Calculus 2

- Fourier Series:
  - $f(x) = \sum_{n=1}^{\infty} a_n e^{inx}$ $\forall x \in \mathbb{C}$
  - $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$ $\quad n = 0, \pm 1, \pm 2, \dots$
- Fourier Transform: Suppose we take Limit of the fourier series of $f(x)$. Then
  - $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$
  - where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$ (Fourier Transform)
  - $\hat{f}(\xi)$ is the fourier transform of $f(x)$.
- $\mathcal{F}^{-1}(\hat{f}(\xi)) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$ (The inverse fourier transform)

(Analogous)
- $\sum n \Rightarrow \int d\xi$
- $\sum c_n \Rightarrow \frac{1}{2\pi}\hat{f}(\xi)$
- $\sum e^{inx} \Rightarrow e^{ix\xi}$

- Fourier transform properties:
  - Linearity:
    - $\mathcal{F}[f(x) + g(x)] = \hat{f}(\xi) + \hat{g}(\xi)$
    - $\mathcal{F}[\lambda f(x)] = \lambda \hat{f}(\xi)$
  - If $f(x) \to \hat{f}(\xi)$
    - Then $f(\lambda x) \to \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$
      - Let $f(x) = \int_{-\infty}^{\infty} f(\lambda x)\, e^{-ix\xi}\, dx$ (i.e.)
      - $= \int_{-\infty}^{\infty} f(\lambda x)\, e^{-i\lambda x \frac{\xi}{\lambda}}\, dx$
      - $= \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$
    - $f(x) \to \hat{f}(\xi) \Rightarrow f(x-a) \to e^{-ia\xi}\hat{f}(\xi)$
      - Let $f(x) = \int_{-\infty}^{\infty} f(x-a)\, e^{-ix\xi}\, dx$
      - $= \int_{-\infty}^{\infty} f(x-a)\, e^{-i(x-a)\xi}\, e^{-ia\xi}\, dx$
      - $= e^{-ia\xi}\int_{-\infty}^{\infty} f(x-a)\, e^{-i(x-a)\xi}\, dx$
      - $= e^{-ia\xi}\hat{f}(\xi)$
  - Differentiation:
    - $\mathcal{F}[f'(x)] = i\xi\hat{f}(\xi)$
      - Proof: use IBP
      - $\mathcal{F}[f'(x)] = \int_{-\infty}^{\infty} f'(x)\, e^{-ix\xi}\, dx$
      - $= \int_{-\infty}^{\infty} f(x)\, \frac{d}{dx}\left(e^{-ix\xi}\right)\, dx$
      - $= \int_{-\infty}^{\infty} f(x)\, (-i\xi)\, e^{-ix\xi}\, dx$
      - $= (-i\xi)\int_{-\infty}^{\infty} f(x)\, e^{-ix\xi}\, dx$
      - $= (-i\xi)\hat{f}(\xi)$
    - $\mathcal{F}[f''(x)] = (i\xi)^2\hat{f}(\xi)$
      - Proof:
        - $\mathcal{F}[f''(x)] = \int_{-\infty}^{\infty} f''(x)\, e^{-ix\xi}\, dx$
        - $= \int_{-\infty}^{\infty} f'(x)\, \frac{d}{dx}\left(e^{-ix\xi}\right)\, dx$
        - $= (-i\xi)\int_{-\infty}^{\infty} f'(x)\, e^{-ix\xi}\, dx$
        - $= (-i\xi)\mathcal{F}[f'(x)] = (-i\xi)^2\hat{f}(\xi)$
  - Convolution properties:
    - $\mathcal{F}_c(f(x)) := f(\xi) = \sqrt{\frac{2}{\pi}}\hat{f}(\xi)$ Back by parts
    - $\mathcal{F}_s(f(x)) := i\sqrt{\frac{2}{\pi}}\hat{f}(\xi)$
    - $\mathcal{F}_c(f''(x)) := -\xi^2\hat{f}_c(\xi)$
    - $\mathcal{F}_s(f''(x)) := i\xi^3\hat{f}_s(\xi)$
  - Suppose $f(x)$ complex-valued and $f(x)^* =$ conjugate.
    - $\mathcal{F}(f(x)^*) = [\hat{f}(-\xi)]^*$
    - Proof:
      - $\hat{f}(-\xi) := \int_{-\infty}^{\infty} f(x)\, e^{-ix(-\xi)}\, dx$
      - $[\hat{f}(-\xi)]^* := \int_{-\infty}^{\infty} [f(x)\, e^{-ix(-\xi)}]^*\, dx$
      - $= \mathcal{F}[f(x)^*]$

