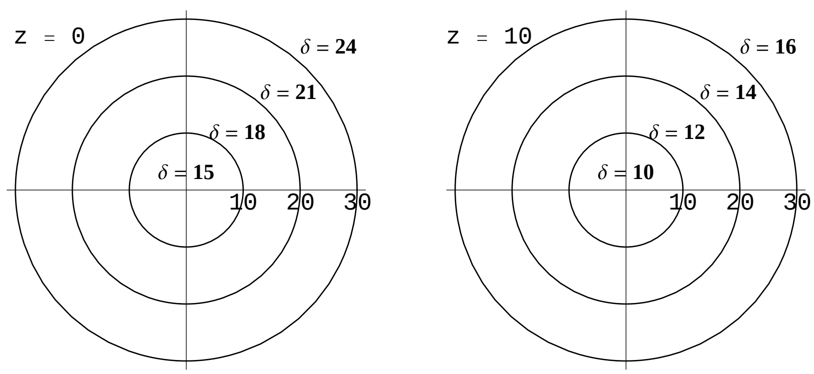
University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #3

1. At the surface, $z = 0$, we have the $z = 0$ section, $\delta(x, y, 0) = \dfrac{\left(50 + \sqrt{x^2 + y^2}\right)(30 - 0)}{100}$.
   The level curves are circles since $\delta(x, y, c)$ is constant whenever $\sqrt{x^2 + y^2}$ is constant.
   The density is $\dfrac{(50)\,(30)}{100} = 15$ kg/m^3 at the center of the pool and increases by 3 kg/m^3 for every 10 m increase in distance from the center of the pool.

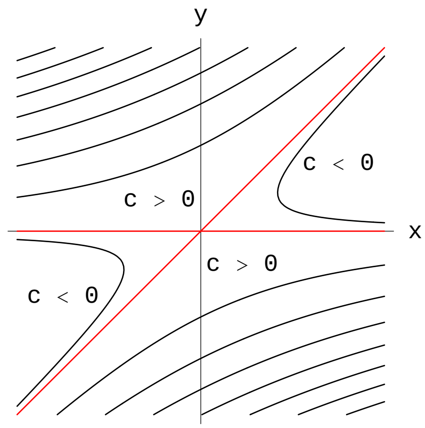
   

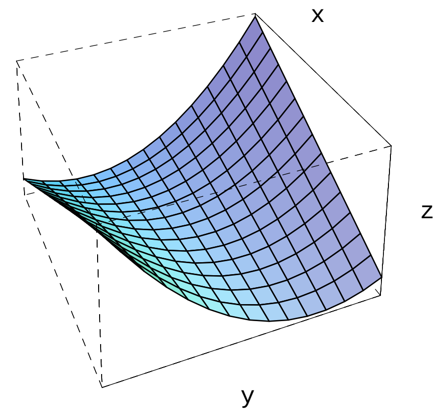
   At 10 m below the surface, $z = 10$, we have $\delta(x, y, 10) = \dfrac{\left(50 + \sqrt{x^2 + y^2}\right)(30 - 10)}{100}$.
   Now the density at the center is $\dfrac{(50)\,(20)}{100} = 10$ kg/m^3 and increases by 2 kg/m^3 for every 10 m increase in distance from the center of the pool.

   This is explained by there being more pollutants near the surface than deeper down and more near the shoreline of the pool than at the center.

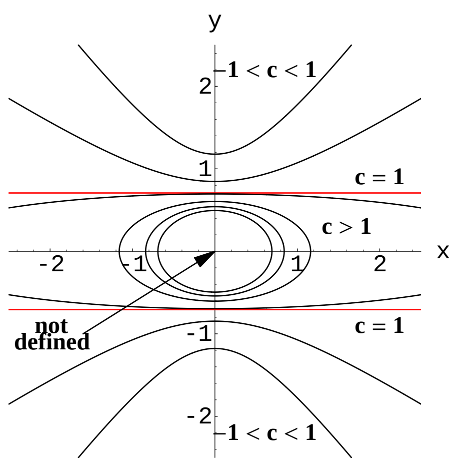
2. (i) $f(x, y) = y^2 - xy$. Domain is $\mathbb{R}^2$. We put $y^2 - xy = c$. If $c = 0$ we have $y(y - x) = 0$ so the level curves are the lines $y = 0$ and $y = x$ (shown in red). If $c \neq 0$, we have $xy = y^2 - c$ or $x = y - \dfrac{c}{y}$, $y \neq 0$. The level curves are hyperbola which are

