Analysis T2

- Open interval: (a,b)
- Closed interval: $[a,b]$
  - usually related to using EVT/IVT, uniform continuity

- Open set: $A \subseteq \mathbb{R}$, $\forall a \in A$, $\exists\, \varepsilon > 0$, $(a-\varepsilon, a+\varepsilon) \subset A$
- Closed set: $B \subseteq \mathbb{R}$, $A = \mathbb{R} \setminus B$ is open (complement of an open set)
  - $\Leftrightarrow$ B closed iff $\{a_n\} \subset B$ $\lim_{n\to\infty} a_n \in B$ $\Rightarrow \lim_{n\to\infty} a_n \in B$)
  - Sometimes used for defining closure, limit points, isolated points.
  - Empty set is both open and closed
  - E.g. $B = \mathbb{Z}$ is closed since $\mathbb{R} \setminus \mathbb{Z}$ is open. Then let $a_n = n$. No subsequence of a_n is convergent.
- Every open set $\subset \mathbb{R}$ is a union of countably many disjoint open intervals.
  - Proof: Let $U \subset \mathbb{R}$ be an open set. Let $x \in U$ and let $I(x) = \bigcup\{S \subset \mathbb{R} \mid S \text{ is open}, x \in S, S \subset U\}$. Show $I(x)$ is open and bounded. Suppose (a_1,b_1) and (a_2,b_2) overlap. Then $(a_1,b_2) \in A$ but $(a_1,b_1) \in A$. Thus all $I(x) \in A$ are disjoint. If I_1 is the union of all $I_1(x) \subset A$. If $x \in I(x_1)$, $x \in I(x_2)$ and $I(x_1) \neq I(x_2)$. Then $I(x_1) \cup I(x_2) \subset A$. Then I_1 is the union of disjoint open intervals.
- Compact set: $K \subset \mathbb{R}$ is compact iff every $\{x_n\} \subset K$ contains a convergent subsequence $\{x_{n_k}\}$ and $\lim_{k\to\infty} x_{n_k} \in K$.
  - Proof:
    - ($\Rightarrow$) Let K be compact. Suppose not closed. Then $\exists\, \{x_n\} \subset K$, $x_n \to L \notin K$ (contradiction). So K is closed. Suppose K is not bdd. Then $\exists\, \{x_n\} \subset K$, $|x_n| \to \infty$. Then $\forall x_{n_k}$, x_{n_k} is also not bdd (contradiction). Then K is bdd.
    - ($\Leftarrow$) Let K be closed and bdd. Then all $\{x_n\} \subset K$ is bdd. So by Bolzano-Weierstrass, we have that $\exists\{x_{n_k}\}$ convergent $\forall \{x_n\} \subset K$. Then K is compact.
- If $A \subset \mathbb{R}$ is a nonempty closed set, $\forall a \in \mathbb{R}\setminus A$ the closest a to a.
  - Proof: $x \in A$ minimal. $x \notin A$. $I = (x-|x|, x+|x|) > 0$. WLOG $I = I_r = (a, b_r)$. $\lim_{r\to 0} |x-a| = 0$ so $a \in A$ $\Rightarrow \Leftarrow$ bdd. By $a_n \in A$, we have $a_n \to x \in A$ since A closed. So $I_r \in I \Rightarrow I_r \cap x = \emptyset$. $\forall x \in \mathbb{R}$, $\exists a \in A$, $|x-a| = \min I$.

- Continuity: $f: I \to \mathbb{R}$ is continuous at $c \in I$ if $\lim_{x\to c} f(x) = f(c)$
  - $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$, $|x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon$
  - Complex version (replace absolute value with modulus)
  - If $f(x)$ and $g(x)$ are continuous at c, then
    - $\to f+g$ continuous
    - $\to f-g$ continuous
    - $\to \frac{f}{g}$ continuous if $g(c) \neq 0$
  - If g is continuous at c and f is continuous at $g(c)$, then $f \circ g$ is continuous at c. (Composition of functions)
    - Proof: $\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x))$ (by limit algebra)
  - Proof: $\lim_{x\to c} f(x)g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$ (by limit algebra)
  - Proof: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$ (by limit algebra)
    - Alt proof: $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} f(x) \cdot \frac{1}{g(x)}$
      - Show $g(x) \neq \frac{1}{g(x)}$
      - Then $\left|\frac{1}{g(x)} - \frac{1}{g(c)}\right| < k \cdot |g(x)-g(c)|$, so
      - $\lim_{x\to c} \frac{1}{g(x)} = \frac{1}{g(c)}$
    - Then $\lim_{x\to c} \frac{f}{g} = \frac{\lim f}{\lim g}$
  - If g continuous at c and f continuous at $g(c)$, then $f \circ g$ is continuous at c.
