طراحی الگوریتم ها (CE221)

جلسه چهاردهم: کوتاه ترین مسیر در گراف

سجاد شیرعلی شهرضا
بهار 1401
شنبه، 27 فروردین 1401

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

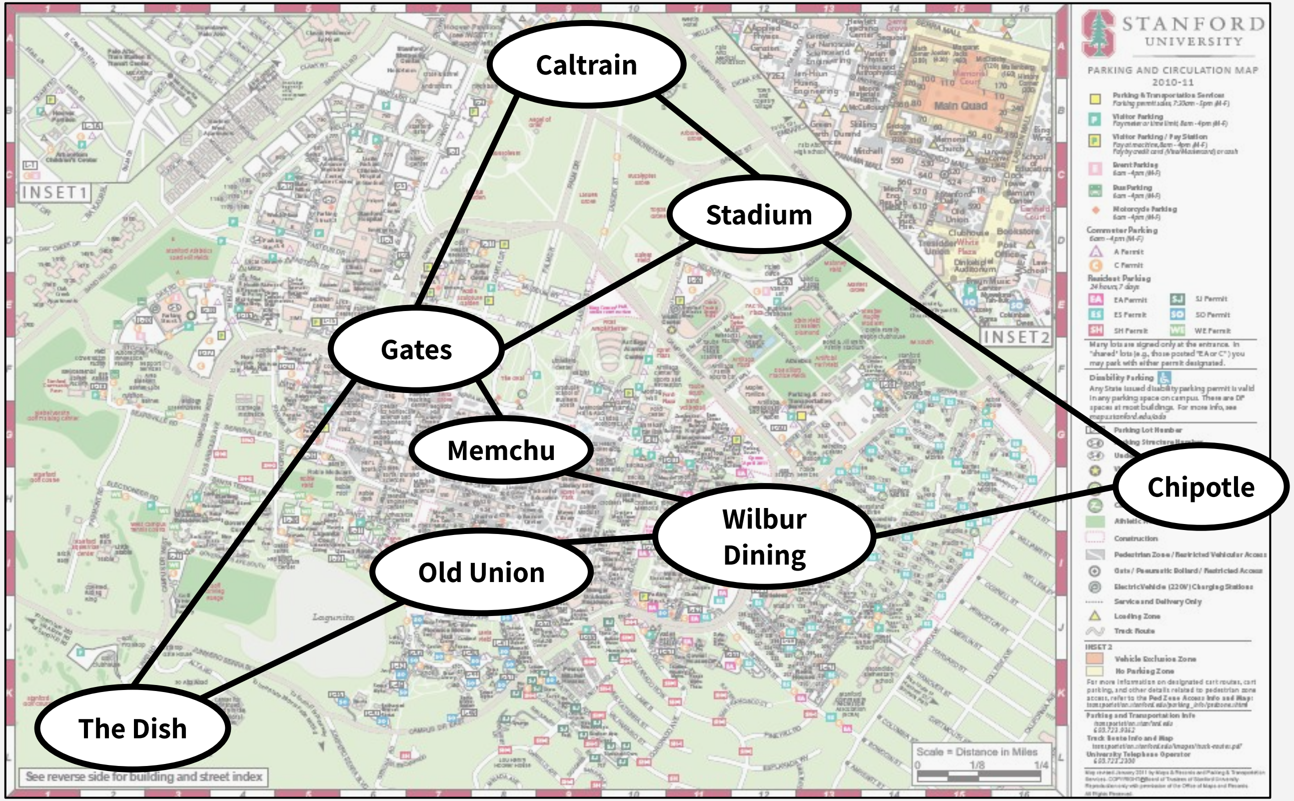
- بخش مرتبط کتاب برای این جلسه: 15.3
- ادغام امتحانک سوم و میان ترم با یکدیگر
- تاریخ امتحان: 24 اردیبهشت (زمان سابق امتحانک سوم)
- 5 نمره (3 نمره میان ترم + 2 نمره امتحانک سوم)
- حضوری در ساعت کلاس
- تمدید مهلت تمرین دوم تا ساعت 8 صبح روز دوشنبه 5 اردیبهشت

یافتن کوتاه ترین مسیر در گراف

تعریف مسئله

SHORTEST PATHS IN WEIGHTED GRAPHS

Suppose you only know your way around campus via certain landmarks

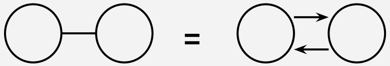


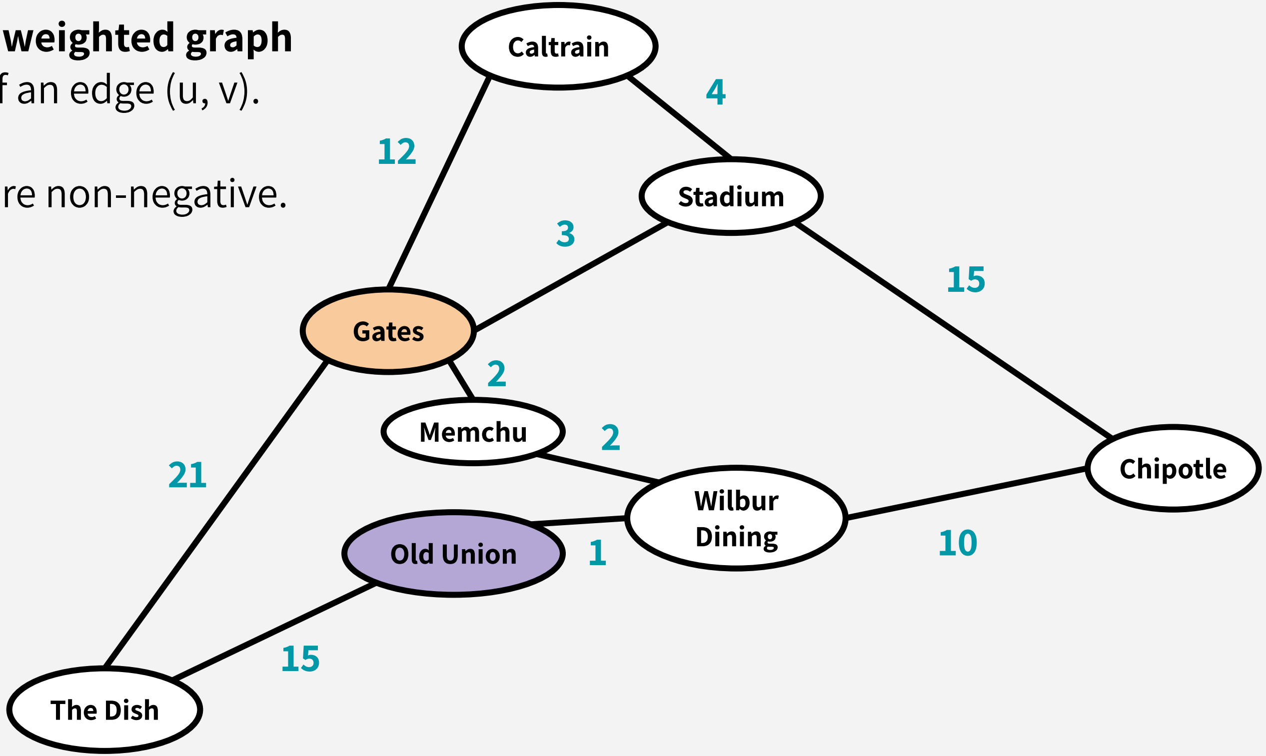
SHORTEST PATHS IN WEIGHTED GRAPHS

We can represent this as a **weighted graph** where **w(u,v)** = weight of an edge (u, v).

For today, these weights are non-negative.

Note: All graphs are directed, but to save the trouble of drawing double arrows everywhere:





SHORTEST PATHS IN WEIGHTED GRAPHS

We can represent this as a **weighted graph** where **w(u,v)** = weight of an edge (u, v).

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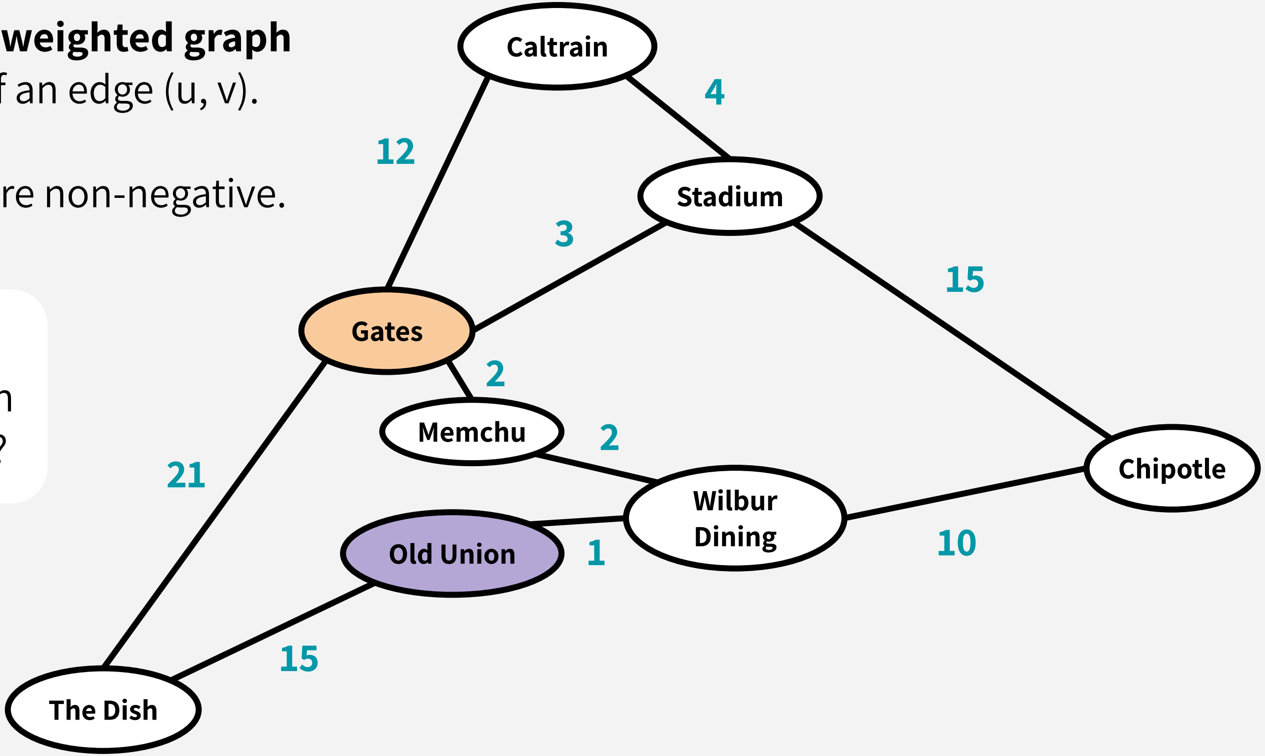
What if we wanted to compute the shortest path from **Gates** to **Old Union**?



