طراحی الگوریتم ها (CE221)

جلسه یازدهم: تحلیل سرشکن

سجاد شیرعلی شهرضا

بهار 1401

شنبه، 21 اسفند 1400

اطلاع رسانی

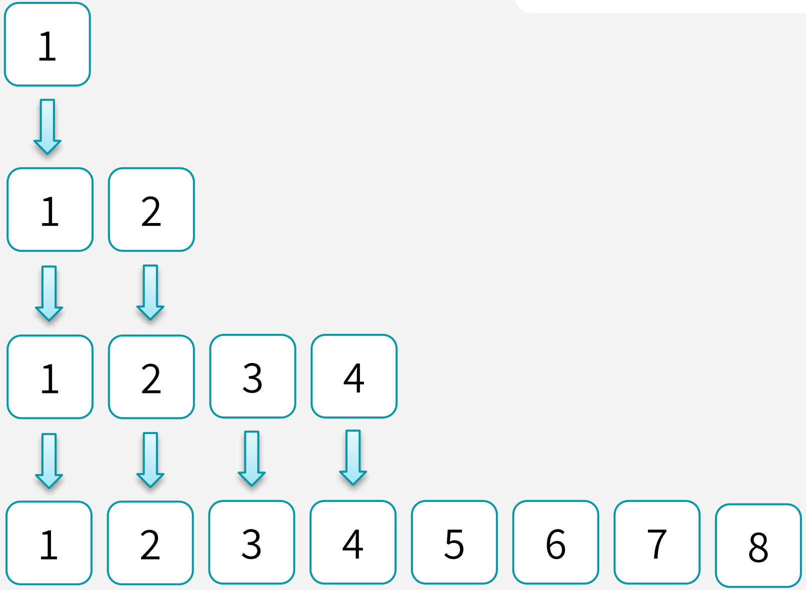
- بخش مرتبط کتاب برای این جلسه: 17
- امتحانک دوم:
  - دوشنبه همین هفته، 23 اسفند 1400
  - در طی ساعت کلاس به صورت برخط (مشابه امتحانک اول)

آرایه پویا

افزایش پویای اندازه آرایه در طول زمان

DYNAMIC ARRAY

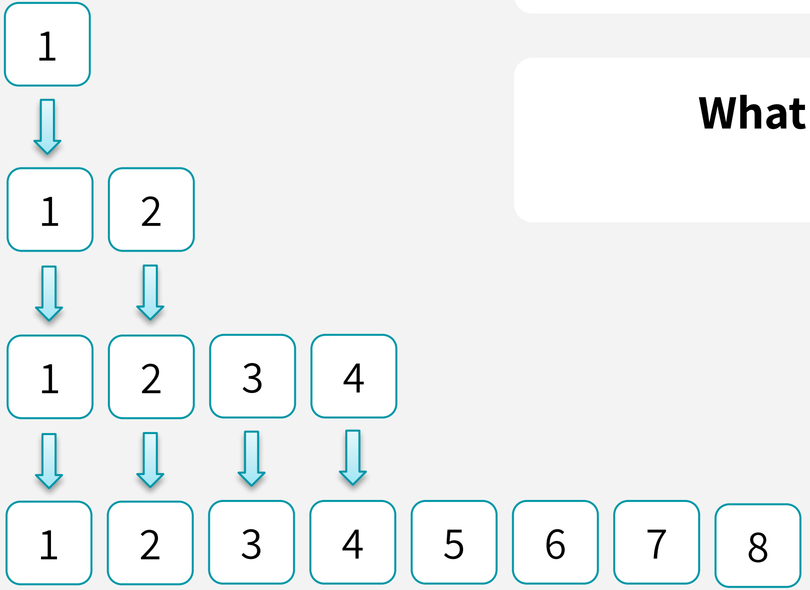
We fill it with n elements. When it is FULL, we replaced it with a new array that has 2*n capacity.



DYNAMIC ARRAY

We fill it with n elements. When it is FULL, we replaced it with a new array that has 2*n capacity.

What is the cost of EACH INSERTION?
What is the WORST CASE?

