University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #8

1. (a) $f(x,y) = x^2 + 2y^2$ subject to the constraint $4x - 6y = 25$. We define $h(x,y,\lambda) = x^2 + 2y^2 - \lambda(4x - 6y - 25)$. The critical points of h will give the constrained critical points of f. Now $h_x = 2x - 4\lambda,\ h_y = 4y - 6\lambda,\ h_\lambda = -(4x - 6y - 25)$. Put $h_x = h_y = h_\lambda = 0$. The first equation gives $\lambda = \dfrac{x}{2}$ ($x = 0 \implies y = 0$ and $(0,0)$ does not satisfy the constraint). Then substitution into the second gives $y = -\dfrac{3\,x}{4}$. Finally substiting into the third gives $4x + \dfrac{9\,x}{2} = 25 \implies 17x = 50 \implies x = \dfrac{50}{17} \implies y = -\dfrac{75}{34}$. There is a single constrained critical point $\left(\dfrac{50}{17}, -\dfrac{75}{34}\right)$.

   (b) $f(x,y) = 2x + 3y$ subject to the constraint $x^2 + y^2 = 4$. We define $h(x,y,\lambda) = 2x + 3y - \lambda(x^2 + y^2 - 4)$. The critical points of h will give the constrained critical points of f. Now $h_x = 2 = 2\lambda x,\ h_y = 3 - 2\lambda y,\ h_\lambda = -(x^2 + y^2 - 4)$. Put $h_x = h_y = h_\lambda = 0$. The first two equations $\implies x,\ y \neq 0$ and $\lambda = \dfrac{1}{x} = \dfrac{3}{2\,y} \implies y = \dfrac{3\,x}{2}$. Substituting into the third we have $x^2 + \dfrac{9\,x^2}{4} = 4 \implies 13x^2 = 16 \implies x = \pm\dfrac{4}{\sqrt{13}}$ giving two constrained critical points: $\left(\dfrac{4}{\sqrt{13}}, \dfrac{6}{\sqrt{13}}\right)$ and $\left(-\dfrac{4}{\sqrt{13}}, -\dfrac{6}{\sqrt{13}}\right)$.

   (c) $f(x,y) = 4x^2 + 9y^2$ subject to the constraint $xy = 4$. We define $h(x,y,\lambda) = 4x^2 + 9y^2 - \lambda(xy - 4)$. The critical points of h will give the constrained critical points of f. Now $h_x = 8x - \lambda y,\ h_y = 18y - \lambda x,\ h_\lambda = -(xy - 4)$. Put $h_x = h_y = h_\lambda = 0$. Solving the first two equations for λ gives $\lambda = \dfrac{8\,x}{y} = \dfrac{18\,y}{x} \implies y = \pm\dfrac{2\,x}{3}$ ($x = 0 \implies y = 0$ and $(0,0)$ does not satisfy the constraint) $\implies y = \dfrac{2\,x}{3}$ (the constraint requires x and y to have the same sign). Now substituting into the third gives $(x)\left(\dfrac{2\,x}{3}\right) = 4 \implies x^2 = 6 \implies x = \pm\sqrt{6}$. Hence there are 2 constrained critical points, $\left(\sqrt{6},\ 2\sqrt{\dfrac{2}{3}}\right)$ and $\left(-\sqrt{6},\ -2\sqrt{\dfrac{2}{3}}\right)$.

   (d) $f(x,y) = x\,y$ subject to the constraint $4x^2 + 9y^2 = 32$. We define $h(x,y,\lambda) = x\,y - \lambda(4x^2 + 9y^2 - 32)$. The critical points of h will give the constrained critical points of f. Now $h_x = y - \lambda 8x,\ h_y = x - \lambda 18y,\ h_\lambda = -(4x^2 + 9y^2 - 32)$. Put

$h_x = h_y = h_\lambda = 0$. Solving the first two for λ we have $\lambda = \dfrac{y}{8\,x} = \dfrac{x}{18\,y} \implies 18y^2 = 8x^2 \implies y = \pm\dfrac{2\,x}{3}$ ($x = 0 \implies y = 0$ and $(0,0)$ does not satisfy the constraint). Now substituting into the third gives $8x^2 = 32 \implies x = \pm 2$. Hence there are four constrainted critical points, $\left(2, \dfrac{4}{3}\right)$, $\left(2, -\dfrac{4}{3}\right)$, $\left(-2, \dfrac{4}{3}\right)$, $\left(-2, -\dfrac{4}{3}\right)$.

