طراحی الگوریتم ها (CE221)

جلسه چهارم:
رابطه بازگشتی و قضیه اصلی

سجاد شیرعلی شهرضا

بهار 1401

دوشنبه، 25 بهمن 1400

Based on Stanford CS161 Slides from Summer 2020 by Karey Shi

اطلاع رسانی

- بخش مرتبط کتاب برای این جلسه: 4.3، 4.4، 4.5

زمان اجرای مرتب سازی ادغامی

چقدر سریع است؟

MERGESORT: IS IT FAST?

```
MERGESORT(A):
    n = len(A)
    if n <= 1:
        return A
    L = MERGESORT(A[0:n/2])
    R = MERGESORT(A[n/2:n])
    return MERGE(L,R)
```

CLAIM: MergeSort runs in time **O(n log n)**

AN ASIDE: O(n log n) vs. O(n^2)?

log(n) grows very slowly! (Much more slowly than n)

AN ASIDE: $O(n \log n)$ vs. $O(n^2)$?

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ALL LOGARITHMS IN THIS COURSE ARE BASE 2

$\log(2) = 1$

$\log(4) = 2$

…

$\log(64) = 6$

$\log(128) = 7$

…

$\log(4096) = 12$

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log(**# particles in the universe**) < 280

AN ASIDE: O(n log n) vs. O(n^2)?

log(n) grows very slowly! (Much more slowly than n)

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log(64) = 6
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log(4096) = 12
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log(**# particles in the universe**) < 280

n log n grows much more slowly than n^2

Punchline: A running time of O(n log n) is a LOT better than O(n^2)

MERGESORT: O(n log n) PROOF

Instead of counting every little operation and tracing all recursive calls, we can think about:

THE RECURSION TREE!

(and we'll add up all the work done across levels to compute the Big-O runtime)

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```
MERGE(L,R):
    result = length n array
    i = 0, j = 0
    for k in [0,...,n-1]:
        if L[i] < R[j]:
            result[k] = L[i]
            i += 1
        else:
            result[k] = R[j]
            j += 1
    return result
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n iterations, O(1) work per iteration

We can see that MERGE is **O(n)**

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MERGE(L,R):
    result = length n array
    ...
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```
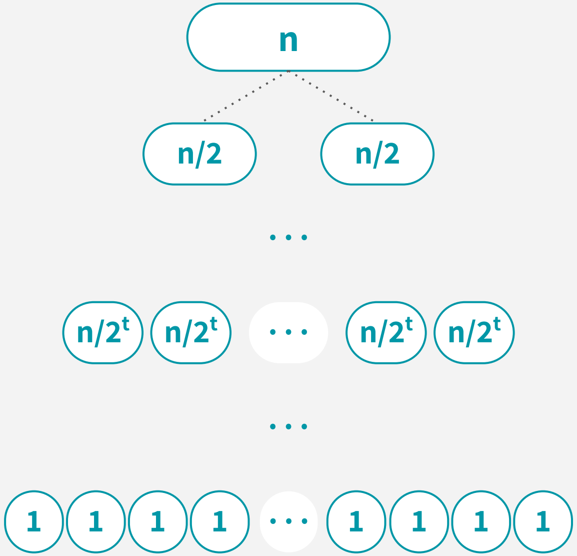

This means that within one recursive call that processes an array/subarray of length ***n***, the work done in that subproblem (creating subproblems & "merging" those results) is **O(n)**.

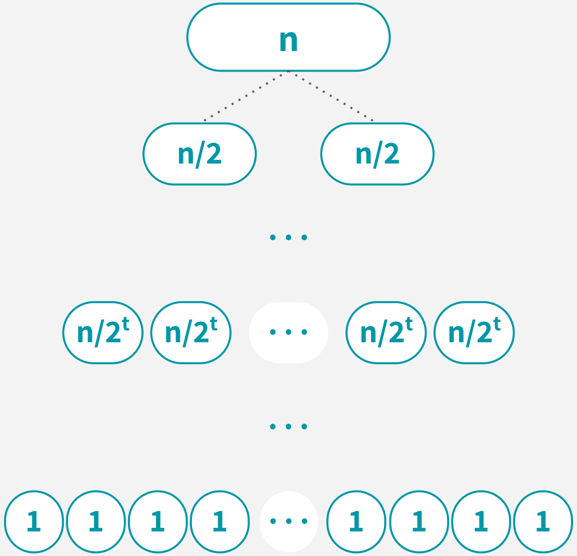
We can see that MERGE is **O(n)**

MERGESORT RECURSION TREE



Level	# of Problems	Size of each Problem	Work done per Problem	Total work on this level
0				
1				
…				
t				
…				
$\log_2 n$				

