# University of Toronto Scarborough
# Department of Computer & Mathematical Sciences

**MAT B41H** **2011/2012**

<u>Term Test Solutions</u>

1. (a) From the lecture notes we have

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that $f$ is *differentiable at* $\boldsymbol{a} \in U$ if the partial derivatives of $f$ exist at $\boldsymbol{a}$ and if
   $$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \frac{\|f(\boldsymbol{x}) - f(\boldsymbol{a}) - Df(\boldsymbol{a})\,(\boldsymbol{x}-\boldsymbol{a})\|}{\|\boldsymbol{x}-\boldsymbol{a}\|} = 0\,,$$
   where $Df(\boldsymbol{a})$ is the $k \times n$ matrix $\left(\dfrac{\partial f_i}{\partial x_j}\right)$ evaluated at $\boldsymbol{a}$.
   $Df(\boldsymbol{a})$ is called the *derivative of* $f$ *at* $\boldsymbol{a}$.

   (b) From the lecture notes we have

   *Chain Rule.* Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subset \mathbb{R}^m \to \mathbb{R}^k$ be given functions such that $f[U] \subset V$ so that $g \circ f$ is defined. Let $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} = f(\boldsymbol{a}) \in \mathbb{R}^m$. If $f$ is differentiable at $\boldsymbol{a}$ and $g$ is differentiable at $\boldsymbol{b}$, then $g \circ f$ is differentiable at $\boldsymbol{a}$ and
   $$D(g\circ f)\,(\boldsymbol{a}) = [Dg(\boldsymbol{b})]\,[Df(\boldsymbol{a})]\ .$$

2. Clearly, the suitable function would be $f(x,y) = \sqrt{x^2+y^2}$ and we wish to approximate $f(4.01,\,2.98)$.
   $f(4,3) = \sqrt{4^2+3^2} = 5$; $f_x = \dfrac{x}{\sqrt{x^2+y^2}}$, $f_x(4,3) = \dfrac{4}{5}$; $f_y = \dfrac{y}{\sqrt{x^2+y^2}}$, $f_y(4,3) = \dfrac{3}{5}$ and the linear approximation is $T_1(x,y) = f(4,3) + f_x(4,3)\,(x-4) + f_y(4,3)\,(y-3)$. Hence $\sqrt{(4.01)^2+(2.98)^2}$ is approximated by $5 + \dfrac{4}{5}\,(4.01-4) + \dfrac{3}{5}\,(2.98-3) = 5 + 0.008 - 0.012 = 4.996$.

3. (a) (i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1-\cos 2x + \sin 2y}{x^2+2y}$. Evaluating along the line $x=0$, the limit reduces to $\displaystyle\lim_{y\to 0}\frac{\sin 2y}{2y} = 1$. On the other hand, evaluating along $y=0$, the limit becomes $\displaystyle\lim_{x\to 0}\frac{1-\cos 2x}{x^2} \overset{\text{l'Hôpital's}}{\underset{\text{Rule}}{=}} \lim_{x\to 0}\frac{2\sin 2x}{2x} = (2)(1) = 2$. Hence the limit does not exist.

   (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3\,x\,y^2}{x^2+y^2}$. We note that $\left|\dfrac{3\,x\,y^2}{x^2+y^2}\right| = 3\,|x|\,\dfrac{y^2}{x^2+y^2} \le 3\,|x| \longrightarrow 0$ as $x \longrightarrow 0$. Since $\displaystyle\lim_{(x,y)\to(0,0)}\left|\frac{3\,x\,y^2}{x^2+y^2}\right| = 0$, we also have $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3\,x\,y^2}{x^2+y^2} = 0$.

(b) $f(x,y) = \begin{cases} \dfrac{xy+2x}{x^2+(y+2)^2} & , \text{ if } (x,y) \neq (0,-2) \\ 0 & , \text{ if } (x,y) = (0,-2) \end{cases}$. For $f$ to be continuous at $(0,-2)$, we need $\displaystyle\lim_{(x,y)\to(0,-2)} f(x,y) = 0 = f(0,-2)$. Evaluating the limit along the line $x = y+2$, we have $\displaystyle\lim_{y\to -2} \frac{(y+2)^2}{2\,(y+2)^2} = \frac{1}{2}$. Since $f(0,-2) = 0 \neq \dfrac{1}{2}$, we can conclude that $f$ is not continuous at $(0,-2)$.

