Solving Systems of ODEs:

A system of ODEs (any degree) has the general form

$$\vec{G}\left(x, y_1, \dots, y_n, \frac{dy_1}{dx}, \dots, \frac{dy_n}{dx}, \dots, \frac{d^k y_1}{dx^k}, \dots, \frac{d^k y_n}{dx^k}\right) = 0$$

A system is ordinary when we only have one independent variable x but has n functions of x $(y_1 \dots y_n)$.

Using substitutions (like $u = \frac{dy}{dx}$), any system of ODEs can be rewritten as a system of first order ODEs:

$$\left.\begin{aligned} \frac{dy_1}{dx} &= F_1(x, y_1, \dots, y_n) \\ &\vdots \\ \frac{dy_n}{dx} &= F_n(x, \dots, y_n) \end{aligned}\right\} \quad \text{In general: } \frac{d\vec{y}}{dx} = \vec{F}_{n\times 1}(x, \vec{y}_{n\times 1})$$

Vector matrix form:

Suppose we have a linear system of ODEs. I.e.

$$\frac{dy_i}{dx} = \sum a_{1i} y_i + g_1(t)$$

$$\vdots$$

Then we can write

$$\frac{d\vec{y}}{dt} = A\vec{y} + \vec{g}(t), \quad A = (a_{ij}), \quad \vec{g}(t) = \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix}$$

$\iff \mathcal{L}[\vec{y}] = \vec{g}(t)$, where $\mathcal{L}[\vec{y}] = \left(\frac{d}{dt} - A\right)\vec{y}$.

Systems of ODEs:

$f: \mathbb{R} \to \mathbb{R}^n$

Homogenous system/solution:

For homogenous system, we have $\mathcal{L}[\vec{y}] = 0$. Then the solution $\vec{y}_{GS}^{\,H}$ can be written as follows:

$$\vec{y}_{GS}^{\,H} = \sum c_i \vec{y}_i, \quad c_i \text{ being constants of integration.}$$

$\{\vec{y}_i\}$ forming a basis for the space.

Non-homogeneous system:

$$\mathcal{L}[\vec{y}] = \vec{g}(t)$$

- Step 1: $\mathcal{L}[\vec{y}_{CF}] = 0$ → Solved by considering $\det(A - \lambda I)$ to find the eigenvalues, then $\vec{y}_{CF}$ is some formula of the eigenvectors and values.
- Step 2: $\mathcal{L}[\vec{y}_{PI}] = \vec{g}(t)$ (Using Ansatz)

Scenarios:

- A has n distinct roots and is diagonalizable:

$$A = V\Lambda V^{-1}, \quad \Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$V = \begin{bmatrix} \vec{v}_1 & \dots & \vec{v}_n \end{bmatrix}, \quad A\vec{v}_i = \lambda \vec{v}_i$$

$$\Rightarrow \vec{y}_{CF} = c_1 e^{\lambda_1 t}\vec{v}_1 + \dots + c_n e^{\lambda_n t}\vec{v}_n$$

- A has repeated λ but still diagonalizable:

Suppose A; $\lambda_1 = \lambda_2 = \lambda$ but $\vec{v}_1, \vec{v}_2$ linearly independent.

Then:

$$\vec{y}_{CF} = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2$$

is the solution to the homogenous system

- A has repeated λ but is not diagonalizable:

Take for instance A a 2 by 2 matrix. So we have $\vec{v}_1 = \vec{v}_2$.

Then $W = [\vec{v}_1, \vec{w}_1]$. Here we need to solve for $\vec{w}_1$.

From this we would have:

$$\vec{y}_{CF} = \left(c_1 e^{\lambda t} + c_2 t e^{\lambda t}\right)\vec{v}_1 + c_2 e^{\lambda t}\vec{w}_1$$

$$= e^{\lambda t}\left(c_1 \vec{v}_1 + c_2(t\vec{v}_1 + \vec{w}_1)\right)$$

Suppose A; $\lambda_1 = \lambda_2 = \lambda$

and $\vec{v}_1, \vec{v}_2$ not linearly independent.