    - Proof: $\exists\, \delta_1 > 0$, $|y-g(c)| < \delta_1 \Rightarrow |f(y)-f(g(c))| < \varepsilon$.
    - $\exists\, \delta_2 > 0$, $|x-c| < \delta_2 \Rightarrow |g(x)-g(c)| < \delta_1$
    - $\therefore |x-c| < \delta_2 \Rightarrow |f(g(x))-f(g(c))| < \varepsilon$
- Continuity of every point:
  - E.g. $\lim_{x\to 0} \frac{1}{x}$ does not exist. Let $x_n = -\frac{1}{n}$, $y_n = \frac{1}{n}$. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. $\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} \to -\infty$. Thus contradicting the uniqueness of limits.
  - E.g. $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist. (Oscillating)
    - If $\lim_{x\to c} f(x) = L$ then $\lim_{x\to c} f(x_n) = L$
      - $\lim_{x\to c} x_n = c \Rightarrow \lim_{n\to\infty} f(x_n) = L$
      - Let $x_n = \frac{1}{\pi/2 + n\pi}$, $y_n = \frac{1}{n\pi}$. (Use sequence and limit, sequence and limits)
      - $\sin x_n \to 1$, $\sin y_n \to 0$
      - $f(x_n) = 1$ & $f(y_n) = 0$ $\forall n$
      - (Contradiction)
  - E.g. $\lim_{x\to 0} \sin(\frac{1}{x}) = 0$
    - $|\sin(\frac{1}{x})| \le 1$
    - $-1 \le \sin\frac{1}{x} \le 1$
    - $\therefore |\sin x| < \varepsilon \Rightarrow |x \sin\frac{1}{x} - 0| < \varepsilon$
    - $\therefore \lim = 0$ $\square$
    - Squeeze
- (Meeting point) Let $f: (a,c) \to (0,1)$ continuous. Then $\exists\, c$: $f(c) = c$.

- Extreme Value Theorem:
  - Let $f: [a,b] \to \mathbb{R}$ be continuous at every point of the closed interval $[a,b]$. f attains a maximum in $[a,b]$, i.e. $\exists\, c$ bdd above in $[a,b]$
    - Proof:
      - Let $A := \{x \in [a,b] \mid f \text{ is bdd above in } [a,x]\}$
      - By continuity, $\exists\, \delta > 0$, f is bdd on $[c, c+\delta]$.
      - Then f is bdd on $[a, c+\frac{\delta}{2}] \Rightarrow c + \frac{\delta}{2} \in A$
      - A is bdd since $A \subset [a,b]$ that is bdd
      - Let $x = \sup(A)$. (Exists by greatest axiom)
      - $x \le b$ by bdd-ness of A.
      - Suppose $x < b$. $\forall \varepsilon > 0$, $\exists\, u \in A$; $x - \varepsilon < u \le x$ (By $x = \sup(A)$)
      - Since f is continuous, $\exists\, \delta > 0$; f is bdd on $(x-\delta, x+\delta)$
      - Let $v \in (u, b)$. Then f is bdd on $[a,v]$ and f is bdd on $(x-\delta, x+\delta)$. Since $v - x < \delta$ can, $f \ni$ bdd on $[a,v] \cup (x-\delta, x+\delta) = [a, x+\delta)$.
      - So f is bdd on $[a, x+\frac{\delta}{2}]$
      - $\Rightarrow x + \frac{\delta}{2} \in A$ (Contradiction)
      - Thus $x = b$ because $f \ni$ bdd above in $[a,b] = (x,b) \cup \{b\}$ $\square$
    - Bad news for finding the gradient of f bdd
    - Stronger than uniform continuity
      - E.g. $\sqrt{x}$ on $[0,\infty)$ is uniform continuous but not Lipschitz.
  - Suppose $f: (a,b) \to \mathbb{R}$. It does not necessarily satisfy EVT. E.g. $f(x) = \frac{1}{x-a}$. Then f continuous on (a,b) but is not bdd above.
    - (Not asking to show minimum exists using using closed bounded or sequence sub proof.)
  - If $X \subseteq \mathbb{R} \to \mathbb{R}$ continuous and X compact $\Rightarrow$ f attains a maximum in X.
