DETERMINANTS CHAP. 3

Determinants: summary of main results

➤ A determinant of an $n \times n$ matrix is a real number associated with this matrix. Its definition is complex for the general case $\rightarrow$ We start with $n = 2$ then list important properties for this case.

- Determinant of a 2×2 matrix is:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- Notation : $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

➤ Next we list the main properties of determinants. Properties also true for $n \times n$ case

➤ Can be defined from GE. Det. $=$ product of pivots in GE when permutation is not used. Adjust signs when permuting. More later

➤ Properties written for columns (easier to write) but are also true for rows

Notation: We let $A = [u, v]$ columns u, and v are in $\mathbb{R}^2$.

1 If $v = \alpha u$ then $\det(A) = 0$.

➤ Determinant of linearly dependent vectors is zero

➤ If any one column is zero then determinant is zero

2 Interchanging columns or rows:

$$\det[v, u] = -\det[u, v]$$

3 Linearity:

$$\det[u, \alpha v + \beta w] = \alpha \det[u, v] + \beta \det[u, w]$$

- $\det(A)$ = linear function of each column (individually)
- $\det(A)$ = linear function of each row (individually)

✍ What is the determinant $\det[u, v + \alpha u]$?

4 Determinant of transpose

$$\boxed{\det(A) = \det(A^T)}$$

5 Determinant of Identity

$$\boxed{\det(I) = 1}$$

6 Determinant of a diagonal:

$$\boxed{\det(D) = d_1 d_2 \cdots d_n}$$

7 Determinant of a triangular matrix (upper or lower)

$$\det(T) = a_{11}a_{22}\cdots a_{nn}$$

8 Determinant of product of matrices [IMPORTANT]

$$\det(AB) = \det(A)\det(B)$$

9 Consequence: Determinant of inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

✍ What is the determinant of αA?

✍ What can you say about the determinant of a matrix which satisfies $A^2 = I$?

✍ Is it true that $\det(A+B) = \det(A) + \det(B)$?

Determinants – general definition

Consider now the general situation of $n \times n$ matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

General idea: det of A is the sum of all possible products of one entry per row of A. Each product has a sign.

- We need to use permutations to define determinants
- $p = \{i_1, i_2, \cdots, i_n\}$ is a permutation of $\{1, 2, \cdots, n\}$ if each if the numbers $1, 2, ..., n$ is represented once and only once in the list p

Example: $\{2, 3, 1, 5, 4\}$ is a permutation of $\{1, 2, \cdots, 5\}$.

Any permutation is the result of a sequence of interchanges (or swaps) applied to $\{1, 2, \cdots, n\}$ in which two keys are exchanged each time.

➤ For the above example: start with $\{1, 2, \cdots, 5\}$, then swap keys in positions 4 and 5 (result $\{1\ 2\ 3\ 5\ 4\ \}$) then those in positions 1 and 2 (result $\{2\ 1\ 3\ 5\ 4\ \}$) and finally keys in positions 2 and 3 to get desired the result $\{2\ 3\ 1\ 5\ 4\ \}$.

✍ Obtain $p = \{3, 1, 4, 2\}$ from $\{1, 2, 3, 4\}$

- The *signature of a permutation* is $(-1)^{n_p}$ where n_p is the number of swaps needed to rearrange $\{1, 2, \cdots, n\}$ into p.

➤ In the above example the signature is -1 since 3 swaps were needed.

- Consider now the case $n = 3$.
- We will need to use all permutations of $\{1, 2, 3\}$.
- Denote by σ the permutations
- Here are all permutations of $\{1, 2, 3\}$ with their signatures

σ	Sign.
{ 1 2 3 }	$+1$
{ 1 3 2 }	-1
{ 2 3 1 }	$+1$
{ 2 1 3 }	-1
{ 3 1 2 }	$+1$
{ 3 2 1 }	-1

- We will denote by $sig(\sigma)$ the signature of σ

General definition of determinants ('Big formula definition')

$$\det\ (A) = \sum_{\sigma} sig(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

Where the sum runs over all $(n!)$ possible permutations σ of $\{1,\ 2\ ,\cdots\ ,n\}$.

Case $n = 3$

σ	Sign.	Det = Sum of:
{ 1 2 3 }	$+1$	$+a_{11}a_{22}a_{33}$
{ 1 3 2 }	-1	$-a_{11}a_{23}a_{32}$
{ 2 3 1 }	$+1$	$+a_{12}a_{23}a_{31}$
{ 2 1 3 }	-1	$-a_{12}a_{21}a_{33}$
{ 3 1 2 }	$+1$	$+a_{13}a_{21}a_{32}$
{ 3 2 1 }	-1	$-a_{13}a_{22}a_{31}$

Computing determinants from defining formula

✍ Compute the following determinant using the 'Big formula definition'

$$\begin{vmatrix} -1 & 2 & 0 \\ 2 & -1 & 3 \\ -1 & 0 & 2 \end{vmatrix}$$

✍ Suppose columns 1 and 2 are swapped. Use the 'big formula definition' to show that the determinant changes signs.