MERGESORT RECURSION TREE



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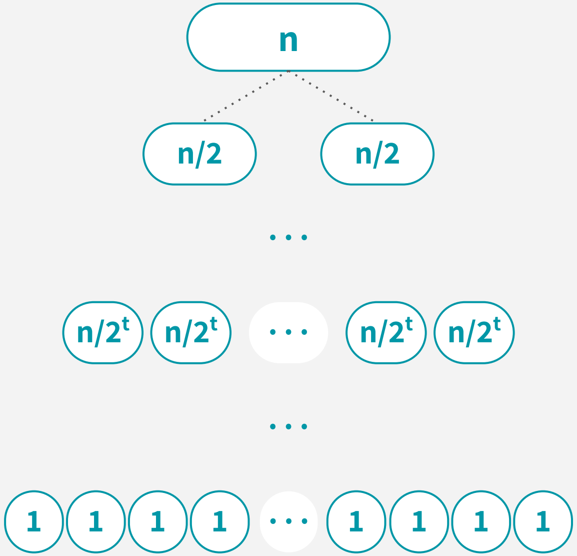
		…		
t	2^t	$n/2^t$		
		…		
$\log_2 n$	$2^{\log_2 n} = n$	1		

MERGESORT RECURSION TREE

If a subproblem is of size **n**, then the work done in that subproblem is **O(n)**.
⇒ **Work ≤ c · n** (c is a constant)



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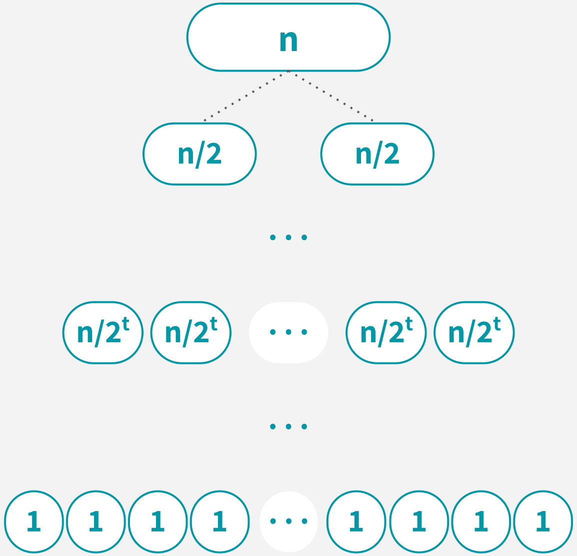

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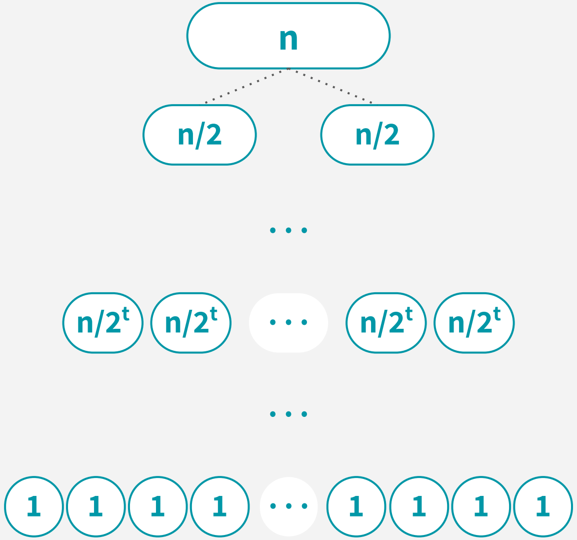

