University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 19, 2011
Duration: 3 hours

1. **[6 points]**

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the Extreme Value Theorem.

2. **[5 points]** Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2 - xy - 6y^2}{x^2 + 3y^2}$ or show that it does not exist.

3. **[5 points]** Give the 4^{th} degree Taylor polynomial about the origin of $f(x, y) = \dfrac{\sin(x + y)}{1 - y^2}$.

4. **[5 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function. Let ℓ be the line which crosses the graph of f at the point $\big(a, b, f(a, b)\big)$ orthogonal to the tangent plane. At what point does ℓ cross the xy–plane?

5. **[6 points]** The temperature of a metal plate at point (x, y) is given by $f(x, y) = x^2 + y + y^2$,

   (a) At $(2, 1)$, in which direction(s) is the temperature

      i. increasing most rapidly?

      ii. decreasing most rapidly?

      iii. not changing?

   (b) Calculate $D_{\boldsymbol{v}}\, f(2, 1)$, when $\boldsymbol{v} = (-12, 5)$.

6. **[14 points]**

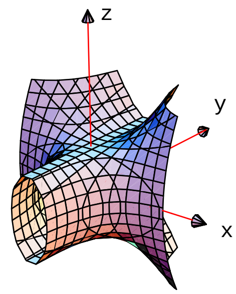
   (a) Carefully state the Chain Rule for functions of more than one variable.

   (b) Let $f(x, y, z) = \ln(x\,y\,z)$, $g(t) = \big(\ln(t),\, \cos(t),\, e^t\big)$ and $h(t) = f \circ g(t)$. Use the Chain Rule to calculate $h'(\pi)$.

(c) Let $z = f(x,y)$ be of class C^2. Let $x = 3u - 2v$ and $y = u + 4v$. Compute a formula for $\dfrac{\partial^2 z}{\partial u\,\partial v}$ in terms of the partial derivatives of z with respect to x and y.

7. [**5 points**] Let $f(x,y,z) = x^2 + xy + x\cos(z) - z^2$. Find and classify the critical points of f.

8. [**8 points**] Find the minimum distance between the origin and the surface $z^2 = x^2y + 4$. Justify your answer.



9. [**9 points**] Find the global extrema of $f(x,y,z) = x^2 - 4x + y^2 - 8y + z^2 - 4z$ on the solid ball $x^2 + y^2 + z^2 \le 16$.

   Justify your answers including an explanation of why global extrema do exist.

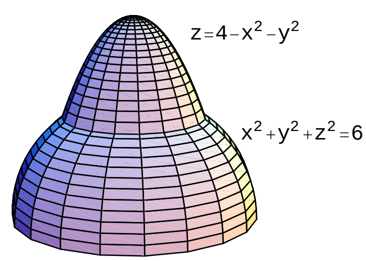
10. [**12 points**]

   (a) Rewrite the integral $\displaystyle\int_0^1 \int_y^{\frac{y+2}{3}} f(x,y)\,dx\,dy$ with the order of integration reversed.

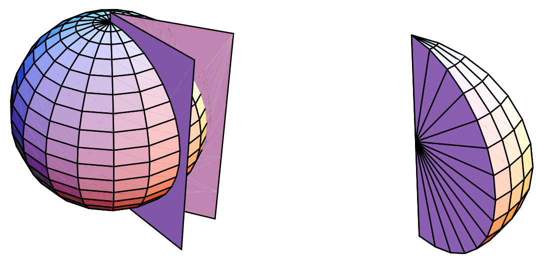
   (b) Evaluate $\displaystyle\int_0^6 \int_{\frac{x}{3}}^2 x\sqrt{y^3+1}\,dy\,dx$.

   (c) Evaluate $\displaystyle\iint_D x\,dA$, where D is the region between $y = x$ and $y = x^3$.

11. [**8 points**] Find the volume of the region above the sphere $x^2 + y^2 + z^2 = 6$ and below the paraboloid $z = 4 - x^2 - y^2$.



12. **[10 points]** Find the volume of the wedge cut from the unit sphere ($x^2+y^2+z^2=1$) by two planes which meet at the z–axis at an angle of $\dfrac{\pi}{6}$ radians.



13. **[12 points]**

(a) Carefully state the Change of Variables Theorem for multiple integrals. Make sure you define your terms.

(b) Use the change of variable $x = \dfrac{u}{v}$, $y = u\,v$ to find the area of D, the first quadrant region bounded by $y = 4x$, $y = x$, $xy = 1$ and $xy = 4$.