Then we try to find W,

$$W^{-1}AW = J,$$

where J is the jordan normal form.

This can solved using simultaneous eqns.

For $\lambda = \lambda_1 = \lambda_2 = \lambda_3$

$$J = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

is formed by "chaining" blocks!

More generally, for one λ repeated n times:

$$\vec{y}_{CF} = e^{\lambda t}\left(c_1\vec{v}_1 + c_2(t\vec{v}_1 + \vec{w}_1) + \dots + c_n\left(\frac{t^{n-1}}{(n-1)!}\vec{v}_1 + \dots + \vec{w}_{n-1}\right)\right)$$

where we have vectors $\vec{v}_1, \vec{w}_1 \dots \vec{w}_{n-1}$

Another example: λ 3 times,

$\vec{v}_1 \neq \vec{v}_2$

Then $\vec{y}_{CF} = c_1 e^{\lambda t}\vec{v}_1 + c_2 e^{\lambda t}\vec{v}_2 + c_3 e^{\lambda t}(t\vec{v}_2 + \vec{w}_2)$

I.e. just carry through each λ repeatedly

Stability Analysis

- Lyapunov Stability: $\forall \varepsilon > 0, \exists \delta > 0$; $\|\vec{y}(t) - \vec{y}^*\| < \delta \Rightarrow \forall t \geq 0, \|\vec{y}(t) - \vec{y}^*\| < \varepsilon$

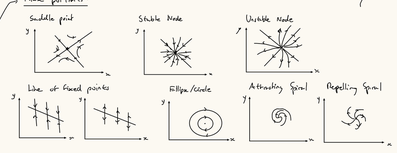
  $\vec{y}^*$ the fixed point. (pert.)

  In words, solution does not blow up.

- Asymptotic Stability (Stronger than Lyapunov): $\exists \delta > 0$; $\|\vec{y}(t) - \vec{y}^*\| < \delta \Rightarrow \lim_{t\to\infty} \|\vec{y}(t) - \vec{y}^*\| = 0$

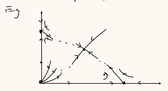
  In words, solution approaches fixed point over long time

Phase portraits:



Global Stability Analysis: Finding the fixed points and their stability for an ODE system (linear or nonlinear)

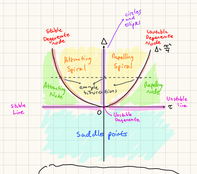
E.g.



Singularities: Points where the ODE goes undefined. E.g. $\frac{dy}{dx} = \frac{1}{x-2} \Rightarrow x = 2$ is a singularity.

Qualitative analysis using the (τ, Δ)-plane

$\tau = tr(A)$, $\Delta = \det(A)$



Bifurcations occur when we switch stability when changing parameter values. These correspond to region shifts using (τ, Δ) specifically $\Delta = \frac{\tau^2}{4}$ and $\tau = 0$ shift points.

→ For linear systems

Values:

$\Delta < 0$, any $\tau \Rightarrow$ Saddle point

$\Delta = 0$
- $\tau < 0$: Stable line of fixed points
- $\tau > 0$: Unstable line of fixed points
- $\tau = 0$: Degenerate, non-stable

$\Delta > 0$
- $\tau < 0$; $\frac{\tau^2}{4} > \Delta$: Stable nodal sink
- $\tau < 0$; $\frac{\tau^2}{4} = \Delta$: Defective nodal sink
- $\tau < 0$; $\frac{\tau^2}{4} < \Delta$: Spiral sink

  (Asymptotically stable)
- $\tau = 0$: Ellipse/circle (Stable)
- $\tau > 0$; $\frac{\tau^2}{4} < \Delta$: Spiral Source
- $\tau > 0$; $\frac{\tau^2}{4} = \Delta$: Defective nodal source
- $\tau > 0$; $\frac{\tau^2}{4} > \Delta$: Unstable nodal source

  (Asymptotically unstable)

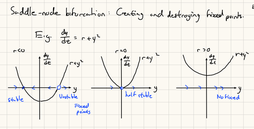
Nonlinear systems and bifurcations (1D systems)

Given $\frac{dy}{dt} = f(x, \mu)$ we construct a plane of (μ, x^*), i.e. the bifurcation diagram, where μ is the param and x^* is the fixed point.

