La Alg

- Linear map w.r.t a basis:
  Let $\mathcal{B}$ be a basis. Then
  $[T]_{\mathcal{B}} = {}_{\mathcal{B}}[T]_{\mathcal{B}}$ is
  the matrix representing T
  with respect to basis $\mathcal{B}$
  - $T: V \to V$, $V = F^n$
    $\Rightarrow [T]_{\mathcal{B}}$ is an $n \times n$ matrix
    $\Rightarrow \det(T) = \det([T]_{\mathcal{B}})$
    $= \det(A)$,
    where $T(x) = Ax$
  - $T: M_n(F) \to M_n(F)$
    $\Rightarrow [T]_{\mathcal{B}}$ is an $n^2 \times n^2$ matrix
    $\Rightarrow \det(T) = \det([T]_{\mathcal{B}}) = (\det(A))^2$
    where $T(X) = \sum_{ij} X_{ij}$
  - Properties:
    Let ω a vector, T a linear map.
    - $[T]_{\mathcal{B}}[\omega]_{\mathcal{B}} = [T\omega]_{\mathcal{B}}$
    - ${}_C[T]_{\mathcal{B}}[\omega]_{\mathcal{B}} = [T\omega]_C$
    - ${}_C[T]_{\mathcal{B}} = {}_C[Id]_{\mathcal{B}}[T]_{\mathcal{B}}[Id]_{\mathcal{B}}$
    - ${}_D[S \circ T]_{\mathcal{B}} = {}_D[S]_C \, {}_C[T]_{\mathcal{B}}$

If $\mathcal{B}$ orthonormal basis $= \{b_1 \dots b_n\}$
Let $A = (a_{ij}) = [T]_{\mathcal{B}} \Rightarrow Tb_j = \sum_i a_{ij} b_i$
Then $v_j = (T(v))^T a_j$. This is useful for showing things like $T = 0 \Leftrightarrow T = 0$ among other points

If we have $[T]_{\mathcal{B}}$, $[T]_C$, $\mathcal{B}, C$ two different matrices, both are similar to T since $\exists Q$: $P^{-1}[T]P = T$

- Change of Basis matrix:
  Supp $\mathcal{B}, C$ are bases.
  Then ${}_C[Id]_{\mathcal{B}}$ is the
  change of basis matrix
  from $\mathcal{B}$ to C. The
  columns of ${}_C[Id]_{\mathcal{B}}$
  correspond to $[Id(b_i)]_C$,
  the basis vector b_i represented
  in the basis C after going
  through the identity transformation.
- Basis:
  - $\mathcal{B}$ is a basis of V
    if it is the minimal
    set of lin indep vecs.

- Orthogonal set of vectors: $w_1 \dots w_n \in \mathbb{R}^n$
  form an orthogonal set if vectors $\forall i \neq j$ $w_i \cdot w_j = 0$.
  If $\|w_i\| = 1$ $\forall i$, then the set is orthonormal.
  - Orthogonal vectors are linearly independent:
    - Proof:
      Let $w_1 \dots w_n$ be an orthogonal set
      of vectors ($w_i \neq 0$).
      Consider $\sum_i a_i w_i = 0$. Take any $w_j \in \{w_1 \dots w_n\}$
      $w_j \left(\sum a_i w_i\right) = \sum a_i (w_j \cdot w_i)$
      $= a_j (w_j \cdot w_j)$
      $= a_j \|w_j\|^2 = 0 \Rightarrow a_j = 0$. ∎

Use $\|w_i\| = 1$ assumed

- Orthogonal Matrix: P; $P^T P = I$
  i.e. P is orthogonal iff the columns
  of P form an orthonormal set in $\mathbb{R}^n$.
  (Primarily orthogonal and length vectors)
  - Proof:
    ($\Rightarrow$) Suppose P orthogonal
    i.e. $P^T P = I$ so $\forall i \in [n]$ $\|P_i\|^2 = 1$
    $\Rightarrow$ columns are all length 1.
    Moreover
    $\forall i \neq j \in [n]$ $0 = I_{ij} = (P^T P)_{ij} = \sum_k P_{ki} P_{kj}$
    $\Rightarrow (P_i \cdot P_j) = 0$ $\forall i \neq j$
    $\Rightarrow$ pairwise orthogonal columns
    ($\Leftarrow$) Suppose columns are pairwise orthogonal and of unit length.
    Then $A^T A = \mathbb{B}$, $b_{ij} = P_i \cdot P_j$.
    $\Rightarrow b_{ij} = 0$ $\forall i \neq j$
    $b_{ij} = 1$ $\forall i = j$
    $\Rightarrow A^T A = I_n$
    $\Leftrightarrow A^T = A^{-1}$.

