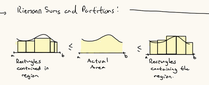
Area, Anti-derivative, Integral

Integral

Riemann Sums and Partitions:



- A partition of $[a,b]$ is a finite collection of points contained in $[a,b]$ and containing both a & b.
  i.e. $P = \{x_0 = a, x_1, \dots, x_n = b\}$
  Equivalently $P = \{x_0, \dots, x_n\}$
- Let $f:[a,b]\to\mathbb{R}$ be a bounded function. Given P a partition of $[a,b]$, let
  $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$
  $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$
  $L(f,P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ (lower sum of f for P)
  $U(f,P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ (upper sum of f for P)
  From the definition: $L(f,P) \le U(f,P)$
- Given partition $P \subseteq P'$, we have
  $L(f,P) \le L(f,P') \le U(f,P') \le U(f,P)$
  Let $P = P_0 \subseteq P_1 \subseteq \dots \subseteq P_n = P'$ such that P_{i+1} contains one more point than P_i.
  Consider the base case of $P \subseteq P_1 \subseteq P_2 = P'$
  Then $P = \{x_0, \dots, x_n\}$
  $P' = \{x_0, \dots, x_{k-1}, x_k, \dots, x_n\}$
  So we have $[x_{k-1}, x_k] \subset [x_{k-1}, x_k]$
  Let $m' := \inf\{f(x) : x \in [x_{k-1}, x_k]\}$
  $m'_1 := \inf\{f(x) : x \in [x_{k-1}, x_k]\}$
  $m'_2 := \inf\{f(x) : x \in [x_k, x_k]\}$
  Then $m' \le m'_1, m'_2$.
  $\Rightarrow L(f,P) = L(f,P)$
  $+ m'(x_k - x_{k-1}) + m'(x_k - x_k) - m_k(x_k - x_{k-1})$
  $\ge m_k(x_k - x_{k-1}) + m_k(x_k - x_k) - m_k(x_k - x_{k-1})$
  ≥ 0
  $\Rightarrow L(f,P) \ge L(f,P)$
  Similarly if $M' := \sup\{f(x) : x \in [x_{k-1}, x_k]\} \le M_k$
  $M'_1 := \sup\{f(x) : x \in [x_{k-1}, x_k]\}$
  $M'_2 := \inf\{f(x) : x \in [x_k, x_k]\}$
  Then $M', M'_1, M'_2 \le M_k$
  $\Rightarrow U(f,P) - U(f,P')$
  $= M_k(x_k - x_{k-1}) - M'(x_k - x_{k-1}) - M'(x_k - x_k)$
  $\ge M_k(x_k - x_{k-1}) - M_k(x_k - x_{k-1}) - M_k(x_k - x_k)$
  ≥ 0
  $\Rightarrow U(f,P) \ge U(f,P')$
  $\therefore L(f,P) \le L(f,P') \le U(f,P') \le U(f,P)$
- Given any partitions P, Q, we have
  $L(f,P) \le U(f,Q)$.
  $\sup\{L(f,P) : P \text{ partition}\} \le \inf\{U(f,Q) : Q \text{ partition}\}$
  Consider $R = P \cup Q$. Then since $P, Q \subset R$, we have $L(f,P) \le L(f,R) \le U(f,R) \le U(f,Q)$

