Error function

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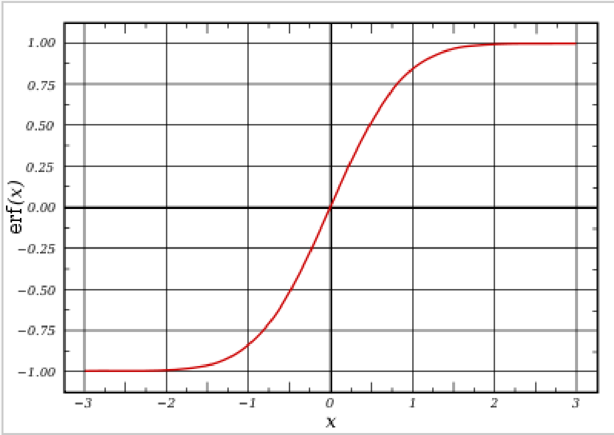
In mathematics, the **error function** (also called the **Gauss error function**) is a special function (non-elementary) of sigmoid shape that occurs in probability, statistics, and partial differential equations describing diffusion. It is defined as:[1][2]

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,\mathrm{d}t.$$



Plot of the error function

The **complementary error function**, denoted *erfc*, is defined as

$$\begin{aligned}\operatorname{erfc}(x) &= 1 - \operatorname{erf}(x) \\ &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,\mathrm{d}t = e^{-x^2} \operatorname{erfcx}(x),\end{aligned}$$

which also defines *erfcx*, the **scaled complementary error function**[3] (which can be used instead of erfc to avoid arithmetic underflow[3][4]).

The **imaginary error function**, denoted *erfi*, is defined as

$$\operatorname{erfi}(x) = -i\,\operatorname{erf}(ix) = \frac{2}{\sqrt{\pi}} e^{x^2} D(x),$$

where $D(x)$ is the Dawson function (which can be used instead of erfi to avoid arithmetic overflow[3]).

When the error function is evaluated for arbitrary complex arguments z, the resulting **complex error function** is usually discussed in scaled form as the Faddeeva function:

$$w(z) = e^{-z^2} \operatorname{erfc}(-iz) = \operatorname{erfcx}(-iz).$$

Contents [hide]

The name "error function" [edit]

The error function is used in measurement theory (using probability and statistics), and although its use in other branches of mathematics has nothing to do with the characterization of measurement errors, the name has stuck.

The error function is related to the cumulative distribution Φ, the integral of the standard normal distribution, by[2]

$$\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(x/\sqrt{2}\right).$$

The error function, evaluated at $\frac{x}{\sigma\sqrt{2}}$ for positive *x* values, gives the probability that a measurement, under the influence of normally distributed errors with standard deviation σ, has a distance less than x from the mean value.[5] This function is used in statistics to predict behavior of any sample with respect to the population mean. This usage is similar to the Q-function, which in fact can be written in terms of the error function.

Properties [edit]

The property $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ means that the error function is an odd function.

For any complex number *z*:

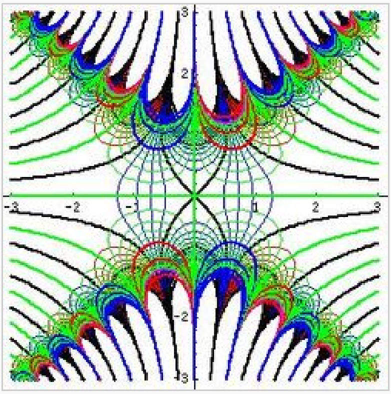
$$\operatorname{erf}(\overline{z}) = \overline{\operatorname{erf}(z)}$$

where $\overline{z}$ is the complex conjugate of *z*.

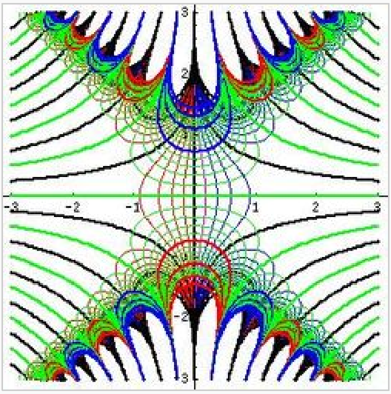
The integrand $f = \exp(-z^2)$ and $f = \operatorname{erf}(z)$ are shown in the complex *z*-plane in figures 2 and 3. Level of Im(f) = 0 is shown with a thick green line. Negative integer values of Im(f) are shown with thick red lines. Positive integer values of Im(f) are shown with thick blue lines. Intermediate levels of Im(f) = constant are shown with thin green lines. Intermediate levels of Re(f) = constant are shown with thin red lines for negative values and with thin blue lines for positive values.

