

Minimum Value of xyz Subject to Exponential-Logarithmic Constraints

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Abstract

We prove that given positive real numbers x , y , and z satisfying the constraints $x^{\log_2(yz)} = 2^8 \cdot 3^4$, $y^{\log_2(zx)} = 2^9 \cdot 3^6$, and $z^{\log_2(xy)} = 2^5 \cdot 3^{10}$, the minimum possible value of xyz is 576. The proof proceeds by transforming the system via logarithms, reducing to a constrained quadratic system, analyzing all sign combinations in the solution space, and establishing uniqueness of the minimum through monotonicity arguments.

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1 Problem Statement and Setup

Theorem 1 (Main Result). *Given that $x, y, z \in \mathbb{R}^+$ such that*

$$x^{\log_2(yz)} = 2^8 \cdot 3^4, \quad (1)$$

$$y^{\log_2(zx)} = 2^9 \cdot 3^6, \quad (2)$$

$$z^{\log_2(xy)} = 2^5 \cdot 3^{10}, \quad (3)$$

the smallest possible value of xyz is 576.

We begin by establishing our notation and fundamental assumptions.

Assumption 1 (Positivity). $x, y, z \in \mathbb{R}^+$ (positive real numbers).

Definition 2 (Logarithmic Variables). Define

$$a = \log_2 x, \quad b = \log_2 y, \quad c = \log_2 z, \quad \alpha = \log_2 3. \quad (4)$$

Remark 3. Since $x, y, z > 0$, the quantities a, b, c are well-defined real numbers. The constant $\alpha = \log_2 3 \approx 1.585$ satisfies $1 < \alpha < 2$ since $2 < 3 < 4$.

2 Logarithmic Transformation

Proposition 4 (Transformed System). *Under Definition 2, the constraints (1)–(3) become:*

$$a(b + c) = 8 + 4\alpha, \quad (5)$$

$$b(c + a) = 9 + 6\alpha, \quad (6)$$

$$c(a + b) = 5 + 10\alpha. \quad (7)$$

Proof. We derive equation (5) in detail; the others follow analogously.

Step 1: Apply \log_2 to both sides of (1):

$$\log_2 \left(x^{\log_2(yz)} \right) = \log_2 (2^8 \cdot 3^4). \quad (8)$$

Step 2: Simplify the left-hand side using the power rule:

$$\log_2 \left(x^{\log_2(yz)} \right) = \log_2(yz) \cdot \log_2(x). \quad (9)$$

Step 3: Apply the product rule to $\log_2(yz)$:

$$\log_2(yz) = \log_2 y + \log_2 z = b + c. \quad (10)$$

Step 4: Substitute Definition 2:

$$\text{LHS} = (b + c) \cdot a = a(b + c). \quad (11)$$

Step 5: Simplify the right-hand side using the product rule:

$$\log_2 (2^8 \cdot 3^4) = \log_2(2^8) + \log_2(3^4). \quad (12)$$

Step 6: Apply the power rule:

$$\log_2(2^8) = 8, \quad \log_2(3^4) = 4 \log_2 3 = 4\alpha. \quad (13)$$

Step 7: Combine:

$$\text{RHS} = 8 + 4\alpha. \quad (14)$$

Therefore $a(b + c) = 8 + 4\alpha$.

For equation (6): Starting from $y^{\log_2(zx)} = 2^9 \cdot 3^6$:

$$\log_2(zx) \cdot \log_2 y = \log_2(2^9 \cdot 3^6) \implies (c + a) \cdot b = 9 + 6\alpha.$$

For equation (7): Starting from $z^{\log_2(xy)} = 2^5 \cdot 3^{10}$:

$$\log_2(xy) \cdot \log_2 z = \log_2(2^5 \cdot 3^{10}) \implies (a + b) \cdot c = 5 + 10\alpha.$$

□

3 Reduction to Quadratic System

Definition 5 (Sum Variable). Let $s = a + b + c$. Then $xyz = 2^a \cdot 2^b \cdot 2^c = 2^s$.

Our goal becomes: minimize s subject to equations (5)–(7).