Integrability

- A bounded function $f:[a,b]\to\mathbb{R}$ is integrable, if we have
  $\sup\{L(f,P) : P \text{ partition}\} = \inf\{U(f,Q) : Q \text{ partition}\}$
  and we denote the value by $\int_a^b f(x)\,dx$.
- If f is bounded on $[a,b]$ then f is integrable on $[a,b]$ iff $\forall \varepsilon > 0$ $\exists P$ a partition;
  $U(f,P) - L(f,P) < \varepsilon$
  If f is integrable then $\sup\{L(f,P) : P\} = \inf\{U(f,P) : P\}$
  So $\forall \varepsilon > 0$, $\exists P, P'$; $U(f,P') - L(f,P) < \varepsilon$.
  Let $P = P \cup P'$. Then
  $U(f,P) - L(f,P) \le U(f,P') - L(f,P') < \varepsilon$
  Suppose $\forall \varepsilon > 0$ $\exists P$; $U(f,P) - L(f,P) < \varepsilon$. Consider $P = P \cup \{x\}$.
  Then $U(f,P) - L(f,P) < \varepsilon$ since $P \subseteq P$.
  Let $P' = P \cap [a,x]$, $P'' = P \cap [x,b]$.
  Then P' is a partition on $[a,x]$ and P'' on $[x,b]$.
  $U(f,P) - L(f,P) = U(f,P') - L(f,P') + U(f,P'') - L(f,P'') < \varepsilon$
  $\Rightarrow U(f,P') - L(f,P') < \varepsilon$ and $U(f,P'') - L(f,P'') < \varepsilon$
  $\therefore$ f is integrable on $[a,x]$ and $[x,b]$.
  (Additivity of the domain)
- Let $a<b<c$. Then f is integrable on $[a,c]$ iff it is integrable on $[a,b]$ and $[b,c]$.
  In $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
  Suppose f is integrable on $[a,b]$ and $[b,c]$.
  Then $\forall \varepsilon > 0$, $\exists P'$ on $[a,b]$ and P'' on $[b,c]$;
  $L(f,P') \le \int_a^b f(x)\,dx \le U(f,P')$
  $U(f,P') - \varepsilon \le \int_a^b f(x)\,dx \le L(f,P') + \varepsilon$
  and
  $L(f,P'') \le \int_b^c f(x)\,dx \le U(f,P'')$
  $U(f,P'') - \varepsilon \le \int_b^c f(x)\,dx \le L(f,P'') + \varepsilon$
  Let $P = P' \cup P''$. Then P is a partition on $[a,c]$.
  $U(f,P) - U(f,P') + U(f,P'')$
  $L(f,P) = L(f,P') + L(f,P'')$
  $\Rightarrow L(f,P) \le \int_a^c f(x)\,dx \le U(f,P)$
  $U(f,P) - 2\varepsilon \le \int_a^c f(x)\,dx + \int_b^c f(x)\,dx \le L(f,P) + 2\varepsilon$.
  This we have f integrable on $[a,c]$ since P partitions $[a,c]$, and $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
  Suppose f is integrable on $[a,c]$; $\forall \varepsilon > 0$ $\exists P$; $U(f,P) - L(f,P) < \varepsilon$. Consider $P = P \cup \{b\}$.
  Then $U(f,P) - L(f,P) < \varepsilon$ since $P \subseteq P$.
  Let $P' = P \cap [a,b]$, $P'' = P \cap [b,c]$.
  Then P' is a partition on $[a,b]$ and P'' on $[b,c]$.
  $U(f,P) - L(f,P) = U(f,P') - L(f,P') + U(f,P'') - L(f,P'') < \varepsilon$
  $\Rightarrow U(f,P') - L(f,P') < \varepsilon$ and $U(f,P'') - L(f,P'') < \varepsilon$
  $\Rightarrow$ f integrable on $[a,b]$ and $[b,c]$
  $\therefore \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
  (Additivity of the integrand)
- If f and g integrable on $[a,b]$ $\Rightarrow$ $f+g$ integrable on $[a,b]$
  Such that $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