(e) $f(x, y) = x^2\,y^4$ subject to the constraint $x^2 + 2y^2 = 6$. We define $h(x, y, \lambda) = x^2\,y^4 - \lambda(x^2 + 2y^2 - 6)$. The critical points of h will give the constrained critical points of f. Now $h_x = 2xy^4 - \lambda 2x$, $h_y = 4x^2y^3 - \lambda 4y$, $h_\lambda = -(x^2 + 2y^2 - 6)$. Put $h_x = h_y = h_\lambda = 0$. First assume x, $y \neq 0$ and solve the first two equations for λ giving $\lambda = \dfrac{x\,y^4}{x} = \dfrac{x^2y^3}{y} \implies y^4 = x^2y^2 \implies y^2 = x^2 \implies y = \pm x$. Substituting into the third gives $3x^2 = 6 \implies x^2 = 2 \implies x = \pm\sqrt{2}$. Hence we have four constrainted critical points, $(\sqrt{2}, \sqrt{2})$, $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$.
If $x = 0, y \neq 0$ or $y = 0, x \neq 0$, we also have constrained critical points $(0, \pm\sqrt{3})$ and $(\pm\sqrt{6}, 0)$.

(f) $f(x, y, z) = xy + xz + yz - xyz$ subject to the constraints $x + y + z = 1$, $x, y, z \geq 0$. We define $h(x, y, z, \lambda) = xy + xz + yz - xyz - \lambda(x + y + z - 1)$. The critical points of h will give the constrained critical points of f. Now $h_x = y + z - yz - \lambda$, $h_y = x + z - xz - \lambda$, $h_z = x + y - xy - \lambda$, $h_\lambda = -(x + y + z - 1)$. Put $h_x = h_y = h_z = h_\lambda = 0$. Solving the first three for λ and equating two pair we have $x - xz = y - yz$ and $x - xy = z - yz$. Rewriting the first of these, we have $x - y - z(x - y) = 0 \implies (x - y)(1 - z) = 0 \implies x = y$ or $z = 1$. Rewriting the second, we have $x - z - y(x - z) = 0 \implies (x - z)(1 - y) = 0 \implies x = z$ or $y = 1$.
If $x = y$, $x = z$ the fourth gives $3z = 1 \implies z = \dfrac{1}{3} \implies x = y = \dfrac{1}{3}$.
If $x = y$, $y = 1$ the fourth gives $2 + z = 1 \implies z = -1$ which is not possible.
If $z = 1$, $x = z$ the fourth gives $y + 2 = 1 \implies y = -1$ which is also not possible.
If $z = 1$, $y = 1$ the fourth gives $x + 2 = 1 \implies x = -1$ which is also not possible.
Hence there is a single constrained critical point, $\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$.

(g) $f(x, y, z) = x$ subject to the constraints $x + \dfrac{y}{2} + \dfrac{z}{3} = 0$ and $x^2 + y^2 + z^2 = 1$. We define $h(x, y, z, \lambda, \mu) = x - \lambda\left(x + \dfrac{y}{2} + \dfrac{z}{3}\right) - \mu(x^2 + y^2 + z^2 - 1)$. The critical points of h will give the constrained critical points of f. Now $h_x = 1 - \lambda - 2\mu x$, $h_y = -\dfrac{\lambda}{2} - 2\mu y$, $h_z = -\dfrac{\lambda}{3} - 2\mu z$, $h_\lambda = -\left(x + \dfrac{y}{2} + \dfrac{z}{3}\right)$. $h_\mu = -(x^2 + y^2 + z^2 - 1)$. Put $h_x = h_y = h_z = h_\lambda = h_\mu = 0$. We first note that $\mu \neq 0$ or the first three equations are inconsistent. Solving equations 2 and 3 for λ and equating we have $-4\mu y = -6\mu z \implies y = \dfrac{3\,z}{2}$. Substituting in the fourth we have $x = -\dfrac{13\,z}{12}$. Now substituting into the last equation we have $\left(-\dfrac{13\,z}{12}\right)^2 + \left(\dfrac{3\,z}{2}\right)^2 + z^2 = 1 \implies$

$\dfrac{637\,x^2}{144} = 1 \implies z = \pm\dfrac{12}{7\sqrt{13}}$. Hence there are two constrained critical points, $\left(-\dfrac{\sqrt{13}}{7}, \dfrac{18}{7\sqrt{13}}, \dfrac{12}{7\sqrt{13}}\right)$ and $\left(\dfrac{\sqrt{13}}{7}, -\dfrac{18}{7\sqrt{13}}, -\dfrac{12}{7\sqrt{13}}\right)$.

