PROOF OF PROPOSITION (INFINITE-Z)

ALETHFELD PROOF ORCHESTRATOR

Proposition 1. *For any random variable $F : \mathbf{Z}^N \to [0, \infty)$,*

$$\mathbb{E}^{\mathrm{X}}(F) \le \frac{N^N}{N!}\,\mathbb{E}^{\mathrm{R}}(F). \tag{1}$$

Suppose $|\mathbf{Z}| \ge N$. The bound is tight and attained on a non-zero indicator function. Moreover, there exists an event A in $\mathbf{Z}^N$ such that

$$\mathbb{P}^{\mathrm{R}}(A) = \frac{N!}{N^N} \le 1 = \mathbb{P}^{\mathrm{X}}(A).$$

Proof. **Setup.** Let $\mathbf{Z}$ be a set with $|\mathbf{Z}| \ge N$, where $N \ge 1$. Define: A1–A4

$$\mathbb{E}^{\mathrm{X}}(F) := \frac{1}{|\operatorname{Bij}([N], \mathbf{Z})|} \sum_{\sigma \in \operatorname{Bij}([N],\mathbf{Z})} F(\sigma),$$
$$\mathbb{E}^{\mathrm{R}}(F) := \frac{1}{|\mathbf{Z}|^N} \sum_{f : [N] \to \mathbf{Z}} F(f),$$

where $\operatorname{Bij}([N], \mathbf{Z})$ denotes the set of bijections (injections) from $[N]$ to $\mathbf{Z}$. Let $F : \mathbf{Z}^N \to [0, \infty)$ be non-negative.

Step 1: Main Inequality. Since bijections are functions, $\operatorname{Bij}([N], \mathbf{Z}) \subseteq \mathbf{Z}^N$. S1–S7
The number of bijections is

$$|\operatorname{Bij}([N], \mathbf{Z})| = \frac{|\mathbf{Z}|!}{(|\mathbf{Z}| - N)!}.$$

Since $F \ge 0$, summing over a subset gives

$$\sum_{\sigma \in \operatorname{Bij}} F(\sigma) \le \sum_{f \in \mathbf{Z}^N} F(f).$$

Thus

$$\mathbb{E}^{\mathrm{X}}(F) = \frac{\sum_\sigma F(\sigma)}{|\operatorname{Bij}|} \le \frac{\sum_f F(f)}{|\operatorname{Bij}|} = \frac{|\mathbf{Z}|^N}{|\operatorname{Bij}|} \cdot \mathbb{E}^{\mathrm{R}}(F).$$

Key bound: For $M = |\mathbf{Z}| \ge N$, we claim

$$\frac{M^N}{M!/(M-N)!} \le \frac{N^N}{N!}.$$

This follows since for each $k \in \{0, \dots, N-1\}$:

$$\frac{M}{M-k} = 1 + \frac{k}{M-k} \le 1 + \frac{k}{N-k} = \frac{N}{N-k},$$

Date: Graph: `graph-29d5d0-dd347c` v33.

where the inequality uses $M \geq N \Rightarrow M-k \geq N-k > 0$. Taking the product over k:

$$\frac{M^N}{M(M-1)\cdots(M-N+1)} = \prod_{k=0}^{N-1} \frac{M}{M-k} \leq \prod_{k=0}^{N-1} \frac{N}{N-k} = \frac{N^N}{N!}.$$

Combining: $\mathbb{E}^{\mathrm{X}}(F) \leq \frac{N^N}{N!}\,\mathbb{E}^{\mathrm{R}}(F)$.

S8–S11 **Step 2: Tightness.** For tightness, take $|\mathbf{Z}| = N$ and define

$$A := \mathrm{Bij}([N], \mathbf{Z}) \subseteq \mathbf{Z}^N.$$

Then:

- $\mathbb{P}^{\mathrm{X}}(A) = 1$ since X is always a bijection.
- $\mathbb{P}^{\mathrm{R}}(A) = \frac{|A|}{|\mathbf{Z}^N|} = \frac{N!}{N^N}$.

For $F = \mathbf{1}_A$:

$$\frac{\mathbb{E}^{\mathrm{X}}(F)}{\mathbb{E}^{\mathrm{R}}(F)} = \frac{\mathbb{P}^{\mathrm{X}}(A)}{\mathbb{P}^{\mathrm{R}}(A)} = \frac{1}{N!/N^N} = \frac{N^N}{N!}.$$

This achieves the bound exactly.

Conclusion. The event $A = \mathrm{Bij}([N], \mathbf{Z})$ satisfies $\mathbb{P}^{\mathrm{R}}(A) = N!/N^N \leq 1 = \mathbb{P}^{\mathrm{X}}(A)$, completing the proof. $\square$