$\Rightarrow \det P = \pm 1$
- Proof:
  $\det(P^T P) = \det(I_n)$

- Properties:
  - $\|\omega\| = \sqrt{\omega \cdot \omega} \Rightarrow \omega = 0 \Leftrightarrow \|\omega\| = 0$
  - $\|\omega\| = 0 \Leftrightarrow \omega = 0$
  - $\|\alpha\omega\| = |\alpha| \|\omega\|$
  - Cauchy Schwarz Inequality: $|\omega \cdot v| \leq \|\omega\| \|v\|$
  - Triangle Inequality: $\|\omega + v\| \leq \|\omega\| + \|v\|$
  - Metric triangle inequality: $\|\omega - v\| \leq \|\omega - u\| + \|u - v\|$

Use the cosine law for angle

$\Rightarrow -1 \leq \frac{\omega \cdot v}{\|\omega\| \|v\|} \leq 1 \Rightarrow \cos\theta = \frac{\omega \cdot v}{\|\omega\| \|v\|}$, θ the angle between ω and v

Orthogonal: when $\theta = \frac{\pi}{2}$.

Useful for proofs

- Proofs:
  1. $\omega \cdot (\alpha v) = \alpha(\omega \cdot v) = \alpha(v \cdot \omega) = v \cdot (\alpha\omega)$
  2. $\|\omega\| = 0 \Leftrightarrow \sqrt{\omega \cdot \omega} = 0 \Rightarrow \omega \cdot \omega = 0 \Rightarrow \omega = 0$ (a "orthogonal" is defined IF $\omega = 0$)
  3. $\|\alpha\omega\| = \sqrt{\omega \cdot \omega} = \sqrt{\alpha^2(\omega \cdot \omega)} = |\alpha| \sqrt{\omega \cdot \omega} = |\alpha| \|\omega\|$
  4. $\|\omega + v\| \leq \|\omega\| + \|v\|$ — dig positive square root
     $\Rightarrow (\omega + v)^2 \leq (\|\omega\| + \|v\|)^2$
     Consider $\|\omega + v\|^2$
     $= (\omega + v) \cdot (\omega + v)$
     $= \|\omega\|^2 + 2\omega \cdot v + \|v\|^2$
     $\leq \|\omega\|^2 + 2|\omega \cdot v| + \|v\|^2$
     $\leq \|\omega\|^2 + 2\|\omega\| \|v\| + \|v\|^2$
     $= (\|\omega\| + \|v\|)^2$
     as needed ∎
  5. $\|\omega - v\| \leq \|\omega - u\| + \|u - v\|$
     By CS θ,
     $\omega \cdot u \leq \|\omega\| \|u\|$
     $\Rightarrow 2\omega \cdot u \leq 2\|\omega\| \|u\|$
     $\Rightarrow \|\omega\|^2 + \|u\|^2 + 2\omega \cdot u \leq \|\omega\|^2 + \|u\|^2 + 2\|\omega\| \|u\|$
     $\Rightarrow (\|\omega - u\|)^2 \leq (\|\omega\| + \|u\|)^2$
     $\Rightarrow \|\omega - u\| \leq \|\omega\| + \|u\|$
     $\Rightarrow \|\omega - v\| \leq \|\omega - u\| + \|u - v\|$
     proof follows directly by replacing triangle.

- Orthogonal Vectors in $\mathbb{R}^n$:
  Supp $\omega = (\omega_1 \dots \omega_n)^T$, $v = (v_1 \dots v_n)^T$.
  - Inner product: $\omega \cdot v = \omega^T v = \sum \omega_i v_i \in \mathbb{R}$
    ω is orthogonal to v if $\omega \cdot v = 0$.
  - Norm: $\|\omega\| = \sqrt{\omega \cdot \omega} = \sqrt{(\sum \omega_i^2)}$
  - Euclidean distance: $\|\omega - v\| = (\sum (\omega_i - v_i)^2)^{1/2}$
    $= \sqrt{(\omega - v) \cdot (\omega - v)}$

- Minor of a matrix A:
  A_{ij} = minor of A is A with
  row i and column j removed.

