University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #7

1. (a) Since f is of class C^3 we must have $\dfrac{\partial^2 f}{\partial x\,\partial y}(\boldsymbol{a}) = \dfrac{\partial^2 f}{\partial y\,\partial x}(\boldsymbol{a})$, so the Hessian matrix must be symmetric. The matrix, $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$, is not symmetric.

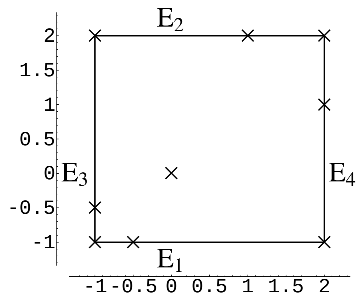
   (b) $\det A_1 = \det(3) = 3 > 0$, $\det A_2 = \det\begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix} = 2 > 0$, $\det A_3 = \det\begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & -2 \end{pmatrix} = -7 < 0$ and $\det A_4 = \det H\,f(\boldsymbol{a}) = \det\begin{pmatrix} 3 & 2 & 0 & 1 \\ 2 & 2 & -1 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & 0 & -3 \end{pmatrix} = 26 > 0$. Since we have the sequence $+\ +\ -\ +$, the critical point yields a saddle point.

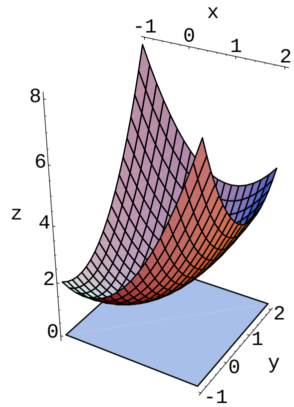
2. (a) Since $f(x, y) = x^2 - xy + y^2 + 1$ is continuous on all of $\mathbb{R}^2$ and the closed square is compact (closed and bounded), f will attain a global maximum and a global minimum on either the interior of the closed square or on its boundary. The boundary of this closed square consists of the sets, $E_1 = \{(x, y) \mid y = -1, -1 \le x \le 2\}$, $E_2 = \{(x, y) \mid y = 2, -1 \le x \le 2\}$, $E_3 = \{x, y) \mid x = -1, -1 \le y \le 2\}$ and $E_4 = \{(x, y) \mid x = 2, -1 \le y \le 2\}$. We need to find critical points.

   (i) <u>on the interior</u> $f_x = 2x - y$ and $f_y = -x + 2y$, so $\nabla f = \mathbf{0}$ only if $x = 0$ and $y = 0$. Since $(0, 0) \in$ interior, we have one critical point $(0, 0)$ on the interior.

   (ii) <u>on the boundary</u>
   <u>on E_1</u> Define $f_1(x) = f(x, -1) = x^2 + x + 2$. Now $f_1'(x) = 2x + 1 = 0$ if $x = -\dfrac{1}{2}$. Since $\left(-\dfrac{1}{2}, -1\right) \in E_1$, it is a critical point for f_1.
   <u>on E_2</u> Define $f_2(x) = f(x, 2) = x^2 - 2x + 5$. Now $f_2'(x) = 2x - 2 = 0$ if $x = 1$. Since $(1, 2) \in E_2$, it is a critical point for f_2.
   <u>on E_3</u> Define $f_3(y) = f(-1, y) = y^2 + y + 2$. Now $f_3'(y) = 2y + 1 = 0$ if $y = -\dfrac{1}{2}$. Since $\left(-1, -\dfrac{1}{2}\right) \in E_3$, it is a critical point for f_3.
   <u>on E_4</u> Define $f_4(y) = f(2, y) = y^2 - 2y + 5$. Now $f_4'(y) = 2y - 2 = 0$ if $y = 1$. Since $(2, 1) \in E_4$, it is a critical point for f_4.

We now evaluate f at all critical points and corners. $f(0,0) = 1$, $f\left(-\dfrac{1}{2}, -1\right) = \dfrac{7}{4}$, $f(1,2) = 4$, $f\left(-1, -\dfrac{1}{2}\right) = \dfrac{7}{4}$, $f(2,1) = 4$, $f(-1,-1) = 2$, $f(2,-1) = 8$, $f(2,2) = 5$ and $f(-1,2) = 8$. The global minimum occurs at $(0,0)$ and the global maximum occurs at both $(2,-1)$ and $(-1,2)$.

