UNIT 3. MATHEMATICAL SYSTEMS

At the end of this module Students will know:

- the notions of Arithmetic, Number Theory and Discrete Mathematics;
- kinds of Arithmetic operations and their precedence;
- fields of Discrete Mathematics study;
- the notion of a theorem and types of proofs.

At the end of this module Students will understand:

- the way Arithmetic and Discrete Mathematics relate to professional spheres;
- how Arithmetic and Discrete Mathematic can be applied in solving practical tasks;
- how to comment actions when solving equations and proving theorems .

At the end of this module Students will be able to:

- express calculations when solving equations;
- use conditional sentences in theorem proofs;
- write mathematical papers, such as proofs, solutions and abstracts.

Unit 6. Arithmetic and Number Theory

Lesson 17

Whole-Class Activity

Task 1. Pre-Assessment

You are going to read questions about Arithmetic. Use your background knowledge to answer them. You may turn to Activity Pack if you need any scaffolds. You have 5 minutes to complete this task.

RATIONAL CONCERN

1. What is Arithmetic?
2. When and where did Arithmetic emerge as a branch of science?
3. What are the common arithmetic operations?

Practical Concern

1. What are the common applications of Arithmetic?
2. What was the first purpose of calculations?
3. In what professions is Arithmetic usually used?

Creative Concern

Analytical Concern

1. What role does Arithmetic play in the analysis?
2. What arithmetic operations are most often used in business?
3. What are the applications of Number Theory?

1. What interesting applications of Arithmetic do you know?
2. How is Arithmetic shown in literature?
3. How do you understand the phrase “Arithmetic of Art”?

Task 2. Reading

Read the text about Arithmetics and Number Theory. Choose the titles (A-E) to the paragraphs (1-5). You have 20 minutes for this activity.

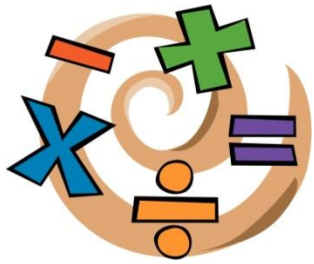
A. Applications
B. Definition
C. Number Theory
D. Integers
E. Arithmetic Operations

Queen of Math

1.

Arithmetic or **arithmetics** (from the Greek word *arithmos* “number”) is the oldest and most elementary branch of mathematics, used very popularly, for tasks ranging from simple day-to-day counting to advanced science and business calculations. It involves the study of quantity, especially as the result of operations that combine numbers. In common usage, it refers to the simpler properties when using the traditional operations of addition, subtraction, multiplication and division with smaller values of numbers. Professional mathematicians sometimes use the term (higher) arithmetic when referring to more advanced results related to number theory, but this should not be confused with elementary arithmetic.

2.



The basic arithmetic operations are addition, subtraction, multiplication and division, although this subject also includes more advanced operations, such as manipulations of percentages, square roots, exponentiation, and logarithmic functions. Arithmetic is performed according to an order of operations. Any set of objects upon which all four arithmetic operations (except division by 0) can be performed, and where these four operations obey the usual laws, is called a field.

3.

The term arithmetic also refers to number theory. Number theory (or arithmetic) is a branch of pure mathematics devoted primarily to the study of the integers, sometimes called “The Queen of Mathematics” because of its foundational place in the discipline. Number theorists study prime numbers as well as the properties of objects made out of integers (e.g., rational numbers) or defined as generalizations of the integers (e.g., algebraic integers).

4.

Integers can be considered either in themselves or as solutions to equations. Questions in number theory are often best understood through the study of analytical objects that encode properties of the integers, primes or other number-theoretic objects in some fashion (analytic number theory). One may also study real numbers in relation to rational numbers, e.g., as approximated by the latter.

5.

The number-theorist Leonard Dickson said “Thank God that number theory is unsullied by any application”. Such a view is no longer applicable to number theory. In 1974 Donald Knuth said “...virtually every theorem in elementary number theory arises in a natural, motivated way in connection with the problem of making computers do high-speed numerical calculations”. Elementary number theory is taught in discrete mathematics courses for computer scientists, and on the other hand number theory also has applications to the continuous in numerical analysis. As well as the well-known applications to cryptography, there are also applications to many other areas of mathematics.

(The text is borrowed and modified from http://en.wikipedia.org/wiki/Arithmetic as of 21st December 2013)

Task 3. Vocabulary Practice

Match the words with their definitions. You have 5 minutes for this task.

