Continuum Limit & Field Theory

Continuous Limit in 2D with weights

Let φ be a complex-valued function of x, y.

2D Lin. Interpolating Lattice: $\mathbb{R}^2 \to \mathbb{R}$

Idea: $\frac{\partial}{\partial x}\varphi(x,y) = \frac{\varphi(x+\Delta x,y)-\varphi(x,y)}{\Delta x}$ exists if φ is complex-diff.

$\Rightarrow \frac{\partial}{\partial x}\varphi = \frac{\partial}{\partial y}\varphi$ is complex-diff in x for all y (Cauchy-Riemann)

$\Rightarrow$ Antiderivative function is a function $\int_{x_0}^{x}\frac{\partial}{\partial x}\varphi = \varphi(x,y)$ is complex-diff in x.

Idea: write φ in polar function $\frac{\partial\varphi}{\partial x}$ the complex integrals in $\mathbb{C}$ and complex-function $\to \int_{\mathbb{R}^2}\frac{\partial\varphi(x,y)}{\partial x}\,dx\,dy$

Partial derivative: $\frac{\partial}{\partial x}$ treat y as constant

Antiderivative: $\int_{x_0}^{x}\frac{\partial}{\partial x}\varphi = \varphi(x,y)$

Fundamental theorem: $\int_{x_0}^{x_1}\frac{\partial}{\partial x}\varphi = \varphi(x_1,y)-\varphi(x_0,y)$

(Discrete potential function from weights h(x))

Circuit: Φ the potential function (voltage at x) when $h(x)$ is weights.

$\Delta\varphi = \frac{\partial}{\partial x}\varphi = \frac{\varphi(x+\Delta x)-\varphi(x)}{\Delta x}$

Let $\Delta x = \frac{1}{h(x)}$, $\Delta x = 1/h(x)$

$\Rightarrow \Delta\varphi = h(x)\Phi$ $\Delta\varphi = h(x)\Phi$

$\Rightarrow \Delta\varphi = \frac{1}{\Delta x}\left(\frac{\varphi(x+\Delta x)}{\Delta x} - \frac{\varphi(x)}{\Delta x}\right)$

$= \frac{1}{\Delta x}\left(\frac{\varphi(x+\Delta x)}{\Delta x} - \frac{\varphi(x)}{\Delta x}\right)$

$\Rightarrow \frac{\Delta\varphi}{\Delta x} = \frac{1}{\Delta x}\left(\frac{\varphi(x+\Delta x)}{\Delta x} - \frac{\varphi(x)}{\Delta x}\right)$

This is exact matrix-vector product of tridiagonal operator with the coefficient of matrix is general weights function $h(x)$

Discrete Laplacian Operator:

write $h(x)\Delta\varphi(x) = h(x)\Phi(x)$ $\varphi = \Phi$

Then $\Phi = \Phi$ where general $\Phi = h(x)\Phi(x)$

$\Rightarrow \left(\frac{\Phi}{\Delta x} - \Phi\right) - \left(\frac{\Phi}{\Delta x} - \Phi\right) = 0$

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Current density vector

$\frac{\partial}{\partial x}\Phi = \frac{\partial}{\partial x}\left(\frac{\Phi}{\Delta x} - \Phi\right)$

$\Rightarrow \frac{\partial}{\partial x}\Phi = \frac{\partial}{\partial x}\left(\frac{\Phi}{\Delta x} - \Phi\right)$

Continuous Limit in 2D Lattice

Idea: Take the rectangular grid or in cubic cell lattice. Then we

Embedded in $\mathbb{R}^2$ (Lattice Points $\mathbb{Z}^2$)

Lattice cell in 2D (unit square)

Gradient in the right: $\frac{\partial\varphi}{\partial x} = \frac{\varphi(x+1,y)-\varphi(x,y)}{1}$

Gradient in the upper: $\frac{\partial\varphi}{\partial y} = \frac{\varphi(x,y+1)-\varphi(x,y)}{1}$

Gradient below: $\frac{\partial\varphi}{\partial x} = \frac{\varphi(x-1,y)-\varphi(x,y)}{1}$

Gradient above: $\frac{\partial\varphi}{\partial y} = \frac{\varphi(x,y-1)-\varphi(x,y)}{1}$

$\nabla\varphi(x,y) = \begin{pmatrix}\frac{\partial\varphi}{\partial x}\\ \frac{\partial\varphi}{\partial y}\end{pmatrix}$

Continuous Chain in 2D (Having 1D grid of order, continuous conductivity σ)

Voltage at (x,y): $\Phi_{x,y}$ $\xrightarrow{\text{grad}}$ $E(x,y) = -\nabla\Phi^{\text{ext}}$ Continuous voltage weighting of Maxwell grid field

$\Phi(x,y) = -\int_{x_0}^{x}E(x,y)\,dx$

Continuous current source density weighting of field

Resistivity $\rho(x,y) = \frac{1}{\sigma(x,y)}$ Ohm's law in continuous form

$E_x = \rho(x,y)\,J_x$

$E(x,y) = \rho(x,y)\,J(x,y)$

Current density and conduction flux:

$\Phi_x(x,y) = \rho(x,y)\frac{\partial J_x}{\partial x}$ (Current flow in x-direction)

where $\Phi_{x,y}$ is the current density associated with flowing in the x-direction

$\Phi_y(x,y) = \rho(x,y)\frac{\partial J_y}{\partial y}$ (Current flow in y-direction)

where $\Phi_{x,y}$ is the current density associated with flowing in the y-direction

$\Phi_x(x,y) = -\int_{x_0}^{x}\frac{\partial\Phi(x,y)}{\partial x}$ Continuous weighting of current density in (x,y) for x-direction

$\Phi_y(x,y) = -\int_{y_0}^{y}\frac{\partial\Phi(x,y)}{\partial y}$ Continuous weighting of current density in (x,y) for y-direction

Charge Law (continuity):

$\Phi_x(x,y) = -\frac{\partial\Phi(x,y)}{\partial x}$ (x-direction) $\Phi_y(x,y) = -\frac{\partial\Phi(x,y)}{\partial y}$ (y-direction)

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Charge Law (continuity)

Current balance / Conservation: $\nabla\cdot(\sigma\nabla\Phi) = 0$

$\Phi_x(x,y) = -\frac{\partial\Phi(x,y)}{\partial x}$ $\Phi_y(x,y) = -\frac{\partial\Phi(x,y)}{\partial y}$

$\Phi_x(x,y) = -\frac{\partial\Phi(x,y)}{\partial x}$ $\Phi_y(x,y) = -\frac{\partial\Phi(x,y)}{\partial y}$

$\Rightarrow \frac{\partial\Phi_x(x,y)}{\partial x} + \frac{\partial\Phi_y(x,y)}{\partial y} = 0$

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$\Rightarrow \frac{\partial}{\partial x}\left(-\frac{\partial\Phi(x,y)}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\Phi(x,y)}{\partial y}\right)