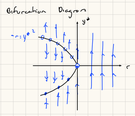
Systems in 1D never oscillate, they approach equilibrium monotonically

Saddle-node bifurcation: Creating and destroying fixed points.

E.g. $\frac{dy}{dt} = r - y^2$

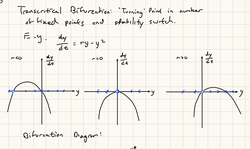


Bifurcation Diagram:

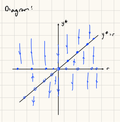


Transcritical Bifurcation: Turning point in number of fixed points and stability switch.

E.g. $\frac{dy}{dt} = ry - y^2$



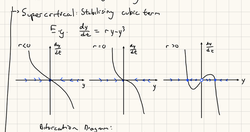
Bifurcation Diagram:



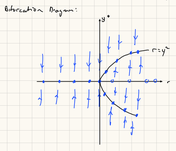
Pitchfork bifurcation: Symmetry

- Supercritical: Stabilizing cubic term

  E.g. $\frac{dy}{dt} = ry - y^3$



Bifurcation Diagram:



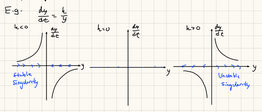
- Subcritical: Cubic term is destabilizing

  E.g. $\frac{dy}{dt} = ry + y^3$

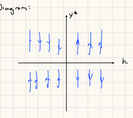
Bifurcation Diagram:

Singular points: Points where $\frac{dy}{dx}$ was undefined

E.g. $\frac{dy}{dx} = \frac{k}{y}$



Bifurcation Diagram:



Linearizing about a fixed point:

E.g. $\frac{dy}{dt} = f(y)$ (1D system)

$\eta = y - y^*$, y^* a fixed sy. } Small perturbation

$$\Rightarrow \frac{d\eta}{dt} = \frac{dy}{dt} = F(y^*) + \eta\frac{df}{dy}(y^*) + \dots \quad \text{(Taylor expansion)}$$

$$\Rightarrow \frac{d\eta}{dt} \approx \eta\frac{df}{dy}(y^*)$$

- If $\frac{df}{dy}(y^*) > 0$, then y^* is unstable w.r.t. perturbations grow.
- If $\frac{df}{dy}(y^*) < 0$, then y^* is stable w.r.t. perturbations decay.

Multivariable functions:

$f: \mathbb{R}^n \to \mathbb{R}$

Sketching Two variable functions:

Find stationary points $\vec{x}^*$:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \text{ at } \vec{x}^*$$

Classifying stationary points:

$$f(\vec{x}^* + \Delta\vec{x}) = f(\vec{x}^*) + \begin{bmatrix} \frac{\partial f}{\partial x}(\vec{x}^*) & \frac{\partial f}{\partial y}(\vec{x}^*) \end{bmatrix}\Delta\vec{x} + \frac{1}{2}\Delta\vec{x}^T\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(\vec{x}^*) & \frac{\partial^2 f}{\partial x \partial y}(\vec{x}^*) \\ \frac{\partial^2 f}{\partial x \partial y}(\vec{x}^*) & \frac{\partial^2 f}{\partial y^2}(\vec{x}^*) \end{bmatrix}\Delta\vec{x} + \dots$$

$$\Rightarrow f(\vec{x}^* + \Delta\vec{x}) - f(\vec{x}^*) = \frac{1}{2}\Delta\vec{x}^T H(\vec{x}^*)\Delta\vec{x}$$

