Convergence of Bregman Alternating Direction Method with Multipliers for Nonconvex Composite Problems

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Abstract

The alternating direction method with multipliers (ADMM) has been one of most powerful and successful methods for solving various convex or nonconvex composite problems that arise in the fields of image & signal processing and machine learning. In convex settings, numerous convergence results have been established for ADMM as well as its varieties. However, there have been few studies on the convergence properties of ADMM under nonconvex frameworks, since the convergence analysis of nonconvex algorithm is generally very difficult. In this paper we study the Bregman modification of ADMM (BADMM), which includes the conventional ADMM as a special case and can significantly improve the performance of the algorithm. Under some assumptions, we show that the iterative sequence generated by BADMM converges to a stationary point of the associated augmented Lagrangian function. The obtained results underline the feasibility of ADMM in applications under nonconvex settings.

Index Terms

nonconvex regularization, nonconvex sparse minimization, alternating direction method, sub-analytic function, K-L inequality, Bregman distance.

I. INTRODUCTION

Many problems arising in the fields of signal & image processing and machine learning [5], [28] involve finding a minimizer of some composite objective functions. More specifically, such problems can be formulated as:

$$\min \ f(x) + g(y)$$
$$\text{s.t.} \ Ax = By, \tag{1}$$

where $A \in \mathbb{R}^{m \times n_1}$ and $B \in \mathbb{R}^{m \times n_2}$ are given matrices, $f : \mathbb{R}^{n_1} \to \mathbb{R}$ is usually a (quadratic, or logistic) loss function, and $g : \mathbb{R}^{n_2} \to \mathbb{R}$ is often a regularizer such as the ℓ_1 norm or $\ell_{1/2}$ quasi-norm.

Because of its separable structure, problem (1) can be efficiently solved by the alternating direction method with multipliers (ADMM), which decomposes the original joint minimization problem into two easily solved subproblems. The standard ADMM for problem (1) takes the form:

$$y^{k+1} = \arg \min_{y \in \mathbb{R}^{n_2}} L_\alpha(x^k, y, p^k) \tag{2}$$

$$x^{k+1} = \arg \min_{x \in \mathbb{R}^{n_1}} L_\alpha(x, y^{k+1}, p^k) \tag{3}$$

$$p^{k+1} = p^k + \alpha(Ax^{k+1} - By^{k+1}), \tag{4}$$

where α is a penalty parameter and

$$L_\alpha(x, y, p) := f(x) + g(y) + \langle p, Ax - By \rangle + \frac{\alpha}{2} \|Ax - By\|^2$$

is the associated augmented Lagrangian function with multiplier p. Generally speaking, ADMM is first minimized with respect to y for fixed values of p, x, then with respect to x with p, y fixed, and finally maximized with respect to p with x, y fixed. Updating the dual variable p^k in the above system is a trivial task, but this is not so simple for the primal variables x^k and y^k. Indeed in many cases, the x-subproblem (3) and y-subproblem (2) cannot easily be solved. Recently, the Bregman

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modification of ADMM (BADMM) has been adopted by several researchers to improve the performance of the conventional ADMM algorithm [16], [35], [36], [47]. BADMM takes the following iterative form: [1]

$$y^{k+1} = \arg \min_{y \in \mathbb{R}^{n_2}} L_\alpha(x^k, y, p^k) + \triangle_\psi(y, y^k) \tag{5}$$

$$x^{k+1} = \arg \min_{x \in \mathbb{R}^{n_1}} L_\alpha(x, y^{k+1}, p^k) + \triangle_\phi(x, x^k) \tag{6}$$

$$p^{k+1} = p^k + \alpha(Ax^{k+1} - By^{k+1}), \tag{7}$$

where $\triangle_\psi$ and $\triangle_\phi$ respectively denote the Bregman distance with respect to function ψ and ϕ. The difference between this algorithm and the standard ADMM is that the objective function in (2)-(3) is replaced by the sum of a Bregman distance function and the augmented Lagrangian function. Moreover, as shown in [26], [36], [47] and the following section, an appropriate choice of Bregman distance does indeed simplify the original subproblems.

ADMM was introduced in the early 1970s [17], [18], and its convergence properties for convex objective functions have been extensively studied. The convergence of ADMM was first established for strongly convex functions [17], [18], before being extended to general convex functions [13], [14]. It has been shown that ADMM converges at a sublinear rate of $\mathcal{O}(1/k)$ [20], [30], or $\mathcal{O}(1/k^2)$ for the accelerated version [19]; furthermore, a linear convergence rate was also shown under certain additional assumptions [12]. The convergence of BADMM for convex objective functions has also been examined with the Euclidean distance [10], Mahalanobis distance [47], and the general Bregman distance [47].

Recent studies on nonnegative matrix factorization, distributed matrix factorization, distributed clustering, sparse zero variance discriminant analysis, polynomial optimization, tensor decomposition, and matrix completion have led to growing interest in ADMM for nonconvex objective functions (see e.g. [21], [27], [38], [44], [46]). It has been shown that the nonconvex ADMM works extremely well for these particular examples.

However, because the convergence analysis of nonconvex algorithms is generally very difficult, there have been few studies on the convergence properties of ADMM under nonconvex frameworks. One major difficulty is that the Féjer monotonicity of iterative sequences does not hold in the absence of convexity. Very recently, [22] analyzed the convergence of ADMM for certain nonconvex consensus and sharing problems. They demonstrated that with A and B set to the identity matrices, ADMM converges to *the set of stationary solutions* as long as the penalty parameter α is sufficiently large. To show the convergence of ADMM to *a stationary point*, additional assumptions are required on the functions involved. For example, if f and g are both semi-algebraic, [26] proved that ADMM converges to a stationary point when B is the identity matrix. This result requires that function f is strongly convex or matrix A has full-column rank.

In this paper, we study the convergencev of BADMM under nonconvex frameworks. First, we extend the convergence of the BADMM from semi-algebraic functions to sub-analytic functions. In particular, this implies that BADMM is convergent for logistic sparse loss functions, which are not semi-algebraic. Second, we establish a global convergence theorem for cases when B has full-column rank. This allows us to choose $\phi \equiv 0$, which covers a recent result in [26]. We also study the case when B does not have full-column rank. In this instance, a suitable Bregman distance also leads to global BADMM convergence. This enhanced flexibility of BADMM enables its application to more general cases. More importantly, the main idea of our convergence analysis is different from that used in [26]. Instead of employing an augmented Lagrangian function at each iteration, we demonstrate global convergence using the descent property of an auxiliary function.

The paper is organized as follows. In Section 2, we recall the definitions of subdifferentials, Bregman distance, and Kurdyka-Łojasiewicz inequality. In Section 3, we establish the global convergence of BADMM to a critical point under certain assumptions. In Section 4, we conduct experimental studies to verify the convergence of BADMM.

II. PRELIMINARIES

In what follows, $\mathbb{R}^n$ will stand for the n-dimensional Euclidean space,

$$\langle x, y \rangle = x^\top y = \sum_{i=1}^n x_i y_i, \ \|x\| = \sqrt{\langle x, x \rangle},$$

where $x, y \in \mathbb{R}^n$ and $\top$ stands for the transpose operation.

A. Subdifferentials

Given a function $f : \mathbb{R}^n \to \mathbb{R}$ we denote by $\mathrm{dom} f$ the domain of f, namely $\mathrm{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. A function f is said to be proper if $\mathrm{dom} f \neq \emptyset$; lower semicontinuous at the point x_0 if

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

If f is lower semicontinuous at every point of its domain of definition, then it is simply called a lower semicontinuous function.

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[1] If the solution to the x or y-subproblem is not unique, then x^k or y^k should be regarded as a selection from their solution sets.

Definition II.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper lower semi-continuous function.*

(i) *Given $x \in \mathrm{dom} f$, the Fréchet subdifferential of f at x, written by $\widehat{\partial} f(x)$, is the set of all elements $u \in \mathbb{R}^n$ which satisfy*

$$\liminf_{\substack{y \neq x \\ y \to x}} \frac{f(y) - f(x) - \langle u, y - x \rangle}{\|x - y\|} \geq 0.$$

(ii) *The limiting subdifferential, or simply subdifferential, of f at x, written by $\partial f(x)$, is defined as*

$$\partial f(x) = \{ u \in \mathbb{R}^n : \exists x^k \to x, f(x^k) \to f(x),$$
$$u^k \in \widehat{\partial} f(x^k) \to u, k \to \infty \}.$$

(iii) *A critical point or stationary point of f is a point x_0 in the domain of f satisfying $0 \in \partial f(x_0)$.*

Definition II.2. *An element $z^* := (x^*, y^*, p^*)$ is called a critical point or stationary point of the Lagrangian function L_α if it satisfies:*

$$\begin{cases} -A^\top p^* = \nabla f(x^*) \\ B^\top p^* \in \partial g(y^*) \\ Ax^* = By^*. \end{cases} \tag{8}$$

Let us now collect some basic properties of the subdifferential (see [31]).

