

# Minimum Value of $xyz$ Subject to Exponential-Logarithmic Constraints

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Formalized by Alethfeld Proof System

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## Abstract

We prove that given positive real numbers  $x$ ,  $y$ , and  $z$  satisfying the constraints  $x^{\log_2(yz)} = 2^8 \cdot 3^4$ ,  $y^{\log_2(zx)} = 2^9 \cdot 3^6$ , and  $z^{\log_2(xy)} = 2^5 \cdot 3^{10}$ , the minimum possible value of  $xyz$  is  $\boxed{576}$ . The proof proceeds by transforming the system via logarithms, reducing to a constrained quadratic system, analyzing all sign combinations in the solution space, and establishing uniqueness of the minimum through monotonicity arguments.

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# 1 Problem Statement and Setup

**Theorem 1** (Main Result). *Given that  $x, y, z \in \mathbb{R}^+$  such that*

$$x^{\log_2(yz)} = 2^8 \cdot 3^4, \quad (1)$$

$$y^{\log_2(zx)} = 2^9 \cdot 3^6, \quad (2)$$

$$z^{\log_2(xy)} = 2^5 \cdot 3^{10}, \quad (3)$$

*the smallest possible value of  $xyz$  is 576.*

We begin by establishing our notation and fundamental assumptions.

**Assumption 1** (Positivity).  $x, y, z \in \mathbb{R}^+$  (positive real numbers).

**Definition 2** (Logarithmic Variables). Define

$$a = \log_2 x, \quad b = \log_2 y, \quad c = \log_2 z, \quad \alpha = \log_2 3. \quad (4)$$

*Remark 3.* Since  $x, y, z > 0$ , the quantities  $a, b, c$  are well-defined real numbers. The constant  $\alpha = \log_2 3 \approx 1.585$  satisfies  $1 < \alpha < 2$  since  $2 < 3 < 4$ .

## 2 Logarithmic Transformation

**Proposition 4** (Transformed System). *Under Definition 2, the constraints (1)–(3) become:*

$$a(b + c) = 8 + 4\alpha, \quad (5)$$

$$b(c + a) = 9 + 6\alpha, \quad (6)$$

$$c(a + b) = 5 + 10\alpha. \quad (7)$$

*Proof.* We derive equation (5) in detail; the others follow analogously.

**Step 1:** Apply  $\log_2$  to both sides of (1):

$$\log_2 \left( x^{\log_2(yz)} \right) = \log_2 (2^8 \cdot 3^4). \quad (8)$$

**Step 2:** Simplify the left-hand side using the power rule:

$$\log_2 \left( x^{\log_2(yz)} \right) = \log_2(yz) \cdot \log_2(x). \quad (9)$$

**Step 3:** Apply the product rule to  $\log_2(yz)$ :

$$\log_2(yz) = \log_2 y + \log_2 z = b + c. \quad (10)$$

**Step 4:** Substitute Definition 2:

$$\text{LHS} = (b + c) \cdot a = a(b + c). \quad (11)$$

**Step 5:** Simplify the right-hand side using the product rule:

$$\log_2 (2^8 \cdot 3^4) = \log_2(2^8) + \log_2(3^4). \quad (12)$$

**Step 6:** Apply the power rule:

$$\log_2(2^8) = 8, \quad \log_2(3^4) = 4 \log_2 3 = 4\alpha. \quad (13)$$

**Step 7:** Combine:

$$\text{RHS} = 8 + 4\alpha. \quad (14)$$

Therefore  $a(b + c) = 8 + 4\alpha$ .

