University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 10, 2010
Duration: 3 hours

1. [**4 points**] Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ or show that it does not exist.

2. [**5 points**] Give the 4^{th} degree Taylor polynomial about the origin of $f(x,y) = \dfrac{e^{-xy}}{1+x^2}$.

3. [**8 points**]

   (a) Let $f(x,y) = \begin{cases} \dfrac{3xy+5y^3}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \ . \end{cases}$ Find $\dfrac{\partial f}{\partial y}(0,0)$.

   (b) Carefully state what it means for a function $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ to be differentiable.

4. [**8 points**] Let $f(x,y,z) = x\,y + y\,z + z\,x$ and let $\boldsymbol{a} = (1,\,2,\,3)$ be a point in $\mathbb{R}^3$.

   (a) Find the equation of the tangent plane to the level set $f(x,y,z) = 11$ at $\boldsymbol{a}$.

   (b) Find the directional derivative of $w = f(x,y,z)$ at $\boldsymbol{a}$ in the direction $\boldsymbol{v} = (2,\,0,\,2)$.

5. [**15 points**]

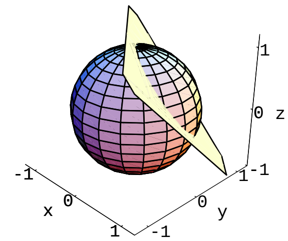
   (a) Carefully state the Chain Rule for functions of more than one variable.

   (b) Let $f(x,y,z)$ be a differentiable function from $\mathbb{R}^3$ to $\mathbb{R}$. If $x = t^2$, $y = t^3$ and $z = t^4$, use the Chain Rule to give a formula for $\dfrac{d\,f}{d\,t}$ at the point where $t = 2$.

   (c) Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be be given by $f(x,y,z) = (yz^2,\, xyz)$ and let $g : \mathbb{R}^2 \to \mathbb{R}^4$ be given by $g(x,y) = (x+y, xy, x, -y)$. Find Df and Dg and use the Chain Rule to find $D(g \circ f)$.

6. **[7 points]** Let $f(x, y) = 2x^4 + x^2 + 2xy + y^2 + x$. Find and classify the critical points of f.

7. **[9 points]** Find the extreme values of $f(x, y, z) = x$ on the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 1$.
   Justify your answers including an explanation of why global extrema do exist.

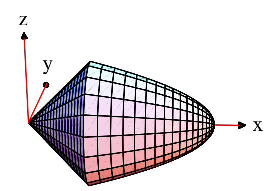


8. **[9 points]** Find the maximum and minimum values of
   $f(x, y, z) = \dfrac{1}{3}x^3 + 5y^2 + 6yz + 5z^2$ on the solid ball $x^2 + y^2 + z^2 \leq 1$.
   Justify your answers including an explanation of why global extrema do exist.

9. **[10 points]**
   (a) Evaluate $\displaystyle\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\, dy\, dx$.
   (b) Evaluate $\displaystyle\iint_D x\, dA$, where D is the first quadrant region between $y = x$ and $y = x^3$.

10. **[8 points]** Find the volume of the first octant solid bounded by the graphs of $z = 1 - y^2$, $y = 2x$ and $x = 3$.

11. **[8 points]** Use a triple integral to find the volume of the solid bounded by the cone $x = \sqrt{y^2 + z^2}$ and the paraboloid $x = 6 - y^2 - z^2$.



12. **[14 points]**
   (a) Carefully state the Change of Variables Theorem for multiple integrals. Make sure you define your terms.
   (b) Use a change of variable to evaluate $\iint_D xy\, dA$, where D is the first quadrant region bounded by $xy = 1$, $xy = 5$, $y = x^2$ and $y = 4x^2$.