- Fourier transform properties:
  - Symmetry:
    - $\mathcal{F}[f(x)] = 2\pi\hat{f}(-\xi)$
      - Proof: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$
      - $\Rightarrow 2\pi f(-x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{-ix\xi}\, d\xi$
      - $\therefore \mathcal{F}[f(x)] = 2\pi\hat{f}(-\xi)$
  - Shifted function:
    - $\mathcal{F}[f(x-a)] = e^{-ia\xi}\hat{f}(\xi)$
      - Proof: $\mathcal{F}[f(x-a)] = \hat{f}(\xi)$
    - $\mathcal{F}[e^{iax}f(x)] = \hat{f}(\xi - a)$
  - Scaling:
    - $\mathcal{F}[f(\lambda x)] = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$
      - Proof: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$
      - $\Rightarrow f(\lambda x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{i\lambda x\xi}\, d\xi$
      - $\Rightarrow f(\lambda x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\left(\frac{\xi}{\lambda}\right)\, e^{ix\xi}\, d\xi$
      - $\therefore \mathcal{F}[f(\lambda x)] = \frac{1}{\lambda}\hat{f}\left(\frac{\xi}{\lambda}\right)$
  - Differentiation:
    - $\mathcal{F}[f'(x)] = i\xi\hat{f}(\xi)$
      - Proof: $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{ix\xi}\, d\xi$
      - $\Rightarrow f'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, i\xi\, e^{ix\xi}\, d\xi$
      - $\therefore \mathcal{F}[f'(x)] = i\xi\hat{f}(\xi)$
    - $\mathcal{F}[f''(x)] = (i\xi)^2\hat{f}(\xi)$
      - Proof:
        - $\mathcal{F}[f''(x)] = \int_{-\infty}^{\infty} f''(x)\, e^{-ix\xi}\, dx$
        - $= \int_{-\infty}^{\infty} f'(x)\, \frac{d}{dx}\left(e^{-ix\xi}\right)\, dx$
        - $= (-i\xi)\int_{-\infty}^{\infty} f'(x)\, e^{-ix\xi}\, dx$
        - $= (-i\xi)\mathcal{F}[f'(x)] = (-i\xi)^2\hat{f}(\xi)$
  - Convolution properties:
    - $\mathcal{F}_c(f(x)) := f(\xi) = \sqrt{\frac{2}{\pi}}\hat{f}(\xi)$ Back by parts
    - $\mathcal{F}_s(f(x)) := i\sqrt{\frac{2}{\pi}}\hat{f}(\xi)$
    - $\mathcal{F}_c(f''(x)) := -\xi^2\hat{f}_c(\xi)$
    - $\mathcal{F}_s(f''(x)) := i\xi^3\hat{f}_s(\xi)$

Parseval's Identity (Fourier transform)
- $\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$
- Proof:
  - $\mathcal{F}[f(x)]^2 = \int_{-\infty}^{\infty} f(x)^2$
  - $\mathcal{F}\left[f(x)\, f(x)\right] = \int_{-\infty}^{\infty} f(x)\, f(x)\, e^{-ix\xi}\, dx$
  - $\mathcal{F}[f(x)\, f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{f}(\xi)\, d\xi$
  - $\mathcal{F}[f(x)\, f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{f}(-\xi)\, d\xi$
  - $\mathcal{F}[f(x)\, f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, \hat{f}(\xi)^*\, d\xi$
  - $\mathcal{F}[f(x)\, f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$
  - $\mathcal{F}[f(x)\, f(x)] = \int_{-\infty}^{\infty} |f(x)|^2\, dx$
  - $\therefore \int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$