- Determinants:
  Suppose $\det$ is defined for $n \times n$ size matrix.
  $\det(A) = \sum a_{ij} (-1)^{i+j} \det(A_{ij})$ (Laplace Expansion)
  - Detects Area Determinant of linear maps
  - Detects linear independence
  - Finds eigenvalues and helps with diagonalization
  - Determinant properties

- Complex matrices:
  - Inner product: $w \cdot v = \bar{w}^T v$
    $w \cdot v = w^T v = \sum w_i \bar{v}_i$
  - Hermitian matrix: $A \in M_n(\mathbb{C})$ (symmetric but for complex)
    is hermitian if $A = \bar{A}^T$
    - denoting the conjugate
    - $(Aw) \cdot v = w \cdot (Av)$ $\forall w, v \in \mathbb{C}^n$
    - All eigenvalues are real
    - A is diagonalizable
    - $PP^T = I_n$, P are diagonalizing matrix
      unitary

- Symmetric Matrices:
  Matrices A; $A = A^T$
  - If A symmetric,
    $(Av) \cdot v = v \cdot (Av)$ (self-adjoint)
    - Proof: $(Av) \cdot v = (Av)^T v$
      $= v^T A^T v = v^T A v = v \cdot (Av)$ ∎
  - If $A \in M_n(\mathbb{R})$ symmetric
    and $\lambda \in \mathbb{C}$ a root of
    $\chi_A(x)$, then $\lambda \in \mathbb{R}$
    $\Rightarrow$ If $A \in M_n(\mathbb{R})$, then there is an eigenvalue of A in $\mathbb{R}$.
    - Proof: $\exists v \neq 0$, $A v = \lambda v$
      Let $\bar{v}$ denote the conjugate of v.
      $v^T (Av) = \lambda v^T v$
      $\bar{v}^T (Av) = v^T (\bar{A}^T \bar{v}) = \overline{(Av)}^T v = \bar{\lambda} v^T \bar{v} v$
      $\Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$. ∎

      Since A symmetric: $A = \bar{A}^T$
      $\det A = \det \bar{A}^T$

      λ is eigenvalue of symmetric matrices are orthogonal.

  - Suppose $A \in M_n(\mathbb{R})$ symmetric. Then
    $\exists$ an orthogonal P; $P^{-1}AP$ is
    diagonal. (Eigenvectors form a basis of A's eigenspaces)
    - Proof:
      $n = 1$ is trivial.
      Suppose we have $n \times n$ case.
      Since $\exists \lambda \in F$; $\chi_A(\lambda) = 0 \Rightarrow \lambda \in \mathbb{R}$,
      we have a real eigenvalue λ and
      real eigenvec v_1. Assume $\|v_1\| = 1$
      Let P_1 be an orthogonal
      matrix with first column v_1.
      $P_1^{-1} = P_1^T$
      $\Rightarrow P_1^{-1} A P_1 = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} (Av_1 \dots Av_n)$
      $= \begin{pmatrix} \lambda & v_1^T A v_2 & \dots & v_1^T A v_n \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix}$

      easier proof

      Can be done by theorem

      $A' \in M_{n-1}(\mathbb{R})$.
      Since $P_1^{-1} A P_1 = P_1^T A P_1$, $P_1^T A P_1$ is symmetric
      as $A = A^T$, thus $v_1^T A v_2, \dots v_1^T A v_n = 0$
      So $P_1^{-1} A P_1 = \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix}$
      and A' is symmetric follows. By
      inductive step, $\exists P'$ orthogonal where
      $(P')^{-1} A' P' = \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $(P')^{-1} = (P')^T$
      Let $P_2 := \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & P' & \\ 0 & & & \end{pmatrix}$ $P_2^{-1} = P_2^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & (P')^{-1} & \\ 0 & & & \end{pmatrix}$
      orthogonal since P' orthogonal. Thus
      $(P_2^{-1})(P_1^{-1} A P_1) P_2 = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \ddots & \\ 0 & & & \end{pmatrix} = \operatorname{diag}(\lambda_1 \dots \lambda_n)$
      So $P := P_1 P_2$ orthogonal and $P^{-1} A P = \operatorname{diag}(\lambda_1 \dots \lambda_n)$. ▷

Fundamental Theorem of Algebra:
$p(x)$ a non-constant polynomial with coefficients in $\mathbb{C}$. Then $\exists \alpha \in \mathbb{C}$; $p(\alpha) = 0$.
another proof

Generalization of using Gram Schmidt to find orthonormal basis of eigenspaces
- Find the distinct eigenvals
  - Find E_{λ_i}, the eigenspace and their bases
  - Apply Gram Schmidt
    - Taking all these together gives an orthonormal basis in $\mathbb{R}^n$.

