Entropy-Influence Bound for Rank-1 Product State Quantum Boolean Functions

A Comprehensive Formal Treatment

Alethfeld Proof System v4

Lean4 Verification: `AlethfeldLean.QBF.Rank1`

Graph ID: `qbf-rank1-entropy-influence`
Version: 2 (Verified) • Sorries: 0

December 2025

Abstract

We establish an explicit upper bound on the entropy-to-influence ratio for rank-1 quantum Boolean functions (QBFs) constructed from product states. For the QBF $U = I - 2|\psi\rangle\langle\psi|$ where $|\psi\rangle = \bigotimes_{k=1}^{n} |\phi_k\rangle$ is a product state, we prove that the influence $I(U) = n \cdot 2^{1-n}$ is independent of the choice of single-qubit states, while the entropy $S(U)$ is maximized when all qubits are in the "magic" state with Bloch vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$. The ratio S/I approaches $\log_2 3 + 4 \approx 5.585$ as $n \to \infty$, establishing a lower bound on any universal constant C satisfying $S(U) \leq C \cdot I(U)$.

All results are formally verified in Lean4 using Mathlib v4.26.0, with 0 sorries remaining in the proof.

Lean4 Verification Status

Mathlib Version: v4.26.0
Lean Version: 4.26.0
Proof Status: Complete
Sorry Count: 0
Last Build: 2025-12-29
Modules: L1Fourier, L2Influence, L3Entropy, ShannonMax, L4Maximum, L5Asymptotic, QBFRank1MasterTheorem

Contents

1 Introduction and Main Result

1.1 The Entropy-Influence Conjecture

The entropy-influence conjecture for Boolean functions posits the existence of a universal constant C such that $S(f) \le C \cdot I(f)$ for all Boolean functions f. For quantum Boolean functions (QBFs), this conjecture extends to unitaries acting on n-qubit systems.

1.2 Main Theorem

Theorem 1.1 (Master Theorem). $^{:theorem}$ *For the rank-1 QBF* $U = I - 2|\psi\rangle\langle\psi|$ *where* $|\psi\rangle = \bigotimes_{k=1}^{n} |\phi_k\rangle$ *is a product state:*

$$\frac{S(U)}{I(U)} \le \log_2 3 + \frac{2^{n-1}}{n}\left[-p_0 \log_2 p_0 + (2n-2)(1-p_0)\right] \tag{1}$$

where $p_0 = (1 - 2^{1-n})^2$. *The maximum is achieved when all qubits are in the magic state with Bloch vector* $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

Lean4 Reference: *QBFRank1MasterTheorem.qbfRank1Master*

The proof proceeds through five lemmas:

1. **L1 (Fourier):** Derive the Fourier coefficient formula
2. **L2 (Influence):** Prove influence is independent of Bloch vectors
3. **L3 (Entropy):** Establish the general entropy formula
4. **L4 (Maximum):** Show the magic state uniquely maximizes entropy
5. **L5 (Asymptotic):** Compute the limit as $n \to \infty$

2 Preliminaries

2.1 Setup and Assumptions

Assumption 2.1 (Product State QBF). $^{:0\text{-}A1}$ *Let* $U = I - 2|\psi\rangle\langle\psi|$ *be a rank-1 QBF where* $|\psi\rangle = \bigotimes_{k=1}^{n} |\phi_k\rangle$ *is a product state with each* $|\phi_k\rangle \in \mathbb{C}^2$.

Definition 2.2 (Bloch Vector). $^{:0\text{-}D1}$ *Each single-qubit state* $|\phi_k\rangle$ *has Bloch vector* $\vec{r}_k = (x_k, y_k, z_k)$ *with*

$$|\vec{r}_k|^2 = x_k^2 + y_k^2 + z_k^2 = 1. \tag{2}$$

Lean4 Reference: *Quantum.Basic.BlochVector*

Definition 2.3 (Extended Bloch Coefficients). $^{:0\text{-}D2}$ *Define the extended coefficients:*

$$q_k^{(0)} = 1, \qquad \left(q_k^{(1)}, q_k^{(2)}, q_k^{(3)}\right) = \left(x_k^2, y_k^2, z_k^2\right). \tag{3}$$

These satisfy $\sum_{\ell=0}^{3} q_k^{(\ell)} = 1 + x_k^2 + y_k^2 + z_k^2 = 2$.

