University of Toronto at Scarborough
Department of Computer & Mathematical Sciences

First Midterm Test

MATB41H3
Techniques of the Calculus of Several Variables I

Examiner: R. Grinnell

Date: June 10, 2004
Duration: 120 minutes

FAMILY NAME: SOLUTIONS

GIVEN NAMES: ______________________

STUDENT NUMBER: ______________________

DAY AND TIME OF YOUR TUTORIAL: ______________________

SIGNATURE: ______________________

DO NOT OPEN THIS BOOKLET UNTIL INSTRUCTED TO DO SO.

NOTES:

- There are 12 numbered pages in the test. It is your responsibility to ensure that, at the start of the test, this booklet has all its pages.
- Answer all questions. Explain and justify your answers.
- **Show all your work.** Credit will not be given for numerical answers if the work is not shown. If you need more space use the back of the page or the blank page.
- Upon receiving your marked test, you will have at most 48 hours to request any re-grading.

FOR MARKERS ONLY	
Question	Marks
1	/ 8
2	/ 18
3	/ 16
4	/ 12
5	/ 8
6	/ 9
7	/ 10
8	/ 12
9	/ 7
TOTAL	**/100**

1. **[8 points]** Find the equation of the tangent plane to the graph of the function $z = f(x,y) = (x-y)^2 + \dfrac{5x^2}{y} - 2ye^x$ at the point $(0,3)$.

General eq$^{\underline{n}}$ of tangent plane is

$$z = T(x,y) = f(0,3) + f_x(0,3)(x-0) + f_y(0,3)(y-3)$$

$$f(0,3) = (0-3)^2 + 0 - 6e^0 = 9-6 = 3$$

$$f_x(x,y) = 2x - 2y + \frac{10x}{y} - 2ye^x \Rightarrow f_x(0,3) = -12$$

$$f_y(x,y) = -2x + 2y - \frac{5x^2}{y^2} - 2e^x \Rightarrow f_y(0,3) = 4$$

$\therefore$ tangent plane is $z = T(x,y) = -12x + 4y - 9$

2. Throughout this question let $f(x,y) = [(y-1)(y-x^2)]^{-\frac{1}{2}}$.

(a) **[2 points]** Compute $f(2,5)$.

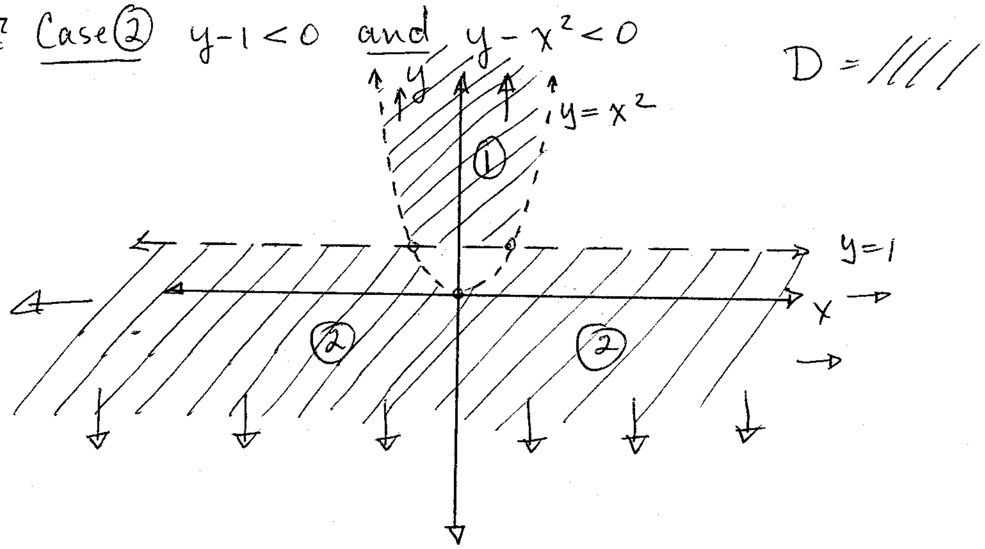
$$f(2,5) = [(5-1)(5-4)]^{-1/2} = \frac{1}{4^{1/2}} = \frac{1}{2}$$

(b) **[6 points]** Assume the domain of f is the set $D = \{(x,y) \mid (y-1)(y-x^2) > 0\}$. Accurately sketch D.