$\forall i \neq j$, $E_{\lambda_i} \cap E_{\lambda_j} = \{0\}$
every $v \in E_{\lambda_i}$ and $v_j \in E_{\lambda_j}$, v_i and v_j are orthogonal.

real symmetric matrices are diagonalizable

- Further properties
  - Let $A, B \in M_n(F)$, $\alpha \in F$,
    $1 \leq i, j \leq n$, $i \neq j$.
    - If B is A with rows i and j interchanged, then
      $\det B = -\det A$.
      - Proof:
        By identical consecutive rows, let R_i, R_{i+1} denote rows
        $0 = \det\begin{pmatrix} R_i + R_{i+1} \\ R_i + R_{i+1} \\ \vdots \end{pmatrix}$ linearity on rows
        $= \det\begin{pmatrix} R_i \\ R_i + R_{i+1} \\ \vdots \end{pmatrix} + \det\begin{pmatrix} R_{i+1} \\ R_i + R_{i+1} \\ \vdots \end{pmatrix} = \det\begin{pmatrix} R_i \\ R_i \\ \vdots \end{pmatrix} + \det\begin{pmatrix} R_{i+1} \\ R_i \\ \vdots \end{pmatrix}$
        $\Rightarrow \det\begin{pmatrix} R_i \\ R_{i+1} \\ \vdots \end{pmatrix} = -\det\begin{pmatrix} R_{i+1} \\ R_i \\ \vdots \end{pmatrix} \Leftrightarrow \det B = -\det A$. ∎
    - Supp A has two equal rows, not necessarily consecutive, then
      $\det A = 0$ (Identical Rows General)
      - Proof: By repeated row interchange n times to A'
        with identical consecutive rows, $\det A = (-1)^n \det A' = 0$ ∎
    - Supp B is A with row i and j interchanged.
      then $\det B = -\det A$. (Row interchange General)
      - Proof: Let R_i, R_j be rows. Then
        $0 = \det\begin{pmatrix} R_i + R_j \\ R_i + R_j \\ \vdots \end{pmatrix}$ (equal rows)
        $= \det\begin{pmatrix} R_i \\ R_i + R_j \\ \vdots \end{pmatrix} + \det\begin{pmatrix} R_j \\ R_i + R_j \\ \vdots \end{pmatrix}$ (split rows out linearity)
        $\Rightarrow \det\begin{pmatrix} R_i \\ R_j \\ \vdots \end{pmatrix} = -\det\begin{pmatrix} R_j \\ R_i \\ \vdots \end{pmatrix} \Leftrightarrow \det B = -\det A$. ∎
    - Supp B is obtained from adding α row i to
      row j. Then $\det B = \det A$. (Constant to row i added to row j, $i \neq j$)
      - Proof: Let R_i and R_j be rows
        $\det\begin{pmatrix} R_i \\ R_i + \alpha R_i \\ \vdots \end{pmatrix} = \det\begin{pmatrix} R_i \\ R_i \\ \vdots \end{pmatrix} + \det\begin{pmatrix} R_i \\ \alpha R_i \\ \vdots \end{pmatrix}$ (linearity and factor)
        $= \det\begin{pmatrix} R_i \\ R_i \\ \vdots \end{pmatrix}$
        $\Leftrightarrow \det B = \det A$. ∎
  - Row Equivalence: A and B are row equivalent if $\det A = \beta \det B$ for some nonzero $\beta \in F$. Thus
    $\det A = 0 \Leftrightarrow \det B = 0$
    - Proof:
      Supp $A, B \in M_{n \times n}(F)$ row equivalent.
      Then B is A with some elementary row operations
      - addition of row i to j
      - multiply row i by α factor
      - interchanging two rows
      $\Rightarrow \det B = \beta \det A$, β the value representing the impacts of interchanges and factoring.
      $\Rightarrow \det B = 0$ iff $\det A = 0$ ∎
    - E.g.
      Vandermonde determinant:
      $\det\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix} = \prod_{i<j} (x_j - x_i)$
      - Proof:
        Applying column operations:
        $\det(V) = \det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & \cdots & x_2^{n-2}(x_2 - x_1) \\ \vdots & & & \vdots \\ 1 & x_n - x_1 & \cdots & x_n^{n-2}(x_n - x_1) \end{pmatrix}$
        $= \det\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & \cdots & x_2^{n-2} \\ \vdots & & & \vdots \\ 1 & x_n - x_1 & \cdots & x_n^{n-2} \end{pmatrix}$
        $= (x_2 - x_1) \dots (x_n - x_1) \det\begin{pmatrix} 1 & \cdots & x_2^{n-2} \\ \vdots & & \vdots \\ 1 & \cdots & x_n^{n-2} \end{pmatrix}$
        $= \prod_{i<j} (x_j - x_i)$ By induction ∎
      - E.g. (Language) Prove that $f(x) = a_0 + \dots + a_{n-1} x^{n-1}$
        has at most n roots.
        - Proof: Let $x_1 \dots x_n$ be roots;
          $\forall i$, $f(x_i) = 0$ $f(x_1) = a_0 + \dots + a_{n-1} x_1^{n-1}$
          $\Rightarrow \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} a_0 + \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} a_1 + \dots + \begin{pmatrix} x_1^{n-1} \\ \vdots \\ x_n^{n-1} \end{pmatrix} a_{n-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$
          $\Rightarrow \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$
          $\Rightarrow \det\begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix} = 0$
          $\Rightarrow x_i = x_j$ for some i, j by dependence of columns.
          $\Rightarrow$ We can't have n distinct roots.

