University of Toronto at Scarborough
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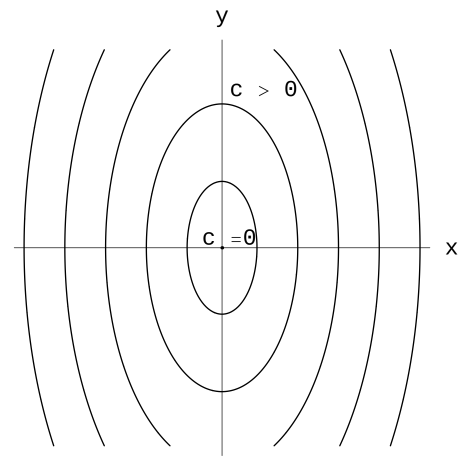
MAT B41H **2007/2008**

Term Test Solutions

1. $f(x,y) = \sqrt{4\,x^2 + y^2}$.
   Domain is $\mathbb{R}^2$. Putting $f(x,y) = c$ we have $\sqrt{4\,x^2+y^2} = c$. For $c = 0$, the level curve is the point $(0,0)$. For $c > 0$, we have $4\,x^2 + y^2 = c^2$, which is a family of ellipses, centered at $(0,0)$ with intercepts $\left(0,\, \pm\dfrac{c}{2}\right)$ and $\big(\pm c,\, 0\big)$. (The graph of f is an ellipitical cone opening upward.)



2. (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$. Evaluating along the line $y = 0$, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{0}{x^2} = 0$, but along the curve $x = y^2$, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{x\to 0} \frac{x^2}{2\,x^2} = \frac{1}{2} \neq 0$. Hence this limit does not exist.

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ when $f(x,y) = \begin{cases} \dfrac{x\,\sin(xy)}{y} & \text{, if } y \neq 0 \\ 0 & \text{, if } y = 0 \end{cases}$. Evaluating along the line $y = 0$, we have $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$. If $x = 0$, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} \frac{0}{y} = 0$. Now, if $y \neq 0$ and $x \neq 0$, we have $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x\,\sin(xy)}{y} = \lim_{(x,y)\to(0,0)} \big(x^2\big)\left(\frac{\sin(xy)}{xy}\right) = (\,0\,)\,(\,1\,) = 0$ since $\displaystyle\lim_{t\to 0} \frac{\sin t}{t} = 1$ from single variable calculus. Hence $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

3. For f to be continuous at $(0,0)$, we need $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 1 = f(0,0)$. Now $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\big(x^2+y^2\big)\big(x^2-y^2\big)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \big(x^2-y^2\big) = 0 \neq 1 = f(0,0)$. Hence we conclude that f is not continuous at $(0,0)$.

4. (a) Let $A \subset \mathbb{R}^n$. A point $\boldsymbol{a} \in A$ is called an *interior point of* A if $B_r(\boldsymbol{a}) \subset A$, for some $r > 0$.
   ($B_r(\boldsymbol{a})$ is an open ball of radius r centered at $\boldsymbol{a}$.)

   (b) $B \subset \mathbb{R}^n$ is *closed* if $\mathbb{R}^n - B$ is open.
   ($X \subset \mathbb{R}^n$ is open if every point in X is an interior point.)

   (c) $C \subset \mathbb{R}^n$ is said to be *bounded* if C can be contained in an open ball, $B_R(\mathbf{0})$, for sufficiently large R, or, if $\|\,\boldsymbol{x}\,\| < M$, for some $M \in \mathbb{R}$, for each $\boldsymbol{x} \in C$.

   (d) $D \subset \mathbb{R}^n$ is said to be *compact* if it is closed and bounded.

5. (a) The rate of change is given by the directional derivative of E at $\boldsymbol{p} = (3,4,5)$ in direction $\boldsymbol{v} = (1,1,-1)$. Now $\nabla E = (10x - 3y + yz,\ -3x + xz,\ xy)$ and $\nabla E\ (3,4,5) = (38,\ 6,\ 12)$ so $D_{\boldsymbol{v}}E(\boldsymbol{p}) = \nabla E\ (3,4,5) \boldsymbol{\cdot} \left(\dfrac{1}{\|(1,1,-1)\|}\right)(1,1,-1) = \dfrac{(38,6,12)\boldsymbol{\cdot}(1,1,-1)}{\sqrt{3}} = \dfrac{32}{\sqrt{3}}.$

   (b) The most rapid change is in the direction of the gradient; i.e., in direction $\nabla E\ (3,4,5) = (38,6,12)$.