  WLOG, Assume $\exists P$ st $[a,b]$;
  $U(f,P) - \varepsilon \le \int_a^b f(x)\,dx \le L(f,P) + \varepsilon$
  $U(g,P) - \varepsilon \le \int_a^b g(x)\,dx \le L(g,P) + \varepsilon$
  Let $M_i := \sup\{f+g : x \in [x_{i-1}, x_i]\}$
  $m_i^f := \inf\{f : x \in [x_{i-1}, x_i]\}$
  $M_i^f := \sup\{f : x \in [x_{i-1}, x_i]\}$
  $m_i^g := \inf\{g : x \in [x_{i-1}, x_i]\}$
  $M_i^g := \sup\{g : x \in [x_{i-1}, x_i]\}$
  $\Rightarrow M_i^f + M_i^g \ge M_i$ and $m_i^f + m_i^g \le m_i$
  $\Rightarrow L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P)$.
  $\Rightarrow U(f+g,P) - L(f+g,P) \le U(f,P) + U(g,P) - L(f,P) - L(g,P)$
  $\le U(f,P) + U(g,P) - L(f,P) - L(g,P)$
  $\le 2\varepsilon$
  $\therefore$ $f+g$ is integrable and $\left|\int_a^b (f+g)(x)\,dx - \int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| \le 2\varepsilon$
  (Scalar multiplication)
- If f is integrable on $[a,b]$ $\Rightarrow$ cf is integrable on $[a,b]$ $\forall c \in \mathbb{R}$
  $\forall \varepsilon > 0$, $\exists P$; $U(f,P) - L(f,P) \le \varepsilon$
  $M_i := \sup\{f : x \in [x_{i-1}, x_i]\}$
  $m_i := \inf\{f : x \in [x_{i-1}, x_i]\}$
  $M_i^c := \sup\{cf : x \in [x_{i-1}, x_i]\}$
  $m_i^c := \inf\{cf : x \in [x_{i-1}, x_i]\}$.
  If $c \ge 0$, $M_i^c = cM_i$; $m_i^c = cm_i$,
  So $L(cf,P) = L(f,P) \le U(f,P) = U(cf,P)$
  $\therefore U(cf,P) - L(cf,P) \le \varepsilon$
  If $c < 0$, $cf = 0$. So $M_i^c = m_i^c = 0$
  If $c < 0$
  $U(cf,P) = cL(f,P)$
  $L(cf,P) = cU(f,P)$
  $\Rightarrow U(cf,P) - L(cf,P) = \frac{L(cf,P) - U(cf,P)}{c} \le \varepsilon$
  $\Rightarrow U(cf,P) - L(cf,P) \le |c|\varepsilon$
- Suppose f is integrable on $[a,b]$. Then let's also integrable on $[c,b]$ and
  $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$
  f integrable on $[a,b]$
  $\Rightarrow \sup\{L(f,P)\} = \inf\{U(f,P)\}$
  Consider $P = \{x_0, \dots, x_n\}$ the partition where x_i is the ith root of f, $x_0 = a$, $x_n = b$.
  Then we can consider uniform sub partitions on $[x_{i-1}, x_i]$ for every i.
  Then $\left|\int_{x_{i-1}}^{x_i} f(x)\,dx\right| = \int_{x_{i-1}}^{x_i} |f(x)|\,dx$.
  (Since either $|f(x)| = f(x)$ $\forall x \in [x_{i-1}, x_i]$ or $|f(x)| = -f(x)$ $\forall x \in [x_{i-1}, x_i]$)
  Then $\left|\int_a^b f(x)\,dx\right| = \left|\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx\right|$
  $= \sum_{i=1}^{n} \left|\int_{x_{i-1}}^{x_i} f(x)\,dx\right|$
  $= \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(x)|\,dx$
  $= \int_a^b |f(x)|\,dx$
  $\therefore \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$ and $|f(x)|$ integrable.
- $f \ge 0$ and integrable and $\int_a^b f(x)\,dx = 0$ $\Rightarrow$ $f(x) = 0$ $\forall x \in [a,b]$.
  Take for instance $f(x) := \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases}$ $(f \ge 0)$