- Linear map determinants:
  Let $T: V \to V$ a linear map with
  $\mathcal{B}$ the basis for V, V finite
  dimensional and $V \cong F^n$.
  - $\det(T) = \det(C(T))$ (Well-defined)
    - Proof: $[T]_{\mathcal{B}} = {}_{\mathcal{B}}[Id]_{\mathcal{B}} T_{\mathcal{B}} [Id]_{\mathcal{B}}$
    Let $P := {}_{\mathcal{B}}[Id]_{\mathcal{B}}$
    $\det([T]_{\mathcal{B}}) = \det T \cdot \det P \cdot \det P^{-1}$
    $= \det T$ ∎

$\forall k$, $\exists A$; $A^k = I_n$.
Proof: Take A rate rotation matrix.

- Eigenvalue eigenvector: Suppose $A \in M_n(F)$
  and $\lambda \in F$. Then λ is an eigenvalue of A if
  there is a nonzero vector $v \in F^n$; $Av = \lambda v$.
  (Source data for $\ker$ map)
  - Suppose V an f.d. vector space over
    F and T a map for V. Let $T: V \to V$.
    Then the eigenvalues for T are in $[T]_{\mathcal{B}}$
    $v \in V$ is an eigenvector for T iff $[v]_{\mathcal{B}}$ is an
    eigenvector for $[T]_{\mathcal{B}}$.
    - Proof:
      $Tv = \lambda v \Leftrightarrow [Tv]_{\mathcal{B}} = [\lambda v]_{\mathcal{B}}$
      $[T]_{\mathcal{B}} [v]_{\mathcal{B}} = \lambda [v]_{\mathcal{B}}$

Multiplicity: The number of times an eigenvalue appears.

Every eigenval has at least one non-zero eigenvec
Proof by subspace of $\{A - \lambda I\}$ being non zero thus $\det = 0$ for given λ.

- Diagonalization:
  Suppose we have T_A; $A \in M_n(F)$
  whereby $v_1 \dots v_n$ are the eigenvectors ($\in F$)
  with eigenvalues $\lambda_1 \dots \lambda_n$. Take
  $Q = [v_1 \dots v_n]$ as a basis
  Then $[T]_{\mathcal{B}} = D = (\lambda_j)$; $d_{ii} = \lambda_i$ and $d_{ij} = 0$ $\forall i \neq j$.
  $\Rightarrow [T]_{\mathcal{B}} = {}_{\mathcal{B}}[Id]_{\mathcal{B}} [T]_{\mathcal{B}} {}_{\mathcal{B}}[Id]_{\mathcal{B}}$ $\Leftrightarrow D = P^{-1} A P$
  $\Leftrightarrow A = PDP^{-1}$, where P is the matrix with columns $v_1 \dots v_n$.
  - $A^k = (PDP^{-1})(\dots)(PDP^{-1})$
    $= PD^k P^{-1}$. $D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}$
    $\Rightarrow A^k = PD^k P^{-1}$, $D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^k \end{pmatrix}$
  - General Expression of nth power matrices:
    Find diagonalize D, P,
    $P^{-1}AP = D$
    As all to the simple, just A^n by expanding eigenpairs

