University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 5, 2014
Start Time: 2:00PM
Duration: 3 hours

1. [**12 points**]

   (a) Carefully complete the following definition:

   > Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the following theorems.

   i. The Chain Rule for functions of more than one variable.

   ii. The Extreme Value Theorem.

   iii. The Change of Variables Theorem for multiple integrals.

2. [**5 points**] Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x^2+2y^2}$ or show that it does not exist.

3. [**5 points**] Give the 4^{th} degree Taylor polynomial about the origin of
$$f(x,y) = \frac{\cos(xy)}{1-x} .$$

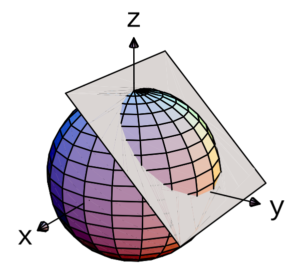
4. [**5 points**] Compute an equation for the tangent plane at the point $(-3,1,0)$ to the graph of $z = f(x,y)$ defined implicitly by $x(y^2+z^2) + y\,e^{x\,z} = -2$.

5. [**5 points**] Let $z = f(x,y)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2. Let $x = 2u + v$ and $y = 2u - 3v$. Compute a formula for $\dfrac{\partial^2 z}{\partial u\,\partial v}$ in terms of the partial derivatives of z with respect to x and y.

6. [**8 points**] Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $f(x,y,z,w) = (yzw, x^2y, xz)$ and let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $g(x,y,z) = (xy, yz)$.
   Find Df and Dg and use the Chain Rule to find $D(g \circ f)$.

7. **[8 points]** Let $f(x,y,z) = x^2 + x^2y + y^2 + y^3 + 3z^2$. Find and classify the critical points of f.

8. **[9 points]** Find the global extrema of $f(x,y,z) = x + y$ on the curve of intersection of the unit sphere, $x^2 + y^2 + z^2 = 1$, and the plane, $y + z = 1$.
   Justify your answer including an explanation of why global extrema do exist.



9. **[9 points]** The temperature at a point (x,y) on the elliptical disk $x^2 + 4y^2 \leq 24$ is given by $T(x,y) = x^2 + 2y + y^2$. Find the maximum and minimum temperatures on the disk and where they occur.

   Justify your answer including an explanation of why global extrema do exist.

10. **[15 points]**

    (a) Evaluate $\displaystyle\iint_D (1 - xy)\, dA$, where D is the triangular region with vertices $(0,0)$, $(2,0)$ and $(0,2)$.

    (b) Evaluate $\displaystyle\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy$.

    (c) Give an integral in the polar coordinates (r,θ) which is equivalent to
    $$\int_0^4 \int_0^{\sqrt{4x-x^2}} \sqrt{x^2+y^2}\, dy\, dx.$$
    (You are NOT required to evaluate this integral.)

11. **[5 points]** Write the polar equation $r = \dfrac{2}{1+\cos\theta}$ in cartesian coordinates.

12. **[8 points]** Compute $\displaystyle\int_B xy\, dV$, where B is the first octant region under the plane $2x + y + z = 4$.

13. **[8 points]** Use a triple integral to find the volume of the pyramid with base in the plane $z = -6$ and sides formed by the planes: $y = 0$, $y - x = 4$ and $2x + y + z = 4$.

14. **[8 points]** Evaluate $\displaystyle\iint_D \frac{dx\, dy}{x+y}$ where D is the region bounded by $x = 0$, $y = 0$, $x + y = 1$ and $x + y = 4$ by making a suitable change of variable.