Lemma 6 (Sum of Products). *Adding equations (5)–(7):*

$$2(ab + bc + ca) = 22 + 20\alpha. \quad (15)$$

Proof.

$$a(b + c) + b(c + a) + c(a + b) = (8 + 4\alpha) + (9 + 6\alpha) + (5 + 10\alpha) = 22 + 20\alpha.$$

The left-hand side equals $ab + ac + bc + ba + ca + cb = 2(ab + bc + ca)$. □

Proposition 7 (Quadratic Reformulation). *Using $b + c = s - a$, etc., the system becomes:*

$$a^2 - sa + (8 + 4\alpha) = 0, \quad (16)$$

$$b^2 - sb + (9 + 6\alpha) = 0, \quad (17)$$

$$c^2 - sc + (5 + 10\alpha) = 0. \quad (18)$$

Proof. From (5): $a(b + c) = 8 + 4\alpha$. Since $b + c = s - a$:

$$a(s - a) = 8 + 4\alpha \implies as - a^2 = 8 + 4\alpha \implies a^2 - sa + (8 + 4\alpha) = 0.$$

The other equations follow identically. □

Corollary 8 (Quadratic Solutions). *By the quadratic formula:*

$$a = \frac{s \pm \sqrt{s^2 - 4(8 + 4\alpha)}}{2} = \frac{s \pm \sqrt{\Delta_1}}{2}, \quad (19)$$

$$b = \frac{s \pm \sqrt{s^2 - 4(9 + 6\alpha)}}{2} = \frac{s \pm \sqrt{\Delta_2}}{2}, \quad (20)$$

$$c = \frac{s \pm \sqrt{s^2 - 4(5 + 10\alpha)}}{2} = \frac{s \pm \sqrt{\Delta_3}}{2}, \quad (21)$$

where we define the discriminants:

$$\Delta_1 = s^2 - 32 - 16\alpha, \quad (22)$$

$$\Delta_2 = s^2 - 36 - 24\alpha, \quad (23)$$

$$\Delta_3 = s^2 - 20 - 40\alpha. \quad (24)$$

4 Discriminant Constraints

Lemma 9 (Non-negativity Requirements). *For real solutions, we require:*

$$s^2 \geq 32 + 16\alpha \quad (\text{from } \Delta_1 \geq 0), \quad (25)$$

$$s^2 \geq 36 + 24\alpha \quad (\text{from } \Delta_2 \geq 0), \quad (26)$$

$$s^2 \geq 20 + 40\alpha \quad (\text{from } \Delta_3 \geq 0). \quad (27)$$

Proposition 10 (Binding Constraint). *Since $\alpha = \log_2 3 \approx 1.585$:*

$$32 + 16\alpha \approx 32 + 25.4 = 57.4,$$

$$36 + 24\alpha \approx 36 + 38.0 = 74.0,$$

$$20 + 40\alpha \approx 20 + 63.4 = 83.4.$$

The binding constraint is $s^2 \geq 20 + 40\alpha$, i.e., $s \geq \sqrt{20 + 40\alpha}$.

5 Testing the Candidate $s = 6 + 2\alpha$

We claim that $s_0 = 6 + 2\alpha = \log_2 576$ achieves the minimum.

Claim 1. $s_0 = 6 + 2\alpha$ satisfies the binding constraint $s^2 \geq 20 + 40\alpha$.

Proof. At $s_0 = 6 + 2\alpha$:

$$s_0^2 = (6 + 2\alpha)^2 = 36 + 24\alpha + 4\alpha^2. \quad (28)$$

We need $36 + 24\alpha + 4\alpha^2 \geq 20 + 40\alpha$, i.e.,

$$4\alpha^2 - 16\alpha + 16 \geq 0 \iff 4(\alpha - 2)^2 \geq 0.$$

This holds for all α , with equality iff $\alpha = 2$. Since $\alpha = \log_2 3 < \log_2 4 = 2$, the inequality is strict. \square

6 Discriminant Perfect Square Factorizations

6.1 Computation of Δ_1 at $s = 6 + 2\alpha$

Lemma 11 (Δ_1 Factorization). *At $s_0 = 6 + 2\alpha$:*

$$\Delta_1 = 4(1 + \alpha)^2, \quad \sqrt{\Delta_1} = 2 + 2\alpha. \quad (29)$$

Proof. Step 1 (Substitution): From (28) and (22):

$$\Delta_1 = s_0^2 - 32 - 16\alpha = (36 + 24\alpha + 4\alpha^2) - 32 - 16\alpha.$$

Step 2 (Grouping):

$$\begin{aligned} \Delta_1 &= (36 - 32) + (24\alpha - 16\alpha) + 4\alpha^2 \\ &= 4 + 8\alpha + 4\alpha^2. \end{aligned}$$

