

# From cube to sphere: Optimizing search space for finding Pythagorean quadruplets

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## Abstract

Being provoked by a tenth grade math question that one supposed to solve with a brute forced method (see algorithm 1) led to the discovery of three conditions of Pythagorean quadruplets. The problem statement regards equation 1.

$$\exists a, b, c, d \in \mathbf{N} \rightarrow a^2 + b^2 + c^2 = d^2 \quad (1)$$

These conditions regards the first two terms (a and b) of the equation. Let p be a factor of  $a^2 + b^2$ , if p satisfies:

1.  $\frac{a^2+b^2}{p} + p \leq 2 * d$
2.  $\frac{a^2+b^2}{p} + p \equiv 0 \mod 2$
3.  $\sqrt{a^2 + 2b^2} - b \geq p$

Then the parametrisation (equation 2) satisfies the equation 1. The rest of the terms (c,d) are determinant by the following expression

$$c = \frac{a^2+b^2-p^2}{2p}$$
$$d = \frac{a^2+b^2+p^2}{2p}$$

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## 1 Naive approche

This adventure started with being presented a task as follows:

A pythagorean quadruple are four numbers such that  $a^2 + b^2 + c^2 = d^2$  and all numbers are integers. Let  $a \leq b \leq c \leq d$ . Show that for  $d \leq 30$  there exist 52 solutions.

This quote has been translated, but the essence is preserved. After a peer review from a committee of retired math teachers it was concluded that it was supposed to be solved using a simple algorithm.

## 2 algorithms

The naive algorithm (algorithm 1) solves the problem in  $O(n^3)$

```
count ← 0;
foreach i = {1...max} do
  foreach j={i...max} do
    foreach k={j...max} do
      possible_solution = a2 + b2 + c2;
      if √possible_solution ∈ ℕ and √possible_solution ≤ 30
        then
          | count ← count + 1;
        end
      end
    end
  end
end
end
```

**Algorithm 1:** Brute force algorithm for finding Pythagorean quadruplets

This algorithm has an time complexity of  $O(n^3)$ .

## 3 Parametrisation

Rather than to tackle the problem head-on it is more appropriate to parametrize the formula 1. The chosen parametrization is derived from [spiraDiophantineEquationX<sup>21962</sup>].

$$a^2 + b^2 + \left(\frac{a^2 + b^2 - p^2}{2p}\right)^2 = \left(\frac{a^2 + b^2 + p^2}{2p}\right)^2 \quad (2)$$

where p divides  $a^2 + b^2$  and a, b are arbitrary number with some restrictions, as will be disused in section 4.

## 4 Optimisations

### 4.1 Bounds

To improve the bounds of valid solution the variable  $k$  will be utilized to denote the new bound. First bound will be to take the formula 1 and assume all integers are equal, giving

$$k^2 + k^2 + k^2 = d^2 \rightarrow 3k^2 = d^2$$

A smaller bound can be achieved with the substitution  $k \rightarrow k - 1$

$$3(k - 1)^2 \leq d^2$$

giving a quadratic equation  $3(k - 1)^2 - d^2 \leq 0$ . The positive solution to the equation is

$$k \leq 1 + \frac{\sqrt{3}d}{3}$$

The negative solution won't be considered as only positive integers get considered. Similar argument can be made for the parameter  $b$ . Instead of considering all values being equal, we will assume that a suitable "a" has been chosen. This gives a similar equation

$$a^2 + 2(k - 1)^2 \leq d^2$$

This gives a function depending on choice of  $a$ .

$$k \leq \sqrt{\frac{d^2 - a^2}{2}} + 1$$

Thereby the set of possible solutions are as follows

$$\{(a, b) | 1 \leq a \leq 1 + \frac{\sqrt{3}d}{3}, a \leq b \leq \sqrt{\frac{d^2 - a^2}{2}} + 1\}$$

### 4.2 limits of a,b

### 4.3 limits of factors

Given the denominator of the equation 2 in  $c, d$  position.

$$\frac{a^2 + b^2 \pm p^2}{2 * p} = c, d$$

This gives two consequences:

1.  $a^2 + b^2 \pm p^2$  has to be even to divide  $2p$
2.  $\frac{a^2 + b^2}{p} \pm p \leq 2d$

assessing the inequality

$$b \leq \frac{a^2 + b^2 - p^2}{2p}$$

will give a bound of factor  $p$ . When moving everything to the right inequality

$$p^2 + 2bp - (a^2 + b^2) \leq 0$$

When solved using the quadratic formula gives the equation 3.

$$p \leq -b + \sqrt{a^2 + 2b^2} \tag{3}$$