1. Addition	a. a mathematical operation that represents the operation of removing objects from a collection
2. Subtraction	b. splitting into equal parts or groups
3. Multiplication	c. a mathematical operation that represents the total amount of objects together in a collection
4. Division	d. a number which produces a specified quantity when multiplied by itself
5. Square Root	e. a quantity representing the power to which a fixed number (the base) must be raised to produce a given number
6. Exponentiation	f. the operation of raising one quantity to the power of another
7. Logarithm	g. an operation that, for positive integers, consists of adding a number (the multiplicand) to itself a certain number of times

Task 4. Vocabulary Practice

Fill in the gaps with the words in bold. You have 10 minutes to complete this task.

expression	**exponents**	**parts**	**order**	**change**
addition	**precedence**	**multiplication**	**right**	**brackets**

In mathematics and computer programming, the (1) .... of operations (sometimes called operator (2) ....) is a rule used to clarify which procedures should be performed first in a given mathematical (3) ....

Since the introduction of modern algebraic notation, (4) .... has taken precedence over addition. When exponents were first introduced in the 16th and 17th centuries, (5) .... took precedence over both addition and multiplication and could be placed only as a superscript to the (6) .... of their base. To (7) .... the order of operations, originally a vinculum (an overline or underline) was used. Today, parentheses or (8) .... are used to explicitly denote precedence by grouping (9) .... of an expression that should be evaluated first. Thus, to force (10) .... to precede multiplication, we write $(2 + 3) \times 4 = 20$, and to force addition to precede exponentiation, we write $(3 + 5)^2 = 64$.

(The text is borrowed and modified from http://en.wikipedia.org/wiki/Order_of_operations as of 21st December 2013)

Task 5. Language in Use

The ability to express common mathematical operations may come in use in many professions. However, mathematicians and theoretical analysts don't restrict themselves to simple 'pluses' and 'minuses'. Study the box explaining the ways of expressing calculations. After that, match the numbers in columns with arithmetic operations to create equations. Make up sentences to express the calculations. You have 10 minutes for this task.

Expressing Calculations

We can express mathematical calculations in many ways. Here are the most often used expressions to denote the basic arithmetic operations:

1. To express addition, we use words **"add", "plus", "sum", "total", "increase":**
   e.g. Two plus two makes four.
   The sum of seven and three is ten.
2. To express subtraction, we use **"subtract", "minus", "less", "difference", "take away", "decrease", "deduct"**:
   e.g. Six minus two makes four.
   Fourteen take away six is equal to eight.
3. We can express multiplication using **"multiply", "by", "times", "lots of"**:
   e.g. Three lots of five is fifteen.
   Nine times seven is 63.
4. To express division, we may use such words as **"divide", "go into", "quotient"**:
   e.g. 22 divided by 5 is 4, with 2 left over.
   The quotient of 40 by 5 is 8.

e.g. Two by four is eight. Nine minus one is eight.

2		**1**		**8**
30	**×**	**2**		**5**

4	**÷**	**3**		**11**
55	**+**	**4**	**=**	**54**
6	**-**	**5**		**12**
63		**6**		**21**
9		**7**		**14**