At the real axis, erf(*z*) approaches unity at $z \to +\infty$ and −1 at $z \to -\infty$. At the imaginary axis, it tends to ±i∞.

Plots in the complex plane



Integrand $\exp(-z^2)$



erf(*z*)

Taylor series [edit]

The error function is an entire function; it has no singularities (except that at infinity) and its Taylor expansion always converges.

The defining integral cannot be evaluated in closed form in terms of elementary functions, but by expanding the integrand e^{-z^2} into its Taylor series and integrating term by term, one obtains the error function's Taylor series as:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}}\left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} - \cdots\right)$$

which holds for every complex number *z*. The denominator terms are sequence A007680 in the OEIS.

For iterative calculation of the above series, the following alternative formulation may be useful:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\left(z\prod_{k=1}^{n}\frac{-(2k-1)z^2}{k(2k+1)}\right) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{z}{2n+1}\prod_{k=1}^{n}\frac{-z^2}{k}$$

because $\frac{-(2k-1)z^2}{k(2k+1)}$ expresses the multiplier to turn the k^{th} term into the $(k+1)^{\text{th}}$ term (considering *z* as the first term).

The error function at +∞ is exactly 1 (see Gaussian integral).

The derivative of the error function follows immediately from its definition:

$$\frac{\mathrm{d}}{\mathrm{d}z}\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}e^{-z^2}.$$

An antiderivative of the error function is

$$z\operatorname{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}.$$

Inverse functions [edit]

The **inverse error function** can be defined in terms of the Maclaurin series

$$\operatorname{erf}^{-1}(z)=\sum_{k=0}^{\infty}\frac{c_k}{2k+1}\left(\frac{\sqrt{\pi}}{2}z\right)^{2k+1},$$

where $c_0 = 1$ and

$$c_k=\sum_{m=0}^{k-1}\frac{c_m c_{k-1-m}}{(m+1)(2m+1)}=\left\{1,1,\frac{7}{6},\frac{127}{90},\frac{4369}{2520},\frac{34807}{16200},\ldots\right\}.$$

So we have the series expansion (note that common factors have been canceled from numerators and denominators):

$$\operatorname{erf}^{-1}(z)=\tfrac{1}{2}\sqrt{\pi}\left(z+\frac{\pi}{12}z^3+\frac{7\pi^2}{480}z^5+\frac{127\pi^3}{40320}z^7+\frac{4369\pi^4}{5806080}z^9+\frac{34807\pi^5}{182476800}z^{11}+\cdots\right).$$

(After cancellation the numerator/denominator fractions are entries A092676/A132467 in the OEIS; without cancellation the numerator terms are given in entry A002067.) Note that the error function's value at ±∞ is equal to ±1.

The **inverse complementary error function** is defined as

$$\operatorname{erfc}^{-1}(1-z)=\operatorname{erf}^{-1}(z).$$

Asymptotic expansion [edit]

A useful asymptotic expansion of the complementary error function (and therefore also of the error function) for large real *x* is

$$\operatorname{erfc}(x)=\frac{e^{-x^2}}{x\sqrt{\pi}}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{(2x^2)^n}\right]=\frac{e^{-x^2}}{x\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2x^2)^n},$$

where $(2n-1)!!$ is the double factorial: the product of all odd numbers up to $(2n-1)$. This series diverges for every finite *x*, and its meaning as asymptotic expansion is that, for any $N\in\mathbb{N}$ one has

$$\operatorname{erfc}(x)=\frac{e^{-x^2}}{x\sqrt{\pi}}\sum_{n=0}^{N-1}(-1)^n\frac{(2n-1)!!}{(2x^2)^n}+R_N(x)$$

where the remainder, in Landau notation, is

$R_N(x)=O(x^{-2N+1}e^{-x^2})$ as $x\to\infty$.