**For equation (6):** Starting from  $y^{\log_2(zx)} = 2^9 \cdot 3^6$ :

$$\log_2(zx) \cdot \log_2 y = \log_2(2^9 \cdot 3^6) \implies (c + a) \cdot b = 9 + 6\alpha.$$

**For equation (7):** Starting from  $z^{\log_2(xy)} = 2^5 \cdot 3^{10}$ :

$$\log_2(xy) \cdot \log_2 z = \log_2(2^5 \cdot 3^{10}) \implies (a + b) \cdot c = 5 + 10\alpha.$$

□

### 3 Reduction to Quadratic System

**Definition 5** (Sum Variable). Let  $s = a + b + c$ . Then  $xyz = 2^a \cdot 2^b \cdot 2^c = 2^s$ .

Our goal becomes: minimize  $s$  subject to equations (5)–(7).

**Lemma 6** (Sum of Products). *Adding equations (5)–(7):*

$$2(ab + bc + ca) = 22 + 20\alpha. \quad (15)$$

*Proof.*

$$a(b + c) + b(c + a) + c(a + b) = (8 + 4\alpha) + (9 + 6\alpha) + (5 + 10\alpha) = 22 + 20\alpha.$$

The left-hand side equals  $ab + ac + bc + ba + ca + cb = 2(ab + bc + ca)$ . □

**Proposition 7** (Quadratic Reformulation). *Using  $b + c = s - a$ , etc., the system becomes:*

$$a^2 - sa + (8 + 4\alpha) = 0, \quad (16)$$

$$b^2 - sb + (9 + 6\alpha) = 0, \quad (17)$$

$$c^2 - sc + (5 + 10\alpha) = 0. \quad (18)$$

*Proof.* From (5):  $a(b + c) = 8 + 4\alpha$ . Since  $b + c = s - a$ :

$$a(s - a) = 8 + 4\alpha \implies as - a^2 = 8 + 4\alpha \implies a^2 - sa + (8 + 4\alpha) = 0.$$

The other equations follow identically. □

**Corollary 8** (Quadratic Solutions). *By the quadratic formula:*

$$a = \frac{s \pm \sqrt{s^2 - 4(8 + 4\alpha)}}{2} = \frac{s \pm \sqrt{\Delta_1}}{2}, \quad (19)$$

$$b = \frac{s \pm \sqrt{s^2 - 4(9 + 6\alpha)}}{2} = \frac{s \pm \sqrt{\Delta_2}}{2}, \quad (20)$$

$$c = \frac{s \pm \sqrt{s^2 - 4(5 + 10\alpha)}}{2} = \frac{s \pm \sqrt{\Delta_3}}{2}, \quad (21)$$

where we define the discriminants:

$$\Delta_1 = s^2 - 32 - 16\alpha, \quad (22)$$

$$\Delta_2 = s^2 - 36 - 24\alpha, \quad (23)$$

$$\Delta_3 = s^2 - 20 - 40\alpha. \quad (24)$$

## 4 Discriminant Constraints

**Lemma 9** (Non-negativity Requirements). *For real solutions, we require:*

$$s^2 \geq 32 + 16\alpha \quad (\text{from } \Delta_1 \geq 0), \quad (25)$$

$$s^2 \geq 36 + 24\alpha \quad (\text{from } \Delta_2 \geq 0), \quad (26)$$

$$s^2 \geq 20 + 40\alpha \quad (\text{from } \Delta_3 \geq 0). \quad (27)$$

**Proposition 10** (Binding Constraint). *Since  $\alpha = \log_2 3 \approx 1.585$ :*

$$32 + 16\alpha \approx 32 + 25.4 = 57.4,$$

$$36 + 24\alpha \approx 36 + 38.0 = 74.0,$$

$$20 + 40\alpha \approx 20 + 63.4 = 83.4.$$

The binding constraint is  $s^2 \geq 20 + 40\alpha$ , i.e.,  $s \geq \sqrt{20 + 40\alpha}$ .

## 5 Testing the Candidate $s = 6 + 2\alpha$

We claim that  $s_0 = 6 + 2\alpha = \log_2 576$  achieves the minimum.

*Claim 1.*  $s_0 = 6 + 2\alpha$  satisfies the binding constraint  $s^2 \geq 20 + 40\alpha$ .