Proposition II.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ be proper lower semi-continuous functions.*

- *$\widehat{\partial} f(x) \subset \partial f(x)$ for each $x \in \mathbb{R}^n$. Moreover, the first set is closed and convex, while the second is closed, and not necessarily convex.*
- *Let (u^k, x^k) be sequences such that $x^k \to x, u^k \to u, f(x^k) \to f(x)$ and $u^k \in \partial f(x^k)$. Then by the definition of the subdifferential, we have $u \in \partial f(x)$.*
- *The Fermat's rule remains true: if $x_0 \in \mathbb{R}^n$ is a local minimizer of f, then x_0 is a critical point or stationary point of f, that is, $0 \in \partial f(x_0)$.*
- *If f is continuously differentiable function, then $\partial(f + g)(x) = \nabla f(x) + \partial g(x)$.*

A function f is said to be ℓ_f-*Lipschitz continuous* $(\ell_f \geq 0)$ if

$$\|f(x) - f(y)\| \leq \ell_f \|x - y\|$$

for any $x, y \in \mathrm{dom} f$; μ-*strongly convex* $(\mu > 0)$ if

$$f(y) \geq f(x) + \langle \xi(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2, \tag{9}$$

for any $x, y \in \mathrm{dom} f$ and $\xi(x) \in \partial f(x)$.

B. Kurdyka-Łojasiewicz inequality

The Kurdyka-Łojasiewicz (K-L) inequality plays an important role in our subsequent analysis. This inequality was first introduced by Łojasiewicz [32] for real real analytic functions, and then was extended by Kurdyka [24] to smooth functions whose graph belongs to an o-minimal structure, and recently was further extended to nonsmooth sub-analytic functions [3].

Definition II.3 (K-L inequality). *A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to satisfy the K-L inequality at x_0 if there exists $\eta > 0, \delta > 0, \varphi \in \mathscr{A}_\eta$, such that for all $x \in \mathcal{O}(x_0, \delta) \cap \{ x : f(x_0) < f(x) < f(x_0) + \eta \}$*

$$\varphi'(f(x) - f(x_0))\mathrm{dist}(0, \partial f(x)) \geq 1,$$

where $\mathrm{dist}(x_0, \partial f(x)) := \inf\{ \|x_0 - y\| : y \in \partial f(x) \}$, and $\mathscr{A}_\eta$ stand for the class of functions $\varphi : [0, \eta) \to \mathbb{R}^+$ such that (a) φ is continuous on $[0, \eta)$; (b) φ is smooth concave on $(0, \eta)$; (c) $\varphi(0) = 0, \varphi'(x) > 0, \forall x \in (0, \eta)$.

The following is an extension of the conventional K-L inequality [4].

Lemma II.2 (K-L inequality on compact subsets). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper lower semi-continuous function and let $\Omega \subseteq \mathbb{R}^n$ be a compact set. If f is a constant on Ω and f satisfies the K-L inequality at each point in Ω, then there exists $\eta > 0, \delta > 0, \varphi \in \mathscr{A}_\eta$, such that for all $x_0 \in \Omega$ and for all $x \in \{ x \in \mathbb{R}^n : \mathrm{dist}(x, \Omega) < \delta \} \cap \{ x \in \mathbb{R}^n : f(x_0) < f(x) < f(x_0) + \eta \}$,*

$$\varphi'(f(x) - f(x_0))\mathrm{dist}(0, \partial f(x)) \geq 1.$$

Typical functions satisfying the K-L inequality include strongly convex functions, real analytic functions, semi-algebraic functions and sub-analytic functions.

A subset $C \subset \mathbb{R}^n$ is said to be *semi-algebraic* if it can be written as

$$C = \bigcup_{j=1}^{r} \bigcap_{i=1}^{s} \{x \in \mathbb{R}^n : g_{i,j}(x) = 0, h_{i,j}(x) < 0\},$$

where $g_{i,j}, h_{i,j} : \mathbb{R}^n \to \mathbb{R}$ are real polynomial functions. Then a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *semi-algebraic* if its graph

$$G(f) := \{(x, y) \in \mathbb{R}^{n+1} : f(x) = y\}$$

is a semi-algebraic subset in $\mathbb{R}^{n+1}$. For example, the ℓ_q quasi norm $\|x\|_q := (\sum_i |x_i|^q)^{1/q}$ with $0 < q \leq 1$, the sup-norm $\|x\|_\infty := \max_i |x_i|$, the Euclidean norm $\|x\|$, $\|Ax - b\|_q^q$, $\|Ax - b\|$ and $\|Ax - b\|_\infty$ are all semi-algebraic functions [4], [39].

A real function on $\mathbb{R}$ is said to be *analytic* if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point. For a real function f on $\mathbb{R}^n$, it is said to be *analytic* if the function of one variable $g(t) := f(x+ty)$ is analytic for any $x, y \in \mathbb{R}^n$. It is readily seen that real polynomial functions such as quadratic functions $\|Ax - b\|^2$ are analytic. Moreover the ε-smoothed ℓ_q norm $\|x\|_{\varepsilon,q} := \sum_i (x_i^2 + \varepsilon)^{q/2}$ with $0 < q \leq 1$ and the logistic loss function $\log(1 + e^{-t})$ are also examples for real analytic functions [39].

A subset $C \subset \mathbb{R}^n$ is said to be *sub-analytic* if it can be written as

$$C = \bigcup_{j=1}^{r} \bigcap_{i=1}^{s} \{x \in \mathbb{R}^n : g_{i,j}(x) = 0, h_{i,j}(x) < 0\},$$

where $g_{i,j}, h_{i,j} : \mathbb{R}^n \to \mathbb{R}$ are real analytic functions. Then a function $f : \mathbb{R}^n \to \mathbb{R}$ is called *sub-analytic* if its graph $G(f)$ is a sub-analytic subset in $\mathbb{R}^{n+1}$. It is clear that both real analytic and semi-algebraic functions are sub-analytic. Generally speaking, the sum of of two sub-analytic functions is not necessarily sub-analytic. As shown in [3], [39], for two sub-analytic functions, if at least one function maps bounded sets to bounded sets, then their sum is also sub-analytic. In particular, the sum of a sub-analytic function and a analytic function is sub-analytic. Some sub-analytic functions that are widely used are as follows:

- $\|Ax - b\|^2 + \lambda \|y\|_q^q$;
- $\|Ax - b\|^2 + \lambda \sum_i (y_i^2 + \varepsilon)^{q/2}$;
- $\frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-c_i(a_i^\top x + b))) + \lambda \|y\|_q^q$;
- $\frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(-c_i(a_i^\top x + b))) + \lambda \sum_i (y_i^2 + \varepsilon)^{q/2}$.

C. Bregman distance

The Bregman distance, first introduced in 1967 [6], plays an important role in various iterative algorithms. As a generalization of squared Euclidean distance, the Bregman distance share many similar nice properties of the Euclidean distance. However, the Bregman distance is not a metric, since it does not satisfy the triangle inequality nor symmetry. For a convex differential function ϕ, the associated Bregman distance is defined as

$$\triangle_\phi(x, y) = \phi(x) - \phi(y) - \langle \nabla\phi(y), x - y \rangle.$$

In particular, if we let $\phi(x) := \|x\|^2$ in the above, then it is reduced to $\|x - y\|^2$, namely the classical Euclidean distance. Some nontrivial examples of Bregman distance include [2]:

- Itakura-Saito distance: $\sum_i x_i(\log x_i/y_i) - \sum_i(x_i - y_i)$;
- Kullback-Leibler divergence: $\sum_i x_i(\log x_i/y_i)$;
- Mahalanobis distance: $\|x - y\|_Q^2 = \langle Qx, x \rangle$ with Q a symmetric positive definite matrix.

Let us now collect some useful properties about Bregman distance.