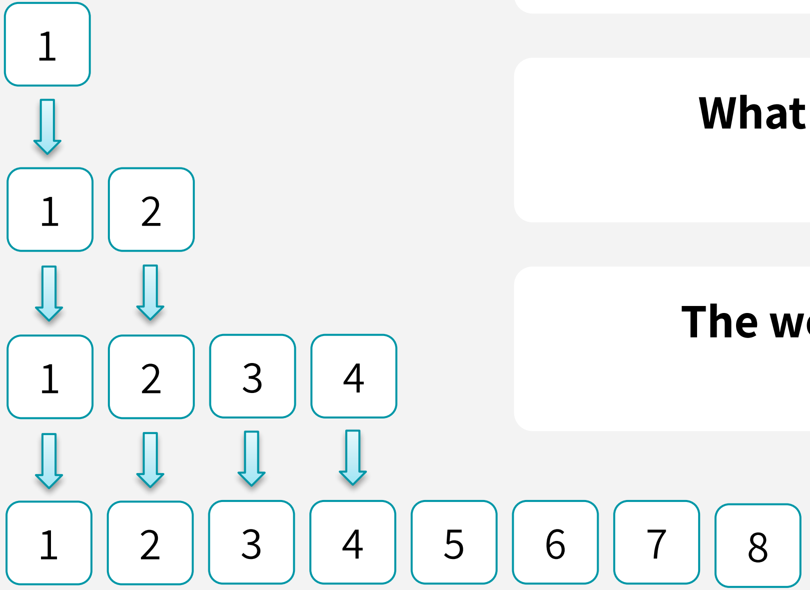


DYNAMIC ARRAY

We fill it with n elements. When it is FULL, we replaced it with a new array that has 2*n capacity.

What is the cost of EACH INSERTION?
What is the WORST CASE?

The worst insertion doubles the array!
So, In worst case O(n)?



DYNAMIC ARRAY

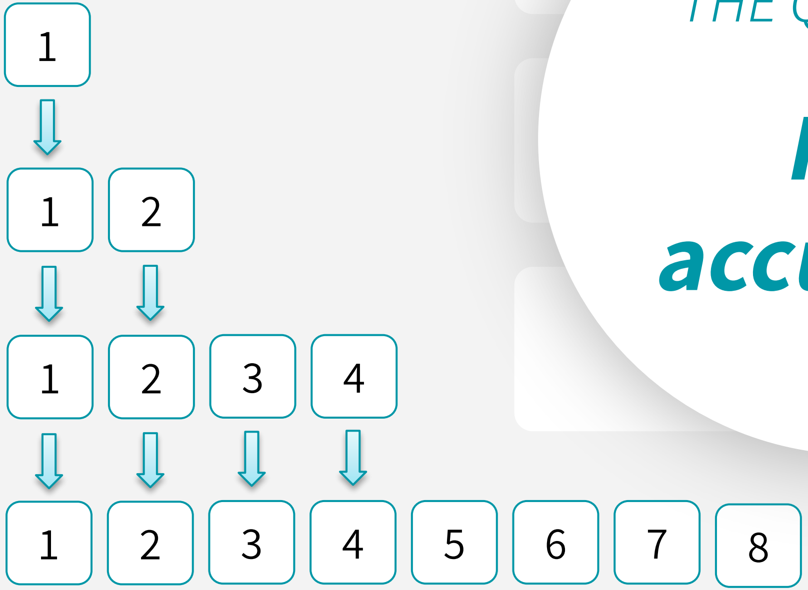
W… …nts. When it is FULL, we
re… …y that has 2*n capacity.

…H INSERTION?
…T CASE?

…oubles the array!
…st case O(n)?

THE QUESTION IS…

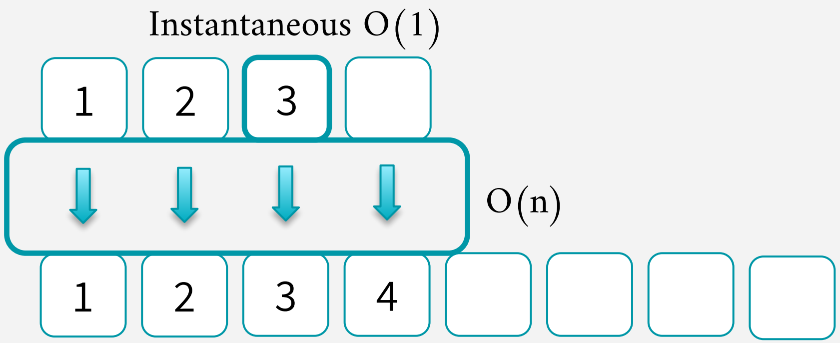
Is it accurate?