Assuming continuity, H is symmetric, so $\exists V$:

$$V^{-1}HV = \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad V^{-1} = V^T$$

$$\Delta f = f(\vec{x}^* + \Delta\vec{x}) - f(\vec{x}^*) = \frac{1}{2}\Delta\vec{x}^T\left[V(\vec{x}^*)\Lambda(\vec{x}^*)V^T(\vec{x}^*)\right]\Delta\vec{x}$$

$$= \frac{1}{2}\Delta\vec{x}^T\Lambda(\vec{x}^*)\Delta\vec{x}, \quad \Delta\vec{x} = V^T(\vec{x}^*)\Delta\vec{x}.$$

$$= \frac{1}{2}(\Delta x_1^2\lambda_1 + \Delta x_2^2\lambda_2)$$

- If $\lambda_1, \lambda_2 \in \mathbb{R}^+$, $\Delta f > 0$ $\Rightarrow \vec{x}^*$ is a minimum
- If $\lambda_1, \lambda_2 \in \mathbb{R}^-$, $\Delta f < 0$, $\Rightarrow \vec{x}^*$ is a maximum
- If $\lambda \in \mathbb{R}^+, \lambda \in \mathbb{R}^-$, $\vec{x}^*$ is a saddle point, Δf can be '+' or '−' depending on $\Delta\vec{x}$.

Let $\Delta = \det H$, $\tau = tr(H)$

- $\Delta > 0, \tau > 0$, then $\lambda_1\lambda_2 > 0$
- $\Delta > 0, \tau < 0$, then $\lambda_1\lambda_2 < 0$
- $\Delta < 0$, then $\vec{x}^*$ saddle.

Representations:

3D curve. E.g.

Contour Plot/level curves: E.g.

E.g. $u(x,y) = (x-y)(x^2+y^2-1)$

$u(x,y) = 0 \Rightarrow x = y$, $x^2+y^2 = 1$

$$\frac{\partial u}{\partial x}(\vec{x}^*) = \frac{\partial u}{\partial y}(\vec{x}^*) = 0$$

$$\Rightarrow \begin{aligned} &x^2+y^2-1 + 2x(x-y) = 0 \\ &-(x^2+y^2-1) + 2y(x-y) = 0 \end{aligned} \Rightarrow 2x^2 = y^2 \text{ or } x^2 = -y^2$$

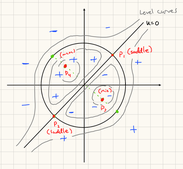
If $x^2 = y^2$, $2x^2 - 1 = 0 \Rightarrow P_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $P_2 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

If $x^2 = -y^2$, $6x^2 - 1 = 0 \Rightarrow P_3 = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$, $P_4 = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$$H(\vec{x}) = \begin{bmatrix} 6x - 2y & 6y - 2x \\ 6y - 2x & 2x - 6y \end{bmatrix}$$

$$H(P_1) = \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & -4\sqrt{2} \end{bmatrix} \quad H(P_2) = \begin{bmatrix} -4\sqrt{2} & 0 \\ 0 & 4\sqrt{2} \end{bmatrix}$$

$$H(P_3) = \begin{bmatrix} \frac{4}{\sqrt{6}} & -\frac{8}{\sqrt{6}} \\ -\frac{8}{\sqrt{6}} & \frac{4}{\sqrt{6}} \end{bmatrix} \quad H(P_4) = \begin{bmatrix} -\frac{4}{\sqrt{6}} & \frac{8}{\sqrt{6}} \\ \frac{8}{\sqrt{6}} & -\frac{4}{\sqrt{6}} \end{bmatrix}$$



$\Delta_1 < 0$, $\Delta_2 < 0$

$\Delta_3 > 0$, $\tau_3 > 0$

$\Delta_4 > 0$, $\tau_4 < 0$

Limits:

$$\lim_{\vec{x} \to \vec{x}^*} f(\vec{x}) = C$$

$\forall \varepsilon > 0\ \exists \delta > 0$; $0 < \|\vec{x} - \vec{x}^*\| < \delta \Rightarrow |f(\vec{x}) - C| < \varepsilon$

Continuity:

$f(\vec{x})$ is continuous at $\vec{x}^*$ iff $\lim_{\vec{x} \to \vec{x}^*} f(\vec{x}) = f(\vec{x}^*)$

Partial Derivatives:

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_n)}{h}$$

E.g.