Proposition II.3. *Let ϕ be a convex differential function and $\triangle_\phi(x, y)$ the associated Bregman distance.*

- *Non-negativity: $\triangle_\phi(x, y) \geq 0, \triangle_\phi(x, x) = 0$ for all x, y.*
- *Convexity: $\triangle_\phi(x, y)$ is convex in x, but not necessarily in y.*
- *Strong Convexity: If ϕ is δ-strongly convex, then $\triangle_\phi(x, y) \geq \frac{\delta}{2}\|x - y\|^2$ for all x, y.*

As shown in the below, an appropriate choice of Bregman distance will simplify the x and y-subproblems, which in turn improve the performance of the algorithm. For example, in y-subproblem (5), when taking $g(y) = \|y\|_{1/2}^{1/2}, \psi \equiv 0$, then the problem is minimizing function

$$\|y\|_{1/2}^{1/2} - \langle p^k, y \rangle + \frac{\alpha}{2}\|By - Ax^k\|^2.$$

In general finding a minimizer of this function is not a easy task. However, if we take $\psi = \frac{\mu}{2}\|y\|^2 - \frac{\alpha}{2}\|By - Ax^k - p^k/\alpha\|^2$ with $\mu > \alpha\|B\|^2$, then it is transformed into minimizing a problem of

$$\|y\|_{1/2}^{1/2} + \frac{\alpha}{2\mu}\|y - (y^k - \mu^{-1}B^\top(By^k - Ax^k - p^k/\alpha))\|^2.$$

Such a problem has a closed form solution (see [40]), and thus it can be very easily solved.

D. Basic assumption

We need the following basic assumptions on problem (1). A basic assumption to guarantee the convergence of the BADMM is that the matrix A has full-row rank. The only difference between Assumptions 1 and 2 is: one needs B having full column rank in Assumption 1, while in Assumption 2 one needs ψ being strongly convex. It worth noting that one can choose $\psi \equiv 0$ under Assumption 1, so that the BADMM includes the standard ADMM as a special case. It is also worth noting that the choice of $\psi \equiv 0$ is not available under Assumption 2.

Assumption 1. *Let* $\min(\mu_0, \mu_1) > 0$, $f : \mathbb{R}^{n_1} \to \mathbb{R}$ *a continuous differential function and* $g : \mathbb{R}^{n_2} \to \mathbb{R}$ *a proper lower semi-continuous functions. Assume that the following hold.*

(a) $AA^\top \succeq \mu_0 I$ *and* B *is injective;*
(b) *either* $L_\alpha(x, y, p)$ *with respect to* x *or* ϕ *is* μ_1 *strongly convex;*
(c) $f + g$ *is a sub-analytic function, and* $\nabla f, \nabla \phi$ *and* $\nabla \psi$ *are Lipshitz continuous.*

In condition (b), the strong convexity of ϕ is easily attained, for example $\phi = \frac{\mu_1}{2} \|x\|^2$, while the strong convexity of $L_\alpha(x, y, p)$ in x can be deduced from some standard assumptions, for example Neumann boundary condition in image processing [15]. Condition (b) will be used to guarantee the sufficient descent property of the augmented Lagrangian functions. More specifically, it implies

$$L_\alpha(x^{k+1}, y^{k+1}, p^k) \le L_\alpha(x^k, y^{k+1}, p^k) - \frac{\mu_1}{2} \|x^{k+1} - x^k\|^2, \tag{10}$$

where (x^k, y^k, p^k) is generated by algorithm (5)-(7). As a matter of fact, if $L_\alpha(x, y, p)$ with respect to x is μ_1-strongly convex, then $L_\alpha(x, y, p) + \triangle_\phi$ is also μ_1-strongly convex because $\triangle_\phi$ is convex from Proposition II.3. Thus the desired inequality will follow from the definition of strong convexity and Proposition II.3. If ϕ is strongly convex, then it follows again from Proposition II.3 that

$$\triangle_\phi(x^{k+1}, x^k) \ge \frac{\mu_1}{2} \|x - x^k\|^2,$$

which together with the definition of x^k yields the desired inequality.

The condition that $f + g$ is sub-analytic in (c) will be used to guarantee the auxiliary function constructed in the following section satisfying the K-L inequality. We notice that all functions mentioned in subsection II-B satisfy assumption (c). The Lipschitz continuity is a standard assumption for various algorithms, even in convex settings.

We also consider the BADMM under another set of conditions listed in Assumption 2 below. The only difference between Assumptions 1 and 2 is that one needs B having full column rank in Assumption 1, where in Assumption 2 we assume that ψ is strongly convex. It is worth noting that one can choose $\psi \equiv 0$ under Assumption 1, so that the BADMM includes the standard ADMM as a special case.

Assumption 2. *Let* $\min(\mu_0, \mu_1) > 0$, $f : \mathbb{R}^{n_1} \to \mathbb{R}$ *a continuous differential function and* $g : \mathbb{R}^{n_2} \to \mathbb{R}$ *a proper lower semi-continuous functions. Assume that the following hold.*

(a') $AA^\top \succeq \mu_0 I$ *and* ψ *is* μ_2-*strongly convex.*
(b) *either* $L_\alpha(x, y, p)$ *with respect to* x *or* ϕ *is* μ_1 *strongly convex.*
(c) $f + g$ *is a sub-analytic function, and* $\nabla f, \nabla \phi$ *and* $\nabla \psi$ *are Lipshitz continuous.*

III. CONVERGENCE ANALYSIS

In this section we prove the convergence of BADMM under two different assumptions. In both assumptions, the parameter α is chosen so that

$$\alpha > \frac{4((\ell_f + \ell_\phi)^2 + \ell_\phi^2)}{\mu_1 \mu_0},$$

where ℓ_f and ℓ_ϕ respectively stand for the Lipshitz constant of functions f and ϕ.

According to a recent work [1], the key point for convergence analysis of nonconvex algorithms is to show the descent property of the augmented Lagrangian function. This is however not easily attained since the dual variable is updated by maximizing the augmented Lagrangian function. As an alternative way, we construct an auxiliary function below, which helps us to deduce the global convergence of BADMM.

A. The case B is injective

Lemma III.1. *Let Assumption 1 be fulfilled. Then there exists* $\sigma_i > 0, i = 0, 1$ *such that*

$$\sigma_1 \|x^{k+1} - x^k\|^2 \le \hat{L}(x^k, y^k, p^k, x^{k-1}) - \hat{L}(x^{k+1}, y^{k+1}, p^{k+1}, x^k),$$

where $\hat{L}(x, y, p, \hat{x}) := L_\alpha(x, y, p) + \frac{\sigma_0}{2} \|x - \hat{x}\|^2$.

Proof: First we show that for each $k \in \mathbb{N}$

$$\|p^{k+1} - p^k\|^2 \leq \frac{2(\ell_f + \ell_\phi)^2}{\mu_0}\|x^{k+1} - x^k\|^2$$
$$+ \frac{2\ell_\phi^2}{\mu_0}\|x^k - x^{k-1}\|^2. \tag{11}$$

Indeed applying Fermat's rule to (6) yields

$$\nabla f(x^{k+1}) + A^\top p^k + \alpha A^\top(Ax^{k+1} - By^{k+1})$$
$$+ \nabla\phi(x^{k+1}) - \nabla\phi(x^k) = 0, \tag{12}$$

which together with (7) implies that

$$A^\top p^{k+1} = A^\top(p^k + \alpha(Ax^{k+1} - By^{k+1}))$$
$$= -\nabla f(x^{k+1}) + \nabla\phi(x^k) - \nabla\phi(x^{k+1}). \tag{13}$$

It then follows that

$$\|A^\top(p^{k+1} - p^k)\|^2$$
$$= \|\nabla f(x^{k+1}) - \nabla f(x^k) + (\nabla\phi(x^{k+1})$$
$$- \nabla\phi(x^k)) + (\nabla\phi(x^{k-1}) - \nabla\phi(x^k))\|^2$$
$$\leq \left(\|\nabla f(x^{k+1}) - \nabla f(x^k)\| + \|\nabla\phi(x^{k+1})$$
$$- \nabla\phi(x^k)\| + \|\nabla\phi(x^{k-1}) - \nabla\phi(x^k)\|\right)^2$$
$$\leq \left(\ell_f\|x^{k+1} - x^k\| + \ell_\phi\|x^k - x^{k+1}\|$$
$$+ \ell_\phi\|x^k - x^{k-1}\|\right)^2$$
$$\leq 2(\ell_f + \ell_\phi)^2\|x^{k+1} - x^k\|^2$$
$$+ 2\ell_\phi^2\|x^k - x^{k-1}\|^2.$$

Since matrix A is surjective, we have

$$\|A^\top(p^{k+1} - p^k)\|^2 = \langle A^\top(p^{k+1} - p^k), A^\top(p^{k+1} - p^k)\rangle$$
$$= \langle AA^\top(p^{k+1} - p^k), p^{k+1} - p^k\rangle$$
$$\geq \mu_0\|p^{k+1} - p^k\|^2,$$

which at once implies (11), as desired.