   (c) The maximum rate of change at $\boldsymbol{p}$ is $D_{\nabla E}E(\boldsymbol{p}) = \dfrac{\nabla E\ (3,4,5)\boldsymbol{\cdot}\nabla E\ (3,4,5)}{\|\nabla E\ (3,4,5)\|} = \|\nabla E\ (3,4,5)\| = \|(38,6,12)\| = \sqrt{38^2 + 6^2 + 12^2} = 2\sqrt{406}.$

6. (a) We put $\boldsymbol{p} = (1,0,4)$. Two direction vectors for π are $(2,-1,0) - (1,0,4) = (1,-1,-4)$ and $(3,1,2) - (1,0,4) = (2,1,-2)$. A normal vector for π is $\boldsymbol{n} = (1,-1,-4)\times(2,1,-2) = (6,-6,3)$. An equation for π is of the form $6x-6y+3z = d$. Since $(1,0,4)$ is a point on π, $d = 6(1) - 6(0) + 3(4) = 18$. An equation for π is $2x - 2y + z = 6$.

   (b) Since ℓ is orthogonal to π, a normal vector for π is a direction vector $\boldsymbol{v}$ for ℓ. A parametric description for ℓ is $(1,1,1) + t\,\boldsymbol{v} = (1,1,1) + t\,(2,-2,1)$, $t \in \mathbb{R}$.

   (c) We note that two planes are parallel if their normals are parallel. A normal for a tangent plane to the ellipsoid is given by the gradient, $\nabla g = (8x,\ 16y,\ 8z)$, of the level surface $g(x,y,z) = 4x^2 + 8y^2 + 4z^2 - 7 = 0$. For the tangent plane to be parallel to π, we need $(8x,\ 16y,\ 8z) = \lambda\,(2,-2,1)$, where $(2,-2,1)$ is a normal for π. Equating components we have $x = \dfrac{\lambda}{4}$, $y = -\dfrac{\lambda}{8}$ and $z = \dfrac{\lambda}{8}$. To solve for λ we substitute these into the equation of the ellipsoid, giving $4x^2 + 8y^2 + 4z^2 = 4\left(\dfrac{\lambda^2}{16}\right) + 8\left(\dfrac{\lambda^2}{64}\right) + 4\left(\dfrac{\lambda^2}{64}\right) = \dfrac{\lambda^2}{4} + \dfrac{\lambda^2}{8} + \dfrac{\lambda^2}{16} = 7 \implies 4\lambda^2 + 2\lambda^2 + \lambda^2 = (7)(16) \implies 7\lambda^2 = (7)(16) \implies \lambda = \pm 4$. The required points are $\left(1,\ -\dfrac{1}{2},\ \dfrac{1}{2}\right)$ and $\left(-1,\ \dfrac{1}{2},\ -\dfrac{1}{2}\right)$.

7. (a) For a function $f : \mathbb{R}^2 \to \mathbb{R}$ the equation of the tangent plane at (a,b) is $z = f(a,b) + \dfrac{\partial f}{\partial x}(a,b)\,(x-a) + \dfrac{\partial f}{\partial y}(a,b)\,(y-b)$. Here we have $f_x = -2x$, $f_x\ (1,1) = -2$; $f_y = -4y$, $f_y\ (1,1) = -4$ and $f(1,1) = -2$. Hence the equation of the tangent plane is $z = -2 + (-2)(x-1) + (-4)(y-1) = -2 - 2x + 2 - 4y + 4 = -2x - 4y + 4$, which can be rewritten as $2x + 4y + z = 4$.