    - Proof:
      - X compact $\Leftrightarrow$ X closed and bdd.
      - Since X closed, X^c bdd and X^c is the union of disjoint open intervals $\Rightarrow$ X is the union of disjoint closed intervals. In every closed interval, we have a c ; $f(c)$ is a maximum. Take the maximum of all $f(c)$ and that f attains a maximum in X.
  - If $f: [a,b] \to \mathbb{R}$ continuous attains a minimum in $[a,b]$.
    - Proof:
      - Let $g(x) = -f(x)$. g is also continuous at all points $x \in [a,b]$.
      - By EVT, $g(x)$ attains a maximum in $[a,b]$. I.e. $\exists c \in [a,b]$;
      - $g(c) \ge g(x)$ $\forall x \in [a,b]$
      - so $-f(c) \ge -f(x)$ $\forall x \in [a,b]$
      - so $f(c) \ge f(x)$ $\forall x \in [a,b]$
      - So $f(c)$ is a minimum $\square$
    - $\Rightarrow$ EVT: $f: [a,b] \to \mathbb{R}$ continuous $\Rightarrow$ $f(c)$ bdd in $[a,b]$
  - Let $f: [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Then $\exists\, f(x_m) \ge f(t)$ $\forall t \in [a,b]$
    - Proof:
      - $F := \{f(x) \mid x \in [a,b]\}$ is bdd above.
      - Let $m = \sup(F)$. $\exists\, \{y_n\} \subset F$;
      - $y_n \to m$. Since $y_n \in F$.
      - $\exists x_n$; $f(x_n) = y_n$, i.e. $f(x_n) \to m$.
      - Since $a \le x_n \le b$, by bolzano weierstrass $\exists$ convergent x_{n_i}.
      - Let $\lim_{i\to\infty} x_{n_i} = x$.
      - $\{f(x_{n_i})\}$ is a subsequence of $\{f(x_n)\}$, so $f(x_{n_i})$ converges to m.
      - Thus $x_{n_i} \to x$, $x \in [a,b]$ (Not at closed side) and $f(x_{n_i}) \to m = \sup F$.
      - Thus $f(x) = m$ $\square$

- Bolzano's Theorem: Let $f: [a,b] \to \mathbb{R}$ be continuous and $f(a) < 0 < f(b)$. Then $\exists\, c \in (a,b)$, $f(c) = 0$.
  - Proof: (Find the least time $f(x)$ is non-negative)
    - Let $A = \{x \in [a,b] \mid f < 0 \text{ on } [a,x]\}$
    - From continuity $\exists \delta > 0$, $f(x) < 0$ $\forall x \in [a, a+\delta)$. Then A is nonempty.
    - From continuity $\exists \delta > 0$, $f(x) > 0$ $\forall x \in (b-\delta, b]$ $\Rightarrow A \cap (b-\delta, b] = \emptyset$ $\Rightarrow A$ is bdd above by $b-\delta$.
    - From nonempty and bdd above, A has a supremum c.
    - Suppose $f(c) < 0$. $\exists \delta$, $f < 0$ $\forall x \in (c-\delta, c+\delta)$. Then $x = \sup(A)$, $x + \frac{\delta}{2} \in A$. $\Rightarrow \Leftarrow$ (Contradiction)
    - Suppose $f(c) > 0$. $\exists \delta$, $f > 0$ $\forall x \in (c-\delta, c+\delta)$. $\Rightarrow \exists\, c - \frac{\delta}{2} \in A$ (Contradiction)
    - By Trichotomy, $f(c) = 0$, $c = \sup(A)$ $\square$
    - $\Rightarrow$ $\sup(A) \in A$ and $\max(A)$
  - Intermediate Value Theorem: Let $f: [a,b] \to \mathbb{R}$ be continuous at every point of the closed interval $[a,b]$. Then f attains all values between $f(a)$ and $f(b)$.
    - Proof: Suppose $f(a) < g(a) < g$, $g(b) < g < f(b)$. Then $f(a) - g(a) < 0$, $f(b) - g(b) > 0$. By Bolzano, $\exists\, c \in (a,b)$, $f(c) = 0$. Then $g(c) = y$ $\square$
    - Example of proof not being satisfactory for IVT: $\{x \mid x \le 0 \cup x \ge 2\}$, $f(x) = x$, $-1 < f(-1) < f(1) < 1$, but $\nexists\, f(c) = 0$, since $0 \notin S$
  - Corollary: Every odd degree polynomial has a real root.