Next we claim that

$$L_\alpha(x^{k+1}, y^{k+1}, p^{k+1}) - L_\alpha(x^k, y^k, p^k)$$
$$\leq -\frac{\mu_1}{2}\|x^{k+1} - x^k\|^2 + \frac{1}{\alpha}\|p^{k+1} - p^k\|^2. \tag{14}$$

To see this, we deduce from (10) and (5)-(7) that

$$L_\alpha(x^k, y^{k+1}, p^k) \leq L_\alpha(x^k, y^k, p^k),$$
$$L_\alpha(x^{k+1}, y^{k+1}, p^k) \leq L_\alpha(x^k, y^{k+1}, p^k)$$
$$- \frac{\mu_1}{2}\|x^{k+1} - x^k\|^2,$$
$$L_\alpha(x^{k+1}, y^{k+1}, p^{k+1}) - L_\alpha(x^{k+1}, y^{k+1}, p^k)$$
$$= \langle p^{k+1} - p^k, Ax^{k+1} - By^{k+1}\rangle$$
$$= \frac{1}{\alpha}\|p^{k+1} - p^k\|^2.$$

Adding up the above formulas at once yields (14).

Finally it follows from (11) and (14) that

$$L_\alpha(x^{k+1}, y^{k+1}, p^{k+1}) - L_\alpha(x^k, y^k, p^k)$$
$$\leq \left(\frac{2(\ell_f + \ell_\phi)^2}{\alpha\mu_0} - \frac{\mu_1}{2}\right)\|x^{k+1} - x^k\|^2$$
$$+ \frac{2\ell_\phi^2}{\alpha\mu_0}\|x^k - x^{k-1}\|^2,$$

which is equivalent to

$$L_\alpha(x^{k+1}, y^{k+1}, p^{k+1}) + \frac{2\ell_\phi^2}{\alpha\mu_0}\|x^{k+1} - x^k\|^2$$

$$\leq L_\alpha(x^k, y^k, p^k) + \frac{2\ell_\phi^2}{\alpha\mu_0}\|x^k - x^{k-1}\|^2$$

$$- \left(\frac{\mu_1}{2} - \frac{2(\ell_f + \ell_\phi)^2}{\alpha\mu_0} - \frac{2\ell_\phi^2}{\alpha\mu_0}\right)\|x^k - x^{k+1}\|^2.$$

Let us now define

$$\sigma_0 = \frac{2\ell_\phi^2}{\alpha\mu_0}, \ \sigma_1 = \left(\frac{\mu_1}{2} - \frac{2(\ell_f + \ell_\phi)^2}{\alpha\mu_0} - \frac{2\ell_\phi^2}{\alpha\mu_0}\right).$$

Clearly both σ_i are positive and thus the desired inequality follows. ∎

Lemma III.2. *If the sequence $z^k := (x^k, y^k, p^k)$ is bounded, then*

$$\sum_{k=0}^\infty \|z^k - z^{k+1}\|^2 < \infty.$$

In particular the sequence $\|z^k - z^{k+1}\|$ is asymptotically regular, namely $\|z^k - z^{k+1}\| \to 0$ as $k \to \infty$. Moreover any cluster point of z^k is a stationary point of L_α.

Proof: Let $\hat{z}^k := (x^k, y^k, p^k, x^{k-1})$. Since $\hat{z}^k$ is clearly bounded, there exists a subsequence $\hat{z}^{k_j}$ so that it is convergent to some element $\hat{z}^*$. By our hypothesis the function $\hat{L}$ is lower semicontinuous, which leads to

$$\liminf_{j\to\infty} \hat{L}(\hat{z}^{k_j}) \geq \hat{L}(\hat{z}^*),$$

so that $\hat{L}(\hat{z}^{k_j})$ is bounded from below. By the previous lemma, $\hat{L}(\hat{z}^k)$ is nonincreasing, so that $\hat{L}(\hat{z}^{k_j})$ is convergent. Moreover $\hat{L}(\hat{z}^k)$ is also convergent and $\hat{L}(\hat{z}^k) \geq \hat{L}(\hat{z}^*)$ for each k.

Now fix $k \in \mathbb{N}$. It then follows from Lemma III.1 that

$$\sigma_1 \sum_{i=0}^k \|x^i - x^{i+1}\|^2$$

$$\leq \sum_{i=0}^k \hat{L}(\hat{z}^i) - \hat{L}(\hat{z}^{i+1})$$

$$= \hat{L}(\hat{z}^0) - \hat{L}(\hat{z}^{k+1})$$

$$\leq \hat{L}(\hat{z}^0) - \hat{L}(\hat{z}^*) < \infty.$$

Since k is chosen arbitrarily, we have $\sum_{k=0}^\infty \|x^k - x^{k+1}\|^2 < \infty$, which with (11) implies $\sum_{k=0}^\infty \|p^k - p^{k+1}\|^2 < \infty$. Since B is injective, it is readily seen that there exists $\mu_B > 0$ so that

$$\alpha^2 \mu_B \|y^k - y^{k+1}\|^2$$

$$\leq \|\alpha B(y^k - y^{k+1})\|^2$$

$$= \|(p^k - p^{k+1}) + (p^k - p^{k-1})$$

$$+ \alpha(Ax^{k+1} - Ax^k)\|^2$$

$$\leq 2(\|p^k - p^{k+1}\|^2 + \|p^k - p^{k-1}\|^2$$

$$+ \alpha^2 \|A\|^2 \|x^{k+1} - x^k\|^2). \tag{15}$$

Hence $\sum_{k=0}^\infty \|y^k - y^{k+1}\|^2 < \infty$, so that $\sum_{k=0}^\infty \|z^k - z^{k+1}\|^2 < \infty$; in particular $\|z^k - z^{k+1}\| \to 0$.

Let $z^* = (x^*, y^*, p^*)$ be any cluster point of z^k and let z^{k_j} be a subsequence of z^k converging to z^*. Since $\|z^k - z^{k+1}\|$ tends to zero as $k \to \infty$, z^{k_j} and z^{k_j+1} have the same limit point z^*. Since $\hat{L}(\hat{z}^k)$ is convergent, it is not hard to see that

$g(y^k)$ is also convergent. It then follows from (5)-(7) that

$$p^{k+1} = p^k + \alpha(Ax^{k+1} - By^{k+1}),$$
$$-\nabla f(x^{k+1}) = A^\top p^{k+1} + \nabla\phi(x^{k+1}) - \nabla\phi(x^k),$$
$$\partial g(y^{k+1}) \ni B^\top p^k + \alpha B^\top(Ax^k - By^{k+1})$$
$$+ \nabla\psi(y^k) - \nabla\psi(y^{k+1})$$
$$= B^\top p^{k+1} + \alpha B^\top(Ax^k - Ax^{k+1})$$
$$+ \nabla\psi(y^k) - \nabla\psi(y^{k+1}).$$

Letting $j \to \infty$ in the above formulas yields

$$A^\top p^* = -\nabla f(x^*), B^\top p^* \in \partial g(y^*), Ax^* = By^*,$$

which implies that z^* is a stationary point. ∎

Lemma III.3. *Let $\hat{z}^{k+1} := (x^{k+1}, y^{k+1}, p^{k+1}, x^k)$. Then there exists $\kappa > 0$ such that for each k*

$$\mathrm{dist}(0, \partial \hat{L}(\hat{z}^{k+1})) \le \kappa(\|x^k - x^{k+1}\| + \|x^k - x^{k-1}\|$$
$$+ \|x^{k-1} - x^{k-2}\|).$$

Proof: By the definitions of $\hat{L}$ and algorithm (5)-(7), we have

$$\partial \hat{L}_x(\hat{z}^{k+1}) = \nabla f(x^{k+1}) + A^\top p^{k+1} + \sigma_0(x^{k+1} - x^k)$$
$$+ \alpha A^\top(Ax^{k+1} - By^{k+1})$$
$$= \nabla\phi(x^k) - \nabla\phi(x^{k+1}) + \sigma_0(x^{k+1} - x^k)$$
$$+ \alpha A^\top(Ax^{k+1} - By^{k+1})$$
$$= \nabla\phi(x^k) - \nabla\phi(x^{k+1}) + \sigma_0(x^{k+1} - x^k)$$
$$+ A^\top(p^{k+1} - p^k),$$

where the last equality follows from (7). On the other hand, it follows from (5) that

$$0 \in \partial g(y^{k+1}) - B^\top p^k - \alpha B^\top(Ax^k - By^{k+1})$$
$$+ \nabla\psi(y^{k+1}) - \nabla\psi(y^k)$$
$$= \partial g(y^{k+1}) - B^\top p^{k+1} - \alpha B^\top(Ax^k - Ax^{k+1})$$
$$+ \nabla\psi(y^{k+1}) - \nabla\psi(y^k),$$

which implies

$$\partial \hat{L}_y(\hat{z}^{k+1})$$
$$= \partial g(y^{k+1}) - B^\top p^{k+1} + \alpha B^\top(By^{k+1} - Ax^{k+1})$$
$$\ni \nabla\psi(y^k) - \nabla\psi(y^{k+1}) + \alpha B^\top(Ax^k - Ax^{k+1})$$
$$- \alpha B^\top(Ax^{k+1} - By^{k+1})$$
$$= \nabla\psi(y^k) - \nabla\psi(y^{k+1}) + \alpha B^\top(Ax^k - Ax^{k+1})$$
$$+ B^\top(p^k - p^{k+1}).$$

