# University of Toronto Scarborough
# Department of Computer & Mathematical Sciences

**MAT B41H** **2010/2011**

Term Test Solutions

1. (a) From the lecture notes we have

Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that $f$ is *differentiable at* $\boldsymbol{a} \in U$ if the partial derivatives of $f$ exist at $\boldsymbol{a}$ and if

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \frac{\|f(\boldsymbol{x}) - f(\boldsymbol{a}) - Df(\boldsymbol{a})\,(\boldsymbol{x}-\boldsymbol{a})\|}{\|\boldsymbol{x}-\boldsymbol{a}\|} = 0\,,$$

where $Df(\boldsymbol{a})$ is the $k \times n$ matrix $\left(\dfrac{\partial f_i}{\partial x_j}\right)$ evaluated at $\boldsymbol{a}$.
$Df(\boldsymbol{a})$ is called the *derivative of* $f$ *at* $\boldsymbol{a}$.

(b) (i) Let $A \subset \mathbb{R}^n$. A point $\boldsymbol{a} \in A$ is called an *interior point of* $A$ if $B_r(\boldsymbol{a}) \subset A$, for some $r > 0$.
($B_r(\boldsymbol{a})$ is an open ball of radius $r$ centered at $\boldsymbol{a}$.)

(ii) Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}$. A point $\boldsymbol{a} \in U$ is called a *local (relative) minimum* of $f$ if there is an open ball $B_r(\boldsymbol{a})$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{a})$ for all $\boldsymbol{x} \in B_r(\boldsymbol{a})$.

2. (a) $$\lim_{(x,y)\to(0,0)} \frac{x\,y - y^2}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{y\left(x-y\right)}{\sqrt{x}+\sqrt{y}} =$$
$$\lim_{(x,y)\to(0,0)} \frac{y\left(\sqrt{x}-\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)}{\sqrt{x}+\sqrt{y}} = \lim_{(x,y)\to(0,0)} y\left(\sqrt{x}-\sqrt{y}\right) = 0.$$

(b) If $(x,y) \neq (0,0)$, $f(x,y) = \dfrac{x^3+2x^2+2xy^2+4y^2}{x^2+2y^2}$. Since rational functions are continuous on their domains, $f(x,y)$ is continuous for $(x,y) \neq (0,0)$.
For $f$ to be continuous at $(0,0)$, we need $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = -2 = f(0,0)$. Now $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3+2x^2+2xy^2+4y^2}{x^2+2y^2} \overset{divide}{=} \lim_{(x,y)\to(0,0)} (x+2) =$ $2 \neq -2 = f(0,0)$. Hence we conclude that $f(x,y)$ is not continuous at $(0,0)$.

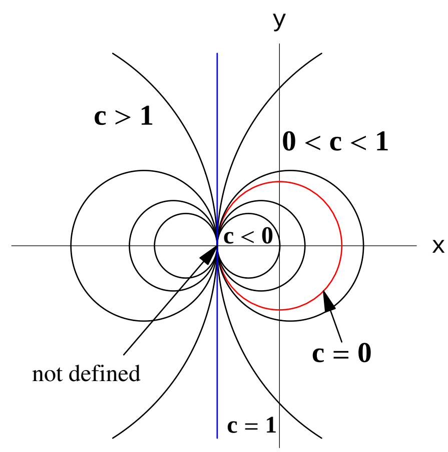
3. $f(x,y) = \dfrac{x^2+y^2-1}{(x+1)^2+y^2}$.

   Domain is $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (-1,0)\}$.

   Putting $f(x,y) = c$ we have $\dfrac{x^2+y^2-1}{(x+1)^2+y^2} = c \iff x^2+y^2-1 = cx^2+2cx+c+cy^2 \iff (1-c)x^2+(1-c)y^2-2cx = c+1$.

   For $c = 1$, we get the line $x = -1$.

