University of Toronto Scarborough
Department of Computer & Mathematical Sciences

Midterm Test

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: October 24, 2015
Start time: 5:00pm
Duration: 110 minutes

1. **[8 points]** In this question, be sure to indicate what type of object each of your symbols represents.

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable* at $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the Chain Rule for functions of more than one variable.

2. **[15 points]**

   (a) Calculate the following limits, showing all your steps, or show that the limit does not exist.

   i. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x^2+y^2}$.

   ii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y}$.

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy-y}{(x-1)^2+y^2} & , \text{ if } (x,y) \neq (1,0) \\ 0 & , \text{ if } (x,y) = (1,0) \end{cases}.$$

   Determine if f is continuous at $(1,0)$. (Explain your answer.)

3. **[11 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \frac{1}{\sqrt{(y-1)(y-x^2)}}$$

   and let D be the domain of f.

   (a) i. Use set notation to describe D.

   ii. Carefully sketch D.

(b) Give a complete explanation as to why f is, or is not, continuous on D. (Indicate the properties and theorems you are using.)

4. **[12 points]** Characterize and sketch several level curves of the function

$$f(x,y) = \frac{x^2}{x-y} \ .$$

Carefully indicate where f is zero, positive, negative and not defined.

5. **[13 points]**

(a) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = \dfrac{x^2}{x-y}$.

i. Find an equation for the tangent plane to the graph of $z = f(x,y)$ at the point $(1, 2, f(1,2))$.

ii. Find the direction of maximum increase in f at the point $(1,2)$.

(b) Find an equation for the tangent plane at the point $(-3,1,0)$ to the graph of $z = f(x,y)$ define implicitly by $x\,(y^2+z^2) + y\,e^{xz} = -2$.

6. **[10 points]**

(a) Determine the rate of change in $f(x,y,z) = y - x^2 + z^2$ as you move from $(-1,0,2)$ towards $(2,4,2)$.

(b) Compute the directional derivative of $f(x,y,z) = x^2y^3z^2$ at the point $(2,1,-1)$ in the direction of the upward normal for the plane $2x + y - 2z = -7$.

7. **[12 points]** Let π be the plane in $\mathbb{R}^3$ passing through the points $(-1,0,2)$, $(1,3,1)$ and $(2,1,-1)$.

(a) Give an equation for π.

(b) Give a parametric description of the line ℓ through $(0,0,1)$ and orthogonal to π. Where does ℓ meet π.

8. **[12 points]** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x,y,z,) = (e^{yz}, xy, x^2z)$ and let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $g(x,y,z) = (y^2 + z,\ xz^2)$.
`USE THE CHAIN RULE` to compute $D(g \circ f)(x,y,z)$.

(**NOTE:** You must use the Chain Rule and show all your steps.)

9. **[6 points]** Give the 6^{th} degree Taylor polynomial about the origin of $f(x,y) = \dfrac{\cos(2xy)}{3-6x}$.

10. **[6 points]** Let $f(x,y) = \ln(1+x+y)$. Use a quadratic approximation to estimate $f(0.2,\ 0.3)$.