University of Toronto Scarborough
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Midterm Test
MATB41H – Techniques of the Calculus of Several Variables I

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Date: October 31, 2008
Duration: 110 minutes

1. [**8 points**]

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the Extreme Value (Min–Max) Theorem for real-valued functions of several variables.

2. [**15 points**]

   (a) If $\displaystyle\lim_{(x,y)\to(1,-1)} f(x,y) = 3$, can you conclude anything about $f(1,-1)$? Give reasons for your answer.

   (b) Calculate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$, showing your steps, or show that the limit does not exist.

   (c) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

   $$f(x,y) = \begin{cases} \dfrac{xy-3y}{(x-3)^2+y^2} & , \text{ if } (x,y) \neq (3,0) \\ 1 & , \text{ if } (x,y) = (3,0)\ . \end{cases}$$

   Is f continuous at $(3,0)$? (Explain your answer.)

3. [**11 points**] Characterize and sketch several level curves of the function

   $$f(x,y) = \frac{x}{x^2+y^2}\ .$$

   Carefully indicate where f is zero, positive, negative and not defined.

4. [**5 points**] Find an equation of the tangent plane to the graph of $f(x,y) = 4x^2 - y^2 + 2y$ at the point $\big(-1,\, 2,\, f(-1,\, 2)\big)$.

5. **[12 points]**

   (a) Give the equation of the tangent plane to the ellipsoid $\dfrac{x^2}{4}+y^2+\dfrac{z^2}{9}=3$ at the point $\boldsymbol{p}=(-2,\ 1,\ -3)$.

   (b) Give a parametric description of the normal line to the ellipsoid in part (a) which passes through $\boldsymbol{p}$ and determine where it meets the coordinate plane $z=0$.

6. **[15 points]** Let $f(x,y,z)=x^2+y^2+z^2$.

   (a) Characterize a typical level surface of this function.

   (b) What is the direction of the maximum rate of increase in f at $\boldsymbol{p}=(1,-1,2)$? What is the maximum rate?

   (c) What is the rate of change in f at $\boldsymbol{p}=(1,-1,2)$ measured in the direction from $\boldsymbol{p}$ towards $(3,1,1)$?

7. **[5 points]** Determine if $f(x,y)=x^4-6\,x^2\,y^2+y^4$ is harmonic.

8. **[6 points]** Let $g(x,y,z)$ be of class C^2 and let $x=2u+3v+w$, $y=u-w$ and $z=2v$. If $f=g(x,y,z)$ is a function of u, v, w, compute $\dfrac{\partial^2 f}{\partial w\,\partial u}(\mathbf{0})$, when $\dfrac{\partial^2 g}{\partial x^2}(\mathbf{0})=1$, $\dfrac{\partial^2 g}{\partial y^2}(\mathbf{0})=2$, $\dfrac{\partial^2 g}{\partial z^2}(\mathbf{0})=3$, $\dfrac{\partial^2 g}{\partial x\,\partial y}(\mathbf{0})=\pi$, $\dfrac{\partial^2 g}{\partial x\,\partial z}(\mathbf{0})=\pi^2$, $\dfrac{\partial^2 g}{\partial y\,\partial z}(\mathbf{0})=\pi^3$.

9. **[11 points]** Let $f:\mathbb{R}^3\to\mathbb{R}^4$ be given by $f(x,y,z)=(xy,\ yz,\ xz,\ xyz)$ and let $g:\mathbb{R}^4\to\mathbb{R}^3$ be given by $g(x,y,z,w)=(x^2y,\ y^2z,\ zw)$.
   `USE THE CHAIN RULE` to compute $D(g\circ f)(x,y,z)$.

   (**NOTE:** You must use the Chain Rule and show all your steps.)

10. **[12 points]**

    (a) Give the 5$^{\text{th}}$ degree Taylor polynomial about the origin of $f(x,y)=\dfrac{\cos(x\,y)}{1+y^2}$.

    (b) Let $f(x,y)=\ln(1+x+y)$. Use a quadratic approximation to estimate $f(0.1,\ 0.2)$.