Energy Theorem:
- Analogous to Parseval's: $\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |f(x)|^2\, dx$
- Proof:
  - $\mathcal{F}\left[f(x)\right]^2 = \int_{-\infty}^{\infty} f(x)^2$
  - $\mathcal{F}\left[f(x)\right]^2 = \int_{-\infty}^{\infty} f(x)\, f(x)$
  - $\mathcal{F}\left[f(x)\right]^2 = \int_{-\infty}^{\infty} f(x)\, f(x)^*$ for real f
  - $\Rightarrow \mathcal{F}\left[f(x)\, f(x)^*\right] = \mathcal{F}[f(x)]\, \mathcal{F}[f(x)]^*$
  - $\Rightarrow \mathcal{F}[f(x)\, f(x)^*] = \hat{f}(\xi)\, \hat{f}(\xi)^*$
  - $\Rightarrow \mathcal{F}[f(x)\, f(x)^*] = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$
  - $\Rightarrow \int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$ $\quad$ (i.e. $\mathcal{F}^{-1}[\hat{f}(\xi)]^2$)
  - $\therefore \int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2\, d\xi$

Fourier transform of $\delta(x)$:
- $\mathcal{F}[\delta(x)] = \int_{-\infty}^{\infty} \delta(x)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, e^{-i0\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, 1\, dx$
- $= 1$
- $\therefore \mathcal{F}^{-1}(1) = \delta(x)$

Fourier of $\delta(x-a)$:
- $\mathcal{F}[\delta(x-a)] = \int_{-\infty}^{\infty} \delta(x-a)\, e^{-ix\xi}\, dx$
- $= e^{-ia\xi}$

Fourier of $\delta(x-a) + \delta(x+a)$:
- $\mathcal{F}[\delta(x-a) + \delta(x+a)] = \int_{-\infty}^{\infty} [\delta(x-a) + \delta(x+a)]\, e^{-ix\xi}\, dx$
- $= e^{-ia\xi} + e^{ia\xi}$
- $= 2\cos(a\xi)$

Fourier of $\delta(x)$ $\cos(ax)$:
- $\mathcal{F}[\delta(x)\cos(ax)] = \int_{-\infty}^{\infty} \delta(x)\cos(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{e^{iax} + e^{-iax}}{2}\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\left(e^{ia(\xi-a)} + e^{-ia(\xi+a)}\right)\, dx$
- $= \frac{1}{2}\left(e^{ia(\xi-a)} + e^{-ia(\xi+a)}\right)$
- $= \frac{1}{2}\left(e^{i\xi a} + e^{-i\xi a}\right)$
- $= \cos(\xi a)$

Fourier of $\delta(x)$ $\sin(ax)$:
- $\mathcal{F}[\delta(x)\sin(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{e^{iax} - e^{-iax}}{2i}\, e^{-ix\xi}\, dx$
- $= \frac{1}{2i}\int_{-\infty}^{\infty} \delta(x)\left(e^{ia(\xi-a)} - e^{-ia(\xi+a)}\right)\, dx$
- $= \frac{1}{2i}\left(e^{i\xi a} - e^{-i\xi a}\right)$
- $= \sin(\xi a)$

Fourier of $\delta(x)$ $\cos^2(ax)$:
- $\mathcal{F}[\delta(x)\cos^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\cos^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{1 + \cos(2ax)}{2}\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\left(1 + \cos(2ax)\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\, e^{-ix\xi}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\cos(2ax)\, e^{-ix\xi}\, dx$
- $= \frac{1}{2} + \frac{1}{2}\cos(\xi a)$

