طراحی الگوریتم ها (CE221)

جلسه نهم:
کران پایین برای مرتب سازی

سجاد شیرعلی شهرضا

بهار 1401

شنبه، 14 اسفند 1400

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 8.1
- امتحانک دوم: دوشنبه هفته آینده، 23 اسفند 1400 در طی ساعت کلاس به صورت برخط (مشابه امتحانک اول)

کران پایین برای مرتب سازی

آیا می توان الگوریتم مرتب سازی بهتر از O(nlgn) هم طراحی کرد؟

O(n log n) ALGORITHMS WE'VE SEEN

- MergeSort
  - Worst-case $\boldsymbol{\Theta}$(n log n) time.
- QuickSort
  - Expected: $\boldsymbol{\Theta}$(n log n)

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THE QUESTION IS…

CAN WE DO BETTER?

INTRODUCING… SPAGHETTI SORT?

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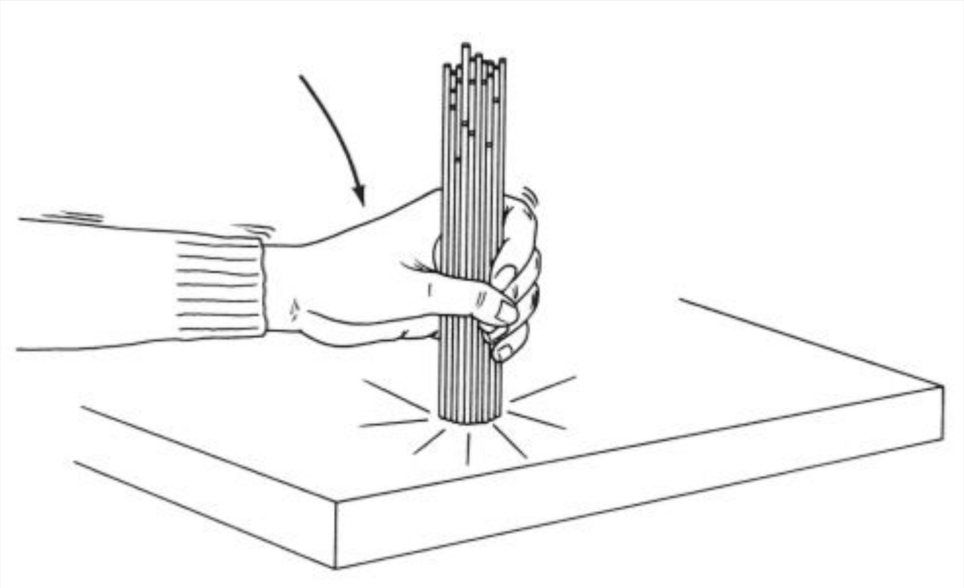
- Take all the spaghetti in your fist, and push their lower sides against the table **O(1)**



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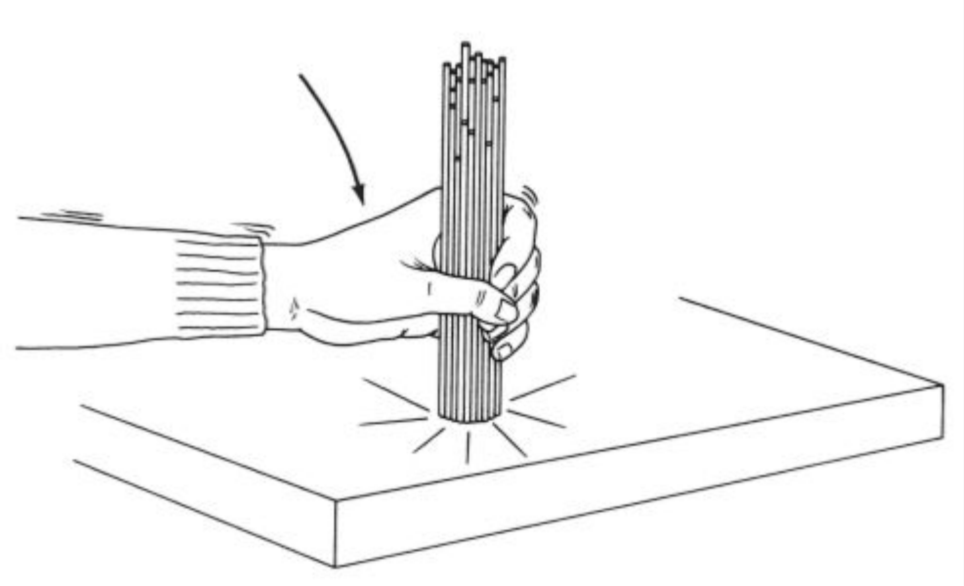
- Take all the spaghetti in your fist, and push their lower sides against the table **O(1)**
- Lower your other hand on the bundle of spaghetti - the first spaghetto you touch is the longest one. Remove it, transcribe its length, and repeat until all spaghetti have been removed. **O(n)**



