 & 0 & -1 & 1 \end{array}\right]$$

- Get the reduced echelon form [use matlab!]

$$\left[\begin{array}{ccccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & \boxed{1} & 2 & 0 & 0 & -8 & 6 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- Basic variables: 1, 3, 5, 6
- Free variables: 2, 4, 7
- Right-hand side : 8

(1): $x_1 + 2x_2 + x_7 = -5$
(2): $x_3 + 2x_4 - 8x_7 = 6$
(3): $x_5 - x_7 = 3$
(4): $x_6 - 2x_7 = 1$
(5): $0 = 0$ (vacuous)

$\rightarrow$

$$\boxed{\begin{array}{l} x_1 = -5 - 2x_2 - x_7 \\ x_3 = 6 - 2x_4 + 8x_7 \\ x_5 = 3 + x_7 \\ x_6 = 1 + 2x_7 \end{array}}$$

Note: It is also possible to use the standard (non-reduced) row-echelon form - Requires back substitution. Result is the same.

✍ What if last component of RHS is zero (i.e., $A(5,8) = 0$)

✍ Below is the standard echelon form for the previous example. Find all solutions.

$$\left[\begin{array}{ccccccc|c} 1 & 2 & 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 2 & 4 & -4 & -4 & -4 & -4 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

✍ Find all solutions for which x_4 and x_7 are zero.

✍ Among these find all solutions for which x_1 is zero.

✍ We seek 5 numbers $x_1, \cdots, x_5$ such that their sum is 50, the sum of 3 of them (e.g. the odd-labeled ones) is 25, and the difference between the other 2 is 5. Write the equations to be satisfied and find the general solution.