University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

<u>Solutions #6</u>

1. (a) Since $f,\, g : \mathbb{R}^n \to \mathbb{R}^n$ are inverse functions, we have $f \circ g =$ identity and $g \circ f =$ identity (identity : $\mathbb{R}^n \to \mathbb{R}^n$ maps each $\boldsymbol{x} \in \mathbb{R}^n$ to itself). Now $D(g \circ f) = D(\text{identity}) = I_n$ (the $n \times n$ identity matrix). Also, by the Chain Rule, we have $D(g \circ f) = \big(Dg(f)\big)\big(Df\big)$. Taking determinants, we have $\big[\det(Dg)\big]\big[\det(Df)\big] = \det(I_n) = 1$. Hence, neither $\det(Dg)$ nor $\det(Df)$ can be zero. We also have $\big(Df(g)\big)\big(Dg\big) = I_n$, so by definition, $Dg = \big(Df\big)^{-1}$.

   (b) We are given that $f,\, g : \mathbb{R}^3 \to \mathbb{R}^3$ are inverse functions. Therefore, since $Df = \begin{pmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 1 & 3 \end{pmatrix}$, $Dg = (\,Df\,)^{-1} = \left(\dfrac{1}{\det Df}\right)\left(\text{cofactor matrix of } Df\right)^T =$
$$\frac{1}{6}\begin{pmatrix} 3 & -6 & -1 \\ 6 & -6 & -4 \\ -3 & 6 & 3 \end{pmatrix}^T = \frac{1}{6}\begin{pmatrix} 3 & 6 & -3 \\ -6 & -6 & 6 \\ -1 & -4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ -1 & -1 & 1 \\ -\frac{1}{6} & -\frac{2}{3} & \frac{1}{2} \end{pmatrix}.$$

2. (a) We are given $A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 9 \end{pmatrix}$. To find the eigenvalues we must solve
$$0 = \det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 0 & -3 \\ 0 & 2-\lambda & 0 \\ -3 & 0 & 9-\lambda \end{pmatrix} = (2-\lambda)\,\det\begin{pmatrix} 1-\lambda & -3 \\ -3 & 9-\lambda \end{pmatrix} =$$
$(2-\lambda)\,\big[9 - 10\lambda + \lambda^2 - 9\big] = (2-\lambda)\,(\lambda^2 - 10\lambda) = (2-\lambda)\,(\lambda - 10)\,(\lambda)$. Hence the eigenvalues are $\lambda = 0$, $\lambda = 2$ and $\lambda = 10$.

   (b) We now find the associated eigenvectors:

   $\underline{\lambda = 0}$. We solve $\begin{pmatrix} 1-0 & 0 & -3 \\ 0 & 2-0 & 0 \\ -3 & 0 & 9-0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -3 & 0 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ giving $\left\{\begin{array}{rrrcl} x & & -3z & = & 0 \\ & 2y & & = & 0 \\ -3x & & +9z & = & 0 \end{array}\right.$ $\Longrightarrow$ $y = 0$ and $x = 3z$. Hence an eigenvector is of the form $(3z,\, 0,\, z)$, $z \in \mathbb{R} - \{0\}$. An eigenvector of unit length is $\left(\dfrac{3}{\sqrt{10}},\, 0,\, \dfrac{1}{\sqrt{10}}\right)$.

$\underline{\lambda = 2}$. We solve $\begin{pmatrix} 1-2 & 0 & -3 \\ 0 & 2-2 & 0 \\ -3 & 0 & 9-2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ giving $\left\{ \begin{array}{rrcl} -x & -z & = & 0 \\ -3x & +7z & = & 0 \end{array} \right. \implies z = 0$, $x = 0$ and $y \in \mathbb{R}$. Hence an eigenvector is of the form $(0,\, y,\, 0)$, $y \in \mathbb{R} - \{0\}$. An eigenvector of unit length is $\big(0,\, 1,\, 0\big)$.

