From cube to sphere: Optimizing search space for finding Pythagorean quadruplets

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Abstract

Being provoked by a tenth grade math question that one supposed to solve with a brute forced method (see algorithm 1) led to the discovery of three conditions of Pythagorean quadruplets. The problem statement regards equation 1.

$$\exists a, b, c, d \in \mathbf{N} \rightarrow a^2 + b^2 + c^2 = d^2 \tag{1}$$

These conditions regards the first two terms (a and b) of the equation. Let p be a factor of $a^2 + b^2$, if p satisfies:

1. $\frac{a^2+b^2}{p} + p \leq 2 * d$
2. $a^2 + b^2 \not\equiv 2 \; mod \; 4$
3. $a^2 + b^2 \equiv 0 \; mod \; 4$
4. $\sqrt{a^2 + 2b^2} - b \geq p$

Then the parametrisation (equation 2) satisfies the equation 1. The rest of the terms (c,d) are determinant by the following expression

$$\begin{aligned} c &= \frac{a^2+b^2-p^2}{2p} \\ d &= \frac{a^2+b^2+p^2}{2p} \end{aligned}$$

Contents

1 Naive approche

This adventure started with being presented a task as follows:

> A pythagorian quadruple are four numbers such that $a^2+b^2+c^2 = d^2$ and all numbers are integers. Let $a \leq b \leq c \leq d$. Show that for $d \leq 30$ there exist 52 solutions.

This quote has been translated, but the essence is preserved. After a peer review from a committee of retired math teachers it was concluded that it was supposed to be solved using a simple algorithm.

2 algorithms

The naive algorithm (algorithm 1) solves the problem in $O(n^3)$

```
count ← 0;
foreach  i = {1...max} do
    foreach j={i...max} do
        foreach k={j...max} do
            possible_solution = a^2 + b^2 + c^2;
            if √possible_solution ∈ N and √possible_solution ≤ 30
             then
                count ← count + 1;
            end
        end
    end
end
```

Algorithm 1: Brute force algorithm for finding Pythagorean quadruplets

This algorithm has an time complexity of $O(max^3)$.

3 Parametrisation

Rather than to tackle the problem head-on it is more appropriate to parametrize the formula 1. The chosen parametrization is derived from [**spiraDiophantineEquationX^21962**] and in section 3.1.

$$a^2 + b^2 + \left(\frac{a^2 + b^2 - p^2}{2p}\right)^2 = \left(\frac{a^2 + b^2 + p^2}{2p}\right)^2 \qquad (2)$$

where p divides a^2+b^2 and a, b are arbitrary number with some restrictions, as will be disused in section 4.

3.1 motivation of parametrisation

When referring to $a^2 + b^2 + c^2 = d^2$ one can rewrite it to the following form

$$a^2 + b^2 = d^2 - c^2 = N$$

Where N is the common number between a^2+b^2 and d^2-c^2. When factorizing the right-side, and substituting $d-c=p$

$$N=(d-c)(d+c) \rightarrow N=p(d+c)$$

tells us that p divides a^2+b^2 and $(d+c)$ is the same as a^2+b^2/p. If we sum up all the factors we can get a parametrization of c and d

$$\begin{aligned}(d+c)+(d-c) &= 2*d\\ \frac{a^2+b^2}{p}+p &= 2d\\ \frac{1}{2}\frac{a^2+b^2}{p}+\frac{1}{2}p &= d\\ \frac{a^2+b^2+p^2}{2p} &= d\end{aligned}$$

$$\begin{aligned}(d+c)-(d-c) &= 2*c\\ \frac{a^2+b^2}{p}-p &= 2c\\ \frac{1}{2}\frac{a^2+b^2}{p}-\frac{1}{2}p &= c\\ \frac{a^2+b^2-p^2}{2p} &= c\end{aligned}$$

4 Optimisations

4.1 Bounds

To improve the bounds of valid solution the variable k will be utelizes to denote the new bound. First bound will be to take the formula 1 and assume all integers are equal, giving

$$k^2+k^2+k^2=d^2 \rightarrow 3k^2=d^2$$

A smaller bound can be achieved with the substitution $k \rightarrow k-1$

$$3(k-1)^2 \leq d^2$$

giving a quadratic equation $3(k-1)^2-d^2 \leq 0$. The positive solution to the equation is

$$k \leq 1+\frac{\sqrt{3}d}{3}$$

The negative solution won't be considered as only positive integers get considered. Similar argument can be made for the parameter b. Instead of considering all values being equal, we will assume that an suitable "a" has been chosen. This gives a similar equation

$$a^2+2(k-1)^2 \leq d^2$$

This gives a function depending on chose of a.

$$k \leq \sqrt{\frac{d^2-a^2}{2}}+1$$

Thereby the set of possible solutions are as follows

$$\{(a,b)|1 \leq a \leq 1 + \frac{\sqrt{3}d}{3}, a \leq b \leq \sqrt{\frac{d^2 - a^2}{2}} + 1\}$$

4.2 limits of a,b

From the alternative equation

$$a^2 + b^2 = d^2 - c^2$$

it can be imposed a factorisation condition where $4|a^2 + b^2$ (4 divides $a^2 + b^2$). This comes from the notion of differences of squares. It can be shown geometrically (see figure 1) that the difference of two arbitrary squares are given as

$$N = p(2 * q + p)$$

where p,q are integers.

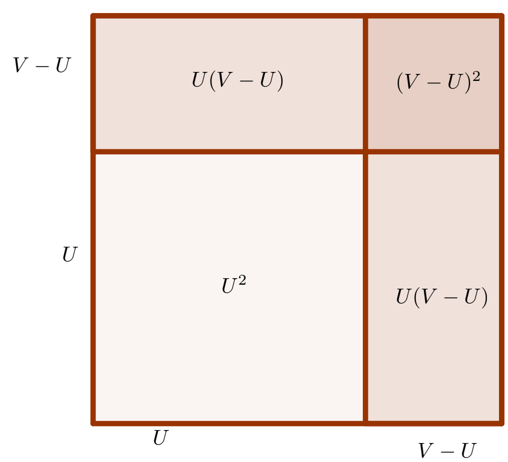

Figure 1: A visual proof of difference of squares $N = 2 * pq + p^2$ where $p = U$ and $q = (V - U)$, $U < V$

4.3 limits of factors

Given the denominator of the equation 2 in c,d position.

$$\frac{a^2 + b^2 \pm p^2}{2p} = c, d$$

This gives two consequences:

1. $a^2 + b^2 \pm p^2$ has to be even to divide 2p

2. $\frac{a^2+b^2}{p} \pm p \leq 2d$

Assessing the inequality

$$b \leq \frac{a^2+b^2-p^2}{2p}$$

will give a bound of factor p. When moving everything to the right inequality

$$p^2 + 2bp - (a^2+b^2) \leq 0$$

When solved using the quadratic formula gives the equation 3.

$$-b - \sqrt{a^2+2b^2} \leq p \leq -b + \sqrt{a^2+2b^2} \tag{3}$$

Since p is a positive integer the lowerbound can be rewriten as 1. This gives a upperbound for p to assure that the constructed c is bigger than b. This comes from that the difference between a^2+b^2 and a chosen factor has to be bigger than $2pb$.

To find a more efficient lower-bound we will assess a new inequality

$$\frac{a^2+b^2-p^2}{2p} \leq d$$

Solving it in the same manner we opptain

$$p \geq -d + \sqrt{d^2+a^2+b^2}$$

To summerize we now have the bound

$$-d + \sqrt{d^2+a^2+b^2} \leq p \leq -b + \sqrt{a^2+2b^2} \tag{4}$$