

# Thermodynamic quantities: Condensate Ansatz

`In[*]:= Clear[t, u, c1, c2]`

`In[*]:= u[t_] = c1 JacobiSN[c1 (-I t + c2), -1]`

`Out[*]= c1 JacobiSN[c1 (c2 - I t), -1]`

`In[*]:= 6 D[u[t], t]^2 + 6 u[t]^4 // Simplify`

`Out[*]= 6 c1^4  
 (-JacobiCN[c1 (c2 - I t), -1]^2 JacobiDN[c1 (c2 - I t), -1]^2 + JacobiSN[c1 (c2 - I t), -1]^4)`

`In[*]:= Integrate[6 c1^4 (-JacobiCN[c1 (c2 - I t), -1]^2 JacobiDN[c1 (c2 - I t), -1]^2 +  
 JacobiSN[c1 (c2 - I t), -1]^4), {t, 0, β}]`

`Out[*]= -2 c1^3 (c1 β - 2 I JacobiCN[c1 c2, -1] JacobiDN[c1 c2, -1] JacobiSN[c1 c2, -1] +  
 2 I JacobiCN[c1 (c2 - I β), -1] JacobiDN[c1 (c2 - I β), -1] JacobiSN[c1 (c2 - I β), -1])`

`In[*]:= 3 D[u[t], t]^2 // Simplify`

`Out[*]= -3 c1^4 JacobiCN[c1 (c2 - I t), -1]^2 JacobiDN[c1 (c2 - I t), -1]^2`

`In[*]:= Integrate[-3 c1^4 JacobiCN[c1 (c2 - I t), -1]^2 JacobiDN[c1 (c2 - I t), -1]^2, {t, 0, β}]`

`Out[*]= -c1^3 (2 c1 β - I JacobiCN[c1 c2, -1] JacobiDN[c1 c2, -1] JacobiSN[c1 c2, -1] +  
 I JacobiCN[c1 (c2 - I β), -1] JacobiDN[c1 (c2 - I β), -1] JacobiSN[c1 (c2 - I β), -1])`

Thermodynamic Potential

`In[*]:= Clear[Nf, μ, β, c1, c2, g, Nc]`

`In[*]:= Ω[T_] =  $\frac{-\pi^2}{45} T^4 (Nc^2 - 1) - 7 \frac{\pi^2}{180} T^4 Nc Nf - \left( \frac{-1}{4 g^2} + \frac{1}{2} \beta \theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] - \frac{1}{(4 \pi)^2} \frac{Nf}{3} \text{Log}[4] \right) 2 c1^3$   
 $\left( c1 - 2 I T \text{JacobiCN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1] + 2 I T \text{JacobiCN}\left[c1 \left(c2 - \frac{I}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{I}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{I}{T}\right), -1\right] \right) - \frac{1}{3} \frac{1}{(4 \pi)^2}$   
 $(Nf - Nc) c1^3 \left( 2 c1 - I T \text{JacobiCN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] + \right.$   
 $\left. I T \text{JacobiCN}\left[c1 \left(c2 - \frac{I}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{I}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{I}{T}\right), -1\right] \right);$`

Pressure

`In[*]:= P[T_] = -Ω[T];`

$\ln[\ast] := \mathbf{P}[\mathbf{T}]$

$$\begin{aligned} \text{Out}[\ast] = & \frac{1}{45} \left( -1 + \mathbf{Nc}^2 \right) \pi^2 \mathbf{T}^4 + \frac{7}{180} \mathbf{Nc} \mathbf{Nf} \pi^2 \mathbf{T}^4 + \frac{1}{48 \pi^2} \\ & \mathbf{c1}^3 \left( -\mathbf{Nc} + \mathbf{Nf} \right) \left( 2 \mathbf{c1} - \frac{\mathbf{i}}{\mathbf{T}} \mathbf{T} \text{JacobiCN}[\mathbf{c1} \mathbf{c2}, -1] \text{JacobiDN}[\mathbf{c1} \mathbf{c2}, -1] \text{JacobiSN}[\mathbf{c1} \mathbf{c2}, -1] + \right. \\ & \quad \left. \frac{\mathbf{i}}{\mathbf{T}} \mathbf{T} \text{JacobiCN} \left[ \mathbf{c1} \left( \mathbf{c2} - \frac{\mathbf{i}}{\mathbf{T}} \right), -1 \right] \text{JacobiDN} \left[ \mathbf{c1} \left( \mathbf{c2} - \frac{\mathbf{i}}{\mathbf{T}} \right), -1 \right] \text{JacobiSN} \left[ \mathbf{c1} \left( \mathbf{c2} - \frac{\mathbf{i}}{\mathbf{T}} \right), -1 \right] \right) + \\ & 2 \mathbf{c1}^3 \left( \mathbf{c1} - 2 \frac{\mathbf{i}}{\mathbf{T}} \mathbf{T} \text{JacobiCN}[\mathbf{c1} \mathbf{c2}, -1] \text{JacobiDN}[\mathbf{c1} \mathbf{c2}, -1] \text{JacobiSN}[\mathbf{c1} \mathbf{c2}, -1] + \right. \\ & \quad \left. 2 \frac{\mathbf{i}}{\mathbf{T}} \mathbf{T} \text{JacobiCN} \left[ \mathbf{c1} \left( \mathbf{c2} - \frac{\mathbf{i}}{\mathbf{T}} \right), -1 \right] \text{JacobiDN} \left[ \mathbf{c1} \left( \mathbf{c2} - \frac{\mathbf{i}}{\mathbf{T}} \right), -1 \right] \text{JacobiSN} \left[ \mathbf{c1} \left( \mathbf{c2} - \frac{\mathbf{i}}{\mathbf{T}} \right), -1 \right] \right) \\ & \left( -\frac{1}{4 g^2} - \frac{\mathbf{Nf} \text{Log}[4]}{48 \pi^2} + \frac{1}{2} \beta \theta \text{Log} \left[ \frac{\mu}{4 \pi \mathbf{T}} \right] \right) \end{aligned}$$