$\underline{\lambda = 10}$. We solve $\begin{pmatrix} 1-10 & 0 & -3 \\ 0 & 2-10 & 0 \\ -3 & 0 & 9-10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $= \begin{pmatrix} -9 & 0 & -3 \\ 0 & -8 & 0 \\ -3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ giving $\left\{ \begin{array}{rrrcl} -9x & & -3z & = & 0 \\ & -8y & & = & 0 \\ -3x & & -z & = & 0 \end{array} \right. \implies$ $y = 0$ and $z = -3x$. Hence an eigenvector is of the form $(x,\, 0,\, -3x)$, $x \in \mathbb{R} - \{0\}$. An eigenvector of unit length is $\left(\dfrac{1}{\sqrt{10}},\, 0,\, \dfrac{-3}{\sqrt{10}} \right)$.

(c) The matrix B is $B = \begin{pmatrix} \dfrac{3}{\sqrt{10}} & 0 & \dfrac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \dfrac{1}{\sqrt{10}} & 0 & -\dfrac{3}{\sqrt{10}} \end{pmatrix}$. To show B is orthogonal it is sufficient to show $B\,B^T = I_3$. $B\,B^T = \begin{pmatrix} \dfrac{3}{\sqrt{10}} & 0 & \dfrac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \dfrac{1}{\sqrt{10}} & 0 & -\dfrac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} \dfrac{3}{\sqrt{10}} & 0 & \dfrac{1}{\sqrt{10}} \\ 0 & 1 & 0 \\ \dfrac{1}{\sqrt{10}} & 0 & -\dfrac{3}{\sqrt{10}} \end{pmatrix} =$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Hence B is orthogonal.

3. We will first show that, if λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n, for all $n \in \mathbb{Z}^+$. Let $\boldsymbol{v}$ be an eigenvector associated to the eigenvalue λ of A; i.e., $A\,\boldsymbol{v} = \lambda\,\boldsymbol{v}$. Now $A^2\,\boldsymbol{v} = A(A\boldsymbol{v}) = A(\lambda\boldsymbol{v}) = \lambda A\boldsymbol{v} = \lambda^2\boldsymbol{v}$. Assume that $A^{k-1}\boldsymbol{v} = \lambda^{k-1}\boldsymbol{v}$, then $A^k\boldsymbol{v} = A(A^{k-1}\boldsymbol{v}) = A(\lambda^{k-1}\boldsymbol{v}) = \lambda^{k-1}A\boldsymbol{v} = \lambda^k\boldsymbol{v}$. By induction, $A^k\boldsymbol{v} = \lambda^k\boldsymbol{v}$, for $k = 1, 2, \cdots$.

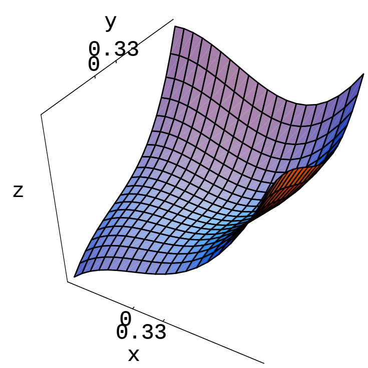
   Now $(A^3 + 2\,A^2 - A - 5\,I)\boldsymbol{v} = A^3\,\boldsymbol{v} + 2\,A^2, \boldsymbol{v} - A, \boldsymbol{v} - 5\,I, \boldsymbol{v} = \lambda^3\,\boldsymbol{v} + 2\,\lambda^2\,\boldsymbol{v} - \lambda\,\boldsymbol{v} - 5\,\boldsymbol{v} = (\lambda^3 + 2\,\lambda^2 - \lambda - 5)\boldsymbol{v}$. So, if -1, 1 and 2 the eigenvalues of A, $(-1)^3 + 2(-1)^2 - (-1) - 5 = -3$, $1^3 + 2(1)^2 - 1 - 5 = -3$ and $2^3 + 2(2)^2 - 2 - 5 = 9$ are the eigenvalues of $A^3 + 2\,A^2 - A - 5\,I$. Since the determinant of a matrix is the product of the eigenvalues, the determinant of $A^3 + 2\,A^2 - A - 5\,I$ is $(-3)(-3)(9) = 81$.

4. Let ℓ, w and h be the length, width and height of the box, let V be the volume and let S be the surface area. We are given $\dfrac{d\ell}{dt} = 1$, $\dfrac{dw}{dt} = 0.5$ and $\dfrac{dh}{dt} = -1$.