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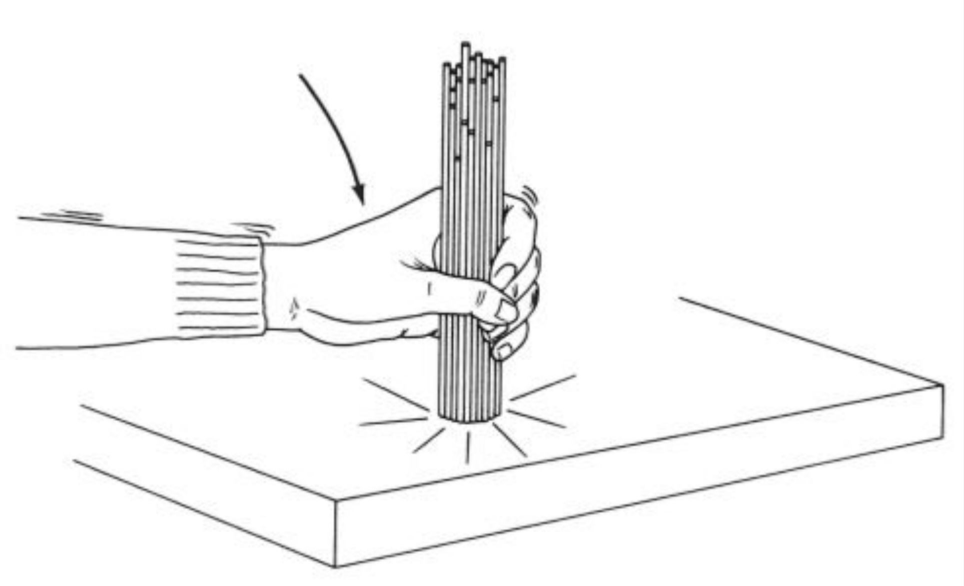

While you shouldn't take this algorithm too seriously… it does raise some important questions!

WHAT IS OUR MODEL OF COMPUTATION?

Input: array of elements

Output: sorted array

Operations allowed: comparisons

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Input: some real numbers

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Operations allowed: breaking spaghetti, dropping on tables, lowering hand

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In a CS class where we're more concerned with what computers can do, the first model seems more reasonable.

سوال؟

COMPARISON-BASED SORTING

- **You want to sort an array of items**
- **You can't access the items' values directly: you can only *compare* two items and find out which is bigger or smaller.**
- Examples: Insertion Sort, MergeSort, QuickSort

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"Comparison-based sorting algorithms" are general-purpose.

The algorithm makes no assumption about the input elements other than that they belong to some totally ordered set.

COMPARISON-BASED SORTING

In other words, the only way you can interact with the array:

For two indices i and j, is A[i] bigger than A[j]?

A[0] A[1] A[2] A[3]

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and you can only ask this genie this YES/NO question to figure out how to sort the items)

A[0] A[1] A[2] A[3]

Is A[1] bigger than A[3]?

Yes!

A Comparison-based
Sorting Algorithm

All-knowing
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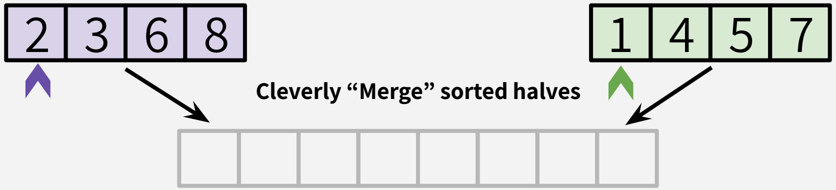

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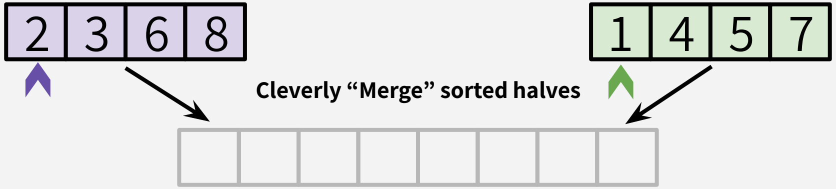

Is 2 bigger than 1 ?