Differentiated Activity

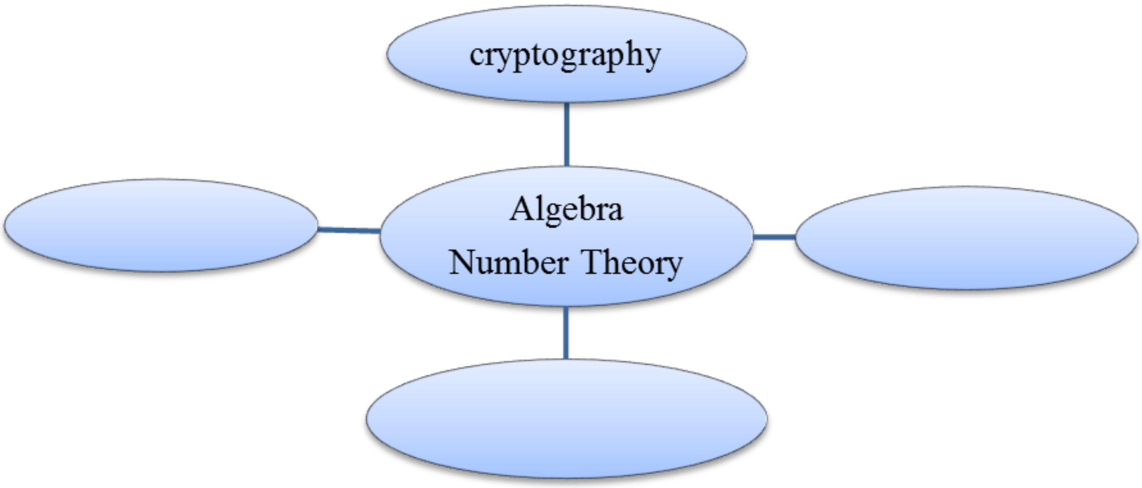
Task 6. Group activity

In groups, work out topical content of the following extract. Report your findings to the class. You have 10 minutes for this task.

Algebra and Number Theory have always been **counted** among the most beautiful mathematical **areas** with deep proofs and elegant results. However, for a long time they were not considered that **important** in view of the lack of real-life **applications**. This has dramatically **changed** with the appearance of new topics such as modern cryptography, coding theory, and wireless communication. Computational and applied mathematics depending on **deep** algebraic and number theoretic methods have become **crucial** for these and other topics of extremely high current interest. Nowadays we find applications of algebra and number theory **frequently** in our **daily** life. We mention security and error detection for internet banking, check digit systems as the bar code on **products** in a supermarket, GPS and radar, pricing options at a stock market, and eliminating noise for mobile phones as examples.

(The text is borrowed and modified from http://www.ricam.oeaw.ac.at/specsem/specsem2013/ as of 26th December 2013)

Group 1. Read the extract and create a scheme which represents the spheres of use of Algebra and Number Theory.



Group 2. Broaden your vocabulary by suggesting synonyms / antonyms to the words in bold.

Group 3. Inside your group, discuss other possible spheres of use of Algebra and Number Theory. Explain, how the knowledge of Algebra can be applied in practice.

Task 7. Listening

You are going to watch a video about solving linear equations. Choose any part of the task you feel confident to complete or do them all. You have 10 minutes for the task. To watch the video, use the following link: http://www.youtube.com/watch?v=wShnYemIr28

Part 1. Answer the questions:

1. Solving what kind of equations is explained in the video?
2. What should be done in the first place to solve such kind of linear equation?
3. What linear equation is given as an example?
4. What is the answer to this equation?

Part 2. Complete the sentences:

1. The first thing to do when solving equations with variables on both sides is to… .
2. The easiest way to avoid negatives when possible is to … .
3. It`s really important to always remember to … .
4. When the variables are on one side, follow the same steps that you follow to ...
5. Always remember to … .

Part 3. Write down the equation which was given as an example. Solve it and give your comments to each stage.

Task 8. Pair work

Using the help box from Task 5, give each other quick tasks to perform basic arithmetic calculations. You may also ask each other to explain the solution of a linear equation. Answer questions as quick as possible. You have 5 minutes for this task.

e.g. *Student A. How much is five by seven?*
Student B. Five by seven makes thirty five.

Task 9. Team Work: Brain Ring

You are going to be divided into three teams. Choose your team name and the captain. The Brain Ring competition consists of two parts:

1. You have to solve and explain the solution of one of the suggested puzzles.

2. You have to prepare the arithmetic puzzle to the rival team and solve theirs.

The team which does both parts of the Brain Ring correctly, wins.

(The puzzles are borrowed from http://www.mathsisfun.com/puzzles/sam-loyd-puzzles-index.html as of 30th December 2013)

Puzzle 1.

An old man goes to a bank with a check of $200 and asks the cashier "Give me some one-dollar bills, ten times as many twos and the balance in fives!".

What will the cashier do?

Puzzle 2.

How much does the baby weigh if the mother weighs 100 pounds more than the combined weight of the baby and the dog, and the dog weighs 60 percent less than the baby?

Puzzle 3.

The five school children in couples weigh 129 pounds, 125 pounds, 124 pounds, 123 pounds, 122 pounds, 121 pounds, 120 pound, 118 pounds, 116 pounds and 114 pounds on a weighing machine. What was the weight of each one of the five little girls if taken separately?

Home Assignment

Do Tasks 1-3 from Workbook section.

Optional Activity

Task 10. Facilitated Task

a) Read the extract and try to explain the meaning of the words in bold.
b) Make up sentences with the words in bold.