2. We need to find the maximum of $T(x,y,z) = 4x^2 + xy + 15$ subject to the constraint $g(x,y,z) = x^2 + y^2 + z^2 - 1 = 0$. Since the constraint, a sphere, is compact and the function T is continuous throughout $\mathbb{R}^3$, a maximum is assured by the Extreme Value Theorem (EVT). We will use Lagrange Multipliers so we define a new function $h(x,y,z,\lambda) = 4x^2 + xy + 15 - \lambda\left(x^2 + y^2 + z^2 - 1\right)$. The maximum point(s) will be found among the critical points of h. Now $h_x = 8x - 2x\lambda = 0$, $h_y = z - 2y\lambda = 0$, $h_z = y - 2z\lambda = 0$ and $h_\lambda = -(x^2 + y^2 + z^2 - 1) = 0$. The first equation gives $x(4-\lambda) = 0 \implies x = 0$ or $\lambda = 4$.

   If $x = 0$, the second and third equations give $y^2 = z^2$ and substituting into the fourth gives $2z^2 = 1 \implies z = \pm\dfrac{1}{\sqrt{2}} \implies y = \pm\dfrac{1}{\sqrt{2}}$. This produces 4 constrained critical points: $\left(0, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$, $\left(0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$, $\left(0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ and $\left(0, -\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$.

   If $\lambda = 4$, the second and third equations imply $y = z = 0$. Now from the fourth equation we have $x^2 = 1 \implies x = \pm 1$ giving 2 more constrained critical points: $(1,0,0)$ and $(-1,0,0)$.

   Evaluating we have $f\left(0, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = 15.5$, $f\left(0, \dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right) = 14.5$, $f\left(0, -\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = 14.5$, $f\left(0, -\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right) = 15.5$, $f\big(1,0,0\big) = 19$ and $f\big(-1,0,0\big) = 19$. The warmest points on the surface of the probe are $\big(1,0,0\big)$ and $\big(-1,0,0\big)$ where the temperature is 19 degrees Kelvin.

3. To find the maximum of $f(s,t) = \dfrac{1}{2}\big(s - 32\big)t^2$ subject to the constraint $g(s,t) = s^2t - 10{,}000 = 0$, we will use Lagrange multipliers. We define a new function $h(s,t,\lambda) = \dfrac{1}{2}\big(s - 32\big) - \lambda\left(s^2t - 10{,}000\right)$. Now $h_s = \dfrac{t^2}{2} - 2st\lambda = 0$, $h_t = (s-32)t - s^2\lambda = 0$ and $h_\lambda = -\left(s^2t - 10{,}000\right) = 0$. From the third equation we know that $s \neq 0$ and $t \neq 0$, so solving the first two equations for λ gives $\lambda = \dfrac{(s-32)\,t}{s^2} = \dfrac{\frac{1}{2}\,t^2}{2st} \implies 2s(s-32)t^2 = \dfrac{1}{2}\,t^2s^2 \implies 4(s-32) = s \implies s = \dfrac{128}{3}$. Substituting into the third equation we have $\left(\dfrac{128}{3}\right)^2 t = 10{,}000 \implies t = \dfrac{5625}{1024}$. Now $f\left(\dfrac{128}{3}, \dfrac{5625}{1024}\right) = \dfrac{10546875}{65536}$.

   To attempt to justify that we have the maximum we look at the physical problem. With limited fuel there will be a maximum attainable height, so we have found it. A thrust of $\dfrac{128}{3}$ ft/sec^2 allows the rocket to climb for $\approx 5.493\,$sec and reach a height $\approx 160.93\,$ft.

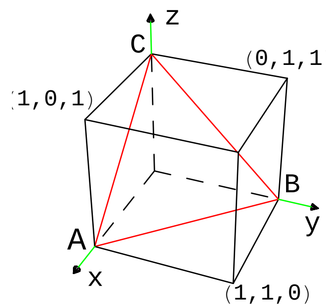
4. Since $f(x,y,z) = x^2+3x+xy+y^2+z^2$ is continuous and the solid sphere $x^2+y^2+z^2 \leq 18$ is compact, f will attain a global maximum and a global minimum on the solid sphere (by EVT). The extrema may occur either on the interior or on the boundary of the solid ball.