*Proof.* At  $s_0 = 6 + 2\alpha$ :

$$s_0^2 = (6 + 2\alpha)^2 = 36 + 24\alpha + 4\alpha^2. \quad (28)$$

We need  $36 + 24\alpha + 4\alpha^2 \geq 20 + 40\alpha$ , i.e.,

$$4\alpha^2 - 16\alpha + 16 \geq 0 \iff 4(\alpha - 2)^2 \geq 0.$$

This holds for all  $\alpha$ , with equality iff  $\alpha = 2$ . Since  $\alpha = \log_2 3 < \log_2 4 = 2$ , the inequality is strict.  $\square$

## 6 Discriminant Perfect Square Factorizations

### 6.1 Computation of $\Delta_1$ at $s = 6 + 2\alpha$

**Lemma 11** ( $\Delta_1$  Factorization). *At  $s_0 = 6 + 2\alpha$ :*

$$\Delta_1 = 4(1 + \alpha)^2, \quad \sqrt{\Delta_1} = 2 + 2\alpha. \quad (29)$$

*Proof. Step 1 (Substitution):* From (28) and (22):

$$\Delta_1 = s_0^2 - 32 - 16\alpha = (36 + 24\alpha + 4\alpha^2) - 32 - 16\alpha.$$

**Step 2 (Grouping):**

$$\begin{aligned} \Delta_1 &= (36 - 32) + (24\alpha - 16\alpha) + 4\alpha^2 \\ &= 4 + 8\alpha + 4\alpha^2. \end{aligned}$$

**Step 3 (Factorization):**

$$4 + 8\alpha + 4\alpha^2 = 4(1 + 2\alpha + \alpha^2) = 4(1 + \alpha)^2.$$

**Step 4 (Square Root):** Since  $\alpha = \log_2 3 > 0$ , we have  $1 + \alpha > 1 > 0$ . Thus:

$$\sqrt{\Delta_1} = \sqrt{4(1 + \alpha)^2} = 2|1 + \alpha| = 2(1 + \alpha) = 2 + 2\alpha.$$

$\square$

## 6.2 Computation of $\Delta_2$ at $s = 6 + 2\alpha$

**Lemma 12** ( $\Delta_2$  Factorization). At  $s_0 = 6 + 2\alpha$ :

$$\Delta_2 = 4\alpha^2, \quad \sqrt{\Delta_2} = 2\alpha. \quad (30)$$

*Proof.* **Step 1 (Substitution):**

$$\Delta_2 = s_0^2 - 36 - 24\alpha = (36 + 24\alpha + 4\alpha^2) - 36 - 24\alpha.$$

**Step 2 (Collection):**

$$\Delta_2 = (36 - 36) + (24\alpha - 24\alpha) + 4\alpha^2 = 0 + 0 + 4\alpha^2 = 4\alpha^2.$$

**Step 3 (Perfect Square):**

$$4\alpha^2 = (2\alpha)^2.$$

**Step 4 (Square Root):** Since  $\alpha = \log_2 3 > 0$ :

$$\sqrt{\Delta_2} = \sqrt{4\alpha^2} = 2|\alpha| = 2\alpha.$$

□

## 6.3 Computation of $\Delta_3$ at $s = 6 + 2\alpha$

**Lemma 13** ( $\Delta_3$  Factorization). At  $s_0 = 6 + 2\alpha$ :

$$\Delta_3 = 4(2 - \alpha)^2, \quad \sqrt{\Delta_3} = 4 - 2\alpha. \quad (31)$$

*Proof.* **Step 1 (Substitution):**

$$\Delta_3 = s_0^2 - 20 - 40\alpha = (36 + 24\alpha + 4\alpha^2) - 20 - 40\alpha.$$

**Step 2 (Collection):**

$$\Delta_3 = (36 - 20) + (24\alpha - 40\alpha) + 4\alpha^2 = 16 - 16\alpha + 4\alpha^2.$$

**Step 3 (Factorization):**

$$16 - 16\alpha + 4\alpha^2 = 4(4 - 4\alpha + \alpha^2) = 4(2 - \alpha)^2.$$

*Verification:*  $(2 - \alpha)^2 = 4 - 4\alpha + \alpha^2. \checkmark$

**Step 4 (Sign Analysis):** Since  $\alpha = \log_2 3$  and  $3 < 4 = 2^2$ , we have  $\log_2 3 < \log_2 4 = 2$ . Thus  $\alpha < 2$ , so  $2 - \alpha > 0$ .