Definition 2.4 (Bloch Entropy). $^{:0\text{-}D3}$ *The Bloch entropy of qubit* k *is*

$$f_k = H(x_k^2, y_k^2, z_k^2) = -\sum_{\ell=1}^{3} q_k^{(\ell)} \log_2 q_k^{(\ell)}. \tag{4}$$

This is the Shannon entropy of the squared Bloch components, viewed as a probability distribution on 3 outcomes.

Lean4 Reference: *Quantum.BlochEntropy.blochEntropy*

Remark. The Bloch entropy f_k measures the "spread" of the Bloch vector across coordinate axes. It is *not* the von Neumann entropy of the qubit state (which is zero for pure states).

3 Lemma L1: Fourier Coefficient Formula

Lemma 3.1 (L1: Fourier Coefficients). *:L1-root For $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$ where $\left|\psi\right\rangle$ is a product state:*

$$\hat{U}(\alpha) = \delta_{\alpha,0} - 2^{1-n}\prod_{k=1}^{n} r_k^{(\alpha_k)} \tag{5}$$

where $r_k^{(0)} = 1$, $r_k^{(1)} = x_k$, $r_k^{(2)} = y_k$, $r_k^{(3)} = z_k$.

Lean4 Reference: `L1Fourier.fourier_coefficient_formula`

Proof. We proceed through four steps.
Step 1 (Definition Expansion) :L1-step1 By definition of the Pauli-Fourier expansion:

$$\hat{U}(\alpha) = 2^{-n}\,\mathrm{Tr}(\sigma^\alpha U) = 2^{-n}\,\mathrm{Tr}(\sigma^\alpha) - 2^{1-n}\,\mathrm{Tr}(\sigma^\alpha\left|\psi\right\rangle\left\langle\psi\right|). \tag{6}$$

This follows from linearity of trace and $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$.
Step 2 (Pauli Trace) :L1-step2 The trace of Pauli strings satisfies:

$$\mathrm{Tr}(\sigma^\alpha) = 2^n\delta_{\alpha,0} \tag{7}$$

2.1 For the single-qubit Pauli matrices: $\mathrm{Tr}(\sigma_0) = \mathrm{Tr}(I_2) = 2$ and $\mathrm{Tr}(\sigma_i) = 0$ for $i \in \{1,2,3\}$.
2.2 For tensor products: $\mathrm{Tr}(\sigma^{\alpha_1}\otimes\cdots\otimes\sigma^{\alpha_n}) = \prod_{k=1}^{n}\mathrm{Tr}(\sigma^{\alpha_k})$.
2.3 Therefore $\mathrm{Tr}(\sigma^\alpha) \neq 0$ only when all $\alpha_k = 0$, giving $\mathrm{Tr}(\sigma^{\vec{0}}) = 2^n$.
Step 3 (Cyclic Property) :L1-step3 By the cyclic property of trace:

$$\mathrm{Tr}(\sigma^\alpha\left|\psi\right\rangle\left\langle\psi\right|) = \left\langle\psi\right|\sigma^\alpha\left|\psi\right\rangle. \tag{8}$$

This is a standard linear algebra identity: $\mathrm{Tr}(A\left|v\right\rangle\left\langle v\right|) = \left\langle v\right|A\left|v\right\rangle$.
Step 4 (Product Factorization) :L1-step4 For a product state $\left|\psi\right\rangle = \bigotimes_k\left|\phi_k\right\rangle$, the expectation value factorizes:

$$\left\langle\psi\right|\sigma^\alpha\left|\psi\right\rangle = \prod_k\left\langle\phi_k\right|\sigma^{\alpha_k}\left|\phi_k\right\rangle = \prod_k r_k^{(\alpha_k)}. \tag{9}$$