MergeSort algorithm

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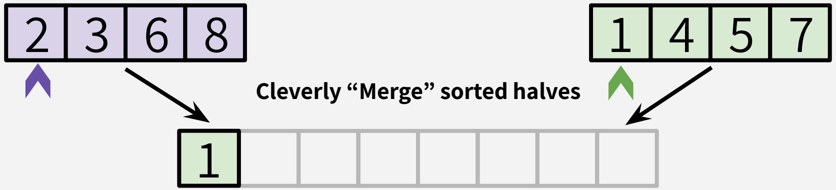

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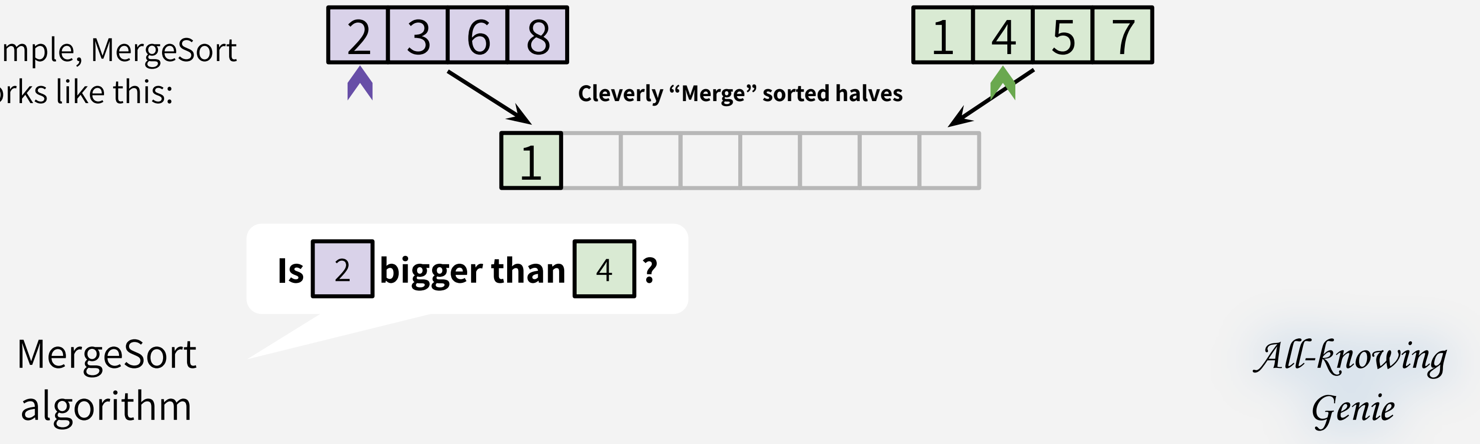

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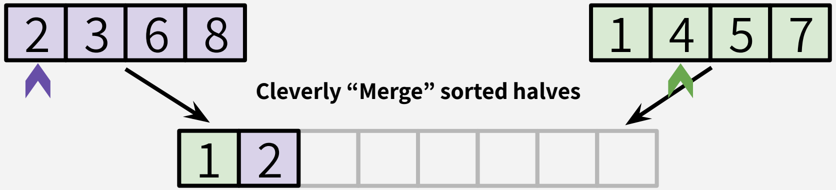

Is 2 **bigger than** 4 **?**

No!