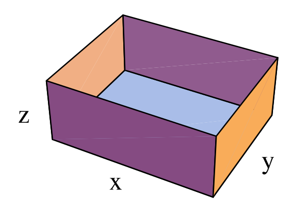




The figures above show the closed square with the critical points indicated and the piece of the graph of $z = f(x,y)$ which lies over the closed square.

(b) Since f is a polynomial, it is continuous on the closed square, $[-1,2] \times [-1,2]$, which is compact in $\mathbb{R}^2$. Hence the Extreme Value Theorem (EVT) ensures that f will attain both a maximum and a minimum on the closed square.

3. Let the length be x, the width be y and the height be z (as shown in the figure on the right). The surface area is $xy + 2xz + 2yz = 108$ and the volume is $V = xyz$.



$\Big($ To do this question we could solve for z giving $z = \dfrac{108 - xy}{2(x+y)}$ and proceed as usual or we could use Lagrange multipliers (as on the next assignment). For this question we are required to treat z as an implicit function. $\Big)$

We think of V as a function of two variables, x and y.

Now $\dfrac{\partial V}{\partial x} = \left(x\,\dfrac{\partial z}{\partial x} + z\right) y$ and $\dfrac{\partial V}{\partial y} = \left(y\,\dfrac{\partial z}{\partial y} + z\right) x$ so, using $xy + 2xz + 2yz = 108$, we have $y + 2\left(x\,\dfrac{\partial z}{\partial x} + z\right) + 2\,y\,\dfrac{\partial z}{\partial x} = 0$ (w.r.t. x) and $x + 2\,x\,\dfrac{\partial z}{\partial y} + 2\left(y\,\dfrac{\partial z}{\partial y} + z\right) = 0$ (w.r.t. y). This gives $\dfrac{\partial z}{\partial x} = -\dfrac{2z + y}{2(x+y)}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{2z + x}{2(x+y)}$. Hence $\dfrac{\partial V}{\partial x} = \left(x\,\dfrac{\partial z}{\partial x} + z\right) y = 0 \implies x\,\dfrac{\partial z}{\partial x} + z = 0$, since $y \neq 0$ and $\dfrac{\partial V}{\partial y} = \left(y\,\dfrac{\partial z}{\partial y} + z\right) x = 0 \implies y\,\dfrac{\partial z}{\partial y} + z = 0$, since $x \neq 0$. Therefore $\dfrac{\partial z}{\partial x} = -\dfrac{z}{x}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{z}{y}$ and, consequently, $-\dfrac{z}{x} = -\dfrac{2z + y}{2(x+y)}$ and $-\dfrac{z}{y} = -\dfrac{2z + x}{2(x+y)}$. Solving for z gives $z = \dfrac{x}{2}$ and $z = \dfrac{y}{2}$. Substituting in the equation for surface area gives $4z^2 + 4z^2 + 4z^2 = 108 \implies z = 3$ and $x = y = 6$. Since we know optimal dimensions exist and this is the only candidate, it must give the maximal volume. The maximal volume is $(6)(6)(3) = 108$ cm^3.

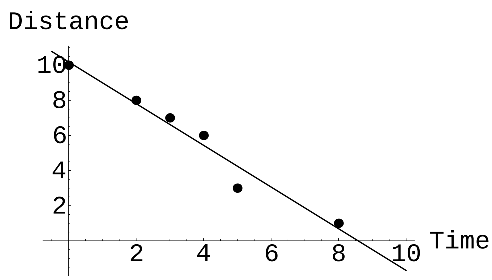
4. Let x and y be as stated in the problem. We wish to maximize the profit function, $P(x,y) = 8x + 10y - (0.001)(x^2 + xy + y^2) - 10{,}000$. Since $P(x,y)$ is continuous for all $(x,y) \in \mathbb{R}^2$, critical points can only occur when $\nabla P = (0,0)$. Now $P_x = 8 - 0.001\,(2x+y)$ and $P_y = 10 - 0.001\,(x+2y)$. Equating these to 0 and simplifying, we have $\begin{cases} 2x + \;\; y &= \;\;8000 \\ \;\; x + 2y &= 10000 \end{cases}$. Solving we get $x = 2000$ and $y = 4000$. The Hessian matrix is $H\,P = \begin{pmatrix} -0.002 & -0.001 \\ -0.001 & -0.002 \end{pmatrix}$ which is independant of the critical point. Since $\det A_1 = \det(-0.002) = -0.002 < 0$ and $\det A_2 = \det H\,P\,(2000, 4000) = .000001 > 0$, we can conclude that we have a local maximum. Since the graph of this profit function is an ellipitical paraboloid opening downward, the sales level of $x = 2000$ units and $y = 4000$ units yields a maximum profit of $\$18,000$.