ANALYZING TIME COMPLEXITY

- Two type of operations
  - Simple operations with $O(1)$
  - Complex operations with $O(n)$

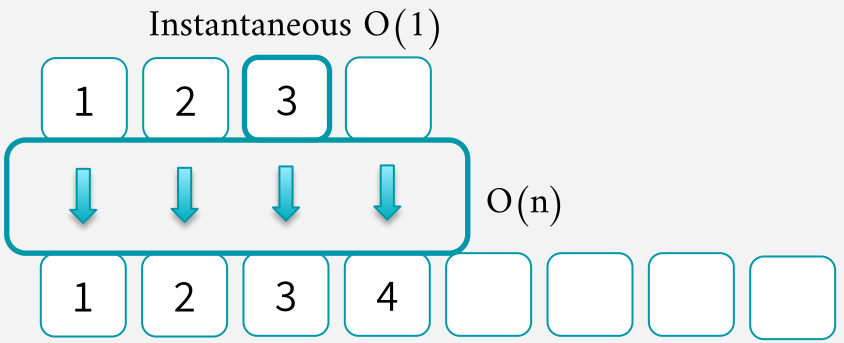
We need new type of analysing that is
Amortized Analysis



ANALYZING TIME COMPLEXITY

- Two type of operations
  - Simple operations with $O(1)$
  - Complex operations with $O(n)$

We need new type of analysing that is
Amortized Analysis



Give a General Solution?

سوال؟

تحلیل سرشکن

AMORTIZED ANALYSIS

- Not just consider one operation, but a sequence of operations
- Average cost over a sequence of operations.

- Example: Dynamic Array

AMORTIZED vs. PROBABILISTIC

- **Probabilistic analysis:**
  - Average case running time: average over all possible inputs for one algorithm (operation)
  - If using probability, called Expected Running Time.

- **Amortized analysis:**
  - No involvement of probability
  - Average performance on a sequence of operations
  - **Guarantee average performance of each operation among the sequence in worst case**

AMORTIZED ANALYSIS METHODS

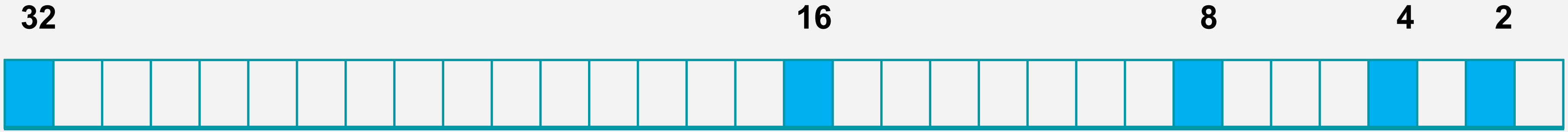
Amortized analysis methods:

- Aggregate analysis:
  - Total cost of n operations/n,
- Accounting Method:
  - Pay extra credit, and save it for expensive operations
- Potential method:
  - Same as accounting method
  - But store the credit as “potential energy” and as a whole

AMORTIZED ANALYSIS METHODS

- **Aggregate analysis**:
    - Total cost of n operations/n,
- **Accounting Method**:
    - Pay extra credit in each operation
    - Save extra credit on elements
    - Use extra credit for expensive operations
- **Potential method**:
    - Same as accounting method
    - But store the credit in one place as **potential energy**

EXPENSIVE INSERT OPERATION



 = 1

 = ?

Becomes more expensive, but happens less frequently

INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

What is the simplest way to determine the cost of each INSERTION?

INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

What is the simplest way to determine the cost of each INSERTION?