   <u>On the interior</u> $f_x = 2x+3+y = 0$, $f_y = x+2y = 0$ and $f_z = 2z = 0$. We see immediately that the only critical point is $(-2,1,0)$. Since $2^2+1^2+0^2 = 5 < 18$, this point is on the interior of the sphere.

   <u>On the boundary</u> We want the constrainted critical points of $f(x,y,z) = x^2+3x+xy+y^2+z^2$ subject to $g(x,y,z) = x^2+y^2+z^2-18 = 0$. We put $h(x,y,z,\lambda) = x^2+3x+xy+y^2+z^2-\lambda\left(x^2+y^2+z^2-18\right)$. Now $h_x = 2x+3+y-2\lambda x = 0$, $h_y = x+2y-2\lambda y = 0$, $h_z = 2z-2\lambda z = 0$ and $h_\lambda = -\left(x^2+y^2+z^2-18\right)$. The third equation gives $z = 0$ or $\lambda = 1$.
   If $\underline{z=0}$, eliminating x from the first two equations gives $2y^2+3y-18 = 0 \implies y = \dfrac{3(-1\pm\sqrt{17})}{4} \implies x = \pm\dfrac{3}{2}\sqrt{\dfrac{7\pm\sqrt{17}}{2}}$. This gives four constrained critical points: $\left(-\dfrac{3}{2}\sqrt{\dfrac{7-\sqrt{17}}{2}}, \dfrac{3(-1-\sqrt{17})}{4}, 0\right)$, $\left(\dfrac{3}{2}\sqrt{\dfrac{7-\sqrt{17}}{2}}, \dfrac{3(-1-\sqrt{17})}{4}, 0\right)$, $\left(-\dfrac{3}{2}\sqrt{\dfrac{7+\sqrt{17}}{2}}, \dfrac{3(-1+\sqrt{17})}{4}, 0\right)$, $\left(\dfrac{3}{2}\sqrt{\dfrac{7+\sqrt{17}}{2}}, \dfrac{3(-1+\sqrt{17})}{4}, 0\right)$.
   If $\underline{\lambda=1}$: the second equation gives $x = 0$, then the first gives $y = -3$ and the fourth gives $z = \pm 3$. The constrained critical points are $(0,-3,3)$ and $(0,-3,-3)$.

   We now evaluate. $f\left(-\dfrac{3}{2}\sqrt{\dfrac{7-\sqrt{17}}{2}}, \dfrac{3(-1-\sqrt{17})}{4}, 0\right) \approx 19.515$, $f\left(\dfrac{3}{2}\sqrt{\dfrac{7-\sqrt{17}}{2}}, \dfrac{3(-1-\sqrt{17})}{4}, 0\right) \approx 16.485$, $f\left(-\dfrac{3}{2}\sqrt{\dfrac{7+\sqrt{17}}{2}}, \dfrac{3(-1+\sqrt{17})}{4}, 0\right) \approx -0.8982$ (minimum), $f\left(\dfrac{3}{2}\sqrt{\dfrac{7+\sqrt{17}}{2}}, \dfrac{3(-1+\sqrt{17})}{4}, 0\right) \approx 36.898$ (maximum), $f(0,-3,3) = 18$ and $f(0,-3,-3) = 18$.

5. The temperature function $T(x,y,z) = 4-2x^2-y^2-z^2$ is continuous and the metal plate is compact so, by EVT, there will be hottest and coldest points on the plate. The extrema may occur either on the interior of the plate or on its edges.



<u>On the interior</u> Here we will use Lagrange multipliers as we need the critical points of $T(x,y,z)$ constrained to the plate $g(x,y,z) = x+y+z-1 = 0$. We define $h(x,y,z,\lambda) = 4-2x^2-y^2-z^2-\lambda(x+y+z-1)$ and seek the critical points of h. Now $h_x = -4x-\lambda = 0$, $h_y = -2y-\lambda = 0$, $h_z = -2z-\lambda = 0$, and $h_\lambda = -(x+y+z-1) = 0$. Solving we get the single point $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$.

<u>On the boundary</u> The boundary is a triangle with vertices $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$. We will consider each side of the triangle.