5. We want to maximize the volume of a box with one corner at the origin and the opposite corner on the paraboloid $z = 1 - \dfrac{x^2}{4} - \dfrac{y^2}{9}$. Using this fact we wish to maximize the volume $V = x\,y\,z = x\,y\left(1 - \dfrac{x^2}{4} - \dfrac{y^2}{9}\right) = xy - \dfrac{x^3\,y}{4} - \dfrac{x\,y^3}{9}$. Since V is a polynomial, critical points will occur when $\nabla V = \mathbf{0}$. Now $V_x = y - \dfrac{3x^2y}{4} - \dfrac{y^3}{9}$, $\;V_y = x - \dfrac{x^3}{4} - \dfrac{xy^2}{3}$. After equating both to zero, the second gives $x\left(1 - \dfrac{x^2}{4} - \dfrac{y^2}{3}\right) = 0 \implies x = 0$, (and $V = 0$) or $y = \sqrt{3 - \dfrac{3\,x^2}{4}}$. Substituting into the first and factoring, we have $\sqrt{3 - \dfrac{3\,x^2}{4}}\left(1 - \dfrac{3\,x^2}{4} - \dfrac{1}{9}\left(3 - \dfrac{3\,x^2}{4}\right)\right) = 0 \implies 3 - \dfrac{3\,x^2}{4} = 0 \implies x = 2$ (only need first octant) or $1 - \dfrac{3\,x^2}{4} - \dfrac{1}{3} + \dfrac{x^2}{12} = 0 \implies x^2 = 1 \implies x = 1$.
$x = 2 \implies y = 0 \implies z = 0 \implies V = 0$.
$x = 1 \implies y = \dfrac{3}{2} \implies z = \dfrac{1}{2} \implies V = \dfrac{3}{4}$.
Since there must be a largest box and symmetry of the paraboloid allows us to only consider the first octant, we have found the largest volume to be $\dfrac{3}{4}$.

6. (a) Let the line be $D = m\,t + b$. We will minimize the sum of the squares of the differences in the "D" values; i.e., we minimize the function $f(m,b) = (10-b)^2 + (8-2m-b)^2 + (7-3m-b)^2 + (6-4m-b)^2 + (3-5m-b)^2 + (1-8m-b)^2$. Now $f_m = 2\,(8-2m-b)(-2) + 2\,(7-3m-b)(-3) + 2\,(6-4m-b)(-4) + 2\,(3-5m-b)(-5) + 2\,(1-8m-b)(-8) = -168 + 236\,m + 44\,b$ and
$f_b = 2\,(10-b)(-1) + 2\,(8-2m-b)(-1) + 2\,(7-3m-b)(-1) + 2\,(6-4m-b)(-1) + 2\,(3-5m-b)(-1) + 2\,(1-8m-b)(-1) = -70 + 44\,m + 12\,b$.

Setting $\nabla f = \mathbf{0}$ gives $\left\{\begin{array}{rcrcr} 236\,m & + & 44\,b & = & 168 \\ 44\,m & + & 12\,b & = & 70 \end{array}\right. \implies$
$\left\{\begin{array}{rcrcr} 2832\,m & + & (12)(44)\,b & = & 2016 \\ 1936\,m & + & (44)(12)\,b & = & 3080 \end{array}\right.$. Subtracting the second from the first gives $896\,m = -1064 \implies m = -\dfrac{19}{16} \implies 12\,b = 70 - 44\left(-\dfrac{19}{16}\right) = \dfrac{489}{4} \implies b = \dfrac{163}{16}$.
Checking the Hessian, we have $H\,f = \begin{pmatrix} 236 & 44 \\ 44 & 12 \end{pmatrix}$ with $\det A_1 = 236 > 0$ and $\det A_2 = \det H\,f = \det \begin{pmatrix} 236 & 44 \\ 44 & 12 \end{pmatrix} = 896 > 0$. Hence we have a local minimum. Since there must be such a line, we have found it. The "best fit" line is $D = -\dfrac{19}{16}\,t + \dfrac{163}{16}$.