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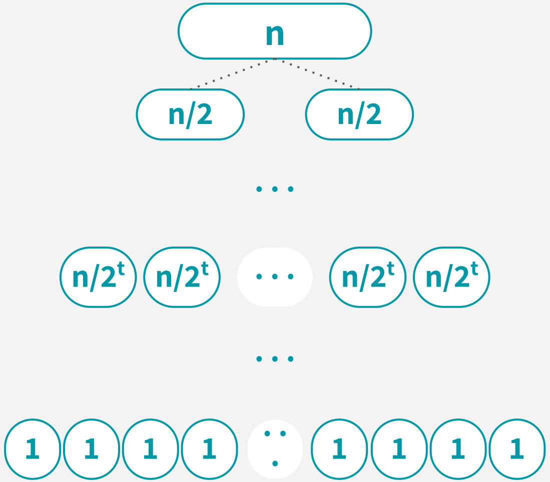
$\log_2 n$	$2^{\log_2 n} = n$	1	$c \cdot (1)$	$n \cdot c \cdot (1) =$ **O(n)**

We have ($\log_2 n + 1$) levels, each level has O(n) work total ⇒ **O(n log n)** work overall!

MERGESORT: O(n log n) RUNTIME

Using the "Recursion Tree Method" (i.e. drawing the tree & filling out the table),
we showed that the runtime of MergeSort is **O(n log n)**



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سوال؟

رابطه بازگشتی

RUNTIMES FOR RECURSIVE ALGOS

Previously, we used the "Recursion Tree Method" (i.e. drawing the tree & filling out the table) to manually add up all the work in the tree and find that the runtime of MergeSort is **O(n log n)**.

Drawing the tree & doing all that adding kind of takes a lot of work...
Here's another way to reason about the runtime of a recursive algorithm like Mergesort:

INTRODUCING...

RECURRENCE RELATIONS

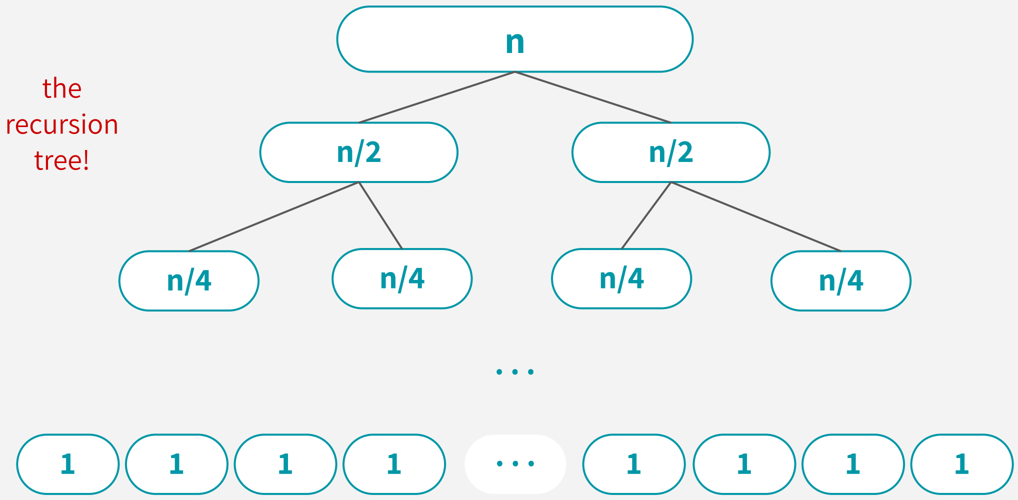
RECURRENCE RELATIONS

Basically, Recurrence Relations give us a *recursive* way to express runtimes for *recursive* algorithms!

We can then employ some math-ier approaches to analyze these recurrence relations.

RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:



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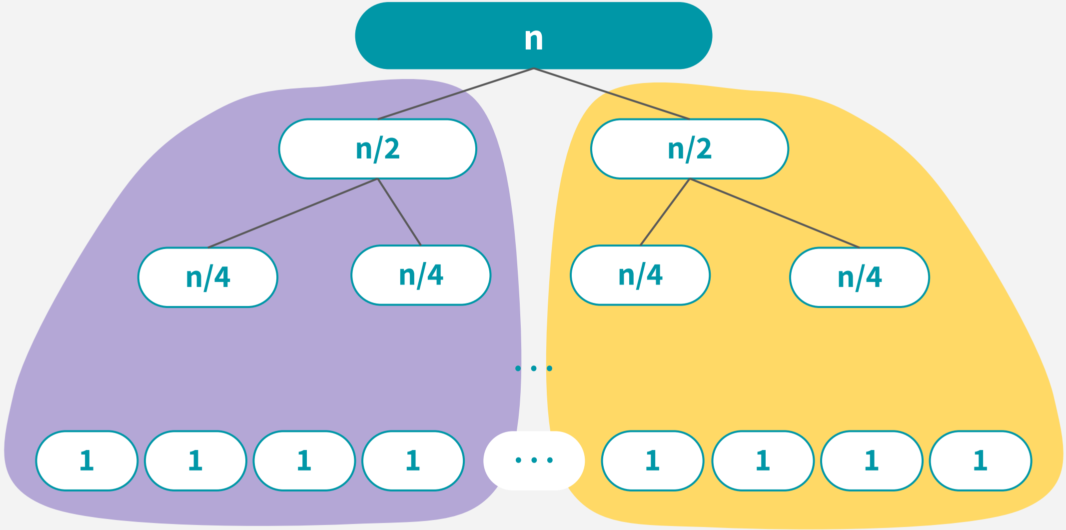

