University of Toronto Scarborough
Department of Computer & Mathematical Sciences

Midterm Test

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: October 28, 2013
Duration: 110 minutes

1. **[8 points]** In this question, be sure to indicate what type of object each of your symbols represents.

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the Chain Rule for functions of more than one variable.

2. **[15 points]**

   (a) Calculate the following limits, showing all your steps, or show that the limit does not exist.

   i. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\,y^2}{x^2+y^2}$.

   ii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3\,y - x\,y^3}{x^4+y^4}$.

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^4-y^4}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0)\ . \end{cases}$$

   Is f continuous at $(0,0)$? (Explain your answer.)

3. **[11 points]** Characterize and sketch several level curves of the function

$$f(x,y) = \frac{x+y}{y^2}\ .$$

   Carefully indicate where f is zero, positive, negative and not defined.

4. **[4 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \frac{\sqrt{x^2-y^2}}{x+3}$$

   and let D be the domain of f.

   (a) Use set notation to describe D.

   (b) Carefully sketch D.

5. **[5 points]** Determine if $f : \mathbb{R}^3 \to \mathbb{R}$ given by

$$f(x, y, z) = 3x^2 + 5y^2 + 4xy - 9xz - 8z^2$$

is harmonic.

6. **[12 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^2\, y - 2\, x\, y + 2\, y^2 - 15\, y - 2\ .$$

(a) Find the equation of the tangent plane to the graph of f at the point $(1, 1, f(1, 1))$.

(b) Find the critical points of f.

7. **[12 points]**

(a) Give the equation of the tangent plane to the surface $z^2 - 2\, x^4 - y^4 = 16$ at the point $\boldsymbol{p} = (2, 2, 8)$.

(b) A particle leaves the surface (from part (a)) at $\boldsymbol{p}$ and travels along the normal line to the xy–plane. Give a parametric description of this line and determine the point where the particle meets the xy–plane.

8. **[11 points]** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x, y, z) = x\, e^{-y^2} + z\, e^{-x^2}$$

and let $\boldsymbol{p} = (0, 0, 1)$.

(a) Give an equation of the level set passing through $\boldsymbol{p}$.

(b) What is the rate of change in f if you move from $\boldsymbol{p}$ towards $(2, 3, 1)$?

(c) i. In what direction from $\boldsymbol{p}$ must you go for the most rapid increase in f?

ii. What is the rate of this increase?

9. **[11 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x, y) = (xy^2, x + 2y, xy)$ and let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $g(x, y, z) = (xy, yz, xz^2)$.
`USE THE CHAIN RULE` to compute $D(g \circ f)(x, y)$.

(**Note**: you must use the chain rule and show all your steps.)

10. **[5 points]** Let $f(x, y)$ be of class C^2. Putting $x = 2u - 3v$ and $y = u + 4v$ makes f into a function of u and v. Compute a formula for $\dfrac{\partial^2 f}{\partial u\, \partial v}$ in terms of the partial derivatives of f with respect to x and y.

11. **[6 points]** Give the 6^{th} degree Taylor polynomial about the origin of $f(x, y) = \dfrac{\sin(xy)}{2 + 4\, x}$.