Average

سوال؟

روش انبوهه

میانگین هزینه های یک سری عملیات

AGGREGATE ANALYSIS

The **simplest** way to perform amortized analysis

How to calculate? $\dfrac{\text{Total cost}}{\text{\# of operations}}$

$$O\left(\sum Cost\ of\ n\ operations\right) = O\left(\begin{array}{c}\sum Cost\ of\ \boldsymbol{Cheap}\ operations \\ + \\ \sum Cost\ of\ \boldsymbol{Expensive}\ operations\end{array}\right)$$

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

After inserting 1 items, total cost of expensive insertions = 1

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

After inserting 2 items, total cost of expensive insertions = 3

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

Upto inserting 3 items, total cost of expensive insertions = 3

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

After inserting 4 items, total cost of expensive insertions = 7

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

Upto inserting 7 items, total cost of expensive insertions = 7

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

After inserting 8 items, total cost of expensive insertions = 15

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

Upto inserting 15 items, total cost of expensive insertions = 15

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

After inserting 16 items, total cost of expensive insertions = 31

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

Upto inserting 31 items, total cost of expensive insertions = 31

TOTAL EXPENSIVE INSERTION COST

16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	Item
16	1	1	1	1	1	1	1	8	1	1	1	4	1	2	1	Cost
32	31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	Item
32	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Cost

After inserting 32 items, total cost of expensive insertions = 63

AGGREGATE ANALYSIS

The **simplest** way to perform amortized analysis

How to calculate? $\frac{\text{Total cost}}{\text{\# of operations}}$

$$O(\sum Cost\ of\ n\ operations) = O\left(\begin{array}{c}\sum Cost\ of\ \boldsymbol{Cheap}\ operations \\ + \\ \sum Cost\ of\ \boldsymbol{Expensive}\ operations\end{array}\right)$$

Total cost of **cheap** operations = n = **O(n)**
Total cost of **expensive** operations < 2 x n = **O(n)**
Total cost = **O(n) + O(n) = O(n)**

AGGREGATE ANALYSIS

Dynamic array insertion cost

$$\text{Amortized cost} = \frac{\text{Total cost}}{\text{\# of operations}} = O(n) / n = O(1)$$

روش حسابداری

جمع آوری هزینه اضافه در حین انجام عملیات ساده

ACCOUNTING METHOD

- Save your money for a rainy day!
- Assign every operation a **cost**
  - Use part of it for the operation
  - Save surplus **beside** new item
- **Cheap** operations will have **extra** cost
  - To support **Expensive** operations
- **Challenge**: Bank balance must always be **0 or greater**