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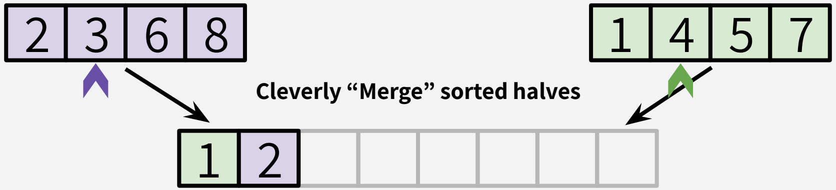

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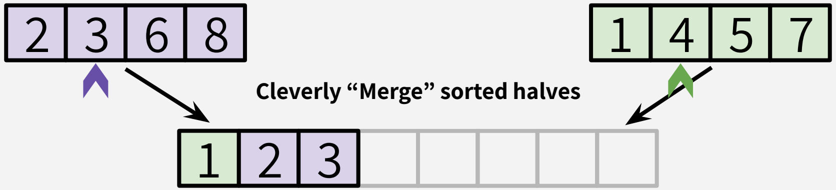

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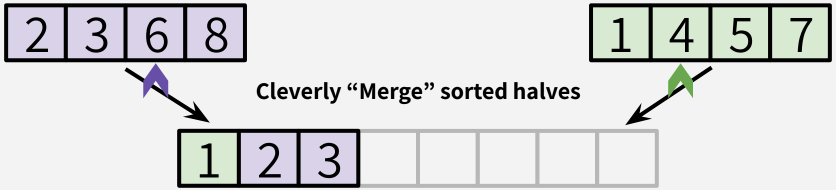

Is 6 bigger than 4 ?

MergeSort algorithm

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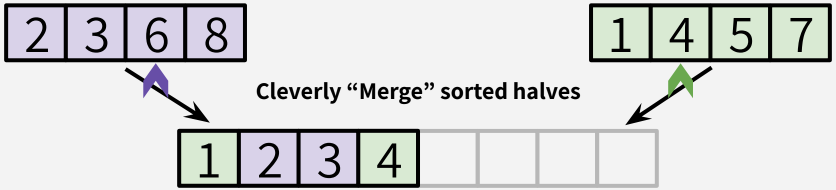

Is 6 bigger than 4 ?

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سوال؟

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Theorem:

Any deterministic comparison-based sorting algorithm must take $\Omega(n \log n)$ time.

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Think about it like this: this is the input format that your algorithm is ready to accept.

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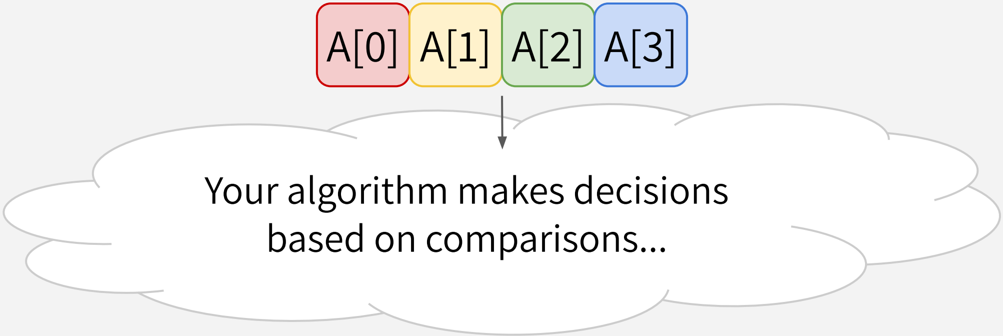

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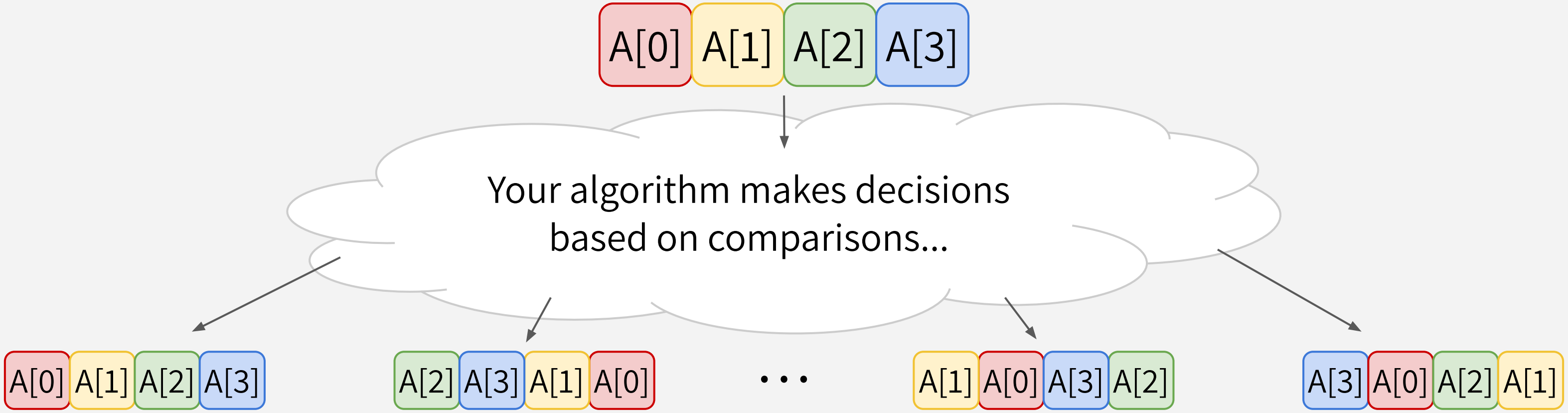

