University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #4

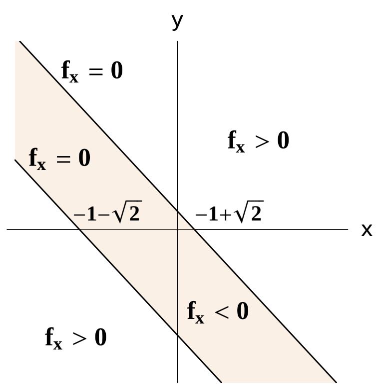
1. $$f(x,y) = \det\begin{pmatrix} e^x & 1 & -1 & 0 \\ e^x y^2 & y^2 & -y^2 & 1 \\ 0 & x+y & 1 & 1 \\ 0 & 1 & x+y & 1 \end{pmatrix} = e^x \det\begin{pmatrix} 1 & 1 & -1 & 0 \\ y^2 & y^2 & -y^2 & 1 \\ 0 & x+y & 1 & 1 \\ 0 & 1 & x+y & 1 \end{pmatrix}$$

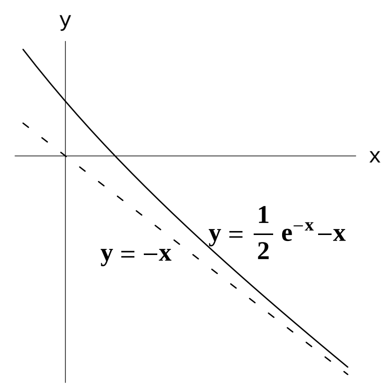
$$\overset{c_2 \mapsto c_2 - c_1}{\underset{c_3 \mapsto c_3 + c_1}{=}} e^x \det\begin{pmatrix} 1 & 0 & 0 & 0 \\ y^2 & 0 & 0 & 1 \\ 0 & x+y & 1 & 1 \\ 0 & 1 & x+y & 1 \end{pmatrix} = e^x \det\begin{pmatrix} 0 & 0 & 1 \\ x+y & 1 & 1 \\ 1 & x+y & 1 \end{pmatrix}$$

$$= e^x \det\begin{pmatrix} x+y & 1 \\ 1 & x+y \end{pmatrix} = e^x((x+y)^2 - 1).$$

(a) $\dfrac{\partial f}{\partial x} = f_x = e^x\big((x+y)^2 - 1\big) + 2(x+y)e^x = e^x\big((x+y+1)^2 - 2\big)$. Since $e^x > 0$ for all x, we have

$$f_x(x,y) \begin{cases} = 0 & , \text{ if } x+y = -1 \pm \sqrt{2} \\ > 0 & , \text{ if } (x+y+1)^2 > 2\,, \text{ i.e. } x+y > \sqrt{2} - 1 \text{ or } x+y < -\sqrt{2} - 1 \\ < 0 & , \text{ if } -1 - \sqrt{2} < x+y < -1 + \sqrt{2}\,. \end{cases}$$





(b) $\dfrac{\partial f}{\partial y} = f_y = 2(x+y)e^x$. For $f_y = 1$ we have $2(x+y)e^x = 1$ or $y = \dfrac{1}{2}e^{-x} - x$, so the required level curve is the graph of $g(x) = \dfrac{1}{2}e^{-x} - x$. Since $g'(x) = -\dfrac{1}{2}e^{-x} - 1 < 0$ for all x, $g(x)$ is always decreasing. Now $g''(x) = \dfrac{1}{2}e^{-x}$, so $g(x)$ is always concave up. We also have $\lim\limits_{x\to-\infty} g(x) = \infty$ and $g(x)$ approaching $y = -x$ as $x \to \infty$. The domain of g is $\mathbb{R}$, $g(0) = \dfrac{1}{2}$, $g(x) = 0$ if $x \approx 0.35173$, $g(x) > 0$ if $x \in (-\infty,\, 0.35173)$ and $g(x) < 0$ if $x \in (0.35173,\, \infty)$.

2. (a) A direction vector for the line is $(2,0,-3)-(-1,1,2) = (3,-1,-5)$, so a parametric description of the line is $(-1,1,2)+t\,(3,-1,-5)$, $t \in \mathbb{R}$.