4.1 The tensor product structure gives: $\langle\bigotimes_k\phi_k|(\bigotimes_k\sigma^{\alpha_k})|\bigotimes_k\phi_k\rangle = \prod_k\left\langle\phi_k\right|\sigma^{\alpha_k}\left|\phi_k\right\rangle$.
4.2 For a pure qubit state with Bloch vector (x_k, y_k, z_k):

$$\left\langle\phi_k\right|\sigma_0\left|\phi_k\right\rangle = 1 = r_k^{(0)} \tag{10}$$
$$\left\langle\phi_k\right|\sigma_1\left|\phi_k\right\rangle = x_k = r_k^{(1)} \tag{11}$$
$$\left\langle\phi_k\right|\sigma_2\left|\phi_k\right\rangle = y_k = r_k^{(2)} \tag{12}$$
$$\left\langle\phi_k\right|\sigma_3\left|\phi_k\right\rangle = z_k = r_k^{(3)} \tag{13}$$

QED:L1-qed: Combining Steps 1–4:

$$\hat{U}(\alpha) = 2^{-n}\cdot 2^n\delta_{\alpha,0} - 2^{1-n}\prod_k r_k^{(\alpha_k)} = \delta_{\alpha,0} - 2^{1-n}\prod_{k=1}^{n} r_k^{(\alpha_k)}. \tag{14}$$

□

Corollary 3.2 (Probability Distribution)**.** :0-L1cor *The Fourier weight distribution is:*

$$p_\alpha = |\hat{U}(\alpha)|^2 = \begin{cases} (1-2^{1-n})^2 & \alpha = 0 \\ 2^{2-2n} \prod_{k=1}^{n} q_k^{(\alpha_k)} & \alpha \neq 0 \end{cases} \tag{15}$$

Proof. For $\alpha = 0$: $|\hat{U}(0)|^2 = |1-2^{1-n}|^2 = (1-2^{1-n})^2 = p_0$.

For $\alpha \neq 0$: $|\hat{U}(\alpha)|^2 = |-2^{1-n}\prod_k r_k^{(\alpha_k)}|^2 = 2^{2-2n}\prod_k |r_k^{(\alpha_k)}|^2 = 2^{2-2n}\prod_k q_k^{(\alpha_k)}$. □

4 Lemma L2: Influence Independence

Lemma 4.1 (L2: Influence Independence)**.** :theorem-L2 *For any rank-1 product state QBF:*

$$I(U) = n \cdot 2^{1-n}. \tag{16}$$

This is ***independent of the choice of Bloch vectors****.*

Lean4 Reference: *L2Influence.total_influence_formula*

Proof. We prove this in five steps.

Step 1 (Influence Definition) :1-step1 The influence of qubit j is:

$$I_j = \sum_{\alpha:\alpha_j \neq 0} p_\alpha. \tag{17}$$

1.1 By definition, influence measures how much the output depends on qubit j.

1.2 For classical Boolean functions: $I_j(f) = \Pr_x[f(x) \neq f(x^{\oplus j})]$ where $x^{\oplus j}$ flips bit j.

1.3 For QBFs, this generalizes to the Fourier form: $I_j = \sum_{\alpha:\alpha_j \neq 0} |\hat{U}(\alpha)|^2$.

Step 2 (Factorization) :1-step2 For $\ell \in \{1,2,3\}$:

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{2-2n} \cdot q_j^{(\ell)} \cdot \prod_{k\neq j}\sum_{m=0}^{3} q_k^{(m)}. \tag{18}$$

2.1 From L1-corollary: $p_\alpha = 2^{2-2n}\prod_{k=1}^{n} q_k^{(\alpha_k)}$ for $\alpha \neq 0$.

2.2 Fixing $\alpha_j = \ell$, the product splits: $\prod_k q_k^{(\alpha_k)} = q_j^{(\ell)} \cdot \prod_{k\neq j} q_k^{(\alpha_k)}$.

2.3 Summing over all $(\alpha_k)_{k\neq j} \in \{0,1,2,3\}^{n-1}$:

$$\sum_{\alpha:\alpha_j=\ell}\prod_{k\neq j} q_k^{(\alpha_k)} = \prod_{k\neq j}\sum_{m=0}^{3} q_k^{(m)}. \tag{19}$$

This uses the distributive law for finite products over sums.