4. (a) $\det A = \det \begin{pmatrix} 1 & 0 & x \\ y & 1 & 2 \\ 0 & 2 & z \end{pmatrix} = (1)\,\det \begin{pmatrix} 1 & 2 \\ 2 & z \end{pmatrix} + (x)\,\det \begin{pmatrix} y & 1 \\ 0 & 2 \end{pmatrix} = z - 4 + 2\,x\,y.$

(b) $f(x,y,z) = \det A = z - 4 + 2\,x\,y$, so $\boldsymbol{\nabla} f = (2\,y,\ 2\,x,\ 1)$.

(c) $\ell_1$ has direction vector $\boldsymbol{v} = \boldsymbol{\nabla} f(2,1,0) = (2,4,1)$ and $\ell_2$ has direction vector $\boldsymbol{w} = (0,1,2) - (2,1,0) = (-2,0,2)$. The angle $\theta$ between $\ell_1$ and $\ell_2$ is the angle between their direction vectors. Therefore, $\theta = \cos^{-1}\left(\dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|}\right) = \cos^{-1}\left(\dfrac{(2,4,1)\cdot(-2,0,2)}{\|(2,4,1)\|\,\|(-2,0,2)\|}\right) = \cos^{-1}\left(\dfrac{-2}{\sqrt{21}\,\sqrt{8}}\right) = \cos^{-1}\left(\dfrac{-1}{\sqrt{42}}\right)$.

5. (a) For a function $f : \mathbb{R}^2 \to \mathbb{R}$ the equation of the tangent plane at $(a,b)$ is $z = f(a,b) + \dfrac{\partial f}{\partial x}(a,b)\,(x-a) + \dfrac{\partial f}{\partial y}(a,b)\,(y-b)$. Here we have $f_x = -2\,\sin(2x+y)$, $f_x\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right) = \dfrac{2}{\sqrt{2}}$, $f_y = -\sin(2x+y)$, $f_y\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ and $f\left(\dfrac{\pi}{2}, \dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$. Hence the equation of the tangent plane is $z = -\dfrac{1}{\sqrt{2}} + \sqrt{2}\left(x - \dfrac{\pi}{2}\right) + \dfrac{1}{\sqrt{2}}\left(y - \dfrac{\pi}{4}\right) = \sqrt{2}\,x + \dfrac{1}{\sqrt{2}}\,y - \dfrac{1}{\sqrt{2}} - \dfrac{5\,\pi}{4\sqrt{2}}$, which can be rewritten as $2x + y - \sqrt{2}\,z = \dfrac{4+5\,\pi}{4}$.

(b) To find an equation for the tangent plane to the graph of the function $z = f(x,y)$ defined implicitly by $x\,z + 2\,x^2\,y + y^2\,z^3 = 11$ at the point $(2,1,1)$, we put $g(x,y,z) = x\,z + 2\,x^2\,y + y^2\,z^3 - 11$. A normal to the level surface $g(x,y,z) = 0$ is $\boldsymbol{\nabla} g = (z + 4xy,\ 2x^2 + 2yz^3,\ x + 3y^2z^2)$. Hence a normal at $(2,1,1)$ is $\boldsymbol{\nabla} g(2,1,1) = (9,\ 10,\ 5)$. Therefore, the tangent plane has normal $(9,\ 10,\ 5)$ and its equation is $9x + 10y + 5z = d$. Since $(2,1,1)$ is point on the tangent plane, we have $9(2) + 10(1) + 5(1) = 33$. Hence the equation of the tangent plane is $9x + 10y + 5z = 33$.