  Then $\forall \varepsilon > 0$, take $P = \{-1, -\frac{\varepsilon}{2}, 1\}$.
  Then $U(f,P) = 1 \cdot \frac{\varepsilon}{2}$, $L(f,P) = 0$
  $\therefore U(f,P) - L(f,P) = \frac{\varepsilon}{2} \le \varepsilon$ (f integrable)
  $\int_a^b f(x)\,dx = \inf\{U(f,P)\} = 0$
  However $f \ne 0$ $\forall x$.
- $f \ge 0$ and integrable and $\int_a^b f(x)\,dx = 0$ and f continuous
  $\Rightarrow f(c) = 0$ $\forall x \in [a,b]$.

  Suppose $\exists x_0$; $f(x_0) > 0$. Then $\exists \delta > 0$; $f(x) > f(x_0)/2$ $\forall x \in (x_0-\delta, x_0+\delta)$ ($x_0 \ne a, b$)
  Then $\int_a^b f(x)\,dx \ge \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx \ge \frac{\delta}{2} f(x_0) > 0$ (Contradiction to $\int_a^b f(x)\,dx = 0$. Similarly for $f(a)$, $f(b)$ $\blacksquare$
- Suppose f is integrable on $[a,b]$. Then f^2 is integrable on $[a,b]$ and
  $\int_a^b f(x)^2\,dx \le \int_a^b f(x)\,dx$
  Let $P = \{x_0, \dots, x_n\}$ be a partition on $[a,b]$.
  Since $f(x) \le M$, f is strictly increasing on $[a,b]$.
  Thus we can consider uniform sub partitions on $[x_{i-1}, x_i]$ for every i.
  Let $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$
  $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$
  $M_i^2 := \sup\{f^2(x) : x \in [x_{i-1}, x_i]\}$
  $m_i^2 := \inf\{f^2(x) : x \in [x_{i-1}, x_i]\}$
  $\Rightarrow M_i^2 - m_i^2 = (M_i - m_i)(M_i + m_i) \le (M_i - m_i)2M$
  $\Rightarrow U(f^2, P) - L(f^2, P) \le 2M\,U(f,P) - 2M\,L(f,P)$
  $\Rightarrow U(f^2, P) - L(f^2, P) \le 2M(U(f,P) - L(f,P))$
  $\Rightarrow U(f^2, P) - L(f^2, P) < 2M\varepsilon$
  $\therefore$ f^2 is integrable.
- $f \ge 0$ and integrable and $\int_a^b f(x)\,dx = 0$ $\Rightarrow$ $f(x) = 0$ $\forall x \in [a,b]$.
  Suppose $\exists x_0$; $f(x_0) > 0$. Then $\exists \delta > 0$; $f(x) > f(x_0)/2$ $\forall x \in (x_0-\delta, x_0+\delta)$.
  $\Rightarrow \int_a^b f(x)\,dx \ge \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx \ge \frac{\delta}{2} f(x_0) > 0$ (Contradiction to $\int_a^b f(x)\,dx = 0$. Similarly for $f(a)$, $f(b)$ $\blacksquare$
- Let $f:[a,b]\to\mathbb{R}$ be bounded. Given P a partition of $[a,b]$, let
  $m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$
  $M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$
  $L(f,P) := \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ (lower sum of f for P)
  $U(f,P) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ (upper sum of f for P)
  From the definition: $L(f,P) \le U(f,P)$
- Given partition $P \subseteq P'$, we have
  $L(f,P) \le L(f,P') \le U(f,P') \le U(f,P)$
  Let $P = P_0 \subseteq P_1 \subseteq \dots \subseteq P_n = P'$ such that P_{i+1} contains one more point than P_i.
  Consider the base case of $P \subseteq P_1 \subseteq P_2 = P'$
  Then $P = \{x_0, \dots, x_n\}$
  $P' = \{x_0, \dots, x_{k-1}, x_k, \dots, x_n\}$
  So we have $[x_{k-1}, x_k] \subset [x_{k-1}, x_k]$
  Let $m' := \inf\{f(x) : x \in [x_{k-1}, x_k]\}$
  $m'_1 := \inf\{f(x) : x \in [x_{k-1}, x_k]\}$
  $m'_2 := \inf\{f(x) : x \in [x_k, x_k]\}$
  Then $m' \le m'_1, m'_2$.
  $\Rightarrow L(f,P) = L(f,P)$
  $+ m'(x_k - x_{k-1}) + m'(x_k - x_k) - m_k(x_k - x_{k-1})$
  $\ge m_k(x_k - x_{k-1}) + m_k(x_k - x_k) - m_k(x_k - x_{k-1})$
  ≥ 0
  $\Rightarrow L(f,P) \ge L(f,P)$
  Similarly if $M' := \sup\{f(x) : x \in [x_{k-1}, x_k]\} \le M_k$
  $M'_1 := \sup\{f(x) : x \in [x_{k-1}, x_k]\}$
  $M'_2 := \sup\{f(x) : x \in [x_k, x_k]\}$
  $\Rightarrow M', M'_1, M'_2 \le M_k$
  $\Rightarrow U(f,P) - U(f,P')$
  $= M_k(x_k - x_{k-1}) - M'(x_k - x_{k-1}) - M'(x_k - x_k)$
  $\ge M_k(x_k - x_{k-1}) - M_k(x_k - x_{k-1}) - M_k(x_k - x_k)$
  ≥ 0
  $\Rightarrow U(f,P) \ge U(f,P')$
  $\therefore L(f,P) \le L(f,P') \le U(f,P') \le U(f,P)$
- Given any partitions P, Q, we have
  $L(f,P) \le U(f,Q)$.
  $\sup\{L(f,P) : P \text{ partition}\} \le \inf\{U(f,Q) : Q \text{ partition}\}$
  Consider $R = P \cup Q$. Then since $P, Q \subset R$, we have $L(f,P) \le L(f,R) \le U(f,R) \le U(f,Q)$ $\blacksquare$