Fourier of $\delta(x)$ $\sin^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{1 - \cos(2ax)}{2}\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\left(1 - \cos(2ax)\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\, e^{-ix\xi}\, dx - \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\cos(2ax)\, e^{-ix\xi}\, dx$
- $= \frac{1}{2} - \frac{1}{2}\cos(\xi a)$

Fourier of $\delta(x)$ $\sin(ax)\cos(ax)$:
- $\mathcal{F}[\delta(x)\sin(ax)\cos(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin(ax)\cos(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{\sin(2ax)}{2}\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\sin(2ax)\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\, \frac{e^{i2ax} - e^{-i2ax}}{2i}\, e^{-ix\xi}\, dx$
- $= \frac{1}{4i}\int_{-\infty}^{\infty} \delta(x)\left(e^{ia(\xi-2a)} - e^{-ia(\xi+2a)}\right)\, dx$
- $= \frac{1}{4i}\left(e^{i\xi a} - e^{-i\xi a}\right)$
- $= \frac{1}{2i}\sin(\xi a)$

Fourier of $\delta(x)$ $\sin^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{1 - \cos(2ax)}{2}\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\left(1 - \cos(2ax)\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\, e^{-ix\xi}\, dx - \frac{1}{2}\int_{-\infty}^{\infty} \delta(x)\cos(2ax)\, e^{-ix\xi}\, dx$
- $= \frac{1}{2} - \frac{1}{2}\cos(\xi a)$

Fourier of $\delta(x)$ $\cos^3(ax)$:
- $\mathcal{F}[\delta(x)\cos^3(ax)] = \int_{-\infty}^{\infty} \delta(x)\cos^3(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{iax} + e^{-iax}}{2}\right)^3\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} + 3e^{iax} + 3e^{-iax} + e^{-3iax}}{8}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{8}\int_{-\infty}^{\infty} \delta(x)\left(e^{3iax} + 3e^{iax} + 3e^{-iax} + e^{-3iax}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{3iax}\, e^{-ix\xi}\, dx + \frac{3}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{iax}\, e^{-ix\xi}\, dx + \frac{3}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{-iax}\, e^{-ix\xi}\, dx + \frac{1}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{-3iax}\, e^{-ix\xi}\, dx$
- $= \frac{1}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{i(3a-\xi)x}\, dx + \frac{3}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{i(a-\xi)x}\, dx + \frac{3}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{i(-a-\xi)x}\, dx + \frac{1}{8}\int_{-\infty}^{\infty} \delta(x)\, e^{i(-3a-\xi)x}\, dx$
- $= \frac{1}{8}\, e^{i(3a-\xi)0} + \frac{3}{8}\, e^{i(a-\xi)0} + \frac{3}{8}\, e^{i(-a-\xi)0} + \frac{1}{8}\, e^{i(-3a-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{8}\, e^{i(3a-\xi)} + \frac{3}{8}\, e^{i(a-\xi)} + \frac{3}{8}\, e^{i(-a-\xi)} + \frac{1}{8}\, e^{i(-3a-\xi)}$

Fourier of $\delta(x)$ $\sin^3(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{iax} - e^{-iax}}{2i}\right)^3\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{3iax}\, e^{-ix\xi}\, dx - \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{iax}\, e^{-ix\xi}\, dx + \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{-iax}\, e^{-ix\xi}\, dx - \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{-3iax}\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\, e^{i(3a-\xi)0} - \frac{3}{-8i}\, e^{i(a-\xi)0} + \frac{3}{-8i}\, e^{i(-a-\xi)0} - \frac{1}{-8i}\, e^{i(-3a-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{-8i}\, e^{i(3a-\xi)} - \frac{3}{-8i}\, e^{i(a-\xi)} + \frac{3}{-8i}\, e^{i(-a-\xi)} - \frac{1}{-8i}\, e^{i(-3a-\xi)}$

