Existence of an n-Copy Quantum Purification Channel: A Structured Proof

Theorem 1 (Existence of n-Copy Purification Channel)**.** *Let $\mathcal{H}_A$ be a finite-dimensional Hilbert space with $\dim(\mathcal{H}_A) = d < \infty$, and let $\mathcal{H}_B \cong \mathcal{H}_A$. For any integer $n \geq 1$, there exists a quantum channel $\Lambda^{(n)}_{\text{purify}} : \mathcal{L}(\mathcal{H}_A^{\otimes n}) \to \mathcal{L}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$ such that for any density operator $\rho_A \in \mathcal{D}(\mathcal{H}_A)$:*

$$\Lambda^{(n)}_{\text{purify}}(\rho_A^{\otimes n}) = \left[\mathbb{E}_U \left[(\mathrm{id}_A \otimes U_B) \left|\psi_\rho\right\rangle \left\langle\psi_\rho\right|_{AB} (\mathrm{id}_A \otimes U_B^\dagger)\right]\right]^{\otimes n}$$

where $|\psi_\rho\rangle_{AB}$ is the canonical purification $|\psi_\rho\rangle = (\sqrt{\rho} \otimes I)|\Omega\rangle$ with (unnormalized) maximally entangled state $|\Omega\rangle = \sum_{i=1}^d |i\rangle_A |i\rangle_B$, and $\mathbb{E}_U$ denotes the expectation over the Haar measure on the unitary group $U(d)$.

Assumptions.

A1: $\dim(\mathcal{H}_A) = d < \infty$.

A2: $\mathcal{H}_B \cong \mathcal{H}_A$.

A3: $n \geq 1$.

External Results.

E1. *Schur–Weyl Duality* [1]: For a representation of $U(d)$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ via $U \mapsto I_A \otimes U_B$, the commutant is $\mathcal{L}(\mathcal{H}_A) \otimes \mathbb{C} \cdot I_B$.

Proof.⟨1⟩1. **Purification Existence.** Every density operator $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ admits a purification $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$. *(Standard result; A1, A2)*

⟨1⟩2. **Canonical Purification.** Define $|\psi_\rho\rangle = (\sqrt{\rho} \otimes I)|\Omega\rangle$ where $|\Omega\rangle = \sum_{i=1}^d |i\rangle_A |i\rangle_B$ (unnormalized). *(Construction; A1, A2)*

⟨2⟩1. Let $\{|i\rangle\}_{i=1}^d$ be an orthonormal basis for $\mathcal{H}_A \cong \mathcal{H}_B$.

⟨2⟩2. Define the (unnormalized) maximally entangled state $|\Omega\rangle = \sum_{i=1}^d |i\rangle_A |i\rangle_B$.

⟨2⟩3. Set $|\psi_\rho\rangle = (\sqrt{\rho} \otimes I)|\Omega\rangle$.

⟨2⟩4. Verify $\mathrm{Tr}_B(|\psi_\rho\rangle\langle\psi_\rho|) = \rho_A$:

⟨3⟩1. $\mathrm{Tr}_B((\sqrt{\rho} \otimes I)|\Omega\rangle\langle\Omega|(\sqrt{\rho} \otimes I))$

⟨3⟩2. $= \sqrt{\rho}\,(\sum_i |i\rangle\langle i|)\,\sqrt{\rho}$ *(partial trace over B)*

⟨3⟩3. $= \sqrt{\rho} \cdot I \cdot \sqrt{\rho} = \rho$. *($\sum_i |i\rangle\langle i| = I$)*

⟨1⟩3. **Purification Uniqueness.** Any two purifications of ρ_A differ by a local unitary $I_A \otimes W$ on $\mathcal{H}_B$. *(Uhlmann's theorem; ⟨1⟩1, ⟨1⟩2)*

⟨2⟩1. Let $|\psi\rangle_{AB}$ and $|\psi'\rangle_{AB}$ be purifications of ρ_A.