Integral properties:

- If f is integrable on $[a,b]$, then
  $F(x) := \int_a^x f(t)\,dt$ is continuous on $[a,b]$
  WLOG, assume $x \le y$.
  $F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt$
  $= \int_x^y f(t)\,dt + \int_x^a f(t)\,dt - \int_x^a f(t)\,dt$
  $= \int_x^y f(t)\,dt$.
  f is bdd by some $M \in \mathbb{R}_{\ge 0}$ by definition.
  So $|f(t)| \le M$
  $\Rightarrow |F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right|$
  $\le \int_x^y |f(t)|\,dt$
  $\le M|y-x|$
  Thus $\forall \varepsilon > 0$, $\exists \delta > 0$; $|y-x| < \delta$ where $\varepsilon > M\delta$
  $\Rightarrow |F(y) - F(x)| \le M|y-x| < M\delta = \varepsilon$
  So F is uniform continuous $\blacksquare$
  F also includes continuous
  ($\forall x$, $\forall \varepsilon_{(x)}$ $|f(x)-f(y)| < |\varepsilon_{(x)}|$)
  F is a primitive of f or $F' = f$
- (Additivity of the integral)
  If f is integrable on $[a,b]$ and $m \le f(x) \le M$ $\forall x \in [a,b]$. Then
  $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
  Let P be any partition.
  Then $U(f,P) = \sum \sup f \cdot (x_i - x_{i-1})$
  $\le \sum M \cdot (x_i - x_{i-1}) = M(b-a)$
  Similarly $L(f,P) = \sum \inf f \cdot (x_i - x_{i-1})$
  $\ge \sum m(x_i - x_{i-1}) = m(b-a)$ $\blacksquare$
  (Inequality of functions)
- If $f(x) \le g(x)$ $\forall x \in [a,b]$, and both on integrable, then
  $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
  Consider any partition P. Then
  $L(g,P) = \sum \inf g \cdot (x_i - x_{i-1})$
  $\ge \sum \inf f \cdot (x_i - x_{i-1})$
  $= L(f,P)$
  Similarly $U(g,P) \ge U(f,P)$.
  $\therefore \int_a^b g\,dx \ge \int_a^b f\,dx$ $\blacksquare$
  (Bounded)
- Let f be integrable on $[a,b]$ with $m \le f(x) \le M$ $\forall x \in [a,b]$. Then
  $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$
  Let P be any partition.
  Then $U(f,P) = \sum \sup f \cdot (x_i - x_{i-1})$
  $\le \sum M \cdot (x_i - x_{i-1}) = M(b-a)$
  Similarly $L(f,P) = \sum \inf f \cdot (x_i - x_{i-1})$
  $\ge \sum m(x_i - x_{i-1}) = m(b-a)$ $\blacksquare$
  (Inequality of functions)
- If $f(x) \le g(x)$ $\forall x \in [a,b]$, and both on integrable, then
  $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$
  Consider any partition P. Then
  $L(g,P) = \sum \inf g \cdot (x_i - x_{i-1})$
  $\ge \sum \inf f \cdot (x_i - x_{i-1})$
  $= L(f,P)$
  Similarly $U(g,P) \ge U(f,P)$.
  $\therefore \int_a^b g\,dx \ge \int_a^b f\,dx$ $\blacksquare$
  (Scalar multiplication)
- If f is integrable on $[a,b]$ $\Rightarrow$ cf is integrable on $[a,b]$ $\forall c \in \mathbb{R}$
  $\forall \varepsilon > 0$, $\exists P$; $U(f,P) - L(f,P) \le \varepsilon$