We have that a point $(x,y) \in D$ iff

Case ① $y-1>0$ and $y-x^2>0$

or Case ② $y-1<0$ and $y-x^2<0$



[Points $(0,0)$, $(\pm 1, 1)$ are excluded as are the lines $y=1$ and curve $y=x^2$]

Question 2 (cont'd)

Throughout this question let $f(x,y) = [(y-1)(y-x^2)]^{-\frac{1}{2}}$.

(c) **[4 points]** Find all values of y so that $f(3,y) = \frac{1}{3}$.

$$[(y-1)(y-x^2)]^{-1/2} = \frac{1}{3} \Leftrightarrow [(y-1)(y-x^2)]^{1/2} = 3$$

$$\Leftrightarrow (y-1)(y-x^2) = 9$$

Put $x=3$ to get $(y-1)(y-9) = 9$

Obtain $y^2 - 10y + 9 = 9 \Leftrightarrow y^2 - 10y = 0$

$$y(y-10) = 0$$

$\therefore$ solutions are $y = 0$ and $y = 10$

(d) **[6 points]** By referencing properties and theorems about continuous functions, explain completely why f is continuous on D.

The function $p(x,y) = (y-1)(y-x^2) = y^2 - yx^2 - y + x^2$ is a polynomial, thus is continuous on $\mathbb{R}^2$ and hence the subset D. If $(x,y) \in D$ then $p(x,y) > 0$. The function $g(t) = \frac{1}{\sqrt{t}}$ is continuous on $(0,\infty)$ because it is a rational-power function ($g(t) = t^{-1/2}$). Since $p(x,y) \in (0,\infty)$ for all $(x,y) \in D$, the function $f = g \circ p$ is continuous on D by the composition theorem for continuous functions.

3. Find the indicated limit or explain why it does not exist. No δ–ε proofs are required.

(a) **[4 points]** $\lim\limits_{(x,y)\to(2,\pi)} \dfrac{3y\cos(xy)}{x^2y-\pi}$

Note 1st that when $x=2$ and $y=\pi$ the denominator is $x^2y-\pi = 3\pi \neq 0$ $\therefore$ Properties of continuous functions imply that $g(x,y) = \dfrac{3y\cos(xy)}{x^2y-\pi}$ is continuous at $(2,\pi)$

$$\therefore \lim_{(x,y)\to(2,\pi)} g(x,y) = \frac{3\pi\cos(2\pi)}{3\pi} = 1$$

(b) **[4 points]** $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+5y^2}{5x^2+y^2}$ Put $f(x,y) = \dfrac{x^2+5y^2}{5x^2+y^2}$

Inspection suggests limit DNE. Details:

As $(x,y)\to(0,0)$ along x-axis: $y=0$, $x\to 0$, $x\neq 0$

$\therefore$ $f(x,0) = \dfrac{x^2}{5x^2} \to \dfrac{1}{5}$ for these points

As $(x,y)\to(0,0)$ along y-axis: $x=0$, $y\to 0$, $y\neq 0$

$\therefore$ $f(0,y) = \dfrac{5y^2}{y^2} \to 5$ for these points

$\therefore$ $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ DNE by Unequal Limit Thm.

Question 3 (cont'd)

Find the indicated limit or explain why it does not exist. No δ–ε proofs are required.