Fourier of $\delta(x)$ $\sin^2(ax)\cos^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^2(ax)\cos^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^2(ax)\cos^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{1 - \cos(2ax)}{2}\, \frac{1 + \cos(2ax)}{2}\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\left(1 - \cos(2ax)\right)\left(1 + \cos(2ax)\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\left(1 - \cos^2(2ax)\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\, e^{-ix\xi}\, dx - \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\cos^2(2ax)\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\, e^{-i\xi 0} - \frac{1}{4}\, e^{-i\xi 0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{4}\, e^{-i\xi 0} - \frac{1}{4}\, e^{-i\xi 0}$

Fourier of $\delta(x)$ $\sin^2(ax)\cos^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^2(ax)\cos^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^2(ax)\cos^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\, \frac{1 - \cos(2ax)}{2}\, \frac{1 + \cos(2ax)}{2}\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\left(1 - \cos(2ax)\right)\left(1 + \cos(2ax)\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\left(1 - \cos^2(2ax)\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\, e^{-ix\xi}\, dx - \frac{1}{4}\int_{-\infty}^{\infty} \delta(x)\cos^2(2ax)\, e^{-ix\xi}\, dx$
- $= \frac{1}{4}\, e^{-i\xi 0} - \frac{1}{4}\, e^{-i\xi 0}$

Fourier of $\delta(x)$ $\sin^3(ax)\cos^3(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)\cos^3(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\cos^3(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{iax} - e^{-iax}}{2i}\right)^3\left(\frac{e^{iax} + e^{-iax}}{2}\right)^3\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{3iax} + 3e^{iax} + 3e^{-iax} + e^{-3iax}}{8}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{6iax}\, e^{-ix\xi}\, dx - \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{4iax}\, e^{-ix\xi}\, dx + \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{2iax}\, e^{-ix\xi}\, dx - \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{0}\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\, e^{i(6a-\xi)0} - \frac{3}{-8i}\, e^{i(4a-\xi)0} + \frac{3}{-8i}\, e^{i(2a-\xi)0} - \frac{1}{-8i}\, e^{i(0-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{-8i}\, e^{i(6a-\xi)} - \frac{3}{-8i}\, e^{i(4a-\xi)} + \frac{3}{-8i}\, e^{i(2a-\xi)} - \frac{1}{-8i}\, e^{i(0-\xi)}$

Fourier of $\delta(x)$ $\sin^3(ax)\cos^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)\cos^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\cos^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{iax} - e^{-iax}}{2i}\right)^3\left(\frac{e^{iax} + e^{-iax}}{2}\right)^2\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{2iax} + 2e^{0} + e^{-2iax}}{4}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{3iax}\, e^{-ix\xi}\, dx - \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{iax}\, e^{-ix\xi}\, dx + \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{-iax}\, e^{-ix\xi}\, dx - \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{-3iax}\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\, e^{i(3a-\xi)0} - \frac{3}{-8i}\, e^{i(a-\xi)0} + \frac{3}{-8i}\, e^{i(-a-\xi)0} - \frac{1}{-8i}\, e^{i(-3a-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{-8i}\, e^{i(3a-\xi)} - \frac{3}{-8i}\, e^{i(a-\xi)} + \frac{3}{-8i}\, e^{i(-a-\xi)} - \frac{1}{-8i}\, e^{i(-3a-\xi)}$

Fourier of $\delta(x)$ $\sin^3(ax)\sin^3(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)\sin^3(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\sin^3(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{iax} - e^{-iax}}{2i}\right)^3\left(\frac{e^{iax} - e^{-iax}}{2i}\right)^3\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{6iax}\, e^{-ix\xi}\, dx - \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{4iax}\, e^{-ix\xi}\, dx + \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{2iax}\, e^{-ix\xi}\, dx - \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{0}\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\, e^{i(6a-\xi)0} - \frac{3}{-8i}\, e^{i(4a-\xi)0} + \frac{3}{-8i}\, e^{i(2a-\xi)0} - \frac{1}{-8i}\, e^{i(0-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{-8i}\, e^{i(6a-\xi)} - \frac{3}{-8i}\, e^{i(4a-\xi)} + \frac{3}{-8i}\, e^{i(2a-\xi)} - \frac{1}{-8i}\, e^{i(0-\xi)}$