DYNAMIC ARRAY

Total Credit

0$

Charge 3 units per operation

DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

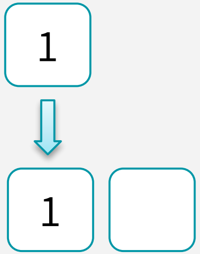
2$	1

DYNAMIC ARRAY

Charge 3 units per operation

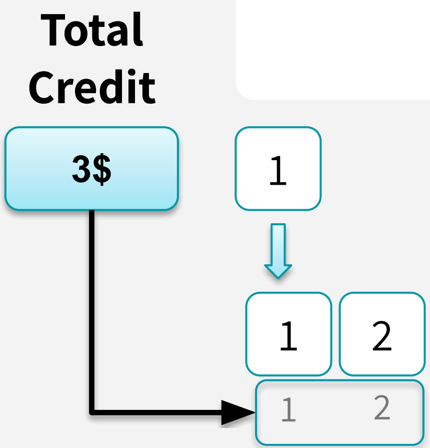
Total Credit

1$



DYNAMIC ARRAY

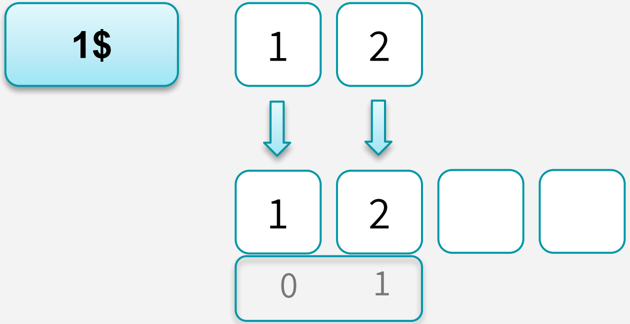
Charge 3 units per operation



DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

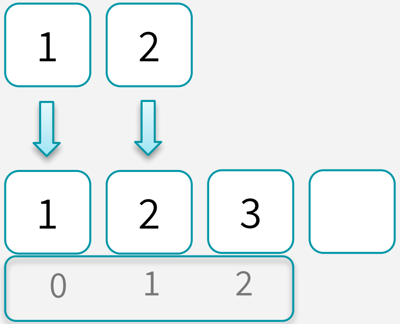


DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

3$

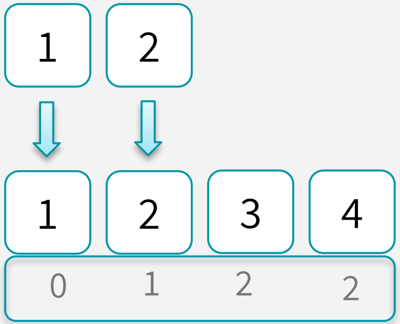


DYNAMIC ARRAY

Total Credit

5$

Charge 3 units per operation

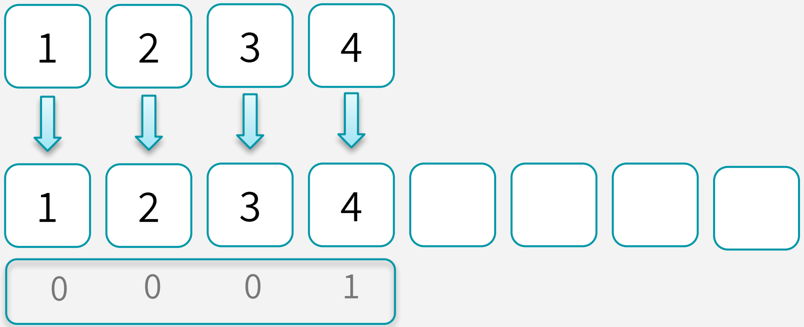


DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

1$

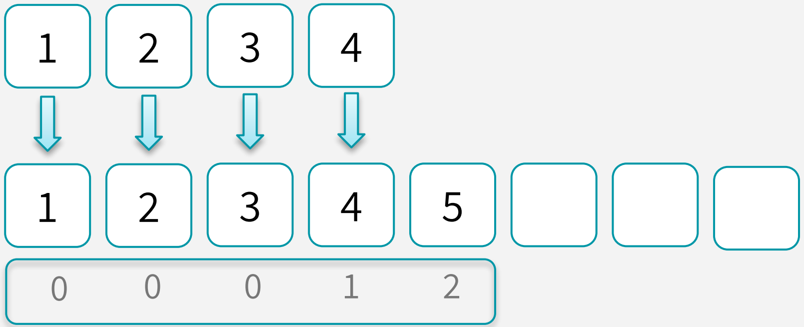


DYNAMIC ARRAY

Total Credit

3$

Charge 3 units per operation

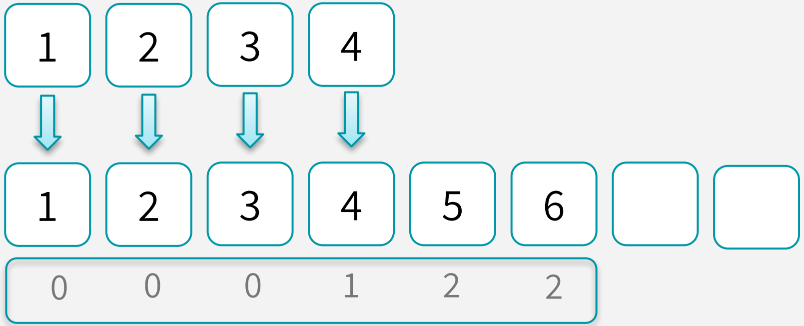


DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

5$

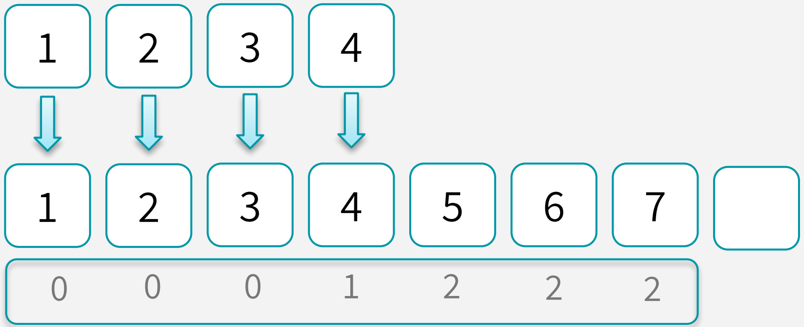


DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

7$

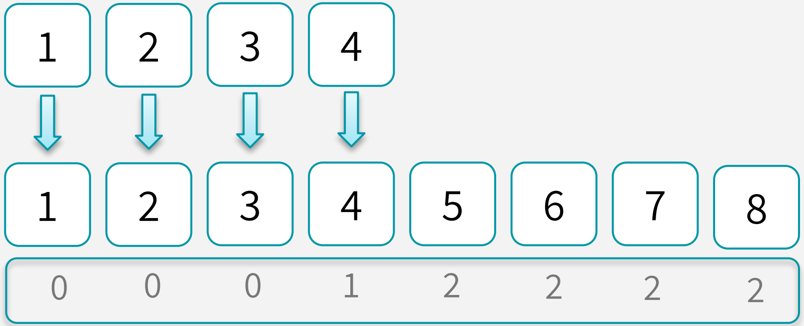


DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

9$

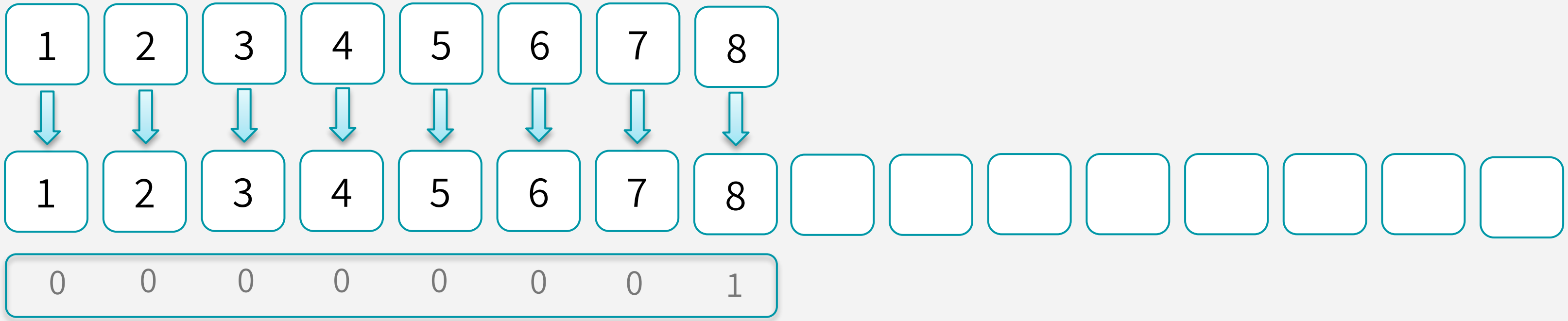


DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

1$



DYNAMIC ARRAY

Charge 3 units per operation

Total Credit

17$

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	0	0	0	0	0	0	1	2	2	2	2	2	2	2	2

Each operation costs 3, i.e., O(1)
Amortized cost = O(1)

روش پتانسیل

حالت بسط داده شده ای از روش حسابداری

POTENTIAL METHOD

Same as Accounting method

Pay extra for cheap operations and store extra credit.
Use stored credit for expensive operations.

Different from Accounting method

The prepaid work not as credit,
but as "potential energy", or "potential"

Potential: associated with the **whole data structure**
Credit: associated with **specific objects** in the data structure

DIFFERENCE FROM ACCOUNTING

In Accounting method, Bank balance of particular state is dependent on previous state

Potential Method uses Potential Function $\Phi(h)$

Potential function:
independently derive the potential at any state

Can compute the **potential difference**:
The change in cost between two operations

