University of Toronto at Scarborough
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MAT B41 — Final Exam

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Date: 13 December 2006
Duration: 180 minutes

1. Define the following two vector subspaces of $\mathbf{R}^4$.

$$V = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 + x_2 - 2x_3 - 2x_4 = 0 \right\}.$$

   (a) Find an orthonormal basis for V. (Hint: you may assume that the vectors defining V are linearly independent.)
   (b) Find a basis for $V \cap W$.
   (c) Find the matrix of the linear transformation for orthogonal projection onto $W^\perp$.

2. Define the function f by $f(x,y) = \dfrac{x^2y^2}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

   (a) What are the partial derivatives at (x,y)?
   (b) Do the partial derivatives exist at $(0,0)$? If so, what are they?
   (c) Are the partial derivatives continuous at $(0,0)$?
   (d) Is f differentiable at $(0,0)$?

3. Suppose a function $z = g(x,y)$ is defined implicitly by the equation $F(x,y,z) = 0$ where F is a smooth function with $\dfrac{\partial F}{\partial z} \neq 0$. Find a formula for the second partial derivative $\dfrac{\partial^2 g}{\partial x\, \partial y}$ at the point $(x,y,g(x,y))$ in terms of the partial derivatives of F at that point.

4. The graph of the function f defined by $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4 + 2$ describes the surface of a mountain range.

   (a) Find the components of the upward-pointing unit normal vector of the surface at the point $(1,1,0)$.
   (b) Find a basis for the tangent space of the surface at the point $(1,1,0)$.
   (c) Find the components of the vector tangent to the surface at the point $(1,1,0)$ and pointing in the direction of steepest ascent. (Hint: this is the vector in $\mathbf{R}^3$ which points in the direction that you move when you decide to climb the mountain in such a way as to increase your altitude most rapidly.)

5. Define the function f by $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$.

   (a) Find all critical points of the function f.
   (b) Find the Hessian of f at the point (x,y).
   (c) Classify each of the critical points f using the second derivative test.

6. (a) Use the method of Lagrange multipliers to find the critical points of the function f from Question 5 subject to the constraint $x^2 + y^2 = 1$.

(b) What is the global maximum value of the function f from Question 5 on the set $\{(x, y) : x^2 + y^2 \leq 1\}$ and justify your answer.

(c) Below is a picture of certain level sets of a function g having only non-degenerate critical points. Indicate on the picture where g has its critical points. Indicate also the type (saddle, maximum, minimum).