Your algorithm needs to be able to output any one of ______ possible orderings

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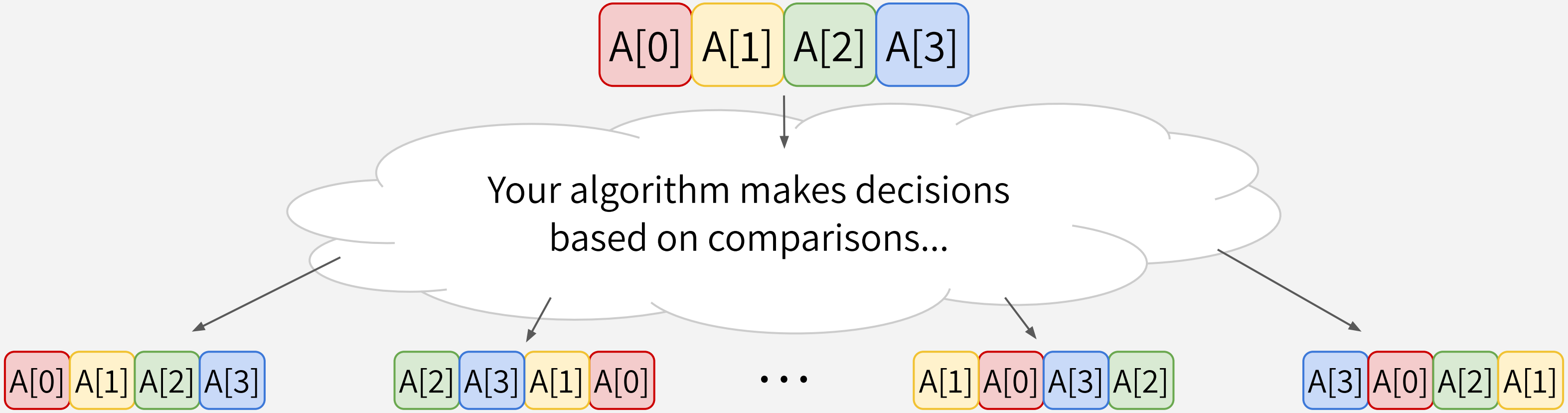

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COMPARISON-BASED SORTING

A[0] A[1] A[2] A[3]

The algorithm's execution "branches" only as a result of comparisons, since this is the only input-specific information that the algorithm receives.

A[0] A[1] A[2] A[3]   A[2] A[3] A[1] A[0]   . . .   A[1] A[0] A[3] A[2]   A[3] A[0] A[2] A[1]