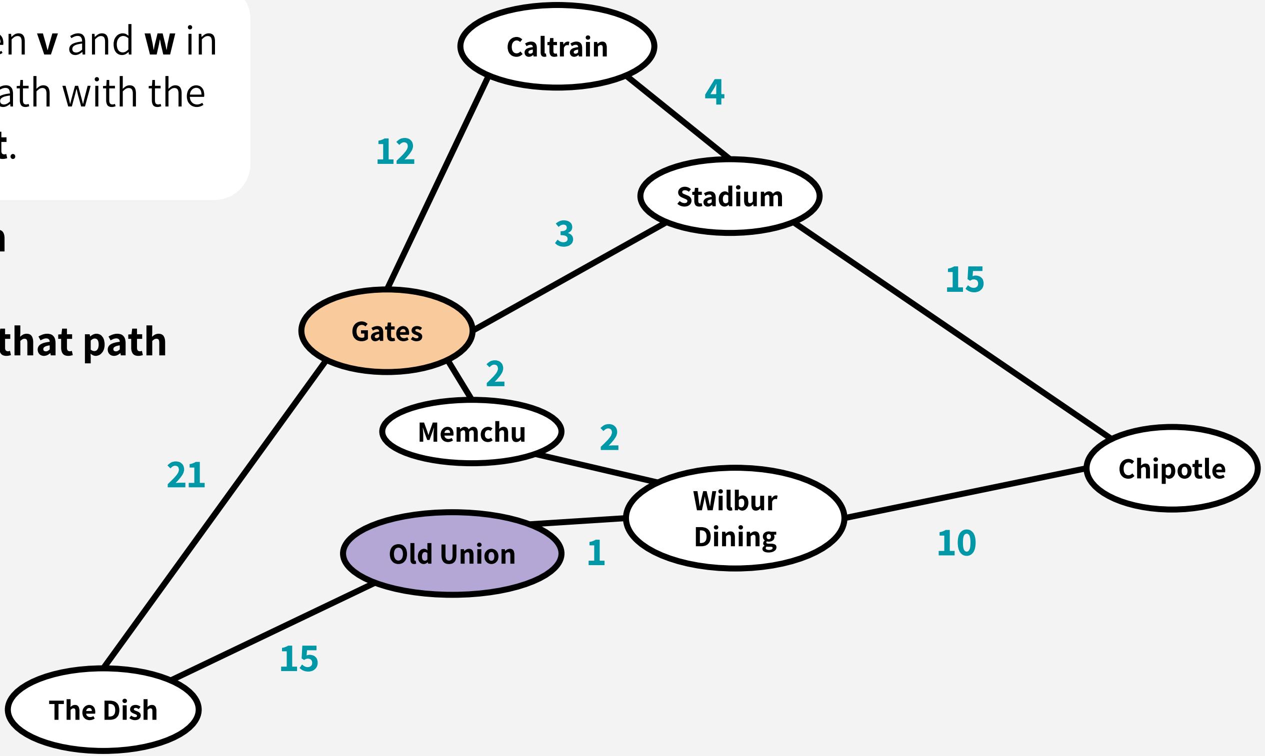
SHORTEST PATHS IN WEIGHTED GRAPHS

The **shortest path** between **v** and **w** in a weighted graph is the path with the **minimum cost**.

Cost of a path

=

sum of weights along that path



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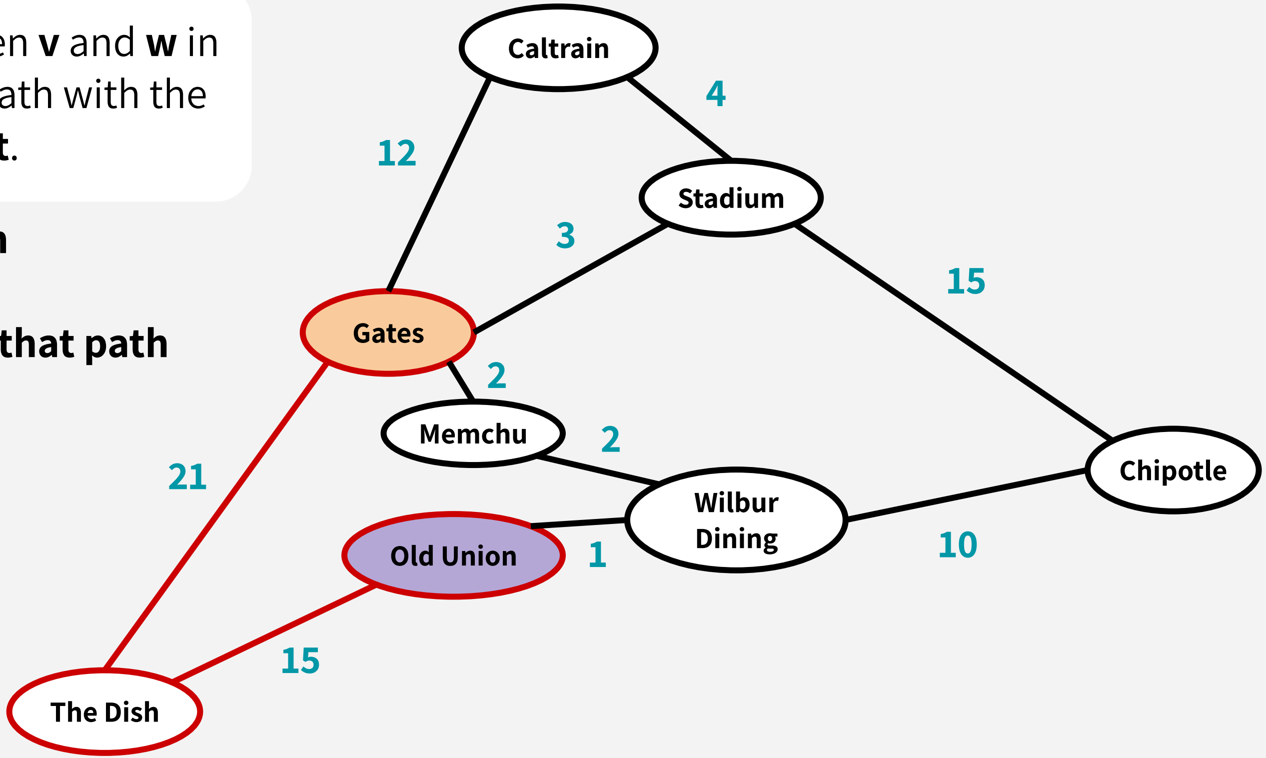

This path from Gates to Old Union has cost 36.

SHORTEST PATHS IN WEIGHTED GRAPHS

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Cost of a path

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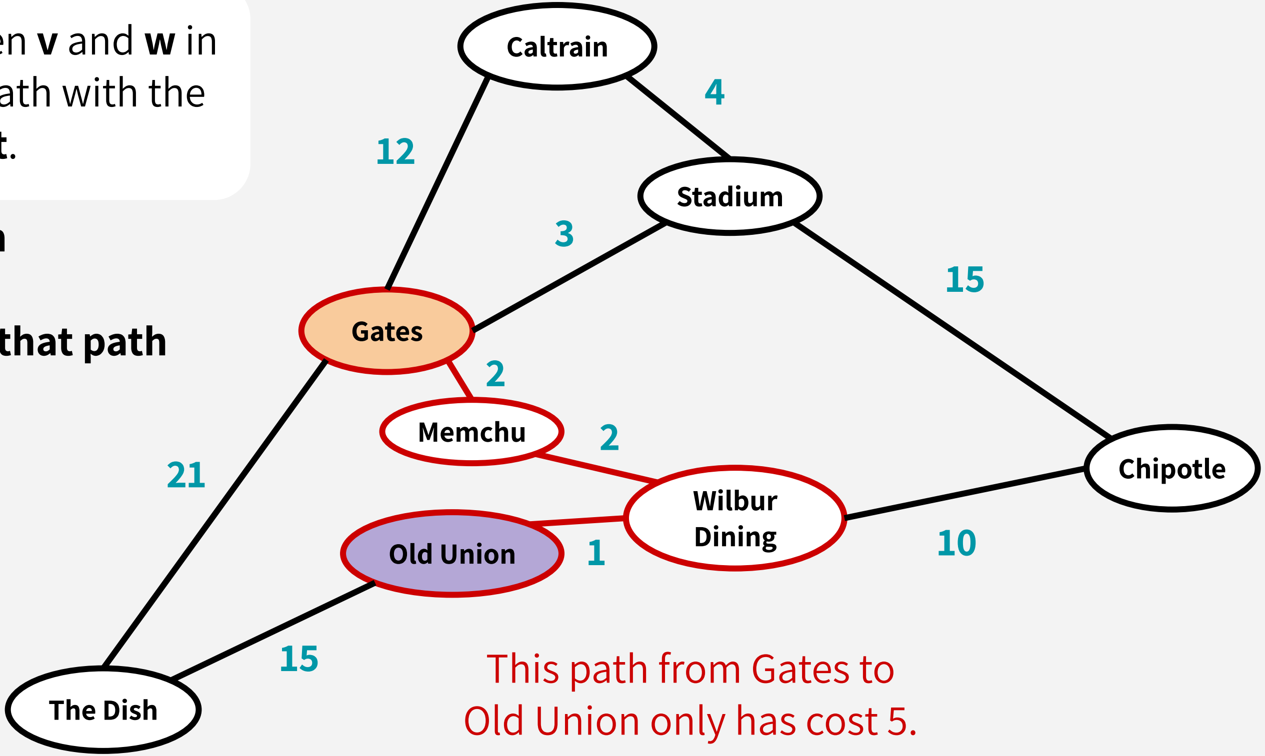
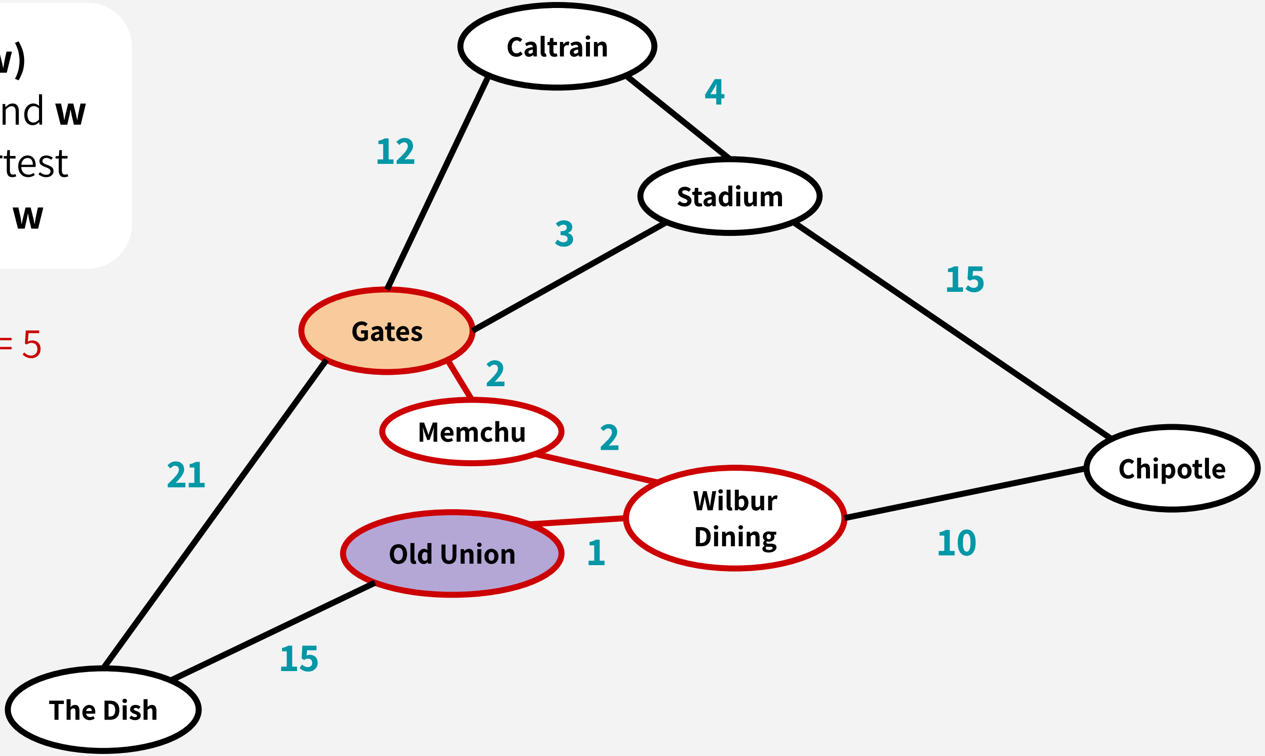