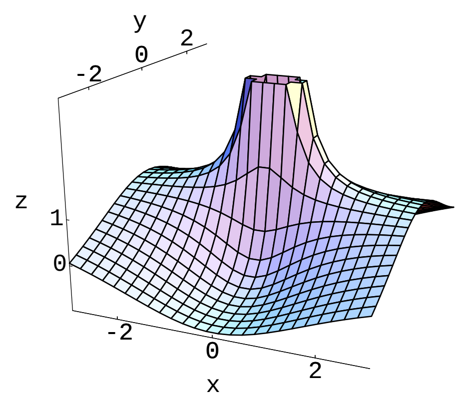
asymptotic to $y = 0$ and $y = x$.





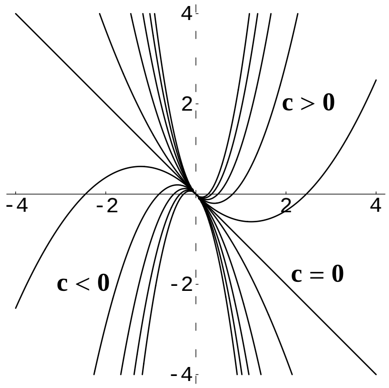
(ii) $f(x, y) = \dfrac{x^2 - y^2 + 1}{x^2 + y^2}$. Domain is $\{(x, y) \in \mathbb{R}^2 \mid (x, y) \neq (0, 0)\}$. Putting $f(x, y) = c$ we have $\dfrac{x^2 - y^2 + 1}{x^2 + y^2} = c \iff (c - 1)\, x^2 + (c + 1)\, y^2 = 1$.
For $c \leq -1$, there is no solution since $c - 1 \leq -2$ and $c + 1 \leq 0$.
For $-1 < c < 1$, we have $c + 1 > 0$, $c - 1 < 0$, so the level curves are hyperbolas crossing the y–axis.
For $c = 1$, we have $2\, y^2 = 1$ or $y = \pm\dfrac{1}{\sqrt{2}}$. The level curve is two lines.
For $c > 1$, we have $c - 1 > 0$, $c + 1 > 0$, so the level curves are ellipses.

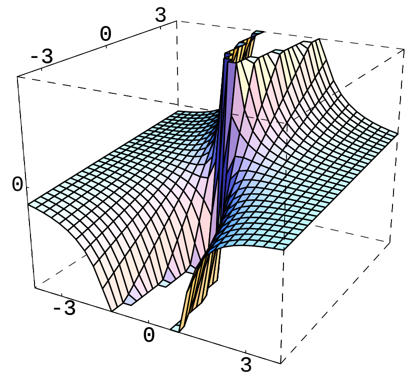




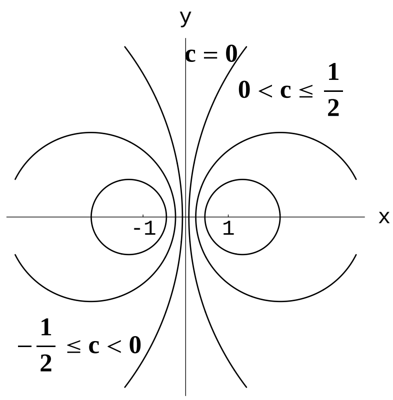
(iii) $f(x, y) = \dfrac{x + y}{x^2}$. Domain is $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$. Putting $f(x, y) = c$ we have $\dfrac{x + y}{x^2} = c$. For $c = 0$ we get the line $y = -x$. For $c \neq 0$, $\dfrac{x + y}{x^2} = c \iff x + y = c\, x^2$ or $y = c\, x^2 - x$. If $c \geq 0$, this gives parabolas with x–intercepts, $x = 0$ and $x = \dfrac{1}{c}$, opening upward. If $c \leq 0$, this gives parabolas with x–intercepts, $x = 0$

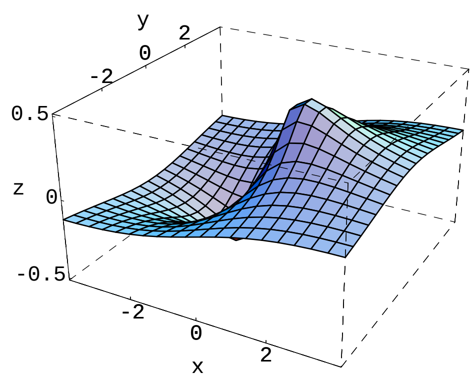
and $x = \dfrac{1}{c}$, opening downward.