$g_1(x,y) = \left(\frac{\partial f}{\partial x}\right)_y$, $g_2(x,y) = \left(\frac{\partial f}{\partial y}\right)_x$, subscript denoting constant

$$\left(\frac{\partial g_1}{\partial x}\right)_y = \frac{\partial^2 f}{\partial x^2}, \quad \left(\frac{\partial g_1}{\partial y}\right)_x = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\left(\frac{\partial g_2}{\partial x}\right)_y = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \left(\frac{\partial g_2}{\partial y}\right)_x = \frac{\partial^2 f}{\partial y^2}$$

Symmetry of mixed derivatives for continuous f. (Schwarz's Theorem)

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Total Differentiation on several variables:

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x,y)$$

$$= \left(\frac{f(x + \Delta x, y + \Delta y) - f(x + \Delta x, y)}{\Delta y}\right)\Delta y + \left(\frac{f(x + \Delta x, y) - f(x,y)}{\Delta x}\right)\Delta x$$

$\Delta x, \Delta y \to 0$,

$$df = \lim_{\Delta x, \Delta y \to 0} \Delta f = \left(\frac{\partial f}{\partial y}\right)_x dy + \left(\frac{\partial f}{\partial x}\right)_y dx$$

In general: $df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} x_i$

Chain rule for multivariate:

E.g. $u = u(x,y)$

$$du = \left(\frac{\partial u}{\partial x}\right)_y dx + \left(\frac{\partial u}{\partial y}\right)_x dy$$

$$= \left(\frac{\partial u}{\partial x}\right)_y \frac{dx}{dt}dt + \left(\frac{\partial u}{\partial y}\right)_x \frac{dy}{dt}dt$$

$$\Rightarrow \frac{du}{dt} = \left(\frac{\partial u}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial u}{\partial y}\right)_x \frac{dy}{dt}$$

In general: $\frac{du}{dt} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)\frac{dx_i}{dt}$

Very Important

Implicit form:

E.g. Take $F(x,y,z) = 0$.

① — $dF = \left(\frac{\partial F}{\partial x}\right)_{y,z} dx + \left(\frac{\partial F}{\partial y}\right)_{x,z} dy + \left(\frac{\partial F}{\partial z}\right)_{x,y} dz = 0$ (Implicit)

② — $z = z(x,y) \Rightarrow dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$ (Explicit)

$$\Rightarrow \left(\frac{\partial F}{\partial x}\right)_{y,z} dx + \left(\frac{\partial F}{\partial y}\right)_{x,z} dy + \left(\frac{\partial F}{\partial z}\right)_{x,y}\left(\left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy\right) = 0$$

$$\left(\left(\frac{\partial F}{\partial x}\right)_{y,z} + \left(\frac{\partial F}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_y\right) dx + \left(\left(\frac{\partial F}{\partial y}\right)_{x,z} + \left(\frac{\partial F}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial y}\right)_x\right) dy = 0$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}\right)_y = -\frac{\left(\frac{\partial F}{\partial x}\right)_{y,z}}{\left(\frac{\partial F}{\partial z}\right)_{x,y}} \qquad \left(\frac{\partial z}{\partial y}\right)_x = -\frac{\left(\frac{\partial F}{\partial y}\right)_{x,z}}{\left(\frac{\partial F}{\partial z}\right)_{x,y}}$$

Relationship between derivatives of implicit and explicit forms.

Important for solving partials when given implicit.

