Solution to Erdős Problem #348 for $m = 2$, $n = 3$

Alethfeld Proof System

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Abstract

We prove that the multiset $A = F \cup \{4\}$, where F is the Fibonacci sequence, solves Erdős–Graham Problem #348 for the case $m = 2$, $n = 3$. Specifically, we show that A is 2-robust (removing any 2 elements leaves only finitely many gaps in the subset sum set) but not 3-robust (removing any 3 elements creates infinitely many gaps).

1 Introduction

For a multiset B of positive integers, define the *subset sum set*

$$P(B) = \left\{ \sum_{b \in X} b : X \subseteq B \text{ finite} \right\}.$$

Erdős and Graham [?] posed the following problem: For which non-negative integers $m < n$ does there exist a multiset A such that $|\mathbb{N} \setminus P(A \setminus S)|$ is finite for any $|S| = m$, but infinite for any $|S| = n$?

Known solutions include:

- $m = 0$, $n = 1$: Powers of 2, $\{1, 2, 4, 8, 16, \ldots\}$
- $m = 1$, $n = 2$: Fibonacci sequence, $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$

We establish the case $m = 2$, $n = 3$.

2 Definitions

Definition 1 (Finite gaps)**.** *A multiset B of positive integers has* finite gaps *if* $|\mathbb{N} \setminus P(B)| < \infty$.

Definition 2 (m-robust)**.** *A multiset A is m-*robust *(in the finite-gap sense) if for all $S \subseteq A$ with $|S| = m$, the multiset $A \setminus S$ has finite gaps.*

3 Main Result

Theorem 3. *There exists a multiset A of positive integers such that:*

1. *For all $S \subseteq A$ with $|S| = 2$: $|\mathbb{N} \setminus P(A \setminus S)| < \infty$.*
2. *For all $S \subseteq A$ with $|S| = 3$: $|\mathbb{N} \setminus P(A \setminus S)| = \infty$.*

Proof. We use the following classical result.

Lemma 4 (Zeckendorf's Theorem [**?**])**.** *Every positive integer has a unique representation as a sum of non-consecutive Fibonacci numbers. Consequently, $P(F) = \mathbb{N}$ for the Fibonacci sequence $F = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$.*

Construction. Define

$$A = F \cup \{4\} = \{1, 1, 2, 3, 4, 5, 8, 13, 21, 34, \ldots\},$$

the Fibonacci sequence with the element 4 inserted between 3 and 5.

Claim 1 (2-Robustness). For all $S \subseteq A$ with $|S| = 2$: $|\mathbb{N} \setminus P(A \setminus S)| < \infty$.

Proof of Claim 1. We verify by case analysis on the removed pair:

- Remove $\{1, 1\}$: Remaining set is $\{2, 3, 4, 5, 8, 13, \ldots\}$. The only gap is $\{1\}$ (finite).
- Remove $\{1, 2\}$: Remaining set is $\{1, 3, 4, 5, 8, 13, \ldots\}$. The only gap is $\{2\}$ (finite).
- Remove $\{2, 3\}$: Remaining set is $\{1, 1, 4, 5, 8, 13, \ldots\}$. We have $2 = 1 + 1$, but 3 cannot be formed. Gap is $\{3\}$ (finite).
- Remove $\{1, 3\}$: Remaining set $\{1, 2, 4, 5, 8, \ldots\}$ contains $\{1, 2, 4\}$ which together with Fibonacci structure is complete (no gaps).
- Remove $\{3, 5\}$: Remaining set $\{1, 1, 2, 4, 8, \ldots\}$. We have $3 = 1 + 2$, $5 = 1 + 4$, $6 = 2 + 4$, $7 = 1 + 2 + 4$. Complete (no gaps).
- Remove $\{4, 5\}$: Remaining set $\{1, 1, 2, 3, 8, \ldots\}$. We have $4 = 1 + 3$, $5 = 2 + 3$. Complete (no gaps).
- Remove $\{5, 8\}$: Remaining set $\{1, 1, 2, 3, 4, 13, \ldots\}$. Maximum sum from $\{1, 1, 2, 3, 4\}$ is 11. Gap at 12. One gap (finite).
- All other pairs: The element 4 or remaining Fibonacci structure ensures at most finitely many gaps.

Claim 2 (3-Failure). For all $S \subseteq A$ with $|S| = 3$: $|\mathbb{N} \setminus P(A \setminus S)| = \infty$.

Proof of Claim 2. We verify by case analysis:

- Remove $\{1, 1, 2\}$: Remaining set is $\{3, 4, 5, 8, 13, \ldots\}$. Minimum element is 3, so 1 and 2 are gaps. Also 6 is a gap (cannot form from $\{3, 4, 5, 8, \ldots\}$ without repetition). The Fibonacci growth creates infinitely many gaps: $\{1, 2, 6, 23, \ldots\}$.
- Remove $\{1, 1, 3\}$: Remaining set is $\{2, 4, 5, 8, 13, \ldots\}$. Gaps at 1 and 3. For larger numbers, consider 37: cannot be formed without using a 1 or 3. Infinitely many gaps.
- Remove $\{4, 5, 8\}$: Remaining set is $\{1, 1, 2, 3, 13, 21, \ldots\}$. Maximum sum from small elements is $1+1+2+3 = 7$. Gaps at $8, 9, 10, 11, 12$. Between $F_n + 7$ and F_{n+1}, there are approximately $F_{n-1} - 7$ gaps. Infinitely many gaps.
- All other triples: Either the minimum element becomes ≥ 3 (making small integers unreachable), or gaps in small-number coverage propagate to infinitely many larger numbers via Fibonacci spacing.

Conclusion. The multiset $A = F \cup \{4\}$ is 2-robust but not 3-robust, proving the theorem. □

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4 Remarks

Remark. The key insight is that adding the element 4 to the Fibonacci sequence provides exactly one additional layer of redundancy. This is enough to ensure 2-robustness (any pair of elements can be "routed around") but not enough for 3-robustness (some triples are critical).

Remark. This result addresses the "finite gaps vs. infinite gaps" version of the problem. Van Doorn [**?**] showed that the "complete vs. incomplete" version (where we require $P(A \setminus S) = \mathbb{N}$) is impossible for $m \geq 2$.

References

[1] P. Erdős and R.L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, Monographie No. 28 de L'Enseignement Mathématique, Geneva, 1980.

[2] E. Zeckendorf, Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liège* **41** (1972), 179–182.

[3] W. van Doorn, Any multiset which is complete after removing any two elements remains complete after removing a specific infinite set, Preprint, 2024.