   (a) The volume is given by $V = \ell\, w\, h$ so, using the product rule, we have $\dfrac{dV}{dt} = w\, h\,\dfrac{d\ell}{dt} + \ell\, h\,\dfrac{dw}{dt} + \ell\, w\,\dfrac{dh}{dt}$. When $\ell = 5$, $w = 4$ and $h = 3$ we have $\dfrac{dV}{dt} = (4)(4)(1) + (5)(4)(0.5) + (5)(4)(-1) = 6\,\mathrm{m}^3/\mathrm{sec}$.

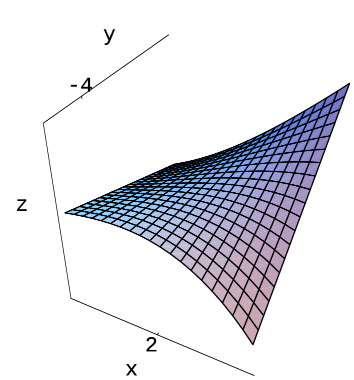
   (b) The surface area is given by $S = \ell\, w + 2\,\ell\, h + 2\, w\, h$ so $\dfrac{dS}{dt} = (w + 2\,h)\,\dfrac{d\ell}{dt} + (\ell + 2\,h)\,\dfrac{dw}{dt} + 2\,(\ell + w)\,\dfrac{dh}{dt}$. When $\ell = 4$, $w = 4$ and $h = 4$ we have $\dfrac{dS}{dt} = (4+2(4))(1) + (5+2(4))(0.5) + 2(5+4)(-1) = 12 + \dfrac{13}{2} - 18 = 0.5\,\mathrm{m}^2/\mathrm{sec}$.

5. (a) $f(x,y) = x^3 - xy + y^3$. Since f is a polynomial, all critical points will occur when $\nabla f = (3x^2 - y, -x + 3y^2) = (0,0)$. The first component gives $y = 3x^2$, then the second gives $0 = -x + 3(3x^2)^2 = -x + 27x^4 = x(-1 + 27x^3) \implies x = 0,\ x = \dfrac{1}{3}$. Hence the critical points are $(0,0)$ and $\left(\dfrac{1}{3}, \dfrac{1}{3}\right)$. To classify we compute $H\,f = \begin{pmatrix} 6x & -1 \\ -1 & 6y \end{pmatrix}$.

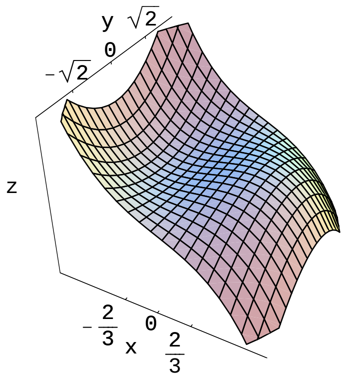


   Now $H\,f(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. Since $\det H f(0,0) = -1$, this must be a saddle point. (Note that, a 2×2 symmetric matrix can have a negative determinant only if one eigenvalue is positive and one is negative.)
   $H\,f\left(\dfrac{1}{3}, \dfrac{1}{3}\right) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Since $\det A_1 = 2$ and $\det A_2 = \det H\,f\left(\dfrac{1}{3}, \dfrac{1}{3}\right) = 3$, this is a local minimum.

   (b) $f(x,y) = x^3y + 12x^2 - 8y$. Since f is a polynomial, all critical points will occur when $\nabla f = (3x^2y + 24x,\ x^3 - 8) = (0,0)$. The second component $\implies x = 2$ and the first becomes $0 = 3(2)^2y + 24(2) \implies 12y = -48 \implies y = -4$. THere is a single critical point, $(2,-4)$. To classify we compute $H\,f = \begin{pmatrix} 6xy + 24 & 3x^2 \\ 3x^2 & 0 \end{pmatrix}$. Now $H\,f(2,-4) = \begin{pmatrix} 24 & 12 \\ 12 & 0 \end{pmatrix}$ so $\det H\,f(2,-4) = -144 < 0$. Hence this critical point yields a saddle point.