Taylor Expansion of multivariate:

E.g. Take $f(\vec{x} + \Delta\vec{x}) = f(x + \Delta x, y + \Delta y)$

Then

$$\begin{aligned} f &= f(x_0, y_0 + \Delta y) + \left(\frac{\partial f}{\partial x}\right)_{x_0, y_0 + \Delta y}\Delta x + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{x_0, y_0 + \Delta y}\Delta x^2 + \dots \\ &= f(x_0, y_0) + \left(\frac{\partial f}{\partial y}\right)_{\vec{x}_0}\Delta y + \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)_{\vec{x}_0}(\Delta y)^2 + \dots \quad \text{Taylor expanded } f(x_0, y_0 + \Delta y) \\ &\quad + \left(\Delta x\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\vec{x}_0} + \frac{\partial^2 f}{\partial x \partial y}\Delta y + \frac{1}{2}\left(\frac{\partial^3 f}{\partial x \partial y^2}\right)(\Delta y)^2 + \dots\right) \quad \text{Taylor expanded } \left(\frac{\partial f}{\partial x}\right)_{x_0, y_0 + \Delta y}\Delta x \\ &\quad + \left(\frac{1}{2}\Delta x^2\left(\left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_{\vec{x}_0} + \left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)\Delta y + \frac{1}{2}\left(\frac{\partial^5 f}{\partial x^2 \partial y^3}\right)(\Delta y)^2 + \dots\right)\right) \quad \text{Taylor expanded } \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{x_0, y_0 + \Delta y}\Delta x^2 \\ &\quad + \dots \end{aligned}$$

$$f(x + \Delta x, y + \Delta y) = f(x,y) + \left(\Delta x f_x(x,y) + \Delta y f_y(x,y)\right) + \frac{1}{2}\left((\Delta x)^2 f_{xx}(x,y) + 2\Delta x \Delta y f_{xy}(x,y) + (\Delta y)^2 f_{yy}(x,y)\right) + \frac{1}{3!}\left((\Delta x)^3 f_{xxx}(x,y) + \dots + (\Delta y)^3 f_{yyy}(x,y)\right) + \dots$$

Gradient of f at $\vec{x}_0$: $\vec{\nabla} f_{\vec{x}_0} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}_{\vec{x}_0}$

Hessian Matrix: $H_{ij}(\vec{x}_0) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{\vec{x}_0}$

$\Rightarrow H_{ii}(\vec{x}_0) = \left(\frac{\partial^2 f}{\partial x_i^2}\right)_{\vec{x}_0}$

General Taylor expansion of n dimensions.

$$f(\vec{x}_0 + \Delta\vec{x}) = \underbrace{f(\vec{x}_0) + \vec{\nabla} f(\vec{x}_0)^T \cdot \Delta\vec{x} + \frac{1}{2}\Delta\vec{x}^T H(\vec{x}_0)\Delta\vec{x}}_{\text{second order}} + \dots$$

- E.g. $A(x,y) = xy$

$$A(\vec{x}_0 + \Delta\vec{x}) = A(\vec{x}_0) + \begin{bmatrix} y_0 & x_0 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \Delta x & \Delta y \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} + \dots$$

$$= x_0 y_0 + y_0 \Delta x + x_0 \Delta y + \Delta x \Delta y + \dots$$

- E.g. (Error Analysis)

$h(r, \theta) = r + r\sin\theta$

$r = r_0 + \Delta r$ $\quad \theta = \theta_0 + \Delta\theta$ $\quad$ Find Δh.

$$h(r, \theta) = h(r_0 + \Delta r, \theta_0 + \Delta\theta)$$

$$= h(\vec{x}_0) + \vec{\nabla} h_{\vec{x}_0}^T \cdot \Delta\vec{x} + \frac{1}{2}\Delta x^T H(\vec{x}_0)\Delta x + \dots$$

$$= h(r_0, \theta_0) + \begin{bmatrix} \left(\frac{\partial h}{\partial r}\right)_{\vec{x}_0} & \left(\frac{\partial h}{\partial \theta}\right)_{\vec{x}_0} \end{bmatrix}\begin{bmatrix} \Delta r \\ \Delta\theta \end{bmatrix} + \dots$$

$$= h(r_0, \theta_0) + \left(\frac{\partial h}{\partial r}\right)_{\vec{x}_0}\Delta r + \left(\frac{\partial h}{\partial \theta}\right)_{\vec{x}_0}\Delta\theta + \dots$$

$$|\Delta h| = |1 - \sin\theta_0||\Delta r| + |r_0\cos\theta_0||\Delta\theta| + \dots$$

$$\approx |1 - \sin\theta_0||\Delta r| + |r_0\cos\theta_0||\Delta\theta| \quad (\text{max error in } h)$$

$$\left|\frac{\Delta h}{h(\vec{x}_0)}\right| \approx \left|\frac{\Delta r}{r_0}\right| + \left|\frac{2\Delta\theta}{\sin 2\theta_0}\right| \quad (\text{relative/percentage error})$$

non uncertainty given multivariate

Change of Coordinates:

E.g. Polar coords: $x = x(r, \theta)$ $\quad y = y(r, \theta)$

where $r = \sqrt{x^2 + y^2}$ $\quad \theta = \arctan\left(\frac{y}{x}\right)$

or $r\cos\theta$ $\quad y = r\sin\theta$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta$$

$$\Rightarrow \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}\begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

(Jacobian) — Let $J = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$

$$dr = \frac{\partial r}{\partial x}dx + \frac{\partial r}{\partial y}dy$$

$$d\theta = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy$$

$$\begin{bmatrix} dr \\ d\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}$$

(Jacobian) — Let $K = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$

$|K| = J^{-1}$

E.g. Approximation (cartesian to polar)

$$ds^2 = dx^2 + dy^2 \quad \text{(length of a curve)}$$

$$\vec{ds} = J\vec{dr}$$

$$\Rightarrow ds^2 = dx^2 + dy^2$$

$$= \begin{bmatrix} dx & dy \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix}$$

$$= \begin{bmatrix} dr & d\theta \end{bmatrix} J^T J \begin{bmatrix} dr \\ d\theta \end{bmatrix}$$

$$ds^2 = dr^2 + r^2(d\theta)^2$$

E.g. Area (cartesian to polar)

$dA = dx\,dy$

$$dA = \|\vec{dx} \times \vec{dy}\|$$

$$= \|\vec{J}dx \times \vec{J}dy\|$$

$$= \left\|\det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ dx & 0 & 0 \\ 0 & dy & 0 \end{bmatrix}\right\|$$

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = J\begin{bmatrix} du \\ dv \end{bmatrix}$$

Transformed area

$$dA' = \|\vec{du} \times \vec{dv}\| = |\det J|\,du\,dv$$

area in new coordinates

$dA \neq dA'$

area scale

$\therefore u = r, v = \theta$,

$$dA' = |\det J|\,dr\,d\theta$$

$$= r\,dr\,d\theta$$

Generalization:

$$d\vec{x}_{n\times 1} = J_{n\times n}\,d\vec{u}_{n\times 1}$$

$$dV = \prod_{i=1}^{n} dx_i$$

$$dV' = |\det J|\prod_{i=1}^{n} du_i$$

Applications:

PDEs: $f(\vec{x}), \vec{x} \in \mathbb{R}^n$;

$$f\left(x_1, \dots, x_n, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}, \frac{\partial^2 f}{\partial x_1 x_2}, \dots\right) = 0$$

- E.g. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Implicit)

  $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial y^2} = 0$

- Change of coordinates for PDEs. → E.g.

$u(x,y) \iff u(r, \theta)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$$

$$= \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right)dr + \left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right)d\theta$$

$$= \frac{\partial u}{\partial r}dr + \frac{\partial u}{\partial \theta}d\theta$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix} = J^T\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}, \quad J = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix}$$

E.g.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial}{\partial x}u = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)u$$

$$\frac{\partial^2}{\partial x^2}u = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)^2 u$$

$$= \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial \theta \partial r} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\sin\theta\cos\theta}{r}\frac{\partial u}{\partial \theta}$$

$$= \cos^2\theta\frac{\partial^2 u}{\partial r^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial u}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Similarly

$$\frac{\partial^2}{\partial y^2}u = \sin^2\theta\frac{\partial^2 u}{\partial r^2} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2 u}{\partial r \partial \theta} + \frac{\cos^2\theta