Dobiński's Formula: A Structured Proof

Theorem 1 (Dobiński's Formula)**.** *For all* $n \geq 0$*,*

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}$$

Notation and Definitions.

B_n: The n-th Bell number.

$S(n, j)$: The Stirling number of the second kind.

$k^{(j)}$: The falling factorial $k(k-1)\cdots(k-j+1)$.

e: Euler's number.

Definitions Used.

D1. $B_n = \sum_{j=0}^{n} S(n, j)$

D3. $k^{(j)} = \dfrac{k!}{(k-j)!}$ for $k \geq j$, and $k^{(j)} = 0$ for $k < j$

D4. $e = \sum_{k=0}^{\infty} \dfrac{1}{k!}$

D5. $k^n = \sum_{j=0}^{n} S(n, j)\, k^{(j)}$

Proof.⟨1⟩1. $k^n = \sum_{j=0}^{n} S(n, j)\, k^{(j)}$ *(by D5, D3)*

⟨2⟩1. $k^{(j)} = k(k-1)\cdots(k-j+1)$ *(by D3)*

⟨2⟩2. $k^{(0)} = 1$ *(by D3)*

⟨2⟩3. $k^{(j)} = 0$ when $k < j$ *(by D3)*

⟨2⟩4. $k^{(j)} = \dfrac{k!}{(k-j)!}$ when $k \geq j$ *(by algebra)*

⟨1⟩2. $\sum_{k=0}^{\infty} \dfrac{k^n}{k!} = \sum_{k=0}^{\infty} \dfrac{1}{k!} \sum_{j=0}^{n} S(n, j)\, k^{(j)}$ *(by substitution from ⟨1⟩1)*

⟨1⟩3. $= \sum_{j=0}^{n} S(n, j) \sum_{k=0}^{\infty} \dfrac{k^{(j)}}{k!}$ *(by Fubini–Tonelli, absolute convergence)*

⟨2⟩8. The inner sum has finitely many nonzero terms ($n+1$ terms). *(finite sum)*

⟨2⟩9. Absolute convergence holds via the bound $k^{(j)} \leq k^j$. *(comparison test)*

$\langle 2\rangle 12.$ Exchange of summation order is justified. *(Fubini–Tonelli)*

$\langle 1\rangle 4.$ **Key Lemma:** $\sum_{k=0}^{\infty} \dfrac{k^{(j)}}{k!} = e$ for all $j \geq 0$. *(Key Lemma)*

$\langle 2\rangle 15.$ Terms with $k < j$ vanish since $k^{(j)} = 0$. *(by D3)*

$\langle 2\rangle 16.$ For $k \geq j$: $\dfrac{k^{(j)}}{k!} = \dfrac{1}{(k-j)!}$. *(by algebra)*

$\langle 2\rangle 18.$ Reindex with $m = k - j$: $\sum_{k=j}^{\infty} \dfrac{1}{(k-j)!} = \sum_{m=0}^{\infty} \dfrac{1}{m!}$. *(substitution)*

$\langle 2\rangle 21.$ $\sum_{m=0}^{\infty} \dfrac{1}{m!} = e$. *(by D4)*

$\langle 1\rangle 5.$ $\sum_{j=0}^{n} S(n,j) \sum_{k=0}^{\infty} \dfrac{k^{(j)}}{k!} = e \sum_{j=0}^{n} S(n,j)$ *(substitute $\langle 1\rangle 4$, factor out e)*

$\langle 1\rangle 6.$ $\sum_{j=0}^{n} S(n,j) = B_n$ *(by D1)*

$\langle 1\rangle 7.$ $\sum_{k=0}^{\infty} \dfrac{k^n}{k!} = e \cdot B_n$ *(chain $\langle 1\rangle 2 \to \langle 1\rangle 3 \to \langle 1\rangle 5 \to \langle 1\rangle 6$)*

$\langle 1\rangle 8.$ $B_n = \dfrac{1}{e} \sum_{k=0}^{\infty} \dfrac{k^n}{k!}$ *(divide by $e \neq 0$)* $\square$