Indeed, the exact value of the remainder is

$$R_N(x):=\frac{(-1)^N}{\sqrt{\pi}}2^{-2N+1}\frac{(2N)!}{N!}\int_x^{\infty}t^{-2N}e^{-t^2}\,\mathrm{d}t,$$

which follows easily by induction, writing $e^{-t^2}=-(2t)^{-1}(e^{-t^2})'$ and integrating by parts.

For large enough values of x, only the first few terms of this asymptotic expansion are needed to obtain a good approximation of erfc(*x*) (while for not too large values of *x* note that the above Taylor expansion at 0 provides a very fast convergence).

Continued fraction expansion [edit]

A continued fraction expansion of the complementary error function is:[6]

$$\operatorname{erfc}(z)=\frac{z}{\sqrt{\pi}}e^{-z^2}\cfrac{a_1}{z^2+\cfrac{a_2}{1+\cfrac{a_3}{z^2+\cfrac{a_4}{1+\cdots}}}}\qquad a_1=1,\quad a_m=\frac{m-1}{2},\quad m\geq 2.$$

Integral of error function with Gaussian density function [edit]

$$\operatorname{erf}\left[\frac{b-ac}{\sqrt{1+2a^2d^2}}\right]=\int_{-\infty}^{\infty}\mathrm{d}x\frac{\operatorname{erf}(ax+b)}{\sqrt{2\pi d^2}}\exp\left[-\frac{(x+c)^2}{2d^2}\right],\quad a,b,c,d\in\mathbb{R}$$

Approximation with elementary functions [edit]

Abramowitz and Stegun give several approximations of varying accuracy (equations 7.1.25–28). This allows one to choose the fastest approximation suitable for a given application. In order of increasing accuracy, they are:

$$\operatorname{erf}(x)\approx 1-\frac{1}{(1+a_1x+a_2x^2+a_3x^3+a_4x^4)^4}$$ (maximum error: 5×10^{-4})

where $a_1 = 0.278393$, $a_2 = 0.230389$, $a_3 = 0.000972$, $a_4 = 0.078108$

$$\mathrm{erf}(x) \approx 1 - (a_1 t + a_2 t^2 + a_3 t^3) e^{-x^2}, \quad t = \frac{1}{1+px}$$ (maximum error: 2.5×10^{-5})

where $p = 0.47047$, $a_1 = 0.3480242$, $a_2 = -0.0958798$, $a_3 = 0.7478556$

$$\mathrm{erf}(x) \approx 1 - \frac{1}{(1 + a_1 x + a_2 x^2 + \cdots + a_6 x^6)^{16}}$$ (maximum error: 3×10^{-7})

where $a_1 = 0.0705230784$, $a_2 = 0.0422820123$, $a_3 = 0.0092705272$, $a_4 = 0.0001520143$, $a_5 = 0.0002765672$, $a_6 = 0.0000430638$

$$\mathrm{erf}(x) \approx 1 - (a_1 t + a_2 t^2 + \cdots + a_5 t^5) e^{-x^2}, \quad t = \frac{1}{1+px}$$ (maximum error: 1.5×10^{-7})

where $p = 0.3275911$, $a_1 = 0.254829592$, $a_2 = -0.284496736$, $a_3 = 1.421413741$, $a_4 = -1.453152027$, $a_5 = 1.061405429$

All of these approximations are valid for $x \geq 0$. To use these approximations for negative x, use the fact that erf(x) is an odd function, so erf(x) = −erf(−x).

Another approximation is given by

$$\mathrm{erf}(x) \approx \mathrm{sgn}(x)\sqrt{1 - \exp\left(-x^2 \frac{4/\pi + ax^2}{1 + ax^2}\right)}$$

where

$$a = \frac{8(\pi - 3)}{3\pi(4 - \pi)} \approx 0.140012.$$

This is designed to be very accurate in a neighborhood of 0 and a neighborhood of infinity, and the error is less than 0.00035 for all x. Using the alternate value $a \approx 0.147$ reduces the maximum error to about 0.00012.[7]

This approximation can also be inverted to calculate the inverse error function:

$$\mathrm{erf}^{-1}(x) \approx \mathrm{sgn}(x)\sqrt{\sqrt{\left(\frac{2}{\pi a} + \frac{\ln(1-x^2)}{2}\right)^2 - \frac{\ln(1-x^2)}{a}} - \left(\frac{2}{\pi a} + \frac{\ln(1-x^2)}{2}\right)}.$$

