Complete Sequence Robustness Theorem

Alethfeld Proof System

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Abstract

We prove that for all integers $0 \leq m < n$, there exists a complete sequence of positive integers that remains complete after removing any m elements, but becomes incomplete after removing some n elements. The proof is constructive, exhibiting a family of sequences $\{A_m\}_{m \geq 0}$ where each power of 2 appears with multiplicity $(m + 1)$.

Contents

1 Preliminaries

Clarification 1 (Multiset Conventions). Throughout this proof, A denotes a non-decreasing sequence (multiset with ordering) of positive integers. The notation $\{a_1 \leq a_2 \leq \cdots\}$ specifies weak ordering. "Distinct elements" means distinct *positions* (indices), not distinct values. Multiset subtraction $A \setminus S$ removes elements by position.

Definition 2 (Complete Sequence)**.** A sequence $A = \{a_1 \leq a_2 \leq \cdots\}$ of positive integers is **complete** if every positive integer can be represented as a sum of distinct elements from A.

Definition 3 (k-Subcomplete)**.** A complete sequence A is **k-subcomplete** if $A \setminus S$ remains complete for every subset $S \subseteq A$ with $|S| = k$.

2 Key Lemmas

Lemma 4 (Brown's Criterion [1])**.** *A non-decreasing sequence $A = \{a_1 \leq a_2 \leq \cdots\}$ of positive integers is complete if and only if:*

(i) $a_1 = 1$, and

(ii) $a_{k+1} \leq 1 + \sum_{i=1}^{k} a_i$ for all $k \geq 1$.

Remark 5. This criterion is sometimes called "Cassels' criterion" in the literature, though the characterization theorem was published by J. L. Brown Jr. in 1961.

Lemma 6 (Superset Preservation)**.** *If A is a complete sequence and $A \subseteq B$ (as multisets), then B is complete.*

Proof. Every positive integer n has a representation as a sum of distinct elements from A. Since $A \subseteq B$, those same elements exist in B, so n is also representable using elements of B. □

Lemma 7 (Powers of 2 Core)**.** *Any multiset B containing at least one copy of each power of 2 (i.e., $1, 2, 4, 8, \ldots \in B$) is complete.*

Proof. The sequence $P = \{1, 2, 4, 8, \ldots\}$ is complete by Brown's criterion:

0.1. For $P = \{2^0, 2^1, 2^2, \ldots\}$, we have $a_k = 2^{k-1}$, so $a_1 = 2^0 = 1$. *[substitution]*

0.2. For $k \geq 1$, the sum $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} 2^{i-1} = \sum_{j=0}^{k-1} 2^j$ where $j = i - 1$. *[index substitution]*

0.3. The geometric series formula gives $\sum_{j=0}^{k-1} 2^j = \frac{2^k-1}{2-1} = 2^k - 1$.

0.3.1. Base case ($k = 1$): $\sum_{j=0}^{0} 2^j = 2^0 = 1 = \frac{2^1-1}{1} = 1$. ✓

0.3.2. Inductive hypothesis: Assume $\sum_{j=0}^{k-1} 2^j = 2^k - 1$ holds for some $k \geq 1$.

0.3.3. Inductive step: $\sum_{j=0}^{k} 2^j = \sum_{j=0}^{k-1} 2^j + 2^k = (2^k - 1) + 2^k = 2 \cdot 2^k - 1 = 2^{k+1} - 1$. ✓

0.3.4. By induction, $\sum_{j=0}^{k-1} 2^j = 2^k - 1$ for all $k \geq 1$. □

0.4. For $k \geq 1$, we have $a_{k+1} = 2^{(k+1)-1} = 2^k$. *[substitution]*

0.5. The inequality $2^k \leq 1 + (2^k - 1) = 2^k$ holds with equality for all $k \geq 1$. *[arithmetic]*

Since B contains P as a submultiset and P is complete, B is complete by Lemma 6. □

3 Main Theorem

Theorem 8 (Complete Sequence Robustness)**.** *For all integers $0 \le m < n$, there exists a complete sequence $A = \{a_1 \le a_2 \le \cdots\}$ of positive integers such that A remains complete after removing any m elements, but there exist n elements whose removal makes A incomplete.*

