University of Toronto Scarborough
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Midterm Test

MATB41H – Techniques of the Calculus of Several Variables I

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Date: October 29, 2010
Duration: 110 minutes

1. [**8 points**]

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Define the following terms

   i. an *interior point* of $A \subset \mathbb{R}^n$.

   ii. a *local minimum* of a function $f : \mathbb{R}^n \to \mathbb{R}$.

2. [**11 points**]

   (a) Calculate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\,y - y^2}{\sqrt{x} + \sqrt{y}}$, showing your steps, or show that the limit does not exist.

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

   $$f(x,y) = \begin{cases} \dfrac{x^3 + 2x^2 + 2xy^2 + 4y^2}{x^2 + 2y^2} & , \text{ if } (x,y) \neq (0,0) \\ -2 & , \text{ if } (x,y) = (0,0)\ . \end{cases}$$

   Find all (x,y) for which $f(x,y)$ is continuous. (Explain your answer.)

3. [**12 points**] Characterize and sketch several level curves of the function

   $$f(x,y) = \frac{x^2 + y^2 - 1}{(x+1)^2 + y^2}\ .$$

   Carefully indicate where f is zero, positive, negative and not defined.

4. [**6 points**] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

   $$f(x,y) = \frac{1}{\sqrt{\big(x + y^2\big)\big(2 - x\big)}}$$

   and let D be the domain of f.

   (a) Use set notation to describe D.

   (b) Carefully sketch D.

5. **[6 points]** Find the equation of the tangent plane at the point $(1, -2, 1)$ to the graph of the function $z = f(x, y)$ defined implicitly by
$$x^2\, y + y^2\, z + z^2\, x + x\, y\, z = 1.$$

6. **[15 points]**

   (a) Find the equation of the tangent plane to the graph of $f(x, y) = y \sin x$ at the point $\left(\frac{\pi}{4},\, 2,\, f\left(\frac{\pi}{4},\, 2\right)\right)$.

   (b) Give the parametric description of the plane which passes through $(0, 1, 4)$, $(-2, -1, 2)$ and $(2, 2, 3)$.

   (c) Find the angle between the planes in part (a) and part (b).

7. **[16 points]**

   (a) Carefully state the Chain Rule for functions of more than one variable.

   (b) Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $f(x, y, z, w) = (y^2 w^2,\, xyw,\, xz^2)$ and let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $g(x, y, z) = (x,\, ze^y,\, xz)$.
   `USE THE CHAIN RULE` to compute $D(g \circ f)(x, y, z, w)$.
   (**NOTE:** You must use the Chain Rule and show all your steps.)

8. **[15 points]** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = x^2 + 2xy + yz + z^2 + 6z$.

   (a) What is the rate of change in f if you move from $(2, -1, 0)$ towards $(0, 2, -1)$.

   (b) What is the direction of the maximum rate of increase in f at $\boldsymbol{p} = (2, -1, 0)$? What is the maximum rate?

   (c) Find the critical points of f.

9. **[5 points]** Let $z = f(x, y)$ be of class C^2. Putting $x = u - v$ and $y = uv$ makes z into a function of u and v. Show that
$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial^2 f}{\partial x^2} + 2\, v\, \frac{\partial^2 f}{\partial x\, \partial y} + v^2\, \frac{\partial^2 f}{\partial y^2}\ .$$
(Be sure to justify your steps.)

10. **[6 points]** Give the 5^{th} degree Taylor polynomial about the origin of $f(x, y) = \dfrac{\sin(x\, y)}{1 - y^2}$.