POTENTIAL FUNCTION

Big challenge

What is the proper
Potential Function $\Phi(h)$

Example: Dynamic array

$$\Phi(h) = 2n - size$$

n is the number of inserted items,
size is the actual size of array

Potential function: must always be non-negative

DYNAMIC ARRAY

Energy
~~Bank~~

0

1	

$\Phi(h) = 2n - size = 2*1 - 2 = 0$

DYNAMIC ARRAY

Energy

0

1	2		

$$\Phi(h) = 2n - size = 2*2-4=0$$

DYNAMIC ARRAY

Energy

2

1	2	3	

$$\Phi(h) = 2n - size = 2*3 - 4 = 2$$

DYNAMIC ARRAY

Energy

0

1	2	3	4				

$$\Phi(h) = 2n - size = 2*4 - 8 = 0$$

DYNAMIC ARRAY

Energy

100

…	112	113	114				…

$$\Phi(h) = 2n - size = 2*114 - 128 = 100$$

POTENTIAL FUNCTION

Amortized cost of the **i**th insertion h_i

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$

Cost of the **i**th insertion

Potential difference of **i**th and **i-1**th state

Example: Dynamic array

Two cases:
Normal case
Expansion case

POTENTIAL FUNCTION

Amortized cost of the **Normal insertion in Dynamic Array**

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$
$$\Phi(h) = 2n - size$$

Normal insertion doesn't change the size

…	i-1	i	…

$$= c_i + (2i - size) - (2(i-1) - size)$$
$$= 1 + 2i - size - 2i + 2 + size$$
$$= 3$$

POTENTIAL FUNCTION

Amortized cost of the **Expansive insertion in Dynamic Array**

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$
$$\Phi(h) = 2n - size$$

Expansion insertion change the size

…	i-1	i	…

$$size_i = 2 * size_{i-1}$$
$$size_{i-1} = i$$

$$= c_i + (2i - size_i) - (2(i-1) - size_{i-1})$$
$$= c_i + (2i - 2i) - (2(i-1) - i)$$
$$= (i+1) + 2i - 2i - 2i + 2 + i$$
$$= 3$$

POTENTIAL FUNCTION

Amortized time of the insertion in **Dynamic Array**

$$c_i + \Phi(h_i) - \Phi(h_{i-1})$$
$$\Phi(h) = 2n - size$$

…	i-1	i	…

Normal
3 operations
(amortized)
O(1)

Amortized Time
O(1)

Expansion
3 operations
(amortized)
O(1)

سوال؟

پشته با حذف چندگانه

حل با استفاده از سه روش معرفی شده

MULTI-POP STACK

PUSH(S,x) **push** x **onto stack** S — $O(1)$

POP(S) **pop top item of** S **and return it** — $O(1)$

MULTIPOP(S,k) **pop top k items of** S **and return them**

While S is not empty and $k \neq 0$
    POP(S)
    $k = k - 1$

$O(n)$???
$\theta(n)$???

AGGREGATE ANALYSIS

- Any sequence of n PUSH, POP, and MULTIPOP operations need $\mathbf{O(n)}$ time
- Average time per operation is $\mathbf{O(n)}/n = \mathbf{O(1)}$
- Amortized cost = $\mathbf{O(1)}$

ACCOUNTING ANALYSIS

	Real cost	Amortized cost
PUSH(S,x)	1	2
POP(S)	1	0
MULTIPOP(S,k)	Min(k,|s|)	0

Number of pushes = Number of pops

POTENTIAL FUNCTION

$$\boldsymbol{\Phi(S) = Number\ of\ items\ on\ stack\ S}$$

Empty stack S_0 gives us that

$$\forall S, \Phi(S) \geq 0 = \Phi(S_0)$$

Assume that S_{i-1} has d items on the stack, So:

PUSH (S,x)

$$\boldsymbol{\Phi(S_i) - \Phi(S_{i-1}) = (d+1) - d = 1}$$

Actual Cost is $c_i = 1$

Amortized Cost is $\widehat{c_i} = c_i + \Phi(S_i) - \Phi(S_{i-1}) = 1 + (d+1) - d = 2$

POTENTIAL FUNCTION

$$\boldsymbol{\Phi(S) = Number\ of\ items\ on\ stack\ S}$$

Empty stack S_0 gives us that

$$\forall S, \Phi(S) \geq 0 = \Phi(S_0)$$

Assume that S_{i-1} has d items on the stack, So:

POP(S)

$$\boldsymbol{\Phi(S_i) - \Phi(S_{i-1}) = (d-1) - d = -1}$$

Actual Cost is $\boldsymbol{c_i = 1}$

Amortized Cost is $\boldsymbol{\hat{c}_i = c_i + \Phi(S_i) - \Phi(S_{i-1}) = 1 - 1 = 0}$

POTENTIAL FUNCTION

$\Phi(S) = \textit{Number of items on stack } S$

Empty stack S_0 gives us that

$\forall S, \Phi(S) \geq 0 = \Phi(S_0)$

Assume that S_{i-1} has d items on the stack, So:

MULTIPOP(S)