**Step 5 (Square Root):**

$$\sqrt{\Delta_3} = \sqrt{4(2 - \alpha)^2} = 2|2 - \alpha| = 2(2 - \alpha) = 4 - 2\alpha.$$

□

## 7 The Sum Constraint

**Proposition 14** (Sum Constraint). *The constraint  $a + b + c = s$  requires:*

$$\epsilon_1\sqrt{\Delta_1} + \epsilon_2\sqrt{\Delta_2} + \epsilon_3\sqrt{\Delta_3} = -s \quad (32)$$

for some  $\epsilon_i \in \{+1, -1\}$ .

*Proof.* From (19)–(21):

$$a + b + c = \frac{3s + \epsilon_1\sqrt{\Delta_1} + \epsilon_2\sqrt{\Delta_2} + \epsilon_3\sqrt{\Delta_3}}{2} = s.$$

Multiplying by 2:

$$3s + \epsilon_1\sqrt{\Delta_1} + \epsilon_2\sqrt{\Delta_2} + \epsilon_3\sqrt{\Delta_3} = 2s.$$

Rearranging:

$$\epsilon_1\sqrt{\Delta_1} + \epsilon_2\sqrt{\Delta_2} + \epsilon_3\sqrt{\Delta_3} = -s.$$

□

*Remark 15.* Since  $s > 0$  (as  $x, y, z > 0$ ), the right-hand side is negative. Therefore, at least one  $\epsilon_i$  must be  $-1$ .

## 8 Exhaustive Sign Combination Analysis

At  $s_0 = 6 + 2\alpha$ , we have computed:

$$\sqrt{\Delta_1} = 2 + 2\alpha, \quad \sqrt{\Delta_2} = 2\alpha, \quad \sqrt{\Delta_3} = 4 - 2\alpha.$$

The target is:

$$-s_0 = -(6 + 2\alpha) = -6 - 2\alpha.$$

We analyze all  $2^3 = 8$  sign combinations  $(\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3$ .

### 8.1 Summary Table

Case	$(\epsilon_1, \epsilon_2, \epsilon_3)$	LHS Expression	LHS Value	Verdict
1	(+, +, +)	$(2 + 2\alpha) + 2\alpha + (4 - 2\alpha)$	$6 + 2\alpha$	Impossible: LHS > 0
2	(+, +, -)	$(2 + 2\alpha) + 2\alpha - (4 - 2\alpha)$	$-2 + 6\alpha$	Req. $\alpha = -\frac{1}{2}$
3	(+, -, +)	$(2 + 2\alpha) - 2\alpha + (4 - 2\alpha)$	$6 - 2\alpha$	Contradiction
4	(+, -, -)	$(2 + 2\alpha) - 2\alpha - (4 - 2\alpha)$	$-2 + 2\alpha$	Req. $\alpha = -1$
5	(-, +, +)	$-(2 + 2\alpha) + 2\alpha + (4 - 2\alpha)$	$2 - 2\alpha$	Contradiction
6	(-, +, -)	$-(2 + 2\alpha) + 2\alpha - (4 - 2\alpha)$	$-6 + 2\alpha$	Req. $\alpha = 0$
7	(-, -, +)	$-(2 + 2\alpha) - 2\alpha + (4 - 2\alpha)$	$2 - 6\alpha$	Req. $\alpha = 2$
8	(-, -, -)	$-(2 + 2\alpha) - 2\alpha - (4 - 2\alpha)$	$-6 - 2\alpha$	<b>VALID</b>

