University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 17, 2013
Duration: 3 hours

1. **[6 points]**

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the Extreme Value Theorem.

2. **[5 points]** Let $f(x, y) = x^2 - y^2$ and $g(x, y) = x\,y$. Show that at any point $\boldsymbol{p}$, the level set of f through $\boldsymbol{p}$ is orthogonal to the level set of g through $\boldsymbol{p}$.

3. **[5 points]** Find an equation of the tangent plane to the surface $xz + 2x^2y + y^2z^3 = 11$ at the point $\boldsymbol{p} = (2, 1, 1)$.

4. **[5 points]** Give the 4^{th} degree Taylor polynomial about the origin of $f(x, y) = (1 + y)\cos(x\,y)$.

5. **[10 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases} .$$

   (a) Determine if $f(x, y)$ is continuous at $(0, 0)$.

   (b) Compute $D_{\boldsymbol{u}}f(0, 0)$, the directional derivative of f at $(0, 0)$ in direction $\boldsymbol{u}$ when (i) $\boldsymbol{u} = (0, 1)$ and (ii) $\boldsymbol{u} = (a, b)$, a unit vector.

   (c) What do you conclude from parts (a) and (b)?

6. **[8 points]**

   (a) Carefully state the Chain Rule for functions of more than one variable.

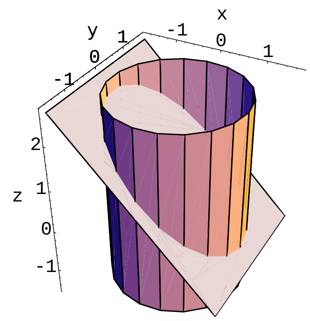
   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = \left(e^{-2xy}, x^2 - z^2 - 4x + \sin(x + y + z)\right)$$

   and let $g : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $g(1, 0) = -1$ and $\nabla g(1, 0) = (1, -3)$. Calculate the gradient of $g \circ f$ at the point $(0, 0, 0)$.

7. **[8 points]** Let $f(x,y,z) = x^3 - z^3 + 3y^2 - 6xy - 9x + 12z$. Find and classify the critical points of f.

8. **[8 points]** Find the global extrema of $f(x,y,z) = x+y+z$ on the curve of intersection of the cylinder $x^2+y^2=2$ and the plane $x+z=1$.
(Justify your answer including an explanation of why global extrema do exist.)



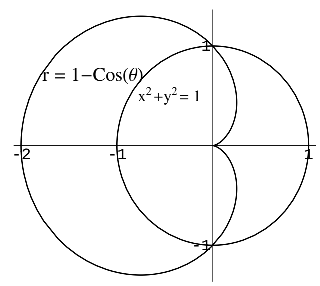
9. **[9 points]** Find the global extrema of $f(x,y) = 5 + 4x - 2x^2 + 3y - y^2$ on the region D bounded by the lines $y=2$, $y=x$ and $y=-x$.

(Justify your answer including an explanation of why global extrema do exist.)

10. **[9 points]**

(a) Write the polar equation $r = 3\sin\theta$ in cartesian coordinates.

(b) Find the area of the region inside the cardioid $r = 1-\cos\theta$ and outside the circle $x^2+y^2=1$.



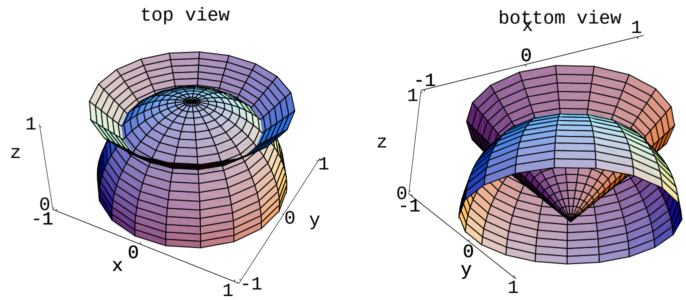
11. **[10 points]**

(a) Evaluate $\displaystyle\int_0^1 \int_{2x}^2 x\sqrt{1+y^3}\,dy\,dx$,

(b) Evaluate $\displaystyle\int_D \frac{dA}{(x+y+1)^2}$, where D is the triangle in the xy–plane with vertices $(0,0)$, $(4,0)$ and $(0,8)$.

12. **[8 points]** A paperweight has a slanted top described by $x+y+z=2$. Its edges are orthogonal to the xy–plane, and the bottom of the paperweight is formed by the triangle with vertices $(1,0,0), (0,-1,0)$ and $(0,1,0)$. Use a triple integral to find the volume of the paperweight.

13. **[9 points]** Find the mass of the solid region that consists of all points that are inside both the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and the cone $z^2 = x^2 + y^2$ if the mass density (mass per unit volume) at (x, y, z) is $\delta(x, y, z) = 1 - z$.



14. **[10 points]**

(a) Carefully state the Change of Variables Theorem for multiple integrals.

(b) Use a change of variable to evaluate $\displaystyle\int_D xy(x^2 + y^2)\, dA$ where D is the first quadrant region bounded by $xy = 1$, $xy = 4$, $y^2 = x^2 - 3$ and $y^2 = x^2 + 3$.