    - Proof:
      - Let $f(x) = a^n x^n + a_{n-1}x^{n-1} + \dots + a_0$
      - n an odd degree polynomial.
      - For $|x| > n \max(|a_0|, \dots, |a_n|)$
        - $|x^n| \ge |a_{n-1}||x|^{n-1} + \dots + |a_0|$
      - $\Rightarrow \operatorname{sign}(f(x)) = \operatorname{sign}(x^n)$ as long as $|x| > n\max(|a_0|, \dots, |a_n|)$.
      - If $f(x)$ is odd degree, $\operatorname{sign}(x^n) = \operatorname{sign}(x)$.
      - If $a_n = \max(|a_0|, \dots, |a_{n-1}|) + 1$
        - $x < 0 \Rightarrow x^n < 0 \Rightarrow f(x) < 0$.
        - Similarly, if $a_n = \max(|a_0|, \dots, |a_{n-1}|) + 1$
        - $x > 0 \Rightarrow x^n > 0 \Rightarrow f(x) > 0$.
      - Then by Bolzano's theorem, $\exists\, c \in [a_n, a_n]$; $f(c) = 0$, since c a real.
      - Thus f has a real root $\square$

- Continuity: $f: I \to \mathbb{R}$ is continuous at $c \in I$ if $\lim_{x\to c} f(x) = f(c)$
  - $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$, $|x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon$
  - Complex version (replace absolute value with modulus)

- Left and Right Continuity:
  - $\lim_{x\to c^+} f(x) = L$ $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x_n$ $c < x_n < c + \delta$, $|f(x) - L| < \varepsilon$
  - $\lim_{x\to c^-} f(x) = L$ $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x_n$ $c - \delta < x_n < c$, $|f(x) - L| < \varepsilon$
    - $\Rightarrow \lim_{x\to c^+} f(x) = L$ or $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$
      - Proof:
        - Supp $\lim_{x\to c^+} f(x) = L$.
        - $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall x_n$ $|x-c| < \delta$, $|f(x) - L| < \varepsilon$.
        - Then $\forall x_n$ $c < x_n < c + \delta$ and $\forall x_n$ $-\delta < x - c < 0 \Rightarrow |f(x) - L| < \varepsilon$
        - $\therefore \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$.
        - Supp $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$.
        - Let $\varepsilon > 0$. $\exists \delta_1$; $x - c < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$.
        - $\exists \delta_2$; $c - x < \delta_2 \Rightarrow |f(x) - L| < \varepsilon$.
        - $\delta = \min(\delta_1, \delta_2)$. Then $|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$ $\square$
  - Continuity on a closed interval: $f: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ if
    - $\lim_{x\to a^+} f(x) = f(a)$ $\forall a \in (a,b)$ (continuous on (a,b))
    - $\lim_{x\to b^-} f(x) = f(b)$
    - $\lim_{x\to a^+} f(x) = f(a)$ (right and left continuous on endpoints)

- Lipschitz Continuity: $\exists\, C$; $|f(x)-f(y)| < C|x-y|$ $\forall x, y$
  - equivalent to having the gradient of f bdd
  - Stronger than uniform continuity
    - E.g. $\sqrt{x}$ on $[0,\infty)$ is uniform continuous but not Lipschitz.
- Uniform Continuity: $f: X \subseteq \mathbb{R} \to \mathbb{R}$ is uniformly continuous on X if $\forall \varepsilon > 0$ $\exists\, \delta > 0$; $\forall x, y \in X$, $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
  - (x, y — the points approximated)
  - uniform continuity on every finite set do uniform continuity is guaranteed
  - e.g. not uniform because bound on x^2 is not continuous on $\mathbb{R}$ (infinite)
  - If not uniform continuous, $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x, y \in X$; $|x-y| < \delta$ and $|f(x) - f(y)| \ge \varepsilon$ } Required expression x and y in round δ δ
    - E.g. $f(x) = \frac{1}{x}$ on $(0,1)$, $\varepsilon = 1$, $\delta > 0$, $|x-y| < \frac{\delta}{2}$, $y = \frac{\delta}{2}$, $x = \delta$. $|f(x) - f(y)| > \frac{1}{\delta}$. $\forall \delta < 2$, we have $|f(x) - f(y)| \ge 1$ $\square$
  - Uniform Continuity theorem: Let a, b, f continuous on $[a,b]$. Then $f \Rightarrow$ uniformly continuous on $[a,b]$
    - Proof: Suppose f not uniformly continuous. $\exists \varepsilon > 0$, $\forall \delta > 0$, $|x-y| < \delta$; $|f(x) - f(y)| \ge \varepsilon$. By BW, $\exists$ sub in $[a,b]$. By Bolzano weierstrass $\exists x_{n_i} \to c \in [a,b]$. By triangle inequality $\forall x_{n_i} \to c$. By triangle inequality, $|f(x_{n_i}) - f(c)| \ge |f(x_{n_i})| - |f(c)| \ge \varepsilon/2$ $\forall x_{n_i}$. But this contradicts continuity $\square$
    - If f not uniformly continuous $\exists\, x_n, y_n$; $|x_n - y_n| \le \frac{1}{n}$, $|f(x_n) - f(y_n)| > \varepsilon$ $\forall n$.