Also it is clear that $\partial \hat{L}_{\hat{x}}(\hat{z}^{k+1}) = -\sigma_0(x^{k+1} - x^k)$ and

$$\partial \hat{L}_p(\hat{z}^{k+1}) = Ax^{k+1} - By^{k+1} = \frac{1}{\alpha}(p^{k+1} - p^k).$$

Consequently, there exists $\kappa_0 > 0$ so that

$$\mathrm{dist}(0, \partial \hat{L}(\hat{z}^{k+1})) \le \kappa_0(\|x^k - x^{k+1}\| + \|y^{k+1} - y^k\| + \|p^{k+1} - p^k\|).$$

On the other hand, it follows from (11) that

$$
\begin{aligned}
\|p^{k+1} - p^k\| &\leq \left[\frac{2(\ell_f + \ell_\phi)^2}{\mu_0}\|x^{k+1} - x^k\|^2 \right.\\
&\quad \left. + \frac{2\ell_\phi^2}{\mu_0}\|x^k - x^{k-1}\|^2\right]^{1/2}\\
&\leq \frac{\sqrt{2}(\ell_f + \ell_\phi)}{\sqrt{\mu_0}}\|x^{k+1} - x^k\|\\
&\quad + \frac{\sqrt{2}\ell_\phi}{\sqrt{\mu_0}}\|x^k - x^{k-1}\|\\
&\leq \frac{\sqrt{2}(\ell_f + \ell_\phi)}{\sqrt{\mu_0}}(\|x^{k+1} - x^k\| + \|x^k - x^{k-1}\|)\\
&= \kappa_1(\|x^{k+1} - x^k\| + \|x^k - x^{k-1}\|),
\end{aligned}
\tag{16}
$$

where we have defined $\kappa_1 := \sqrt{2}(\ell_f + \ell_\phi)/\sqrt{\mu_0}$. Furthermore, it follows from (15) that

$$
\begin{aligned}
\|y^k - y^{k+1}\| &\leq \frac{\sqrt{2}}{\alpha\sqrt{\mu_B}}(\|p^k - p^{k+1}\|^2 + \|p^k - p^{k-1}\|^2\\
&\quad + \alpha^2\|A\|^2\|x^{k+1} - x^k\|^2)^{1/2}\\
&\leq \frac{\sqrt{2}}{\alpha\sqrt{\mu_B}}(\|p^k - p^{k+1}\| + \|p^k - p^{k-1}\|\\
&\quad + \alpha\|A\|\|x^{k+1} - x^k\|)\\
&\leq \frac{\sqrt{2}}{\alpha\sqrt{\mu_B}}((\kappa_1 + \alpha\|A\|)\|x^k - x^{k+1}\|\\
&\quad + 2\kappa_1\|x^k - x^{k-1}\| + \kappa_1\|x^{k-1} - x^{k-2}\|)\\
&= \kappa_2(\|x^k - x^{k+1}\| + \|x^k - x^{k-1}\|\\
&\quad + \|x^{k-1} - x^{k-2}\|),
\end{aligned}
\tag{17}
$$

where we have defined $\kappa_2 := \sqrt{2}(2\kappa_1 + \alpha\|A\|)/\alpha\sqrt{\mu_B}$. Hence, with $\kappa := \kappa_0(\kappa_1 + \kappa_2)$, we immediately obtain the inequality as desired. ∎

Theorem III.4. *Let Assumption 1 be fulfilled. If $z^k := (x^k, y^k, p^k)$ is bounded, then*

$$
\sum_{k=0}^{\infty} \|z^k - z^{k+1}\| < \infty.
$$

Moreover the sequence (z^k) converges to a stationary point of problem (1).

Proof: Let $\hat{z}^{k+1} = (x^{k+1}, y^{k+1}, p^{k+1}, x^k)$ and let Ω denote the cluster point set of $\hat{z}^k$. By Lemma III.2, the sequence x^k is asymptotically regular, then the sequence x^k and x^{k+1} share the the same cluster points. Hence we can take $\hat{z}^* := (x^*, y^*, q^*, x^*) \in \Omega$ and let $\hat{z}^{k_j}$ be a subsequence of $\hat{z}^k$ converging to $\hat{z}^*$. By our hypothesis on g, we have that $\hat{L}(\hat{z}^{k_j}) \to \hat{L}(\hat{z}^*)$. Since by Lemma III.2 the sequence $\hat{L}(\hat{z}^k)$ is convergent, this implies that $\hat{L}(\hat{z}^k) \to \hat{L}(\hat{z}^*)$; hence the function $\hat{L}(\cdot)$ is a constant on Ω.

Let us now consider two possible cases on $\hat{L}(\hat{z}^k)$. First assume that there exists $k_0 \in \mathbb{N}$ such that $\hat{L}_{k_0} = \hat{L}(\hat{z}^*)$. Then we deduce from Lemma III.1 that for any $k > k_0$

$$
\begin{aligned}
\sigma_1\|x^{k+1} - x^k\|^2 &\leq \hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^{k+1})\\
&\leq \hat{L}(\hat{z}^{k_0}) - \hat{L}(\hat{z}^*) = 0,
\end{aligned}
$$

where we have used the fact that $\hat{L}(\hat{z}^k)$ is nonincreasing. This together with (16) and (17) implies that z^k is a constant sequence except for some finite terms, and thus it is a convergent sequence.

Let us now assume that $\hat{L}(\hat{z}^k) > \hat{L}(\hat{z}^*)$ for each $k \in \mathbb{N}$. By our hypothesis on f and g, it is clear that $\hat{L}(\cdot)$ is a sub-analytic function and thus satisfies the K-L inequality. Thus by Lemma II.2 there exists $\eta > 0, \delta > 0, \varphi \in \mathscr{A}_\eta$, such that for all $\hat{z}$ satisfying $\text{dist}(\hat{z}, \Omega) < \delta$ and $\hat{L}(\hat{z}^*) < \hat{L}(\hat{z}) < \hat{L}(\hat{z}^*) + \eta$, there holds the inequality

$$
\varphi'(\hat{L}(\hat{z}) - \hat{L}(\hat{z}^*))\text{dist}(0, \partial\hat{L}(\hat{z})) \geq 1.
$$

By the definition of Ω, we have that $\lim_k \text{dist}(\hat{z}^k, \Omega) = 0$. This together with the fact that $\hat{L}(\hat{z}^k) \to \hat{L}(\hat{z}^*)$ implies that there exists $k_1 \in \mathbb{N}$ such that $\text{dist}(\hat{z}^k, \Omega) < \delta$ and $\hat{L}(\hat{z}^k) < \hat{L}(\hat{z}^*) + \eta$ for all $k \geq k_1$.