(c) **[4 points]** $\lim_{(x,y)\to(1,-2)} \frac{xy+2x-y-2}{(x^2-1)(y+2)}$

Let $g(x,y) = \frac{xy+2x-y-2}{(x^2-1)(y+2)}$

$$= \lim_{(x,y)\to(1,-2)} \frac{x(y+2)-1(y+2)}{(x-1)(x+1)(y+2)}$$

$$= \lim_{(x,y)\to(1,-2)} \frac{\cancel{(x-1)}\cancel{(y+2)}}{\cancel{(x-1)}\cancel{(y+2)}(x+1)}$$

$$= \lim_{(x,y)\to(1,-2)} \frac{1}{x+1} = \frac{1}{2}$$

Since $(x,y)\to(1,-2)$ we have that g is defined only if $x \neq 1$ and $y \neq -2$

Since the f^n $h(x,y) = \frac{1}{x+1}$ is cts @ $(1,-2)$.

(d) **[4 points]** $\lim_{(x,y)\to(0,0)} \frac{3xy^2}{x^2+y^4}$

Let $f(x,y) = \frac{3xy^2}{x^2+y^4}$

Careful preliminary inspection suggests limit DNE. Details:

As $(x,y)\to(0,0)$ along the x-axis: $y=0$, $x\to 0$, $x\neq 0$

$\therefore$ $f(x,0) = 0$ for these points.

As $(x,y)\to(0,0)$ along the curve $y=\sqrt{x}$ where $x>0$, $x\to 0$, $f(x,\sqrt{x}) = \frac{3x^2}{x^2+x^2} = \frac{3}{2}$ for these points

$\therefore$ $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE again by the Unequal Limit Theorem.

4. Let $f(u, v, w) = (2u - vw, 5w^2)$ and $g(x, y, z) = (\underbrace{x^2y}_{u}, \underbrace{x + 3z}_{v}, \underbrace{\sin(z)}_{w})$.

(a) **[4 points]** Find the composition $(f \circ g)(x, y, z)$.

$$(f \circ g)(x, y, z) = f(x^2y, x+3z, \sin(z))$$
$$= (2x^2y - (x+3z)\sin(z), 5\sin^2(z))$$

(b) **[8 points]** Use the chain rule to calculate $D(f \circ g)(x, y, z)$.
[A direct calculation without using the chain rule will not earn any points.]

$$D(f \circ g)(x, y, z) = [Df(u,v,w)][Dg(x,y,z)]$$

$$= \begin{pmatrix} 2 & -w & -v \\ 0 & 0 & 10w \end{pmatrix} \begin{pmatrix} 2xy & x^2 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & \cos(z) \end{pmatrix}$$

$$= \begin{pmatrix} 4xy - w & 2x^2 & -3w - v\cos(z) \\ 0 & 0 & 10w\cos(z) \end{pmatrix}$$

$$= \begin{pmatrix} 4xy - \sin(z) & 2x^2 & -3\sin(z) - (x+3z)\cos(z) \\ 0 & 0 & 10\sin(z)\cos(z) \end{pmatrix}$$

5. [**8 points**] Assume z is defined implicitly as a function of x and y by the equation

$$x\sin(y^2z) + ze^x = 2y - x + z\ .$$

Assume x nor y is a function of any other variable. Find $z_x(2,1,0)$.

Write $\frac{\partial z}{\partial x} = z_x$ Differentiate implicitly :

$$\frac{\partial}{\partial x}\left(x\sin(y^2z)\right) + \frac{\partial}{\partial x}\left(ze^x\right) = \frac{\partial}{\partial x}\left(2y - x + z\right)$$

$$\sin(y^2z) + x\cos(y^2z)(y^2z_x) + z_xe^x + ze^x = -1 + z_x \quad (*)$$

$$z_x\left[xy^2\cos(y^2z) + e^x - 1\right] = -1 - ze^x - \sin(y^2z)$$

$$\therefore z_x = \frac{-1 - ze^x - \sin(y^2z)}{xy^2\cos(y^2z) + e^x - 1} \qquad \therefore z_x(2,1,0) = \frac{-1}{1+e^2}$$

[Note: At line (*) we could've subbed-in (2,1,0) and solved for $z_x(2,1,0)$ directly]

6. Let $\mathbf{c}(t) = (\underbrace{\cos(t)}_{x}, \underbrace{\sin(t)}_{y}, \underbrace{e^t}_{z})$ where $t \in \mathbb{R}$.

