HMMT February 2025 Problem 3
Minimum Value of xyz

Alethfeld Proof Orchestrator

Graph Version 81

Theorem 1 (HMMT Feb 2025 #3)**.** *Given that x, y, and z are positive real numbers such that*

$$x^{\log_2(yz)} = 2^8 \cdot 3^4, \quad y^{\log_2(zx)} = 2^9 \cdot 3^6, \quad z^{\log_2(xy)} = 2^5 \cdot 3^{10},$$

the smallest possible value of xyz is $\boxed{576}$.

Proof

Setup

A1 $x, y, z \in \mathbb{R}^+$ (positive real numbers). *[assumption]*

A2 $x^{\log_2(yz)} = 2^8 \cdot 3^4$. *[assumption]*

A3 $y^{\log_2(zx)} = 2^9 \cdot 3^6$. *[assumption]*

A4 $z^{\log_2(xy)} = 2^5 \cdot 3^{10}$. *[assumption]*

D1 Define $a = \log_2 x$, $b = \log_2 y$, $c = \log_2 z$, and $\alpha = \log_2 3$. *[definition]*

Logarithmic Transformation

Claim 2 (Step 1)**.** *Taking $\log_2$ of both sides of the three equations:*

$$a(b + c) = 8 + 4\alpha, \quad b(c + a) = 9 + 6\alpha, \quad c(a + b) = 5 + 10\alpha.$$

Proof. Apply $\log_2$ to equation A2: $\log_2(yz) \cdot \log_2(x) = \log_2(2^8 \cdot 3^4)$, giving $(b+c) \cdot a = 8 + 4 \log_2 3 = 8 + 4\alpha$. Similarly for the other equations. *[algebraic-rewrite from A2, A3, A4, D1]* □

Claim 3 (Step 2)**.** *Let $s = a + b + c$. Then $xyz = 2^s$, and we seek to minimize s.*

Proof. Since $a = \log_2 x$, we have $x = 2^a$, similarly $y = 2^b$, $z = 2^c$. Thus $xyz = 2^{a+b+c} = 2^s$. *[definition-expansion from D1]* □

Quadratic System

Claim 4 (Step 4). *The system can be rewritten as quadratic equations:*

$$a^2 - sa + (8 + 4\alpha) = 0, \quad b^2 - sb + (9 + 6\alpha) = 0, \quad c^2 - sc + (5 + 10\alpha) = 0.$$

Proof. Since $b + c = s - a$, the equation $a(b + c) = 8 + 4\alpha$ becomes $a(s - a) = 8 + 4\alpha$, which rearranges to $a^2 - sa + (8 + 4\alpha) = 0$. Similarly for b and c. *[algebraic-rewrite from Step 1, Step 2]* □

Claim 5 (Step 5). *By the quadratic formula:*

$$a = \frac{s \pm \sqrt{s^2 - 4(8 + 4\alpha)}}{2}, \quad b = \frac{s \pm \sqrt{s^2 - 4(9 + 6\alpha)}}{2}, \quad c = \frac{s \pm \sqrt{s^2 - 4(5 + 10\alpha)}}{2}.$$

Claim 6 (Step 6). *For real solutions, we require:*

$$s^2 \geq 32 + 16\alpha, \quad s^2 \geq 36 + 24\alpha, \quad s^2 \geq 20 + 40\alpha.$$

Constraint Analysis

Claim 7 (Step 7). *Define $\Delta_1 = s^2 - 32 - 16\alpha$, $\Delta_2 = s^2 - 36 - 24\alpha$, $\Delta_3 = s^2 - 20 - 40\alpha$. The constraint $a + b + c = s$ requires:*

$$\frac{3s + \epsilon_1\sqrt{\Delta_1} + \epsilon_2\sqrt{\Delta_2} + \epsilon_3\sqrt{\Delta_3}}{2} = s$$

for some $\epsilon_i \in \{\pm 1\}$.