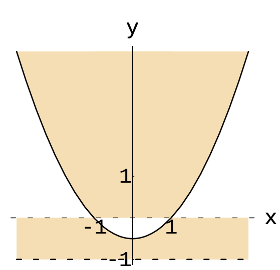


(iv) $f(x,y) = \dfrac{x}{1+x^2+y^2}$. Domain is $\mathbb{R}^2$. Putting $f(x,y) = c$ we have $\dfrac{x}{1+x^2+y^2} = c$ or $x = c(1+x^2+y^2)$.
For $c = 0$, the level curve is the y–axis
For $c \neq 0$ we have $x^2 - \dfrac{1}{c}\,x + y^2 = -1$ and, after completing the square, $\left(x - \dfrac{1}{2\,c}\right)^2 + y^2 = \dfrac{1}{4\,c^2} - 1 = \dfrac{1-4c^2}{4\,c^2}$. (We note that this is only valid for $4c^2 \leq 1$ or $-\dfrac{1}{2} \leq c \leq \dfrac{1}{2} \implies$ range is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.) Hence for $c \neq 0$ the level curve is a circle centered at $\left(\dfrac{1}{2\,c}, 0\right)$, radius $\sqrt{\dfrac{1-4c^2}{4\,c^2}}$.

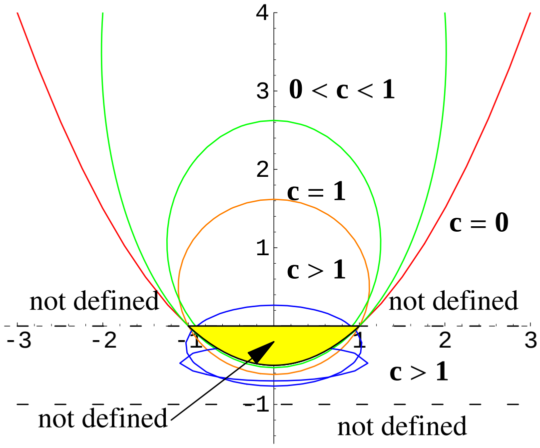


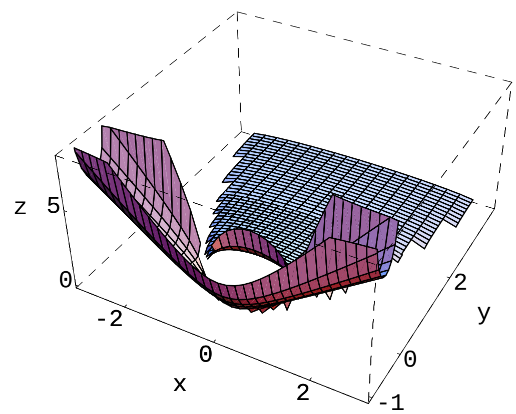


(v) $f(x,y) = \sqrt{\dfrac{1+2y-x^2}{y^2+y}}$. For f to be defined we need $\dfrac{1+2y-x^2}{y^2+y} \geq 0$ and $y \neq 0,\ y \neq -1$. Hence the domain is $\{(x,y) \in \mathbb{R}^2\}$ where $\begin{array}{rcl} 1+2y-x^2 & \geq & 0 \\ y^2+y & > & 0 \end{array}$ and $\begin{array}{rcl} 1+2y-x^2 & \leq & 0 \\ y^2+y & < & 0 \end{array}$. Hence the domain is the shaded region shown on the right. The range is $[0,\infty)$.



Putting $f(x,y)=c$ we have $\dfrac{1+2y-x^2}{y^2+y}=c^2$ or $1+2y-x^2=c^2\,(y^2+y)$.
For $c=0$, the level curve is the parabola $x^2-2y-1=0$.
For $c>0$, we have an ellipse $x^2+c^2\left(y+\dfrac{1}{2}-\dfrac{1}{c^2}\right)^2=1+\left(\dfrac{1}{2}-\dfrac{1}{c^2}\right)^2$ centered at $\left(0,\ \dfrac{1}{c^2}-\dfrac{1}{2}\right)$, axes $\sqrt{1+\left(\dfrac{1}{2}-\dfrac{1}{c^2}\right)^2}$, $\dfrac{1}{c}\sqrt{1+\left(\dfrac{1}{2}-\dfrac{1}{c^2}\right)^2}$.
We should note that when $c=1$, the level curve is a circle.