This path from Gates to Old Union only has cost 5.

SHORTEST PATHS IN WEIGHTED GRAPHS

The **distance d(v,w)** between 2 vertices **v** and **w** is the cost of the shortest path between **v** and **w**

d(Gates, Old Union) = 5



SINGLE-SOURCE SHORTEST PATH

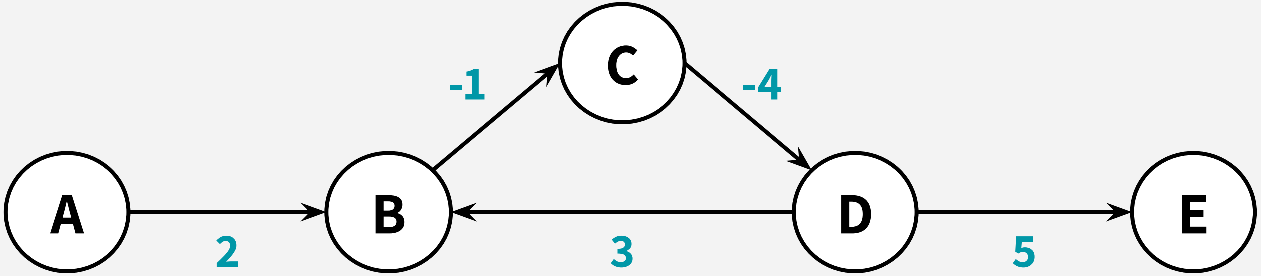
Applications:

Finding the shortest/most efficient path from point A to point B via bike, walking, Uber, Lyft, train, etc.
(Edge weights could be time, money, hassle, effort)

Finding the shortest path to send packets from my computer to some desired server using the Internet
(Edge weights could be link length, traffic, etc.)

NEGATIVE CYCLES

If negative cycles exist in the graph, we'll say *no solution exists*. Why?



What's the shortest path from A to E?

Is it: A → B → C → D → E? Cost = 2 - 1 - 4 + 5 = **2**.

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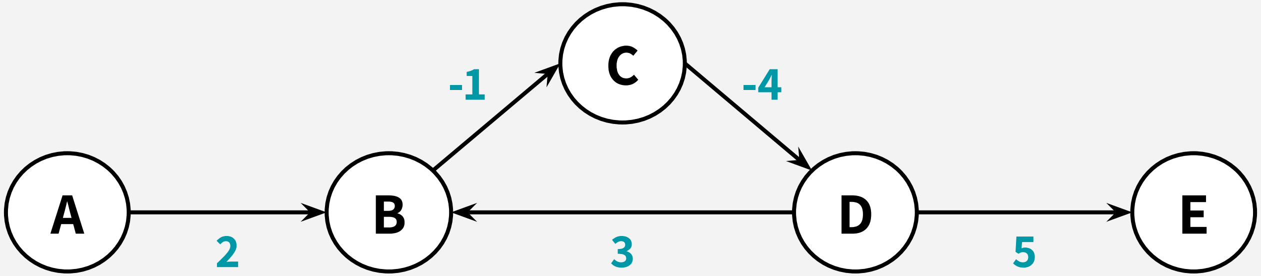


What's the shortest path from A to E?

Is it: A → B → C → D → E? Cost = 2 - 1 - 4 + 5 = **2**.

Or is it: A → B → C → D → B → C → D → B → C → D → B → C → D → B → C → D → E ?

Basically, shortest paths aren't defined if there are negative cycles!

سوال؟

الگوریتم بلمن-فورد

پیدا کردن کوتاه ترین مسیر از یک راس به تمام رئوس

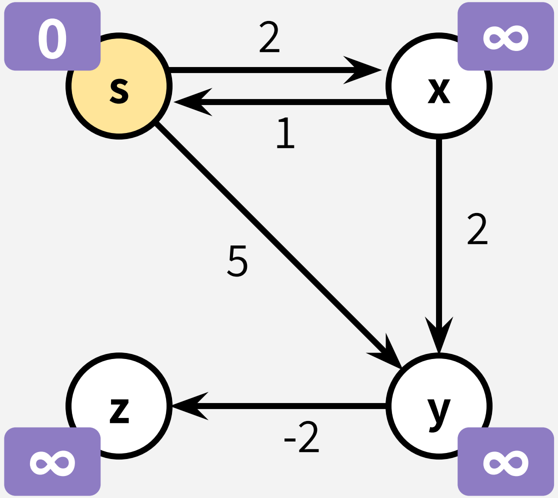
BELLMAN-FORD

We maintain a list $\mathbf{d}^{(k)}$ of length n, for each k = 0, 1, …, n–1.

$\mathbf{d}^{(k)}[\mathbf{b}]$ = the cost of the shortest path from s to b *with at most k edges.*

	s	x	y	z
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$				
$\mathbf{d}^{(2)}$				
$\mathbf{d}^{(3)}$				

We know how to fill in $\mathbf{d}^{(0)}$ -- the costs of shortest paths to each vertex with at most **k = 0** edges in it



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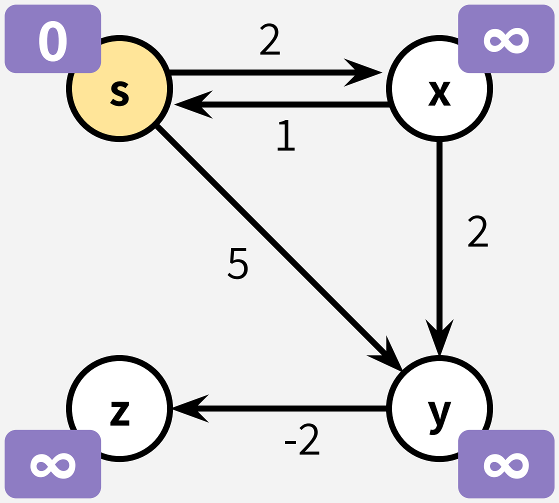
We will use table $\mathbf{d}^{(0)}$ to fill in $\mathbf{d}^{(1)}$. More generally, we will use table $\mathbf{d}^{(k-1)}$ to fill in $\mathbf{d}^{(k)}$.



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Case 1: the shortest path from s to b with at most k edges could be one with at most k–1 edges! In other words, allowing k edges is not going to change anything. Then:

$$\mathbf{d}^{(k)}[\mathbf{b}] = \mathbf{d}^{(k-1)}[\mathbf{b}]$$

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Case 2: the shortest path from s to b with at most k edges could be one with exactly k edges!
I.e. this length-k shortest path is [length k–1 shortest path to some incoming neighbor a] + w(a,b).
Which of b's incoming neighbors will offer this shortest path? Let's check them all:

$$\mathbf{d}^{(k)}[\mathbf{b}] = \mathbf{min}_{\text{a in b's incoming neighbors}} \{ \mathbf{d}^{(k-1)}[\mathbf{a}] + \mathbf{w(a,b)} \}$$

BELLMAN-FORD PSEUDOCODE

```
BELLMAN_FORD(G,s):
    $d^{(k)}$ = [] for k = 0, ..., n-1
    $d^{(0)}$[v] = ∞ for all v in V (except s)
    $d^{(0)}$[s] = 0
    for k = 1, ..., n-1:
        for b in V:
            $d^{(k)}[b] \leftarrow \min\{ d^{(k-1)}[b], \min_a\{d^{(k-1)}[a] + w(a,b)\} \}$
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Keeping all n–1 rows is a simplification to make the pseudocode straightforward. In practice, we'd only keep 2 of them at a time!

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CASE 2

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Take the minimum over all incoming neighbors a (i.e. all a s.t. (a, b) ∈ E)

This takes O(deg(b))!!!

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CASE 1

CASE 2

Runtime: O(m·n)

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We store a list $\mathbf{d}^{(k)}$ of length n, for each k = 0, 1, …, n–1.

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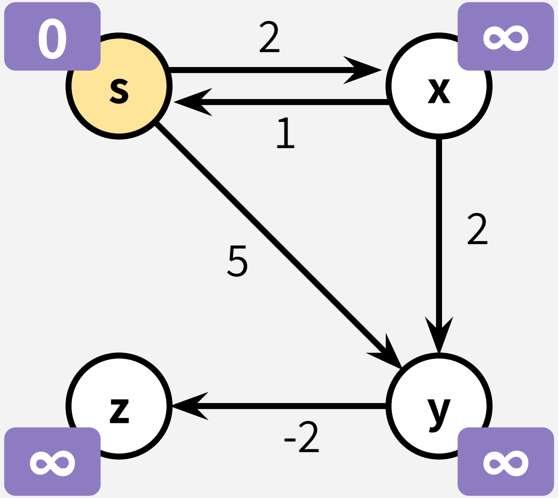

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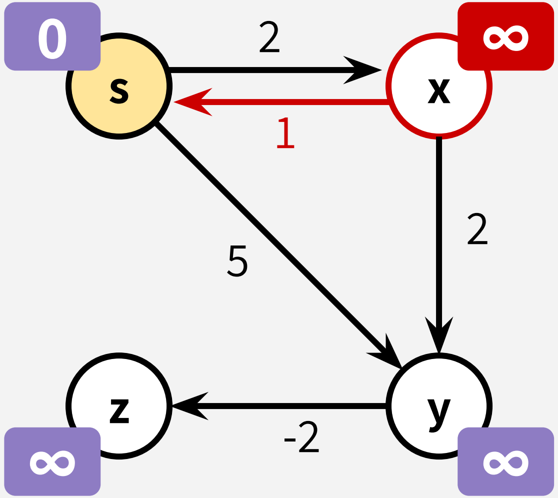
```

	s	x	y	z
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$	0			
$\mathbf{d}^{(2)}$				
$\mathbf{d}^{(3)}$				