Exponential bounds and a pure exponential approximation for the complementary error function are given by [8]

$$\mathrm{erfc}(x) \leq \frac{1}{2}e^{-2x^2} + \frac{1}{2}e^{-x^2} \leq e^{-x^2}, \qquad x > 0$$
$$\mathrm{erfc}(x) \approx \frac{1}{6}e^{-x^2} + \frac{1}{2}e^{-\frac{4}{3}x^2}, \qquad x > 0\,.$$

Numerical approximation [edit]

Over the complete range of values, there is an approximation with a maximal error of 1.2×10^{-7}, as follows:[9]

$$\mathrm{erf}(x) = \begin{cases} 1 - \tau & \text{for } x \geq 0 \\ \tau - 1 & \text{for } x < 0 \end{cases}$$

with

$$\begin{aligned} \tau = {} & t \cdot \exp\big(-x^2 - 1.26551223 + 1.00002368 \cdot t + 0.37409196 \cdot t^2 + 0.09678418 \cdot t^3 \\ & -0.18628806 \cdot t^4 + 0.27886807 \cdot t^5 - 1.13520398 \cdot t^6 + 1.48851587 \cdot t^7 \\ & -0.82215223 \cdot t^8 + 0.17087277 \cdot t^9\big) \end{aligned}$$

and

$$t = \frac{1}{1 + 0.5\,|x|}$$

Applications [edit]

When the results of a series of measurements are described by a normal distribution with standard deviation σ and expected value 0, then $\mathrm{erf}\left(\frac{a}{\sigma\sqrt{2}}\right)$ is the probability that the error of a single measurement lies between −a and +a, for positive a. This is useful, for example, in determining the bit error rate of a digital communication system.

The error and complementary error functions occur, for example, in solutions of the heat equation when boundary conditions are given by the Heaviside step function.

Related functions [edit]

The error function is essentially identical to the standard normal cumulative distribution function, denoted Φ, also named norm(x) by software languages, as they differ only by scaling and translation. Indeed,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-t^2}{2}}\, dt = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right] = \frac{1}{2}\operatorname{erfc}\left(-\frac{x}{\sqrt{2}}\right)$$

or rearranged for erf and erfc:

$$\operatorname{erf}(x) = 2\Phi\left(x\sqrt{2}\right) - 1$$
$$\operatorname{erfc}(x) = 2\Phi\left(-x\sqrt{2}\right) = 2\left(1 - \Phi\left(x\sqrt{2}\right)\right).$$

Consequently, the error function is also closely related to the Q-function, which is the tail probability of the standard normal distribution. The Q-function can be expressed in terms of the error function as

$$Q(x) = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right).$$

The inverse of Φ is known as the normal quantile function, or probit function and may be expressed in terms of the inverse error function as

$$\operatorname{probit}(p) = \Phi^{-1}(p) = \sqrt{2}\operatorname{erf}^{-1}(2p-1) = -\sqrt{2}\operatorname{erfc}^{-1}(2p).$$

The standard normal cdf is used more often in probability and statistics, and the error function is used more often in other branches of mathematics.

The error function is a special case of the Mittag-Leffler function, and can also be expressed as a confluent hypergeometric function (Kummer's function):

$$\operatorname{erf}(x) = \frac{2x}{\sqrt{\pi}}\, {}_1F_1\left(\tfrac{1}{2}, \tfrac{3}{2}, -x^2\right).$$

It has a simple expression in terms of the Fresnel integral.[further explanation needed]

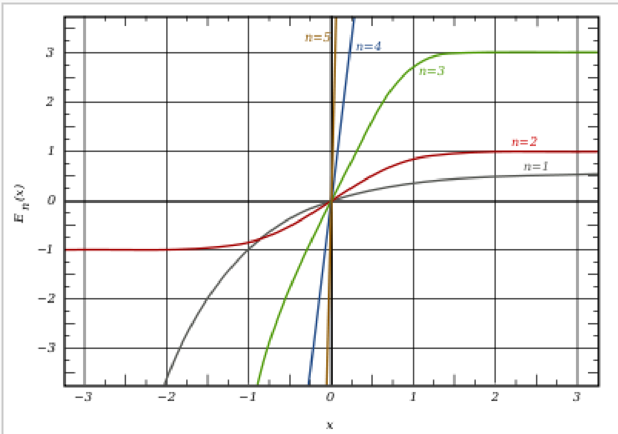
In terms of the Regularized Gamma function P and the incomplete gamma function,

$$\operatorname{erf}(x) = \operatorname{sgn}(x) P\left(\tfrac{1}{2}, x^2\right) = \frac{\operatorname{sgn}(x)}{\sqrt{\pi}}\gamma\left(\tfrac{1}{2}, x^2\right).$$

