University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 15, 2008
Duration: 3 hours

1. [**4 points**] Evaluate $\lim_{(x,y)\to(0,0)} \dfrac{\cos(x\,y)\ -1}{x^2\,y^2}$ or show that it does not exist.

2. [**5 points**] Let $f(x,y)$ be a real-valued function of class C^2.
If $f_y = e^x + xy^3 + 5x$, which of the following,if any, can be f_x ? Explain your answer.

   (a) $e^y + \dfrac{y^4}{4} + 5y + 5$

   (b) $y\,e^x + \dfrac{y^4}{4} + \dfrac{x}{2} + 5y$

   (c) $y\,e^y + \dfrac{y^4}{4} + \dfrac{x}{2} + 5y$

   (d) $e^x + \dfrac{y^4}{4} + \dfrac{x}{2} + 5$

3. [**9 points**]

   (a) Carefully state what it means for a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ to be differentiable.

   (b) Carefully state the Chain Rule for functions of more than one variable.

   (c) Carefully state the Extreme Value Theorem for real-valued functions of several variables.

4. [**4 points**] Give the 5^{th} degree Taylor polynomial about the origin of
$f(x,y) = (x+1)\,\cos(x\,y)$.

5. [**8 points**] Let $f(x,y,z) = x^2 - 2xy + 2y^2 - 3xz + z^3$ and let $\boldsymbol{p} = (2,1,-1)$ be a point in $\mathbb{R}^3$.

   (a) Determine the direction and magnitude of the maximal increase in f at $\big(\boldsymbol{p},\ f(\boldsymbol{p})\big)$.

   (b) Find an equation for the tangent plane to the level surface of f that passes through $\boldsymbol{p}$.

6. **[5 points]** Let $z = f(x, y)$, where $f : \mathbb{R}^2 \to \mathbb{R}$ is of class C^2. Let $x = u + v$ and $y = 2u - 2v$. Compute a formula for $\dfrac{\partial^2 z}{\partial u\, \partial v}$ in terms of the partial derivatives of z with respect to x and y.

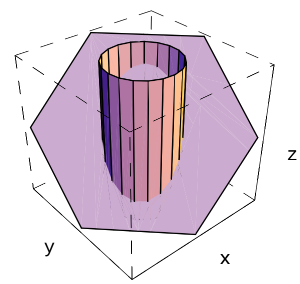
7. **[9 points]** Let $f(x, y, z) = x^3 + y^2 - z^2 - 6xy + 6x + 3y + 1$. Find and classify the critical points of f.

8. **[10 points]** Find the minimum value of $x^2 - 2x + y^2 - 4y + z^2 - 4z + 1$ on the solid ball $x^2 + y^2 + z^2 \leq 9$.

   Justify your answer including an explanation of why global extrema do exist.

9. **[10 points]** The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse. Find the points on this ellipse that lie closest to and farthest from the origin.

   Justify your answer including an explanation of why global extrema do exist.



10. **[10 points]**

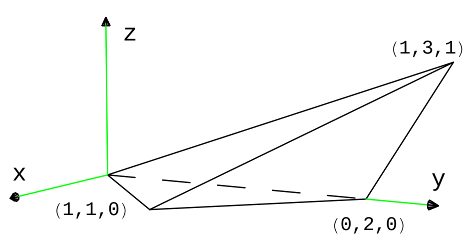
   (a) Rewrite the integral $\displaystyle\int_0^1 \int_y^{\sqrt{2-y^2}} f(x, y)\, dx\, dy$ with the order of integration reversed.

   (b) Use a double integral to find the area of the region D, where

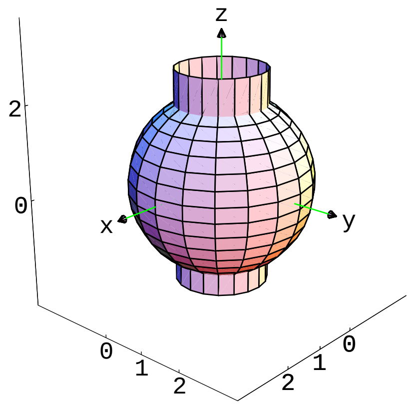
   $$D = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; 4x^2 + 12xy + 13y^2 + 40y \leq -75 \right\} .$$

   (Hint: complete the squares.)

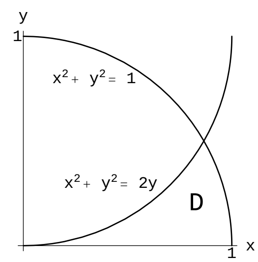
11. **[8 points]** Evaluate $\displaystyle\int_B (x + y + 2z)\, dV$ where B is the tetrahedron with vertices $(0, 0, 0)$, $(0, 2, 0)$, $(1, 3, 1)$ and $(1, 1, 0)$.



12. **[9 points]** A central cylinder of radius 1 is drilled out of a sphere of radius 2 (see figure on the right). Let B be the region inside the sphere, but outside the cylinder. Evaluate $\displaystyle\int_B \frac{1}{x^2+y^2+z^2}\,dV$.



13. **[9 points]** Let D be the first quadrant region inside $x^2+y^2=1$ but outside $x^2+y^2=2y$. Let $u=x^2+y^2$ and $v=x^2+y^2-2y$.



   (a) Give an integral in the variables u and v which is equivalent to $\displaystyle\int_D x\,e^y\,dx\,dy$.

   (b) Evaluate the integral.