Entropy Density

$\ln[\ast] := \mathbf{s}[\mathbf{T\_}] = -\mathbf{D}[\Omega[\mathbf{T}], \mathbf{T}];$

$\ln[\ast] := \mathbf{s}[T]$

$$\begin{aligned}
 \text{Out}[\ast] = & \frac{4}{45} \left( -1 + \text{Nc}^2 \right) \pi^2 T^3 + \frac{7}{45} \text{Nc Nf} \pi^2 T^3 - \frac{1}{T} \\
 & c1^3 \beta \theta \left( c1 - 2 \frac{i}{T} \text{JacobiCN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1] + \right. \\
 & \quad \left. 2 \frac{i}{T} \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \right) + \\
 & \frac{1}{48 \pi^2} c1^3 (-\text{Nc} + \text{Nf}) \left( -\frac{c1 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{T} - \right. \\
 & \quad \frac{i \text{JacobiCN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1] +}{T} \\
 & \quad \frac{i \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] -}{T} \\
 & \quad \frac{c1 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{T} + \\
 & \quad \left. \frac{c1 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{T} \right) + \\
 & 2 c1^3 \left( -\frac{2 c1 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{T} - \right. \\
 & \quad 2 \frac{i}{T} \text{JacobiCN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1] + \\
 & \quad 2 \frac{i}{T} \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] - \\
 & \quad \frac{2 c1 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{T} + \\
 & \quad \left. \frac{2 c1 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{T} \right) \\
 & \left( -\frac{1}{4 g^2} - \frac{\text{Nf Log}[4]}{48 \pi^2} + \frac{1}{2} \beta \theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right)
 \end{aligned}$$

Energy Density

$$\ln[\ast] := \epsilon[T_-] = -T^2 D\left[\frac{\Omega[T]}{T}, T\right];$$

$\ln[*]:= \epsilon[T] \text{ // Simplify}$

$$\begin{aligned}
 \text{Out}[*]:= & \frac{1}{240} \left( \frac{120 c1^4}{g^2} + \frac{10 c1^4 Nc}{\pi^2} - \frac{10 c1^4 Nf}{\pi^2} - 16 \pi^2 T^4 + 16 Nc^2 \pi^2 T^4 + 28 Nc Nf \pi^2 T^4 - 240 c1^4 \beta 0 + \right. \\
 & 480 i c1^3 T \beta 0 \text{JacobiCN}\left[c1 c2, -1\right] \text{JacobiDN}\left[c1 c2, -1\right] \text{JacobiSN}\left[c1 c2, -1\right] - 480 i c1^3 T \\
 & \beta 0 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] - \\
 & \frac{240 c1^4 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{g^2} - \\
 & \frac{5 c1^4 Nc \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{\pi^2} + \\
 & \frac{5 c1^4 Nf \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{\pi^2} + \frac{10 c1^4 Nf \text{Log}[4]}{\pi^2} - \\
 & \frac{20 c1^4 Nf \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{Log}[4]}{\pi^2} - \\
 & 240 c1^4 \beta 0 \text{Log}\left[\frac{\mu}{4 \pi T}\right] + 480 c1^4 \beta 0 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \\
 & \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{Log}\left[\frac{\mu}{4 \pi T}\right] - \frac{1}{g^2 \pi^2} 5 c1^4 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \\
 & \left( \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 + \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \right) \\
 & \left. \left( -48 \pi^2 + g^2 (-Nc + Nf - 4 Nf \text{Log}[4]) + 96 g^2 \pi^2 \beta 0 \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) \right)
 \end{aligned}$$

Trace Anomaly

$$\ln[*]:= \Theta[T_-] = -T^5 D\left[\frac{\Omega[T]}{T^4}, T\right];$$

$\ln[*]:= \Theta[T] \text{ // Simplify}$

$$\begin{aligned} \text{Out}[*]:= & \frac{1}{48 g^2 \pi^2} c1^3 \left( 8 c1 \left( 12 \pi^2 + g^2 (Nc - 6 \pi^2 \beta\theta + Nf (-1 + \text{Log}[4])) - 24 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) + \right. \\ & c1 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \\ & \left. \left( -48 \pi^2 + g^2 (-Nc + Nf - 4 Nf \text{Log}[4]) + 96 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) - c1 \right. \\ & \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \left( \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 + \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \right) \\ & \left. \left( -48 \pi^2 + g^2 (-Nc + Nf - 4 Nf \text{Log}[4]) + 96 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) + \right. \\ & 3 i T \text{JacobiCN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1] \\ & \left. \left( -48 \pi^2 + g^2 (-Nc + Nf + 32 \pi^2 \beta\theta - 4 Nf \text{Log}[4]) + 96 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) \right] - \\ & 3 i T \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \\ & \left. \left( -48 \pi^2 + g^2 (-Nc + Nf + 32 \pi^2 \beta\theta - 4 Nf \text{Log}[4]) + 96 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) \right] \right) \end{aligned}$$