This is min of:

$d^{(0)}[s]$

$d^{(0)}[\mathbf{x}] + w(\mathbf{x}, s)$



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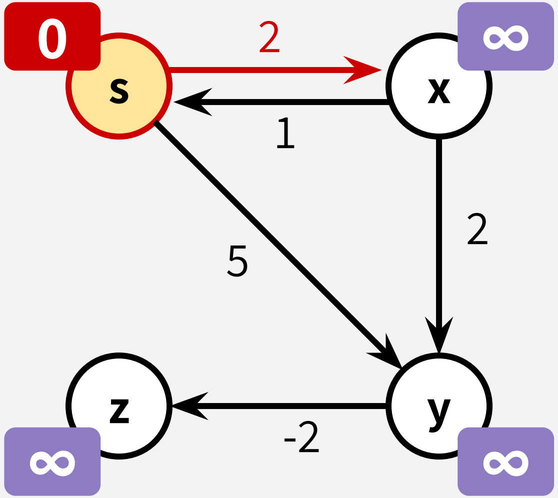
$\mathbf{d}^{(2)}$				
$\mathbf{d}^{(3)}$				

This is min of:

$d^{(0)}[x]$

$d^{(0)}[\mathbf{s}] + w(\mathbf{s}, x)$



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```

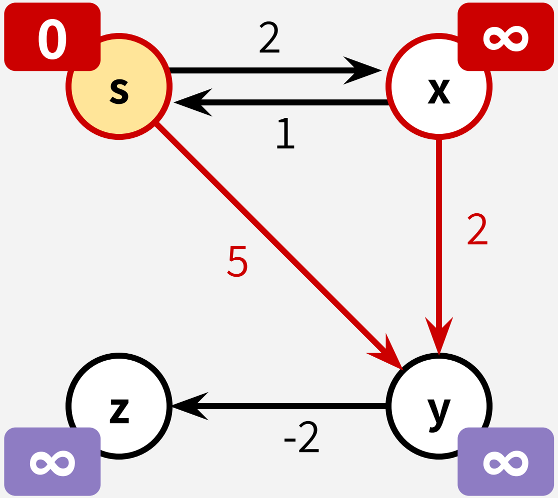
	s	x	y	z
$\mathbf{d^{(0)}}$	0	∞	∞	∞
$\mathbf{d^{(1)}}$	0	2	5	
$\mathbf{d^{(2)}}$				
$\mathbf{d^{(3)}}$				

This is min of:

$d^{(0)}[y]$

$d^{(0)}[\mathbf{s}] + w(\mathbf{s}, y)$

$d^{(0)}[\mathbf{x}] + w(\mathbf{x}, y)$



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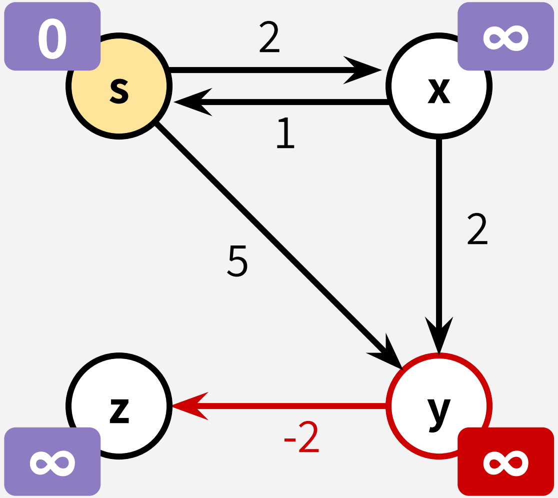
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$				
$\mathbf{d}^{(3)}$				

This is min of:

$d^{(0)}[z]$

$d^{(0)}[\mathbf{y}] + w(\mathbf{y}, z)$



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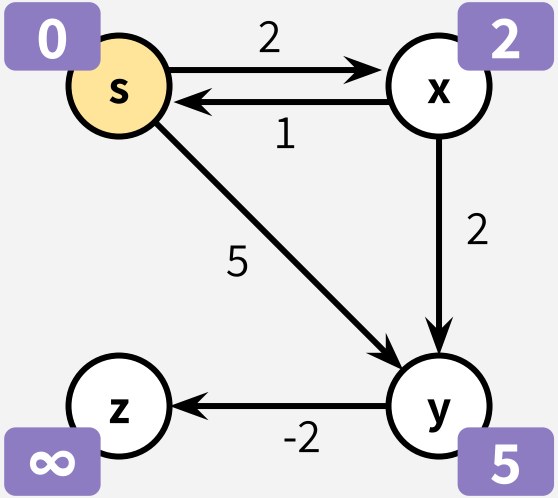
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$				
$\mathbf{d}^{(3)}$				

Now, fill in $\mathbf{d}^{(2)}$!!!



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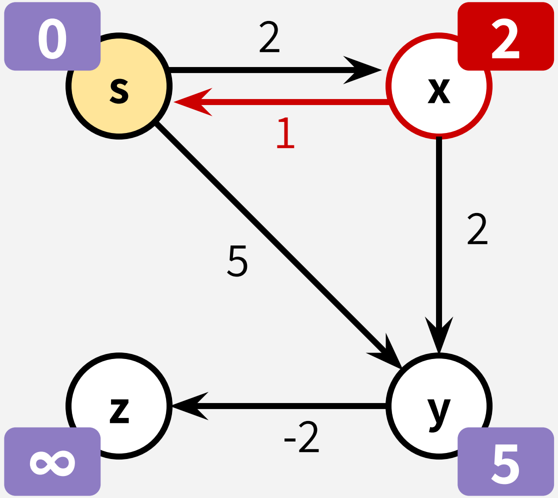

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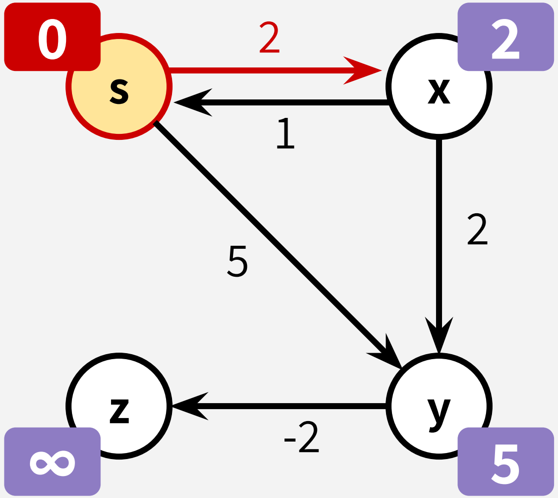

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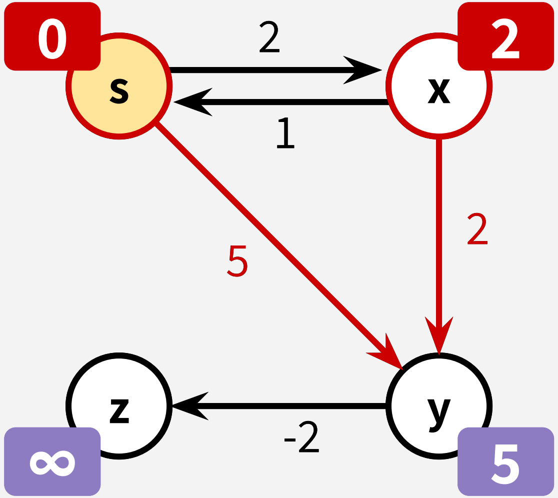
	s	x	y	z
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$	0	2	4	
$\mathbf{d}^{(3)}$				

This is min of:

$d^{(1)}[y]$

$d^{(1)}[\mathbf{s}] + w(\mathbf{s}, y)$

$d^{(1)}[\mathbf{x}] + w(\mathbf{x}, y)$



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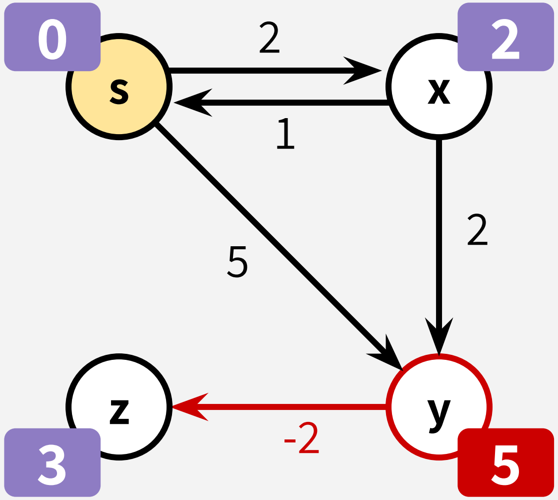
$d^{(0)}$	**0**	**∞**	**∞**	**∞**
$d^{(1)}$	**0**	**2**	**5**	**∞**
$d^{(2)}$	**0**	**2**	**4**	**3**
$d^{(3)}$				

This is min of:

$d^{(1)}[z]$

$d^{(1)}[\mathbf{y}] + w(\mathbf{y}, z)$



BELLMAN-FORD

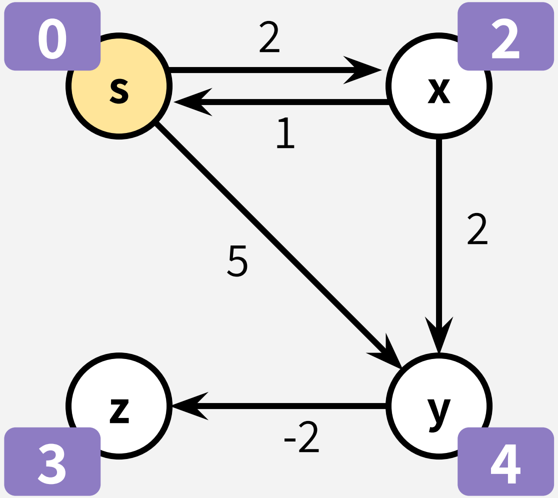
We store a list $\mathbf{d}^{(k)}$ of length n, for each k = 0, 1, …, n–1.