   If (x,y,z) is on the line we have $(x,y,z) = (-1,1,2)+t\,(3,-1,-5)$ for some t. This gives $x = -1+3t$, $y = 1-t$ and $z = 2-5t$, so a rectangular description is $\big(t\ =\big)\quad \dfrac{x+1}{3} = 1-y = \dfrac{2-z}{5}$.

   (b) A pair of direction vectors in the plane are $\boldsymbol{v} = (2,0,-3)-(-1,1,2) = (3,-1,-5)$ and $\boldsymbol{w} = (2,-1,2)-(-1,1,2) = (3,-2,0)$. A parametric description of π is $(-1,1,2)+t\,(3,-1,-5)+s\,(3,-2,0)$, $s,t \in \mathbb{R}$.

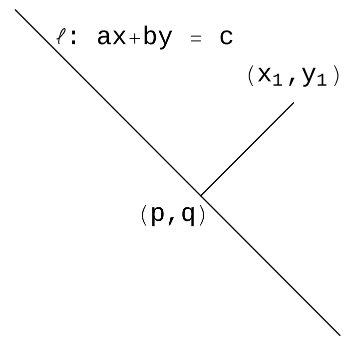
   If (x,y,z) is on the plane π, we have $\left\{\begin{array}{rcrl} x &=& -1 & +\ 3t+3s \\ y &=& 1 & -\ \ t-2s \\ z &=& 2 & -\ 5t \end{array}\right.$. The third equation gives $t = \dfrac{2-z}{5}$. If we substitute this into the second equation we get $y = 1-\dfrac{2-z}{5}-2s$ which gives $s = -\dfrac{1}{2}y+\dfrac{3}{10}+\dfrac{z}{10}$. We substitute the values for t and s into the first equation giving $x = -1+3\left(\dfrac{2-z}{5}\right)+3\left(-\dfrac{1}{2}y+\dfrac{3}{10}+\dfrac{z}{10}\right)$. Rewriting we get $10x+15y+3z = 11$ as an equation for π.

   (An alternate approach would be to find $\boldsymbol{n} = \boldsymbol{v}\times\boldsymbol{w}$.)

   (c) A direction vector for this line is a normal vector for the plane π, $\boldsymbol{n} = (10,15,3)$. A parametric description of the line is $(0,1,0)+t\,(10,15,3)$, $t \in \mathbb{R}$.

   To see where the line and the plane meet we must find a t such that $(10t, 1+15t, 3t)$ satisfies the equation of the plane. For this we need $10(10t)+15(1+15t)+3(3t) = 11 \implies 334t = -4 \implies t = \dfrac{-2}{167}$. The point of intersection is $\left(-\dfrac{20}{167}, \dfrac{137}{167}, -\dfrac{6}{167}\right)$.

3. Let (p,q) be the point where the perpendicular from (x_1,y_1) meets the line ℓ. A normal for ℓ is $\boldsymbol{n}=(a,b)$. Hence $(p-x,\ q-y)=t\,(a,b)$, for some $t\in\mathbb{R}$. The required distance is $\|\,t\,(a,b)\,\|$. We can also write (p,q) as $(p,q)=(x_1+t\,a,\ y_1+t\,b)$ and as $a\,p+b\,q=c$ (since (p,q) is a point on ℓ). Substituting we have $a\,x_1+t\,a^2+b\,y_1+t\,b^2=c \implies t=\dfrac{c-ax_1-by_1}{a^2+b^2}$. Now the distance is $\big\|\,t\,(a,\,b)\,\big\|=\dfrac{|c-ax_1-by_1|}{a^2+b^2}\,\big\|(a,\,b)\big\|=\dfrac{|c-ax_1-by_1|}{a^2+b^2}\sqrt{a^2+b^2}=\dfrac{|c-ax_1-by_1|}{\sqrt{a^2+b^2}}$.