Arithmetic is the oldest and branch of mathematics which is used very popularly. The four basic arithmetic operations are **addition, subtraction, multiplication, and division**. These are the first two natural hyperoperations and their inverses.
Harder arithmetic includes working with signed numbers, **fractions**, and decimals, and taking **powers** and **roots**. Arithmetic is performed according to an **order of operations**.

The term arithmetic also refers to **number theory**. Number theory is a branch of pure mathematics devoted to the study of the **integers**. Number theorists study prime numbers as well as the properties of objects made out of integers. Elementary number theory is taught in **discrete mathematics** courses for computer scientists. Number theory also has applications to numerical analysis, **cryptography** and many other areas of mathematics.

(The text is borrowed and modified from http://en.wikipedia.org/wiki/Arithmetic as of 29th December 2013)

Task 11. Complex Task

Read the article from "Forbes" magazine about Math myopia. Try to explain the meaning of the words in bold. Formulate the problem which the article deals with. State two approaches to the problem mentioned by the author. Which conclusion does the author draw?

Math Myopia

"Ambition, Distraction, Uglification, and Derision" is how Lewis Carroll referred to addition, subtraction, multiplication and division. Although most people **resonate with this repugnance toward computation**, most would also grant its frequent necessity.

This tension underlies the latest **skirmish in the simmering Math War**. The issue is the proper place of computation and algorithms (step-by-step procedures) in the school curriculum. What, in particular, is their relation to such often neglected skills as understanding graphs, interpreting probability, modeling situations, applying mathematical concepts in other domains or estimating and comparing magnitudes?

Textbooks and curriculums that attempt **to foster the skills** mentioned above have been criticized as insufficiently rigorous. When the Department of Education endorsed some of these new curriculums as “exemplary,” a group of prominent mathematicians published a letter to Education Secretary claiming that many of the recommended books and programs neglect basic algorithms.

This might seem a **parochial controversy** were it not for the social cost of our arithmetical failings – clerks who are perplexed by discounts and sales taxes, medical personnel who have difficulty reckoning correct dosages, quality control managers who don’t understand simple statistics, voters who can’t recognize trade-offs between contrary desiderata and journalists who are sometimes oblivious to serious risks but apoplectic over trivial ones.

Although there is no real opposition between understanding concepts and mastering algorithms, extreme positions are easy to parody. Assigning 500 long-division problems to elementary school students is **a sure way to stultify them**. So is requiring older students to factor 500 polynomials in algebra class or to differentiate 500 functions in calculus class.

On the other hand, the reformist endeavor to tell stories, describe applications, play games and naturally embed mathematical insights and ideas into everyday life can also be mocked. **A mere glimmer of the idea** generally isn’t sufficient to secure numerical answers.

The proper balance depends on the student’s age and background, and the specific algorithm. **There is no royal road to mathematical education**, certainly not one capable of being reduced to a column. Despite common belief, arithmetic is not easy; nor are “higher-level” subjects necessarily difficult. Some “elementary” algorithms, such as those for dealing with fractions, may be **drudgery** if they are not presented well, but they are mathematically significant and essential to real understanding. No stories about combining parts of pies or salaries, for example, can replace the formal rules for finding 2/7 + 3/11.

We should no more be teaching our children to try to compete with $5 calculators than we should be training them **to dig ditches with hand shovels**. In arithmetic the stories and applications should set the stage and provide motivation for understanding the algorithms. The many good people on opposite sides of the Math War should **recompute their strategies**.

(The text is borrowed and modified from http://www.forbes.com/forbes/2000/0124/6502036a.html as of 29th December 2013)

WORKBOOK

Task 1. Tiered Task

Part 1. Insert the following words in the gaps. Translate the sentences into your native language.

division	**Properties**	**operations**	**discrete**
relation	**Counting**	**applied**	**logarithmic**

1. Arithmetic is used in day-to-day …………. , advanced science and business calculations.
2. Traditional arithmetic operations include addition, subtraction, multiplication and …………. .
3. Advanced arithmetic operations are manipulations of percentages, square roots, exponentiation, and ……………… functions.
4. Arithmetic is performed according to an order of …………. .
5. Number theorists study prime numbers as well as the ………… of objects made out of integers.
6. Real numbers may be studied in ………… to rational numbers.
7. Elementary number theory is taught in …………. mathematics courses for computer scientists.
8. Number theory may be …………. to cryptography.

Part 2. Fill in the gaps with the words from the Unit.

