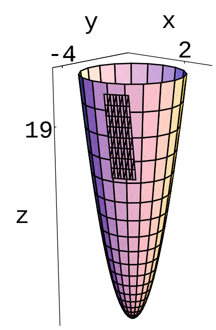
University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #5

1. A normal vector to the plane $4x-8y-z=3$ is $(4,-8,-1)$ and a normal vector to the tangent plane to the graph of $f(x,y)=x^2+y^2-1$ is $(2x,\,2y,\,-1)$. For the planes to be parallel we need $(2x,\,2y,\,-1)=k\,(4,-8,-1)$, for some k. From the third component we see that $k=1$, so we have $x=2$ and $y=-4$. The tangent plane passes through $(2,-4,f(2,-4))=(2,-4,19)$ and has equation $4x-8y-z=21$.



2. (a) We have $f(0,0)=0$, so we can compute the partial derivatives from the definition.
$$\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$
$$\frac{\partial f}{\partial y}(0,0)=\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h}=\lim_{h\to 0}\frac{0-0}{h}=0$$

(b) Recall that for a function f to be differentiable at a point $\boldsymbol{a}$, we need
$$\lim_{\boldsymbol{x}\to\boldsymbol{a}}\frac{\|f(\boldsymbol{x})-f(\boldsymbol{a})-D\,f(\boldsymbol{a})\big(\boldsymbol{x}-\boldsymbol{a}\big)\|}{\|\boldsymbol{x}-\boldsymbol{a}\|}=0.$$
Here we need to evaluate $\displaystyle\lim_{(x,y)\to(0,0)}\frac{|x^{1/3}y^{1/3}-0-(0,0)\cdot(x,y)|}{\|\,(x,y)\,\|}=\lim_{(x,y)\to(0,0)}\frac{|x^{1/3}y^{1/3}|}{\sqrt{x^2+y^2}}$. If we approach along the line $y=x$, with $x>0$, this limit reduces to $\displaystyle\lim_{x\to 0^+}\frac{x^{2/3}}{\sqrt{2\,x^2}}=\lim_{x\to 0^+}\frac{x^{2/3}}{x\,\sqrt{2}}=\lim_{x\to 0^+}\frac{1}{x^{1/3}\,\sqrt{2}}=\infty$. Since this limit is not 0, f is not differentiable at $(0,0)$.

3. The rate of change in depth is the directional derivative.

(a) The depth will increase most rapidly in the direction of the gradient; i.e., in direction $\nabla D\,(1,-2)$. Now $\nabla D=(-6xy^2,\,-6x^2y)$ so the rubber duck swims in direction $\nabla D\,(1,-2)=(-24,\,12)$.

(b) The depth will stay constant if the duck stays on the level set which passes through $(1,-2)$. Since the gradient is orthogonal to level sets it can proceed in direction $(1,2)$ or direction $(-1,-2)$ since $\pm 1\,(1,2)\cdot\nabla D\,(1,-2)=\pm 1\,(1,2)\cdot(-24,12)=0$.

4. (a) Let $x = u^2 - v^2$ and $y = v^2 - u^2$. Then $g(u,v) = f(x,y)$ and the Chain Rule can be applied to give $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u} = \dfrac{\partial f}{\partial x}\big(2u\big) + \dfrac{\partial f}{\partial y}\big(-2u\big)$ and $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} = \dfrac{\partial f}{\partial x}\big(-2v\big) + \dfrac{\partial f}{\partial y}\big(2v\big)$. Hence we have $v\,\dfrac{\partial g}{\partial u} + u\,\dfrac{\partial g}{\partial v} = \left(2uv\,\dfrac{\partial f}{\partial x} - 2uv\,\dfrac{\partial f}{\partial y}\right) + \left(-2uv\,\dfrac{\partial f}{\partial x} + 2uv\,\dfrac{\partial f}{\partial y}\right) = 0$.

(b) $\dfrac{\partial w}{\partial v} \overset{\text{Chain}}{\underset{\text{Rule}}{=}} \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} = \dfrac{\partial w}{\partial x}\big(1\big) + \dfrac{\partial w}{\partial y}\big(-1\big) = \dfrac{\partial w}{\partial x} - \dfrac{\partial w}{\partial y}$.

$$\frac{\partial^2 w}{\partial u\,\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial u}\left(\frac{\partial w}{\partial y}\right) \overset{\text{Chain}}{\underset{\text{Rule}}{=}} \left(\frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\left(\frac{\partial w}{\partial x}\right)}{\partial y}\frac{\partial y}{\partial u}\right) - \left(\frac{\partial\left(\frac{\partial w}{\partial y}\right)}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial\left(\frac{\partial w}{\partial y}\right)}{\partial y}\frac{\partial y}{\partial u}\right) = \frac{\partial^2 w}{\partial x^2}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y\,\partial x}\frac{\partial y}{\partial u} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial x}{\partial u} + \frac{\partial^2 w}{\partial y^2}\frac{\partial y}{\partial u}\right) = \frac{\partial^2 w}{\partial x^2}(1) + \frac{\partial^2 w}{\partial y\,\partial x}(1) - \frac{\partial^2 w}{\partial x\,\partial y}(1) - \frac{\partial^2 w}{\partial y^2}(1) \overset{f\text{ is of}}{\underset{\text{class } C^2}{=}} \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