Step 3 (Factorization):

$$4 + 8\alpha + 4\alpha^2 = 4(1 + 2\alpha + \alpha^2) = 4(1 + \alpha)^2.$$

Step 4 (Square Root): Since $\alpha = \log_2 3 > 0$, we have $1 + \alpha > 1 > 0$. Thus:

$$\sqrt{\Delta_1} = \sqrt{4(1 + \alpha)^2} = 2|1 + \alpha| = 2(1 + \alpha) = 2 + 2\alpha.$$

\square

6.2 Computation of Δ_2 at $s = 6 + 2\alpha$

Lemma 12 (Δ_2 Factorization). At $s_0 = 6 + 2\alpha$:

$$\Delta_2 = 4\alpha^2, \quad \sqrt{\Delta_2} = 2\alpha. \quad (30)$$

Proof. **Step 1 (Substitution):**

$$\Delta_2 = s_0^2 - 36 - 24\alpha = (36 + 24\alpha + 4\alpha^2) - 36 - 24\alpha.$$

Step 2 (Collection):

$$\Delta_2 = (36 - 36) + (24\alpha - 24\alpha) + 4\alpha^2 = 0 + 0 + 4\alpha^2 = 4\alpha^2.$$

Step 3 (Perfect Square):

$$4\alpha^2 = (2\alpha)^2.$$

Step 4 (Square Root): Since $\alpha = \log_2 3 > 0$:

$$\sqrt{\Delta_2} = \sqrt{4\alpha^2} = 2|\alpha| = 2\alpha.$$

□

6.3 Computation of Δ_3 at $s = 6 + 2\alpha$

Lemma 13 (Δ_3 Factorization). At $s_0 = 6 + 2\alpha$:

$$\Delta_3 = 4(2 - \alpha)^2, \quad \sqrt{\Delta_3} = 4 - 2\alpha. \quad (31)$$

Proof. **Step 1 (Substitution):**

$$\Delta_3 = s_0^2 - 20 - 40\alpha = (36 + 24\alpha + 4\alpha^2) - 20 - 40\alpha.$$

Step 2 (Collection):

$$\Delta_3 = (36 - 20) + (24\alpha - 40\alpha) + 4\alpha^2 = 16 - 16\alpha + 4\alpha^2.$$

Step 3 (Factorization):

$$16 - 16\alpha + 4\alpha^2 = 4(4 - 4\alpha + \alpha^2) = 4(2 - \alpha)^2.$$

Verification: $(2 - \alpha)^2 = 4 - 4\alpha + \alpha^2$. ✓

Step 4 (Sign Analysis): Since $\alpha = \log_2 3$ and $3 < 4 = 2^2$, we have $\log_2 3 < \log_2 4 = 2$. Thus $\alpha < 2$, so $2 - \alpha > 0$.

Step 5 (Square Root):

$$\sqrt{\Delta_3} = \sqrt{4(2 - \alpha)^2} = 2|2 - \alpha| = 2(2 - \alpha) = 4 - 2\alpha.$$

□

7 The Sum Constraint

Proposition 14 (Sum Constraint). *The constraint $a + b + c = s$ requires:*

$$\epsilon_1 \sqrt{\Delta_1} + \epsilon_2 \sqrt{\Delta_2} + \epsilon_3 \sqrt{\Delta_3} = -s \quad (32)$$

for some $\epsilon_i \in \{+1, -1\}$.

Proof. From (19)–(21):

$$a + b + c = \frac{3s + \epsilon_1 \sqrt{\Delta_1} + \epsilon_2 \sqrt{\Delta_2} + \epsilon_3 \sqrt{\Delta_3}}{2} = s.$$

Multiplying by 2:

$$3s + \epsilon_1 \sqrt{\Delta_1} + \epsilon_2 \sqrt{\Delta_2} + \epsilon_3 \sqrt{\Delta_3} = 2s.$$

Rearranging:

$$\epsilon_1 \sqrt{\Delta_1} + \epsilon_2 \sqrt{\Delta_2} + \epsilon_3 \sqrt{\Delta_3} = -s.$$

□

Remark 15. Since $s > 0$ (as $x, y, z > 0$), the right-hand side is negative. Therefore, at least one ϵ_i must be -1 .