Arithmetic is a branch of mathematics usually (1) .... with the four **operations** (adding, subtracting, multiplication and division) of positive (2) .... Arithmetic will include **concepts** like (3) .... , identifying numbers and amounts, learning the **basic** mental math facts. It is the basic **day to day** math. Arithmetic is the (4) .... of math.

The (5) .... between arithmetic and algebra that **underscores** the use of letters is valid on an elementary level. Elementary arithmetic does not (6) .... symbols other than numerals and those of basic (7) .... and equality. Elementary algebra, a **step** ahead of arithmetic, does use letters for **formulating** and (8) .... problems and to **annunciate** properties of the arithmetic operations in a general form. However, in *Higher Arithmetic*, which is another **name** for (9) .... *Theory*, letters are used **extensively** as they are in the (10) .... of mathematics.

(The text is borrowed and modified from http://www.cut-the-knot.org/WhatIs/WhatIsArithmetic.shtml and http://math.about.com/od/glossaryofterms/g/The-Definition-Of-Arithmetic.htm as of 8th January 2014)

Part 3. Complete the previous part of the Task and suggest your synonyms to the words in bold.

Task 2. Tiered Task

Read the text about mathematical proofs and complete at least one part of the task.

Mathematical Proof

In mathematics, a proof is a deductive argument for a mathematical statement. Proofs are examples of deductive reasoning and are distinguished from inductive or empirical arguments; a proof must demonstrate that a statement is always true, rather than enumerate many confirmatory cases. An unproven statement that is believed true is known as a conjecture. Proofs may be viewed as aesthetic objects, admired for their mathematical beauty. There exist several methods of proofs:

Direct proof

In direct proof, the conclusion is established by logically combining the axioms, definitions, and earlier theorems. This proof uses the definition of even integers, the integer properties of closure under addition and multiplication, and distributivity.

Proof by mathematical induction

In proof by mathematical induction, a single "base case" is proved, and an "induction rule" is proved, which establishes that a certain case implies the next case. Applying the induction rule repeatedly, starting from the independently proved base case, proves many, often infinitely many, other cases. Since the base case is true, the infinity of other cases must also be true, even if all of them cannot be proved directly because of their infinite number.

Proof by contradiction

In proof by contradiction (also known as *reductio ad absurdum*, Latin for "by reduction to the absurd"), it is shown that if some statement were true, a logical contradiction occurs, hence the statement must be false.

Proof by construction

Proof by construction, or proof by example, is the construction of a concrete example with a property to show that something having that property exists. Joseph Liouville, for instance, proved the existence of transcendental numbers by constructing an explicit example. It can also be used to construct a counterexample to disprove a proposition that all elements have a certain property.

Proof by exhaustion

In proof by exhaustion, the conclusion is established by dividing it into a finite number of cases and proving each one separately. The number of cases sometimes can become very large. For example, the first proof of the four color theorem was a proof by exhaustion with 1,936 cases. This proof was controversial because the majority of the cases were checked by a computer program, not by hand. The shortest known proof of the four colour theorem as of 2011 still has over 600 cases.

(The text is borrowed and modified from http://en.wikipedia.org/wiki/Mathematical_proof as of 8th January 2014)

Part 1. Decide whether the following statements are true (T) or false (F). Justify your answer.

1. Proof is an inductive argument for a mathematical statement. □
2. Proof shall not enumerate many confirmatory cases. □
3. Direct proof uses logical combinations to establish a conclusion. □
4. In proof by mathematical induction only the base case is proved. □
5. In proof by contradiction, the statement is proved by showing that opposite statement is false. □
6. Proof by example is another name for proof by exhaustion. □
7. In proof by exhaustion the statement is divided into cases. □

Part 2. On the basis of the text, give definitions to the following terms:

Proof …………………………………………………………………………..…………..
Conjecture…………………………………………………………………………………
Direct proof……………………………………………………………………………….
Proof by induction………………………………………………………………………...
Proof by contradiction…………………………………………………………………….
Proof by construction……………………………………..……………………………….
Proof by exhaustion……………………………………………………………………….

Part 3. Complete the following table. Use additional sources of information, if necessary:

Title	Methodology	Example of application
Direct proof	*The conclusion is established by logically combining the axioms, definitions, and earlier theorems*	*Proof that the sum of two even integers is always even*
Proof by induction		
Proof by contradiction		
Proof by construction		
Proof by exhaustion		

Task 3. Internet Search

Using sites recommended by your teacher and sites that you can find yourself, try to find information about famous mathematicians. The search should be done in pairs. Be ready to report your findings to the class.