      - Proof: $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists x_n, y_n$; $|x_n - y_n| \le \frac{1}{n}$, $|f(x_n) - f(y_n)| > \varepsilon$. $S = \frac{1}{n}$. Then $\forall x_n, y_n$ the x_n and y_n form a Cauchy. Then $\forall x_n$, $|x_n - y_n| < \frac{1}{n}$, $|f(x_n) - f(y_n)| > \varepsilon$ and x_n, y_n Cauchy.
  - If X is compact and $f: X \to \mathbb{R}$ continuous, it is uniformly continuous
    - Simpler Proof dep $[a,b]$ closed

- Convergence of a function to a point:
  - Let $f: (a,b) \to \mathbb{R}$, $c < x < b$, $y \in \mathbb{R}$. Then, $\lim_{x\to c} f(x) = y$ $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists \delta := \delta(\varepsilon) > 0$; $|x - c| < \delta$ and $x \ne c$ $\Rightarrow |f(x) - y| < \varepsilon$
    - $\Leftrightarrow$ $\forall$ open intervals $J \subseteq \mathbb{R}$ containing y, $\exists$ an open interval I containing c, $f(I \setminus \{c\}) \subset J$
    - $(a,b) \cap c$ Open interval definition: $J \subset \mathbb{R}$ open, $y \in J$. Let $c < c$, $(y-\varepsilon, y+\varepsilon) \subset J$. $\exists\, \delta > 0$, $(c < |x| < \delta \Rightarrow |f(x) - y| < \varepsilon)$ $\Leftrightarrow (f((c, c+\delta) \cap I) \subset (y-\varepsilon, y+\varepsilon) \subset J)$ $\Leftrightarrow (f(I \setminus \{c\}) \subset J$, $I = (c, c+\delta, c+\delta))$ $\square$
  - Sequence definition of function limit: $\lim_{x\to c} f(x) = y$ iff $\forall x_n \in \mathbb{R} \setminus \{c\}$; $x_n \to x_n$, $f(x_n) \to y$
    - Proof:
      - Supp $\lim_{x\to c} f(x) = y$ and $x_n \to c$.
      - $\Leftrightarrow$ $x_n \in (x_n - \varepsilon, x_n + \varepsilon)$
      - Let $\varepsilon > 0$. $\exists\, \delta > 0$, $\forall |x - x_n| < \delta$, $|f(x) - y| < \varepsilon$. Then $\exists N$, $\forall n \ge N$, $|x_n - x| < \varepsilon, \varepsilon > 0$. Then $\forall n \ge N$, $|f(x_n) - y| < \varepsilon$. Then $f(x_n) \to y$.
      - Supp $\exists x_n \to c$, $f(x_n) \to y$. Using the above steps, we could get a contradiction with the ε-δ definition of $\lim_{x\to c} f(x) = y$ $\square$
  - Squeeze Theorem: Let f, g, h be functions on $(a,b) \setminus \{c\}$. $\forall x \in (a,b)$, $f(x) \le g(x) \le h(x)$. If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = y$ then $\lim_{x\to c} g(x) = y$
    - Proof:
      - $f(x) - a \le g(x) - a \le h(x) - a$
      - Let $\varepsilon > 0$. $\exists \delta_1 > 0$, $|x - a| < \delta_1 \Rightarrow |f(x) - a| < \varepsilon$
      - $\Rightarrow -\varepsilon < f(x) - a < \varepsilon$
      - $\exists \delta_2 > 0$; $|x - y_n < \delta_2 \Rightarrow |h(x) - a| < \varepsilon$
      - $\Rightarrow h(x) - a < \varepsilon$
      - $\therefore$ Let $\delta = \min(\delta_1, \delta_2)$, then $|x - a| < \delta$
      - $\Rightarrow -\varepsilon < g(x) - a < \varepsilon$
      - $\Leftrightarrow |g(x) - a| < \varepsilon$ $\square$
  - If $\lim_{x\to c} f = a$ and $\lim_{x\to c} f = b$, then $a = b$ (Uniqueness of limits)
    - Proof: Supp $a \ne b$. Let $\varepsilon = \frac{|a-b|}{2} > 0$. Then $\exists \delta_1, \delta_2$; $|x - c| < \delta_1 \Rightarrow |f(x) - a| < \frac{|a-b|}{2}$, $|x - c| < \delta_2 \Rightarrow |f(x) - b| < \frac{|a-b|}{2}$. Choose $\delta := \min(\delta_1, \delta_2)$, then $|x - c| < \delta$ $\Rightarrow |f(x) - a| < \frac{|a-b|}{2}$ and $|f(x) - b| < \frac{|a-b|}{2}$. $|a-b| = |a - f(x) + f(x) - b| \le |f(x) - a| + |f(x) - b| < |a-b|$ (Contradiction). So $a = b$; $\frac{|a-b|}{2} = 0$.