Specific heat

$$\ln[*]:= C_v[T_-] = \left( 4 \frac{\epsilon[T]}{T^4} + 3 \frac{\Theta[T]}{T^4} + T D\left[\frac{\Theta[T]}{T^4}, T\right] \right) T^3;$$

$\ln[*]:= C_v[T] \text{ // Simplify}$

$$\begin{aligned} \text{Out}[*]:= & \frac{1}{60 g^2 \pi^2 T^2} \left( 240 c1^4 g^2 \pi^2 T \beta\theta \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \right. \\ & \left. \left( \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 + \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \right) + \right. \\ & 4 g^2 \pi^2 T \left( -4 \pi^2 T^4 + 4 Nc^2 \pi^2 T^4 + 7 Nc Nf \pi^2 T^4 + 15 c1^4 \beta\theta + \right. \\ & 30 i c1^3 T \beta\theta \text{JacobiCN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1] - \\ & 60 c1^4 \beta\theta \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \left. \right) - \\ & 5 i c1^5 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^3 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \\ & \left. \left( -48 \pi^2 + g^2 (-Nc + Nf - 4 Nf \text{Log}[4]) + 96 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) + \right. \\ & 5 i c1^3 \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \\ & \left. \left( -24 g^2 \pi^2 T^2 \beta\theta + c1^2 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \left( -48 \pi^2 + g^2 (-Nc + Nf - 4 Nf \text{Log}[4]) + \right. \right. \right. \\ & 96 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \left. \right) + c1^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \\ & \left. \left. \left( -48 \pi^2 + g^2 (-Nc + Nf - 4 Nf \text{Log}[4]) + 96 g^2 \pi^2 \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) \right) \right) \right) \end{aligned}$$

## Speed of Sound

$$\text{In[*]:= } C_s[T\_]=\frac{s[T]}{C_v[T]};$$

$$\text{In[*]:= } C_s[T] \text{ // Simplify}$$

$$\begin{aligned}
\text{Out}[*]= & \left( T \left( -64 g^2 \pi^4 T^4 + 64 g^2 N_c^2 \pi^4 T^4 + 112 g^2 N_c N_f \pi^4 T^4 - 720 c_1^4 g^2 \pi^2 \beta \theta - \right. \right. \\
& 15 c_1^4 g^2 N_c \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 + \\
& 15 c_1^4 g^2 N_f \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 - \\
& 720 c_1^4 \pi^2 \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 - \\
& 60 c_1^4 g^2 N_f \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{Log}[4] + 1440 c_1^4 \\
& g^2 \pi^2 \beta \theta \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{Log} \left[ \frac{\mu}{4 \pi T} \right] - 15 c_1^4 \\
& \text{JacobiCN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \left( \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 + \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \right) \\
& \left( -48 \pi^2 + g^2 (-N_c + N_f - 4 N_f \text{Log}[4]) + 96 g^2 \pi^2 \beta \theta \text{Log} \left[ \frac{\mu}{4 \pi T} \right] \right) - \\
& 15 i c_1^3 T \text{JacobiCN} [c_1 c_2, -1] \text{JacobiDN} [c_1 c_2, -1] \text{JacobiSN} [c_1 c_2, -1] \\
& \left( -48 \pi^2 - g^2 (N_c - N_f + 96 \pi^2 \beta \theta + 4 N_f \text{Log}[4]) + 96 g^2 \pi^2 \beta \theta \text{Log} \left[ \frac{\mu}{4 \pi T} \right] \right) + 15 i c_1^3 T \\
& \text{JacobiCN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \\
& \left( -48 \pi^2 - g^2 (N_c - N_f + 96 \pi^2 \beta \theta + 4 N_f \text{Log}[4]) + 96 g^2 \pi^2 \beta \theta \text{Log} \left[ \frac{\mu}{4 \pi T} \right] \right) \Big) \Big) / \\
& \left( 12 \left( 240 c_1^4 g^2 \pi^2 T \beta \theta \text{JacobiCN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \right. \right. \\
& \left. \left( \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 + \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \right) + \right. \\
& 4 g^2 \pi^2 T \left( -4 \pi^2 T^4 + 4 N_c^2 \pi^2 T^4 + 7 N_c N_f \pi^2 T^4 + 15 c_1^4 \beta \theta + \right. \\
& 30 i c_1^3 T \beta \theta \text{JacobiCN} [c_1 c_2, -1] \text{JacobiDN} [c_1 c_2, -1] \text{JacobiSN} [c_1 c_2, -1] - \\
& 60 c_1^4 \beta \theta \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \Big) - \\
& 5 i c_1^5 \text{JacobiCN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^3 \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \\
& \left( -48 \pi^2 + g^2 (-N_c + N_f - 4 N_f \text{Log}[4]) + 96 g^2 \pi^2 \beta \theta \text{Log} \left[ \frac{\mu}{4 \pi T} \right] \right) + \\
& 5 i c_1^3 \text{JacobiCN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right] \\
& \left( -24 g^2 \pi^2 T^2 \beta \theta + c_1^2 \text{JacobiDN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \left( -48 \pi^2 + g^2 (-N_c + N_f - 4 N_f \text{Log}[4]) + \right. \right. \\
& 96 g^2 \pi^2 \beta \theta \text{Log} \left[ \frac{\mu}{4 \pi T} \right] \Big) + c_1^2 \text{JacobiSN} \left[ c_1 \left( c_2 - \frac{i}{T} \right), -1 \right]^2 \\
& \left. \left. \left( -48 \pi^2 + g^2 (-N_c + N_f - 4 N_f \text{Log}[4]) + 96 g^2 \pi^2 \beta \theta \text{Log} \left[ \frac{\mu}{4 \pi T} \right] \right) \right) \right) \Big) \Big)
\end{aligned}$$