$\mathbf{d}^{(k)}[\mathbf{b}]$ = cost of shortest path from s to b *w/ at most k edges.*

```
for k = 1, ..., n-1:
    for b in V:
        $d^{(k)}[b] \leftarrow \min\{d^{(k-1)}[b], \min_a\{d^{(k-1)}[a] + w(a,b)\}\}$
```

	s	x	y	z
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$				

Now, fill in $\mathbf{d}^{(3)}$!!!



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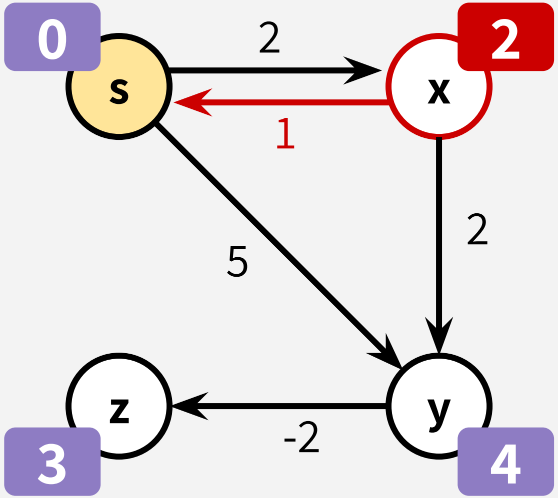
```

	s	x	y	z
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$	0	2	4	3
$\mathbf{d}^{(3)}$	0			

This is min of:

$d^{(2)}[s]$

$d^{(2)}[\mathbf{x}] + w(\mathbf{x}, s)$



BELLMAN-FORD

We store a list $\mathbf{d}^{(k)}$ of length n, for each k = 0, 1, …, n–1.

$\mathbf{d}^{(k)}[\mathbf{b}]$ = cost of shortest path from s to b *w/ at most k edges.*

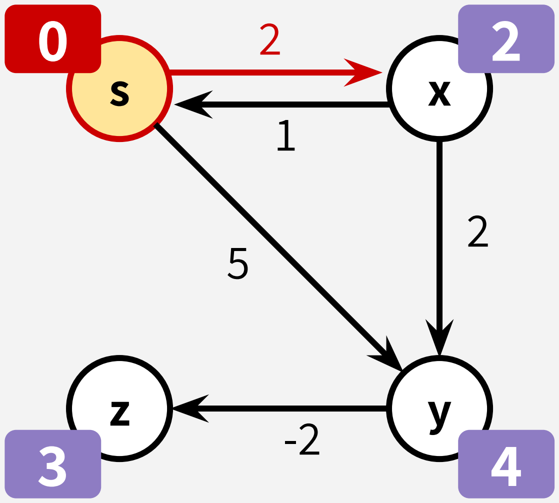
```
for k = 1, ..., n-1:
    for b in V:
        $d^{(k)}[b] \leftarrow \min\{d^{(k-1)}[b], \min_a\{d^{(k-1)}[a] + w(a,b)\}\}$
```

	s	x	y	z
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$	0	2	4	3
$\mathbf{d}^{(3)}$	0	2		

This is min of:

$d^{(2)}[x]$

$d^{(2)}[\mathbf{s}] + w(\mathbf{s}, x)$



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```

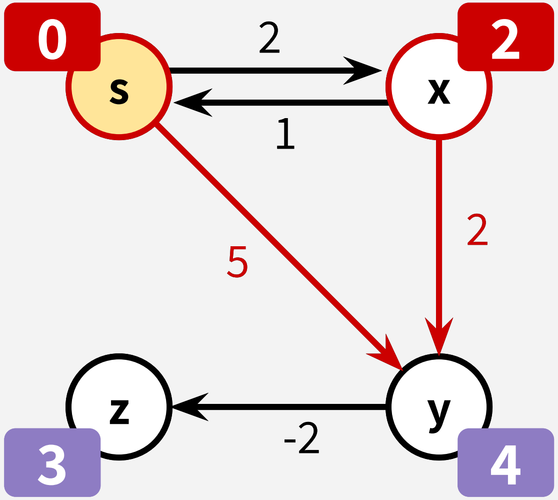
	s	x	y	z
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$	0	2	4	3
$\mathbf{d}^{(3)}$	0	2	4	

This is min of:

$d^{(2)}[y]$

$d^{(2)}[\mathbf{s}] + w(\mathbf{s}, y)$

$d^{(2)}[\mathbf{x}] + w(\mathbf{x}, y)$



BELLMAN-FORD

We store a list $\mathbf{d^{(k)}}$ of length n, for each k = 0, 1, …, n−1.

$\mathbf{d^{(k)}[b]}$ = cost of shortest path from s to b *w/ at most k edges.*

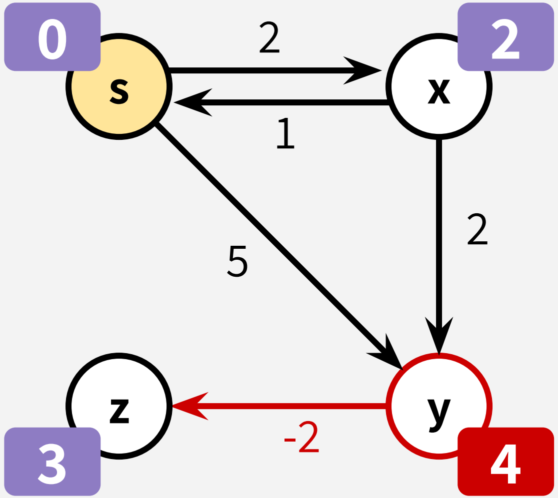
```
for k = 1, ..., n-1:
    for b in V:
        $d^{(k)}[b] \leftarrow \min\{d^{(k-1)}[b], \min_a\{d^{(k-1)}[a] + w(a,b)\}\}$
```

	s	**x**	**y**	**z**
$\mathbf{d^{(0)}}$	0	∞	∞	∞
$\mathbf{d^{(1)}}$	0	2	5	∞
$\mathbf{d^{(2)}}$	0	2	4	3
$\mathbf{d^{(3)}}$	0	2	4	2

This is min of:

$d^{(2)}[z]$

$d^{(2)}[\mathbf{y}] + w(\mathbf{y}, z)$



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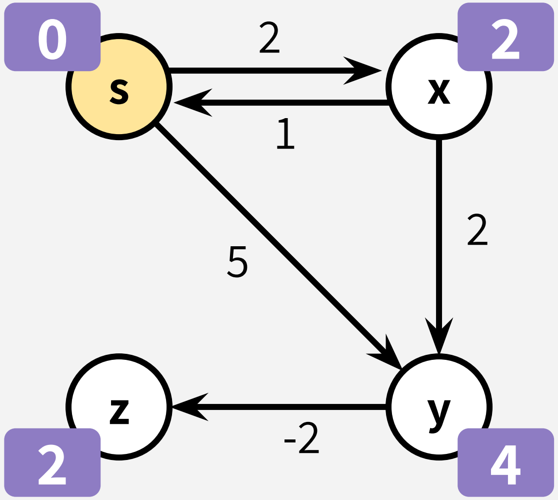
	s	**x**	**y**	**z**
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$	0	2	4	3
$\mathbf{d}^{(3)}$	0	2	4	2

We're done!

We can double check the entry for **z** in each $d^{(i)}$:



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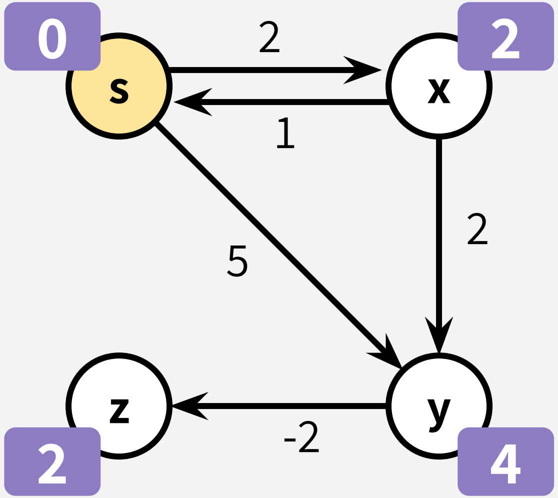
```
for k = 1, ..., n-1:
    for b in V:
        $d^{(k)}[b]$ ← min{$d^{(k-1)}[b]$, $\min_a\{d^{(k-1)}[a] + w(a,b)\}$}
```

Just to double check:

cost of shortest path from s to z with **1** edge = **∞**

	s	x	y	z
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
$d^{(3)}$	0	2	4	2



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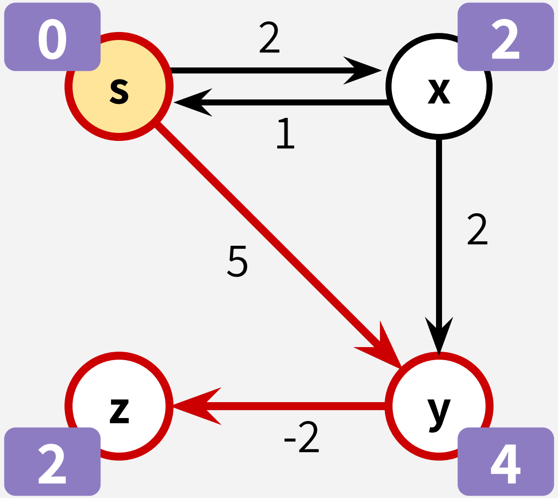
        $d^{(k)}[b] \leftarrow \min\{d^{(k-1)}[b], \min_a\{d^{(k-1)}[a] + w(a,b)\}\}$
```

Just to double check:

cost of shortest path from s to z with **1** edge = **∞**

cost of shortest path from s to z with **2** edges = **3**

	s	x	y	z
$\mathbf{d}^{(0)}$	0	∞	∞	∞
$\mathbf{d}^{(1)}$	0	2	5	∞
$\mathbf{d}^{(2)}$	0	2	4	3
$\mathbf{d}^{(3)}$	0	2	4	2



BELLMAN-FORD

We store a list **d**$^{(k)}$ of length n, for each k = 0, 1, …, n−1.