3. (a) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{e^{xy}}{3x+5y+2}=\frac{e^0}{3(0)+5(0)+2}=\frac{1}{2}$, since elementary functions are continuous on their domains.

(b) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{5y^2}{2x^2+y^2}$. If we restrict to the x–axis, we have $\displaystyle\lim_{(x,y)\to(0,0)}\frac{5y^2}{2x^2+y^2}=\lim_{(x,0)\to(0,0)}\frac{0}{2x^2}=0$. On the other hand, if we restrict to the y–axis, we have $\displaystyle\lim_{(x,y)\to(0,0)}\frac{5y^2}{2x^2+y^2}=\lim_{(0,y)\to(0,0)}\frac{5y^2}{y^2}=5$. Since the two values are different, the limit does not exist.

(c) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3xy}{2x^2-y^2}$. If we restrict to the x–axis, we have $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3xy}{2x^2-y^2}=\lim_{(x,0)\to(0,0)}\frac{0}{2x^2}=0$. On the other hand, if we restrict to the line $y=x$, we have $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3xy}{2x^2-y^2}=\lim_{(x,x)\to(0,0)}\frac{3x^2}{x^2}=3$. Since the two values are different, the limit does not exist.

(d) $\displaystyle\lim_{(x,y)\to(1,-2)}\frac{xy+2x-y-2}{(x^2-1)(y+2)}=\lim_{(x,y)\to(1,-2)}\frac{x(y+2)-1(y+2)}{(x-1)(x+1)(y+2)}\overset{x\neq1}{\underset{y\neq-2}{=}}\lim_{(x,y)\to(1,-2)}\frac{1}{x+1}=\frac{1}{2}$ (continuous function at $(1,-2)$).

(e) $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}\;-1}=\lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)(\sqrt{x^2+y^2+1}\;+1)}{(\sqrt{x^2+y^2+1}\;-1)(\sqrt{x^2+y^2+1}\;+1)}=$

$\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}\ +1)}{(x^2+y^2+1)-1} = \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(\sqrt{x^2+y^2+1}\ +1)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} (\sqrt{x^2+y^2+1}\ +1) = 2.$

(f) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|}$. We will first rewrite in terms of $u = x-1$ and $v = y-1$. We now have $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2+y^2-2}{|x-1|+|y-1|} = \lim_{(u,v)\to(0,0)} \frac{(u+1)^2+(v+1)^2-2}{|u|+|v|} = \lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|}$. If we restrict to $v = u = |u|$, we have $\displaystyle\lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|} = \lim_{u\to 0} \frac{2u^2+4u}{2u} = \lim_{u\to 0}(u+2) = 2$. On the other hand, if we restrict to $v = -u = |u|$, we have $\displaystyle\lim_{(u,v)\to(0,0)} \frac{u^2+2u+v^2+2v}{|u|+|v|} = \lim_{u\to 0} \frac{u^2+2u+u^2-2u}{-u-u} = \lim_{u\to 0} \frac{2u^2}{-2u} = \lim_{u\to 0} -u = 0$. Since the limits along two distinct paths are different, the original limit does not exist.

4. The function would be continuous at $(0,0)$, if the limit, as $(x,y)\to(0,0)$, is equal to $\ln 2$, the value of the function at $(0,0)$. Now $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(2^x-1)(\sin y)}{xy} = \lim_{(x,y)\to(0,0)} \left(\frac{(2^x-1)}{x}\right)\left(\frac{\sin y}{y}\right) = \left(\lim_{x\to 0} \frac{(2^x-1)}{x}\right)\left(\lim_{y\to 0} \frac{\sin y}{y}\right) = \left(\lim_{x\to 0} \frac{(\ln 2)\, 2^x}{1}\right)(1) = \ln 2$. (Because the left limit involves only the single variable x, we were able to use L'Hopital's Rule on that limit.) Since $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, we have $f(x,y)$ continuous at $(0,0)$.