✍ Let B be the matrix obtained from a matrix A by multiplying a certain row (or column) of A by a scalar α. Use the 'big formula definition' to show that: $det(B) = \alpha det(A).$

✍ What is the computational cost of evaluating the determinant using the 'big formula definition'? [Hint: It is big!]

Cofactors

- Let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained from A by deleting its i-th row and its j-th column.

- Define C the matrix of cofactors, having entries:

$$c_{ij} = (-1)^{i+j} \det\ A_{ij}$$

- We can expand $\det(A)$ with respect to i-th row as follows:

$$\det(A) = \sum_{j=1}^{n} a_{ij} c_{ij}$$

- Note i is fixed. Can be done for any i [same result each time]

- Similar expressions for expanding column-wise

➤ This gives a second definition of determinants – a recursive one.

➤ We know how do define determinants for $n = 2$. For $n > 2$ define determinant by expanding with respect to the first row.

Recursive definition of determinants: For $n > 2$, the determinant of a matrix $A = [a_{ij}]$ is the sum of the n terms $a_{1j}c_{1j}$, i.e.,

$$\det(A) = a_{11}c_{11} + a_{12}c_{12} + \cdots + a_{1n}c_{1n}$$
$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

✍ Establish a recurrence relation that gives the cost of computing $\det(A)$ using co-factors. Show that the cost is $\approx 2(n!)$

✍ Prove the above result for $n = 3$ [Hint: list permutations in a certain order]

✍ Compute the following determinant by using co-factors. Expand with respect to 1st row.

$$\begin{vmatrix} -1 & 2 & 0 \\ 2 & -1 & 3 \\ -1 & 0 & 2 \end{vmatrix}$$

✍ Compute the above determinant by using co-factors. Expand with respect to last row. Then expand with respect to last column.

✍ Compute F_2, F_3, F_4 when F_n is the n-th dimensional determinant:

$$F_n = \begin{vmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & -1 \\ & & & 1 & -1 \end{vmatrix}$$

✍ (continuation) Challenge: Show a recurrence relation between F_n, F_{n-1} and F_{n-2}. Do you recognize this relation? Compute the first 8 values of F_n

Cramer's rule

Notation: For any $n \times n$ matrix A and any b in $\mathbb{R}^n$ let $A_i(b)$ be the matrix obtained from A by remplacing its i-th column by b:

$$A_i(b) = [a_1, a_2, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n]$$

Cramer's rule Let A be an invertible $n \times n$ matrix and b in $\mathbb{R}^n$. The unique solution of the system $Ax = b$ has entries given by:

$$\boxed{x_i = \frac{\det(A_i(b))}{\det(A)}}$$

- In addition the following formula for the inverse holds:

$$\boxed{A^{-1} = \frac{1}{\det(A)} C^T}$$

where C be the matrix of cofactors.

✍ Find the inverse of $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{4} \\ \mathbf{3} & \mathbf{1} & \mathbf{-2} \end{bmatrix}$

✍ Determine how $\boldsymbol{x}_1(\boldsymbol{\alpha})$ depends on $\boldsymbol{\alpha}$ when $\boldsymbol{x}_1(\boldsymbol{\alpha})$ is the first component of the solution of the system $\boldsymbol{Ax} = \boldsymbol{b}$, where

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} & \mathbf{4} \\ \mathbf{3} & \mathbf{1} & \mathbf{-2} \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} \mathbf{-2} \\ \boldsymbol{\alpha} \\ \mathbf{1} \end{bmatrix}$$

Areas and volumes

- Area of a parallelogram in $\mathbb{R}^2$ spanned by points $(0,0)$, (a,b), (c,d), $(a+c, b+d)$ is:

$$\left|\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right|$$

- Area of triangle in $\mathbb{R}^2$ spanned by the points (x_1, y_1), (x_2, y_2), (x_3, y_3) is:

$$\left|\frac{1}{2}\det \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}\right|$$

- Volume of a parallelogram in $\mathbb{R}^3$ spanned by points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$\left|\det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}\right|$$

How to compute determinants in practice?

➤ Co-factor expansion?? *Not practical*. Instead:

➤ Perform an LU factorization of A with pivoting.

➤ If a zero column is encountered LU fails but $det(A) = 0$

➤ If not get det $=$ product of diagonal entries multiplied by a sign ± 1 depending on how many times we interchanged rows.

✍ Compute the determinants of the matrices

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 5 & 9 \\ 1 & 0 & -12 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & -1 & 1 & 2 \\ 1 & -2 & -1 & 1 \\ 2 & 0 & 2 & 0 \\ -1 & 1 & -1 & -1 \end{bmatrix}$$