University of Toronto Scarborough
Department of Computer & Mathematical Sciences

Midterm Test

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: October 28, 2009
Duration: 110 minutes

1. **[8 points]** Let $f(x,y) = x^3 - 6xy + y^3$.

   (a) Find an equation of the tangent plane to the graph of $f(x,y)$ at the point $\big(1, 2, f(1,2)\big)$.

   (b) Use a linear approximation to estimate $f(0.99,\, 2.01)$.

2. **[15 points]**

   (a) Calculate the following limits, showing all your steps, or show that the limit does not exist.

   i. $\displaystyle\lim_{(x,y)\to(2,0)} \frac{(x-2)^2}{(x-2)^2 + y^2}$

   ii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\|\,(x,y)\,\|}$

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , \text{ if } (x,y) \neq (0,0) \\ 0 & , \text{ if } (x,y) = (0,0) \ . \end{cases}$$

   Is f continuous at $(0,0)$? (Explain your answer.)

3. **[12 points]** Characterize and sketch several level curves of the function

$$f(x,y) = \frac{x^2}{x+y+1} \ .$$

   Carefully indicate where f is zero, positive, negative and not defined.

4. **[10 points]** Let π be the plane which passes through $(0,1,-1)$, $(-2,-1,2)$ and $(1,-1,-1)$.

   (a) Find an equation for π.

   (b) Find the points on the hyperboloid $x^2 - y^2 + 4z^2 = 4$ where the tangent plane is parallel to π.

5. **[10 points]** Let $g(x,y,z) = xy + yz + zx$.

   (a) Give the equation of the level surface for g which passes through the point $\boldsymbol{p} = (1,1,0)$.

   (b) Find an equation of the tangent plane to this level surface at the point $\boldsymbol{p}$.

   (c) Give a parametric description of the normal line to this level surface which passes through the point $\boldsymbol{p}$.

6. **[10 points]** Let $f(x,y,z) = x^2\, y^3\, z^2$.

   (a) What is the direction of the maximum rate of increase in f at $\boldsymbol{p} = (2,1,-1)$? What is the maximum rate?

   (b) Compute the directional derivative of f at the point $\boldsymbol{p} = (2,1,-1)$ in the direction of the normal line for the plane $x + 3y + 2z = -2$.

7. **[5 points]** Carefully state the Extreme Value (Min–Max) Theorem for real-valued functions of several variables.

8. **[16 points]**

   (a) Carefully state the Chain Rule for functions of more than one variable.

   (b) Let $f : \mathbb{R}^3 \to \mathbb{R}^4$ be given by $f(x,y,z) = (x^2y,\ y^2z^2,\ xyz^2,\ xy)$ and let $g : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $g(x,y,z,w) = (y\,e^z,\ xzw)$.
   `USE THE CHAIN RULE` to compute $D(g \circ f)(x,y,z)$.
   (**NOTE:** You must use the Chain Rule and show all your steps.)

9. **[8 points]** Let $z = f(x,y)$ be of class C^2. Putting $x = 2u - 3v$ and $y = 4u + 5v$ makes z into a function of u and v. Compute a formula for $\dfrac{\partial^2 z}{\partial v\, \partial u}$ in terms of the partial derivatives of z with respect to x and y.

10. **[6 points]** Give the 5^{th} degree Taylor polynomial about the origin of $f(x,y) = e^{-xy} \arctan y$.