In what follows let us fix $k > k_1$. It then follows that

$$\hat{z}^k \in \{\hat{z} : \text{dist}(\hat{z}, \Omega) < \delta)\} \cap \{\hat{z} : \hat{L}(\hat{z}^*) < \hat{L}(\hat{z}) < \hat{L}(\hat{z}^*) + \eta\}.$$

Hence $\text{dist}(0, \partial\hat{L}(\hat{z}^k))\varphi\prime(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) \geq 1$, which with Lemma III.3 yields

$$\frac{1}{\varphi\prime(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*))} \leq \kappa(\|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| + \|x^{k-2} - x^{k-3}\|).$$

By the concavity of φ, this further implies

$$\begin{aligned}
&\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^{k+1}) \\
&= (\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - (\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*)) \\
&\leq \frac{\varphi(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))}{\varphi\prime(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*))} \\
&\leq \kappa(\|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| + \|x^{k-2} - x^{k-3}\|) \\
&\quad \times [\varphi(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))].
\end{aligned}$$

Hence we deduce from Lemma III.1 that

$$\begin{aligned}
&\|x^{k+1} - x^k\|^2 \\
&\leq \frac{\kappa}{\sigma_1}(\|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| + \|x^{k-2} - x^{k-3}\|) \\
&\quad \times [\varphi(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&4\|x^k - x^{k+1}\| \\
&\leq 2(\|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| + \|x^{k-2} - x^{k-3}\|)^{1/2} \\
&\quad \times 2\sqrt{\frac{\kappa}{\sigma_1}}[\varphi(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))]^{1/2}.
\end{aligned}$$

On the other hand, using the inequality $2ab \leq a^2 + b^2$, we get

$$\begin{aligned}
&2(\|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| + \|x^{k-2} - x^{k-3}\|)^{1/2} \\
&\quad \times 2\sqrt{\frac{\kappa}{\sigma_1}}[\varphi(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))]^{1/2} \\
&\leq \|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| + \|x^{k-2} - x^{k-3}\| \\
&\quad + 4\frac{\kappa}{\sigma_1}[\varphi(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))],
\end{aligned}$$

so that

$$\begin{aligned}
&4\|x^k - x^{k+1}\| \\
&\leq \|x^k - x^{k-1}\| + \|x^{k-1} - x^{k-2}\| + \|x^{k-2} - x^{k-3}\| \\
&\quad + 4\frac{\kappa}{\sigma_1}[\varphi(\hat{L}(\hat{z}^k) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))].
\end{aligned}$$

Consequently we have

$$\begin{aligned}
&\sum_{i=k_1}^{k} 4\|x^i - x^{i+1}\| \\
&\leq \sum_{i=k_1}^{k} (\|x^i - x^{i-1}\| + \|x^{i-1} - x^{i-2}\| + \|x^{i-2} - x^{i-3}\|) \\
&\quad + 4\frac{\kappa}{\sigma_1} \sum_{i=k_1}^{k} [\varphi(\hat{L}(\hat{z}^i) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{i+1}) - \hat{L}(\hat{z}^*))],
\end{aligned}$$

which is equivalent to

$$\sum_{i=k_1}^{k} \|x^i - x^{i+1}\|$$

$$\leq \sum_{i=k_1}^{k} (\|x^i - x^{i-1}\| - \|x^i - x^{i+1}\|)$$

$$+ \sum_{i=k_1}^{k} (\|x^{i-1} - x^{i-2}\| - \|x^i - x^{i+1}\|)$$

$$+ \sum_{i=k_1}^{k} (\|x^{i-2} - x^{i-3}\| - \|x^i - x^{i+1}\|)$$

$$+ 4\frac{\kappa}{\sigma_1} \sum_{i=k_1}^{k} [\varphi(\hat{L}(\hat{z}^i) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{i+1}) - \hat{L}(\hat{z}^*))]$$

$$\leq 3\|x^{k_1} - x^{k_1-1}\| + 2\|x^{k_1-1} - x^{k_1-2}\| + \|x^{k_1-2} - x^{k_1-3}\|$$

$$+ 4\frac{\kappa}{\sigma_1} [\varphi(\hat{L}(\hat{z}^{k_1}) - \hat{L}(\hat{z}^*)) - \varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*))]$$

$$\leq 3\|x^{k_1} - x^{k_1-1}\| + 2\|x^{k_1-1} - x^{k_1-2}\| + \|x^{k_1-2} - x^{k_1-3}\|$$

$$+ 4\frac{\kappa}{\sigma_1} \varphi(\hat{L}(\hat{z}^{k_1}) - \hat{L}(\hat{z}^*)),$$

where the last inequality follows from the fact that $\varphi(\hat{L}(\hat{z}^{k+1}) - \hat{L}(\hat{z}^*)) \geq 0$. Since k is chosen arbitrarily, we deduce that $\sum_{k=0}^{\infty} \|x^k - x^{k+1}\| < \infty$. It follows from the previous lemma that

$$\|q^{k+1} - q^k\| \leq \kappa_1(\|x^{k+1} - x^k\| + \|x^k - x^{k-1}\|$$
$$+ \|y^{k+1} - y^k\|),$$
$$\|y^k - y^{k+1}\| \leq \kappa_2(\|x^k - x^{k+1}\| + \|x^k - x^{k-1}\|$$
$$+ \|x^{k-1} - x^{k-2}\|).$$

Hence $\sum_{k=0}^{\infty}(\|y^k - y^{k+1}\| + \|q^k - q^{k+1}\|) < \infty$. Moreover we note that

$$\|z^k - z^{k+1}\| = (\|x^k - x^{k+1}\|^2 + \|y^k - y^{k+1}\|^2$$
$$+ \|q^{k+1} - q^k\|^2)^{1/2}$$
$$\leq \|x^k - x^{k+1}\| + \|y^k - y^{k+1}\|$$
$$+ \|q^{k+1} - q^k\|,$$

so that we can conclude $\sum_{k=0}^{\infty} \|z^k - z^{k+1}\| < \infty$. Consequently (z^k) is a Cauchy sequence and thus is convergent, which together with Lemma III.2 completes the proof. ∎

Remark 1. We can deduce from (13) that p^k is bounded if x^k is. So in the above theorem, it suffices to assume that the primal variables x^k and y^k are bounded, which can be automatically fulfilled in many particular cases. For example, the boundedness of x^k or y^k can be obtained by assuming the coerciveness of f or g.

B. The case that B is not injective

Lemma III.5. Let Assumption 2 be fulfilled. For each $k \in \mathbb{N}$ there exists $\sigma_i > 0, i = 0, 1$ such that

$$\sigma_1(\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2)$$
$$\leq \tilde{L}(x^k, y^k, p^k, x^{k-1}) - \tilde{L}(x^{k+1}, y^{k+1}, p^{k+1}, x^k),$$

where $\tilde{L}(x, y, p, \tilde{x}) := L_\alpha(x, y, p) + \frac{\sigma_0}{2}\|x - \tilde{x}\|^2$.

Proof: Since ψ is strongly convex, we have

$$L_\alpha(x^k, y^{k+1}, p^k) \leq L_\alpha(x^k, y^k, p^k) - \Delta_\psi(y^{k+1}, y^k)$$
$$\leq L_\alpha(x^k, y^k, p^k) - \frac{\mu_2}{2}\|y^{k+1} - y^k\|^2,$$

which implies

$$L_\alpha(x^{k+1}, y^{k+1}, p^k) - L_\alpha(x^k, y^k, p^k)$$
$$\leq -\frac{\mu_1}{2}\|x^{k+1} - x^k\|^2 - \frac{\mu_2}{2}\|y^{k+1} - y^k\|^2.$$

Moreover we deduce form (11) and (7) that

$$L_\alpha(x^{k+1}, y^{k+1}, p^{k+1}) - L_\alpha(x^k, y^k, p^k)$$
$$= L_\alpha(x^{k+1}, y^{k+1}, p^{k+1}) - L_\alpha(x^{k+1}, y^{k+1}, p^k)$$
$$+ L_\alpha(x^{k+1}, y^{k+1}, p^k) - L_\alpha(x^k, y^k, p^k)$$
$$\leq -\frac{\mu_1}{2}\|x^{k+1} - x^k\|^2 - \frac{\mu_2}{2}\|y^{k+1} - y^k\|^2$$
$$+ \frac{1}{\alpha}\|p^{k+1} - p^k\|^2$$
$$\leq -\frac{\mu_1}{2}\|x^{k+1} - x^k\|^2 - \frac{\mu_2}{2}\|y^{k+1} - y^k\|^2$$
$$+ \frac{2(\ell_f + \ell_\phi)^2}{\alpha\mu_0}\|x^{k+1} - x^k\|^2$$
$$+ \frac{2\ell_\phi^2}{\alpha\mu_0}\|x^k - x^{k-1}\|^2,$$

which is equivalent to

$$L_\alpha(x^{k+1}, y^{k+1}, p^{k+1}) + \frac{2\ell_\phi^2}{\alpha\mu_0}\|x^{k+1} - x^k\|^2$$
$$\leq L_\alpha(x^k, y^k, p^k) + \frac{2\ell_\phi^2}{\alpha\mu_0}\|x^k - x^{k-1}\|^2$$
$$- \frac{\mu_2}{2}\|y^{k+1} - y^k\|^2$$
$$- \left(\frac{\mu_1}{2} - \frac{2(\ell_f + \ell_\phi)^2}{\alpha\mu_0} - \frac{2\ell_\phi^2}{\alpha\mu_0}\right)\|x^k - x^{k+1}\|^2.$$

Let us now define

$$\sigma_0 = \frac{4\ell_\phi^2}{\alpha\mu_0}, \quad \sigma_1 = \min\left(\frac{\mu_2}{2}, \frac{\mu_1}{2} - \frac{2(\ell_f + \ell_\phi)^2}{\alpha\mu_0} - \frac{2\ell_\phi^2}{\alpha\mu_0}\right).$$

Clearly both σ_i are positive and thus the desired inequality follows. ∎

Lemma III.6. *If the sequence $z^k := (x^k, y^k, p^k)$ is bounded, then*

$$\sum_{k=0}^{\infty}\|z^k - z^{k+1}\|^2 < \infty.$$

In particular the sequence $\|z^k - z^{k+1}\|$ is asymptotically regular, namely $\|z^k - z^{k+1}\| \to 0$ as $k \to \infty$. Moreover any cluster point of z^k is a stationary point of L_α.