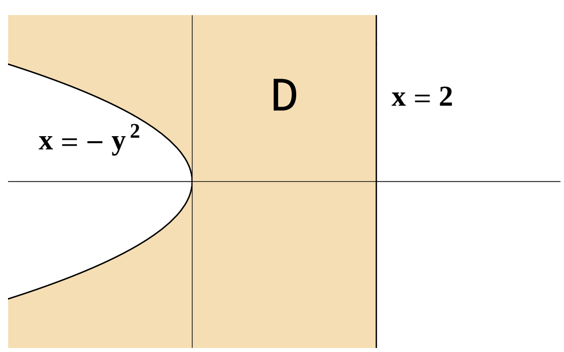
   For $c \neq 1$, we get $(1-c)\left(x^2 - \dfrac{2c}{1-c}x + \dfrac{c^2}{(1-c)^2}\right) + (1-c)y^2 = \dfrac{1}{1-c} \iff \left(x - \dfrac{c}{1-c}\right)^2 + y^2 = \left(\dfrac{1}{1-c}\right)^2$. Hence the level curves are circles centered at $\left(\dfrac{c}{1-c}, 0\right)$ with radius $\left|\dfrac{1}{1-c}\right|$.




4. $f(x,y) = \dfrac{1}{\sqrt{(x+y^2)(2-x)}}$.

   (a) The domain $D$ is given by $D = \{\ (x,y) \in \mathbb{R}^2 \ \mid\ (x+y^2)(2-x) > 0\ \}$ $= \{\ (x,y) \in \mathbb{R}^2 \ \mid\ x > -y^2 \text{ and } x < 2\ \}$.

   (b) The domain $D$ is the shaded region below. It does not include the parabola $x = -y^2$ and the line $x = 2$.




5. To find an equation for the tangent plane to the graph of the function $z = f(x,y)$ defined implicitly by $x^2\,y + y^2\,z + z^2\,x + x\,y\,z = 1$ at the point $(1,-2,1)$, we put $g(x,y,z) = x^2\,y + y^2\,z + z^2\,x + x\,y\,z - 1$. A normal to the level surface $g(x,y,z) = 0$ is $\nabla g = \big(2xy + z^2 + yz,\ x^2 + 2yz + xz,\ y^2 + 2zx + xy\big)$. Hence a normal at $(1,-2,1)$ is $\nabla g\,(1,-2,1) = \big(-5,-2,4\big)$. Therefore, the tangent plane has normal $\big(-5,-2,4\big)$ and its equation is $-5x - 2y + 4z = d$. Since $(1,-2,1)$ is a point on the tangent plane, we have $-5(1) - 2(-2) + 4(1) = 3$. Hence the equation of the tangent plane is $-5x - 2y + 4z = 3$ or $5x + 2y - 4z = -3$.

6. (a) For a function $f : \mathbb{R}^2 \to \mathbb{R}$ the equation of the tangent plane at $(a,b)$ is $z = f(a,b) + \dfrac{\partial f}{\partial x}(a,b)\,(x-a) + \dfrac{\partial f}{\partial y}(a,b)\,(y-b)$. Here we have $f_x = y\,\cos x$, $f_x\left(\dfrac{\pi}{4}, 2\right) = \sqrt{2}$, $f_y = \sin x$, $f_y\left(\dfrac{\pi}{4}, 2\right) = \dfrac{1}{\sqrt{2}}$ and $f\left(\dfrac{\pi}{4}, 2\right) = \sqrt{2}$. Hence the equation of the tangent plane is $z = \sqrt{2} + \sqrt{2}\left(x - \dfrac{\pi}{4}\right) + \dfrac{1}{\sqrt{2}}\left(y-2\right) = \sqrt{2}\,x + \dfrac{y}{\sqrt{2}} - \dfrac{\sqrt{2}\,\pi}{4}$, which can be rewritten as $2x + y - \sqrt{2}\,z = \dfrac{\pi}{2}$.

(b) We put $\boldsymbol{p} = (0,1,4)$. Two direction vectors for the plane are $\boldsymbol{v} = (-2,-1,2) - (0,1,4) = (-2,-2,-2)$ and $\boldsymbol{w} = (2,2,3) - (0,1,4) = (2,1,-1)$. Hence a parametric description is $\boldsymbol{p} + t\,\boldsymbol{v} + s\,\boldsymbol{w} = (0,1,4) + t\,(-2,-2,-2) + s\,(2,1,-1)$, $s,\, t \in \mathbb{R}$.