- Limit algebra:
  - Supp $f(x)$ and $g(x)$ have limits at x_0
    - $\lim_{x\to x_0} f(x) + g(x) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$
    - $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x)$
    - $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$ ($g(x) \ne 0$ on some N_δ)
      - Proof: Let $f(x) = a$, $g(x) = b$ on some $\left|\frac{f(x)}{g(x)} - \frac{a}{b}\right| = \left|\frac{f(x)b - g(x)a}{g(x)b}\right|$
        - $= \frac{1}{|g(x)b|} \cdot \frac{1}{|b|} \cdot |f(x)b - ab + ab - g(x)a|$
        - $\le \frac{1}{|g(x)b|} \cdot \frac{1}{|b|} \cdot (|b||f(x) - a| + |a||g(x) - b|)$
        - Since $g(x) \to b$, non-zero on some $\varepsilon = \frac{|b|}{2}$. $\exists \delta_1$, $\forall x$ $|x - x_0| < \delta_1$, $|g(x) - b| < \frac{|b|}{2}$
        - $\Rightarrow \frac{|b|}{2} < |g(x)| < \frac{3|b|}{2}$
        - So $\frac{1}{|g(x)|} < \frac{2}{|b|}$. $\exists \delta_2$, $\forall x$ $|x - x_0| < \delta_2$, $|f(x) - a| < \frac{\varepsilon}{2|b|}$
        - $\Rightarrow \frac{\varepsilon}{2|g(x)|} \exists \delta_3$, $\forall x$ $|x - x_0| < \delta_3$, $|g(x) - b| < \frac{\varepsilon}{2|g(x)|}$
        - Let $\varepsilon > 0$
        - $\delta := \min(\delta_1, \delta_2, \delta_3)$
        - $\forall x$ $|x - x_0| < \delta$
        - $\left|\frac{f(x)}{g(x)} - \frac{a}{b}\right| < \frac{2}{|b|} \cdot \frac{1}{|b|} \left(|b| \cdot \frac{\varepsilon}{2|b|} + |a| \cdot \frac{\varepsilon}{2|g(x)|}\right)$
        - $= \left(\frac{1}{|b|} \cdot \frac{\varepsilon}{|b|} \left(\frac{1}{2} + \frac{|a|}{2|g(x)|}\right)\right) \varepsilon$
        - $= C\varepsilon$, $C > 0$.
        - $\therefore \left|\frac{f(x)}{g(x)} - \frac{a}{b}\right| < C\varepsilon$ $\square$
    - $\lim_{x\to x_0} f(x)g(x) = \lim_{x\to x_0} f(x) \cdot \lim_{x\to x_0} g(x) = ab$
      - Proof: Supp $\lim_{x\to x_0} f(x) = a$, $\lim_{x\to x_0} g(x) = b$. Let $\varepsilon, \delta_1 > 0$, $\delta_2 = \delta_3 = \delta = \min(\delta_1, \delta_2)$. $\forall x_n$, $|x_n - x_0| < \delta$.
