University of Toronto Scarborough
Department of Computer & Mathematical Sciences
FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 16, 2015
Start Time: 2:00PM
Duration: 3 hours

1. [**12 points**]

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the following theorems.

   i. The Chain Rule for functions of more than one variable.
   ii. The Extreme Value Theorem.
   iii. The Change of Variables Theorem for multiple integrals.

2. [**5 points**] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{2x^2 - 2xy + 4y^2}{x^2 + 2y^2} & , \text{ if } (x,y) \neq (0,0) \\ 2 & , \text{ if } (x,y) = (0,0) \ . \end{cases}$$

   Determine all values of (x,y) for which $f(x,y)$ is continuous.

3. [**5 points**] Let S be the surface defined by the equation $z = \sqrt{\dfrac{5 + x^2 + 2y^4}{y^2 + x^4}}$. Find the tangent plane to S at the point $(1,-1,2)$.

4. [**7 points**] Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

   (a) Find the eigenvalues of A.

   (b) If A is the Hessian matrix for $f : \mathbb{R}^2 \to \mathbb{R}$ at a critical point in the domain of f, use the eigenvalues from part (a) to classify the critical point.

5. [**10 points**]

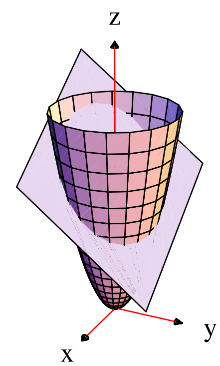
   (a) Compute the 4^{th} degree Taylor polynomial about the origin of $f(x,y) = e^{y^2} \sin(x+y)$.

   (b) Find the linear approximation to the function $f(x,y) = \dfrac{x+2}{4y-2}$ at the point $(2,3)$ and use it to estimate $f(2.1, 2.9)$.

6. **[8 points]** Let $f(x,y,z) = x^3 + x^2 + y^2 + z^2 - xy + xz$. Find all the critical points of f. Characterize each critical point as a local maximum, a local minimum, or a saddle point.

7. **[9 points]** Find the points on the intersection of the paraboloid $z = x^2 + y^2$ and the plane $x + y + z = 12$ that are closest to and farthest from the origin.

   Justify your answer including an explanation of why global extrema do exist.



8. **[10 points]** Find the maximum value of $x^2 - 4x + y^2 - 2y + z^2 - 4z - 1$ on the solid ball $x^2 + y^2 + z^2 \leq 9$.

   Justify your answer including an explanation of why global extrema do exist.

9. **[5 points]** Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $f(x,y,z) = (xy^2, yz^2, x^2z, xyz)$ and let $g : \mathbb{R}^4 \to \mathbb{R}$ be given by $g(x,y,z,w) = xy + zw$. Use the Chain Rule to compute the gradient of $g \circ f$.

10. **[15 points]**

    (a) Evaluate $\displaystyle\int_D e^{x+y}\, dA$, where D is the region bounded by $y = x - 1$ and $y = 12 - x$ for $2 \leq y \leq 4$.

    (b) Evaluate $\displaystyle\int_0^1 \int_{x^{2/3}}^1 x\, e^{y^4}\, dy\, dx$.

    (c) Integrate $f(x,y) = x + 1$ over the interior of the triangle with vertices $(1,1)$, $(3,5)$ and $(5,3)$.

11. **[8 points]** Find the volume of the solid B bounded by the parabolic cylinder $x = (y-4)^2 + 3$ and the planes $z = x + 2y - 4$, $z = x + 4y - 7$ and $x + 2y = 11$.

12. **[8 points]** Evaluate $\displaystyle\int_B z\, dV$, where B is the region bounded by the planes $z = 0$ and $z = 1$ and the surface $(z+1)\sqrt{x^2+y^2} = 1$.

13. **[8 points]** Let B be the first octant region bounded by $z = x^2 + y^2 + 16$, $z = \sqrt{x^2+y^2}$ and $x^2 + y^2 = 4$. Use cylindrical or spherical polars to describe B and set up a triple integral to find the volume of B. What is the volume of B?