Lemma L2: Influence Independence

Alethfeld Verified Proof

Alethfeld Proof Orchestrator v4

Verification Status: **VERIFIED** | Taint: **CLEAN**

Rigor: **STRICTEST**

Dependencies

Assumption 1 (A1: Product State QBF)**.** Let $U = I - 2\left|\psi\right\rangle\left\langle\psi\right|$ be a rank-1 quantum Boolean function where $\left|\psi\right\rangle = \bigotimes_{k=1}^{n}\left|\phi_k\right\rangle$ is a product state.

Definition 1 (D1: Bloch Vector)**.** For each qubit $k \in \{1, \ldots, n\}$, the Bloch vector $\vec{r}_k = (x_k, y_k, z_k)$ satisfies the unit sphere constraint:

$$x_k^2 + y_k^2 + z_k^2 = 1.$$

Definition 2 (D2: Squared Bloch Components)**.** Define the squared Bloch components:

$$q_k^{(0)} = 1, \quad q_k^{(1)} = x_k^2, \quad q_k^{(2)} = y_k^2, \quad q_k^{(3)} = z_k^2.$$

Lemma 1 (L1: Fourier Coefficient Formula — External Reference)**.** *For any multi-index $\alpha \in \{0,1,2,3\}^n$:*

$$\hat{U}(\alpha) = \delta_{\alpha,0} - 2^{1-n}\prod_{k=1}^{n} r_k^{(\alpha_k)}.$$

Lemma 2 (L1-Corollary: Probability Distribution)**.** *For $\alpha \neq 0$, the probability $p_\alpha = |\hat{U}(\alpha)|^2$ satisfies:*

$$p_\alpha = 2^{2-2n}\prod_{k=1}^{n} q_k^{(\alpha_k)}.$$

Statement

Lemma 3 (L2: Influence Independence)**.** *For any rank-1 product state QBF on n qubits:*

$$I(U) = n \cdot 2^{1-n}.$$

This value is independent of the choice of single-qubit states (Bloch vectors).

Proof

Step 1: Definition of Influence

Claim 1 (1-step1)**.** *The influence of qubit j is:*

$$I_j = \sum_{\alpha:\alpha_j \neq 0} p_\alpha.$$

Proof. We establish this in substeps.

(1a) By definition, the influence I_j measures how much the output of the function depends on qubit j.

(1b) For a classical Boolean function $f : \{0,1\}^n \to \{0,1\}$, the influence is defined as:

$$I_j(f) = \Pr_x[f(x) \neq f(x^{\oplus j})]$$

where $x^{\oplus j}$ denotes x with bit j flipped.

(1c) For quantum Boolean functions, this generalizes via the Pauli-Fourier expansion.

(1c.1) The Pauli-Fourier expansion is $U = \sum_\alpha \hat{U}(\alpha)\sigma^\alpha$.

(1c.2) The index $\alpha_j \neq 0$ captures when qubit j is acted upon non-trivially by $\sigma^{\alpha_j} \in \{\sigma_x, \sigma_y, \sigma_z\}$.

Therefore, $I_j = \sum_{\alpha:\alpha_j \neq 0} |\hat{U}(\alpha)|^2 = \sum_{\alpha:\alpha_j \neq 0} p_\alpha$. □

Step 2: Factorization of Partial Sum

Claim 2 (1-step2)**.** *For each $\ell \in \{1,2,3\}$:*

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{2-2n} \cdot q_j^{(\ell)} \cdot \prod_{k \neq j} \sum_{m=0}^{3} q_k^{(m)}.$$

Proof. **(2a)** From Lemma **??**, for $\alpha \neq 0$:

$$p_\alpha = 2^{2-2n} \prod_{k=1}^{n} q_k^{(\alpha_k)}.$$

(2b) Fixing $\alpha_j = \ell$, the product factors:

(2b.1) Products over disjoint index sets factor: for $A \cap B = \emptyset$,

$$\prod_{k \in A \cup B} f_k = \prod_{k \in A} f_k \cdot \prod_{k \in B} f_k.$$

(2b.2) Setting $A = \{j\}$ and $B = \{1, \ldots, n\} \setminus \{j\}$:

$$\prod_{k=1}^{n} q_k^{(\alpha_k)} = q_j^{(\ell)} \cdot \prod_{k \neq j} q_k^{(\alpha_k)}.$$

(2c) Summing over all multi-indices with $\alpha_j = \ell$:

(2c.1) Each index α_k for $k \neq j$ ranges independently over $\{0, 1, 2, 3\}$.