Fourier of $\delta(x)$ $\sin^3(ax)\sin^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)\sin^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\sin^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{iax} - e^{-iax}}{2i}\right)^3\left(\frac{e^{iax} - e^{-iax}}{2i}\right)^2\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{2iax} - 2e^{0} + e^{-2iax}}{-4}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{3iax}\, e^{-ix\xi}\, dx - \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{iax}\, e^{-ix\xi}\, dx + \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{-iax}\, e^{-ix\xi}\, dx - \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{-3iax}\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\, e^{i(3a-\xi)0} - \frac{3}{-8i}\, e^{i(a-\xi)0} + \frac{3}{-8i}\, e^{i(-a-\xi)0} - \frac{1}{-8i}\, e^{i(-3a-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{-8i}\, e^{i(3a-\xi)} - \frac{3}{-8i}\, e^{i(a-\xi)} + \frac{3}{-8i}\, e^{i(-a-\xi)} - \frac{1}{-8i}\, e^{i(-3a-\xi)}$

Fourier of $\delta(x)$ $\sin^3(ax)\sin^3(ax)\sin^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)\sin^3(ax)\sin^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\sin^3(ax)\sin^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{2iax} - 2e^{0} + e^{-2iax}}{-4}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{6iax}\, e^{-ix\xi}\, dx - \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{4iax}\, e^{-ix\xi}\, dx + \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{2iax}\, e^{-ix\xi}\, dx - \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{0}\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\, e^{i(6a-\xi)0} - \frac{3}{-8i}\, e^{i(4a-\xi)0} + \frac{3}{-8i}\, e^{i(2a-\xi)0} - \frac{1}{-8i}\, e^{i(0-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{-8i}\, e^{i(6a-\xi)} - \frac{3}{-8i}\, e^{i(4a-\xi)} + \frac{3}{-8i}\, e^{i(2a-\xi)} - \frac{1}{-8i}\, e^{i(0-\xi)}$

Fourier of $\delta(x)$ $\sin^3(ax)\sin^3(ax)\sin^3(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)\sin^3(ax)\sin^3(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\sin^3(ax)\sin^3(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{9iax}\, e^{-ix\xi}\, dx - \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{7iax}\, e^{-ix\xi}\, dx + \frac{3}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{5iax}\, e^{-ix\xi}\, dx - \frac{1}{-8i}\int_{-\infty}^{\infty} \delta(x)\, e^{3iax}\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\, e^{i(9a-\xi)0} - \frac{3}{-8i}\, e^{i(7a-\xi)0} + \frac{3}{-8i}\, e^{i(5a-\xi)0} - \frac{1}{-8i}\, e^{i(3a-\xi)0}$
- $\Rightarrow$ Evaluating $\delta(x)$: $\frac{1}{-8i}\, e^{i(9a-\xi)} - \frac{3}{-8i}\, e^{i(7a-\xi)} + \frac{3}{-8i}\, e^{i(5a-\xi)} - \frac{1}{-8i}\, e^{i(3a-\xi)}$

Fourier of $\delta(x)$ $\sin^3(ax)\sin^3(ax)\sin^3(ax)\sin^2(ax)$:
- $\mathcal{F}[\delta(x)\sin^3(ax)\sin^3(ax)\sin^3(ax)\sin^2(ax)] = \int_{-\infty}^{\infty} \delta(x)\sin^3(ax)\sin^3(ax)\sin^3(ax)\sin^2(ax)\, e^{-ix\xi}\, dx$
- $= \int_{-\infty}^{\infty} \delta(x)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{3iax} - 3e^{iax} + 3e^{-iax} - e^{-3iax}}{-8i}\right)\left(\frac{e^{2iax} - 2e^{0} + e^{-2iax}}{-4}\right)\, e^{-ix\xi}\, dx$
- $= \frac{1}{-8i}\int_{-\infty}^{