Claim 8 (Step 8). *This simplifies to:*

$$\epsilon_1\sqrt{\Delta_1} + \epsilon_2\sqrt{\Delta_2} + \epsilon_3\sqrt{\Delta_3} = -s.$$

Finding the Minimum

Claim 9 (Step 10). *Testing $s = \log_2 576 = \log_2(2^6 \cdot 3^2) = 6 + 2\alpha$: We have $s^2 = 36 + 24\alpha + 4\alpha^2$.*

Claim 10 (Steps 11-13). *At $s = 6 + 2\alpha$, the discriminants are perfect squares:*

$$\begin{aligned}
&\Delta_1 = 4 + 8\alpha + 4\alpha^2 = 4(1 + \alpha)^2, \quad \sqrt{\Delta_1} = 2 + 2\alpha \\
&\Delta_2 = 4\alpha^2, \quad \sqrt{\Delta_2} = 2\alpha \\
&\Delta_3 = 16 - 16\alpha + 4\alpha^2 = 4(2 - \alpha)^2, \quad \sqrt{\Delta_3} = 4 - 2\alpha
\end{aligned}$$

(Note: $\sqrt{\Delta_3} = 2(2 - \alpha)$ since $\alpha = \log_2 3 \approx 1.585 < 2$.)

Claim 11 (Step 15). *With $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, -1)$:*

$$-(2 + 2\alpha) - 2\alpha - (4 - 2\alpha) = -6 - 2\alpha = -(6 + 2\alpha) = -s.$$

This satisfies the constraint.

Claim 12 (Step 16). *The solution is:*

$$a = \frac{(6 + 2\alpha) - (2 + 2\alpha)}{2} = 2, \quad b = \frac{(6 + 2\alpha) - 2\alpha}{2} = 3, \quad c = \frac{(6 + 2\alpha) - (4 - 2\alpha)}{2} = 1 + 2\alpha.$$

Claim 13 (Step 17). *Therefore:*

$$x = 2^2 = 4, \quad y = 2^3 = 8, \quad z = 2^{1+2\alpha} = 2 \cdot 3^2 = 18.$$

Verification

Claim 14 (Steps 18-20). *Checking the original equations:*

- $yz = 144$, $\log_2(144) = 4 + 2\alpha$, *so* $x^{\log_2(yz)} = 4^{4+2\alpha} = 2^{8+4\alpha} = 2^8 \cdot 3^4$. ✓
- $zx = 72$, $\log_2(72) = 3 + 2\alpha$, *so* $y^{\log_2(zx)} = 8^{3+2\alpha} = 2^{9+6\alpha} = 2^9 \cdot 3^6$. ✓
- $xy = 32$, $\log_2(32) = 5$, *so* $z^{\log_2(xy)} = 18^5 = 2^5 \cdot 3^{10}$. ✓

Claim 15 (Step 21). *Therefore* $(x, y, z) = (4, 8, 18)$ *is a valid solution with* $xyz = 576$.

Minimality

Claim 16 (Step 31). *Define* $f(s) = \sqrt{\Delta_1} + \sqrt{\Delta_2} + \sqrt{\Delta_3} - s$ *for* $s \geq \sqrt{20 + 40\alpha}$. *We have* $f(s_0) = 0$ *where* $s_0 = 6 + 2\alpha$.

Claim 17 (Step 32). *Computing the derivative:*

$$f'(s) = \frac{s}{\sqrt{\Delta_1}} + \frac{s}{\sqrt{\Delta_2}} + \frac{s}{\sqrt{\Delta_3}} - 1.$$

Since each $\sqrt{\Delta_i} < s$ *(for positive* a, b, c*), each fraction exceeds* 1, *so* $f'(s) > 2 > 0$.

Claim 18 (Step 33). *Since* $f'(s) > 0$, *the function* f *is strictly increasing. With* $f(s_0) = 0$*:*

- *For* $s < s_0$*:* $f(s) < 0$, *so* $\sqrt{\Delta_1} + \sqrt{\Delta_2} + \sqrt{\Delta_3} < s$
- *For* $s > s_0$*:* $f(s) > 0$, *so* $\sqrt{\Delta_1} + \sqrt{\Delta_2} + \sqrt{\Delta_3} > s$

Thus s_0 *is the unique solution where all-minus signs work.*

Claim 19 (Step 34). *For other sign combinations: if any* $\epsilon_i = +1$, *the LHS increases, making it impossible to achieve* $-s < 0$ *for* $s \leq s_0$.

Conclusion

Claim 20 (QED). *The minimum value of* xyz *is:*

$$2^{a+b+c} = 2^{6+2\alpha} = 2^6 \cdot 2^{2\log_2 3} = 64 \cdot 9 = \boxed{576}.$$

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