8 Exhaustive Sign Combination Analysis

At $s_0 = 6 + 2\alpha$, we have computed:

$$\sqrt{\Delta_1} = 2 + 2\alpha, \quad \sqrt{\Delta_2} = 2\alpha, \quad \sqrt{\Delta_3} = 4 - 2\alpha.$$

The target is:

$$-s_0 = -(6 + 2\alpha) = -6 - 2\alpha.$$

We analyze all $2^3 = 8$ sign combinations $(\epsilon_1, \epsilon_2, \epsilon_3) \in \{+1, -1\}^3$.

8.1 Summary Table

Case	$(\epsilon_1, \epsilon_2, \epsilon_3)$	LHS Expression	LHS Value	Verdict
1	$(+, +, +)$	$(2 + 2\alpha) + 2\alpha + (4 - 2\alpha)$	$6 + 2\alpha$	Impossible: LHS > 0
2	$(+, +, -)$	$(2 + 2\alpha) + 2\alpha - (4 - 2\alpha)$	$-2 + 6\alpha$	Req. $\alpha = -\frac{1}{2}$
3	$(+, -, +)$	$(2 + 2\alpha) - 2\alpha + (4 - 2\alpha)$	$6 - 2\alpha$	Contradiction
4	$(+, -, -)$	$(2 + 2\alpha) - 2\alpha - (4 - 2\alpha)$	$-2 + 2\alpha$	Req. $\alpha = -1$
5	$(-, +, +)$	$-(2 + 2\alpha) + 2\alpha + (4 - 2\alpha)$	$2 - 2\alpha$	Contradiction
6	$(-, +, -)$	$-(2 + 2\alpha) + 2\alpha - (4 - 2\alpha)$	$-6 + 2\alpha$	Req. $\alpha = 0$
7	$(-, -, +)$	$-(2 + 2\alpha) - 2\alpha + (4 - 2\alpha)$	$2 - 6\alpha$	Req. $\alpha = 2$
8	$(-, -, -)$	$-(2 + 2\alpha) - 2\alpha - (4 - 2\alpha)$	$-6 - 2\alpha$	VALID

8.2 Case-by-Case Analysis

8.2.1 Case 1: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, +1, +1)$

$$\begin{aligned} \text{LHS} &= (2 + 2\alpha) + 2\alpha + (4 - 2\alpha) \\ &= 2 + 2\alpha + 2\alpha + 4 - 2\alpha \\ &= 6 + 2\alpha > 0. \end{aligned}$$

Verdict: Impossible since target $= -6 - 2\alpha < 0$.

8.2.2 Case 2: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, +1, -1)$

$$\begin{aligned}\text{LHS} &= (2 + 2\alpha) + 2\alpha - (4 - 2\alpha) \\ &= 2 + 2\alpha + 2\alpha - 4 + 2\alpha \\ &= -2 + 6\alpha.\end{aligned}$$

Setting $\text{LHS} = -6 - 2\alpha$:

$$-2 + 6\alpha = -6 - 2\alpha \implies 8\alpha = -4 \implies \alpha = -\frac{1}{2}.$$

Verdict: Impossible since $\alpha = \log_2 3 > 0 \notin \{-\frac{1}{2}\}$.

8.2.3 Case 3: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, -1, +1)$

$$\begin{aligned}\text{LHS} &= (2 + 2\alpha) - 2\alpha + (4 - 2\alpha) \\ &= 2 + 2\alpha - 2\alpha + 4 - 2\alpha \\ &= 6 - 2\alpha.\end{aligned}$$

Setting $\text{LHS} = -6 - 2\alpha$:

$$6 - 2\alpha = -6 - 2\alpha \implies 6 = -6.$$

Verdict: Impossible (algebraic contradiction).

8.2.4 Case 4: $(\epsilon_1, \epsilon_2, \epsilon_3) = (+1, -1, -1)$

$$\begin{aligned}\text{LHS} &= (2 + 2\alpha) - 2\alpha - (4 - 2\alpha) \\ &= 2 + 2\alpha - 2\alpha - 4 + 2\alpha \\ &= -2 + 2\alpha.\end{aligned}$$

Setting $\text{LHS} = -6 - 2\alpha$:

$$-2 + 2\alpha = -6 - 2\alpha \implies 4\alpha = -4 \implies \alpha = -1.$$

Verdict: Impossible since $\alpha > 0$.