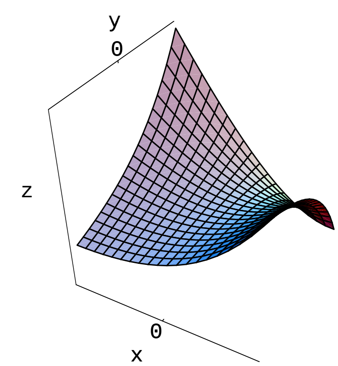


(c) $f(x,y) = 4x - 3x^3 - 2xy^2$. Computing the partials we have $f_x = 4 - 9x^2 - 2y^2 = 0$ and $f_y = -4xy = 0$. The second $\Longrightarrow$ $x = 0$ or $y = 0$.
If $x = 0$, the first becomes $4 - 2y^2 = 0 \Longrightarrow y = \pm\sqrt{2}$. If $y = 0$, the first becomes $4 - 9x^2 = 0 \Longrightarrow x = \pm\dfrac{2}{3}$. There are 4 critical points: $(0, \sqrt{2})$, $(0, -\sqrt{2})$, $\left(\dfrac{2}{3}, 0\right)$, $\left(\dfrac{-2}{3}, 0\right)$. To classify we compute $H\,f = \begin{pmatrix} -18x & -4y \\ -4y & -4x \end{pmatrix}$.



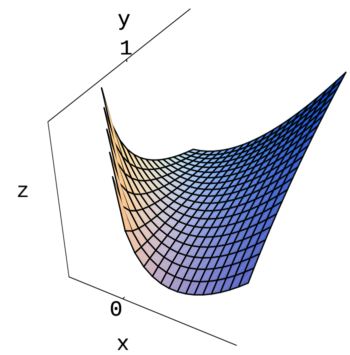
Now $H\,f(0, \sqrt{2}) = \begin{pmatrix} 0 & -4\sqrt{2} \\ -4\sqrt{2} & 0 \end{pmatrix}$. Since $\det \begin{pmatrix} 0 & -4\sqrt{2} \\ -4\sqrt{2} & 0 \end{pmatrix} = -32 < 0$, we have a saddle.
$H\,f(0, -\sqrt{2}) = \begin{pmatrix} 0 & 4\sqrt{2} \\ 4\sqrt{2} & 0 \end{pmatrix}$. Since $\det \begin{pmatrix} 0 & 4\sqrt{2} \\ 4\sqrt{2} & 0 \end{pmatrix} = -32 < 0$, we have a saddle.
$H\,f\left(\dfrac{2}{3}, 0\right) = \begin{pmatrix} -12 & 0 \\ 0 & -\frac{8}{3} \end{pmatrix}$. Since $\det A_1 = -12 < 0$ and $\det A_2 = 32 > 0$, this is a local maximum.
$H\,f\left(\dfrac{-2}{3}, 0\right) = \begin{pmatrix} 12 & 0 \\ 0 & \frac{8}{3} \end{pmatrix}$. Since $\det A_1 = 12 > 0$ and $\det A_2 = 32 > 0$, this is a local minimum.

(d) $f(x,y) = e^x - x\,e^y$. Computing the partials we have $f_x = e^x - e^y = 0$, $f_y = -x\,e^y = 0$. Since $e^y \neq 0$, the second $\Longrightarrow$ $x = 0$. The first now becomes $e^0 - e^y = 0 \Longrightarrow e^y = 1 \Longrightarrow y = 0$. The only critical point is $(0,0)$. To classify the critical point we compute $H\,f = \begin{pmatrix} e^x & -e^y \\ -e^y & -x\,e^y \end{pmatrix}$. Now $H\,f(0,0) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$ and $\det \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = -1 < 0$. Hence this is a saddle point.