$\operatorname{sgn}(x)$ is the sign function.

Generalized error functions [edit]

Some authors discuss the more general functions:[citation needed]



Graph of generalised error functions $E_n(x)$:
grey curve: $E_1(x) = (1 - e^{-x})/\sqrt{\pi}$
red curve: $E_2(x) = \operatorname{erf}(x)$
green curve: $E_3(x)$
blue curve: $E_4(x)$
gold curve: $E_5(x)$.

$$E_n(x) = \frac{n!}{\sqrt{\pi}} \int_0^x e^{-t^n}\, dt = \frac{n!}{\sqrt{\pi}} \sum_{p=0}^{\infty} (-1)^p \frac{x^{np+1}}{(np+1)p!}.$$

Notable cases are:

- $E_0(x)$ is a straight line through the origin: $E_0(x) = \frac{x}{e\sqrt{\pi}}$
- $E_2(x)$ is the error function, erf(x).

After division by $n!$, all the E_n for odd n look similar (but not identical) to each other. Similarly, the E_n for even n look similar (but not identical) to each other after a simple division by $n!$. All generalised error functions for $n > 0$ look similar on the positive x side of the graph.

These generalised functions can equivalently be expressed for $x > 0$ using the Gamma function and incomplete Gamma function:

$$E_n(x) = \frac{\Gamma(n)\left(\Gamma\left(\frac{1}{n}\right) - \Gamma\left(\frac{1}{n}, x^n\right)\right)}{\sqrt{\pi}}, \qquad x > 0.$$

Therefore, we can define the error function in terms of the incomplete Gamma function:

$$\operatorname{erf}(x) = 1 - \frac{\Gamma\left(\frac{1}{2}, x^2\right)}{\sqrt{\pi}}.$$

Iterated integrals of the complementary error function [edit]

The iterated integrals of the complementary error function are defined by

$$\mathrm{i}^n \operatorname{erfc}(z) = \int_z^\infty \mathrm{i}^{n-1} \operatorname{erfc}(\zeta)\, \mathrm{d}\zeta.$$

They have the power series

$$\mathrm{i}^n \operatorname{erfc}(z) = \sum_{j=0}^\infty \frac{(-z)^j}{2^{n-j} j!\, \Gamma\left(1 + \frac{n-j}{2}\right)},$$

from which follow the symmetry properties

$$\mathrm{i}^{2m} \operatorname{erfc}(-z) = -\mathrm{i}^{2m} \operatorname{erfc}(z) + \sum_{q=0}^m \frac{z^{2q}}{2^{2(m-q)-1}(2q)!(m-q)!}$$

and

$$\mathrm{i}^{2m+1} \operatorname{erfc}(-z) = \mathrm{i}^{2m+1} \operatorname{erfc}(z) + \sum_{q=0}^m \frac{z^{2q+1}}{2^{2(m-q)-1}(2q+1)!(m-q)!}.$$

Implementations [edit]