### 8.2 Case-by-Case Analysis

#### 8.2.1 Case 1: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, +1, +1)$

$$\begin{aligned} \text{LHS} &= (2 + 2\alpha) + 2\alpha + (4 - 2\alpha) \\ &= 2 + 2\alpha + 2\alpha + 4 - 2\alpha \\ &= 6 + 2\alpha > 0. \end{aligned}$$

**Verdict:** Impossible since target  $= -6 - 2\alpha < 0$ .

### 8.2.2 Case 2: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, +1, -1)$

$$\begin{aligned}\text{LHS} &= (2 + 2\alpha) + 2\alpha - (4 - 2\alpha) \\ &= 2 + 2\alpha + 2\alpha - 4 + 2\alpha \\ &= -2 + 6\alpha.\end{aligned}$$

Setting LHS =  $-6 - 2\alpha$ :

$$-2 + 6\alpha = -6 - 2\alpha \implies 8\alpha = -4 \implies \alpha = -\frac{1}{2}.$$

**Verdict:** Impossible since  $\alpha = \log_2 3 > 0 \notin \{-\frac{1}{2}\}$ .

### 8.2.3 Case 3: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, -1, +1)$

$$\begin{aligned}\text{LHS} &= (2 + 2\alpha) - 2\alpha + (4 - 2\alpha) \\ &= 2 + 2\alpha - 2\alpha + 4 - 2\alpha \\ &= 6 - 2\alpha.\end{aligned}$$

Setting LHS =  $-6 - 2\alpha$ :

$$6 - 2\alpha = -6 - 2\alpha \implies 6 = -6.$$

**Verdict:** Impossible (algebraic contradiction).

### 8.2.4 Case 4: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, -1, -1)$

$$\begin{aligned}\text{LHS} &= (2 + 2\alpha) - 2\alpha - (4 - 2\alpha) \\ &= 2 + 2\alpha - 2\alpha - 4 + 2\alpha \\ &= -2 + 2\alpha.\end{aligned}$$

Setting LHS =  $-6 - 2\alpha$ :

$$-2 + 2\alpha = -6 - 2\alpha \implies 4\alpha = -4 \implies \alpha = -1.$$

**Verdict:** Impossible since  $\alpha > 0$ .

### 8.2.5 Case 5: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, +1, +1)$

$$\begin{aligned}\text{LHS} &= -(2 + 2\alpha) + 2\alpha + (4 - 2\alpha) \\ &= -2 - 2\alpha + 2\alpha + 4 - 2\alpha \\ &= 2 - 2\alpha.\end{aligned}$$

Setting LHS =  $-6 - 2\alpha$ :

$$2 - 2\alpha = -6 - 2\alpha \implies 2 = -6.$$

**Verdict:** Impossible (algebraic contradiction).

### 8.2.6 Case 6: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, +1, -1)$

$$\begin{aligned}\text{LHS} &= -(2 + 2\alpha) + 2\alpha - (4 - 2\alpha) \\ &= -2 - 2\alpha + 2\alpha - 4 + 2\alpha \\ &= -6 + 2\alpha.\end{aligned}$$

Setting LHS =  $-6 - 2\alpha$ :

$$-6 + 2\alpha = -6 - 2\alpha \implies 4\alpha = 0 \implies \alpha = 0.$$

**Verdict:** Impossible since  $\alpha = \log_2 3 > 0$  (boundary excluded).

### 8.2.7 Case 7: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, +1)$

$$\begin{aligned}\text{LHS} &= -(2 + 2\alpha) - 2\alpha + (4 - 2\alpha) \\ &= -2 - 2\alpha - 2\alpha + 4 - 2\alpha \\ &= 2 - 6\alpha.\end{aligned}$$