Proof: Analogously, we can deduce as in Lemma III.2 that the sequence $\tilde{L}(\tilde{z}^k)$ is convergent and $\tilde{L}(\tilde{z}^k) \geq \tilde{L}(\tilde{z}^*)$ for each k, where $\tilde{z}^k := (x^k, y^k, p^k, x^{k-1})$ and $\tilde{L}$ is defined as in Lemma III.5. Now fix any $k \in \mathbb{N}$. It then follows from Lemma III.5 that

$$\sigma_1 \sum_{i=0}^{k}(\|x^i - x^{i+1}\|^2 + \|y^i - y^{i+1}\|^2)$$
$$\leq \sum_{i=0}^{k}(\tilde{L}(\tilde{z}^i) - \tilde{L}(\tilde{z}^{i+1})) = \tilde{L}(\tilde{z}^0) - \tilde{L}(\tilde{z}^{k+1})$$
$$\leq \tilde{L}(\tilde{z}^0) - \tilde{L}(\tilde{z}^*) < \infty.$$

Since k is chosen arbitrarily, we can deduce that $\sum_{k=0}^{\infty}(\|x^k - x^{k+1}\|^2 + \|y^k - y^{k+1}\|^2) < \infty$, which with (11) implies $\sum_k \|p^k - p^{k+1}\|^2 < \infty$, so that $\sum_{k=0}^{\infty}\|z^k - z^{k+1}\|^2 < \infty$; in particular $\|z^k - z^{k+1}\| \to 0$. It is clear that any cluster point of z^k is a stationary point of function L_α. ∎

The proof of the following lemma is similar to that of Lemma III.3, so we omit the details.

Lemma III.7. *Let $\tilde{z}^{k+1} = (x^{k+1}, y^{k+1}, p^{k+1}, x^k)$. Then for each k there exists $\kappa > 0$ such that*

$$\operatorname{dist}(0, \partial \tilde{L}(\tilde{z}^{k+1})) \leq \kappa(\|x^k - x^{k+1}\| + \|y^k - y^{k+1}\| + \|x^k - x^{k-1}\|).$$

Theorem III.8. *Assume that Assumption 2 is fulfilled. If the sequence $z^k := (x^k, y^k, q^k)$ is bounded, then*

$$\sum_{k=0}^{\infty} \|z^k - z^{k+1}\| < \infty.$$

In particular the sequence (z^k) converges to a stationary point of L_α.

Proof: Let $\tilde{z}^{k+1} = (x^{k+1}, y^{k+1}, q^{k+1}, x^k)$ and let Ω be the cluster point set of $\tilde{z}^k$. Similar to the proof of Theorem III.4, we can find a sufficient large k_1 such that for all $k > k_1$

$$\tilde{z}^k \in \{\tilde{z} : \operatorname{dist}(\tilde{z}, \Omega) < \delta\} \cap \{\tilde{z} : \tilde{L}(\tilde{z}^*) < \tilde{L}(\tilde{z}) < \tilde{L}(\tilde{z}^*) + \eta\}.$$

In what follows, let us fix $k > k_1$. Then the K-L inequality

$$\operatorname{dist}(0, \partial \tilde{L}(\tilde{z}^k)) \varphi\prime(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) \geq 1$$

together with Lemma III.7 implies

$$\frac{1}{\varphi\prime(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*))}$$
$$\leq \kappa(\|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|x^{k-2} - x^{k-1}\|),$$

so that the concavity of φ yields

$$\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^{k+1})$$
$$= (\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - (\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))$$
$$\leq \frac{\varphi(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))}{\varphi\prime(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*))}$$
$$\leq \kappa(\|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|x^{k-2} - x^{k-1}\|)$$
$$\times [\varphi(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))].$$

From Lemma III.5, this implies

$$\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2$$
$$\leq \frac{\kappa}{\sigma_1}(\|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|x^{k-2} - x^{k-1}\|)$$
$$\times [\varphi(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))],$$

which is equivalent to

$$3(\|x^k - x^{k+1}\| + \|y^k - y^{k+1}\|)$$
$$\leq 3\sqrt{2}(\|x^{k+1} - x^k\|^2 + \|y^{k+1} - y^k\|^2)^{1/2}$$
$$\leq 2(\|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|x^{k-2} - x^{k-1}\|)^{1/2}$$
$$\times \sqrt{\frac{9\kappa}{2\sigma_1}}[\varphi(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))]^{1/2}.$$

It is readily seen that

$$2(\|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|x^{k-2} - x^{k-1}\|)^{1/2}$$
$$\times \sqrt{\frac{9\kappa}{2\sigma_1}}[\varphi(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))]^{1/2}$$
$$\leq \|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|x^{k-2} - x^{k-1}\|$$
$$+ \frac{9\kappa}{2\sigma_1}[\varphi(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))],$$

so that

$$3(\|x^k - x^{k+1}\| + \|y^k - y^{k+1}\|)$$
$$\leq \|x^k - x^{k-1}\| + \|y^k - y^{k-1}\| + \|x^{k-2} - x^{k-1}\|$$
$$+ \frac{9\kappa}{2\sigma_1}[\varphi(\tilde{L}(\tilde{z}^k) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))].$$

Hence we have

$$\sum_{i=k_1}^k 3(\|x^i - x^{i+1}\| + \|y^i - y^{i+1}\|)$$
$$\leq \sum_{i=k_1}^k (\|x^i - x^{i-1}\| + \|y^i - y^{i-1}\| + \|x^{i-1} - x^{i-2}\|)$$
$$+ \frac{9\kappa}{2\sigma_1} \sum_{i=k_1}^k [\varphi(\tilde{L}(\tilde{z}^i) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{i+1}) - \tilde{L}(\tilde{z}^*))],$$

from which it follows that

$$\sum_{i=k_1}^k \|x^i - x^{i+1}\| + 2\sum_{i=k_1}^k \|y^i - y^{i+1}\|$$
$$\leq \sum_{i=k_1}^k (\|x^i - x^{i-1}\| - \|x^i - x^{i+1}\|)$$
$$+ \sum_{i=k_1}^k (\|x^{i-1} - x^{i-2}\| - \|x^i - x^{i+1}\|)$$
$$+ \sum_{i=k_1}^k (\|y^i - y^{i-1}\| - \|y^i - y^{i+1}\|)$$
$$+ \frac{9\kappa}{2\sigma_1} \sum_{i=k_1}^k [\varphi(\tilde{L}(\tilde{z}^i) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{i+1}) - \tilde{L}(\tilde{z}^*))]$$
$$= \|x^{k_1-1} - x^{k_1-2}\| + 2\|x^{k_1} - x^{k_1-1}\|$$
$$- \|x^{k-1} - x^{k_1-2}\| - 2\|x^k - x^{k+1}\|$$
$$+ \|y^{k_1-1} - y^{k_1}\| - \|y^k - y^{k+1}\|$$
$$+ \frac{9\kappa}{2\sigma_1}[\varphi(\tilde{L}(\tilde{z}^{k_1}) - \tilde{L}(\tilde{z}^*)) - \varphi(\tilde{L}(\tilde{z}^{k+1}) - \tilde{L}(\tilde{z}^*))]$$
$$\leq \|x^{k_1-1} - x^{k_1-2}\| + 2\|x^{k_1} - x^{k_1-1}\| + \|y^{k_1} - y^{k_1-1}\|$$
$$+ \frac{9\kappa}{2\sigma_1}\varphi(\tilde{L}(\tilde{z}^{k_1}) - \tilde{L}(\tilde{z}^*)),$$

where the last inequality follows from the fact that $\varphi(\tilde{L}(\tilde{z}^{k_1}) - \tilde{L}(\tilde{z}^*)) \geq 0$. Since k is chosen arbitrarily, we can deduce that $\sum_{k=0}^\infty (\|x^k - x^{k+1}\| + \|y^k - y^{k+1}\|) < \infty$, which together with (16) enables us to deduce that $\sum_{k=0}^\infty \|q^k - q^{k+1}\| < \infty$, and moreover $\sum_{k=0}^\infty \|z^k - z^{k+1}\| < \infty$. Consequently (z^k) is convergent, which together with Lemma III.5 completes the proof. ∎