        - $|f(x)g(x) - ab| = |f(x)g(x) - af(x) + af(x) - ab|$
        - $\le |f(x)||g(x) - b| + |b||f(x) - a|$
        - $\le |f(x)||g(x) - b| + |b||f(x) - a|$
        - Take $\varepsilon = 1$, $\exists \delta_1$, $\forall x_n$ $|x - x_0| < \delta_1$, $|f(x)| \le |f(x) - a| + |a| < 1 + |a|$
        - Take $\varepsilon = \frac{\varepsilon}{2(1+|a|)}$, $\exists \delta_2$, $\forall x_n$ $|x - x_0| < \delta_2$, $|f(x) - a| < \frac{\varepsilon}{2(1+|a|)}$
        - Take $\varepsilon = \frac{\varepsilon}{2|b|+1}$, $\exists \delta_3$, $\forall x_n$ $|x - x_0| < \delta_3$, $|g(x) - b| < \frac{\varepsilon}{2(2|b|+1)}$
        - $\therefore$ Let $\delta = \min(\delta_1, \delta_2, \delta_3)$
        - $\forall x$ $|x - x_0| < \delta$
        - $|f(x)g(x) - ab| \le |f(x)(g(x) - b)| + |b(f(x) - a)|$
        - $< |f(x)||g(x) - b| + (1+|b|)|f(x) - a|$
        - $< (1+|a|)\left(\frac{\varepsilon}{2(1+|a|)}\right) + (1+|b|)\left(\frac{\varepsilon}{2(1+|b|)}\right)$
        - $= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$
        - $\therefore |f(x)g(x) - ab| < \varepsilon$ $\square$
    - Squeeze Approach Similar

- Sequence Theorem: Let f, g, h be functions on $(a,b) \setminus \{c\}$.
  - $\forall x \in (a,b)$, $f(x) \le g(x) \le h(x)$.
  - If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = y$ then $\lim_{x\to c} g(x) = y$
  - Proof:
    - $f(x) - a \le g(x) - a \le h(x) - a$
    - Let $\varepsilon > 0$. $\exists \delta_1 > 0$, $|x - a| < \delta_1 \Rightarrow |f(x) - a| < \varepsilon$
    - $\Rightarrow -\varepsilon < g(x) - a < \varepsilon$
    - $\exists \delta_2 > 0$; $|x - y_n < \delta_2 \Rightarrow |h(x) - a| < \varepsilon$
    - $\Rightarrow h(x) - a < \varepsilon$
    - $\therefore$ Let $\delta = \min(\delta_1, \delta_2, \delta_3)$
    - $\forall x$ $|x - x_0| < \delta$
    - $-\varepsilon < g(x) - a < \varepsilon$
    - $\Leftrightarrow |g(x) - a| < \varepsilon$ $\square$

- Differentiability/existence of derivative:
  - f is differentiable at x_0 if f is defined in an open interval containing x_0 and $\exists L$; $\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = L$
    - $\Leftrightarrow$ $f'(x_0) = \frac{df}{dx}(x_0) = L$.
    - $\Leftrightarrow$ $\lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} = L$ $(x_0 \to x + h)$
  - Derivative from limit definition
    - $f(x_0) = x$; $f'(x_0) = \lim_{h\to 0} \frac{x + h - x}{h} = 0$
    - $f(x_0) = x$; $f'(x_0) = \lim_{h\to 0} \frac{x + h - x}{h} = 1$
    - $f(x_0) = x^n$; $f'(x_0) = \lim_{h\to 0} \frac{f(x) - f(x_0)}{x - x_0}$
      - $= \lim_{x\to x_0} \frac{x^n - x_0^n}{x - x_0}$
      - $= \lim_{x\to x_0} x^{n-1} + x x^{n-2} + \dots + x^{n-1}$
      - $= n x_0^{n-1}$
  - Set of differentiable functions: $C^1(I) = \{f: I \to \mathbb{R} \mid f^{(1)} \text{ continuous on } I\}$
    - $C^1(I)$ is a vector space: $(f+g)' = f' + g'$, $(cf)' = cf'$
      - $\frac{d}{dx}(f+g) = \frac{d}{dx} f + \frac{d}{dx} g$
      - $\frac{d}{dx}(cf) = c\frac{d}{dx} f$
  - f is k times differentiable at x_0 if
    - $f^{(1)}(x_0) = \lim_{h\to 0} \frac{h^{-1}(x_0 h) - h^{-1}(x)}{h}$
    - $\vdots$
    - $f^{(k)}(x) = \lim_{h\to 0} \frac{f^{(k-1)}(x_0 + h) - f^{(k-1)}(x)}{h}$ all except.