  $M_i := \sup\{f : x \in [x_{i-1}, x_i]\}$
  $m_i := \inf\{f : x \in [x_{i-1}, x_i]\}$
  $M_i^c := \sup\{cf : x \in [x_{i-1}, x_i]\}$
  $m_i^c := \inf\{cf : x \in [x_{i-1}, x_i]\}$.
  If $c \ge 0$, $M_i^c = cM_i$; $m_i^c = cm_i$,
  So $L(cf,P) = L(f,P) \le U(f,P) = U(cf,P)$
  $\therefore U(cf,P) - L(cf,P) \le \varepsilon$
  If $c < 0$, $cf = 0$. So $M_i^c = m_i^c = 0$
  If $c < 0$
  $U(cf,P) = cL(f,P)$
  $L(cf,P) = cU(f,P)$
  $\Rightarrow U(cf,P) - L(cf,P) = \frac{L(cf,P) - U(cf,P)}{c} \le \varepsilon$
  $\Rightarrow U(cf,P) - L(cf,P) \le |c|\varepsilon$
- (Product of functions)
  If f, g are integrable on $[a,b]$ then fg is integrable on $[a,b]$
  Such that $\int_a^b (fg)(x)\,dx = \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$
  WLOG, assume $\exists P$ st $[a,b]$;
  $U(f,P) - \varepsilon \le \int_a^b f(x)\,dx \le L(f,P) + \varepsilon$
  $U(g,P) - \varepsilon \le \int_a^b g(x)\,dx \le L(g,P) + \varepsilon$
  Let $M_i := \sup\{f+g : x \in [x_{i-1}, x_i]\}$
  $m_i^f := \inf\{f : x \in [x_{i-1}, x_i]\}$
  $M_i^f := \sup\{f : x \in [x_{i-1}, x_i]\}$
  $m_i^g := \inf\{g : x \in [x_{i-1}, x_i]\}$
  $M_i^g := \sup\{g : x \in [x_{i-1}, x_i]\}$
  $\Rightarrow M_i^f M_i^g \ge M_i$ and $m_i^f m_i^g \le m_i$
  $\Rightarrow L(f,P) \cdot L(g,P) \le L(fg,P) \le U(fg,P) \le U(f,P) \cdot U(g,P)$.
  $\Rightarrow U(fg,P) - L(fg,P) \le U(f,P) \cdot U(g,P) - L(f,P) \cdot L(g,P)$
  $\le U(f,P) \cdot U(g,P) - L(f,P) \cdot L(g,P)$
  $\le 2\varepsilon$
  $\therefore$ fg is integrable and $\left|\int_a^b (fg)(x)\,dx - \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx\right| \le 2\varepsilon$
- (Change of variables)
  Let $\phi:[a,b]\to\mathbb{R}$, $f:\mathbb{R}\to\mathbb{R}$.
  Assume f, ϕ' continuous everywhere.
  Then
  $\int_a^b (f \circ \phi)(x)\,\phi'(x)\,dx = \int_{\phi(a)}^{\phi(b)} f(x)\,dx$
  Let $F(x) := \int_c^x f(\phi(x))\,\phi'(x)\,dx$
  $G(x) := \int_{\phi(c)}^{x} f(x)\,dx$
  $\frac{d}{dx}\left(\int_c^x f(\phi(x))\,dx\right) = (f \circ \phi)(x)\,\phi'(x)$ by chain rule
  $\Rightarrow F'(x) = \frac{d}{dx}\left(\int_{\phi(c)}^{\phi(x)} f(x)\,dx\right)$
  $= \frac{d}{dx}\left(\int_a^{\phi(x)} f(x)\,dx - \int_a^{\phi(c)} f(x)\,dx\right)$
  $= (f \circ \phi)(x)\,\phi'(x)$
  $\frac{d}{dx}(G \circ \phi)(x) = \frac{d}{dx}\int_{\phi(c)}^{\phi(x)} f(x)\,dx$
  $= (f \circ \phi)(x)\,\phi'(x)$
  $\Rightarrow \int_c^x F'(x) - (G \circ \phi)'(x)\,dx = \text{constant}$
  i.e. $F(x) - (G \circ \phi)(x) = \text{constant}$ $\forall x$.
  Since $F(c) = 0$, $(G \circ \phi)(c) = 0$, we have
  $F(x) = (G \circ \phi)(x)$,
  i.e. $\int_c^x f(\phi(x))\,\phi'(x)\,dx = \int_{\phi(c)}^{\phi(x)} f(x)\,dx$ (Implying above)