(c) Let $(a,b,c)$ be a point on the surface $z^2 = x^2 + y^2$. To find a tangent plane to this surface at $(a,b,c)$, we put $g(x,y,z) = x^2 + y^2 - z^2$. A normal to the level surface $g(x,y,z) = 0$ is $\boldsymbol{\nabla} g = (2x, 2y, -2z)$. Hence a normal at $(a,b,c)$ is $\boldsymbol{\nabla} g(a,b,c) = (2a,\ 2b,\ -2c)$ Therefore, the tangent plane can be given by $2a\,(x - a) + 2b\,(y-b) - 2c(z-c) = 0$ and, because $(a,b,c)$ is a point on the surface, rewritten as $a\,x + b\,y - c\,z = a^2 + b^2 - c^2 = 0$, which is a plane through the origin.

6. (a) We know that the value of a function is constant on level sets. Here, to stay at the same temperature, the bug must stay on the level set which passes through $(1,1,1)$; i.e., on the surface $x^2 + y\,z + x\,z^2 = 1^2 + (1)(1) + (1)(1^2) = 3$.

   (b) The direction of the maximum rate of increase is the direction of the gradient of $T$ at $(1,1,1)$. Now $\boldsymbol{\nabla} T = (2x + z^2,\ z,\ y + 2xz)$, so the temperature would increase fastest in direction $\boldsymbol{\nabla} T(1,1,1) = (3,1,3)$.

   The maximum rate of increase is the magnitude of the gradient. Hence the maximum rate is $\|\boldsymbol{\nabla} T(1,1,1)\| = \|(3,1,3)\| = \sqrt{9+1+9} = \sqrt{19}$.

   (c) To reach the food, the bug would travel in direction $(3,-2,1) - (1,1,1) = (2,-3,0)$. The rate of change in temperature in this direction is given by the directional derivative, $D_{(2,-3,0)}T(1,1,1) = \boldsymbol{\nabla} T(1,1,1) \boldsymbol{\cdot} \dfrac{(2,-3,0)}{\|(2,-3,0)\|} = \dfrac{(3,1,3) \boldsymbol{\cdot} (2,-3,0)}{\sqrt{4+9}} = \dfrac{3}{\sqrt{13}}$.

7. $f(x,y,z) = x^2 - 3y^2 + 2z^2$ , so $\dfrac{\partial f}{\partial x} = 2\,x$, $\dfrac{\partial f}{\partial y} = -6\,y$ and $\dfrac{\partial f}{\partial z} = 4\,z$. Now $\dfrac{\partial^2 f}{\partial x^2} = 2$, $\dfrac{\partial^2 f}{\partial y^2} = -6$ and $\dfrac{\partial^2 f}{\partial z^2} = 4$. Since the $2^{\text{nd}}$ partials are continuous and $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 2 + (-1)6 + 4 = 0$, we conclude that $f(x,y,z)$ is harmonic.

8. $f : \mathbb{R}^4 \to \mathbb{R}^4$ is given by $f(x,y,z,w) = (xw,\ yz,\ xy,\ zw)$ so $Df = \begin{pmatrix} w & 0 & 0 & x \\ 0 & z & y & 0 \\ y & x & 0 & 0 \\ 0 & 0 & w & z \end{pmatrix}$.

   $g : \mathbb{R}^4 \to \mathbb{R}^2$ is given by $g(x,y,z,w) = (wx^2,\ wyz))$ so $Dg = \begin{pmatrix} 2xw & 0 & 0 & x^2 \\ 0 & wz & wy & yz \end{pmatrix}$

   and $D\,g(f(x,y,z,w)) = \begin{pmatrix} 2xzw^2 & 0 & 0 & x^2w^2 \\ 0 & xyzw & yz^2w & xy^2z \end{pmatrix}$.

