How to Program a Calculator

A look at the geometric cosine method

Jake Darby

Project Overview

Introduction

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- Numerical Analysis of Common Functions

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  - GNU & MPFR libraries

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Geometric Method Figure 1

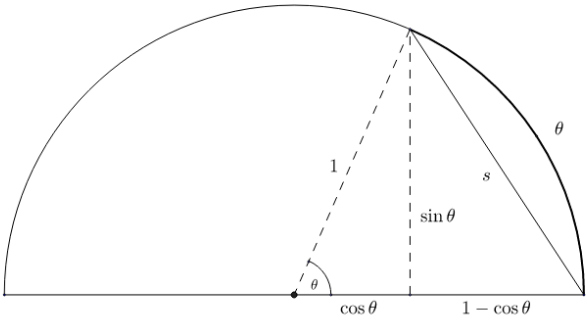
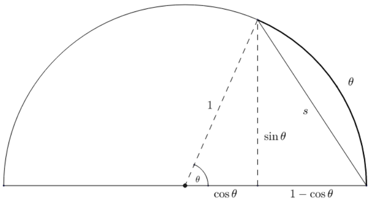


Figure: First diagram in developing the geometric method

Geometric Method Derivation 1



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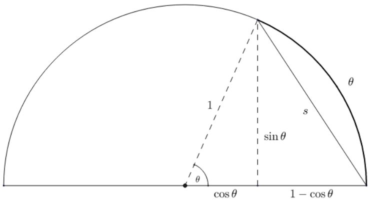

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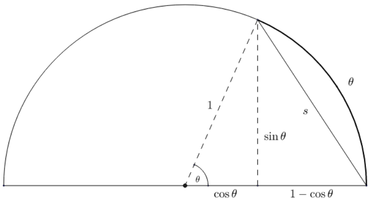

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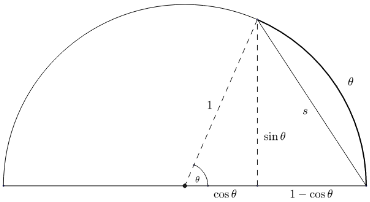

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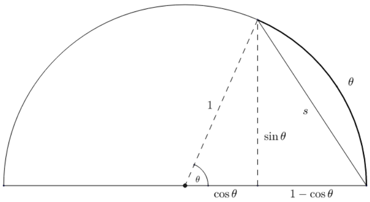

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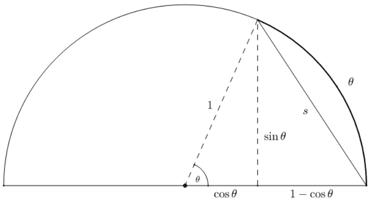


Geometric Method Diagrams 2

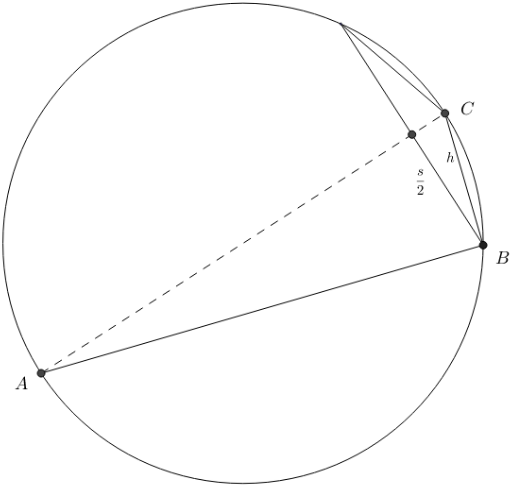
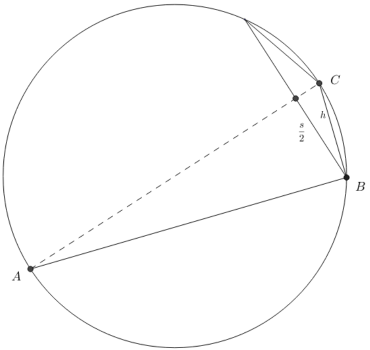