5. (a) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$. Here we must use the definition to compute the partials. Hence, $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \dfrac{\dfrac{h\,(0)^2}{h^2+0^2} - 0}{h} = \lim\limits_{h\to 0}\dfrac{0}{h} = 0$, and $\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{h\to 0} \dfrac{\dfrac{(0)\,(h)^2}{0^2+h^2} - 0}{h} = \lim\limits_{h\to 0}\dfrac{0}{h} = 0$ showing that the partials exist. This is a case where the partials exist but the derivative does not.

(b) We will now see why the chain rule requires the existence of the derivative. Put $g(t) = (a\,t,\, b\,t)$ and we have $(f\circ g)(t) = \dfrac{(at)\,(bt)^2}{(at)^2+(bt)^2} = \dfrac{a\,b^2}{a^2+b^2}\,t$, which is defined even when $t = 0$. Hence $(f\circ g)'(t) = \dfrac{a\,b^2}{a^2+b^2}$. From part (a) we have $\left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)(0,0) = (0,0)$. We can differentiate $g(t)$ to get $g'(t) = (a,b)$. Now taking the dot product gives us $\left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right)\cdot g'(t) = (0,0)\cdot(a,b) = 0 \neq \dfrac{a\,b^2}{a^2+b^2}$.

6. Put $g(t) = f\big(x(t),\, y(t)\big)$. Now the chain rule gives $g'(t) = x'(t)\,f_x + y'(t)\,f_y$. Since we are given that $x'(t)\,f_x + y'(t)\,f_y \le 0$, $g'(t) \le 0$. Hence $\displaystyle\int_0^1 g'(t)\,dt \le 0$. The

Fundamental Theorem of Calculus (FTC) gives $g(1) - g(0) = \displaystyle\int_0^1 g'(t)\,dt$. Hence $g(1) - g(0) \leq 0$. Hence we have $f\big(x(1),\, y(1)\big) = g(1) \leq g(0) = f\big(x(0),\, y(0)\big)$.

7. We have $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $f(x, y, z) = (x + y + z,\, x^3 - e^{yz},\, xz)$ and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by $g(x, y, z) = (xy,\, yz,\, zx)$ so $Df = \begin{pmatrix} 1 & 1 & 1 \\ 3x^2 & -ze^{yz} & -ye^{yz} \\ z & 0 & x \end{pmatrix}$ and $Dg = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix}$. The chain rule gives $D(g \circ f)(x, y, z) = Dg(f(x, y, z))\, Df(x, y, z) =$

$$\begin{pmatrix} x^3 - e^{yz} & x + y + z & 0 \\ 0 & xz & x^3 - e^{yz} \\ xz & 0 & x + y + z \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 3x^2 & -ze^{yz} & -ye^{yz} \\ z & 0 & x \end{pmatrix} =$$

$$\begin{pmatrix} 4x^3 + 3x^2y + 3x^2z - e^{yz} & x^3 - (1 + xz + yz + z^2)\, e^{yz} & x^3 - (1 + xy + y^2 + yz)\, e^{yz} \\ 4x^z - z\, e^{yz} & -xz^2 e^{yz} & x^4 - x(yz + 1)\, e^{yz} \\ 2xz + yz + z^2 & xz & 2xz + x^2 + xy \end{pmatrix}$$

and $D(f \circ g)(x, y, z) = Df(g(x, y, z))\, Dg(x, y, z) =$

$$\begin{pmatrix} 1 & 1 & 1 \\ 3x^2y^2 & -zx\, e^{xyz^2} & -yz\, e^{xyz^2} \\ zx & 0 & xy \end{pmatrix} \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \end{pmatrix} =$$

$$\begin{pmatrix} y + z & x + z & x + y \\ 3x^2y^3 - yz^2 e^{xyz^2} & 3x^3y^2 - xz^2 e^{xyz^2} & -2xyz\, e^{xyz^2} \\ 2xyz & x^2 z & x^2 y \end{pmatrix}.$$

Now $f \circ g : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is given by $f \circ g(x, y, z) = f(g(x, y, z)) = (xy + yz + zx,\, x^3y^3 - e^{xyz^2},\, x^2yz)$ and $D(f \circ g)(x, y, z) = \begin{pmatrix} y + z & x + z & x + y \\ 3x^2y^3 - yz^2 e^{xyz^2} & 3x^3y^2 - xz^2 e^{xyz^2} & -2xyz\, e^{xyz^2} \\ 2xyz & x^2 z & x^2 y \end{pmatrix}$ which is the same as we had from the chain rule.