   Now $D(g \circ f)(x,y,z,w) = [D\,g(f(x,y,z,w))]\,[D\,f(x,y,z,w)]$

$$= \begin{pmatrix} 2xzw^2 & 0 & 0 & x^2w^2 \\ 0 & xyzw & yz^2w & xy^2z \end{pmatrix} \begin{pmatrix} w & 0 & 0 & x \\ 0 & z & y & 0 \\ y & x & 0 & 0 \\ 0 & 0 & w & z \end{pmatrix}$$

$$= \begin{pmatrix} 2xzw^3 & 0 & x^2w^3 & 3x^2zw^2 \\ y^2z^2w & 2xyz^2w & 2xy^2zw & xy^2z^2 \end{pmatrix}.$$

9. $\dfrac{\partial^2 f}{\partial v\,\partial u} = \dfrac{\partial}{\partial v}\left(\dfrac{\partial f}{\partial u}\right) \overset{Chain}{\underset{Rule}{=}} \dfrac{\partial}{\partial v}\left[\dfrac{\partial f}{\partial x}\,\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\,\dfrac{\partial y}{\partial u}\right] = \dfrac{\partial}{\partial v}\left[\dfrac{\partial f}{\partial x}\,(1) + \dfrac{\partial f}{\partial y}\,(-3)\right] = \dfrac{\partial}{\partial v}\left(\dfrac{\partial f}{\partial x}\right) - 3\,\dfrac{\partial}{\partial v}\left(\dfrac{\partial f}{\partial y}\right) \overset{Chain}{\underset{Rule}{=}} \dfrac{\partial^2 f}{\partial x^2}\,\dfrac{\partial x}{\partial v} + \dfrac{\partial^2 f}{\partial y\,\partial x}\,\dfrac{\partial y}{\partial v} - 3\left[\dfrac{\partial^2 f}{\partial x\,\partial y}\,\dfrac{\partial x}{\partial v} + \dfrac{\partial^2 f}{\partial y^2}\,\dfrac{\partial y}{\partial v}\right] = \dfrac{\partial^2 f}{\partial x^2}\,(-1) + \dfrac{\partial^2 f}{\partial y\,\partial x}\,(2) - 3\left[\dfrac{\partial^2 f}{\partial x\,\partial y}\,(-1) + \dfrac{\partial^2 f}{\partial y^2}\,(2)\right] = -\dfrac{\partial^2 f}{\partial x^2} + 2\,\dfrac{\partial^2 f}{\partial y\,\partial x} + 3\,\dfrac{\partial^2 f}{\partial x\,\partial y} - 6\,\dfrac{\partial^2 f}{\partial y^2} \overset{f\ is\ of}{\underset{class\ C^2}{=}} -\dfrac{\partial^2 f}{\partial x^2} + 5\,\dfrac{\partial^2 f}{\partial x\,\partial y} - 6\,\dfrac{\partial^2 f}{\partial y^2}$

10. Recall $e^t = \displaystyle\sum_{k=0}^{\infty} \frac{t^k}{k\,!}$, $|t| < \infty$, so $e^{2\,x} = 1 + 2\,x + \dfrac{1}{2}(2\,x)^2 + \dfrac{1}{3\,!}(2\,x)^3 + \dfrac{1}{4\,!}(2\,x)^4 + \ \cdots\ $, $|2\,x| < \infty$ (by replacement). We also recall $\ln(1+t) = \displaystyle\sum_{k=0}^{\infty} (-1)^k\,\frac{t^{k+1}}{k+1}$, $|t| < 1$, so $\ln(1+xy) = x\,y - \dfrac{(x\,y)^2}{2} + \dfrac{(x\,y)^3}{3} - \ \cdots\ $, $|x\,y| < 1$ (by replacement). We now obtain a Taylor series for $f(x,y) = e^{2\,x}\,\ln(1+x\,y)$,

$$T = \left(1 + 2\,x + \frac{1}{2}(2\,x)^2 + \frac{1}{3\,!}(2\,x)^3 + \ \cdots\ \right)\left(x\,y - \frac{(x\,y)^2}{2} + \frac{(x\,y)^3}{3} - \ \cdots\ \right),$$

using multiplication of series. Hence the $4^{\text{th}}$ degree Taylor polynomial for $f$ about the origin is

$$T_4 = x\,y + 2\,x^2\,y + 2\,x^3\,y - \frac{1}{2}\,x^2\,y^2.$$