Step 3 (Unit Sphere Simplification) :1-step3 Since $\sum_{m=0}^{3} q_k^{(m)} = 2$:

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{1-n} q_j^{(\ell)}. \tag{20}$$

3.1 By Definition 2.3: $\sum_{m=0}^{3} q_k^{(m)} = 1 + x_k^2 + y_k^2 + z_k^2$.

3.2 By the Bloch constraint (Definition 2.2): $x_k^2 + y_k^2 + z_k^2 = 1$.

3.3 Therefore $\sum_{m=0}^{3} q_k^{(m)} = 1 + 1 = 2$.

3.4 Substituting: $2^{2-2n} \cdot q_j^{(\ell)} \cdot 2^{n-1} = 2^{2-2n+n-1} q_j^{(\ell)} = 2^{1-n} q_j^{(\ell)}$.

Step 4 (Single-Qubit Influence) :1-step4

$$I_j = 2^{1-n} \sum_{\ell=1}^{3} q_j^{(\ell)} = 2^{1-n} \tag{21}$$

This is independent of j and the Bloch vector.

4.1 From Step 1: $I_j = \sum_{\alpha:\alpha_j \neq 0} p_\alpha = \sum_{\ell=1}^{3} \sum_{\alpha:\alpha_j=\ell} p_\alpha$.

4.2 Applying Step 3: $I_j = \sum_{\ell=1}^{3} 2^{1-n} q_j^{(\ell)} = 2^{1-n} \sum_{\ell=1}^{3} q_j^{(\ell)}$.

4.3 From Definition 2.3 and 2.2: $\sum_{\ell=1}^{3} q_j^{(\ell)} = x_j^2 + y_j^2 + z_j^2 = 1$.

4.4 Therefore $I_j = 2^{1-n} \cdot 1 = 2^{1-n}$.

Step 5 (Total Influence) :1-qed

$$I(U) = \sum_{j=1}^{n} I_j = n \cdot 2^{1-n}. \tag{22}$$

5.1 Total influence sums individual qubit influences: $I(U) = \sum_{j=1}^{n} I_j$.

5.2 From Step 4: each $I_j = 2^{1-n}$, independent of j and Bloch vectors.

5.3 Summing n copies: $I(U) = n \cdot 2^{1-n}$. □

5 Lemma L3: General Entropy Formula

Lemma 5.1 (L3: Entropy Formula). *:theorem-L3*

$$S = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \sum_{k=1}^{n} f_k \tag{23}$$

where $f_k = H(x_k^2, y_k^2, z_k^2)$ is the Bloch entropy of qubit k.

Lean4 Reference: *L3Entropy.entropy_formula*

Proof. We proceed through seven steps.

Step 1 (Shannon Entropy Definition) :L3-lem4

$$S = -p_0 \log_2 p_0 - \sum_{\alpha \neq 0} p_\alpha \log_2 p_\alpha. \tag{24}$$

1.1 Shannon entropy: $S = -\sum_\alpha p_\alpha \log_2 p_\alpha$.

1.2 The index set $\{0,1,2,3\}^n$ partitions as $\{\vec{0}\} \cup \{\alpha : \alpha \neq 0\}$.

1.3 Splitting the sum gives the stated form.

Step 2 (Logarithm Expansion) :L3-step1 For $\alpha \neq 0$:

$$-p_\alpha \log_2 p_\alpha = p_\alpha(2n-2) - p_\alpha \sum_k \log_2 q_k^{(\alpha_k)}. \tag{25}$$

2.1 From Corollary 3.2: $\log_2 p_\alpha = \log_2(2^{2-2n} \prod_k q_k^{(\alpha_k)})$.

2.2 By log rules: $\log_2(2^{2-2n} \prod_k q_k) = (2-2n) + \sum_k \log_2 q_k^{(\alpha_k)}$.

2.3 Multiplying by $-p_\alpha$: $-p_\alpha \log_2 p_\alpha = -p_\alpha(2-2n) - p_\alpha \sum_k \log_2 q_k$.

2.4 Sign simplification: $-(2-2n) = 2n-2$.

Step 3 (Constant Factor Sum) :L3-step2

$$\sum_{\alpha \neq 0} p_\alpha(2n-2) = (2n-2)(1-p_0). \tag{26}$$

3.1 Factor out constant: $\sum_{\alpha \neq 0} p_\alpha(2n-2) = (2n-2)\sum_{\alpha \neq 0} p_\alpha$.

3.2 Probability normalization: $\sum_\alpha p_\alpha = 1$.

3.3 Therefore $\sum_{\alpha \neq 0} p_\alpha = 1 - p_0$.

Step 4 (Case Split on α_j) :L3-step3 For $\alpha_j = 0$: $\log_2 q_j^{(0)} = 0$, so only $\alpha_j \neq 0$ contributes.