Functions from the integral

If f is integrable on $[a,b]$, then $F(x) := \int_a^x f(t)\,dx$ is continuous on $[a,b]$

WLOG, assume $a \le x \le y \le b$.
$F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt$
$= \int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt$
$= \int_x^y f(t)\,dt$.
f is bdd by some $M \in \mathbb{R}_{\ge 0}$ by definition.
So $|f(t)| \le M$
$\Rightarrow |F(y) - F(x)| = \left|\int_x^y f(t)\,dt\right|$
$\le \int_x^y |f(t)|\,dt$
$\le M|y-x|$
Thus $\forall \varepsilon > 0$, $\exists \delta > 0$; $|y-x| < \delta$ where $\varepsilon > M\delta$
$\Rightarrow |F(y) - F(x)| \le M|y-x| < M\delta = \varepsilon$
So F is uniform continuous $\blacksquare$
F also includes continuous
($\forall x$, $\forall \varepsilon_{(x)}$ $|f(x)-f(y)| < |\varepsilon_{(x)}|$)

Fundamental Theorem of calculus:

Let f be integrable on $[a,b]$ and $F(x) := \int_a^x f(t)\,dt$.
If f is continuous at $c \in [a,b]$ then F is differentiable at c and $F'(c) = f(c)$.

(FTC 1)

Consider c continuous f and $h > 0$
$F(c+h) - F(c) = \int_a^{c+h} f(t)\,dt - \int_a^c f(t)\,dt$
$= \int_c^{c+h} f(t)\,dt$
$m_h \le \frac{F(c+h) - F(c)}{h} \le M_h$
$\Rightarrow \lim_{h\to 0^+} m_h = \lim_{h\to 0^+} M_h = \lim_{h\to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$
$\Rightarrow \lim_{h\to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$.
Consider
$F(c) - F(c-h) = \int_a^c f(t)\,dt - \int_a^{c-h} f(t)\,dt$
$= \int_{c-h}^{c} f(t)\,dt$
$m_h' \le \frac{F(c) - F(c-h)}{h} \le M_h'$
$\Rightarrow m_h' \le \frac{F(c) - F(c-h)}{h} \le M_h'$
$\lim_{h\to 0^+} m_h' = \lim_{h\to 0^+} m_h = \lim_{h\to 0^+} M_h' = \lim_{h\to 0^+} M_h' = f(c)$
$\therefore \lim_{h\to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)$
$\therefore$ If f is continuous at $c \in [a,b]$,
$F'(c) = f(c)$ $\blacksquare$

If f is continuous on $[a,b]$ and F, g for some function g, then $\int_a^b f(x)\,dx = g(b) - g(a)$ (continuity of f, g not necessary)
Consider
$F(x) := \int_a^x f(t)\,dt$ $\Rightarrow$ $F(a) = 0$
$G(x) := \int_a^x f(t)\,dt$ $\quad G(a) = 0$
Since F, g continuous on $[a,b]$,
$F'(x) = f(x) = G'(x)$.
Moreover $G'(x) = g'(x)$, so
$(F - G)$ is a constant
Since $(F - G)(a) = 0$, we $(F - G)(x) = 0$
$\Rightarrow F(x) = G(x)$ $\forall x \in [a,b]$
$\therefore \int_a^b f(x)\,dx = g(b) - g(a)$

Consider $u = -x$ $\int_a^b f(x)\,dx$, swaping the linear limit
$\int_a^b f(x)\,dx = \int_{-a}^{-b} f(-u)\,du$
$\Rightarrow \frac{d}{dx}\int_a^b f(x)\,dx \Big|_{x=c} = -f(-c)$ when f is continuous at c.
Thus we have $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (Flipping domain)
where $\frac{d}{dx}\int_a^b f(x)\,dx\Big|_{x=c} = f(c)$ by continuity.

Examples:
Let $F_k(x) := x^k$. Then $F_k'(x) = kx^{k-1}$
$\Rightarrow x^{k-1} = \frac{1}{k}F_k'(x)$
$\Rightarrow \int_a^b x^{k-1}\,dx = \frac{1}{k}\int_a^b F_k'(x)\,dx$
$= \frac{1}{k}(b^k - a^k)$

(FTC Part 2) If g' is integrable on $[a,b]$ then $\int_a^b g'(t)\,dt = g(b) - g(a)$.
Let $P = \{x_0 = a, x_1, \dots, x_n = b\}$ be a partition on $[a,b]$. Then by the mean value theorem $\exists$ a point $x_i \in [x_{i-1}, x_i]$;
$g'(x_i) = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}$,
$\inf_{[x_{i-1},x_i]} g' \le g'(x_i) \le \sup_{[x_{i-1},x_i]} g'$
$\Rightarrow m_i(x_i - x_{i-1}) \le g(x_i)(x_i - x_{i-1}) \le M_i(x_i - x_{i-1})$
$\Rightarrow m_i(x_i - x_{i-1}) \le g(x_i) - g(x_{i-1}) \le M_i(x_i - x_{i-1})$
$\Rightarrow L(g,P) \le g(b) - g(a) \le U(g,P)$ $\forall P$.
$\Rightarrow \int_a^b g'(t)\,dt \le g(b) - g(a)$ $\blacksquare$