(a) [**4 points**] Describe the curve C traced out by the path $\mathbf{c}(t)$.

$\cos^2(t) + \sin^2(t) = 1 \quad \forall t \in \mathbb{R}$ $\therefore$ C is a circular helix wrapped around the cylinder $x^2+y^2=1$ in $\mathbb{R}^3$. Since $z = e^t > 0 \quad \forall t \in \mathbb{R}$, we have that C always lies above the xy-plane. As $t \to \infty$, $e^t \to \infty$, so there is no bound on the height to which C winds up. As $t \to -\infty$, $e^t \to 0$, so C wraps towards the xy-plane but never touches. It would be accurate to describe C as a "spring" or a "slinky"®

(b) [**5 points**] Find the equation of the tangent line to C at the point where $t = 0$.

$\underline{c}(0) = (1,0,1) \qquad \underline{c}'(t) = (-\sin(t), \cos(t), e^t)$

$\Rightarrow \underline{c}'(0) = (0, 1, 1)$

$\therefore$ Equation of the tangent line is

$$\underline{l}(t) = \underline{c}(0) + t\,\underline{c}'(0) = (1,0,1) + t(0,1,1)$$
$$= (1, t, 1+t) \quad t \in \mathbb{R}$$

7. **[10 points]** Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{5xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \quad \leftarrow \text{4 points} \\ 0 & \text{if } (x,y) = (0,0) \quad \leftarrow \text{6 points} \end{cases}$$

Use properties of continuous functions and the appropriate δ–ε proof to completely show that f is continuous on all of $\mathbb{R}^2$.

(A) <u>Continuity on $\mathbb{R}^2 \setminus \{(0,0)\}$</u> If $(x,y) \neq (0,0)$ then $x^2+y^2 \neq 0$.
This means $f(x,y)$ is the rational function
$f(x,y) = \dfrac{5xy^2}{x^2+y^2}$ on $\mathbb{R}^2 - \{(0,0)\}$, hence is continuous by the theorem about continuity of rational f^{ns} on their domain.

(B) <u>Continuity of f @ $(0,0)$</u> We must show (by δ-ε) that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Let $\varepsilon > 0$ be given. We must prove $\exists \delta > 0$ such that if $0 < \sqrt{x^2+y^2} < \delta$ then $|f(x,y) - 0| < \varepsilon$.

We estimate :

$$|f(x,y) - 0| = \left| \frac{5xy^2}{x^2+y^2} \right| = \frac{5y^2|x|}{x^2+y^2} \leq 5|x| = 5\sqrt{x^2}$$
$$\leq 5\sqrt{x^2+y^2} \quad (*)$$

Line $(*)$ suggests we define $\delta = \frac{\varepsilon}{5}$. For this δ, if $0 < \sqrt{x^2+y^2} < \delta$, then

$$|f(x,y) - 0| \leq 5\sqrt{x^2+y^2} < 5\delta = 5\left(\frac{\varepsilon}{5}\right) = \varepsilon$$

This shows continuity @ $(0,0)$

By (A) & (B), f is now proved to be continuous on all of $\mathbb{R}^2$.

8. Let $f(x,y) = x + \frac{y^2}{x}$.

(a) **[8 points]** Completely describe the level curves of $f(x,y)$ by using appropriate concepts from analytic geometry.