4. $\boldsymbol{u}\times\boldsymbol{w}=\det\begin{pmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ 2 & -1 & 1\\ 3 & -4 & -2\end{pmatrix}=(2+4)\,\boldsymbol{e}_1-(-4-3)\,\boldsymbol{e}_2+(-8+3)\,\boldsymbol{e}_3=6\,\boldsymbol{e}_1+7\,\boldsymbol{e}_2-5\,\boldsymbol{e}_3=\big(6,\,7,\,-5\big)$.

5. The tangent plane to the graph of $z=f(x,y)$ at the point $(a,b,f(a,b))$ is $z=f(a,b)+\dfrac{\partial f}{\partial x}(a,b)\,(x-a)+\dfrac{\partial f}{\partial y}(a,b)\,(y-b)$.

   (a) $f(x,y)=y^2-xy$. $\dfrac{\partial f}{\partial x}=-y$, $\dfrac{\partial f}{\partial x}(2,3)=-3$; $\dfrac{\partial f}{\partial y}=2y-x$, $\dfrac{\partial f}{\partial y}(2,3)=4$ and $f(2,3)=3$.
   The equation of the tangent plane is $z=f(2,3)+\dfrac{\partial f}{\partial x}(2,3)\,(x-2)+\dfrac{\partial f}{\partial y}(2,3)\,(y-3)=3-3\,(x-2)+4\,(y-3)=-3x+4y-4$, which we can rewrite as $3\,x-4\,y+z=-3$.

   (b) $f(x,y)=\dfrac{x^2-y^2+1}{x^2+y^2}$. $\dfrac{\partial f}{\partial x}=\dfrac{(x^2+y^2)\,(2x)-(x^2-y^2+1)\,(2x)}{(x^2+y^2)^2}$, $\dfrac{\partial f}{\partial x}(2,3)=\dfrac{68}{169}$; $\dfrac{\partial f}{\partial y}=\dfrac{(x^2+y^2)\,(-2y)-(x^2-y^2+1)\,(2y)}{(x^2+y^2)^2}$, $\dfrac{\partial f}{\partial y}(2,3)=-\dfrac{54}{169}$ and $f(2,3)=-\dfrac{4}{13}$.
   The equation of the tangent plane is $z=f(2,3)+\dfrac{\partial f}{\partial x}(2,3)\,(x-2)+\dfrac{\partial f}{\partial y}(2,3)\,(y-3)=-\dfrac{4}{13}+\dfrac{68}{169}\,(x-2)-\dfrac{54}{169}\,(y-3)=\dfrac{68}{169}\,x-\dfrac{54}{169}\,y-\dfrac{2}{13}$, which we can rewrite as $68\,x-54\,y-169\,z=26$.

   (c) $f(x,y)=\dfrac{x+y}{x^2}$. $f_x=-\dfrac{x+2y}{x^3}$, $f_x(2,3)=-1$; $f_y=\dfrac{1}{x^2}$, $f_y(2,3)=\dfrac{1}{4}$ and $f(2,3)=\dfrac{5}{4}$.
   The equation of the tangent plane is $z=\dfrac{5}{4}-1\,\big(x-2\big)+\dfrac{1}{4}\,\big(y-3\big)$ which can be rewritten as $4x-y+4z=10$.

(d) $f(x,y) = \dfrac{x}{1+x^2+y^2}$. $f_x = \dfrac{1-x^2+y^2}{(1+x^2+y^2)^2}$, $f_x(2,3) = \dfrac{3}{98}$; $f_y = \dfrac{-2\,x\,y}{(1+x^2+y^2)^2}$, $f_y(2,3) = -\dfrac{3}{49}$ and $f(2,3) = \dfrac{1}{7}$.

The equation of the tangent plane is $z = \dfrac{1}{7} + \dfrac{3}{98}\,(x-2) - \dfrac{3}{49}\,(y-3) = \dfrac{3}{98}\,x - \dfrac{3}{49}\,y + \dfrac{13}{49}$ which can be rewritten as $3\,x - 6\,y - 98\,z = -26$.