Work in the whole tree =

total work in LEFT recursive call
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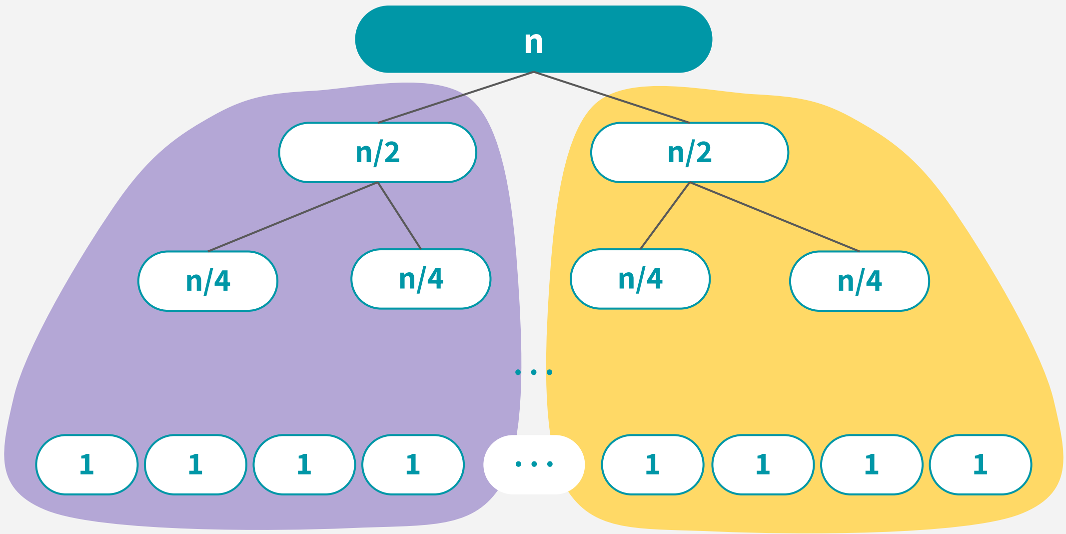

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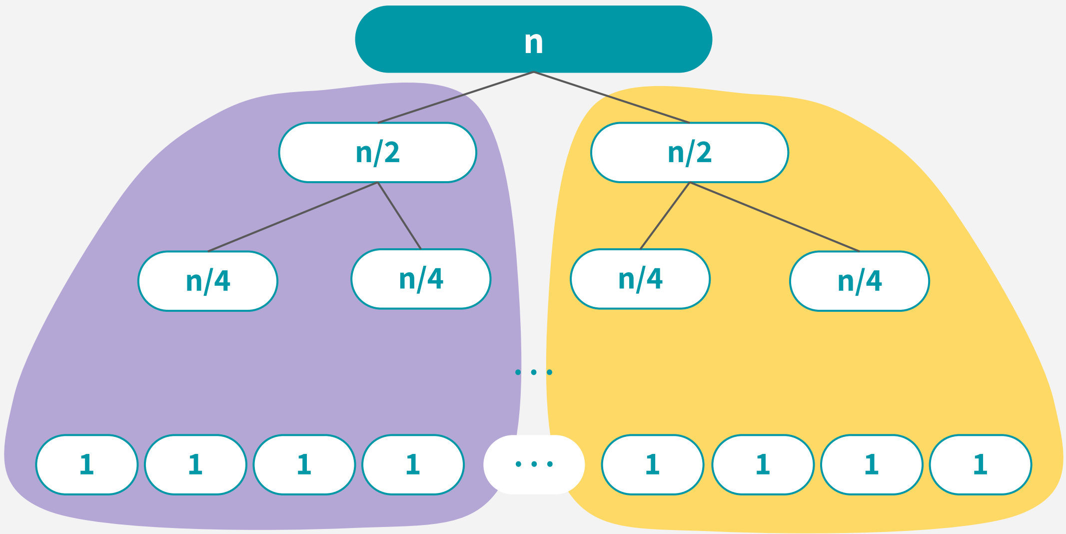

