University of Toronto Scarborough
Department of Computer & Mathematical Sciences
FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 20, 2012
Duration: 3 hours

1. **[9 points]** Carefully state the following theorems. Make sure that you define your terms.

   (a) The Chain Rule for functions of more than one variable.

   (b) The Extreme Value (Min-Max) Theorem for real valued functions of several variables.

   (c) The Change of Variables Theorem for multiple integrals.

2. **[5 points]** Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\|(x,y)\|}$ or show that it does not exist.

3. **[5 points]** Give the 4^{th} degree Taylor polynomial about the origin of $f(x,y) = \dfrac{\cos(x\,y)}{1+y}$.

4. **[6 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x,y) = (xy, x^2y, xy^3)$ and let $g : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $g(x,y,z) = (xy, xz, z^2, y\sin x)$. Find Df and Dg and use the chain rule to find $D(g\circ f)$.

5. **[5 points]** Suppose that the concentration in mg/cm^3 of a chemical at position (x,y,z) is given by

$$C(x,y,z) = 50 + z\cos(2\pi x)\sin(2\pi y)\ .$$

   In what direction is the concentration increasing most rapidly at the point $\left(\dfrac{1}{6}, \dfrac{1}{8}, 3\right)$? What is the rate of change in this direction?

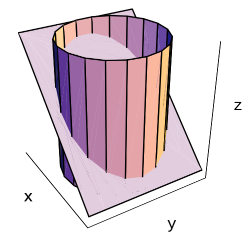
6. **[8 points]** Let $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$.

   (a) Find the eigenvalues of A.

   (b) If A is the Hessian matrix for $f : \mathbb{R}^2 \to \mathbb{R}$ at a critical point a in the domain of f, use part (a) to determine if the critical point is a local maximum, a local minimum, or a saddle.

7. **[7 points]** Let $f(x,y) = x^3 + 2xy - 2y^2 - 10x$. Find and classify the critical points of f.

8. **[8 points]** The cylinder $x^2 + y^2 = 1$ intersects the plane $x + z = 1$ in an ellipse. Find the point on that ellipse that is the furthest from the origin.
(Justify your answer including an explanation of why global extrema do exist.)



9. **[9 points]** Find the maximum value of $f(x, y, z) = xz + yz$ on the solid ellipsoid $x^2 + 2y^2 + 6z^2 \leq 12$.

   (Justify your answer including an explanation of why global extrema do exist.)

10. **[15 points]**

    (a) Rewrite the integral $\displaystyle\int_{-1}^{2} \int_{y^2-2}^{y} f(x, y)\, dx\, dy$ with the order reversed.

    (b) Compute $\displaystyle\int_0^1 \int_y^{y^{1/3}} e^{y/x}\, dx\, dy$.

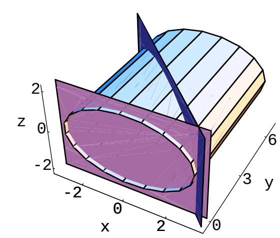
    (c) Give an integral in the Cartesian coordinates (x, y) which is equivalent to $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \int_0^{\sin\theta} r^2\, dr\, d\theta$
    .
    (You are **NOT** required to evaluate this integral.)

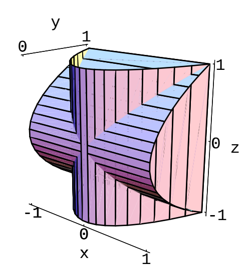
11. **[9 points]**

    (a) Sketch the curve given by the polar equation $r = \sin 2\theta$.

    (b) Use a double integral to find the area enclosed by one loop of the curve sketched in part (a).

12. **[8 points]** Evaluate $\displaystyle\int_B (x + y)\, dV$ when B is the region bounded by the cylinder $x^2 + 3z^2 = 9$ and the planes $y = 0$ and $x + y = 3$.





13. **[8 points]** Find the volume of the solid bounded by the cylinders $x^2 = y$ and $z^2 = y$ and the plane $y = 1$.

14. **[8 points]** Find the volume of the first octant region under the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$.