(c) The angle between two planes is the angle between their normal vectors. A normal for the tangent plane in part (a) is $\boldsymbol{n}_1 = (2,1,-\sqrt{2})$.
Using the direction vectors from part (b) we have $\boldsymbol{v} \times \boldsymbol{w} = (-2,-2,-2) \times (2,1,-1) = (4,-6,2)$. Hence a normal for part (b) is $\boldsymbol{n}_2 = (2,-3,1)$.
Now the angle between the planes is $\cos^{-1}\left(\dfrac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{\|\boldsymbol{n}_1\|\,\|\boldsymbol{n}_2\|}\right) =$
$\cos^{-1}\left(\dfrac{(2,1,-\sqrt{2})\cdot(2,-3,1)}{\|(2,1,-\sqrt{2})\|\,\|(2,-3,1)\|}\right) = \cos^{-1}\left(\dfrac{4-3-\sqrt{2}}{\sqrt{7}\,\sqrt{14}}\right) = \cos^{-1}\left(\dfrac{1-\sqrt{2}}{7\,\sqrt{2}}\right)$.

7. (a) From the lecture notes we have

> *Chain Rule.* Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subset \mathbb{R}^m \to \mathbb{R}^k$ be given functions such that $f[U] \subset V$ so that $g \circ f$ is defined. Let $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} = f(\boldsymbol{a}) \in \mathbb{R}^m$. If $f$ is differentiable at $\boldsymbol{a}$ and $g$ is differentiable at $\boldsymbol{b}$, then $g \circ f$ is differentiable at $\boldsymbol{a}$ and
> $$D(g \circ f)\,(\boldsymbol{a}) = [Dg(\boldsymbol{b})]\,[Df(\boldsymbol{a})]\ .$$

(b) $f : \mathbb{R}^4 \to \mathbb{R}^3$ is given by $f(x,y,z,w) = (y^2w^2,\ xyw,\ xz^2)$ so
$$Df = \begin{pmatrix} 0 & 2yw^2 & 0 & 2y^2w \\ yw & xw & 0 & xy \\ z^2 & 0 & 2xz & 0 \end{pmatrix}.$$

$g : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $g(x,y,z) = (x,\ ze^y,\ xz)$ so $Dg = \begin{pmatrix} 1 & 0 & 0 \\ 0 & ze^y & e^y \\ z & 0 & x \end{pmatrix}$

and $D\,g(f(x,y,z,w)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & xz^2e^{xyw} & e^{xyw} \\ xz^2 & 0 & y^2w^2 \end{pmatrix}$.

Now $D(g \circ f)(x,y,z,w) = [D\,g(f(x,y,z,w))]\,[D\,f(x,y,z,w)]$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & xz^2e^{xyw} & e^{xyw} \\ xz^2 & 0 & y^2w^2 \end{pmatrix} \begin{pmatrix} 0 & 2yw^2 & 0 & 2y^2w \\ yw & xw & 0 & xy \\ z^2 & 0 & 2xz & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2yw^2 & 0 & 2y^2w \\ xywz^2e^{xyw} + z^2e^{xyw} & x^2wz^2e^{xyw} & 2xze^{xyw} & x^2yz^2e^{xyw} \\ y^2z^2w^2 & 2xyz^2w^2 & 2xy^2zw^2 & 2xy^2z^2w \end{pmatrix}.$$

8. $f(x,y,z) = x^2 + 2xy + yz + z^2 + 6z$. Now $\nabla f(x,y,z) = (2x+2y,\ 2x+z,\ y+2z+6)$ and $\nabla f(2,-1,0) = (2,4,5)$.

