The First Arclength Formula: A Structured Proof

Theorem 1 (First Arclength Formula)**.** *Let $p : \mathbb{C} \to \mathbb{C}$ be a non-zero polynomial, and let $\Omega \subseteq \mathbb{C}$ be a semialgebraic open set that does not contain any critical points of p in its closure. Define the lemniscate*

$$E_1(p) = \{z \in \mathbb{C} : |p(z)| \leq 1\}.$$

Then

$$\ell(\partial E_1(p) \cap \Omega) = \int_{-\pi}^{\pi} \sum_{z \in \Omega : p(z) = e^{i\alpha}} \frac{1}{|p'(z)|} \, d\alpha.$$

Hypotheses.

H1: $p : \mathbb{C} \to \mathbb{C}$ is a non-zero polynomial.

H2: $\Omega \subseteq \mathbb{C}$ is a semialgebraic open set.

H3: $p'(z) \neq 0$ for all $z \in \overline{\Omega}$.

External Results.

E1. *Whitney Stratification* [1]: One-dimensional semialgebraic sets are rectifiable with finite arclength.

E2. *Area Formula* [2]: For a local diffeomorphism $f : M \to N$ between equidimensional Riemannian manifolds,

$$\int_M g \, dV_M = \int_N \sum_{x \in f^{-1}(y)} \frac{g(x)}{|Jf(x)|} \, dV_N.$$

Proof. $\langle 1 \rangle 1$. $p|_\Omega$ is a local diffeomorphism. *(Inverse Function Theorem; H1, H3)*

$\langle 2 \rangle 1$. By **H3**, $p'(z) \neq 0$ on $\overline{\Omega}$.

$\langle 2 \rangle 2$. The inverse function theorem implies p is a local diffeomorphism at each point of Ω.

$\langle 1 \rangle 2$. $\partial E_1(p) \cap \Omega$ is a smooth 1-dimensional submanifold. *(Implicit Function Theorem; H1, H3)*

$\langle 2 \rangle 1$. Define $f(z) = |p(z)|^2 - 1$. Then $\partial E_1(p) \cap \Omega = f^{-1}(0) \cap \Omega$.

$\langle 2 \rangle 2$. Compute $|\nabla f|^2 = 4|p(z)|^2 |p'(z)|^2$.

$\langle 2 \rangle 3$. On $\partial E_1(p)$, we have $|p(z)| = 1$, so $|\nabla f|^2 = 4|p'(z)|^2$.

$\langle 2 \rangle 4$. By **H3**, $|p'(z)|^2 \neq 0$ on $\overline{\Omega}$, hence $|\nabla f| \neq 0$ on $\partial E_1(p) \cap \overline{\Omega}$.

$\langle 2 \rangle 5$. By the implicit function theorem, $f^{-1}(0) \cap \Omega$ is a smooth 1-submanifold.

$\langle 1 \rangle 3$. $\partial E_1(p) \cap \Omega$ is rectifiable with finite arclength. *(Whitney Stratification; H1, H2, $\langle 1 \rangle 2$)*

⟨2⟩1. The set $\partial E_1(p) = \{z : |p(z)|^2 = 1\}$ is semialgebraic (polynomial condition).
⟨2⟩2. By **H2**, Ω is semialgebraic.
⟨2⟩3. Therefore $\partial E_1(p) \cap \Omega$ is semialgebraic (intersection of semialgebraic sets).
⟨2⟩4. By **E1**, one-dimensional semialgebraic sets are rectifiable with finite arclength.

⟨1⟩4. For each $w \in S^1$, the fiber $p^{-1}(\{w\}) \cap \Omega$ is finite. *(Fundamental Theorem of Algebra; H1)*

⟨2⟩1. By the Fundamental Theorem of Algebra, $p(z) = w$ has at most $\deg(p)$ solutions.
⟨2⟩2. Hence $p^{-1}(\{w\}) \cap \Omega \subseteq p^{-1}(\{w\})$ has at most $\deg(p)$ elements.