⟨2⟩2. Write spectral decomposition $\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|$.

⟨2⟩3. Express $|\psi\rangle = \sum_i \sqrt{\lambda_i}|e_i\rangle_A |f_i\rangle_B$ and $|\psi'\rangle = \sum_i \sqrt{\lambda_i}|e_i\rangle_A |f'_i\rangle_B$.

⟨2⟩4. There exists unitary W on $\mathcal{H}_B$ with $W|f_i\rangle = |f'_i\rangle$ for all i.

⟨2⟩5. Hence $|\psi'\rangle = (I_A \otimes W)|\psi\rangle$.

⟨1⟩4. **Haar Twirl Identity.** For any pure state $|\psi\rangle_{AB}$:

$$\mathbb{E}_U\left[(I_A \otimes U)|\psi\rangle\langle\psi|(I_A \otimes U^\dagger)\right] = \rho_A \otimes \frac{I_B}{d}$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$. *(Schur–Weyl; E1)*

⟨2⟩1. Define $\mathcal{T}(\sigma) = \mathbb{E}_U[(I \otimes U)\sigma(I \otimes U^\dagger)]$ (the twirl map).

⟨2⟩2. By Schur–Weyl duality (E1), the commutant of $\{I \otimes U : U \in U(d)\}$ is $\mathcal{L}(\mathcal{H}_A) \otimes \mathbb{C} \cdot I_B$.

⟨2⟩3. Therefore $\mathcal{T}(|\psi\rangle\langle\psi|) = X_A \otimes I_B$ for some $X_A \in \mathcal{L}(\mathcal{H}_A)$.

⟨2⟩4. The twirl map commutes with partial trace: $\mathrm{Tr}_B \circ \mathcal{T} = \mathrm{Tr}_B$.

⟨2⟩5. Compute: $\mathrm{Tr}_B(X_A \otimes I_B) = d \cdot X_A$.

⟨2⟩6. Also: $\mathrm{Tr}_B(|\psi\rangle\langle\psi|) = \rho_A$.

⟨2⟩7. Hence $d \cdot X_A = \rho_A$, giving $X_A = \rho_A/d$.

⟨2⟩8. Therefore $\mathcal{T}(|\psi\rangle\langle\psi|) = \rho_A \otimes I_B/d$.

⟨1⟩5. **Single-Copy Purification Map.** Define $P : \mathcal{D}(\mathcal{H}_A) \to \{\text{pure states on } \mathcal{H}_A \otimes \mathcal{H}_B\}$ by $P(\rho) = |\psi_\rho\rangle\langle\psi_\rho|$. *(⟨1⟩2)*

⟨1⟩6. **Single-Copy Channel Definition.** Define $\Lambda : \mathcal{L}(\mathcal{H}_A) \to \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ by:

$$\Lambda(\rho) = \rho_A \otimes \frac{I_B}{d}$$

(⟨1⟩4)

⟨1⟩7. **Linear Extension.** Λ extends linearly to all of $\mathcal{L}(\mathcal{H}_A)$. *(linearity of tensor product)*

⟨1⟩8. **Λ is Completely Positive (CP).** *(Kraus representation)*

⟨2⟩1. Write $\Lambda(\rho) = \rho \otimes I_B/d$.

⟨2⟩2. Define Kraus operators $K_i = \frac{1}{\sqrt{d}}|i\rangle_B$ for $i = 1, \ldots, d$.

⟨2⟩3. Then $\Lambda(\rho) = \sum_i (I_A \otimes K_i)\rho(I_A \otimes K_i^\dagger)$.

⟨2⟩4. Kraus representation implies complete positivity.

⟨1⟩9. **Λ is Trace-Preserving (TP).** *(direct computation)*

⟨2⟩1. $\mathrm{Tr}(\Lambda(\rho)) = \mathrm{Tr}(\rho_A \otimes I_B/d) = \mathrm{Tr}(\rho_A) \cdot \mathrm{Tr}(I_B/d)$.