(e) $f(x,y) = \sqrt{\dfrac{1+2y-x^2}{y^2+y}}$. $f_x = \dfrac{x\sqrt{\frac{1+2y-x^2}{y^2+y}}}{1+2y-x^2}$, $f_x(2,3) = -\dfrac{1}{3}$;
$f_y = \dfrac{-1-2y-2y^2+x^2(1+2y)}{2y^2(1+y^2)\sqrt{\frac{1+2y-x^2}{y^2+y}}}$, $f_y(2,3) = \dfrac{1}{48}$ and $f(2,3) = \dfrac{1}{2}$.

The equation of the tangent plane is $z = \dfrac{1}{2} - \dfrac{1}{3}\,(x-2) + \dfrac{1}{48}\,(y-3) = -\dfrac{1}{3}\,x + \dfrac{1}{48}\,y + \dfrac{53}{48}$ which can be rewritten as $16\,x - y + 48\,z = 53$.

6. (a) (i) Here we want the equation of the tangent plane to the level surface $g(x,y,z) = x^2 + y^2 + z - 7 = 0$ at the point $(1,-2,2)$. The tangent plane is given by $\nabla g\,(1,-2,2) \cdot ((x,y,z) - (1,-2,2)) = 0$. Now $\nabla g = (2x, 2y, 1)$ and $\nabla g\,(1,-2,2) = (2,-4,1)$. Hence the equation of the tangent plane is $(2,-4,1) \cdot (x-1, y+2, z-2) = 2x - z - 4y - 8 + z - 2 = 2x - 4y + z - 12 = 0$, which we can rewrite as $2\,x - 4\,y + z = 12$.

(ii) The tangent plane to the level surface $g(x,y,z) = (\cos x)\,(\sin y)\,e^z = 0$ at the point $\left(\dfrac{\pi}{2}, 1, 0\right)$ is given by $\nabla g\,\left(\dfrac{\pi}{2}, 1, 0\right) \cdot \left((x,y,z) - \left(\dfrac{\pi}{2}, 1, 0\right)\right) = 0$. Now $\nabla g = ((-\sin x)\,(\sin y)\,e^z,\ (\cos x)\,(\cos y)\,e^z,\ (\cos x)\,(\sin y)\,e^z)$, so $\nabla g\,\left(\dfrac{\pi}{2}, 1, 0\right) = (-\sin 1, 0, 0)$. The equation of the tangent plane is given by $(-\sin 1, 0, 0) \cdot \left(x - \dfrac{\pi}{2},\ y-1,\ z\right) = (-\sin 1)\,x + \dfrac{\pi}{2}\sin 1 = 0$, or $x = \dfrac{\pi}{2}$.

(b) To find an equation for the tangent plane to the graph of the function $z = f(x,y)$ defined implicitly by $x^2y + yz^2 + x\,e^{xz} = -4$ at the point $(1,-5,0)$, we put $g(x,y,z) = x^2y + yz^2 + x\,e^{xz} + 4$. A normal to the level surface $g(x,y,z) = 0$ is $\nabla g = (2xy + e^{xz} + xz\,e^{xz},\ x^2 + z^2,\ 2yz + x^2\,e^{xz})$. Hence a normal at $(1,-5,0)$ is $\nabla g(1,-5,0) = (-9,1,1)$. Therefore, the tangent plane has normal $(-9,1,1)$ and its equation is $-9x + y + z = d$. Since $(1,-5,0)$ is a point on the tangent plane, we have $-9(1) + (-5) + (0) = -14$. Hence the equation of the tangent plane is $-9x + y + z = -14$ or $9x - y - z = 14$.

7. (a) For $f(x,y,z) = xz + y^2z^2$, $\nabla f = (z,\ 2yz^2,\ x + 2y^2z)$ and $\nabla f\,(3,-1,2) = (2,-8,7)$. The directional derivative is $D_{(0,-3,4)}f\,(3,-1,2) = \nabla f\,(3,-1,2) \cdot \dfrac{(0,-3,4)}{\|(0,-3,4)\|} = \dfrac{(2,-8,7)\cdot(0,-3,4)}{\sqrt{0+9+16}} = \dfrac{52}{\sqrt{25}} = \dfrac{52}{5}$.