8. (a) $f(x, y) = x^2 + xy - y^2$ so $\dfrac{\partial f}{\partial x} = 2x + y$ and $\dfrac{\partial f}{\partial y} = x - 2y$. Now $\dfrac{\partial^2 f}{\partial x^2} = 2$, and $\dfrac{\partial^2 f}{\partial y^2} = -2$. Since the 2^{nd} order partials are continuous and $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 2 + (-2) = 0$, we conclude that $f(x, y)$ is harmonic.

   (b) $f(x, y) = x^3 + 3xy^2$ so $\dfrac{\partial f}{\partial x} = 3x^2 + 3y^2$ and $\dfrac{\partial f}{\partial y} = 6xy$. Now $\dfrac{\partial^2 f}{\partial x^2} = 6x$, and $\dfrac{\partial^2 f}{\partial y^2} = 6x$. Since $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 6x + 6x = 12x \neq 0$, we conclude that $f(x, y)$ is not harmonic.

(c) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$ so $\dfrac{\partial f}{\partial x} = -\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$, $\dfrac{\partial f}{\partial y} = -\dfrac{y}{(x^2+y^2+z^2)^{3/2}}$ and $\dfrac{\partial f}{\partial z} = -\dfrac{z}{(x^2+y^2+z^2)^{3/2}}$. Now $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$, $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}$ and $\dfrac{\partial^2 f}{\partial z^2} = \dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}}$. Since the 2^{nd} order partials are continuous and $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}+\dfrac{\partial^2 f}{\partial z^2} = \dfrac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\dfrac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{5/2}}+\dfrac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{5/2}} = 0$, we conclude that $f(x,y,z)$ is harmonic.

9. (a) $f(x,y) = x^3 - x^2y^2 + y^3$. Now $f(tx,ty) = t^3x^3 - t^2x^2t^2y^2 + t^3y^3 = t^3\,(x^3 - tx^2y^2 + y^3) \neq t^3\, f(x,y)$. Hence f is not homogeneous.

   (b) $f(x,y,z) = 3x^3y + 5x^2z^2 - xyz^2 + z^4$. Now $f(tx,ty,tz) = 3t^3x^3ty + 5t^2x^2t^2z^2 - txtyt^2z^2 + t^4z^4 = t^4\,(3x^3y + 5x^2z^2 - xyz^2 + z^4) = t^4\, f(x,y,z)$. Hence f is homogeneous of degree 4.

10. (a) $f(x,y) = (\sin x)\ln(1+y)$. From single variable calculus we have $\sin t = \displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\,\frac{t^{2k+1}}{(2k+1)\,!}$, $\quad |t| < \infty$, so $\sin x = x - \dfrac{1}{3\,!}\,x^3 + \dfrac{1}{5\,!}\,x^5 - \cdots$, and $\ln(1+t) = \displaystyle\sum_{k=0}^{\infty}(-1)^{k}\,\frac{t^{k+1}}{k+1}$, $\quad |t| < 1$, so $\ln(1+y) = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \cdots$. Now, for (x,y) near $(0,0)$, we have $T = \left(\, x - \dfrac{1}{3\,!}\,x^3 + \dfrac{1}{5\,!}\,x^5 - \cdots \,\right)\left(\, y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \cdots \,\right)$ and $T_3 = x\,y - \dfrac{1}{2}\,x\,y^2$. (The product will only yield 2 terms with degree less than or equal to 3.)

    (b) $f(x,y) = \dfrac{e^{xy}}{1+x} = \left(e^{xy}\right)\left(\dfrac{1}{1+x}\right)$. From single variable calculus we have $e^t = \displaystyle\sum_{k=0}^{\infty}\frac{t^k}{k\,!}$, $\quad |t| < \infty$, so $e^{x\,y} = 1 + x\,y + \dfrac{x^2\,y^2}{2} + \dfrac{x^3\,y^3}{3\,!} + \cdots$, and $\dfrac{1}{1+t} = \displaystyle\sum_{k=0}^{\infty}(-1)^k\,t^k$, $|t| < 1$, so $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$. Now, for (x,y) near $(0,0)$, we have $T = \left(\, 1 + x\,y + \dfrac{x^2\,y^2}{2} + \cdots \,\right)\left(\, 1 - x + x^2 - x^3 + \cdots \,\right)$ and $T_3 = 1 + x\,y - x - x^2\,y + x^2 - x^3 = 1 - x + \left(x^2 + x\,y\right) - \left(x^3 + x^2\,y\right)$. Again, we only want terms with degree less than or equal to 3.