4.1 From Definition 2.3: $q_j^{(0)} = 1$.

4.2 $\log_2(1) = 0$ by definition.

4.3 Therefore $-p_\alpha \log_2 q_j^{(0)} = -p_\alpha \cdot 0 = 0$.

Step 5 (Application of L2) :L3-step4 From Lemma 4.1:

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{1-n} q_j^{(\ell)} \quad \text{for } \ell \in \{1,2,3\}. \tag{27}$$

Step 6 (Bloch Entropy Identification) :L3-step5

$$-\sum_{\alpha:\alpha_j \neq 0} p_\alpha \log_2 q_j^{(\alpha_j)} = 2^{1-n} f_j. \tag{28}$$

6.1 Partition: $\sum_{\alpha:\alpha_j \neq 0} = \sum_{\ell=1}^{3} \sum_{\alpha:\alpha_j=\ell}$.

6.2 For fixed ℓ, $\log_2 q_j^{(\ell)}$ is constant across the inner sum.

6.3 Applying Step 5: $-\sum_{\ell=1}^{3} 2^{1-n} q_j^{(\ell)} \log_2 q_j^{(\ell)}$.

6.4 By Definition 2.4: $-\sum_{\ell=1}^{3} q_j^{(\ell)} \log_2 q_j^{(\ell)} = f_j$.

6.5 Therefore: $2^{1-n} \cdot f_j$.

Step 7 (Sum Over All Qubits) :L3-step6

$$-\sum_{\alpha \neq 0} p_\alpha \sum_k \log_2 q_k^{(\alpha_k)} = 2^{1-n} \sum_k f_k. \tag{29}$$

7.1 Exchange order of summation: $\sum_\alpha \sum_k = \sum_k \sum_\alpha$ (Fubini for finite sums).

7.2 When $\alpha_k = 0$: $\log_2 q_k^{(0)} = 0$ (by Step 4), so these terms contribute zero.

7.3 Applying Step 6 for each k: $\sum_{k=1}^{n} 2^{1-n} f_k = 2^{1-n} \sum_{k=1}^{n} f_k$.

QED:L3-qed: Combining Steps 1–7:

$$S = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \sum_k f_k. \tag{30}$$

□

6 Lemma L4: Maximum at Magic State

Lemma 6.1 (Shannon Maximum Entropy). *:ext-shannon* *For any probability distribution $(p_1, \ldots, p_m)$ with $\sum_i p_i = 1$:*

$$H(p_1, \ldots, p_m) = -\sum_{i=1}^{m} p_i \log_2 p_i \leq \log_2 m \tag{31}$$

with equality if and only if $p_i = 1/m$ for all i (uniform distribution).

Lean4 Reference: *ShannonMax.shannon_entropy_le_log*

Lemma 6.2 (L4: Maximum Ratio). *:theorem-L4* *The ratio S/I is maximized when all qubits are in the magic state:*

$$(x_k^2, y_k^2, z_k^2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{32}$$

Lean4 Reference: *L4Maximum.l4_maximum_entropy*

Proof. We proceed through six steps.

Step 1 (Influence Constancy) :1-main Maximizing S/I is equivalent to maximizing S.

1.1 By Lemma 4.1: $I(U) = n \cdot 2^{1-n}$ is constant w.r.t. Bloch vectors.

1.2 For $n \geq 1$: $I(U) = n \cdot 2^{1-n} > 0$.

1.3 For $c > 0$ constant: $\arg\max f/c = \arg\max f$. Apply with $c = I(U)$, $f = S(U)$.

Step 2 (Bloch Dependence) :2-main Only $2^{1-n} \sum_k f_k$ depends on Bloch vectors in S.

2.1 $-p_0 \log_2 p_0$ depends only on n (since $p_0 = (1 - 2^{1-n})^2$).

2.2 $(2n-2)(1-p_0)$ depends only on n.

2.3 $\sum_k f_k$ depends on Bloch vectors via $f_k = H(x_k^2, y_k^2, z_k^2)$.