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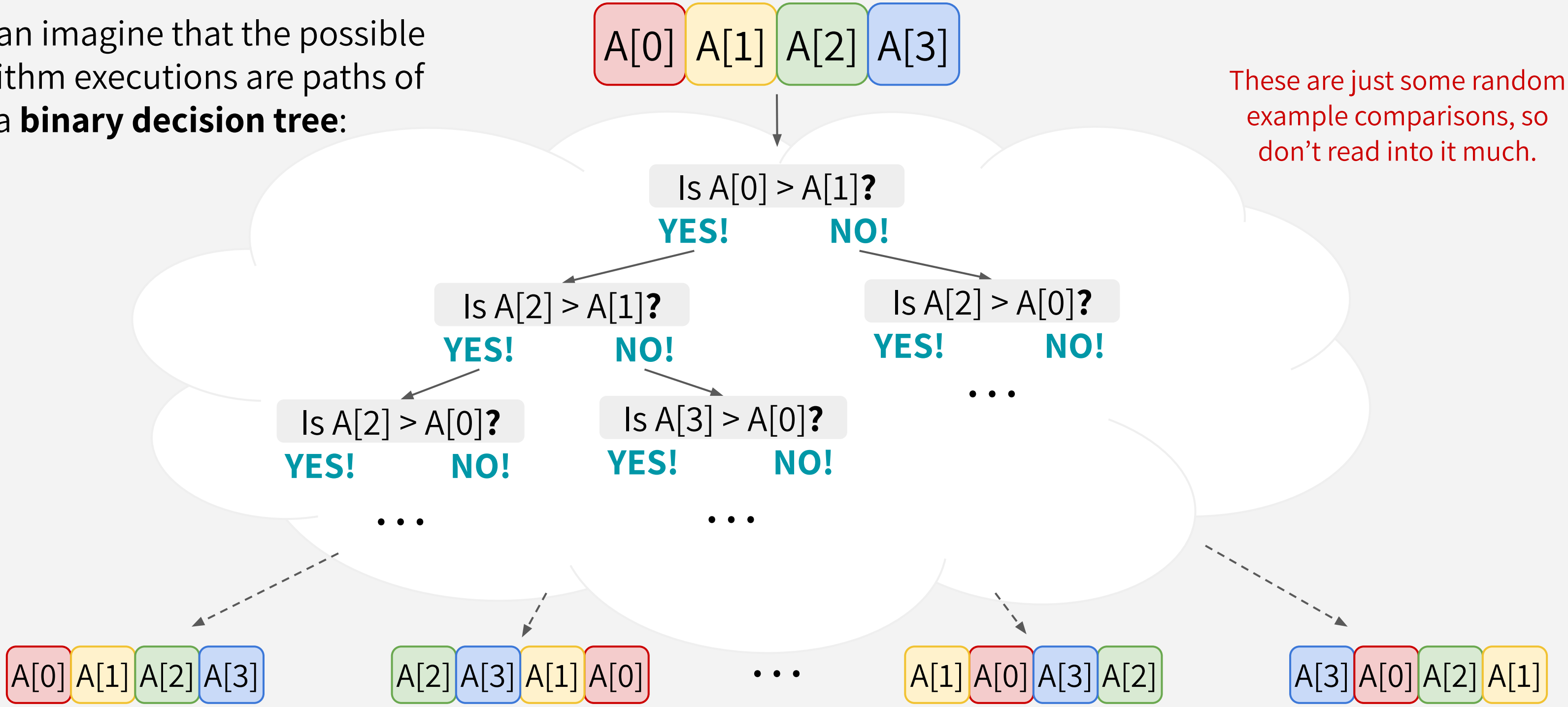
COMPARISON-BASED SORTING

You can imagine that the possible algorithm executions are paths of a **binary decision tree**:



These are just some random example comparisons, so don't read into it much.

