University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Assignment #5

The Midterm Test will be written on **Monday, October 28, 5:00 pm – 7:00 pm.**

This assignment is due at the start of your tutorial in the period October 21 – October 25, 2013.

A. Suggested reading:

1. Marsden & Tromba, Chapter 2, sections 2.5 and 2.6.
2. Marsden & Tromba, Chapter 3, sections 3.1 and 3.2.

B. Problems:

1. Find the point(s) on the graph of the function $f(x,y) = x^2 + y^2 - 1$ where the tangent plane is parallel to the plane $4x - 8y - z = 3$. What is the equation of the tangent plane at this point.

2. Let $f(x,y) = x^{\frac{1}{3}}\, y^{\frac{1}{3}}$.

   (a) Evaluate $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$.

   (b) Show that f is not differentiable at $(0,0)$.

3. The surface of back-yard water garden can be represented by a region in the xy–plane such that the depth (in centimeters) at the point (x,y) is given by $100 - 3x^2y^2$. A rubber duck is in the water at the point $(1,-2)$. In which direction should the rubber duck swim

   (a) so that the depth increases most rapidly?

   (b) so that the depth remains constant?

4. (a) If $g(u,v) = f(u^2 - v^2, v^2 - u^2)$ and f is differentiable, show that g satisfies $v\,\dfrac{\partial g}{\partial u} + u\,\dfrac{\partial g}{\partial v} = 0\,.$

   (b) Let $w = f(x,y)$ be a C^2 function of two variables and let $x = u + v$, $y = u - v$. Show that $\dfrac{\partial^2 w}{\partial u\,\partial v} = \dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^2 w}{\partial y^2}$.
   (cf page 157, # 22).

5. Marsden & Tromba, page 134, #22.

   (This question is an example of the fact that the chain rule is not applicable if f is not differentiable.)

6. Let $g : [0,1] \subset \mathbb{R} \to \mathbb{R}^2$ be given by $g(t) = (x(t), y(t))$, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function (all partial derivatives exist and are continuous). Assume that $\left(\dfrac{dx}{dt}\right) f_x + \left(\dfrac{dy}{dt}\right) f_y \le 0$. Show that $f\left(x(1), y(1)\right) \le f\left(x(0), y(0)\right)$.

   (cf page 145, #28)

7. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x,y,z) = (x+y+z, x^3 - e^{yz}, xz)$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $g(x,y,z) = (xy, yz, zx)$. Find Df and Dg. Use the chain rule to find $D(g \circ f)$ and $D(f \circ g)$. Compute $f \circ g$ and $D(f \circ g)$ directly.

8. A function $u = u\big(x_1, x_2, \cdots, x_n\big)$ with continuous second partial derivatives satisfying *Laplace's equation*

   $$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0$$

   is called a *harmonic function.*

   Determine whether or not the following functions are harmonic.

   (a) $f(x,y) = x^2 + xy - y^2$.
   (b) $f(x,y) = x^3 + 3xy^2$.
   (c) $f(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$.

9. A function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ is said to be *homogeneous of degree* p if, for all $\boldsymbol{x} \in U$ and all $t \in \mathbb{R}$ such that $t\,\boldsymbol{x} \in U$, we have $f(t\,\boldsymbol{x}) = t^p\, f(\boldsymbol{x})$.
   Determine whether or not the following functions are homogeneous. For those that are, indicate the degree of homogeneity.

   (a) $f(x,y) = x^3 - x^2y^2 + y^3$.
   (b) $f(x,y,z) = 3x^3y + 5x^2z^2 - xyz^2 + z^4$.

10. Find the 3$^{\text{rd}}$ degree Taylor polynomial about the origin of the following.

   (a) $f(x,y) = (\sin x)\ln(1+y)$
   (b) $f(x,y) = \dfrac{e^{xy}}{1+x}$