⟨1⟩5. $p : (\partial E_1(p) \cap \Omega) \to S^1$ is a local diffeomorphism. *(tangent space analysis; ⟨1⟩2, H3)*

⟨2⟩1. At $z \in \partial E_1(p) \cap \Omega$, the tangent space $T_z(\partial E_1(p))$ consists of vectors v with

$$\mathrm{Re}(\overline{p(z)}p'(z)v) = 0.$$

⟨2⟩2. The differential $dp_z(v) = p'(z)v$ maps this isomorphically onto $T_{p(z)}S^1 = i \cdot p(z) \cdot \mathbb{R}$.
⟨2⟩3. This holds because $|p(z)| = 1$ implies $\overline{p(z)} = 1/p(z)$.

⟨1⟩6. For arclength-parametrized γ, $|(p \circ \gamma)'(t)| = |p'(\gamma(t))|$. *(Chain Rule; ⟨1⟩2, ⟨1⟩5)*

⟨2⟩1. By the chain rule, $(p \circ \gamma)'(t) = p'(\gamma(t)) \cdot \gamma'(t)$.
⟨2⟩2. Since γ is arclength-parametrized, $|\gamma'(t)| = 1$.
⟨2⟩3. Therefore $|(p \circ \gamma)'(t)| = |p'(\gamma(t))| \cdot |\gamma'(t)| = |p'(\gamma(t))|$.

⟨1⟩7. **Area Formula:**

$$\ell(\partial E_1(p) \cap \Omega) = \int_{S^1} \sum_{z \in p^{-1}(w) \cap \Omega} \frac{1}{|p'(z)|} \, d\ell(w).$$

(E2; ⟨1⟩3, ⟨1⟩4, ⟨1⟩5, ⟨1⟩6)

⟨2⟩1. Apply **E2** with $M = \partial E_1(p) \cap \Omega$, $N = S^1$, $f = p$, and $g \equiv 1$.
⟨2⟩2. The Jacobian is $|Jf| = |p'|$ by step ⟨1⟩6.
⟨2⟩3. The fibers are finite by step ⟨1⟩4.
⟨2⟩4. The domain is rectifiable by step ⟨1⟩3.
⟨2⟩5. The map is a local diffeomorphism by step ⟨1⟩5.

⟨1⟩8. Parametrizing S^1 by $\alpha \mapsto e^{i\alpha}$, $\alpha \in [-\pi, \pi]$, we have $d\ell = d\alpha$. *(arclength computation)*

⟨2⟩1. Let $\phi(\alpha) = e^{i\alpha}$ for $\alpha \in [-\pi, \pi]$.
⟨2⟩2. Then $\phi'(\alpha) = ie^{i\alpha}$, so $|\phi'(\alpha)| = |ie^{i\alpha}| = 1$.
⟨2⟩3. Hence ϕ is arclength-parametrized.
⟨2⟩4. The pullback satisfies $\phi^*(d\ell) = d\alpha$.

⟨1⟩9. **Conclusion:**

$$\ell(\partial E_1(p) \cap \Omega) = \int_{-\pi}^{\pi} \sum_{z \in \Omega : p(z) = e^{i\alpha}} \frac{1}{|p'(z)|} \, d\alpha.$$

(substitution; ⟨1⟩7, ⟨1⟩8)

⟨2⟩1. Substitute the parametrization from step ⟨1⟩8 into step ⟨1⟩7.
⟨2⟩2. Under $w = e^{i\alpha}$, we have $p^{-1}(\{w\}) \cap \Omega = \{z \in \Omega : p(z) = e^{i\alpha}\}$.
⟨2⟩3. Direct substitution yields the claimed formula. □

References

[1] J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, Ergebnisse der Mathematik, vol. 36, Springer-Verlag, 1998.

[2] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.