Setting LHS =  $-6 - 2\alpha$ :

$$2 - 6\alpha = -6 - 2\alpha \implies -4\alpha = -8 \implies \alpha = 2.$$

**Verdict:** Impossible since  $\alpha = \log_2 3 < 2$  (boundary excluded).

### 8.2.8 Case 8: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, -1)$

This is the key case. We compute:

$$\begin{aligned}\text{LHS} &= -(2 + 2\alpha) - 2\alpha - (4 - 2\alpha) \\ &= (-2 - 2\alpha) + (-2\alpha) + (-4 + 2\alpha).\end{aligned}$$

#### Step-by-step:

1. Distribute signs:  $(-1)(2 + 2\alpha) = -2 - 2\alpha$
2. Distribute signs:  $(-1)(2\alpha) = -2\alpha$
3. Distribute signs:  $(-1)(4 - 2\alpha) = -4 + 2\alpha$
4. Group constants:  $(-2) + (-4) = -6$
5. Group  $\alpha$  terms:  $(-2\alpha) + (-2\alpha) + (2\alpha) = -2\alpha$
6. Combine: LHS =  $-6 - 2\alpha$

#### Verification:

$$\text{LHS} = -6 - 2\alpha = -(6 + 2\alpha) = -s_0. \quad \checkmark$$

**Verdict:** This is the unique valid solution!

**Theorem 16** (Unique Sign Combination). *By exhaustive enumeration of all 8 sign combinations, only  $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, -1)$  yields  $\text{LHS} = -s$  for  $\alpha \in (0, 2)$ .*

## 9 Explicit Solution Values

**Proposition 17** (Values of  $a, b, c$ ). *With  $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$  and  $s = 6 + 2\alpha$ :*

$$a = \frac{(6 + 2\alpha) - (2 + 2\alpha)}{2} = \frac{4}{2} = 2, \tag{33}$$

$$b = \frac{(6 + 2\alpha) - 2\alpha}{2} = \frac{6}{2} = 3, \tag{34}$$

$$c = \frac{(6 + 2\alpha) - (4 - 2\alpha)}{2} = \frac{2 + 4\alpha}{2} = 1 + 2\alpha. \tag{35}$$

*Proof.* Using (19)–(21) with the minus signs:

$$\begin{aligned} a &= \frac{s - \sqrt{\Delta_1}}{2} = \frac{(6 + 2\alpha) - (2 + 2\alpha)}{2} = \frac{4}{2} = 2. \\ b &= \frac{s - \sqrt{\Delta_2}}{2} = \frac{(6 + 2\alpha) - 2\alpha}{2} = \frac{6}{2} = 3. \\ c &= \frac{s - \sqrt{\Delta_3}}{2} = \frac{(6 + 2\alpha) - (4 - 2\alpha)}{2} = \frac{2 + 4\alpha}{2} = 1 + 2\alpha. \end{aligned}$$

□

**Corollary 18** (Values of  $x, y, z$ ).

$$x = 2^a = 2^2 = 4, \quad (36)$$

$$y = 2^b = 2^3 = 8, \quad (37)$$

$$z = 2^c = 2^{1+2\alpha} = 2 \cdot 2^{2\alpha} = 2 \cdot (2^\alpha)^2 = 2 \cdot 3^2 = 18. \quad (38)$$

**Proposition 19** (Verification of Sum).

$$a + b + c = 2 + 3 + (1 + 2\alpha) = 6 + 2\alpha = s_0. \quad \checkmark$$

## 10 Verification of Original Constraints

We verify that  $(x, y, z) = (4, 8, 18)$  satisfies all three original constraints.

### 10.1 Verification of Constraint 1

*Claim 2.*  $x^{\log_2(yz)} = 2^8 \cdot 3^4$ .

*Proof.*

$$\begin{aligned} yz &= 8 \cdot 18 = 144 = 16 \cdot 9 = 2^4 \cdot 3^2. \\ \log_2(yz) &= \log_2(2^4 \cdot 3^2) = 4 + 2\alpha. \\ x^{\log_2(yz)} &= 4^{4+2\alpha} = (2^2)^{4+2\alpha} = 2^{8+4\alpha}. \end{aligned}$$

Since  $2^\alpha = 3$ :

$$2^{8+4\alpha} = 2^8 \cdot 2^{4\alpha} = 2^8 \cdot (2^\alpha)^4 = 2^8 \cdot 3^4. \quad \checkmark$$

□

### 10.2 Verification of Constraint 2

*Claim 3.*  $y^{\log_2(zx)} = 2^9 \cdot 3^6$ .