Figure: Diagram to show how to recursively calculate s

Geometric Method Derivation 2



Geometric Method Derivation 2

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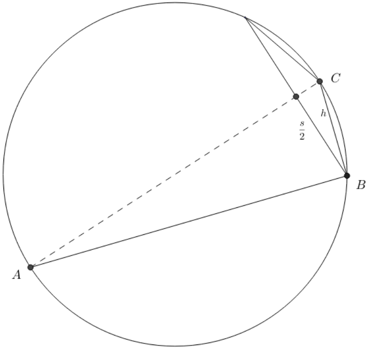

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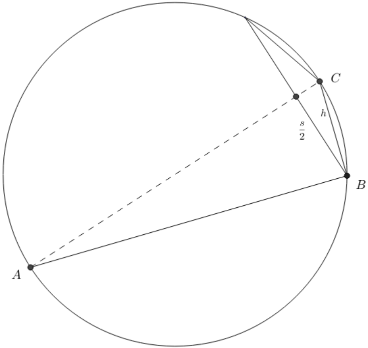

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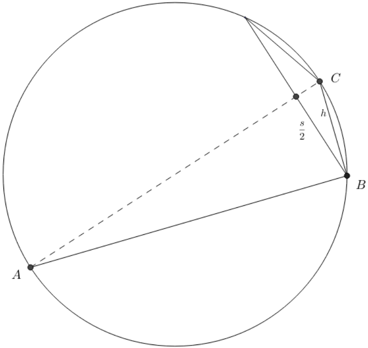

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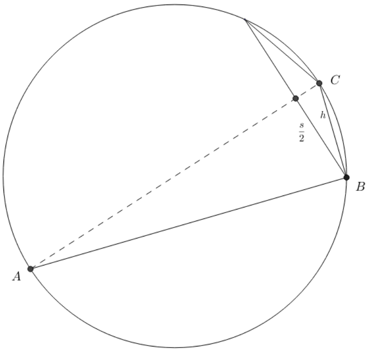

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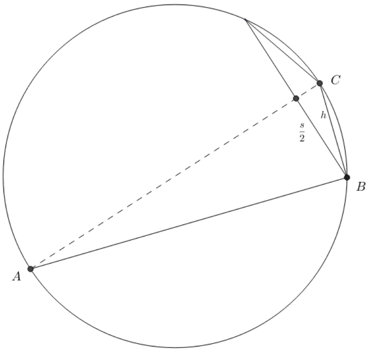

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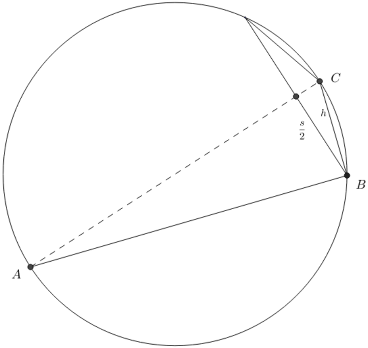

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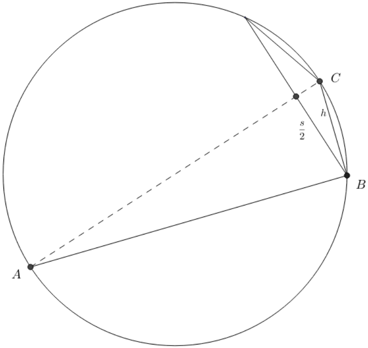

Geometric Method Algorithm


```
geometric_cos(θ ∈ [0, π/2], k ∈ ℕ):
    h_0 := θ2^(-k)
    n := 0
    while n < k:
        h_{n+1}^2 := h_n^2 · (4 − h_n^2)
        n ↦ n + 1
    return 1 − (1/2) h_k^2
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$$\text{geometric\_cos}(\theta \in [0, \tfrac{\pi}{2}], k \in \mathbb{N}):$$
$$h_0 := \theta 2^{-k}$$
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$$n \mapsto n + 1$$
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Geometric Method Implementation

```c
#include <assert.h>
#include <math.h>

double geometric_cos(double theta,
                     unsigned int k){
    //k > 0
    assert(k);
    //0 <= theta < pi/2
    assert(0 <= theta < 1.57079632679);

    double h = theta * pow(2, -k);
    h *= h;

    for(int i = 0; i < k; ++i)
        h = h * (4 - h);

    return 1 - h/2;
}
```

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N	k
5	6
10	14
50	80
100	163
1000	1658

Figure: This table shows the minimum k required to guarantee N digits of accuracy for our approximation of $\cos(0.5)$

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Any questions?