(b) $D = -\dfrac{19}{16}\,t + \dfrac{163}{16}$.



(c) We use the "best fit" line to estimate the time until the distance is 0. $D = -\dfrac{19}{16}\,t + \dfrac{163}{16} \implies \dfrac{19}{16}\,t = \dfrac{163}{16} \implies t = \dfrac{163}{19}$. The distance will be 0 when t is approximately 8.58 sec.

7. (3) $f(x, y, z) = x - y + z$ subject to the constraint $x^2 + y^2 + z^2 = 2$. We define $h(x, y, z, \lambda) = x - y + z - \lambda(x^2 + y^2 + z^2 - 2)$. The critical points of h will give the constrainted critical points of f. Now $h_x = 1 - 2\,\lambda\,x,\ h_y = -1 + 2\,\lambda\,y,\ 1 - 2\,\lambda\,z,\ h_\lambda = -(x^2 + y^2 + z^2 - 2)$. Put $h_x = h_y = h_z = h_\lambda = 0$. The first three give $x = y = z = \dfrac{1}{2\lambda}$, then the fourth gives $\dfrac{3}{4\,\lambda^2} = 2 \implies \lambda = \pm\left(\dfrac{1}{2}\right)\sqrt{\dfrac{3}{2}} \implies$ $(x, y, z) = \left(\pm\sqrt{\dfrac{2}{3}}, \mp\sqrt{\dfrac{2}{3}}, \pm\sqrt{\dfrac{2}{3}}\right)$. Now $f\left(\sqrt{\dfrac{2}{3}}, -\sqrt{\dfrac{2}{3}}, \sqrt{\dfrac{2}{3}}\right) = \sqrt{6}$ (maximum) and $f\left(-\sqrt{\dfrac{2}{3}}, \sqrt{\dfrac{2}{3}}, -\sqrt{\dfrac{2}{3}}\right) = -\sqrt{6}$ (minimum).

(4) $f(x, y) = x - y$ subject to the constraint $x^2 - y^2 = 2$. If we proceed as usual, define $h(x, y, \lambda) = x - y - \lambda(x^2 - y^2 - 2)$, compute $h_x = 1 - 2\,\lambda\,x,\ h_y = -1 + 2\,\lambda\,y,\ h_\lambda = -(x^2 - y^2 - 2)$ and equate to zero. The first two give $x = \dfrac{1}{2\lambda},\ y = \dfrac{1}{2\lambda}$, the substituting into the third gives $\dfrac{1}{4\lambda^2} - \dfrac{1}{4\lambda^2} = 2 \implies 0 = 2$ which is impossible. This happens because the graph of f and the constraint are never tangent.

(5) $f(x,y) = x$ subject to the constraint $x^2 + 2y^2 = 3$. We define $h(x,y,\lambda) = x - \lambda\,(x^2 + 2y^2 - 3)$. The critical points of h will give the constrained critical points of f. Now $h_x = 1 - 2\,\lambda\,x$, $h_y = -4\,\lambda\,y$, $h_\lambda = -(x^2 + 2y^2 - 3)$. Put $h_x = h_y = h_\lambda = 0$. The first gives $\lambda \neq 0$ and $x = \dfrac{1}{2\,\lambda}$ and the second gives $y = 0$. Substituting into the third gives $\dfrac{1}{4\,\lambda^2} = 3 \implies \lambda = \pm\dfrac{2}{\sqrt{3}} \implies x = \pm\sqrt{3}$. Now $f(\sqrt{3},\,0) = \sqrt{3}$ (maximum) and $f(-\sqrt{3},\,0) = -\sqrt{3}$ (minimum).