8.2.5 Case 5: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, +1, +1)$

$$\begin{aligned}\text{LHS} &= -(2 + 2\alpha) + 2\alpha + (4 - 2\alpha) \\ &= -2 - 2\alpha + 2\alpha + 4 - 2\alpha \\ &= 2 - 2\alpha.\end{aligned}$$

Setting $\text{LHS} = -6 - 2\alpha$:

$$2 - 2\alpha = -6 - 2\alpha \implies 2 = -6.$$

Verdict: Impossible (algebraic contradiction).

8.2.6 Case 6: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, +1, -1)$

$$\begin{aligned}\text{LHS} &= -(2 + 2\alpha) + 2\alpha - (4 - 2\alpha) \\ &= -2 - 2\alpha + 2\alpha - 4 + 2\alpha \\ &= -6 + 2\alpha.\end{aligned}$$

Setting $\text{LHS} = -6 - 2\alpha$:

$$-6 + 2\alpha = -6 - 2\alpha \implies 4\alpha = 0 \implies \alpha = 0.$$

Verdict: Impossible since $\alpha = \log_2 3 > 0$ (boundary excluded).

8.2.7 Case 7: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, +1)$

$$\begin{aligned} \text{LHS} &= -(2 + 2\alpha) - 2\alpha + (4 - 2\alpha) \\ &= -2 - 2\alpha - 2\alpha + 4 - 2\alpha \\ &= 2 - 6\alpha. \end{aligned}$$

Setting $\text{LHS} = -6 - 2\alpha$:

$$2 - 6\alpha = -6 - 2\alpha \implies -4\alpha = -8 \implies \alpha = 2.$$

Verdict: Impossible since $\alpha = \log_2 3 < 2$ (boundary excluded).

8.2.8 Case 8: $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, -1)$

This is the key case. We compute:

$$\begin{aligned} \text{LHS} &= -(2 + 2\alpha) - 2\alpha - (4 - 2\alpha) \\ &= (-2 - 2\alpha) + (-2\alpha) + (-4 + 2\alpha). \end{aligned}$$

Step-by-step:

1. Distribute signs: $(-1)(2 + 2\alpha) = -2 - 2\alpha$
2. Distribute signs: $(-1)(2\alpha) = -2\alpha$
3. Distribute signs: $(-1)(4 - 2\alpha) = -4 + 2\alpha$
4. Group constants: $(-2) + (-4) = -6$
5. Group α terms: $(-2\alpha) + (-2\alpha) + (2\alpha) = -2\alpha$
6. Combine: $\text{LHS} = -6 - 2\alpha$

Verification:

$$\text{LHS} = -6 - 2\alpha = -(6 + 2\alpha) = -s_0. \quad \checkmark$$

Verdict: This is the unique valid solution!

Theorem 16 (Unique Sign Combination). *By exhaustive enumeration of all 8 sign combinations, only $(\epsilon_1, \epsilon_2, \epsilon_3) = (-1, -1, -1)$ yields $\text{LHS} = -s$ for $\alpha \in (0, 2)$.*

9 Explicit Solution Values

Proposition 17 (Values of a, b, c). *With $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$ and $s = 6 + 2\alpha$:*

$$a = \frac{(6 + 2\alpha) - (2 + 2\alpha)}{2} = \frac{4}{2} = 2, \tag{33}$$

$$b = \frac{(6 + 2\alpha) - 2\alpha}{2} = \frac{6}{2} = 3, \tag{34}$$

$$c = \frac{(6 + 2\alpha) - (4 - 2\alpha)}{2} = \frac{2 + 4\alpha}{2} = 1 + 2\alpha. \tag{35}$$

Proof. Using (19)–(21) with the minus signs:

$$\begin{aligned} a &= \frac{s - \sqrt{\Delta_1}}{2} = \frac{(6 + 2\alpha) - (2 + 2\alpha)}{2} = \frac{4}{2} = 2. \\ b &= \frac{s - \sqrt{\Delta_2}}{2} = \frac{(6 + 2\alpha) - 2\alpha}{2} = \frac{6}{2} = 3. \\ c &= \frac{s - \sqrt{\Delta_3}}{2} = \frac{(6 + 2\alpha) - (4 - 2\alpha)}{2} = \frac{2 + 4\alpha}{2} = 1 + 2\alpha. \end{aligned}$$

□

Corollary 18 (Values of x, y, z).

$$x = 2^a = 2^2 = 4, \tag{36}$$

$$y = 2^b = 2^3 = 8, \tag{37}$$

$$z = 2^c = 2^{1+2\alpha} = 2 \cdot 2^{2\alpha} = 2 \cdot (2^\alpha)^2 = 2 \cdot 3^2 = 18. \tag{38}$$

Proposition 19 (Verification of Sum).

$$a + b + c = 2 + 3 + (1 + 2\alpha) = 6 + 2\alpha = s_0. \quad \checkmark$$

10 Verification of Original Constraints

We verify that $(x, y, z) = (4, 8, 18)$ satisfies all three original constraints.