Ideal gas limit

$\text{In}[*]:= \text{Nc} = 2; \text{Nf} = 2;$

$$\text{Pideal} = \frac{\pi^2}{45} T^4 (\text{Nc}^2 - 1) + \frac{7 \pi^2}{180} T^4 \text{Nc Nf};$$

$$\text{Eideal} = \frac{\pi^2}{15} T^4 (\text{Nc}^2 - 1) + \frac{7 \pi^2}{60} T^4 \text{Nc Nf};$$

$$\text{Sideal} = 4 \frac{\pi^2}{45} T^3 \left( \text{Nc}^2 - 1 + \frac{7}{4} \text{Nc Nf} \right);$$

$\text{In}[*]:= \text{Pideal}$

$$\text{Out}[*]= \frac{2 \pi^2 T^4}{9}$$

$\text{In}[*]:= \text{Eideal}$

$$\text{Out}[*]= \frac{2 \pi^2 T^4}{3}$$

$\text{In}[*]:= \text{Sideal}$

$$\text{Out}[*]= \frac{8 \pi^2 T^3}{9}$$

(\*Replace g with  $4\pi \alpha_S[T]$  for plotting\*)

Pressure plot (Normalised)

$\text{In}[*]:= \text{Clear}[\text{c1}, \text{c2}, \mu, \beta\theta]$

$$\Delta_{\text{MS}} = 176; \mu = 2 \pi T; \beta\theta = \frac{1}{(4 \pi)^2} \left( \frac{11}{3} \text{Nc} - \frac{2}{3} \text{Nf} \right);$$

$$\text{t}[T\_]= 2 \text{Log}\left[\frac{\mu}{\Delta_{\text{MS}}}\right];$$

$$\alpha_S[T\_]= \frac{6 \pi}{11 * \text{t}[T]} (1); (*\text{For only loop} *)$$

$$\text{Press}[\text{c1}_-, \text{c2}_-, T\_]= \left( \frac{1}{45} (-1 + \text{Nc}^2) \pi^2 T^4 + \frac{7}{180} \text{Nc Nf} \pi^2 T^4 + \frac{1}{48 \pi^2} \text{c1}^3 (-\text{Nc} + \text{Nf}) \right.$$

$$\left( 2 \text{c1} - \frac{i}{T} T \text{JacobiCN}[\text{c1} \text{c2}, -1] \text{JacobiDN}[\text{c1} \text{c2}, -1] \text{JacobiSN}[\text{c1} \text{c2}, -1] + \frac{i}{T} T \right.$$

$$\left. \text{JacobiCN}\left[\text{c1} \left( \text{c2} - \frac{i}{T} \right), -1\right] \text{JacobiDN}\left[\text{c1} \left( \text{c2} - \frac{i}{T} \right), -1\right] \text{JacobiSN}\left[\text{c1} \left( \text{c2} - \frac{i}{T} \right), -1\right] \right) +$$

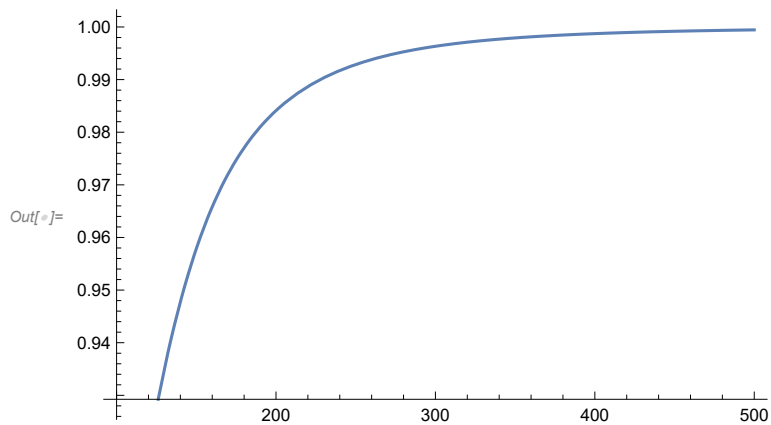
$$2 \text{c1}^3 \left( \text{c1} - 2 \frac{i}{T} T \text{JacobiCN}[\text{c1} \text{c2}, -1] \text{JacobiDN}[\text{c1} \text{c2}, -1] \text{JacobiSN}[\text{c1} \text{c2}, -1] + 2 \frac{i}{T} T \right.$$

$$\left. T \text{JacobiCN}\left[\text{c1} \left( \text{c2} - \frac{i}{T} \right), -1\right] \text{JacobiDN}\left[\text{c1} \left( \text{c2} - \frac{i}{T} \right), -1\right] \text{JacobiSN}\left[\text{c1} \left( \text{c2} - \frac{i}{T} \right), -1\right] \right)$$