d$^{(k)}$**[b]** = cost of shortest path from s to b *w/ at most k edges.*

```
for k = 1, ..., n-1:
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```

Just to double check:

cost of shortest path from s to z with **1** edge = **∞**

cost of shortest path from s to z with **2** edges = **3**

cost of shortest path from s to z with **3** edges = **2**

	s	x	y	z
$d^{(0)}$	0	∞	∞	∞
$d^{(1)}$	0	2	5	∞
$d^{(2)}$	0	2	4	3
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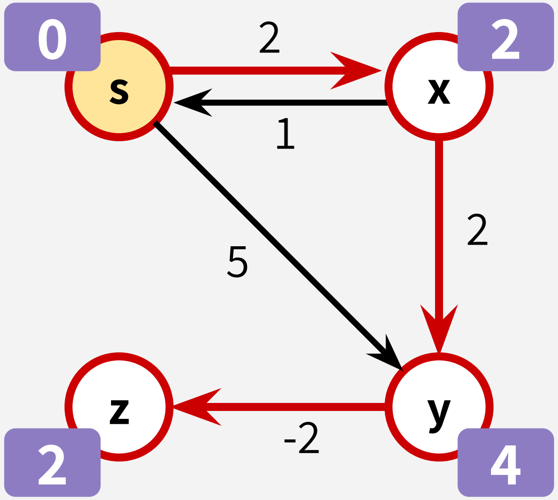

سوال؟

پیاده سازی دیگری از الگوریتم بلمن-فورد

پیدا کردن کوتاه ترین مسیر از یک راس به تمام رئوس

DYNAMIC PROGRAMMING

Two approaches for DP

(2 different ways to think about and/or implement DP algorithms)

Bottom-up: iterates through problems by size and solves the small problems first (kind of like taking care of base cases first & building up).
e.g. **Bellman-Ford (as we will see shortly!) computes $d^{(0)}$, then $d^{(1)}$, then $d^{(2)}$, etc.**

Top-down: instead uses recursive calls to solve smaller problems, while using memoization/caching to keep track of small problems that you've already computed answers for (simply fetch the answer instead of re-solving that problem and waste computational effort)
We will see a way later to implement **Bellman-Ford** using a top-down approach.

TOP-DOWN BELLMAN-FORD

```
RECURSIVE_BELLMAN_FORD(G,s):
    $d^{(k)}$ = [None] * n for k = 0, ..., n-1
    $d^{(0)}$[v] = $\infty$ for all v in V (except s)
    $d^{(0)}$[s] = 0
    for b in V:
        $d^{(n-1)}$[b] ← RECURSIVE_BF_HELPER(G, b, n-1)

RECURSIVE_BF_HELPER(G, b, k):
    A = {a such that (a,b) in E} ∪ {b}    // b's in-neighbors
    for a in A:
        if $d^{(k-1)}$[a] is None:                  // not yet solved
            $d^{(k-1)}$[a] ← RECURSIVE_BF_HELPER(G, a, k-1)
    return min{ $d^{(k-1)}$[b], $\min_a$\{$d^{(k-1)}$[a] + w(a,b)} }
```

TOP-DOWN BELLMAN-FORD

```
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            $d^{(k-1)}$[a] ← RECURSIVE_BF_HELPER(G, a, k-1)
    return $\min\{ d^{(k-1)}[b], \min_a\{d^{(k-1)}[a] + w(a,b)\} \}$
```

Think of this as a table/cache that holds the computed answers of our subproblems.

if the answer to this subproblem hasn’t been computed yet, then we’ll first solve it! It immediately gets saved in our cache, so we won’t ever solve it twice.

TOP-DOWN BELLMAN-FORD

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```

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Runtime: O(m·n)

الگوریتم فلوید-وارشال

پیدا کردن کوتاه ترین مسیر بین هر دو راس

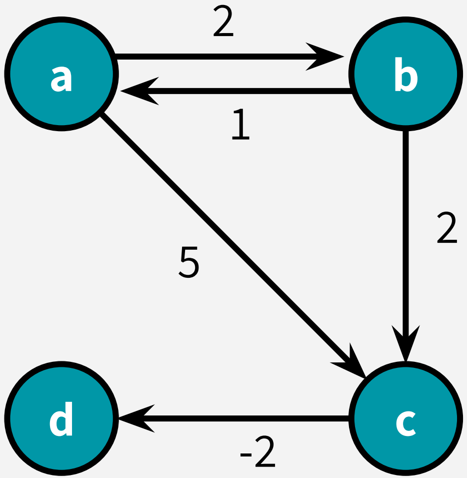
ALL-PAIRS SHORTEST PATHS (APSP)

Find the shortest paths from **v** to **w** for ALL pairs **v**, **w** of vertices in the graph (not just shortest paths from a special single source **s**)

Destination

Source

	a	**b**	**c**	**d**
a	**0**	**2**	**4**	**2**
b	**1**	**0**	**2**	**0**
c	**∞**	**∞**	**0**	**-2**
d	**∞**	**∞**	**∞**	**0**



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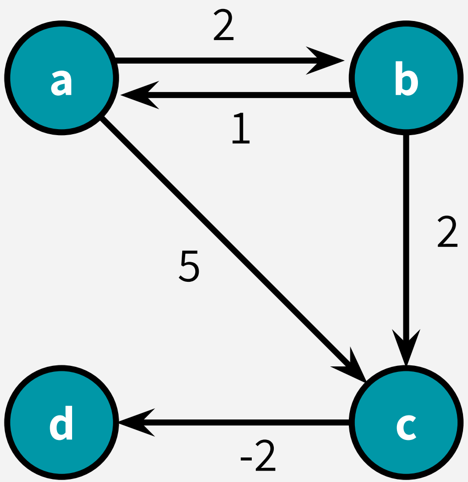
c	**∞**	**∞**	**0**	**-2**
d	**∞**	**∞**	**∞**	**0**



What's a naive algorithm?

ALL-PAIRS SHORTEST PATHS (APSP)

Find the shortest paths from **v** to **w** for ALL pairs **v**, **w** of vertices in the graph

Naive algorithm (if we want to handle negative edge weights):