$D = \text{domain of } f = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0\}$

For $c \in \mathbb{R}$, level curve is $L_c = \{(x,y) \in \mathbb{R}^2 \mid f(x,y) = c\}$

$\underline{c=0}$ $\quad 0 = x + \frac{y^2}{x} \iff 0 = x^2 + y^2 \iff x = y = 0$. But $(0,0) \notin D$

$\therefore L_0 = \emptyset$

$\underline{c \neq 0}$ $\quad c = x + \frac{y^2}{x} \iff x^2 + y^2 = cx$

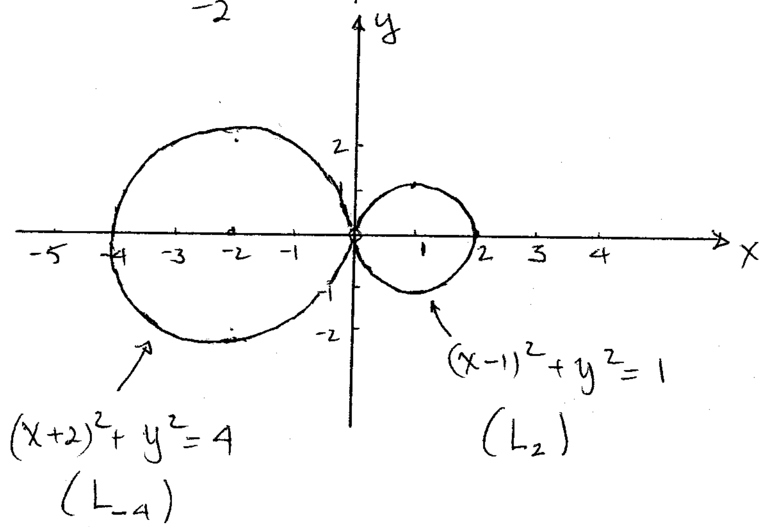
$x^2 - cx + y^2 = 0 \implies x^2 - cx + \frac{c^2}{4} + y^2 = \frac{c^2}{4}$

$\implies \left(x - \frac{c}{2}\right)^2 + y^2 = \left(\frac{|c|}{2}\right)^2$

$\therefore L_c$ is the circle with center $\left(\frac{c}{2}, 0\right)$, radius $\frac{|c|}{2}$, but excludes $(0,0)$ (This is the case for $c \neq 0$ only)

(b) **[4 points]** On the same axis, draw the level curve for $c = 2$ and the level curve that passes through the point $(-2, 2)$.

$f(-2,2) = -2 + \frac{4}{-2} = -4 = c$



9. **[7 points]** Use the definition of differentiability to show that the function $f(x, y) = -3x + 5y^2$ is differentiable at every point (a, b) in $\mathbb{R}^2$. A δ–ε proof is not required to earn full points.

To show that f is differentiable @ an arbitrary point $(a,b) \in \mathbb{R}$, we check: (i) partials are cts @ (a,b)

(ii) $$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - T(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}} = 0$$ $T(x,y)$ = tangent plane to f @ (a,b)

<u>For (i)</u> $f_x(x,y) = -3$ $\quad f_y(x,y) = 10y$

f_x and f_y are polynomials so they are continuous on $\mathbb{R}^2$ and thus @ (a,b). ✓

<u>For (ii)</u> $$T(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= -3a + 5b^2 - 3(x-a) + 10b(y-b)$$
$$= -\cancel{3a} + 5b^2 - 3x + \cancel{3a} + 10by - 10b^2$$
$$= -3x + 10by - 5b^2$$

$$\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - T(x,y)|}{\sqrt{(x-a)^2 + (y-b)^2}}$$

$$= \lim_{(x,y)\to(a,b)} \frac{|-\cancel{3x} + 5y^2 + \cancel{3x} - 10by + 5b^2|}{\sqrt{(x-a)^2 + (y-b)^2}}$$

$$= \lim_{(x,y)\to(a,b)} \frac{5|(y-b)^2|}{\sqrt{(x-a)^2 + (y-b)^2}} \leq \lim_{(x,y)\to(a,b)} \frac{5[(x-a)^2 + (y-b)^2]}{\sqrt{(x-a)^2 + (y-b)^2}}$$

$$= \lim_{(x,y)\to(a,b)} 5\sqrt{(x-a)^2 + (y-b)^2} = 0$$ ✓

Since (i) & (ii) have been shown, we are done.

$\boxed{\text{END}}$