IV. A DEMONSTRATION EXAMPLE

In compressed sensing, a fundamental problem is recovering an n-dimensional sparse signal x from a set of m incomplete measurements with $m << n$. It is possible as long as the number of nonzero elements of x is small enough. In such case one needs to find the sparsest solution of a linear system, which can be modeled as

$$\min_{x \in \mathbb{R}^n} \|x\|_0$$
$$\text{s.t. } Dx = b,$$

where $D \in \mathbb{R}^{m \times n}$ is the measurement matrix, $b \in \mathbb{R}^m$ is the observed data, and $\|x\|_0$ denotes the number of nonzero elements of x. In most cases, the sparsity is usually demonstrated under a linear transformation, for example in total variation denoising

[34]. This then requires to solve:

$$\min_{x \in \mathbb{R}^n} \|Ax\|_0$$
$$\text{s.t. } Dx = b,$$

or its regularization version:

$$\min_{x \in \mathbb{R}^n} \|Dx - b\|^2 + \lambda\|Ax\|_0, \tag{18}$$

where $\lambda > 0$ is a regularization parameter and $A \in \mathbb{R}^{(n-1) \times n}$ is the difference matrix, say, defined by

$$A_{ij} = \begin{cases} 1, & j = i + 1 \\ -1, & j = i \\ 0, & otherwise. \end{cases} \tag{19}$$

It is clear that the difference matrix has full-row rank.

In general, the above-mentioned problems are intractable because it is in fact a NP-hard problem. To overcome this difficulty, one may relax the ℓ_0 norm to the ℓ_1 norm as in (18), which then leads to a convex composite problem:

$$\min \|Dx - b\|^2 + \lambda\|y\|_1$$
$$\text{s.t. } Ax = y. \tag{20}$$

where $\|x\|_1 = \sum_i |x_i|$ stands for the ℓ_1 norm. Applying BADMM to problem (20) with $\phi(x) = \psi(x) = \mu\|x\|^2/2$ yields

$$y^{k+1} = H(Ax^k + p^k/\alpha; \lambda/\alpha)$$
$$x^{k+1} = (2D^\top D + \alpha A^\top A + \mu I)^{-1}w^{k+1} \tag{21}$$
$$p^{k+1} = p^k + \alpha(Ax^{k+1} - By^{k+1}),$$

where $w^{k+1} = \mu x^k + \alpha A^\top y^{k+1} + 2D^\top b - A^\top p^k$ and $S(\cdot; \mu)$ is the soft shrinkage operator.

Nevertheless, the ℓ_1 regularization has been shown to be suboptimal in many cases; in particular it cannot enforce further sparsity, since the ℓ_1 norm is a loose approximation of the ℓ_0 norm and often leads to an overpenalized problem. To overcome the drawback caused by the ℓ_1 regularization, an alternative way is to replace the ℓ_1 norm by the $\ell_{1/2}$ quasi norm in problem (18) (see e.g. [40]–[43]). This then leads to the following nonconvex composite problem:

$$\min \|Dx - b\|^2 + \lambda\|y\|_{1/2}^{1/2}$$
$$\text{s.t. } Ax = y. \tag{22}$$

Applying BADMM to problem (22) also with $\phi(x) = \psi(x) = \mu\|x\|^2/2$ yields

$$y^{k+1} = H(Ax^k + p^k/\alpha; 2\lambda/\alpha)$$
$$x^{k+1} = (2D^\top D + \alpha A^\top A + \mu I)^{-1}w^{k+1} \tag{23}$$
$$p^{k+1} = p^k + \alpha(Ax^{k+1} - By^{k+1}).$$

Here $w^{k+1} = \mu x^k + \alpha A^\top y^{k+1} + 2D^\top b - A^\top p^k$ and $H(\cdot; \mu)$ is the half shrinkage operator [40] defined as $H(x; \mu) = \{h_\mu(x_1), h_\mu(x_2) \cdots h_\mu(x_n)\}^\top$ with

$$h_\mu(x_i) = \begin{cases} \frac{2x_i}{3}\left(1 + \cos\frac{2}{3}(\pi - \varphi(|x_i|))\right), & |x_i| > \frac{\sqrt[3]{54}}{4}\mu^{2/3}; \\ 0, & h_\mu(x_i) = 0, \end{cases}$$

with $\varphi(x) = \arccos(\frac{\mu}{8}(\frac{|x_i|}{3})^{-3/2})$.

For simplicity, we denote algorithms (23) and (21) by HADMM and SADMM, respectively. We now conduct an experiment to verify convergence of the nonconvex BADMM, and reveal its advantages in sparsity-inducing and efficiency through comparing the performance of HADMM and SADMM. In the experiment, the difference matrix $A \in \mathbb{R}^{511 \times 512}$ was generated according to (19), and $D \in \mathbb{R}^{256 \times 512}$ was randomly generated with Gaussian $\mathcal{N}(0, 1/256)$ i.i.d. entries. We applied the HADMM and SADMM with the same parameters $\lambda = 0.015, \alpha = 10$ and $\mu_1 = \mu_2 = 10$.

The experimental results are shown in Figure 1, where the restoration accuracy is measured by means of the mean squared error

$$\text{MSE}(\|x^* - x^k\|) = \frac{1}{n}\|x^* - x^k\|,$$
$$\text{MSE}(\|y^* - y^k\|) = \frac{1}{n}\|y^* - y^k\|.$$

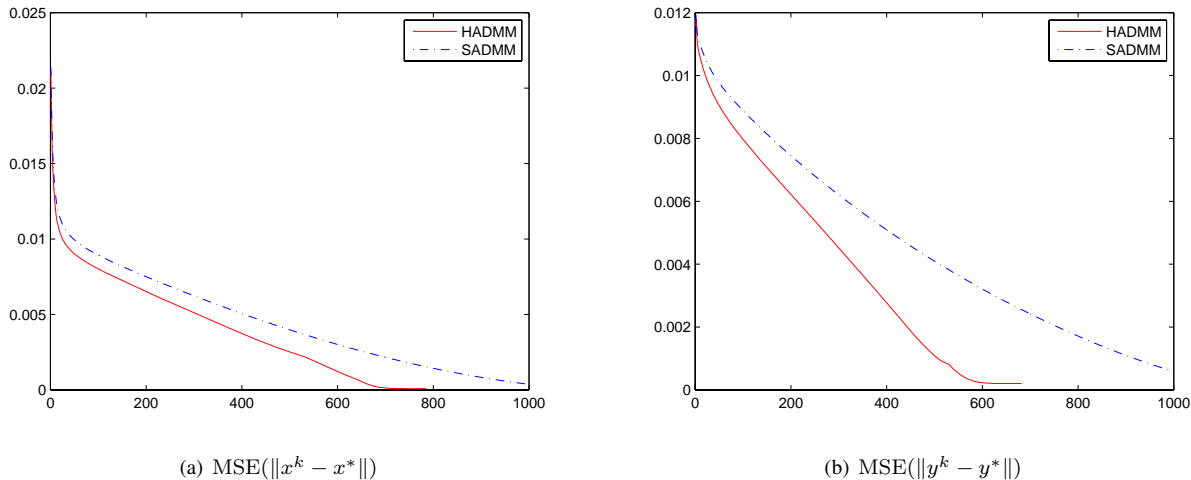(a) MSE($\|x^k - x^*\|$)   (b) MSE($\|y^k - y^*\|$)

Fig. 1.   Comparison the performance of HADMM and SADMM

Here (x^*, y^*) is the true solution of the problem. As shown in Figure 1, both sequences x^k and y^k were fairly near the true solution. i.e., the convergence is justified. It is readily seen that HADMM converges faster than SADMM does. Moreover, this difference is particularly notable for y^k. This supports in partial the advantage of the nonconvex model (22) over the convex model (20) for the considered problem.

V. Conclusion

In this paper, we conducted a convergence analysis on BADMM in the absence of convexity. We have shown that under certain conditions, the BADMM algorithm can converge to a stationary point for sub-analytic functions. More importantly, our analysis is based on the sufficient descent property of the auxiliary function, instead of the augmented Lagrangian function.

It is worth noting that the order for updating the primal variables x^k and y^k plays a key role in our convergence analysis. If we change the order, namely first update x^k and then y^k, this may lead to a difficulty to derive an relation between x^k and p^k. Thus how to establish the convergence results under this case is our next subject to study.

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