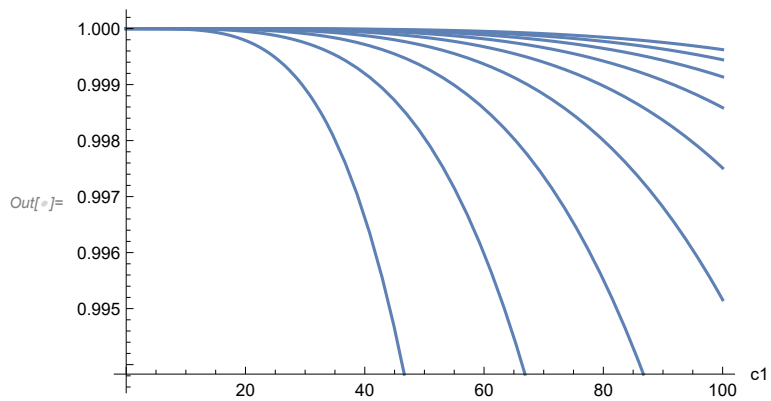
$$\left( -\frac{1}{4 (4 \pi) \alpha_S[T]} - \frac{\text{Nf} \text{Log}[4]}{48 \pi^2} + \frac{1}{2} \beta\theta \text{Log}\left[\frac{\mu}{4 \pi T}\right] \right) \Big/ \text{Pideal};$$



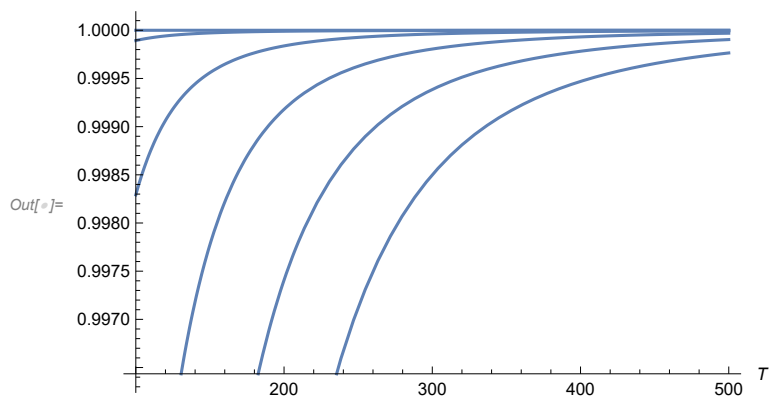
```
In[ ]:= Plot[Re[Press[110, 1, T]], {T, 100, 500}]
```



```
In[ ]:= Plot[Table[Press[c1, 0, T], {T, 100, 500, 50}],  
{c1, 0, 100}, AxesLabel -> Automatic, PlotLegends -> "Expressions"]
```



```
In[ ]:= Plot[Table[Press[c1, 0, T], {c1, 0, 50, 10}],  
{T, 100, 500}, AxesLabel -> Automatic, PlotLegends -> "Expressions"]
```



Energy density plot (Normalised)

```
In[ ]:= Clear[c1, c2]
```

$$\ln[*]:= \Delta ms = 176; \mu = 2 \pi T; \beta\theta = \frac{1}{(4\pi)^2} \left( \frac{11}{3} Nc - \frac{2}{3} Nf \right);$$