Diagonalizable when D and P are matrix

- Eigenspace: $E_\lambda = \{v \in F^n \mid Av = \lambda v\} \subseteq \ker((\lambda I_n - A)v)$
  is the eigenspace for eigenvalue λ.
  - E_λ: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\lambda = 1$, $E_\lambda = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

- Characteristic polynomial:
  Suppose $A \in M_n(F)$ and x a variable/indeterminate
  The characteristic polynomial is
  $\chi_A(x) = \det(xI_n - A)$.
  (Similar definition for $T = L(v)$ map)
  - Properties:
    - $\chi_A(x) = \det(xI_n - A)$
      is an n degree polynomial
      with 1 as the coefficient for x^n
    - $xI_n - A \in M_n(F[x])$ where $F[x]$ denotes
      the field of rational functions $\frac{p(x)}{q(x)}$, p, q polynomials.
    - Basis dependent affect $\chi_A(x)$
      - Proof: A; PAP
        $\det(xI - A) = \det(P^{-1}(xI - A)P)$ ∎
    - $\lambda \in F$ is an eigenvalue iff $\chi_A(\lambda) = 0$ $\Leftrightarrow$ $A \in M_n(F)$ runs A has
      $\leq n$ real n eigenvalues.
      - Proof:
        Supp λ an eigenvalue. Then $(\lambda I - A)v = 0$
        for non-zero $v \Rightarrow \lambda I_n - A$ non-invertible
        $\Leftrightarrow \det(\lambda I_n - A) = 0 \Rightarrow \chi_A(\lambda) = 0$ ∎

roots of $\chi_A(x)$

Proof:
Let $Q_i = xI - A$
Since $\det(x) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod b_{i,\sigma(i)}$
- $\operatorname{sgn}(\sigma) \prod b_{ii} = k$ + $\operatorname{sgn}(\sigma) \prod (x - a_{ii}) = k$.
Since by σ definition, k is of degree $n-1$,
we have coeff of x^n is 1.

$\chi_{AB} = \chi_{BA}$ $\forall A, B \in M_n(\mathbb{R})$
Disprove:

- Properties:
  - If A and B are similar,
    $\chi_A = \chi_B$.

Cayley Hamilton:
$\chi_A(A) = 0$
Proof:

- Suppose $T \sim$ lin map. Then
  $v \in \ker \circ T \Rightarrow v$ is an eigenvector
  of T with eigenvalue 1.

- Diagonalization:
  Suppose $T \sim$ lin map. Then
  $\exists \mathcal{B}$ basis for V where $[T]_{\mathcal{B}}$ is diagonal
  - T is diagonalizable iff $\exists \mathcal{B} \in \ker$;
    $[T]_{\mathcal{B}} = D$ a diagonal matrix.
    $\Leftrightarrow$ A diagonalizable iff $\exists$ invertible P;
    $P^{-1}AP = D$ a diagonal matrix.
    - Extending a field: F_0 runs from
      $\mathbb{R} \to \mathbb{C}$, then $\exists \lambda \in \mathbb{C}$
      $\chi_A(\lambda) = \lambda^n + 1 = 0$ while
      $\nexists \lambda \in \mathbb{R}$; $\chi_A(\lambda) = 0$.
  - If $v_1 \dots v_n$ are eigenvectors with distinct
    eigenvalues $\lambda_1 \dots \lambda_n$, then $v_1 \dots v_n$ are
    linearly independent.
    - Having n distinct eigenvalues $\Rightarrow$ diagonalizability
      $\Updownarrow$
      If $\chi_A(x)$ has n distinct roots A is diagonalizable.
  - If P invertible, we have that $v_1 \dots v_n$ the
    columns of P form a basis $\mathcal{B}$ of F^n.
    $\Rightarrow P = [Id]_{\mathcal{B}}$, E the standard basis.
    - Proof:
      $\Leftrightarrow LI$ vecs basis follows.

all distinct eigenvals $\Rightarrow$ LI eigenvecs

- Proof (Contradiction):
  Suppose $\exists \omega_i$; $\sum \omega_i v_i = 0$
  $T(\sum \omega_i v_i) = 0$
  $T \circ T(\sum \omega_i v_i) = 0$
  Solving for ω we get
  $\omega = 0$ by non-zero lin. ∎

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  Solving for ω we get
  $\omega = 0$ by
  non-zero lin. ∎