2.4 Therefore $\max S \Leftrightarrow \max \sum_k f_k$.

Step 3 (Shannon Entropy on 3 Outcomes) :3-main Each f_k is Shannon entropy on 3 outcomes.

3.1 Let $q_1 = x_k^2$, $q_2 = y_k^2$, $q_3 = z_k^2$. Then $f_k = H(q_1, q_2, q_3)$.

3.2 (q_1, q_2, q_3) is a probability distribution: $q_i \geq 0$ and $q_1 + q_2 + q_3 = 1$ (by Definition 2.2).

Step 4 (Shannon Bound) :4-main $f_k \leq \log_2 3$ with equality iff $(x_k^2, y_k^2, z_k^2) = (1/3, 1/3, 1/3)$.

4.1 By Lemma 6.1: $H(p_1, \ldots, p_m) \leq \log_2 m$ with equality iff uniform.

4.2 Apply with $m = 3$: $f_k = H(x_k^2, y_k^2, z_k^2) \leq \log_2 3$.

4.3 Equality requires uniform: $x_k^2 = y_k^2 = z_k^2 = 1/3$.

4.4 This is the magic state: $(x_k, y_k, z_k) = \pm(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

Step 5 (Independence of Qubits) :5-main $\sum_k f_k$ is maximized when *all* qubits are at the magic state.

5.1 Terms f_k are independent: each depends only on (x_k, y_k, z_k).

5.2 For independent terms: $\max \sum f_k = \sum \max f_k$ (separable optimization).

5.3 Each $\max f_k = \log_2 3$ at magic (Step 4).

5.4 Therefore $\max \sum_k f_k = n \log_2 3$, achieved when all qubits are at magic.

Step 6 (QED) :final-qed S/I is maximized when all qubits are at the magic state $(1/3, 1/3, 1/3)$.

Combining Steps 1–5: $\max S/I \Leftrightarrow \max S \Leftrightarrow \max \sum_k f_k \Leftrightarrow$ all qubits magic. □

Corollary 6.3 (Explicit Maximum). *:corollary-L4 At the magic state:*

$$\frac{S_{\max}}{I} = \log_2 3 + \frac{2^{n-1}}{n} \left[-p_0 \log_2 p_0 + (2n-2)(1-p_0)\right] \tag{33}$$

where $p_0 = (1 - 2^{1-n})^2$.

Proof. At magic: $f_k = \log_2 3$ for all k. From Lemma 5.1:

$$S_{\max} = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \cdot n \cdot \log_2 3. \tag{34}$$

Dividing by $I = n \cdot 2^{1-n}$ and rearranging yields the result. □

7 Lemma L5: Asymptotic Analysis

Lemma 7.1 (L5: Limiting Behavior). *:theorem-L5*

$$\lim_{n\to\infty} \frac{S_{\max}}{I} = \log_2 3 + 4 \approx 5.585. \tag{35}$$

Lean4 Reference: *L5Asymptotic.l5_asymptotic_ratio*

Proof. Let $\varepsilon = 2^{1-n}$.

Setup :L5-assume Define $\varepsilon = 2^{1-n}$. Then:

- $\varepsilon > 0$ for all $n \geq 1$
- $\varepsilon < 1$ for $n \geq 2$
- $p_0 = (1-\varepsilon)^2$
- $1 - p_0 = 2\varepsilon - \varepsilon^2$

Step 1 (Taylor Expansion for Entropy Term) :L5-step1

$$-p_0 \log_2 p_0 = \frac{2\varepsilon}{\ln 2} + O(\varepsilon^2). \tag{36}$$

1.1 Taylor series: $\ln(1-x) = -x - x^2/2 - \cdots$ for $|x| < 1$ (Mercator series).
1.2 Change of base: $\log_2(1-\varepsilon) = \ln(1-\varepsilon)/\ln 2 = (-\varepsilon + O(\varepsilon^2))/\ln 2$.
1.3 $\log_2(p_0) = \log_2((1-\varepsilon)^2) = 2\log_2(1-\varepsilon) = -2\varepsilon/\ln 2 + O(\varepsilon^2)$.
1.4 $-p_0 \log_2 p_0 = -p_0 \cdot (-2\varepsilon/\ln 2) + O(\varepsilon^2) = 2p_0\varepsilon/\ln 2 + O(\varepsilon^2)$.
1.5 Since $p_0 = 1 - 2\varepsilon + \varepsilon^2 = 1 + O(\varepsilon)$: $p_0\varepsilon = \varepsilon + O(\varepsilon^2)$.
1.6 Therefore: $-p_0 \log_2 p_0 = 2\varepsilon/\ln 2 + O(\varepsilon^2)$.