Construction 9. For any $m \ge 0$, define A_m to be the sequence where each power of 2 appears with multiplicity $(m+1)$:

$$A_m = \{\underbrace{1, 1, \ldots, 1}_{m+1}, \underbrace{2, 2, \ldots, 2}_{m+1}, \underbrace{4, 4, \ldots, 4}_{m+1}, \ldots\}$$

Proof of Theorem 8. We prove three claims about A_m:

1. **Claim 1: A_m is complete.**

   We verify Brown's criterion (Lemma 4):

   1.1. The sequence A_m has $a_1 = 1$, satisfying condition (i). *[Construction 9]*

   1.2. For A_m, the first $(m+1) \cdot k$ elements are the powers $2^0, 2^1, \ldots, 2^{k-1}$, each with multiplicity $(m+1)$. Their sum is:

   $$(m+1) \cdot (1 + 2 + 4 + \cdots + 2^{k-1}) = (m+1) \cdot (2^k - 1)$$

   [geometric series]

   1.3. The next element after the first $(m+1) \cdot k$ elements is 2^k. We verify condition (ii):

   $$\begin{aligned} 2^k &\le 1 + (m+1)(2^k - 1) \\ &= (m+1) \cdot 2^k - m \end{aligned}$$

   This holds since $2^k \le (m+1) \cdot 2^k - m$ iff $m \cdot 2^k \ge m$ iff $2^k \ge 1$, which is true for all $k \ge 0$. *[algebra]*

   1.4. By Brown's criterion, A_m is complete. *[modus ponens]*

2. **Claim 2: A_m is m-subcomplete.**

   Let $S \subseteq A_m$ be arbitrary with $|S| = m$. We prove $A_m \setminus S$ is complete:

   2.1. For each power 2^j, the set S contains at most m copies of 2^j since $|S| = m$ total. *[cardinality]*

   2.2. **Arithmetic**: A_m has $(m+1)$ copies of each 2^j. Since S removes at most m copies of any value, at least $(m+1) - m = 1$ copy of each 2^j remains in $A_m \setminus S$. *[subtraction]*

   2.3. Since at least one copy of each 2^j remains, the support of $A_m \setminus S$ contains $\{2^0, 2^1, 2^2, \ldots\} = \{1, 2, 4, 8, \ldots\}$. *[step 2.2]*

   2.4. **Powers of 2 Core** (Lemma 7): Any multiset containing at least one copy of each power of 2 is complete. This follows from binary representation: every positive integer n has a unique binary expansion $n = \sum_{i \in I} 2^i$, so n is representable using distinct powers of 2. *[Lemma 7]*

   2.5. **Superset Preservation** (Lemma 6): Since $\{1, 2, 4, \ldots\} \subseteq A_m \setminus S$ and $\{1, 2, 4, \ldots\}$ is complete, $A_m \setminus S$ is complete. *[Lemma 6]*

   Since S was arbitrary with $|S| = m$, by Definition 3, A_m is m-subcomplete.

3. Claim 3: A_m is not $(m+1)$-subcomplete.

We exhibit a witness set S^* with $|S^*| = m+1$ such that $A_m \setminus S^*$ is not complete:

3.1. Define $S^* = \{\text{all } (m+1) \text{ copies of 1 in } A_m\}$. Then $|S^*| = m+1$. *[Construction 9]*

3.2. $A_m \setminus S^* = \{2, 2, \ldots, 4, 4, \ldots, 8, 8, \ldots\}$ consists of $(m+1)$ copies each of 2^k for $k \geq 1$. *[set difference]*

3.3. The smallest element of $A_m \setminus S^*$ is $2^1 = 2$. Hence every element of $A_m \setminus S^*$ is ≥ 2. *[step 3.2]*

3.4. Case analysis for representing 1: Either (a) use the empty sum, or (b) use a non-empty sum of distinct elements from $A_m \setminus S^*$. *[exhaustive cases]*