*Proof.*

$$\begin{aligned} zx &= 18 \cdot 4 = 72 = 8 \cdot 9 = 2^3 \cdot 3^2. \\ \log_2(zx) &= \log_2(2^3 \cdot 3^2) = 3 + 2\alpha. \\ y^{\log_2(zx)} &= 8^{3+2\alpha} = (2^3)^{3+2\alpha} = 2^{9+6\alpha}. \\ 2^{9+6\alpha} &= 2^9 \cdot 2^{6\alpha} = 2^9 \cdot (2^\alpha)^6 = 2^9 \cdot 3^6. \quad \checkmark \end{aligned}$$

□

### 10.3 Verification of Constraint 3

*Claim 4.*  $z^{\log_2(xy)} = 2^5 \cdot 3^{10}$ .

*Proof.*

$$\begin{aligned} xy &= 4 \cdot 8 = 32 = 2^5. \\ \log_2(xy) &= \log_2(2^5) = 5. \\ z^{\log_2(xy)} &= 18^5 = (2 \cdot 3^2)^5 = 2^5 \cdot 3^{10}. \quad \checkmark \end{aligned}$$

□

## 11 Proof of Minimality

We now prove that  $s_0 = 6 + 2\alpha$  is indeed the minimum value of  $s = a + b + c$ .

### 11.1 Definition of the Constraint Function

**Definition 20.** For  $s \geq \sqrt{20 + 40\alpha}$ , define:

$$f(s) = \sqrt{s^2 - 32 - 16\alpha} + \sqrt{s^2 - 36 - 24\alpha} + \sqrt{s^2 - 20 - 40\alpha} - s. \quad (39)$$

The constraint equation (for the all-minus case) is  $f(s) = 0$ .

### 11.2 Verification that $f(s_0) = 0$

**Lemma 21.**  $f(s_0) = 0$  where  $s_0 = 6 + 2\alpha$ .

*Proof.* Substituting the computed discriminant roots:

$$\begin{aligned} f(s_0) &= \sqrt{\Delta_1} + \sqrt{\Delta_2} + \sqrt{\Delta_3} - s_0 \\ &= (2 + 2\alpha) + 2\alpha + (4 - 2\alpha) - (6 + 2\alpha). \end{aligned}$$

Simplifying the first three terms:

$$(2 + 2\alpha) + 2\alpha + (4 - 2\alpha) = 2 + 2\alpha + 2\alpha + 4 - 2\alpha = 6 + 2\alpha.$$

Therefore:

$$f(s_0) = (6 + 2\alpha) - (6 + 2\alpha) = 0. \quad \checkmark$$

□

### 11.3 Strict Monotonicity of $f$

**Lemma 22.**  $f'(s) > 0$  for all  $s$  in the domain.

*Proof.* Write  $f(s) = \sqrt{s^2 - c_1} + \sqrt{s^2 - c_2} + \sqrt{s^2 - c_3} - s$  where:

$$c_1 = 32 + 16\alpha, \quad c_2 = 36 + 24\alpha, \quad c_3 = 20 + 40\alpha.$$

By the chain rule:

$$\frac{d}{ds} \sqrt{s^2 - c} = \frac{1}{2\sqrt{s^2 - c}} \cdot 2s = \frac{s}{\sqrt{s^2 - c}}.$$

Therefore:

$$f'(s) = \frac{s}{\sqrt{s^2 - c_1}} + \frac{s}{\sqrt{s^2 - c_2}} + \frac{s}{\sqrt{s^2 - c_3}} - 1. \quad (40)$$

**Claim:** Each fraction  $\frac{s}{\sqrt{s^2 - c_i}} > 1$ .

**Proof of claim:** For  $\frac{s}{\sqrt{s^2 - c_i}} > 1$ , we need  $s > \sqrt{s^2 - c_i}$ , i.e.,  $s^2 > s^2 - c_i$ , i.e.,  $c_i > 0$ .