- C: C99 provides the functions `double erf(double x)` and `double erfc(double x)` in the header math.h. The pairs of functions {`erff()`,`erfcf()`} and {`erfl()`,`erfcl()`} take and return values of type `float` and `long double` respectively. For `complex double` arguments, the function names `cerf` and `cerfc` are "reserved for future use"; the missing implementation is provided by the open-source project libcerf, which is based on the Faddeeva package.
- C++: C++11 provides `erf()` and `erfc()` in the header `cmath`. Both functions are overloaded to accept arguments of type `float`, `double`, and `long double`. For `complex<double>`, the Faddeeva package provides a C++ `complex<double>` implementation.
- Excel: Microsoft Excel provides the `erf`, and the `erfc` functions, nonetheless both inverse functions are not in the current library.[10]
- Fortran: The Fortran 2008 standard provides the `ERF`, `ERFC` and `ERFC_SCALED` functions to calculate the error function and its complement for real arguments. Fortran 77 implementations are available in SLATEC.
- Google search: Google's search also acts as a calculator and will evaluate "erf(...)" and "erfc(...)" for real arguments.
- Haskell: An erf package[11] exists that provides a typeclass for the error function and implementations for the native (real) floating point types.
- IDL: provides both erf and erfc for real and complex arguments.
- Java: Apache commons-math[12] provides implementations of erf and erfc for real arguments.
- Maple: Maple implements both erf and erfc for real and complex arguments.
- MathCAD provides both erf(x) and erfc(x) for real arguments.
- Mathematica: erf is implemented as Erf and Erfc in Mathematica for real and complex arguments, which are also available in Wolfram Alpha.
- Matlab provides both erf and erfc for real arguments, also via W. J. Cody's algorithm.[13]
- Maxima provides both erf and erfc for real and complex arguments.
- PARI/GP: provides erfc for real and complex arguments, via tanh-sinh quadrature plus special cases.
- Perl: erf (for real arguments, using Cody's algorithm[13]) is implemented in the Perl module Math::SpecFun
- Python: Included since version 2.7 as `math.erf()` for real arguments. For previous versions or for complex arguments, SciPy includes implementations of erf, erfc, erfi, and related functions for complex arguments in `scipy.special`.[14] A complex-argument erf is also in the arbitrary-precision arithmetic mpmath library as `mpmath.erf()`
- R: "The so-called 'error function'"[15] is not provided directly, but is detailed as an example of the normal cumulative

distribution function (`?pnorm`), which is based on W. J. Cody's rational Chebyshev approximation algorithm.[13]
- Ruby: Provides `Math.erf()` and `Math.erfc()` for real arguments.

See also [edit]

Related functions [edit]

- Gaussian integral, over the whole real line
- Gaussian function, derivative
- Dawson function, renormalized imaginary error function

In probability [edit]

- Normal distribution
- Normal cumulative distribution function, a scaled and shifted form of error function
- Probit, the inverse or quantile function of the normal CDF
- Q-function, the tail probability of the normal distribution

References [edit]

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2. ^ a b Greene, William H.; *Econometric Analysis* (fifth edition), Prentice-Hall, 1993, p. 926, fn. 11
3. ^ a b c W. J. Cody, "Algorithm 715: SPECFUN—A portable FORTRAN package of special function routines and test drivers," *ACM Trans. Math. Soft.* **19**, pp. 22–32 (1993).
4. ^ M. R. Zaghloul, "On the calculation of the Voigt line profile: a single proper integral with a damped sine integrand," *Monthly Notices of the Royal Astronomical Society* **375**, pp. 1043–1048 (2007).
5. ^ Van Zeghbroeck, Bart; *Principles of Semiconductor Devices*, University of Colorado, 2011. [1]
6. ^ Cuyt, Annie A. M.; Petersen, Vigdis B.; Verdonk, Brigitte; Waadeland, Haakon; Jones, William B. (2008). *Handbook of Continued Fractions for Special Functions*. Springer-Verlag. ISBN 978-1-4020-6948-2.
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9. ^ Numerical Recipes in Fortran 77: The Art of Scientific Computing (ISBN 0-521-43064-X), 1992, page 214, Cambridge University Press.
10. ^ These results can however be obtained using the `NormSInv` function as follows: `erf_inverse(p)` = -`NormSInv`((1 - p)/2)/`SQRT`(2); `erfc_inverse(p)` = -`NormSInv`(p/2)/`SQRT`(2). See [2].
11. ^ http://hackage.haskell.org/package/erf
12. ^ http://commons.apache.org/math
13. ^ a b c Cody, William J. (1969). "Rational Chebyshev Approximations for the Error Function". *Math. Comp.* **23** (107): 631–637. doi:10.1090/S0025-5718-1969-0247736-4.
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15. ^ R Development Core Team (25 February 2011), *R: The Normal Distribution*

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External links [edit]

- MathWorld – Erf
- Error-function numerical table and calculator

Categories: Special hypergeometric functions | Gaussian function | Functions related to probability distributions | Analytic functions