   (a) The rate of change in $f$ as you move from $(2,-1,0)$ towards $(0,2,-1)$ is given by the directional derivative $D_{\boldsymbol{v}}\, f\,(2,-1,0)$ where $\boldsymbol{v} = (0,2,-1) - (2,-1,0) = (-2,3,-1)$. Now $D_{(-2,3,-1)}\, f\,(2,-1,0) = \nabla f(2,-1,0) \cdot \dfrac{(-2,3,-1)}{\|(-2,3,-1)\|} = \dfrac{(2,4,5)\cdot(-2,3,-1)}{\sqrt{14}} = \dfrac{3}{\sqrt{14}}$.

   (b) The direction of the maximum rate of increase is the direction of the gradient of $f$ at $(2,-1,0)$. Here that direction is $\nabla f(2,-1,0) = (2,4,5)$.
   The maximum rate is the magnitude of the gradient. Hence the maximum rate is $\|\nabla f(2,-1,0)\| = \|(2,4,5)\| = \sqrt{4+16+25} = \sqrt{45} = 3\sqrt{5}$.

   (c) Since $f$ is a polynomial, it is differentiable for all $(x,y,z) \in \mathbb{R}^3$. Hence to find critical points we need only consider those points where $\nabla f(x,y,z) = \mathbf{0}$. To find those points we solve $\begin{aligned} 2x+2y &= 0 \\ 2x+z &= 0 \\ y+2z+6 &= 0 \end{aligned}$. The first gives $y = -x$ and the second gives $z = -2x$. Substituting into the third gives $-x - 4x + 6 = 0 \implies 5x = 6 \implies x = \dfrac{6}{5}$. The only critical point is $\left(\dfrac{6}{5}, -\dfrac{6}{5}, -\dfrac{12}{5}\right)$.

9. $\dfrac{\partial^2 f}{\partial u^2} = \dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial u}\right) \underset{Rule}{\overset{Chain}{=}} \dfrac{\partial}{\partial u}\left[\dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}\right] = \dfrac{\partial}{\partial u}\left[\dfrac{\partial f}{\partial x}(1) + \dfrac{\partial f}{\partial y}(v)\right] = \dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial x}\right) + v\,\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial y}\right) \underset{Rule}{\overset{Chain}{=}} \dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial x}{\partial u} + \dfrac{\partial^2 z}{\partial y\,\partial x}\dfrac{\partial y}{\partial u} + v\left(\dfrac{\partial^2 z}{\partial x\,\partial y}\dfrac{\partial x}{\partial u} + \dfrac{\partial^2 z}{\partial y^2}\dfrac{\partial y}{\partial u}\right) = \dfrac{\partial^2 z}{\partial x^2} + v\,\dfrac{\partial^2 z}{\partial y\,\partial x} + v\,\dfrac{\partial^2 z}{\partial x\,\partial y} + v^2\,\dfrac{\partial^2 z}{\partial y^2} \underset{class\ C^2}{\overset{f\ is\ of}{=}} \dfrac{\partial^2 z}{\partial x^2} + 2\,v\,\dfrac{\partial^2 z}{\partial x\,\partial y} + v^2\,\dfrac{\partial^2 z}{\partial y^2}$.

10. Recall $\sin t = \displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k+1}}{(2k+1)!}$ , $|t| < \infty$ so $\sin(xy) = xy - \dfrac{x^3\,y^3}{3\,!} + \dfrac{x^5\,y^5}{5\,!} + \cdots$ , $|xy| < \infty$ (by replacement). We also recall that $\dfrac{1}{1+t} = \displaystyle\sum_{k=0}^{\infty}(-1)^k t^k$ , $|t| < 1$ so $\dfrac{1}{1-y^2} = 1 + y^2 + y^4 + y^6 + \cdots$ , $|y| < 1$ (by replacement). We now obtain a Taylor series for $f(x,y) = \dfrac{\sin(x\,y)}{1-y^2}$,

$$T = \left(xy - \frac{x^3\,y^3}{3\,!} + \cdots\right)\left(1 + y^2 + y^4 + \cdots\right).$$

using multiplication of series. Hence the $5^{\text{th}}$ degree Taylor polynomial for $f$ about $(0,0)$ is

$$T_5 = xy + xy^3.$$

(There are no degree 5 terms.)