5. (a) $f(x,y) = \dfrac{\sin x}{xy}$. $\dfrac{\partial f}{\partial x} = \dfrac{xy\,\cos x - y\,\sin x}{x^2y^2}$. $\left.\dfrac{\partial f}{\partial x}\right|_{(\frac{\pi}{2},2)} = \dfrac{\pi\cos\frac{\pi}{2} - 2\sin\frac{\pi}{2}}{\frac{\pi^2}{4}\,4} = -\dfrac{2}{\pi^2}$.

(b) $f(x,y,z) = xy + y\cos z - x\sin yz$. $\dfrac{\partial f}{\partial x} = y - \sin yz$. $\left.\dfrac{\partial f}{\partial x}\right|_{(2,-1,\pi)} = -1 - \sin(-\pi) = -1$.

(c) $f(x,y,z) = \ln\sqrt{2z - xy}$. $\dfrac{\partial f}{\partial x} = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{2z-xy}\right)(-y)$.
$\left.\dfrac{\partial f}{\partial x}\right|_{(2,1,3)} = \left(\dfrac{1}{2}\right)\left(\dfrac{1}{4}\right)(-1) = -\dfrac{1}{8}$.

(d) $f(x,y) = \|(x,y)\| = \sqrt{x^2+y^2}$. $\dfrac{\partial f}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2}}$. $\left.\dfrac{\partial f}{\partial x}\right|_{(-1,2)} = \dfrac{-1}{\sqrt{1+4}} = \dfrac{-1}{\sqrt{5}}$.

(e) $f(x,y) = \begin{cases} \dfrac{3xy+5y^3}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$. $\dfrac{\partial f}{\partial x}(0,0) = \displaystyle\lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0}{h^2}-0}{h} = \lim_{h\to 0} \frac{0}{h^3} = 0.$

6. (a) $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x + y \sin x,\ x^2 e^{yx},\ 2^{xy})$.

$$Df = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 + y \cos x & \sin x \\ 2\, x\, e^{yx} + x^2\, y\, e^{yx} & x^3\, e^{yx} \\ y\, 2^{xy}\ \ln 2 & x\, 2^{xy}\ \ln 2 \end{pmatrix}.$$

(f is differentiable for all $(x, y) \in \mathbb{R}^2$.)

(b) $f : \mathbb{R}^5 \to \mathbb{R}^3$, $f(x_1, x_2, x_3, x_4, x_5) = \left(x_1 x_2^2 x_3^3 x_4^4,\ x_5 \tan(x_3 x_4),\ \dfrac{x_1 x_2}{x_3 x_5} \right)$.

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \nabla f_3 \end{pmatrix} =$$

$$\begin{pmatrix} x_2^2 x_3^3 x_4^4 & 2x_1 x_2 x_3^3 x_4^4 & 3x_1 x_2^2 x_3^2 x_4^4 & 4x_1 x_2^2 x_3^3 x_4^3 & 0 \\ 0 & 0 & x_4 x_5 \sec^2(x_3 x_4) & x_3 x_5 \sec^2(x_3 x_4) & \tan(x_3 x_4) \\ \dfrac{x_2}{x_3 x_5} & \dfrac{x_1}{x_3 x_5} & -\dfrac{x_1 x_2}{x_3^2 x_5} & 0 & -\dfrac{x_1 x_2}{x_3 x_5^2} \end{pmatrix}.$$

(f is differentiable for all $\big(x_1, x_2, x_3, x_4, x_5\big)$ such that $x_3 x_4 \neq \dfrac{k\,\pi}{2}$, k an odd integer and $x_3 x_5 \neq 0$.)

(c) $f : \mathbb{R}^n \to \mathbb{R}^k$, $\quad f(\boldsymbol{x}) = A\boldsymbol{x}$ where $A \in M_{k,n}(\mathbb{R})$.

Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$, so $f(x_1, x_2, \ldots, x_n) = (a_{11}x_1 + \cdots + a_{1k}x_n\,, \cdots\,, a_{k1}x_1 + \cdots + a_{kn}x_n)$.

$$Df = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_k \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \cdots & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix} = A.$$

(f is differentiable for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.)

(d) $f : \mathbb{R}^n \to \mathbb{R}$, $\quad f(\boldsymbol{x}) = \|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

$Df = \nabla f$

$$= \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}, \cdots, \frac{x_k}{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}} \right)$$

$$= \frac{1}{\|\boldsymbol{x}\|}\,\boldsymbol{x}.$$

(f is differentiable for all non zero $\boldsymbol{x} \in \mathbb{R}^n$).