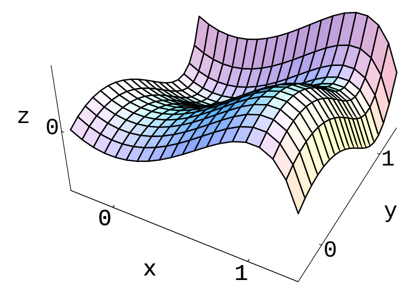
(e) $f(x,y) = x\,\ln(x+y)$. Computing the partials we have $f_x = \ln(x+y) + \dfrac{x}{x+y} = 0$, $f_y = \dfrac{x}{x+y} = 0$. The second equation gives $x = 0$ so the first becomes

$\ln y = 0 \implies y = 1$. Thus $(0,1)$ is a critical point. Because f_x and f_y are defined through out the domain, it is the only critical point. To classify we compute $H\,f = \begin{pmatrix} \dfrac{x+2y}{(x+y)^2} & \dfrac{y}{(x+y)^2} \\ \dfrac{y}{(x+y)^2} & -\dfrac{x}{(x+y)^2} \end{pmatrix}$. Now $H\,f(0,1) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\det \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = -1$, $(0,1)$ is a saddle point.



(f) $f(x,y) = \displaystyle\int_x^y (e^{t^2} - e^t)\,dt$. Using FTC, we have $\begin{cases} \dfrac{\partial f}{\partial x} = e^x - e^{x^2} = 0 \\ \dfrac{\partial f}{\partial y} = e^{y^2} - e^y = 0 \end{cases}$,

which we rewrite as $\begin{cases} e^x\,(1 - e^{x^2-x}) = 0 \\ e^y\,(e^{y^2-y} - 1) = 0 \end{cases}$. Hence, $\nabla f = \mathbf{0}$ if $\begin{cases} x^2 - x = 0 \\ y^2 - y = 0 \end{cases} \implies x = 0$ or 1 and $y = 0$ or 1. Hence there are four critical points: $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. After computing second partials we have $H\,f = \begin{pmatrix} e^x - 2xe^{x^2} & 0 \\ 0 & 2ye^{y^2} - e^y \end{pmatrix}$. Now $H\,f(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \implies$ saddle,

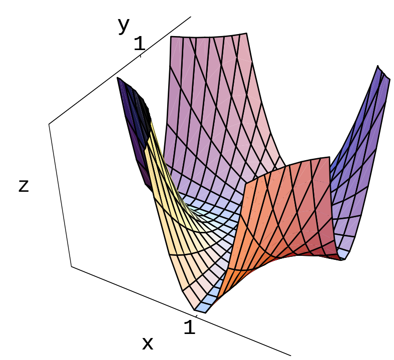


$H\,f(0,1) = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \implies$ local minimum, $H\,f(1,0) = \begin{pmatrix} -e & 0 \\ 0 & -1 \end{pmatrix} \implies$ local maximum, and $H\,f(1,1) = \begin{pmatrix} -e & 0 \\ 0 & e \end{pmatrix} \implies$ saddle.

(g) $f(x,y,z) = x^3 + xz^2 - 3x^2 + y^2 + 2z^2$. We first compute the partials: $\dfrac{\partial f}{\partial x} = 3x^2 + z^2 - 6x$, $\dfrac{\partial f}{\partial y} = 2y$ and $\dfrac{\partial f}{\partial z} = 2xz + 4z = 2z(x+2)$, so $\nabla f = \mathbf{0}$ if $\begin{cases} 3x^2 + z^2 - 6x = 0 \\ 2y = 0 \\ 2xz + 4z = 0 \end{cases}$. From second we have $y = 0$ and from the third we have either $z = 0$ or $x = -2$. If $z = 0$ the first becomes $0 = 3x^2 - 6x = 3x(x-2)$ so we have $x = 0$ or $x = 2$. If $x = -2$ then $z^2 = -24$ so we have no real solutions. Therefore the critical points are $(0,0,0)$ and $(2,0,0)$. The Hessian matrix is $H\,f = \begin{pmatrix} 6x-6 & 0 & 2z \\ 0 & 2 & 0 \\ 2z & 0 & 2x+4 \end{pmatrix}$. $H\,f(0,0,0) = \begin{pmatrix} -6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ so we have the sequence $-\ +\ +$ giving a saddle point. $H\,f(2,0,0) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$ so we have the sequence $+\ +\ +$ giving a local minimum.