10.1 Verification of Constraint 1

Claim 2. $x^{\log_2(yz)} = 2^8 \cdot 3^4$.

Proof.

$$\begin{aligned} yz &= 8 \cdot 18 = 144 = 16 \cdot 9 = 2^4 \cdot 3^2. \\ \log_2(yz) &= \log_2(2^4 \cdot 3^2) = 4 + 2\alpha. \\ x^{\log_2(yz)} &= 4^{4+2\alpha} = (2^2)^{4+2\alpha} = 2^{8+4\alpha}. \end{aligned}$$

Since $2^\alpha = 3$:

$$2^{8+4\alpha} = 2^8 \cdot 2^{4\alpha} = 2^8 \cdot (2^\alpha)^4 = 2^8 \cdot 3^4. \quad \checkmark$$

□

10.2 Verification of Constraint 2

Claim 3. $y^{\log_2(zx)} = 2^9 \cdot 3^6$.

Proof.

$$\begin{aligned} zx &= 18 \cdot 4 = 72 = 8 \cdot 9 = 2^3 \cdot 3^2. \\ \log_2(zx) &= \log_2(2^3 \cdot 3^2) = 3 + 2\alpha. \\ y^{\log_2(zx)} &= 8^{3+2\alpha} = (2^3)^{3+2\alpha} = 2^{9+6\alpha}. \\ 2^{9+6\alpha} &= 2^9 \cdot 2^{6\alpha} = 2^9 \cdot (2^\alpha)^6 = 2^9 \cdot 3^6. \quad \checkmark \end{aligned}$$

□

10.3 Verification of Constraint 3

Claim 4. $z^{\log_2(xy)} = 2^5 \cdot 3^{10}$.

Proof.

$$\begin{aligned} xy &= 4 \cdot 8 = 32 = 2^5. \\ \log_2(xy) &= \log_2(2^5) = 5. \\ z^{\log_2(xy)} &= 18^5 = (2 \cdot 3^2)^5 = 2^5 \cdot 3^{10}. \quad \checkmark \end{aligned}$$

□

11 Proof of Minimality

We now prove that $s_0 = 6 + 2\alpha$ is indeed the minimum value of $s = a + b + c$.

11.1 Definition of the Constraint Function

Definition 20. For $s \geq \sqrt{20 + 40\alpha}$, define:

$$f(s) = \sqrt{s^2 - 32 - 16\alpha} + \sqrt{s^2 - 36 - 24\alpha} + \sqrt{s^2 - 20 - 40\alpha} - s. \quad (39)$$

The constraint equation (for the all-minus case) is $f(s) = 0$.

11.2 Verification that $f(s_0) = 0$

Lemma 21. $f(s_0) = 0$ where $s_0 = 6 + 2\alpha$.

Proof. Substituting the computed discriminant roots:

$$\begin{aligned} f(s_0) &= \sqrt{\Delta_1} + \sqrt{\Delta_2} + \sqrt{\Delta_3} - s_0 \\ &= (2 + 2\alpha) + 2\alpha + (4 - 2\alpha) - (6 + 2\alpha). \end{aligned}$$

Simplifying the first three terms:

$$(2 + 2\alpha) + 2\alpha + (4 - 2\alpha) = 2 + 2\alpha + 2\alpha + 4 - 2\alpha = 6 + 2\alpha.$$

Therefore:

$$f(s_0) = (6 + 2\alpha) - (6 + 2\alpha) = 0. \quad \checkmark$$

□

11.3 Strict Monotonicity of f

Lemma 22. $f'(s) > 0$ for all s in the domain.