```
For all s in G:
    Run Bellman-Ford on G starting at s
```

Runtime: O(n ·mn) = **O(mn²)**… this may be as bad as n^4 if $m = n^2$

Can we do better?

What's a naive algorithm?

FLOYD-WARSHALL: A DP APPROACH

We need to define the optimal substructure: Figure out what your subproblems are, and how you'll express an optimal solution in terms of optimal solutions to subproblems.

FLOYD-WARSHALL: A DP APPROACH

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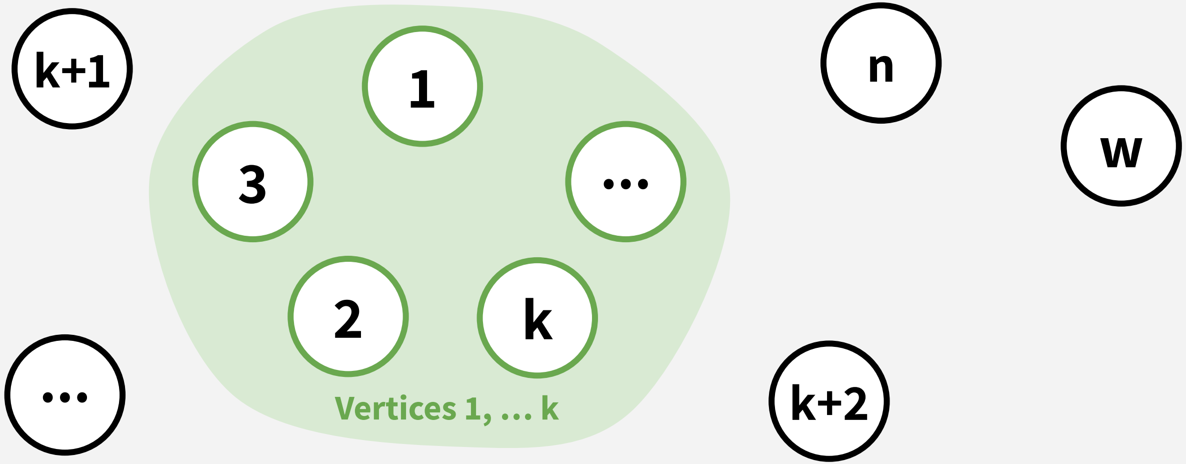
Subproblem(k): for all pairs **v**, **w**, find the cost of the shortest path from **v** to **w** so that all the internal vertices on that path are in {1, …, k}

Let $\mathbf{D^{(k)}[v, w]}$ be the solution to Subproblem(k)

Assign each vertex a number in {1, 2..., n}



(This diagram omits many edges and nodes, so it's not a full example)

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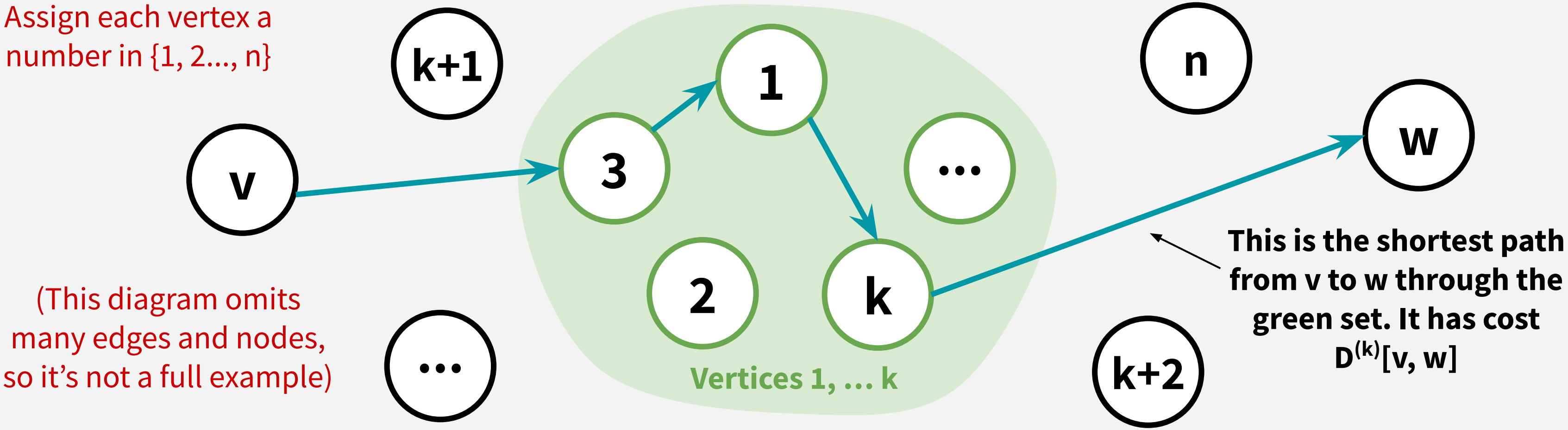

Assign each vertex a number in {1, 2..., n}

(This diagram omits many edges and nodes, so it's not a full example)

This is the shortest path from v to w through the green set. It has cost $D^{(k)}[v, w]$

FLOYD-WARSHALL: A DP APPROACH

We need to define the optimal substructure: Figure out what your subproblems are, and how you'll express an optimal solution in terms of optimal solutions to subproblems.

Subproblem(k): for all pairs **v**, **w**, find the cost of the shortest path from **v** to **w** so that all the internal vertices on that path are in {1, …, k}

Let **$D^{(k)}$[v, w]** be the solution to Subproblem(k)

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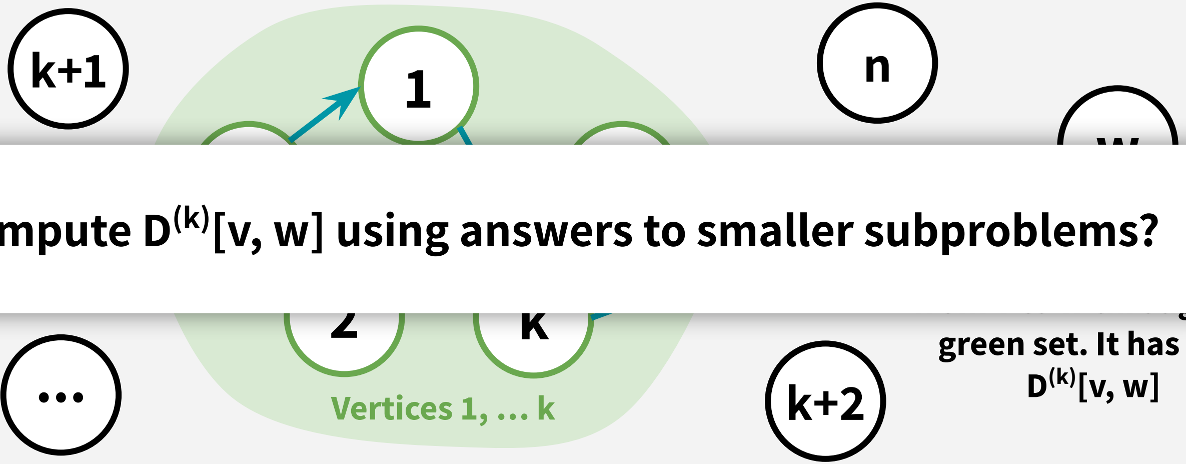


How do I compute $D^{(k)}$[v, w] using answers to smaller subproblems?

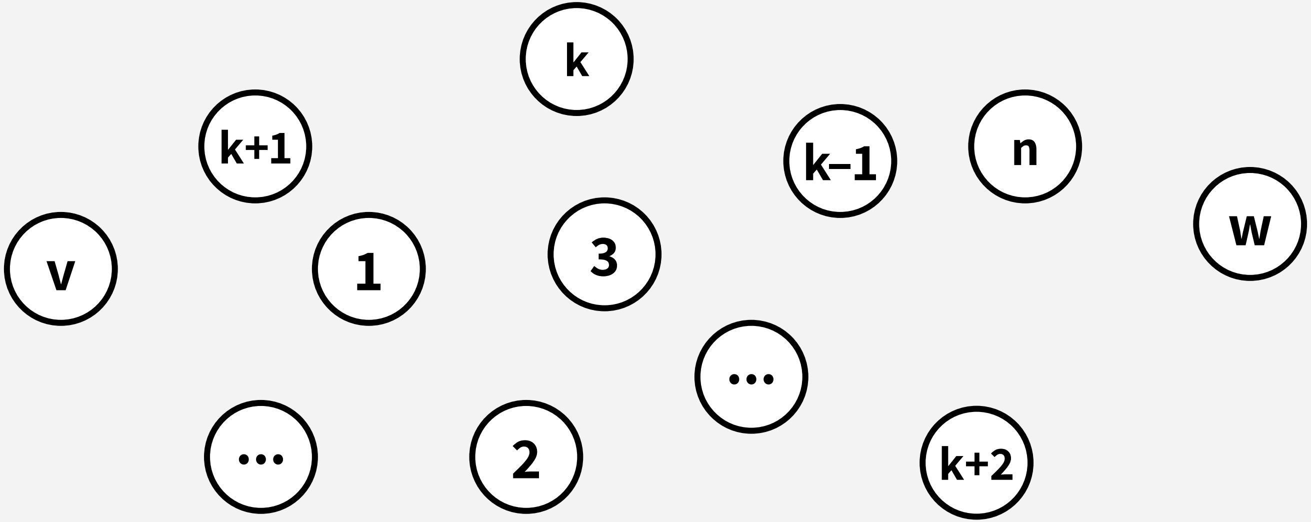
(This diagram omits many edges and nodes, so it's not a full example)

ath
the
green set. It has cost $D^{(k)}$[v, w]

FLOYD-WARSHALL: A DP APPROACH

$D^{(k)}$[v, w] is the cost of the shortest path from **v** to **w**, s.t. all of the internal vertices on the path are in the set of vertices {1, …, k}.

Two cases to consider: vertex k *is not* included in that path, or it *is*.

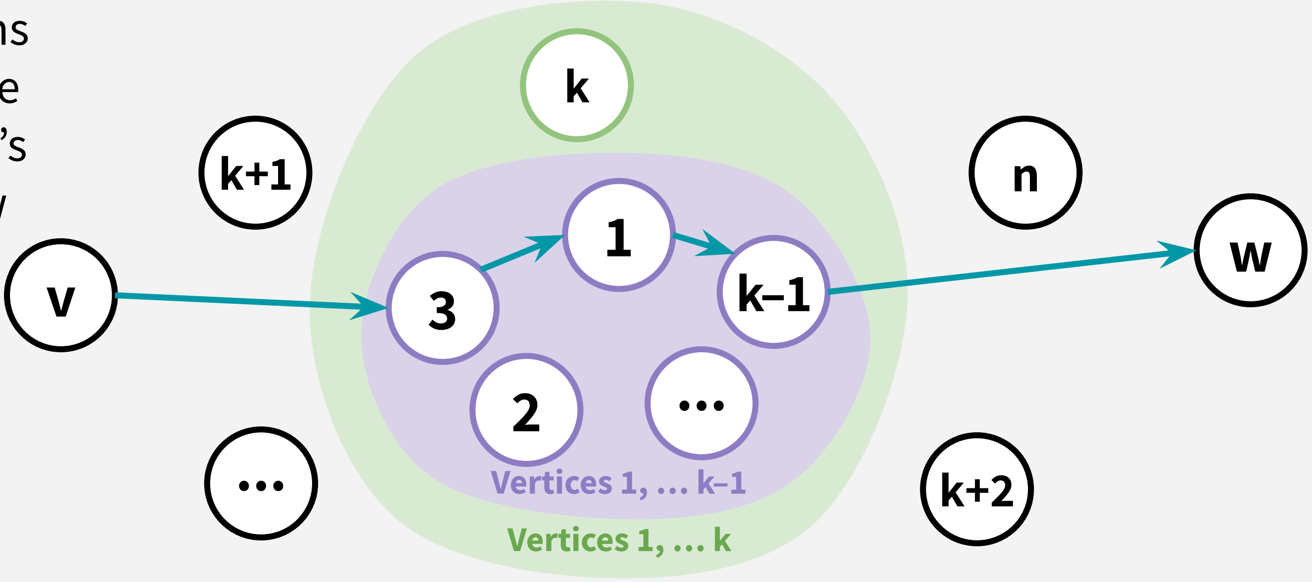


FLOYD-WARSHALL: A DP APPROACH

$D^{(k)}[v, w]$ = cost of the shortest path from **v** to **w**, s.t. all the internal vertices on the path are in the set of vertices {1,…, k}.

CASE 1: We don't need vertex k! So, $D^{(k)}[v, w] = D^{(k-1)}[v, w]$

In this case, this means that **this path** was the shortest before *and* it's still the shortest now



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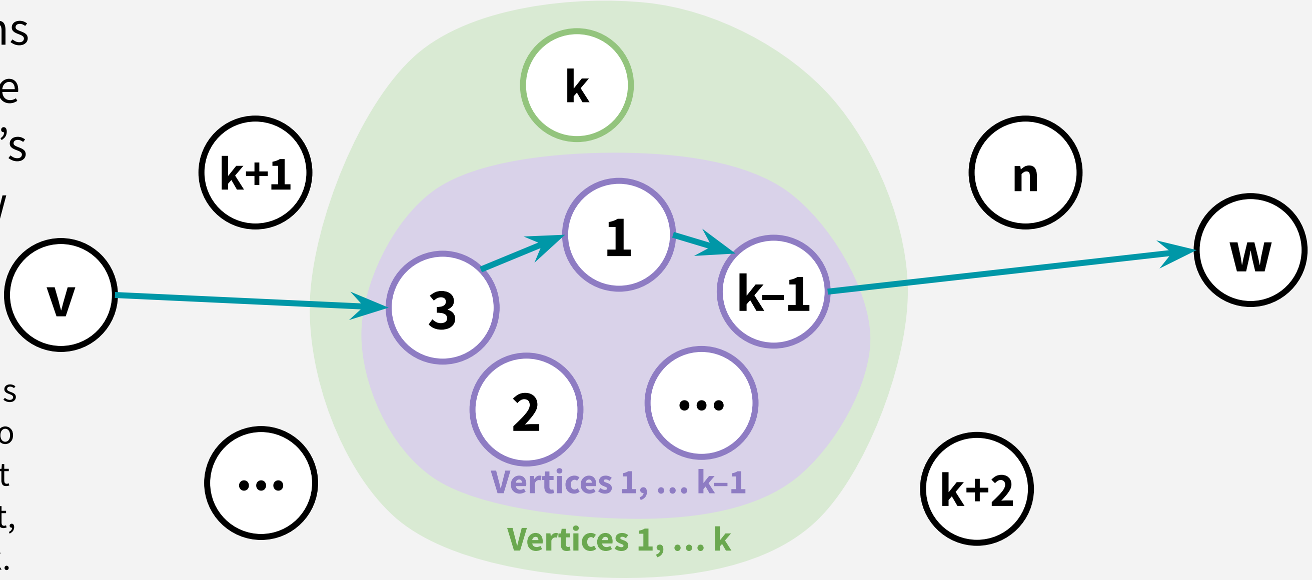


In other words, allowing paths to go through k (in addition to nodes 1, …, k–1) now doesn't change the shortest path cost, since it doesn't need to use k.

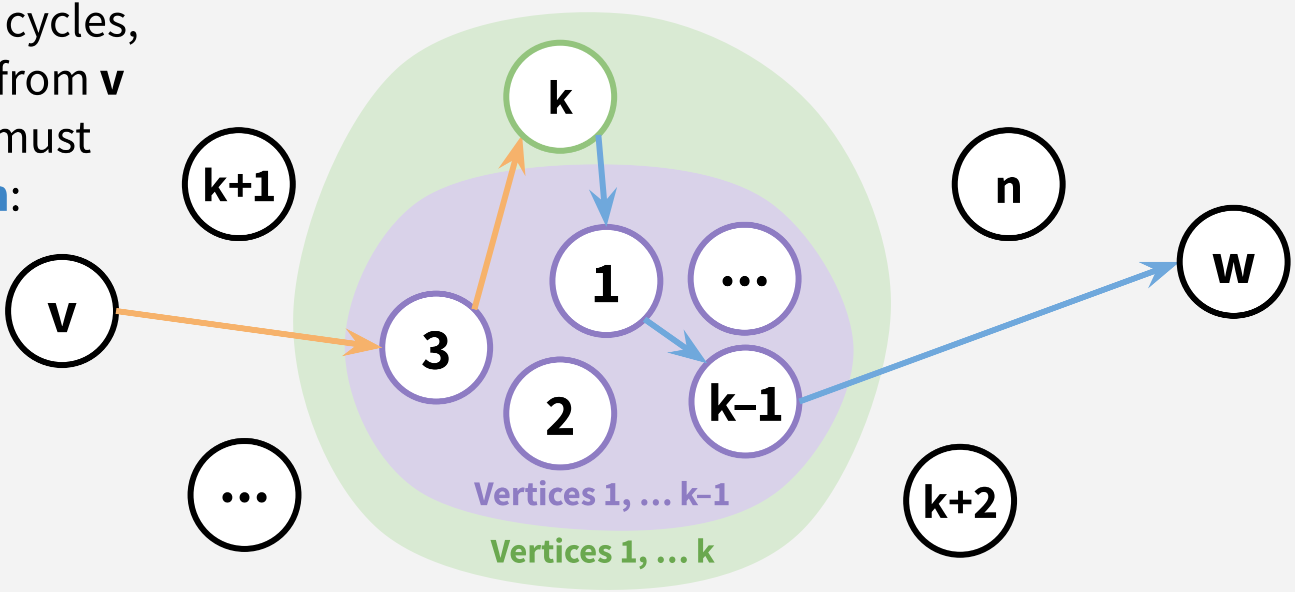
FLOYD-WARSHALL: A DP APPROACH

$D^{(k)}[v, w]$ = cost of the shortest path from **v** to **w**, s.t. all the internal vertices on the path are in the set of vertices {1,…, k}.

CASE 2: We need vertex k! So, $D^{(k)}[v, w] = D^{(k-1)}[v, k] + D^{(k-1)}[k, w]$

If there are no negative cycles, then the shortest path from **v** to **w** is *simple*, and it must look like **this path**:

(we also know that neither of these subpaths contains nodes greater than k–1.)



FLOYD-WARSHALL: A DP APPROACH

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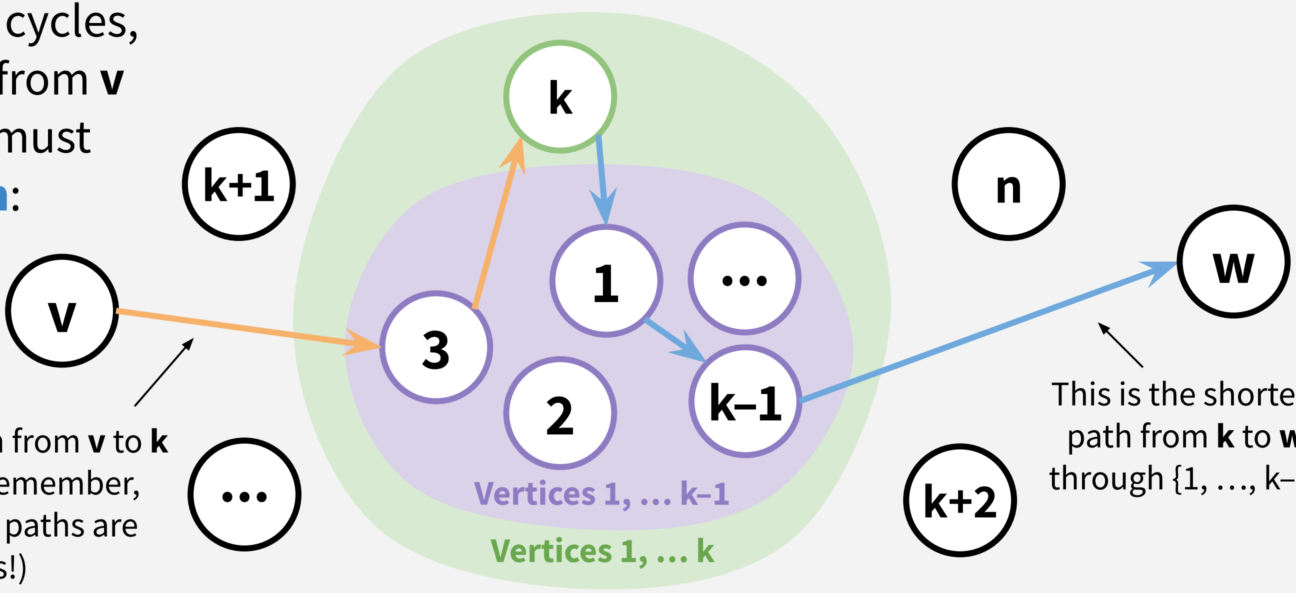


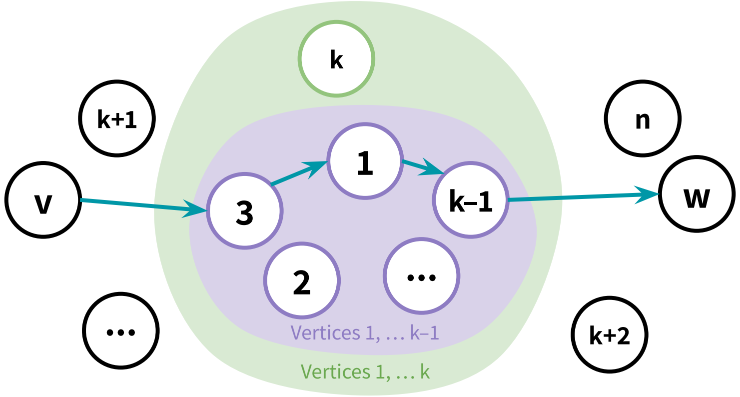
This is the shortest path from **v** to **k** through {1, …, k–1} (remember, sub-paths of shortest paths are shortest paths!)

This is the shortest path from **k** to **w** through {1, …, k–1}

FLOYD-WARSHALL: A DP APPROACH

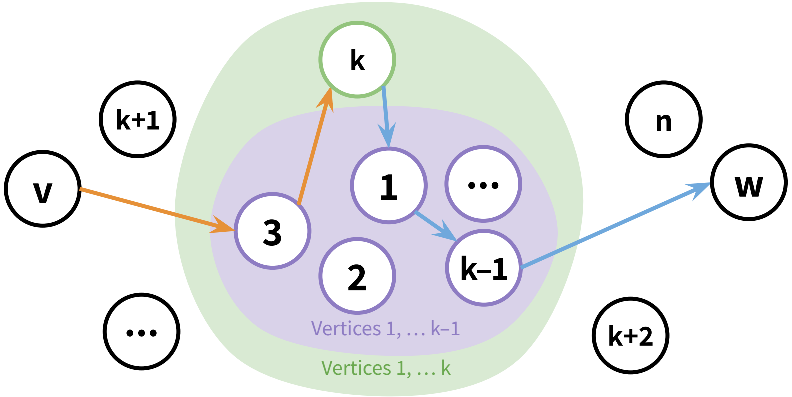
How do we find $D^{(k)}[v, w]$ using $D^{(k-1)}$? Choose the minimum of these 2 cases:

CASE 1: We don't need vertex **k**



$$D^{(k)}[v, w] = D^{(k-1)}[v, w]$$

CASE 2: We need vertex **k**



$$D^{(k)}[v, w]= D^{(k-1)}[v, k] + D^{(k-1)}[k, w]$$

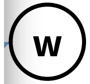
FLOYD-WARSHALL: A DP APPROACH

How do we find $D^{(k)}[v, w]$ using $D^{(k-1)}$? Choose the minimum of these 2 cases:

This is our optimal substructure: We know what our subproblems are (finding costs of shortest paths through a restricted set of vertices), and we know how to express our optimal solution in terms of these subproblem results (get the minimum of these two cases).

These subproblems are also overlapping: Memoization/caching can be useful here! For example, $D^{(k-1)}[k, w]$ can be used to help compute $D^{(k)}[v, w]$ for a lot of different starting points as v!

Now that we've settled this, we can write the algorithm!



$$D^{(k)}[v, w] = D^{(k-1)}[v, w]$$

$$D^{(k)}[v, w]= D^{(k-1)}[v, k] + D^{(k-1)}[k, w]$$

FLOYD-WARSHALL: A DP APPROACH