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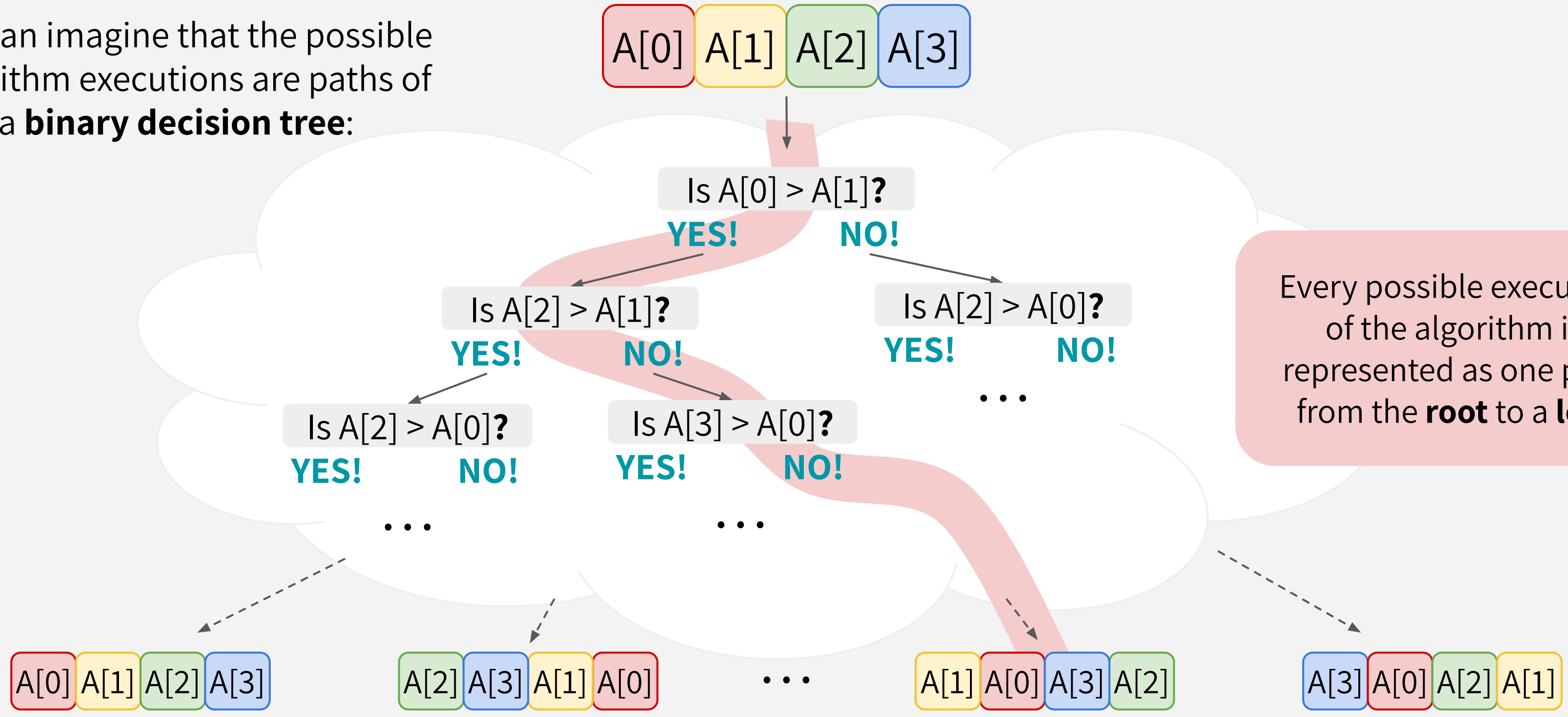

Every possible execution of the algorithm is represented as one path from the **root** to a **leaf**

Your algorithm needs to be able to output any one of **n!** possible orderings

COMPARISON-BASED SORTING

You can imagine that the possible algorithm executions are paths of a **binary decision tree:**

A[0] A[1] A[2] A[3]

This is a binary tree with at least **n!** leaves.

What is the length of the longest possible path?