$$t[T_] = 2 \operatorname{Log}\left[\frac{\mu}{\Delta ms}\right];$$

$$\alpha s[T_] = \frac{6\pi}{11 * t[T]} \quad (1); (*\text{For one loop}*)$$

$$\text{Eden}[c1_, c2_, T_] =$$

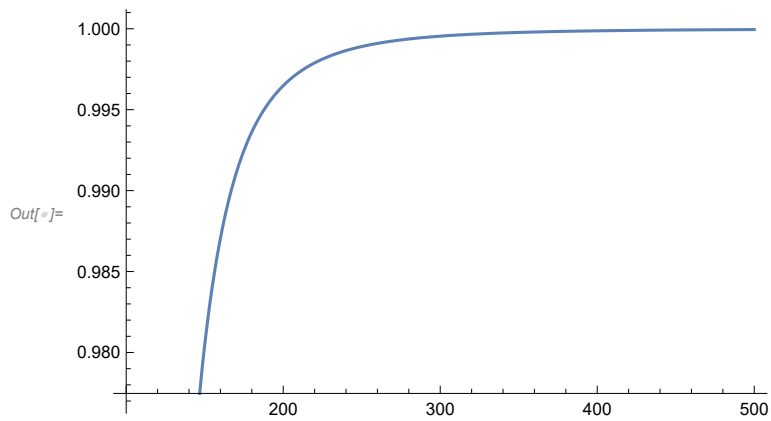
$$\begin{aligned} & \frac{1}{240} \left( \frac{120 c1^4}{4\pi \alpha s[T]} + \frac{10 c1^4 Nc}{\pi^2} - \frac{10 c1^4 Nf}{\pi^2} - 16 \pi^2 T^4 + 16 Nc^2 \pi^2 T^4 + 28 Nc Nf \pi^2 T^4 - 240 c1^4 \beta\theta + \right. \\ & 480 \pi c1^3 T \beta\theta \operatorname{JacobiCN}[c1 c2, -1] \operatorname{JacobiDN}[c1 c2, -1] \operatorname{JacobiSN}[c1 c2, -1] - 480 \pi c1^3 T \\ & \beta\theta \operatorname{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \operatorname{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \operatorname{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] - \\ & \frac{240 c1^4 \operatorname{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \operatorname{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{4\pi \alpha s[T]} - \\ & \frac{5 c1^4 Nc \operatorname{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \operatorname{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{\pi^2} + \\ & \frac{5 c1^4 Nf \operatorname{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \operatorname{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2}{\pi^2} + \frac{10 c1^4 Nf \operatorname{Log}[4]}{\pi^2} - \\ & \frac{20 c1^4 Nf \operatorname{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \operatorname{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \operatorname{Log}[4]}{\pi^2} - \\ & 240 c1^4 \beta\theta \operatorname{Log}\left[\frac{\mu}{4\pi T}\right] + 480 c1^4 \beta\theta \operatorname{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \\ & \operatorname{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \operatorname{Log}\left[\frac{\mu}{4\pi T}\right] - \frac{1}{4\pi^3 \alpha s[T]} 5 c1^4 \operatorname{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \\ & \left( \operatorname{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 + \operatorname{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \right) \\ & \left. \left( -48 \pi^2 + 4\pi \alpha s[T] (-Nc + Nf - 4 Nf \operatorname{Log}[4]) + 96 \times 4 \pi^3 \alpha s[T] \beta\theta \operatorname{Log}\left[\frac{\mu}{4\pi T}\right] \right) \right) / \text{Eideal} \end{aligned}$$

$$\begin{aligned}
\text{Out}[*]= & \frac{1}{160 \pi^2 T^4} \left( -\frac{90 c1^4}{\pi^2} + 160 \pi^2 T^4 + \right. \\
& \frac{180 \, i \, c1^3 T \text{JacobiCN}[c1 c2, -1] \text{JacobiDN}[c1 c2, -1] \text{JacobiSN}[c1 c2, -1]}{\pi^2} - \\
& \frac{180 \, i \, c1^3 T \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right] \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]}{\pi^2} \\
& + \frac{90 c1^4 \text{Log}[2]}{\pi^2} - \\
& \frac{180 c1^4 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{Log}[2]}{\pi^2} + \frac{20 c1^4 \text{Log}[4]}{\pi^2} - \\
& \frac{40 c1^4 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{Log}[4]}{\pi^2} + \frac{110 c1^4 \text{Log}\left[\frac{\pi T}{88}\right]}{\pi^2} - \\
& \frac{220 c1^4 \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \text{Log}\left[\frac{\pi T}{88}\right]}{\pi^2} - \frac{1}{12 \pi^4} 55 c1^4 \\
& \text{JacobiCN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \left( \text{JacobiDN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 + \text{JacobiSN}\left[c1 \left(c2 - \frac{i}{T}\right), -1\right]^2 \right) \\
& \left( -48 \pi^2 - \frac{432 \pi^2 \text{Log}[2]}{11 \text{Log}\left[\frac{\pi T}{88}\right]} - \frac{96 \pi^2 \text{Log}[4]}{11 \text{Log}\left[\frac{\pi T}{88}\right]} \right) \text{Log}\left[\frac{\pi T}{88}\right] \Big)
\end{aligned}$$