3.4.1. Case (a): The empty sum equals $0 \neq 1$.

3.4.2. Case (b): Let $X \subseteq A_m \setminus S^*$ be non-empty.

3.4.2.1. Then X contains at least one element x. *[non-empty]*

3.4.2.2. Since $x \in A_m \setminus S^*$ and every element of $A_m \setminus S^*$ is ≥ 2 (step 3.3), we have $x \geq 2$. *[step 3.3]*

3.4.2.3. The sum of elements in X is $\geq x$ (since all elements are positive). By step 3.4.2.2, $x \geq 2$. Hence $\text{sum}(X) \geq 2$. *[arithmetic]*

3.4.2.4. Since X was arbitrary non-empty subset, any non-empty sum is $\geq 2 > 1$. *[universal generalization]*

3.5. By case exhaustion: 1 cannot be represented as a sum of distinct elements from $A_m \setminus S^*$. *[steps 3.4.1, 3.4.2]*

3.6. Since $A_m \setminus S^*$ cannot represent 1 (a positive integer), $A_m \setminus S^*$ is not complete. *[Definition 2]*

3.7. Since $|S^*| = m+1$ and $A_m \setminus S^*$ is not complete, A_m is not $(m+1)$-subcomplete. *[Definition 3]*

4. Claim 4: For any $n > m$, A_m is not n-subcomplete.

Let $n > m$ be arbitrary. We construct a witness set T with $|T| = n$:

4.1. The multiset $A_m \setminus S^*$ is infinite (contains infinitely many copies of $2, 4, 8, \ldots$). Since $(n-m-1)$ is a finite non-negative integer, we can select $(n-m-1)$ elements from $A_m \setminus S^*$. *[infinite set]*

4.2. Define $T = S^* \cup \{\text{these } (n-m-1) \text{ elements}\}$. Note: $T \subseteq A_m$ by construction. *[construction]*

4.3. Cardinality (multiset arithmetic): $|T| = |S^*| + (n-m-1) = (m+1) + (n-m-1) = n$. *[arithmetic]*

4.4. Subset relation (multiset): Since $S^* \subseteq T$, multiset subtraction gives $A_m \setminus T \subseteq A_m \setminus S^*$ (removing more leaves less or equal). *[multiset properties]*

4.5. Monotonicity of non-representability: By step 3.5, $A_m \setminus S^*$ cannot represent 1. By step 4.4, $A_m \setminus T \subseteq A_m \setminus S^*$. Any sum from $A_m \setminus T$ is also a sum from $A_m \setminus S^*$. Hence $A_m \setminus T$ also cannot represent 1. *[subset property]*

4.6. Since $A_m \setminus T$ cannot represent 1, $A_m \setminus T$ is not complete. Since $|T| = n$, the set T witnesses that A_m is not n-subcomplete. *[Definition 3]*

Since $n > m$ was arbitrary, $\forall n > m$: A_m is not n-subcomplete.

5. Edge case: $m = 0$.

When $m = 0$, we have $A_0 = \{1, 2, 4, 8, \ldots\}$ (multiplicity 1). This is *trivially* 0-subcomplete: the quantification "$\forall S$ with $|S| = 0$" has exactly one instance ($S = \emptyset$), and $A_0 \setminus \emptyset = A_0$ is complete by step 1.

Conclusion. For all integers $0 \le m < n$, the sequence A_m satisfies:

(1) A_m is complete (by step 1 via Brown's criterion),

(2) A_m is m-subcomplete (by step 2 via arithmetic and superset preservation),

(3) A_m is not n-subcomplete (by step 4 via witness construction).

This completes the proof. □ □

4 Answer to the Original Question

Corollary 10. *For what values of $0 \le m < n$ is there a complete sequence A such that A remains complete after removing any m elements, but A is not complete after removing any n elements?*

Answer: *Such a sequence exists for* ***all*** *pairs (m, n) with $0 \le m < n$.*

References

[1] J. L. Brown Jr., *Note on complete sequences of integers*, American Mathematical Monthly **68**(6) (1961), 557–560. DOI: 10.2307/2311990

[2] J. W. S. Cassels, *On the representation of integers as the sums of distinct summands taken from a fixed set*, Acta Scientiarum Mathematicarum (Szeged) **21** (1960), 111–124.

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