(h) $f(x,y,z) = x^2y + y^2z + z^2 - 2x$. We first compute the partials: $\dfrac{\partial f}{\partial x} = 2xy - 2$, $\dfrac{\partial f}{\partial y} = x^2 + 2yz$ and $\dfrac{\partial f}{\partial z} = y^2 + 2z$, so $\nabla f = \mathbf{0}$ if $\left\{ \begin{array}{rcl} 2xy - 2 & = & 0 \\ x^2 + 2yz & = & 0 \\ y^2 + 2z & = & 0 \end{array} \right.$. The third equation gives $z = -\dfrac{y^2}{2}$ so the second $\implies$ $y^3 = x^2$ so the first $\implies$ $y = 1 \implies z = -\dfrac{1}{2}$. Since $xy = 1$ we have $x = 1$. The only critical point is $\left(1,\, 1,\, -\dfrac{1}{2}\right)$. The Hessian matrix is $Hf = \begin{pmatrix} 2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 2 \end{pmatrix}$ and $Hf(1,1,-\frac{1}{2}) = \begin{pmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix}$. Evaluating the chain of determinants we have $\det A_1 = 2 > 0$, $\det A_2 = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} = -6 < 0$ and $\det A_3 = \det \begin{pmatrix} 2 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix} = -20 < 0$. Since we have the pattern $+\ -\ -$, this is a saddle point.

6. (a) $f(x,y) = (x-1)^2\,(y-1)^2$. Since f is a polynomial, all critical points will occur where $\nabla f = \big(2(x-1)(y-1)^2,\, 2(x-1)^2(y-1)\big) = (0,\,0)$. This occurs if $x = 1$ or $y = 1$; that is, every point along the line $x = 1$ or the line $y = 1$ is a critical point. Now $H\,f = \begin{pmatrix} 2(y-1)^2 & 4(x-1)(y-1) \\ 4(x-1)(y-1) & 2(x-1)^2 \end{pmatrix}$ so $\det H\,f(1,y) = \det \begin{pmatrix} 2(y-1)^2 & 0 \\ 0 & 0 \end{pmatrix} = 0$ and $\det H\,f(x,1) = \det \begin{pmatrix} 0 & 0 \\ 0 & 2(x-1)^2 \end{pmatrix} = 0 \implies$ the test fails.



Since $f(x,y) = 0$ for every point along the line $x = 1$ or the line $y = 1$ and $f(x,y) = (x-1)^2(y-1)^2 > 0$ for all other points, each critical point yields a local (and global) minimum.

(b) $f(x,y,z) = (x-1)^2\,(y-1)^2\,(z-1)^2$. Since f is a polynomial, all critical points will occur where $\nabla f = \big(2(x-1)(y-1)^2(z-1)^2,\, 2(x-1)^2(y-1)(z-1)^2,\, 2(x-1)^2(y-1)^2(z-1)\big) = (0,0,0)$. This occurs if $x = 1$ or $y = 1$ or $z = 1$; that is, every point on the plane $x = 1$ or on the plane $y = 1$ or on the plane $z = 1$ is a critical point. Now

$$H\,f = \begin{pmatrix} 2(y-1)^2(z-1)^2 & 4(x-1)(y-1)(z-1)^2 & 4(x-1)(y-1)^2(z-1) \\ 4(x-1)(y-1)(z-1)^2 & 2(x-1)^2(z-1)^2 & 4(x-1)^2(y-1)(z-1) \\ 4(x-1)(y-1)^2(z-1) & 4(x-1)^2(y-1)(z-1) & 2(x-1)^2(y-1)^2 \end{pmatrix}$$

so $\det H\,f(1,y,z) = \det \begin{pmatrix} (y-1)^2(z-1)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$, $\det H\,f(x,1,y) =$

$\det \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2(x-1)^2(z-1)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$ and $\det H\,f(x,y,1) =$

$\det \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2(x-1)^2(y-1)^2 \end{pmatrix} = 0 \implies$ the test fails.

Since $f(x,y,z) = 0$ for every point on the plane $x = 1$, on the plane $y = 1$ or on the plane $z = 1$ and $f(x,y,z) = (x-1)^2\,(y-1)^2\,(z-1)^2 > 0$ for all other points, each critical point yields a local (and global) minimum.