$k' = min\ (k, d)$

$\Phi(S_i) - \Phi(S_{i-1}) = (d - k') - d = -k'$

Actual Cost is $c_i = k'$

Amortized Cost is $\hat{c}_i = c_i + \Phi(S_i) - \Phi(S_{i-1}) = k' - k' = 0$

POTENTIAL FUNCTION

	Real cost	Amortized cost
PUSH(S,x)	1	2
POP(S)	1	0
MULTIPOP(S,k)	Min(k,|s|)	0

Same as Accounting method

شمارنده بیتی

حل با استفاده از سه روش معرفی شده

BINARY COUNTER

Binary Counter

```
INCREMENT(A)
    i ← 0
    while i < length[A] and A[i] = 1 do
        A[i] ← 0
        i ← i + 1
    if i < length[A] then
        A[i] ← 1
```

A[7] A[6] A[5] A[4] A[3] A[2] A[1] A[0]

digit	A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]
0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0
4	0	0	1	0	0	0	0	0
5	1	0	1	0	0	0	0	0
6	0	1	1	0	0	0	0	0
7	1	1	1	0	0	0	0	0
8	0	0	0	1	0	0	0	0
9	1	0	0	1	0	0	0	0

EXAMPLE

digit	A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	cost	Total cost
0	0	0	0	0	0	0	0	0	0	0
1	**1**	0	0	0	0	0	0	0	1	1
2	0	**1**	0	0	0	0	0	0	2	3
3	**1**	**1**	0	0	0	0	0	0	1	4
4	0	0	**1**	0	0	0	0	0	3	7
5	**1**	0	**1**	0	0	0	0	0	1	8
6	0	**1**	**1**	0	0	0	0	0	2	10
7	**1**	**1**	**1**	0	0	0	0	0	1	11
8	0	0	0	**1**	0	0	0	0	4	15
9	**1**	0	0	**1**	0	0	0	0	1	16

EXAMPLE

digit	A[0]	A[1]	A[2]	A[3]	A[4]	A[5]	A[6]	A[7]	cost	Total cost
0	0	0	0	0	0	0	0	0	0	0
1	**1**	0	0	0	0	0	0	0	1	1
2	0	**1**	0	0	0	0	0	0	2	3
3	**1**	**1**	0	0	0	0	0	0	1	4
4	0	0	**1**	0	0	0	0	0	3	7
5	**1**	0	**1**	0	0	0	0	0	1	8
6	0	**1**	**1**	0	0	0	0	0	2	10
7	**1**	**1**	**1**	0	0	0	0	0	1	11
8	0	0	0	**1**	0	0	0	0	4	15
9	**1**	0	0	**1**	0	0	0	0	1	16

AGGREGATE ANALYSIS

Each A[i] flipped after 2^i increments

So the total number of bits flipped after n increments will be:

$$\sum_{i=0}^{k} \left\lfloor \frac{n}{2^i} \right\rfloor \le n \sum_{i=0}^{k} \frac{1}{2^i} < 2n$$

So, every operation requires at most 2n/n bit flips on average, i.e., has an amortized cost of O(1)

ACCOUNTING ANALYSIS

Binary Counter

```
INCREMENT(A)
    i ← 0
    while i < length[A] and A[i] = 1 do
        A[i] ← 0
        i ← i + 1
    if i < length[A] then
        A[i] ← 1
```

Real cost	Amortized cost
1	0
1	2

Every increment flips exactly one 0 to be a 1
Every 1 that is flipped to be a 0 was originally made into a 1 in a previous operation

POTENTIAL FUNCTION

$$\boldsymbol{\Phi(D) = Number\ of\ 1's\ in\ the\ counter}$$

Suppose that the i^{th} increment operation flips t_i 1 bits to 0
let b_i be the number of 1s in the counter after the operation

Actual cost is $c_i \leq t_i + 1$

If $b_i = 0$ then increment totally resets the counter and $b_{i-1} = t_i = k$
If $b_i > 0$ then $b_i = b_{i-1} - t_i + 1$
In both cases $b_i = b_{i-1} - t_i + 1$ so

$$\Phi(D_i) - \Phi(D_{i-1}) \leq b_{i-1} - t_i + 1 - b_{i-1} = 1 - t_i$$

Amortized Cost is $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1}) \leq (t_i + 1) + (1 - t_i) = 2$

سوال؟

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