University of Toronto Scarborough
Department of Computer & Mathematical Sciences

FINAL EXAMINATION

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 17, 2007
Duration: 3 hours

1. **[5 points]** Let $f(x,y) = \begin{cases} \dfrac{\sin(x\,y)}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \end{cases}$. Determine the set of all points (x,y) where f is continuous.

2. **[5 points]** Give the 4$^{\text{th}}$ degree Taylor polynomial about the origin of $f(x,y) = \dfrac{\sin(xy)}{1+x+y}$.

3. **[15 points]** Let $f(x,y,z) = y\,e^{-x^2}\sin z$ and let $\boldsymbol{a} = \left(0,\ 1,\ \dfrac{\pi}{3}\right)$ be a point in $\mathbb{R}^3$.

   (a) Find an equation for the tangent plane to the level surface of f that passes through $\boldsymbol{a}$.

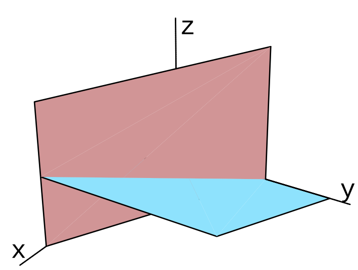
   (b) Determine the direction and magnitude of the maximal increase in f at $\big(\boldsymbol{a},\ f(\boldsymbol{a})\big)$.

   (c) Compute the directional derivative of f at $\boldsymbol{a}$ in the direction of the line segment from $\boldsymbol{a}$ to $(1,\ 2,\ -1)$.

4. **[5 points]** Sketch the curve given by the polar equation $r = 1 + 2\,\cos(2\,\theta)$.

5. **[7 points]** Let $f(x,y,z) = x^2 + xy - z^2 + x\,\cos z$. Find and classify all the critical points of f.

6. **[10 points]** Let $f(x,y,z) = xy + yz$. Use Lagrange Multipliers to find the maximum of f on the intersection of the planes $x + 2y = 6$ and $x - 3z = 0$.
(Justify your answer.)



7. **[10 points]** Find the maximum value of $f(x, y, z) = x\,y\,z$ on the solid ball $x^2 + y^2 + z^2 \leq 1$.

   Justify your answer including an explanation of why global extrema do exist.

8. **[10 points]**

   (a) Compute $\displaystyle\int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y}\, dy\, dx$.

   (b) Compute $\displaystyle\int_D y^2\, dA$, where D is the region bounded by the lines $2x - y = 0$, $5x - y = 0$ and $x = 2$.

9. **[7 points]** Use a triple integral to find the volume of the solid bounded by $z = 9 - x^2$, $z = 0$, $y = 0$ and $y = 2x$.

10. **[12 points]** A hemispherical solid B of radius a has density depending on the distance d from the center of the base disk. The density is given by $k(2a - d)$, where k is a constant.

    (a) Give the integral you would use to find the mass of B in terms of (i) cartesian coordinates, (ii) cylindrical coordinates and (iii) spherical coordinates.

    (b) Choose one of the integrals from part (a) and calculate the mass of B.

11. **[14 points]**

    (a) Carefully state the "change of variables" theorem for multiple integrals . Make sure you define your terms.

    (b) Use a suitable change of variable to evaluate the integral $\displaystyle\iint_D x^2\, y^2\, dA$, where D is the first quadrant region bounded by the parabolas $y = x^2$ and $y = 2\,x^2$ and the hyperbolas $xy = 1$ and $xy = 2$.