- Cofactor: Let $A \in M_n(F)$. If $i, j \in [n]$,
  $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is the ijth cofactor of A.
  Let $C = (C_{ij})$
  $\Rightarrow \det A = \sum_j c_{ij} a_{ij} = ith$ entry of $C^T A$.
  Suppose $a_{ik} = a_{jk}$ $\forall k \in [n]$. Then
  $\det A = \sum_j c_{ij} a_{ij} = \sum_j c_{ij} a_{kj} = 0$ (identical rows)
  $\Rightarrow A^{-1} = \frac{1}{\det A} C^T$, where $\det A \neq 0$.
  - $\frac{1}{\det A} \operatorname{adj}(A)$, $\operatorname{adj}(A)$ the adjugate of A
    $\Leftrightarrow \operatorname{adj}(A) \cdot A^{-1} = \det A$
    $\Leftrightarrow (\operatorname{adj}(A)) A = (\det A) I_n$

matrix of signed cofactors

Let $GL(F) \subset M_n(F)$ the set of invertible matrices. Then $\cdot : GL(F) \to GL(F)$
where $\cdot(A) = A^{-1}$ is of the form
$\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$

- Cramer's Rule: Let $A \in M_n(F)$ and
  $b = (b_1, \dots, b_n)^T \in F^n$.
  $x = (x_1, \dots, x_n)^T = A^{-1} b$ is
  a unique solution to $Ax = b$.
  $\Updownarrow$
  $x_i := \det(A_i)/\det(A)$
  where A_i is A with the ith
  column replaced by b
  - Proof:

row subsets or rows

- Adjugate properties:
  - $\operatorname{adj}(\lambda A) = \lambda^n \operatorname{adj}(A)$
  - $\operatorname{adj}(A^T) = \operatorname{adj}(A)^T$
  - $\operatorname{adj}(-\operatorname{adj}(A)) = (\det A)^{n-2} \cdot A$
  - Proofs:
    - $\operatorname{adj}(\lambda A) = C^T$
      $C = (c_{ij})$
      $c_{ij} = \sum (-1)^{i+j} \det(\lambda A_{ij})$
      $= \lambda^{n-1} \sum (-1)^{i+j} \det(A_{ij})$
      $\therefore \operatorname{adj}(\lambda A) = \lambda^{n-1} \operatorname{adj} A$ ∎
    - $\operatorname{adj}(A^T) = (\det A^T) \det(A^{-1})$
      $= A \cdot \frac{1}{\det(A)}$ ∎
    - $\operatorname{adj}(\operatorname{adj}(A)) = \operatorname{adj}(\det(A) \cdot A^{-1})$
      $= (\det A)^{n-1} \operatorname{adj}(A^{-1})$
      $= (\det A)^{n-1} \cdot A$

- Singular matrix A is
  singular if $\exists$ nonzero $v \in F^n$;
  $Av = 0$. Otherwise it is
  non-singular
  - Equivalent definitions: $\Rightarrow$ Proof:
    1. A is singular
    2. A invertible
    3. row of A are LI
    4. A is row equivalent to I_n ($\det A = \beta \det I_n$)
    5. $\det A \neq 0$

    1. $\Rightarrow$ 5. by row equivalence property
    5. $\Rightarrow$ 1. all equivalent by previous discussions.
    5. $\Rightarrow$ 3. Supp A row row equivalent to I_n. Then it has a row of zeros either row elimination so $\det(A) = 0$. (Proof by contradiction)

Singular matrices definitions

- Other determinant expansions:
  - (i^{th} row expansion)
    $\det A = \sum_j (-1)^{i+j} a_{ij} \det A_{ij}$
    - Proof:
      Let $i = 1$. Let B be A with
      rows R_1 and R_2 interchanged
      $\det A = -\det B = -\sum (-1)^{1+j} a_{1j} \det(B_{1j})$
      $= \sum (-1)^{1+j} a_{1j} \det(B_{1j})$
      $A_{1j} = R_1 R_2 \dots R_{j-1} R_{j+1} \dots$
      $B_{1j} = R_2 R_3 \dots R_{j-1} R_1 R_{j+1} \dots$
      so $\det A_{1j} = (-1)^{j-1} \det(B_{1j})$ (By row interchange)
      $\Rightarrow \det A = \sum (-1)^{1+j} a_{1j} \det(A_{1j})$ ∎

$\begin{pmatrix} - & + & - & + \\ + & - & + & - \\ - & + & - & + \\ + & - & + & - \end{pmatrix}$