Proof. Write $f(s) = \sqrt{s^2 - c_1} + \sqrt{s^2 - c_2} + \sqrt{s^2 - c_3} - s$ where:

$$c_1 = 32 + 16\alpha, \quad c_2 = 36 + 24\alpha, \quad c_3 = 20 + 40\alpha.$$

By the chain rule:

$$\frac{d}{ds} \sqrt{s^2 - c} = \frac{1}{2\sqrt{s^2 - c}} \cdot 2s = \frac{s}{\sqrt{s^2 - c}}.$$

Therefore:

$$f'(s) = \frac{s}{\sqrt{s^2 - c_1}} + \frac{s}{\sqrt{s^2 - c_2}} + \frac{s}{\sqrt{s^2 - c_3}} - 1. \quad (40)$$

Claim: Each fraction $\frac{s}{\sqrt{s^2 - c_i}} > 1$.

Proof of claim: For $\frac{s}{\sqrt{s^2 - c_i}} > 1$, we need $s > \sqrt{s^2 - c_i}$, i.e., $s^2 > s^2 - c_i$, i.e., $c_i > 0$.

Since $\alpha > 0$:

$$c_1 = 32 + 16\alpha > 32 > 0,$$

$$c_2 = 36 + 24\alpha > 36 > 0,$$

$$c_3 = 20 + 40\alpha > 20 > 0.$$

Therefore, for $s > 0$ in the domain:

$$f'(s) > 1 + 1 + 1 - 1 = 2 > 0.$$

□

11.4 Uniqueness of the Minimum

Theorem 23 (Uniqueness). $s_0 = 6 + 2\alpha$ is the unique solution to $f(s) = 0$.

Proof. 1. The domain of f is $[s_{\min}, \infty)$ where $s_{\min} = \sqrt{20 + 40\alpha}$ ensures all radicands are non-negative.

2. f is continuous on its domain as a composition of continuous functions.

3. By Lemma 22, $f'(s) > 0$ for all s in the domain.

4. By the Mean Value Theorem, continuous f with $f'(s) > 0$ everywhere implies f is strictly increasing.

5. For strictly increasing f : if $f(s_0) = 0$, then:

- $f(s) < 0$ for $s < s_0$, and
- $f(s) > 0$ for $s > s_0$.

6. By Lemma 21, $f(s_0) = 0$ at $s_0 = 6 + 2\alpha$.

Therefore $s_0 = 6 + 2\alpha$ is the unique zero of f , and no solution exists for $s < s_0$. □

Corollary 24 (Minimum Product). $s = a + b + c \geq s_0 = 6 + 2\alpha$, with equality achieved. Therefore:

$$xyz = 2^s \geq 2^{6+2\alpha} = 2^6 \cdot 2^{2\alpha} = 64 \cdot (2^\alpha)^2 = 64 \cdot 9 = 576.$$

12 Conclusion

Proof of Theorem 1. We have shown:

1. The constraints transform to a quadratic system in logarithmic variables.
2. The system admits real solutions only when $s = a + b + c \geq \sqrt{20 + 40\alpha}$.
3. At $s_0 = 6 + 2\alpha$, all discriminants are perfect squares.
4. Exhaustive analysis of all 8 sign combinations shows only $(-1, -1, -1)$ is valid.
5. The constraint function $f(s) = 0$ has a unique solution at $s_0 = 6 + 2\alpha$ by strict monotonicity.
6. The explicit solution $(x, y, z) = (4, 8, 18)$ satisfies all original constraints.

7. Therefore, $xyz = 4 \cdot 8 \cdot 18 = 576$ is the minimum.

□

$$\boxed{xyz_{\min} = 576} \tag{41}$$

A Numerical Verification

For reference, $\alpha = \log_2 3 \approx 1.5849625007211563$.

$$\begin{aligned} s_0 &= 6 + 2\alpha \approx 9.1699250014423126 \\ 2^{s_0} &= 2^{6+2\log_2 3} = 2^6 \cdot 3^2 = 64 \cdot 9 = 576 \quad \checkmark \end{aligned}$$

B Graph Metadata

This proof was formalized using the Alethfeld proof system from graph **graph-51acde-43ac9a**, containing 40 nodes at depth 0–1, with 67 expansion nodes at depths 2–3.

Key verified nodes:

- :1-000001 – Logarithmic transformation
- :1-000011, :1-000012, :1-000013 – Discriminant factorizations
- :1-000015 – Valid sign combination verification
- :1-000034 – Exhaustive sign analysis (8 cases)
- :1-000031, :1-000032, :1-000033 – Minimality via monotonicity