(2c.2) Products distribute over sums (Fubini for finite sums):

$$\sum_{\alpha_1,\ldots,\alpha_{n-1}} \prod_k f_k(\alpha_k) = \prod_k \sum_{\alpha_k} f_k(\alpha_k).$$

Combining:

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{2-2n} \cdot q_j^{(\ell)} \cdot \prod_{k\neq j} \sum_{m=0}^{3} q_k^{(m)}. \qquad \square$$

Step 3: Unit Sphere Simplification

Claim 3 (1-step3). *Since $\sum_{m=0}^{3} q_k^{(m)} = 2$, we have:*

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{1-n} q_j^{(\ell)}.$$

Proof. **(3a)** By Definition **??**:

(3a.1) Substituting $q_k^{(0)} = 1$, $q_k^{(1)} = x_k^2$, $q_k^{(2)} = y_k^2$, $q_k^{(3)} = z_k^2$:

$$\sum_{m=0}^{3} q_k^{(m)} = 1 + x_k^2 + y_k^2 + z_k^2.$$

(3b) By Definition **??** (unit sphere constraint):

(3b.1) The Bloch vector satisfies $|\vec{r}_k|^2 = x_k^2 + y_k^2 + z_k^2 = 1$.

Therefore: $\sum_{m=0}^{3} q_k^{(m)} = 1 + 1 = 2$.

(3c) Substituting into Step 2:

(3c.1) The product $\prod_{k\neq j} 2 = 2^{n-1}$ since there are $n - 1$ factors.

(3c.2) Exponent arithmetic: $(2 - 2n) + (n - 1) = 2 - 2n + n - 1 = 1 - n$.

$$\sum_{\alpha:\alpha_j=\ell} p_\alpha = 2^{2-2n} \cdot q_j^{(\ell)} \cdot 2^{n-1} = 2^{1-n} q_j^{(\ell)}. \qquad \square$$

Step 4: Single-Qubit Influence

Claim 4 (1-step4). *For all $j \in \{1, \ldots, n\}$:*

$$I_j = 2^{1-n}$$

independent of the choice of Bloch vector $\vec{r}_j$.

Proof. **(4a)** From Step 1, partitioning by the value of α_j:

(4a.1) The condition $\alpha_j \neq 0$ is equivalent to $\alpha_j \in \{1, 2, 3\}$.

$$I_j = \sum_{\alpha:\alpha_j \neq 0} p_\alpha = \sum_{\ell=1}^{3} \sum_{\alpha:\alpha_j=\ell} p_\alpha.$$

(4b) Applying Step 3:

(4b.1) The constant 2^{1-n} factors out of the sum over ℓ.

$$I_j = \sum_{\ell=1}^{3} 2^{1-n} q_j^{(\ell)} = 2^{1-n} \sum_{\ell=1}^{3} q_j^{(\ell)}.$$

(4c) By Definitions **??** and **??**:

(4c.1) From D2: $q_j^{(1)} + q_j^{(2)} + q_j^{(3)} = x_j^2 + y_j^2 + z_j^2$.

(4c.2) From D1: $x_j^2 + y_j^2 + z_j^2 = 1$ (unit sphere).

$$I_j = 2^{1-n} \cdot 1 = 2^{1-n}.$$

□

Conclusion

Proof of Lemma **??**. **(QED-a)** By definition, total influence is:

(a.1) Total influence sums individual qubit influences over all n qubits.

$$I(U) = \sum_{j=1}^{n} I_j.$$

(QED-b) From Step 4:

(b.1) The value 2^{1-n} depends only on n, not on any Bloch vector.

Each $I_j = 2^{1-n}$, independent of j and the choice of single-qubit states.
(QED-c) Therefore:

(c.1) Summing the constant 2^{1-n} over $j \in \{1, \ldots, n\}$ gives $n \cdot 2^{1-n}$.

(c.2) This is the unique total influence for all rank-1 product state QBFs on n qubits.

$$I(U) = \sum_{j=1}^{n} 2^{1-n} = n \cdot 2^{1-n}. \quad \square$$

□

Remarks

This is a remarkable result: the total influence $I(U) = n \cdot 2^{1-n}$ is *completely independent* of the choice of single-qubit states. This universality arises because:

1. The unit sphere constraint forces $\sum_{\ell=1}^{3} q_k^{(\ell)} = 1$ for every qubit.
2. This exact cancellation occurs for each qubit independently.
3. The only remaining dependence is on n, the number of qubits.

This universality is what makes the entropy-influence ratio analysis tractable in the main theorem.

Alethfeld Verification Report

Graph ID:	L2-influence-verify-2024
Total Nodes:	43 (5 original + 38 expanded)
Max Depth:	3
Admitted Steps:	0
Rigor Level:	STRICTEST
Taint Status:	CLEAN