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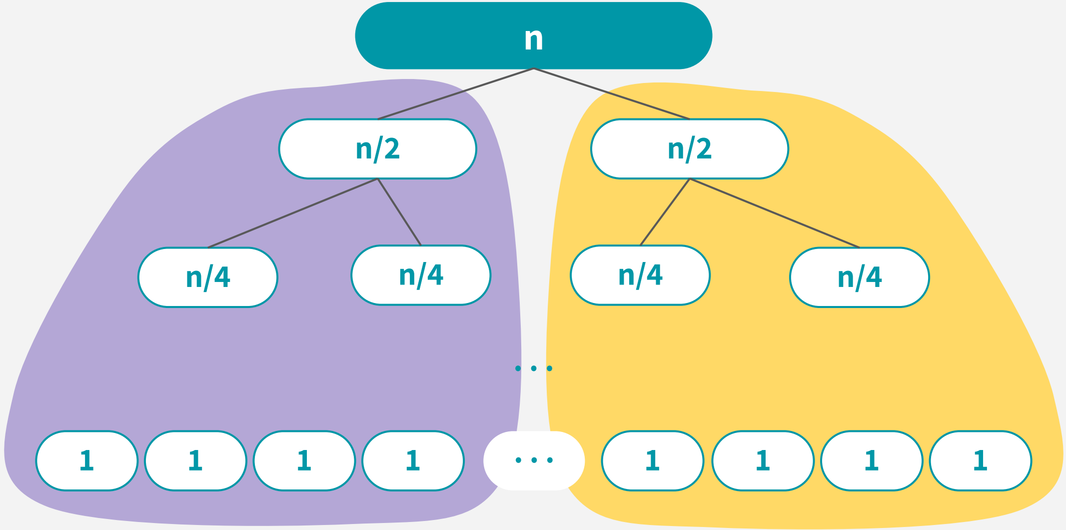

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work to create suproblems & "merge" their solutions

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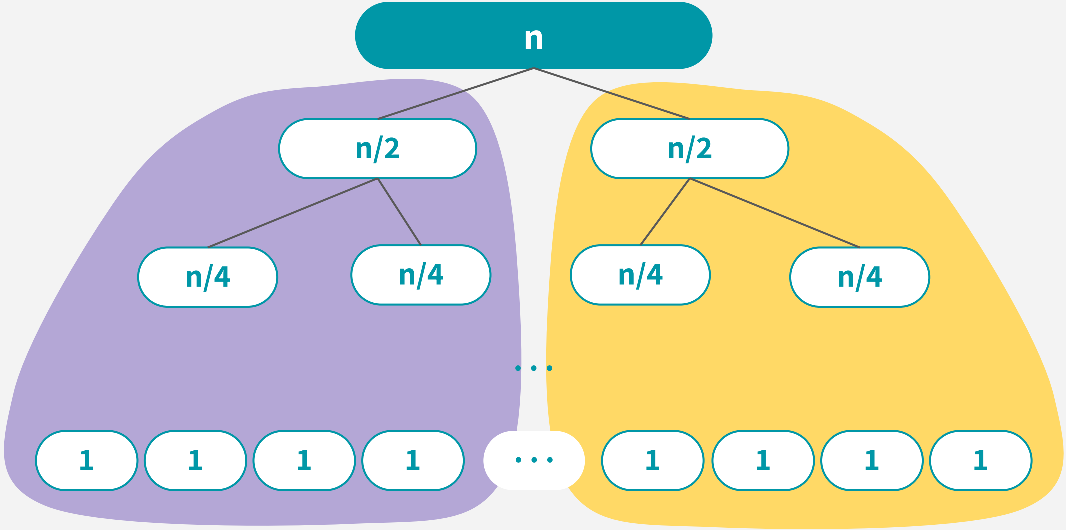

T(n) =

T(n/2)

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RECURRENCE RELATIONS

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A note:

We're making a simplifying assumption here that **n** is a perfect power of two (otherwise, we should use floors and ceilings).