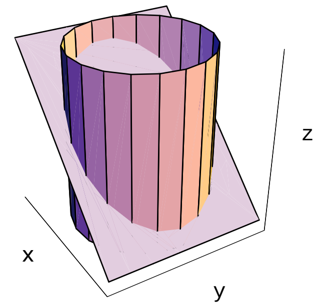
<u>Edge AC: $\{(x,y,z) \mid x+z=1,\ y=0\}$</u> Here the temperature is given by $t(x) = T(x,0,1-x) = 3-3x^2+2x,\ 0 \le x \le 1$. Now $t'(x) = -6x+2 = 0$ if $x = \frac{1}{3}$. There is a critical point $\left(\frac{1}{3}, 0, \frac{2}{3}\right)$.

<u>Edge AB: $\{(x,y,z) \mid x+y=1,\ z=0\}$</u> Here the temperature is given by $t(x) = T(x,1-x,0) = 3-3x^2+2x,\ 0 \le x \le 1$. Now $t'(x) = -6x+2 = 0$ if $x = \frac{1}{3}$. There is a critical point $\left(\frac{1}{3}, \frac{2}{3}, 0\right)$.

<u>Edge BC: $\{(x,y,z) \mid y+z=1,\ x=0\}$</u> Here the temperature is given by $t(y) = T(0,y,1-y) = 3+2y-2y^2,\ 0 \le y \le 1$. Now $t'(y) = 2-4y = 0$ if $y = \frac{1}{2}$. There is a critical point $\left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Evaluating at critical points and endpoints, we have $T\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right) = \frac{18}{5}$, $T\left(\frac{1}{3}, 0, \frac{2}{3}\right) = \frac{10}{3}$, $T\left(\frac{1}{3}, \frac{2}{3}, 0\right) = \frac{10}{3}$, $T\left(0, \frac{1}{2}, \frac{1}{2}\right) = \frac{7}{2}$, $T(1,0,0) = 2$, $T(0,1,0) = 3$, and $T(0,0,1) = 3$. Hence the hottest point is $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$ where the temperature is $360\,^oC$ and the coldest is $(1,0,0)$ where the temperature is $200\,^oC$.

6. We need to maximize the distance function (sufficient to use distance squared) subject to two constraints $x^2 + y^2 = 1$ and $x + z = 1$. We define $h(x, y, z, \lambda, \mu) = x^2 + y^2 + z^2 - \lambda(x^2 + y^2 - 1) - \mu(x + z - 1)$. Now $h_x = 2x - 2\lambda x - \mu = 0$, $h_y = 2y - 2\lambda y = 0$, $h_z = 2z - \mu = 0$, $h_\lambda = -(x^2 + y^2 - 1) = 0$, $h - \mu = -(x + z - 1) = 0$. The third equation gives $\mu = 2z$,and the second gives $y = 0$ or $\lambda = 1$. If $\lambda = 1$ the first gives $\mu = 0 \implies z = 0$ then the last gives $x = 1$ and the fourth gives $y = 0$. If $y = 0$, the fourth gives $x^2 = 1 \implies x = \pm 1$. If $x = 1$, the fifth gives $z = 0$. If $x = -1$, the fifth gives $z = 2$. Hence there are two constrained critical points: $(1, 0, 0)$ and $(-1, 0, 2)$.



Now $\sqrt{1 + 0 + 0} = 1$ and $\sqrt{(-1)^2 + 0 + 2^2} = \sqrt{5}$ so the furthest point is $(-1, 0, 2)$ at distance $\sqrt{5}$ from the origin.
Since the distance squared is a continuous function and the curve of intersection (an ellipse) is compact in $\mathbb{R}^3$, the Extreme Value Theorem (EVT) ensures the existence of extrema.

7. We need to minimize $f(q_1, q_2) = 0.1\, q_1^2 + 7\, q_1 + 15\, q_2 + 1000$ subject to the constraint $q_1 + q_2 = 100$. We define $h(q_1, q_2, \lambda) = 0.1\, q_1^2 + 7\, q_1 + 15\, q_2 + 1000 - \lambda(q_1 + q_2 - 100)$. The minimum cost will be found in the critical points of h. Now $h_{q_1} = 0.2q_1 + 7 - \lambda = 0$, $h_{q_2} = 15 - \lambda = 0$, $h_\lambda = -(q_1 + q_2 - 100) = 0$. The second gives $\lambda = 15$ and substituting into the first gives $0.2q_1 + 7 - 15 = 0 \implies q_1 = 40$. Finally substituting into the third gives $40 + q_2 = 100 \implies q_2 = 60$. Since there must be a minimum cost, it occurs when plant 1 produces 40 units and plant 2 produces 60 units.