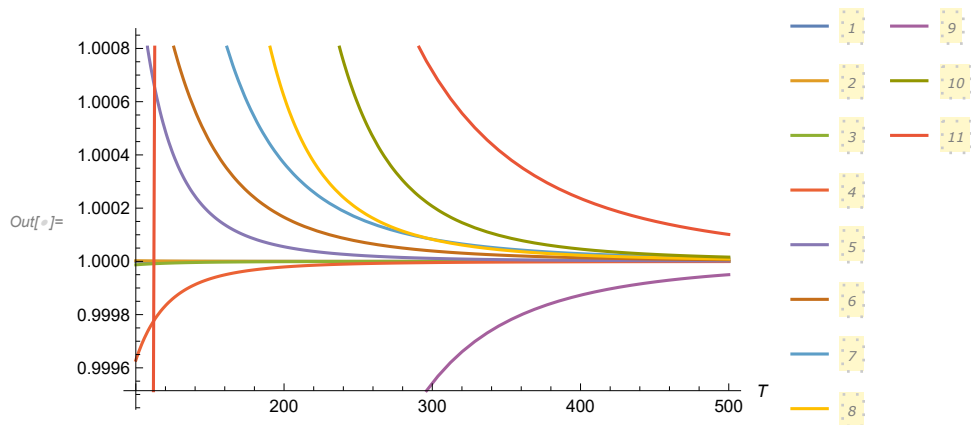
$\text{In}[*]=$  **Eden[1, 0, T]**

$$\begin{aligned}
\text{Out}[*]= & \frac{1}{160 \pi^2 T^4} \left( -\frac{90}{\pi^2} + 160 \pi^2 T^4 + \frac{180 \, i \, T \text{JacobiCN}\left[\frac{i}{T}, -1\right] \text{JacobiDN}\left[\frac{i}{T}, -1\right] \text{JacobiSN}\left[\frac{i}{T}, -1\right]}{\pi^2} + \right. \\
& \frac{90 \text{Log}[2]}{\pi^2} - \frac{180 \text{JacobiDN}\left[\frac{i}{T}, -1\right]^2 \text{JacobiSN}\left[\frac{i}{T}, -1\right]^2 \text{Log}[2]}{\pi^2} + \\
& \frac{20 \text{Log}[4]}{\pi^2} - \frac{40 \text{JacobiDN}\left[\frac{i}{T}, -1\right]^2 \text{JacobiSN}\left[\frac{i}{T}, -1\right]^2 \text{Log}[4]}{\pi^2} + \\
& \frac{110 \text{Log}\left[\frac{\pi T}{88}\right]}{\pi^2} - \frac{220 \text{JacobiDN}\left[\frac{i}{T}, -1\right]^2 \text{JacobiSN}\left[\frac{i}{T}, -1\right]^2 \text{Log}\left[\frac{\pi T}{88}\right]}{\pi^2} - \\
& \frac{1}{12 \pi^4} 55 \text{JacobiCN}\left[\frac{i}{T}, -1\right]^2 \left( \text{JacobiDN}\left[\frac{i}{T}, -1\right]^2 + \text{JacobiSN}\left[\frac{i}{T}, -1\right]^2 \right) \\
& \left( -48 \pi^2 - \frac{432 \pi^2 \text{Log}[2]}{11 \text{Log}\left[\frac{\pi T}{88}\right]} - \frac{96 \pi^2 \text{Log}[4]}{11 \text{Log}\left[\frac{\pi T}{88}\right]} \right) \text{Log}\left[\frac{\pi T}{88}\right] \Big)
\end{aligned}$$

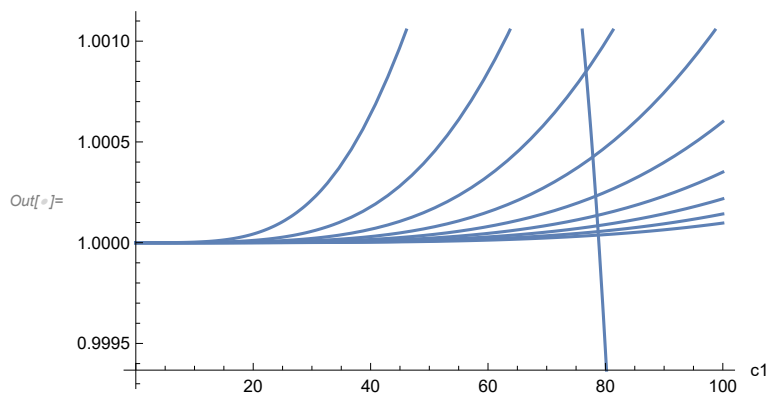
```
In[ ]:= Plot[Re[Eden[80, 1, T]], {T, 100, 500}]
```



```
In[ ]:= Plot[Evaluate[Table[Re[Eden[c1, 1, T]], {c1, 0, 100, 10}]],
{ T, 100, 500}, AxesLabel -> Automatic, PlotLegends -> Automatic]
```



```
In[ ]:= Plot[Table[Eden[c1, 0, T], {T, 100, 500, 50}],
{c1, 0, 100}, AxesLabel -> Automatic, PlotLegends -> "Expressions"]
```



Entropy density plots

$$\ln[\ast] := \Delta ms = 176; \mu = 2 \pi T; \beta\theta = \frac{1}{(4 \pi)^2} \left( \frac{11}{3} Nc - \frac{2}{3} Nf \right);$$

$$t[T\_] = 2 \operatorname{Log} \left[ \frac{\mu}{\Delta ms} \right];$$

$$\alpha s[T\_] = \frac{6 \pi}{11 \ast t[T]} (1); (*\text{For one loop}*)$$

$$Sden[c1\_ , c2\_ , T\_ ] = \left( \frac{4}{45} (-1 + Nc^2) \pi^2 T^3 + \frac{7}{45} Nc Nf \pi^2 T^3 - \frac{1}{T} \right.$$

$$c1^3 \beta\theta \left( c1 - 2 \pm T \operatorname{JacobiCN}[c1 c2, -1] \operatorname{JacobiDN}[c1 c2, -1] \operatorname{JacobiSN}[c1 c2, -1] + 2 \pm T \right. \\ \left. \operatorname{JacobiCN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] \operatorname{JacobiDN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] \operatorname{JacobiSN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] \right) +$$

$$\frac{1}{48 \pi^2} c1^3 (-Nc + Nf) \left( - \frac{c1 \operatorname{JacobiCN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2 \operatorname{JacobiDN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2}{T} - \right.$$

