University of Toronto Scarborough
Department of Computer & Mathematical Sciences

Midterm Test

MATB41H – Techniques of the Calculus of Several Variables I

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Date: October 27, 2012
Duration: 110 minutes

1. [**12 points**]

   (a) Define the following terms:

   i. An **open set** A, where $A \subset \mathbb{R}^n$.
   ii. A **bounded set** B, where $B \subset \mathbb{R}^n$.
   iii. A **compact set** C, where $C \subset \mathbb{R}^n$.
   iv. A **local maximum** of a function $f : \mathbb{R}^n \to \mathbb{R}$.

   (b) Carefully state the **Extreme Value** (Min-Max) Theorem for real-valued functions of several variables.

2. [**15 points**]

   (a) Calculate the following limits, showing all your steps, or show that the limit does not exist.

   i. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 4\,x\,y + 4\,y^2}{x^2 + 2y^2}$.

   ii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 + 4\,x\,y + 4\,y^2}{x + 2y}$.

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

   $$f(x,y) = \begin{cases} \dfrac{x\,\sin(xy)}{y} & , \text{ if } y \neq 0 \\ 0 & , \text{ if } y = 0\ . \end{cases}$$

   Is f continuous at $(0,0)$? (Explain your answer.)

3. [**10 points**] Characterize and sketch several level curves of the function

   $$f(x,y) = \frac{2\,x + y}{x - 2\,y}\ .$$

   Carefully indicate where f is zero, positive, negative and not defined.

4. **[11 points]**

   (a) Find an equation for the tangent plane to the graph of

   $$f(x,y) = x^2 - 2x + 3y^2$$

   at the point $\boldsymbol{p} = (2, 1, f(2,1))$.

   (b) Give a parametric description of the normal line to the graph in part (a) which passes through $\boldsymbol{p}$ and determine where it meets the coordinate plane $z = 0$.

5. **[15 points]** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x,y,z) = 2x^2 + 2xz + y^2 + 4y + yz$.

   (a) What is the rate of change in f if you move from $(1,0,1)$ towards $(1,2,3)$.

   (b) What is the direction of the maximum rate of increase in f at $(1,0,1)$? What is the magnitude of the maximum increase.

   (c) Find the critical points of f.

6. **[7 points]** Find an equation for the tangent plane at the point $(1,-5,0)$ to the graph of the function $z = f(x,y)$ defined implicitly by

   $$x^2y + yz^2 + x\,e^{x\,z} = -4$$

   .

7. **[16 points]**

   (a) Carefully state the Chain Rule for functions of more than one variable.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x,y,z) = (xy^2,\, yz^2,\, x^2z)$ and let $g : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $g(x,y,z) = (xz,\, xyz,\, x+z,\, y^2)$.
   `USE THE CHAIN RULE` to compute $D(g \circ f)(x,y,z)$.
   (**NOTE:** You must use the Chain Rule and show all your steps.)

8. **[8 points]** Let $f(x,y,z)$ be of class C^2. Putting $x = u + v - w$, $y = 2u - 3v$ and $z = v + 2w$ makes f into a function of u, v and w. Compute a formula for $\dfrac{\partial^2 f}{\partial v\,\partial w}$ in terms of the partial derivatives of f with respect to x, y and z.

9. **[6 points]** Give the 6^{th} degree Taylor polynomial about the origin of $f(x,y) = \cos(x\,y)\,\ln(1-x^2)$.