⟨2⟩2. $= \mathrm{Tr}(\rho_A) \cdot 1 = \mathrm{Tr}(\rho_A)$.

⟨2⟩3. For $\rho \in \mathcal{D}(\mathcal{H}_A)$, $\mathrm{Tr}(\rho_A) = 1$, so $\mathrm{Tr}(\Lambda(\rho)) = 1$.

⟨1⟩10. **Λ is CPTP.** Λ is a valid quantum channel. *(⟨1⟩8, ⟨1⟩9)*

⟨1⟩11. **Output Equals Twirled Purification.** For any purification $|\psi\rangle$ of ρ:

$$\Lambda(\rho) = \mathbb{E}_U \left[(I_A \otimes U) |\psi\rangle \langle\psi| (I_A \otimes U^\dagger) \right]$$

(⟨1⟩4, ⟨1⟩6)

⟨2⟩1. By ⟨1⟩4, the Haar twirl of any purification gives $\rho_A \otimes I_B/d$.

⟨2⟩2. By ⟨1⟩6, $\Lambda(\rho) = \rho_A \otimes I_B/d$.

⟨2⟩3. Hence $\Lambda(\rho) = \mathbb{E}_U[(I \otimes U) |\psi\rangle \langle\psi| (I \otimes U^\dagger)]$.

⟨2⟩4. **Haar Invariance:** For $|\psi'\rangle = (I \otimes W) |\psi\rangle$:

⟨3⟩1. $\mathbb{E}_U[(I \otimes U) |\psi'\rangle \langle\psi'| (I \otimes U^\dagger)]$

⟨3⟩2. $= \mathbb{E}_U[(I \otimes UW) |\psi\rangle \langle\psi| (I \otimes W^\dagger U^\dagger)]$

⟨3⟩3. $= \mathbb{E}_U[(I \otimes V) |\psi\rangle \langle\psi| (I \otimes V^\dagger)]$ *(right-invariance: $V = UW$)*

⟨3⟩4. Result is independent of purification choice.

⟨1⟩12. **Output Independent of Purification.** The Haar twirl gives the same result for any purification of ρ. *(⟨1⟩3, ⟨1⟩11)*

⟨1⟩13. **n-Copy Channel Definition.** Define $\Lambda^{(n)} = \Lambda^{\otimes n}$:

$$\Lambda^{(n)} : \mathcal{L}(\mathcal{H}_A^{\otimes n}) \to \mathcal{L}((\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n})$$

(A3)

⟨1⟩14. **$\Lambda^{(n)}$ is CPTP.** *(tensor product of CPTP maps)*

⟨2⟩1. Λ is CPTP by ⟨1⟩10.

⟨2⟩2. The tensor product of CPTP maps is CPTP.

⟨2⟩3. Hence $\Lambda^{(n)} = \Lambda^{\otimes n}$ is CPTP.

⟨1⟩15. **Final Result.**

$$\Lambda^{(n)}(\rho^{\otimes n}) = \left[\mathbb{E}_U \left[(I_A \otimes U) |\psi_\rho\rangle \langle\psi_\rho| (I_A \otimes U^\dagger) \right] \right]^{\otimes n}$$

(⟨1⟩11, ⟨1⟩13, ⟨1⟩14)

⟨2⟩1. By definition, $\Lambda^{(n)}(\rho^{\otimes n}) = \Lambda(\rho)^{\otimes n}$.

⟨2⟩2. By ⟨1⟩11, $\Lambda(\rho) = \mathbb{E}_U[(I \otimes U) |\psi_\rho\rangle \langle\psi_\rho| (I \otimes U^\dagger)]$.

⟨2⟩3. Substituting: $\Lambda^{(n)}(\rho^{\otimes n}) = \left[\mathbb{E}_U[(I \otimes U) |\psi_\rho\rangle \langle\psi_\rho| (I \otimes U^\dagger)] \right]^{\otimes n}$.

⟨2⟩4. This completes the proof. □

References

[1] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.