$$\pm \operatorname{JacobiCN}[c1 c2, -1] \operatorname{JacobiDN}[c1 c2, -1] \operatorname{JacobiSN}[c1 c2, -1] +$$

$$\pm \operatorname{JacobiCN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] \operatorname{JacobiDN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] \operatorname{JacobiSN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] -$$

$$\frac{c1 \operatorname{JacobiCN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2 \operatorname{JacobiSN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2}{T} +$$

$$\frac{c1 \operatorname{JacobiDN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2 \operatorname{JacobiSN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2}{T} \right) +$$

$$2 c1^3 \left( - \frac{2 c1 \operatorname{JacobiCN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2 \operatorname{JacobiDN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2}{T} - \right.$$

$$2 \pm \operatorname{JacobiCN}[c1 c2, -1] \operatorname{JacobiDN}[c1 c2, -1] \operatorname{JacobiSN}[c1 c2, -1] + 2 \pm$$

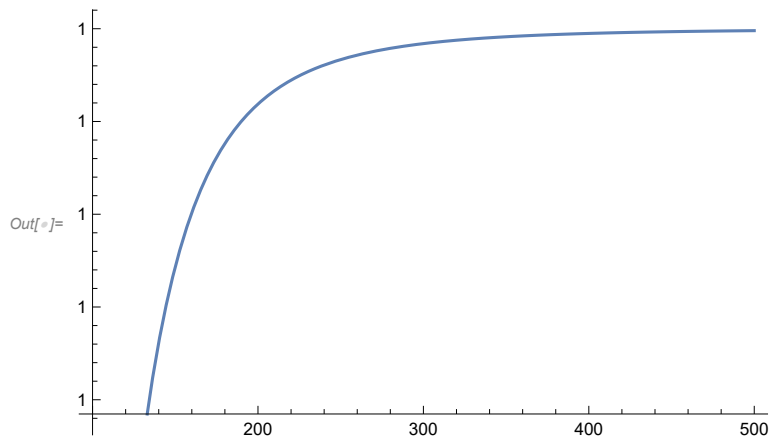
$$\operatorname{JacobiCN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] \operatorname{JacobiDN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] \operatorname{JacobiSN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right] -$$

$$\frac{2 c1 \operatorname{JacobiCN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2 \operatorname{JacobiSN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2}{T} +$$

$$\frac{2 c1 \operatorname{JacobiDN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2 \operatorname{JacobiSN} \left[ c1 \left( c2 - \frac{i}{T} \right), -1 \right]^2}{T} \right)$$

$$\left( - \frac{1}{4 (4 \pi) \alpha s[T]} - \frac{Nf \operatorname{Log}[4]}{48 \pi^2} + \frac{1}{2} \beta\theta \operatorname{Log} \left[ \frac{\mu}{4 \pi T} \right] \right) / \text{Sideal};$$

```
In[ ]:= Plot[Sden[1, 0, T], {T, 100, 500}]
```



HTL perturbation theory plots (Free energy) (for comparison only)

$\Delta m s = 176$ ;  $\Lambda = 2 \pi T$ ;

```
In[ ]:= Clear[Λ];
```

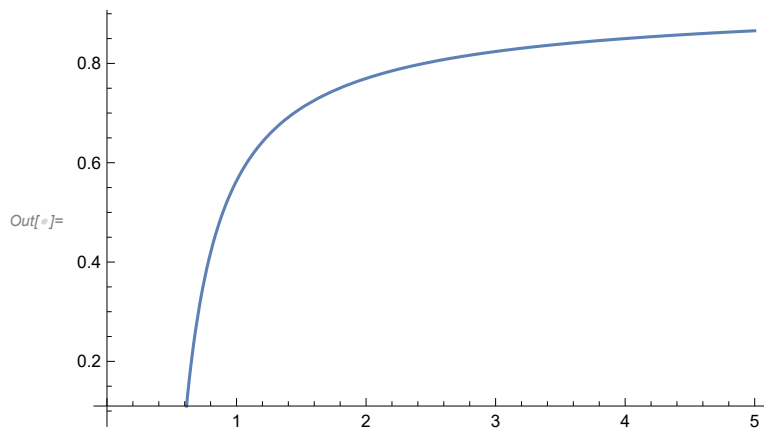
```
t[Λ_] = 2 Log[2 π * 1.03 Λ];
```

$$\alpha s[\Lambda] = \frac{4 \pi}{11 * t[\Lambda]} \left( 1 - \frac{102}{121} \frac{\text{Log}[t[\Lambda]]}{t[\Lambda]} \right);$$

```
Fn[Λ_] =
```

$$1 - \frac{15}{4 \pi} \alpha s[\Lambda] + \frac{30}{\pi^{3/2}} \alpha s[\Lambda]^{3/2} + \frac{135}{2} (\text{Log}[\alpha s[\Lambda] / \pi] + 3.51) \frac{1}{\pi^2} \alpha s[\Lambda]^2 - \frac{799.2}{\pi^{5/2}} \alpha s[\Lambda]^{5/2};$$

```
In[ ]:= Plot[Fn[T], {T, 0, 5}]
```




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## Special functions expansions at zero

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### For third coefficient of HK expansion:

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### Rough