# Determinant of a Rank-One Perturbation of a Diagonal Matrix

## Preliminaries

**Theorem 1** (Matrix Determinant Lemma)**.** *Let $M$ be an invertible $n \times n$ matrix and let $u, v \in \mathbb{R}^n$ be column vectors. Then*

$$\det(M + uv^T) = (1 + v^T M^{-1} u) \det(M).$$

This is a special case of the Sylvester determinant identity; see Horn & Johnson, *Matrix Analysis*, 2nd ed., Theorem 1.3.22.

## Main Result

**Proposition 2.** *Let $a_1, a_2, \ldots, a_n \neq 0$. Then*

$$d_n = \begin{vmatrix} 1+a_1 & 1 & \cdots & 1 \\ 1 & 1+a_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1+a_n \end{vmatrix} = \left(1 + \sum_{k=1}^{n} \frac{1}{a_k}\right) a_1 a_2 \cdots a_n.$$

**Notation.**

- $D = \operatorname{diag}(a_1, a_2, \ldots, a_n)$: the $n \times n$ diagonal matrix with entries $a_1, \ldots, a_n$.
- $\mathbf{1} = (1, 1, \ldots, 1)^T$: the column vector of all ones.
- $\delta_{ij}$: Kronecker delta ($\delta_{ij} = 1$ if $i = j$, else 0).

*Proof.* Let $A$ denote the matrix whose determinant is $d_n$, so $A_{ij} = 1 + a_i \delta_{ij}$.

⟨1⟩1. **Matrix decomposition:** $A = D + \mathbf{1}\mathbf{1}^T$ *(algebraic verification)*

⟨2⟩1. The $(i, j)$-entry of $A$ is $A_{ij} = 1 + a_i \delta_{ij}$. *(by definition)*
⟨2⟩2. The $(i, j)$-entry of $D$ is $D_{ij} = a_i \delta_{ij}$. *(diagonal matrix)*
⟨2⟩3. The $(i, j)$-entry of $\mathbf{1}\mathbf{1}^T$ is $(\mathbf{1}\mathbf{1}^T)_{ij} = 1 \cdot 1 = 1$. *(outer product)*
⟨2⟩4. Therefore $(D + \mathbf{1}\mathbf{1}^T)_{ij} = a_i \delta_{ij} + 1 = A_{ij}$. *(entry-wise equality)*

⟨1⟩2. **Invertibility:** $D$ is invertible with $D^{-1} = \operatorname{diag}(1/a_1, \ldots, 1/a_n)$. *(since all $a_k \neq 0$)*

⟨2⟩1. Since $a_k \neq 0$ for all $k$, every diagonal entry of $D$ is nonzero. *(hypothesis)*
⟨2⟩2. A diagonal matrix is invertible iff all diagonal entries are nonzero. *(standard theorem)*
⟨2⟩3. Therefore $D$ is invertible. *(modus ponens)*

⟨2⟩4. The inverse of $\operatorname{diag}(a_1, \ldots, a_n)$ is $\operatorname{diag}(1/a_1, \ldots, 1/a_n)$. *(diagonal inverse)*

⟨1⟩3. **Determinant of $D$:** $\det(D) = a_1 a_2 \cdots a_n$. *(diagonal matrix property)*

⟨2⟩1. For any diagonal matrix, det = product of diagonal entries. *(standard theorem)*

⟨2⟩2. The diagonal entries of $D$ are $a_1, a_2, \ldots, a_n$. *(by construction)*

⟨2⟩3. Therefore $\det(D) = \prod_{k=1}^{n} a_k = a_1 a_2 \cdots a_n$. *(substitution)*

⟨1⟩4. **Apply Matrix Determinant Lemma:**

$$\det(D + \mathbf{1}\mathbf{1}^T) = (1 + \mathbf{1}^T D^{-1} \mathbf{1}) \det(D).$$

*(Theorem **??** with $M = D$, $u = v = \mathbf{1}$)*

⟨2⟩1. By Theorem **??**: $\det(M + uv^T) = (1 + v^T M^{-1} u) \det(M)$. *(cited)*

⟨2⟩2. Set $M = D$ and $u = v = \mathbf{1}$, so $uv^T = \mathbf{1}\mathbf{1}^T$. *(substitution)*

⟨2⟩3. The matrix $D$ is invertible by step ⟨1⟩2. *(verified hypothesis)*

⟨2⟩4. Therefore $\det(D + \mathbf{1}\mathbf{1}^T) = (1 + \mathbf{1}^T D^{-1} \mathbf{1}) \det(D)$. *(lemma application)*

⟨1⟩5. **Compute the quadratic form:** $\mathbf{1}^T D^{-1} \mathbf{1} = \displaystyle\sum_{k=1}^{n} \frac{1}{a_k}$. *(matrix computation)*

⟨2⟩1. $D^{-1} = \operatorname{diag}(1/a_1, 1/a_2, \ldots, 1/a_n)$. *(from ⟨1⟩2)*

⟨2⟩2. $D^{-1}\mathbf{1} = (1/a_1, 1/a_2, \ldots, 1/a_n)^T$. *(diagonal times vector)*

⟨3⟩1. The $k$-th component is $(D^{-1}\mathbf{1})_k = (D^{-1})_{kk} \cdot 1 = \dfrac{1}{a_k}$. *(definition)*

⟨2⟩3. $\mathbf{1}^T D^{-1} \mathbf{1} = \mathbf{1}^T \cdot (1/a_1, \ldots, 1/a_n)^T$. *(substitution)*

⟨2⟩4. $= \dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n} = \displaystyle\sum_{k=1}^{n} \frac{1}{a_k}$. *(dot product)*

⟨1⟩6. **Combine results:**

$$d_n = \det(A) = \left(1 + \sum_{k=1}^{n} \frac{1}{a_k}\right) a_1 a_2 \cdots a_n.$$

*(substitution into ⟨1⟩4)*

⟨2⟩1. $d_n = \det(A)$ by definition.

⟨2⟩2. $\det(A) = \det(D + \mathbf{1}\mathbf{1}^T)$ by step ⟨1⟩1.

⟨2⟩3. $= (1 + \mathbf{1}^T D^{-1} \mathbf{1}) \det(D)$ by step ⟨1⟩4.

⟨2⟩4. Substitute $\mathbf{1}^T D^{-1} \mathbf{1} = \sum_{k=1}^{n} \frac{1}{a_k}$ from step ⟨1⟩5.

⟨2⟩5. Substitute $\det(D) = a_1 a_2 \cdots a_n$ from step ⟨1⟩3.

⟨2⟩6. Therefore $d_n = \left(1 + \sum_{k=1}^{n} \frac{1}{a_k}\right) a_1 a_2 \cdots a_n$. □