  - C^{th} column expansion)
    $\det A = \sum (-1)^{j+i} a_{ij} \det A_{ij}$
    - Proof:
      $\det A = \det A^T$. Column expansion is equivalent to
      row expansion on A^T ∎
  - (permutation expansion)
    $\det A = \sum \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, $\operatorname{sgn}(\sigma)$ depends on even/odd-ness of σ

(Triangular Matrix Determinant)
Suppose A is upper triangular:
$\det(A) = \prod a_{ii}$
- Proof:
  By row expansion from the bottom up,
  $\det(A) = a_{nn} a_{n-1,n-1} \cdots a_{11}$
  $= \prod a_{ii}$ ∎
- Suppose A is lower triangular
  $\det A = \prod a_{ii}$
  - Proof: A^T is upper triangular ∎

- D1 (Taking Out Factors):
  Let $A \in M_n(F)$, $\alpha \in F$, $i \leq n$ and $\mathcal{B}$,
  $\begin{cases} b_{ij} = \alpha \cdot a_{ij} & \forall j \\ b_{ij} = a_{ij} & \forall i \neq j \end{cases}$
  Then $\det(\mathcal{B}) = \alpha \cdot \det(A)$
  - Proof:
    Case 1: $i = 1$. Then $b_{1j} = \alpha a_{1j}$
    $\therefore \det \mathcal{B} = \sum (-1)^{1+j} b_{1j} \det(\mathcal{B}_{1j})$
    $= \alpha \sum (-1)^{1+j} a_{1j} \det(A_{1j})$
    $= \alpha \det A$
    Case 2: $i > 1$:
    $\det(\mathcal{B}) = \sum (-1)^{1+j} b_{1j} \det(\mathcal{B}_{1j})$
    $= \sum (-1)^{1+j} \alpha a_{1j} \det(\mathcal{B}_{1j})$
    For each j, the $(i-1)^{th}$ row of $\mathcal{B}_{1j}$ = $\alpha \cdot (i-1)^{th}$ row of A_{1j}. Since rows are $(n-1) \times (n-1)$, by induction we get $\mathcal{B}_{1j} = \alpha A_{1j}$. Let row α of $A_{(n-1)j}$
    so $\det A_{1j} = \det \mathcal{B}_{1j}$
    $\therefore \det \mathcal{B} = \sum (-1)^{1+j} a_{1j} \det A_{1j}$
    $= \alpha \det A$ ∎

- D2 (Linearity on row):
  Let $A, B, C \in M_n(F)$ and $1 \leq i \leq n$
  Suppose A, B, C all identical except at i.
  $\begin{cases} c_{ij} = b_{ij} + c_{ij}, & \forall i \neq i \\ c_{ij} = a_{ij} + b_{ij} & \forall j \end{cases}$
  Then $\det C = \det A + \det B$
  - Proof:
    Case 1, $i = 1$:
    $\det C = \sum (-1)^{1+j} (a_{1j} + b_{1j}) \det C_{1j}$
    $= \det A + \det B$
    since $\det C_{1j} = \det A_{1j} = \det B_{1j}$
    as all entries are equal.
    Case 2, $i > 1$:
    $\det C = \sum (-1)^{1+j} (a_{1j} + b_{1j}) \det C_{1j}$
    $(i-1)^{th}$ row of C_{1j} is equal to sum of
    $(i-1)^{th}$ row of A_{1j} and B_{1j}. By
    induction $\det C_{(1,j)} = \det A_{(1,j)} + \det B_{(1,j)}$
    $\Rightarrow \det C = \det A + \det B$.

- D3 (Identical consecutive row)
  Let $A \in M_n(F)$, $1 \leq k \leq n$
  Suppose the kth and $(k+1)^{th}$ rows
  in A same. Then
  $\det(A) = 0$
  - Proof:
    Suppose $n = 2$.
    Then $A = \begin{pmatrix} a_1 & a_2 \\ a_1 & a_2 \end{pmatrix}$
    $\Rightarrow \det A = a_1 a_2 - a_1 a_2 = 0$
    Suppose $n \geq 3$ and that
    the claim is true for $(n-1) \times (n-1)$
    matrices.
    Case 1: $i > j > 1$, then
    $\det A = \sum (-1)^{i+j} a_{ij} \det A_{ij}$
    $\Rightarrow$ rows k and k are the
    same in A_{ij} $\forall j$
    $\Rightarrow \det A_{ij} = 0$ by indu