Since  $\alpha > 0$ :

$$c_1 = 32 + 16\alpha > 32 > 0,$$

$$c_2 = 36 + 24\alpha > 36 > 0,$$

$$c_3 = 20 + 40\alpha > 20 > 0.$$

Therefore, for  $s > 0$  in the domain:

$$f'(s) > 1 + 1 + 1 - 1 = 2 > 0.$$

□

## 11.4 Uniqueness of the Minimum

**Theorem 23** (Uniqueness).  $s_0 = 6 + 2\alpha$  is the unique solution to  $f(s) = 0$ .

*Proof.* 1. The domain of  $f$  is  $[s_{\min}, \infty)$  where  $s_{\min} = \sqrt{20 + 40\alpha}$  ensures all radicands are non-negative.

2.  $f$  is continuous on its domain as a composition of continuous functions.
3. By Lemma 22,  $f'(s) > 0$  for all  $s$  in the domain.
4. By the Mean Value Theorem, continuous  $f$  with  $f'(s) > 0$  everywhere implies  $f$  is strictly increasing.
5. For strictly increasing  $f$ : if  $f(s_0) = 0$ , then:
  - $f(s) < 0$  for  $s < s_0$ , and
  - $f(s) > 0$  for  $s > s_0$ .

6. By Lemma 21,  $f(s_0) = 0$  at  $s_0 = 6 + 2\alpha$ .

Therefore  $s_0 = 6 + 2\alpha$  is the unique zero of  $f$ , and no solution exists for  $s < s_0$ . □

**Corollary 24** (Minimum Product).  $s = a + b + c \geq s_0 = 6 + 2\alpha$ , with equality achieved. Therefore:

$$xyz = 2^s \geq 2^{6+2\alpha} = 2^6 \cdot 2^{2\alpha} = 64 \cdot (2^\alpha)^2 = 64 \cdot 9 = 576.$$

## 12 Conclusion

*Proof of Theorem 1.* We have shown:

1. The constraints transform to a quadratic system in logarithmic variables.
2. The system admits real solutions only when  $s = a + b + c \geq \sqrt{20 + 40\alpha}$ .
3. At  $s_0 = 6 + 2\alpha$ , all discriminants are perfect squares.
4. Exhaustive analysis of all 8 sign combinations shows only  $(-1, -1, -1)$  is valid.
5. The constraint function  $f(s) = 0$  has a unique solution at  $s_0 = 6 + 2\alpha$  by strict monotonicity.
6. The explicit solution  $(x, y, z) = (4, 8, 18)$  satisfies all original constraints.

7. Therefore,  $xyz = 4 \cdot 8 \cdot 18 = 576$  is the minimum.

□

$$xyz_{\min} = 576 \quad (41)$$

## A Numerical Verification

For reference,  $\alpha = \log_2 3 \approx 1.5849625007211563$ .

$$\begin{aligned} s_0 &= 6 + 2\alpha \approx 9.1699250014423126 \\ 2^{s_0} &= 2^{6+2\log_2 3} = 2^6 \cdot 3^2 = 64 \cdot 9 = 576 \quad \checkmark \end{aligned}$$

## B Graph Metadata

This proof was formalized using the Alethfeld proof system from graph `graph-51acde-43ac9a`, containing 40 nodes at depth 0–1, with 67 expansion nodes at depths 2–3.

Key verified nodes:

- :1-000001 – Logarithmic transformation
- :1-000011, :1-000012, :1-000013 – Discriminant factorizations
- :1-000015 – Valid sign combination verification
- :1-000034 – Exhaustive sign analysis (8 cases)
- :1-000031, :1-000032, :1-000033 – Minimality via monotonicity