  - (Strength of differentiability)
    - If f is differentiable at x then it is continuous at x_0
      - Proof:
        - $\lim_{h\to 0} f(x_0 + h) - f(x)$ (Continuity)
        - $\Leftrightarrow \lim_{h\to 0} (f(x_0 + h) - f(x)) = 0$
        - If $\lim_{h\to 0} \frac{f(x_0 + h) - f(x)}{h}$ exists.
        - $\lim_{h\to 0} \frac{f(x_0 + h) - f(x)}{h} \cdot h = \lim_{h\to 0} (f(x_0 + h) - f(x)) = 0$
  - Derivative Properties:
    - If f, g differentiable at $x_0 \in \mathbb{R}$. then $f + g$ differentiable at x_0 and $(f+g)' = f' + g'$
    - (Leibniz/product) If f and g are differentiable at x then $f \cdot g$ is differentiable at x_0 and $(f \cdot g)' = f' \cdot g + g' \cdot f$
    - If g differentiable at x_0 and $g(x_0) \ne 0$, then $\frac{1}{g(x)}$ differentiable at x_0 and $\left(\frac{1}{g}\right)' = -\frac{g'}{g^2}$
    - (Quotient rule) If f and g are differentiable at x_0 with $g(x_0) \ne 0$, then $\frac{f}{g}$ is differentiable at x_0 and $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - g' \cdot f}{g^2}$
    - If $c \in \mathbb{R}$ and f differentiable at x_0, then $(cf)' = c \cdot f'$.
    - Proof: $\lim_{h\to 0} \frac{f(x_0 + h) + g(x_0 + h) - f(x_0) - g(x_0)}{h}$
      - $= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} + \frac{g(x_0 + h) - g(x_0)}{h}$
      - $= f'(x_0) + g'(x_0)$
    - Proof: $\lim_{h\to 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0)}{h}$
      - $= \lim_{h\to 0} \frac{f(x_0 + h)g(x_0 + h) - f(x_0)g(x_0 + h) + f(x_0)g(x_0 + h) - f(x_0)g(x_0)}{h}$
      - $= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h} g(x_0 + h) + f(x_0) \frac{g(x_0 + h) - g(x_0)}{h}$
      - $= f' \cdot g + g' \cdot f$
    - Proof: $\lim_{h\to 0} \frac{\frac{1}{g(x_0 + h)} - \frac{1}{g(x_0)}}{h} = \lim_{h\to 0} \frac{g(x_0) - g(x_0 + h)}{g(x_0 + h) \cdot h}$
      - $= \frac{1}{g^2} \cdot (-g') = -\frac{g'}{g^2}$
    - Proof: By Leibniz,
      - $f' \left(\frac{1}{g}\right) + \left(\frac{1}{g}\right)' f$
      - $f' \left(\frac{1}{g}\right) - \frac{g'}{g^2} f$
      - $\frac{f' \cdot g - g' \cdot f}{g^2}$
  - Elementary function derivatives:
    - Polynomials
    - Trig
    - Exponential: $f(x) = e^x$; $f(x+y) = f(x) f(y)$, $f(0) = 1$.
      - $f(x) = f(x)$
      - $f: \mathbb{R} \to (0, \infty)$
      - f is continuous, monotonic
      - f is bijective.
      - Proof: $\lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$
        - $= e^x \lim_{h\to 0} \frac{e^h - 1}{h}$
        - $= e^x \left(\lim_{h\to 0} e^h\right)$
        - $= e^x$
    - Logarithm
      - $f(x) = \log x$
      - $f(e^x) = x$
      - $f(1) = 0$
      - $\mathbb{R} \xrightarrow{e^x} (0, \infty) \xrightarrow{\log x} \mathbb{R}$
      - Since e^x and $\log x$ are inverses, let $g = \log x$. $f \circ e^x$
        - $g'(f(x)) = \frac{1}{f'(x)}$ $(f \circ f(x))'$
        - $\Rightarrow g'(x) = \frac{1}{x}$
        - $\Rightarrow \frac{d}{dx} \log x = \frac{1}{x}$ $\square$
  - Function inverse.
    - Suppose f and g are inverses and $f(x) \ne 0$. Then g differentiable at $g(y)$. where $g'(y) = \frac{1}{f'(x)}$
      - Proof:
        - $g \circ f(x) = x$
        - $\Rightarrow g'(f(x)) \cdot f'(x) = 1$
        - $\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$
        - $\Rightarrow g'(f(x)) = \frac{1}{f'(g(y))}$
  - Function Composition:
    - Let $f \circ g$ be functions, g differentiable at x and f at $g(x)$.
    - Then $(f \circ g)' = (f' \circ g) \cdot g'$ (Chain