```
FLOYD_WARSHALL(G):
    Initialize n x n arrays $D^{(k)}$ for k = 0,...,n
        $D^{(k)}$[v,v] = 0 for all v, for all k
        $D^{(k)}$[v,w] = ∞ for all v ≠ w, for all k
        $D^{(0)}$[v,w] = weight(v,w) for all (v,w) in E
    for k = 1,...,n:
        for pairs v,w in $V^2$:
            $D^{(k)}$[v,w] = min{ $D^{(k-1)}$[v,w], $D^{(k-1)}$[v,k] + $D^{(k-1)}$[k,w] }
    return $D^{(n)}$
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Keeping all these n x n arrays would be a waste of space. In practice, only need to store 2!

Take the minimum over our two cases!

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Take the minimum over our two cases!

Runtime: $O(n^3)$

(Better than running Bellman-Ford n times!)

WHAT ABOUT NEGATIVE CYCLES?

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}
    for v in V:
        if $D^{(n)}$[v,v] < 0:
            return "NEGATIVE CYCLE!"
    return $D^{(n)}$
```

SHORTEST-PATH ALGORITHMS

$n = |V|$
$m = |E|$

BFS	**DFS**	**DIJKSTRA**	**BELLMAN-FORD**	**FLOYD-WARSHALL**
$O(m+n)$	$O(m+n)$	$O(m+n\log n)$*	$O(mn)$	$O(n^3)$
Unweighted (or weights don't matter)	Unweighted (or weights don't matter)	Weighted (weights must be *non-negative*)	Weighted (can handle *negative* weights)	Weighted (can handle *negative* weights)
Single source shortest path Test bipartiteness Find connected components	Path finding (s,t) Toposort (DAG!!) Find SCC's Find connected components	***Single source shortest paths:*** Compute shortest path from a source s to all other nodes	***Single source shortest paths:*** Compute shortest path from source s to all other nodes Detect negative cycles	***All pairs shortest paths**:* Compute shortest path between every pair of nodes (v,w)