Turns out that if we do incorporate floors and ceilings, we still get constant size subproblems at level $\lfloor \log_b n \rfloor$, and generally, the stuff we'll do in this class with Recurrence Relations will still work if we forget about floors and ceilings here.

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RECURRENCE RELATIONS

To build the recurrence relation for MergeSort, we can think of its runtime as follows:

$$\mathbf{T(n)} = \underbrace{\boxed{T(n/2)} + \boxed{T(n/2)}}_{} + \boxed{\mathbf{O(n)}}$$

since the subproblems are equal sizes, we can also write this as ***2 · T(n/2)***

This is a *recursive* definition for T(n), so we also need a BASE CASE:

$$\mathbf{T(1) = O(1)}$$

No matter what T is, T(1) = O(1). If it's greater than O(1), then the problem size wouldn't actually be 1.

Since we already used the Recursion Tree to compute the runtime of MergeSort, we know that **T(n) = O(n log n)**.

EXAMPLE RECURRENCE RELATIONS

Useless Divide-and-Conquer Multiplication

$$T(n) = 4 \cdot T(n/2) + O(n)$$
$$T(n) = O(n^{\log_2 4}) = \mathbf{O(n^2)}$$

Karatsuba Integer Multiplication

$$T(n) = 3 \cdot T(n/2) + O(n)$$
$$T(n) = O(n^{\log_2 3}) \approx \mathbf{O(n^{1.6})}$$

MergeSort

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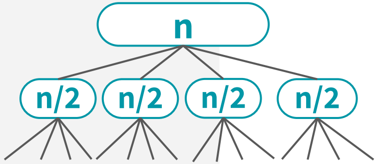

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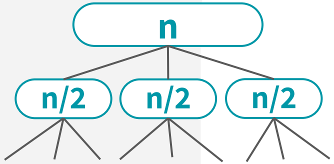

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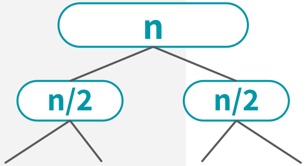

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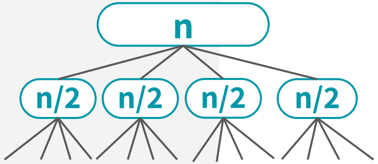

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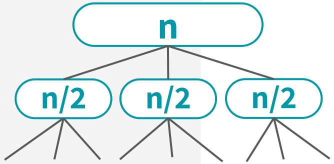

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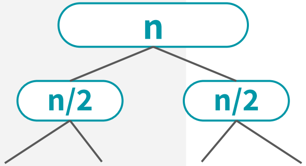

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IS THERE A
PATTERN
???

سوال؟

قضیه اصلی

فرمولی برای حل بسیاری از روابط بازگشتی
(اما نه همه آنها!)

THE MASTER THEOREM

Suppose that $\mathbf{a \geq 1}$, $\mathbf{b > 1}$, and $\mathbf{d}$ are constants (i.e. independent of $\mathbf{n}$).

Suppose $\mathbf{T(n) = a \cdot T(n/b) + O(n^d)}$. The Master Theorem states:

THE MASTER THEOREM

Suppose that **$a \geq 1$**, **$b > 1$**, and **d** are constants (i.e. independent of **n**).

Suppose **$T(n) = a \cdot T(n/b) + O(n^d)$**. The Master Theorem states:

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

a: number of subproblems (branching factor)

b: factor by which input size shrinks (shrinking factor)

d: need to do $O(n^d)$ work to create subproblems + “merge” their solutions

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USELESS DIVIDE & CONQUER MULTIPLICATION

$$T(n) = 4 \cdot T(n/2) + O(n)$$

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سوال؟

اثبات قضیه اصلی

MASTER THEOREM "INTUITION"

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

amount of work at each level ~ same

highest level "dominates": work per level decreases (subproblem work shrinks more!)

leaves "dominate": work per level increases (branch more!)

a: number of subproblems (branching factor)

b: factor by which input size shrinks (shrinking factor)

d: need to do $O(n^d)$ work to create subproblems + "merge" their solutions

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To see why this is true:

We need to look at a "generalized" recursion tree

increases (branch more!)

a: number of subproblems (branching factor)

b: factor by which input size shrinks (shrinking factor)

d: need to do $O(n^d)$ work to create subproblems + "merge" their solutions

MASTER THEOREM "INTUITION"

Original set up: $T(n) = a \cdot T(n/b) + O(n^d)$

We're going to suppose that $T(n) \leq a \cdot T(n/b) + c \cdot n^d$

Wait a minute: Is it okay to just replace $O(n^d)$ with $c \cdot n^d$? What about n_0? Do we just drop that?
For simplicity, we're going to assume that we're using $\mathbf{n_0 = 1}$ in the definition of big-O. (The proof of the Master Theorem" will work for larger n_0 too - verify this yourself!)

GENERALIZED RECURRENCE TREE

1. Draw the tree for **$T(n) = a \cdot T(n/b) + c \cdot n^d$**
2. Fill out the table & sum up last column (from $t = 0$ to $t = \log_b n$)

LEVEL	# OF PROBLEMS	SIZE OF EACH PROBLEM	WORK PER PROBLEM	TOTAL WORK AT THIS LEVEL
0	1	n	$c \cdot n^d$	$1 \cdot c \cdot n^d$
1	a	n/b	$c \cdot (n/b)^d$	$a \cdot c \cdot (n/b)^d$
. . .				
t	a^t	n/b^t	$c \cdot (n/b^t)^d$	$a^t \cdot c \cdot (n/b^t)^d$
. . .				
$\log_b n$	$a^{\log_b n}$	$n/b^{\log_b n} = 1$	$c \cdot (n/b^{\log_b n})^d$	$a^{\log_b n} \cdot c \cdot \overbrace{(n/b^{\log_b n})^d}^{\text{This} = 1}$

Total amount of work:

$$\boxed{c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}$$

(add work across all levels up, then factor out the c & n^d terms & write in summation form)

GENERALIZED RECURRENCE TREE

1. Draw the tree for **$T(n) = a \cdot T(n/b) + c \cdot n^d$**
2. Fill out the table & sum up last column (from $t = 0$ to $t = \log_b n$)

So $T(n) \leq$ $$\boxed{c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\frac{a}{b^d}\right)^t}$$

We can verify that for each of the three cases (**a =, <, or > b^d**), this equation above gives us the desired results:

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

CASE 1: $a = b^d$

$$T(n) = \begin{cases} \boldsymbol{\Theta(n^d \log n)} & \textbf{if } \boldsymbol{a = b^d} \\ \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

$$T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\boxed{\frac{a}{b^d}}\right)^t$$

This is equal to 1!

$$= c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} 1$$

$$= c \cdot n^d \cdot (\log_b(n) + 1)$$

$$= c \cdot n^d \cdot \left(\frac{\log(n)}{\log(b)} + 1\right)$$

$$= \Theta(n^d \log(n))$$

CASE 2: a < b^d

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a = b^d \\ \mathbf{\Theta(n^d)} & \textbf{if } \mathbf{a < b^d} \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

This is less than 1!

$$T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\boxed{\frac{a}{b^d}} \right)^t$$

$$= c \cdot n^d \cdot [\text{ some constant }]$$

$$= \Theta(n^d)$$

Geometric series with the "multiplier" < 1 & constant!

CASE 2: $a > b^d$

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^d) & \text{if } a < b^d \\ \boldsymbol{\Theta(n^{\log_b(a)})} & \textbf{if } \boldsymbol{a > b^d} \end{cases}$$

$$T(n) = c \cdot n^d \cdot \sum_{t=0}^{\log_b(n)} \left(\boxed{\frac{a}{b^d}} \right)^t$$

$$= \Theta\left(n^d \left(\frac{a}{b^d} \right)^{\log_b(n)} \right)$$

$$= \Theta\left(n^{\log_b(a)} \right)$$

This is greater than 1!

Use the geometric series formula to convince yourself that this is legitimate!

The n^d term cancels with (b^d)^{$\log_b n$}!
And $\boldsymbol{a^{\log_b n} = n^{\log_b a}}$!

WE CHECKED ALL THREE CASES!

$$T(n) = a \cdot T(n/b) + O(n^d)$$

$$T(n) = \begin{cases} \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^{\log_b(a)}) & \text{if } a > b^d \end{cases}$$

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