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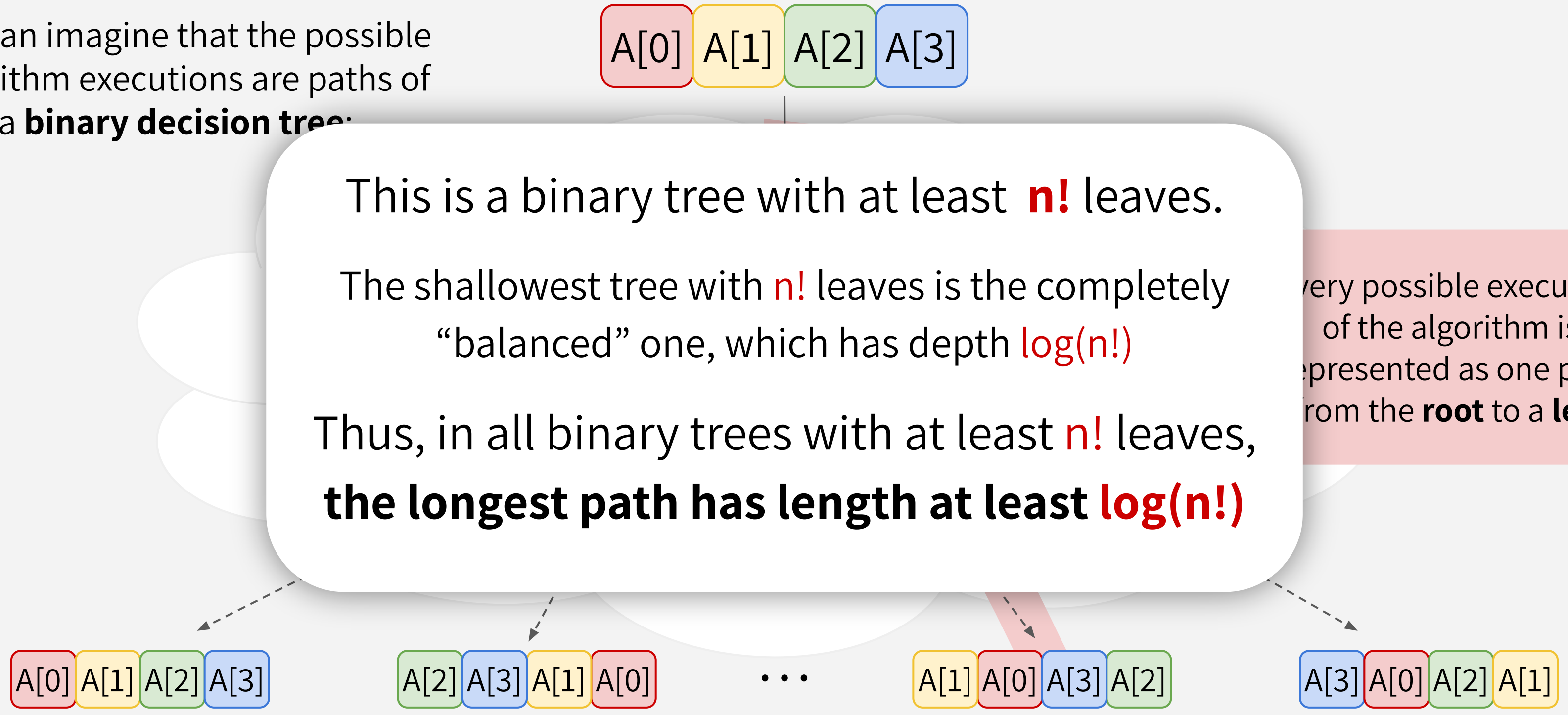

This is a binary tree with at least **n!** leaves.

The shallowest tree with n! leaves is the completely "balanced" one, which has depth log(n!)

Thus, in all binary trees with at least n! leaves, **the longest path has length at least log(n!)**

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COMPARISON-BASED SORTING

The longest path has length at least log(n!)

Consequently, any execution of a comparison-based sorting algorithm has to perform at least log(n!) steps.

The worst-case runtime is at least $\log(n!) = \Omega(n \log n)$.

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$$
\begin{aligned}
\log(n!) &= \log 1 + \log 2 + \cdots + \log(n-1) + \log n \\
&\geq \log\left(\frac{n}{2} + 1\right) + \log\left(\frac{n}{2} + 2\right) + \cdots + \log(n-1) + \log n \\
&\geq \left(\frac{n}{2}\right) \log\left(\frac{n}{2}\right) \\
&= \frac{1}{2} n \big(\log n - \log 2\big) \\
&= \Omega(n \log n)
\end{aligned}
$$

PROOF RECAP

Theorem:

Any deterministic comparison-based sorting algorithm must take $\Omega(n \log n)$ time.

- Any deterministic comparison-based algorithm can be represented as a decision tree with n! leaves
- The worst-case runtime is at least the length of the longest path in the decision tree
- All decision trees with n! leaves have a longest path with length at least $\log(n!) = \Omega(n \log n)$
- So, any comparison-based sorting algorithm must have worst-case runtime at least $\Omega(n \log n)$

THE GOOD NEWS

Theorem:

Any deterministic comparison-based sorting algorithm must take $\Omega(n \log n)$ time.

This bound also applies to the expected runtime of *randomized* comparison-based sorting algorithms!
The proof is out of scope of this class, but it relies on this theorem.

This means that MergeSort is optimal!

(This is one of the cool things about proving lower bounds - we know when we can declare victory!)

THE GOOD NEWS

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This bound also applies to -based sorting algorithms!
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CAN WE DO BETTER?

**using a model of computation that's less silly than spaghetti?*

سوال؟