Step 2 (Influence Term) :L5-step2

$$(2n-2)(1-p_0) = 4(n-1)\varepsilon + O(n\varepsilon^2). \tag{37}$$

2.1 From Setup: $1 - p_0 = 2\varepsilon - \varepsilon^2$.
2.2 $(2n-2)(2\varepsilon - \varepsilon^2) = (2n-2) \cdot 2\varepsilon - (2n-2)\varepsilon^2$.
2.3 $(2n-2) \cdot 2\varepsilon = 4(n-1)\varepsilon$.
2.4 Error: $(2n-2)\varepsilon^2 \leq 2n \cdot 4^{1-n} = O(n \cdot 4^{-n})$.

Step 3 (Substitution into $g(n)$) :L5-step3 Define $g(n) = S/I - \log_2 3$:

$$g(n) = \frac{2^{n-1}}{n} \cdot \varepsilon \cdot \left[\frac{2}{\ln 2} + 4(n-1)\right] + O(\text{error}). \tag{38}$$

3.1 From Corollary 6.3: $g(n) = \frac{2^{n-1}}{n}[-p_0 \log_2 p_0 + (2n-2)(1-p_0)]$.
3.2 Substitute Step 1: $-p_0 \log_2 p_0 = 2\varepsilon/\ln 2 + O(\varepsilon^2)$.
3.3 Substitute Step 2: $(2n-2)(1-p_0) = 4(n-1)\varepsilon + O(n\varepsilon^2)$.
3.4 Factor out ε: $g(n) = \frac{2^{n-1}}{n} \cdot \varepsilon \cdot [2/\ln 2 + 4(n-1)] + O(\text{error})$.

Step 4 (Key Cancellation and Limit) :L5-step4

$$g(n) = \frac{2}{n \ln 2} + 4 - \frac{4}{n} + O(\varepsilon) \to 4 \quad \text{as } n \to \infty. \tag{39}$$

4.1 **Key identity:** $2^{n-1} \cdot \varepsilon = 2^{n-1} \cdot 2^{1-n} = 2^0 = 1$.
4.2 Therefore: $g(n) = \frac{1}{n}[2/\ln 2 + 4(n-1)] + O(\varepsilon) = \frac{2}{n \ln 2} + 4 - \frac{4}{n} + O(\varepsilon)$.
4.3 As $n \to \infty$: $2/(n \ln 2) \to 0$.
4.4 As $n \to \infty$: $4/n \to 0$.
4.5 As $n \to \infty$: $\varepsilon = 2^{1-n} \to 0$.
4.6 By limit arithmetic: $\lim_{n\to\infty} g(n) = 0 + 4 - 0 + 0 = 4$.

QED:L5-qed:

$$\frac{S_{\max}}{I} = \log_2 3 + g(n) \to \log_2 3 + 4 \approx 1.585 + 4 = 5.585. \tag{40}$$

□

8 Finite-n Values and Numerical Verification

Theorem 8.1 (Finite n Values)**.** *The ratio $S_{\max}/I$ for small n:*

n	*Formula*	*Numerical Value*
1	$\log_2 3$	*1.585*
2	$2+\log_2 3$	*3.585*
3	*explicit*	*4.541*
4	*explicit*	*4.987*
5	*explicit*	*5.209*
10	*explicit*	*5.469*
20	*explicit*	*5.529*
∞	$\log_2 3+4$	*5.585*

Proof. Direct substitution into the formula from Corollary 6.3. □

9 Implications for the Entropy-Influence Conjecture

Theorem 9.1 (Supremum)**.**

$$\sup_{n,\,\text{product states}} \frac{S}{I} = \log_2 3 + 4 \approx 5.585. \tag{41}$$

This supremum is achieved in the limit $n \to \infty$ with all qubits in the magic state.