Exponent and logarithm using integrals:

Consider $f(x) := \frac{1}{x}$, $f:(0,\infty)\to(0,\infty)$, continuous.
Then $F(x) := \int_1^x \frac{1}{t}\,dt$ is a primitive of $\frac{1}{x}$.
Use define $\ln(x) := \int_1^x \frac{1}{t}\,dt$, $x > 0$ (Logarithm)
and $\exp(x) := \ln^{-1}(x)$, $y \in \mathbb{R}$. (Exponential)

Properties:
- F is strictly monotone increasing
  Since $\frac{1}{t}$ continuous, $F'(x) = \frac{1}{x} > 0$ $\forall x > 0$
  $\Rightarrow F$ monotone increasing
- $\ln(1) = 0$
  Take $x \le 1$, $x \ge 1$.

  $F(x^{-1}) = \int_1^{x^{-1}} \frac{1}{t}\,dt = \int_x^1 \frac{1}{t}\,dt$
  $\ge \int_x^1 \frac{1}{1}\,dt$
  Taking same $F(x) \ge \lim_{x\to 0} \frac{1}{x} = \infty$ (Contradiction)
- If $a, b > 1$, then $F(a \cdot b) = F(a) + F(b)$
  $F(a \cdot b) = \int_1^{a \cdot b} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{a \cdot b} \frac{1}{t}\,dt$
  $= \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{sa}\,ds$
  $= \int_1^a \frac{1}{t}\,dt + \int_1^b \frac{1}{s}\,ds = F(a) + F(b)$
- If $x > 1$, then $F(x^{-1}) = -F(x)$
  $F(x^{-1}) = \int_1^{x^{-1}} \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$
  $= -\int_1^x \frac{1}{t}\,dt = -F(x)$
- $-\lim_{x\to 0^+} F(x) = -\infty$
  $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} F\left(\frac{1}{x}\right) = \lim_{x\to\infty} -F(x)$
  $= -\infty$
- Let $G: \mathbb{R} \to (0,\infty)$ be the inverse of F. Then
  $\begin{cases} G'(0) = 1 & (1) \\ G(x+y) = G(x) \cdot G(y) & (2) \end{cases}$
  Since F is monotone increasing, it is a bijection onto its image, $\mathbb{R}$. By the inverse function theorem, it
  G is its inverse
  $G'(F(x)) = \frac{1}{F'(x)}$
  (1) So $G'(F(0)) = G(0) = \frac{1}{F'(0)} = \frac{1}{\ln^{-1}(1)} = 1$
  (2) $F: G(x+y) = x+y$
  $F: (G(x) \cdot G(y)) = F(G(x) \cdot G(y))$
  $= x+y$
  By bijectivity of F, $G(x+y) = G(x) \cdot G(y)$ $\blacksquare$

$e^y > |y|$ $\forall y \ge 1$

Derivative and Integrals

Derivative:

Assuming f is defined on an open interval $I \ni x_0$.
- $f'(x_0) := \frac{df(x)}{dx}\Big|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$
- $f'(x_0)$ is the derivative of f at x_0
- f is differentiable at x_0 if the limit $\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists.

Equivalent limits: $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$, $h = x - x_0$

- If f is differentiable at $x \in \mathbb{R}$, it is continuous at x. (Differentiability is stronger)
  Suppose f differentiable at x
  $\Rightarrow \exists L$; $\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = L$
  Since $\lim_{h\to 0} h = 0$ exists,
  $\Rightarrow \lim_{h\to 0} f(x+h) - f(x) = L \cdot \lim_{h\to 0} h$
  By the algebra of limits.
  $\Rightarrow \lim_{h\to 0} f(x+h) - f(x) = 0$
  $\Leftrightarrow \lim_{h\to 0} f(x+h) = f(x)$
  $\Leftrightarrow \lim_{x\to x_0} f(x) = f(x_0)$

Examples of derivatives:
- $f(x) = c$ $\forall x \in \mathbb{R}$. (Constant)
  $\Rightarrow f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0$ $\blacksquare$
- $f(x) = x$ $\forall x \in \mathbb{R}$. (Identity)
  $\