   (b) To find the equation of the tangent plane to the surface given by $z^2 - 2x^2 - 2y^2 = 12$ at $(1,-1,4)$ we regard the surface as a level set of $g(x,y,z) = z^2 - 2x^2 - 2y^2 - 12$. Now $\nabla g = (-4x, -4y, 2z)$; hence a normal vector at $(1,-1,4)$ is $\nabla g\ (1,-1,4) = (-4,4,8)$. The equation of the tangent plane is given by $0 = \nabla g\ (1,-1,4) \cdot \big((x,y,z) - (1,-1,4)\big) = (-4,4,8) \cdot (x-1,\ y+1,\ z-4) = -4x + 4y + 8z - 24$ which we can rewrite as $-x + y + 2z = 6$.

8. (a) From the lecture notes we have

   Chain Rule. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subset \mathbb{R}^m \to \mathbb{R}^k$ be given functions such that $f[U] \subset V$ so that $g \circ f$ is defined. Let $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} = f(\boldsymbol{a}) \in \mathbb{R}^m$. If f is differentiable at $\boldsymbol{a}$ and g is differentiable at $\boldsymbol{b}$, then $g \circ f$ is differentiable at $\boldsymbol{a}$ and
$$D(g \circ f)\,(\boldsymbol{a}) = [Dg(\boldsymbol{b})]\,[Df(\boldsymbol{a})]\ .$$

   (b) $f : \mathbb{R}^4 \to \mathbb{R}^3$ is given by $f(x,y,z,w) = (xzw,\ y^2w^3,\ x^2z)$ so
$$Df = \begin{pmatrix} zw & 0 & xw & xz \\ 0 & 2yw^3 & 0 & 3y^2w^2 \\ 2xz & 0 & x^2 & 0 \end{pmatrix}.$$
$g : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $g(x,y,z) = (ye^x,\ yz^2, x + yz)$ so $D\,g = \begin{pmatrix} ye^x & e^x & 0 \\ 0 & z^2 & 2yz \\ 1 & z & y \end{pmatrix}$

   and $D\,g(f(x,y,z)) = \begin{pmatrix} y^2w^3e^{xzw} & e^{xzw} & 0 \\ 0 & x^4z^2 & 2x^2y^2zw^3 \\ 1 & x^2z & y^2w^3 \end{pmatrix}$.

   Now $D(g \circ f)(x,y,z,w) = [D\,g(f(x,y,z,w))]\,[D\,f(x,y,z,w)]$
$$= \begin{pmatrix} y^2w^3e^{xzw} & e^{xzw} & 0 \\ 0 & x^4z^2 & 2x^2y^2zw^3 \\ 1 & x^2z & y^2w^3 \end{pmatrix} \begin{pmatrix} zw & 0 & xw & xz \\ 0 & 2yw^3 & 0 & 3y^2w^2 \\ 2xz & 0 & x^2 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} y^2zw^4e^{xzw} & 2yw^3e^{xzw} & xy^2w^4e^{xzw} & xy^2zw^3e^{xzw} + 3y^2w^2e^{xzw} \\ 4x^3y^2z^2w^3 & 2x^4yz^2w^3 & 2x^4y^2zw^3 & 3x^4y^2z^2w^2 \\ zw + 2xy^2zw^3 & 2x^2yzw^3 & xw + x^2y^2w^3 & xz + 3x^2y^2zw^2 \end{pmatrix}.$$

9. Using the Chain Rule we have $\dfrac{\partial f}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} = e^w \dfrac{\partial f}{\partial x} + w\, e^v \dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial w} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial w} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial w} = v\, e^w \dfrac{\partial f}{\partial x} + e^v \dfrac{\partial f}{\partial y}$.

10. Recall $\cos t = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)\,!}$, $|t| < \infty$, so $\cos(xy) = 1 - \dfrac{x^2y^2}{2\,!} + \dfrac{x^4y^4}{4\,!} - \cdots$, and also recall $\ln(1+t) = \displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{t^{k+1}}{k+1}$, $|t| < 1$, so $\ln(1+x^2) = x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{3} - \cdots$. Hence a Taylor series for $\cos(xy)\,\ln(1+x^2)$ is $T = \left(1 - \dfrac{x^2y^2}{2\,!} + \dfrac{x^4y^4}{4\,!} - \cdots\right)\left(x^2 - \dfrac{x^4}{2} + \dfrac{x^6}{3} - \cdots\right)$. The 4^{th} degree Taylor polynomial about $(0,0)$ is $T_4 = x^2 - \dfrac{x^4}{2}$.