Theorem 9.2 (Conjecture Lower Bound)**.** *For the entropy-influence conjecture $S(U) \leq C \cdot I(U)$ to hold for all rank-1 product state QBFs:*

$$\boxed{C \geq \log_2 3 + 4 \approx 5.585} \tag{42}$$

Proof. For any $C < \log_2 3 + 4$, there exists n sufficiently large such that $S_{\max}/I > C$ (since the limit is $\log_2 3 + 4$). Hence any universal bound requires $C \geq \log_2 3 + 4$. □

10 Summary of Results

For rank-1 QBFs from product states, we have proven:

1. **Fourier Coefficients (L1):**

$$\hat{U}(\alpha) = \delta_{\alpha,0} - 2^{1-n} \prod_{k=1}^{n} r_k^{(\alpha_k)}. \tag{43}$$

2. **Influence Independence (L2):**

$$I(U) = n \cdot 2^{1-n} \quad \text{(independent of Bloch vectors)}. \tag{44}$$

3. **Entropy Formula (L3):**

$$S(U) = -p_0 \log_2 p_0 + (2n-2)(1-p_0) + 2^{1-n} \sum_{k=1}^{n} f_k. \tag{45}$$

4. **Maximum at Magic State (L4):**

$$\max_{(\vec{r}_k)} \frac{S}{I} \text{ achieved when } (x_k^2, y_k^2, z_k^2) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ for all } k. \tag{46}$$

5. **Asymptotic Limit (L5):**

$$\lim_{n\to\infty} \frac{S_{\max}}{I} = \log_2 3 + 4 \approx 5.585. \tag{47}$$

6. **Conjecture Bound:**

$$C \geq \log_2 3 + 4 \approx 5.585. \tag{48}$$

References

References

[1] C. E. Shannon, *A Mathematical Theory of Communication*, Bell System Technical Journal, vol. 27, pp. 379–423, 623–656, 1948.

A Lean4 Verification Details

A.1 Module Structure

```
AlethfeldLean.QBF.Rank1/
 L1Fourier.lean          -- Fourier coefficient formula
 L2Influence.lean        -- Influence independence
 L3Entropy.lean          -- General entropy formula
 L4Maximum.lean          -- Maximum at magic state
 L5Asymptotic/
    Step1_Setup.lean
    Step2_EpsilonSetup.lean
    Step3_TaylorExpansion.lean
    Step4_InfluenceTerm.lean
    Step5_GnSubstitution.lean
    Step6_Cancellation.lean
    Step7_LimitComputation.lean
    Step8_MainTheorem.lean
 ShannonMax.lean         -- Shannon maximum entropy theorem
 QBFRank1MasterTheorem.lean  -- Master theorem combining L1-L5
```

A.2 Main Theorem Structure

```
structure QBFRank1MasterResult where
  influence_constant :  {n : } (bloch : Fin n → BlochVector),
    totalInfluence bloch = n * (2 : )^(1 - (n : ))
  influence_universal :  {n : } (bloch bloch : Fin n → BlochVector),
    totalInfluence bloch = totalInfluence bloch
  entropy_formula :  {n : } (bloch : Fin n → BlochVector) (hq_all) (hp),
    totalEntropy bloch = entropyTerm (p_zero n) + ...
  blochEntropy_bound :  (v : BlochVector), blochEntropy v  log2 3
  magic_optimal :  (v : BlochVector) (hq),
```

```
    blochEntropy v = log2 3  isMagicState v
  asymptotic_ratio : Tendsto entropy_influence_ratio atTop (nhds (log2 3 + 4))

def qbfRank1Master : QBFRank1MasterResult := { ... }
```

A.3 Alethfeld Graph Metadata

```
Graph ID:          qbf-rank1-entropy-influence
Version:           2
Proof Mode:        formal-physics
Status:            VERIFIED

Nodes:             42 (42 verified, 0 proposed, 0 admitted)
Lemmas:            5 (L1-L5)
External Refs:     1 (Shannon entropy theorem)
Taint:             ALL CLEAN
Obligations:       NONE

Lean4 Verification:
  - Mathlib v4.26.0
  - All modules: 0 sorries
  - Build status: SUCCESS
  - Last verified: 2025-12-29

Verification Summary:
  - Total nodes verified: 42
  - Initially accepted:   39
  - Challenged:           3
  - Revisions applied:    3
  - Final status:         ALL VERIFIED
```