(b) For $f(x,y,z) = xy^2z$, $\nabla f = \left(y^2z,\, 2xyz,\, xy^2\right)$ and $\nabla f\,(3,4,5) = (80,120,48)$. A normal to the surface $2x^2+2y^2-z^2 = 25$ is given by the gradient $(4x,\, 4y,\, -2z)$ so a normal at $(3,4,5)$ would be $\lambda\,(12,16,-10)$ for some λ. Since $(3,4,5)$ is in the first octant, an outward normal requires positive x–component; hence, a suitable normal would be $(6,8,-5)$. The directional derivative is $D_{(6,8,-5)}f(3,4,5) = \nabla f\,(3,4,5)\boldsymbol{\cdot}\dfrac{(6,8,-5)}{\|(6,8,-5)\|} = \dfrac{(80,120,48)\boldsymbol{\cdot}(6,8,-5)}{\sqrt{36+64+25}} = \dfrac{1200}{5\sqrt{5}} = 48\sqrt{5}$.

8. (a) $f : \mathbb{R}^4 \to \mathbb{R}^3$ is given by $f(x,y,z,w) = (xzw,\, y^2w^3,\, x^2z)$ so

$$Df = \begin{pmatrix} zw & 0 & xw & xz \\ 0 & 2yw^3 & 0 & 3y^2w^2 \\ 2xz & 0 & x^2 & 0 \end{pmatrix}.$$

$g : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $g(x,y,z) = (ye^x,\, yz^2, x+yz)$ so $D\,g = \begin{pmatrix} ye^x & e^x & 0 \\ 0 & z^2 & 2yz \\ 1 & z & y \end{pmatrix}$

and $D\,g(f(x,y,z)) = \begin{pmatrix} y^2w^3e^{xzw} & e^{xzw} & 0 \\ 0 & x^4z^2 & 2x^2y^2zw^3 \\ 1 & x^2z & y^2w^3 \end{pmatrix}$.

Now $D(g\circ f)(x,y,z,w) = [D\,g(f(x,y,z,w))]\,[D\,f(x,y,z,w)]$

$$= \begin{pmatrix} y^2w^3e^{xzw} & e^{xzw} & 0 \\ 0 & x^4z^2 & 2x^2y^2zw^3 \\ 1 & x^2z & y^2w^3 \end{pmatrix} \begin{pmatrix} zw & 0 & xw & xz \\ 0 & 2yw^3 & 0 & 3y^2w^2 \\ 2xz & 0 & x^2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} y^2zw^4e^{xzw} & 2yw^3e^{xzw} & xy^2w^4e^{xzw} & xy^2zw^3e^{xzw} + 3y^2w^2e^{xzw} \\ 4x^3y^2z^2w^3 & 2x^4yz^2w^3 & 2x^4y^2zw^3 & 3x^4y^2z^2w^2 \\ zw + 2xy^2zw^3 & 2x^2yzw^3 & xw + x^2y^2w^3 & xz + 3x^2y^2zw^2 \end{pmatrix}.$$

(b) To compute directly, we have $g\circ f : \mathbb{R}^4 \to \mathbb{R}^3$ and $g\circ f(x,y,z) = g(f(x,y,z))$ $g(xzw,\, y^2w^3,\, x^2z) = (y^2w^3e^{xzw},\, x^4y^2w^3z^2,\, xzw{+}x^2y^2zw^3)$. Now $D(g{\circ}f)(x,y,z,w)$

$$= \begin{pmatrix} y^2zw^4e^{xzw} & 2yw^3e^{xzw} & xy^2w^4e^{xzw} & xy^2zw^3e^{xzw} + 3y^2w^2e^{xzw} \\ 4x^3y^2z^2w^3 & 2x^4yz^2w^3 & 2x^4y^2zw^3 & 3x^4y^2z^2w^2 \\ zw + 2xy^2zw^3 & 2x^2yzw^3 & xw + x^2y^2w^3 & xz + 3x^2y^2zw^2 \end{pmatrix} \text{ which}$$

is the same as we had when we used the Chain Rule.