PROOF OF THE IMPLICIT FUNCTION THEOREM

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Theorem 1 (Implicit Function Theorem). *Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets, and let $F : U \times V \to \mathbb{R}^m$ be continuously differentiable. Suppose $(a, b) \in U \times V$ satisfies $F(a, b) = 0$ and the partial derivative $D_y F(a, b) : \mathbb{R}^m \to \mathbb{R}^m$ is invertible. Then there exist open neighborhoods $U_0 \subseteq U$ of a and $V_0 \subseteq V$ of b, and a unique continuously differentiable function $g : U_0 \to V_0$ such that:*

(i) $g(a) = b$,
(ii) $F(x, g(x)) = 0$ *for all* $x \in U_0$, *and*
(iii) $Dg(x) = -[D_y F(x, g(x))]^{-1} D_x F(x, g(x))$ *for all* $x \in U_0$.

Proof. **Assumptions.** Let $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ be open sets. Let $F : U \times V \to \mathbb{R}^m$ A1–A4
be a continuously differentiable (C^1) function. Let $(a, b) \in U \times V$ satisfy $F(a, b) = 0$. Assume the partial derivative $D_y F(a, b) : \mathbb{R}^m \to \mathbb{R}^m$ is an invertible linear map.

Step 1: Auxiliary Map. Define the auxiliary map $\Phi : U \times V \to \mathbb{R}^n \times \mathbb{R}^m$ by D1, C1–C2

$$\Phi(x, y) := (x, F(x, y)).$$

The map Φ is continuously differentiable (C^1) on $U \times V$, since F is C^1 and the identity map $x \mapsto x$ is smooth.

The derivative of Φ at (x, y) is the block matrix

$$D\Phi(x, y) = \begin{pmatrix} I_n & 0 \\ D_x F(x, y) & D_y F(x, y) \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix.

Step 2: Invertibility of $D\Phi(a, b)$. At the point (a, b): C3–C5

$$D\Phi(a, b) = \begin{pmatrix} I_n & 0 \\ D_x F(a, b) & D_y F(a, b) \end{pmatrix}.$$

For a block lower-triangular matrix $\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$, the determinant equals $\det(A) \cdot \det(D)$, and invertibility holds if and only if both A and D are invertible.

Since I_n is invertible and $D_y F(a, b)$ is invertible by assumption, $D\Phi(a, b)$ is invertible.

Step 3: Application of Inverse Function Theorem. Note that C6–C8

$$\Phi(a, b) = (a, F(a, b)) = (a, 0).$$

By the **Inverse Function Theorem** applied to Φ at (a, b): since Φ is C^1 and $D\Phi(a, b)$ is invertible, there exist open neighborhoods $W_1 \subseteq U \times V$ of (a, b) and $W_2 \subseteq \mathbb{R}^n \times \mathbb{R}^m$ of $(a, 0)$ such that $\Phi : W_1 \to W_2$ is a C^1 diffeomorphism with C^1 inverse $\Psi : W_2 \to W_1$.

There exists an open neighborhood $U_0 \subseteq \mathbb{R}^n$ of a such that $U_0 \times \{0\} \subseteq W_2$.

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D2, C9 **Step 4: Definition of Implicit Function.** For $x \in U_0$, define

$$g(x) := \pi_y(\Psi(x, 0)),$$

where $\pi_y : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is the projection onto the second factor.

The function $g : U_0 \to \mathbb{R}^m$ is C^1, since Ψ is C^1 and projection is smooth.

C10 **Step 5: Verification of Property (i).**

$$g(a) = \pi_y(\Psi(a, 0)) = \pi_y((a, b)) = b,$$

since $\Phi(a, b) = (a, 0)$ implies $\Psi(a, 0) = (a, b)$.

C11–C15 **Step 6: Verification of Property (ii).** For all $x \in U_0$, we have $\Psi(x, 0) = (\pi_x(\Psi(x, 0)), g(x))$ by definition of g.

Since $\Phi \circ \Psi = \mathrm{id}$ on W_2:

$$\Phi(\Psi(x, 0)) = (x, 0) \quad \text{for all } x \in U_0.$$

Write $\Psi(x, 0) = (\xi(x), g(x))$ for some function $\xi : U_0 \to \mathbb{R}^n$. Then

$$\Phi(\xi(x), g(x)) = (\xi(x), F(\xi(x), g(x))) = (x, 0).$$

Comparing first components: $\xi(x) = x$ for all $x \in U_0$.

Comparing second components: $F(x, g(x)) = 0$ for all $x \in U_0$.

C16–C18 **Step 7: Verification of Property (iii).** Differentiating $F(x, g(x)) = 0$ with respect to x using the chain rule:

$$D_x F(x, g(x)) + D_y F(x, g(x)) \cdot Dg(x) = 0.$$

By continuity of $D_y F$ and invertibility at (a, b), shrinking U_0 if necessary, $D_y F(x, g(x))$ is invertible for all $x \in U_0$.

Solving the chain rule equation:

$$Dg(x) = -[D_y F(x, g(x))]^{-1} D_x F(x, g(x)) \quad \text{for all } x \in U_0.$$

C19–C20 **Step 8: Uniqueness.** Let $V_0 := g(U_0) \subseteq V$. Since g is continuous and U_0 is open containing a with $g(a) = b$, V_0 contains a neighborhood of b.

Suppose $h : U_0 \to V$ is another C^1 function with $h(a) = b$ and $F(x, h(x)) = 0$ for all x in some neighborhood of a. Then for x in this neighborhood, $(x, h(x)) \in W_1$ and

$$\Phi(x, h(x)) = (x, F(x, h(x))) = (x, 0),$$

so $(x, h(x)) = \Psi(x, 0) = (x, g(x))$, hence $h(x) = g(x)$.

QED **Conclusion.** The Implicit Function Theorem is established: there exist open neighborhoods U_0 of a and V_0 of b, and a unique C^1 function $g : U_0 \to V_0$ satisfying $g(a) = b$, $F(x, g(x)) = 0$ for all $x \in U_0$, and

$$Dg(x) = -[D_y F(x, g(x))]^{-1} D_x F(x, g(x)).$$

$\square$