University of Toronto Scarborough
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MAT B41H **2013/2014**

Solutions #2

1. (a) We first note that the angle formed by $\boldsymbol{u} = (u_1, u_2, u_3)$ and the x–axes is the same as the angle between $\boldsymbol{u}$ and $\boldsymbol{e}_1$. From our discussion of projection and orthonormal bases, we have $u_1 = \boldsymbol{u}\cdot\boldsymbol{e}_1 = \|\boldsymbol{u}\| \cos\alpha$. Similarly we have $u_2 = \|\boldsymbol{u}\| \cos\beta$ and $u_3 = \|\boldsymbol{u}\| \cos\gamma$. Now $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = \dfrac{u_1^2}{\|\boldsymbol{u}\|^2}+\dfrac{u_2^2}{\|\boldsymbol{u}\|^2}+\dfrac{u_3^2}{\|\boldsymbol{u}\|^2} = \dfrac{u_1^2+u_2^2+u_3^2}{\|\boldsymbol{u}\|^2} = 1$.

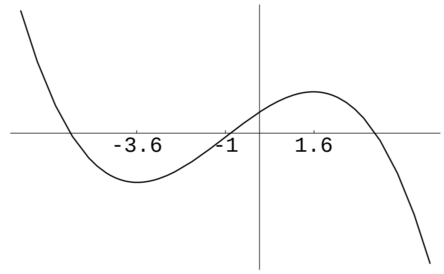
   (b) (i) Since $\boldsymbol{x} = \|\boldsymbol{w}\|\,\boldsymbol{v} + \|\boldsymbol{v}\|\,\boldsymbol{w}$ is described in terms of $\boldsymbol{v}$ and $\boldsymbol{w}$, we know that $\boldsymbol{x}$ is in the same plane as $\boldsymbol{v}$ and $\boldsymbol{w}$. To show that $\boldsymbol{x}$ bisects the angle θ between $\boldsymbol{v}$ and $\boldsymbol{w}$, we need to show that the angle θ_a between $\boldsymbol{v}$ and $\boldsymbol{x}$, and the angle θ_b between $\boldsymbol{x}$ and $\boldsymbol{w}$ are each $\dfrac{1}{2}\theta$. Since the components of $\boldsymbol{x}$ along both $\boldsymbol{v}$ and $\boldsymbol{w}$ are positive, it is sufficient to show that $\cos\theta_a = \cos\theta_b$. Now $\cos\theta_a = \dfrac{\boldsymbol{v}\cdot\boldsymbol{x}}{\|\boldsymbol{v}\|\,\|\boldsymbol{x}\|} = \dfrac{\boldsymbol{v}\cdot(\|\boldsymbol{w}\|\,\boldsymbol{v}+\|\boldsymbol{v}\|\,\boldsymbol{w})}{\|\boldsymbol{v}\|\,\|\,\|\boldsymbol{w}\|\,\boldsymbol{v}+\|\boldsymbol{v}\|\,\boldsymbol{w}\,\|} = \dfrac{\|\boldsymbol{v}\|^2\,\|\boldsymbol{w}\|+\|\boldsymbol{v}\|\,\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\,\|\boldsymbol{w}\|\,\boldsymbol{v}+\|\boldsymbol{v}\|\,\boldsymbol{w}\,\|} = \dfrac{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|+\boldsymbol{v}\cdot\boldsymbol{w}}{\|\,\|\boldsymbol{w}\|\,\boldsymbol{v}+\|\boldsymbol{v}\|\,\boldsymbol{w}\,\|} = \dfrac{\|\boldsymbol{w}\|^2\,\|\boldsymbol{v}\|+\|\boldsymbol{w}\|\,\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{w}\|\,\|\,\|\boldsymbol{w}\|\,\boldsymbol{v}+\|\boldsymbol{v}\|\,\boldsymbol{w}\,\|} = \dfrac{(\|\boldsymbol{w}\|\,\boldsymbol{v}+\|\boldsymbol{v}\|\,\boldsymbol{w})\cdot\boldsymbol{w}}{\|\,\|\boldsymbol{w}\|\,\boldsymbol{v}+\|\boldsymbol{v}\|\,\boldsymbol{w}\,\|\,\|\boldsymbol{w}\|} = \dfrac{\boldsymbol{x}\cdot\boldsymbol{w}}{\|\boldsymbol{x}\|\,\|\boldsymbol{w}\|} = \cos\theta_b$, provided $\|\boldsymbol{x}\| \neq 0$, but this follows since $\boldsymbol{v}$, $\boldsymbol{w} \neq \boldsymbol{0}$ and $\boldsymbol{w} \neq -\boldsymbol{v}$.

   (ii) Using the properties of the dot product we have $(\|\boldsymbol{w}\|\boldsymbol{v}+\|\boldsymbol{v}\|\boldsymbol{w})\cdot(\|\boldsymbol{w}\|\boldsymbol{v}-\|\boldsymbol{v}\|\boldsymbol{w}) = \|\boldsymbol{w}\|^2\boldsymbol{v}\cdot\boldsymbol{v} - \|\boldsymbol{w}\|\|\boldsymbol{v}\|\boldsymbol{v}\cdot\boldsymbol{w} + \|\boldsymbol{v}\|\|\boldsymbol{w}\|\boldsymbol{w}\cdot\boldsymbol{v} - \|\boldsymbol{v}\|^2\boldsymbol{w}\cdot\boldsymbol{w} = \|\boldsymbol{w}\|^2\boldsymbol{v}\cdot\boldsymbol{v} - \|\boldsymbol{v}\|^2\boldsymbol{w}\cdot\boldsymbol{w} = \|\boldsymbol{w}\|^2\|\boldsymbol{v}\|^2 - \|\boldsymbol{v}\|^2\|\boldsymbol{w}\|^2 = 0$. Hence the vectors are orthogonal.

2. Let $\boldsymbol{x} = (x, y, z)$. Now $\boldsymbol{x}\cdot\boldsymbol{x} - \boldsymbol{x}\cdot(1,-2,3) = x^2+y^2+z^2-x+2y-3z \leq 1$. After completing the square we have $\left(x-\dfrac{1}{2}\right)^2 + \left(y+1\right)^2 + \left(z-\dfrac{3}{2}\right)^2 \leq \dfrac{9}{2}$. This inequality describes the solid ball (sphere together with its interior) centered at $\left(\dfrac{1}{2}, -1, \dfrac{3}{2}\right)$ with radius $\dfrac{3}{\sqrt{2}}$.

3. $\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & -2 & 3 \\ -2 & -\lambda & 0 \\ 3 & 0 & -4-\lambda \end{pmatrix} \overset{\textit{expand on}}{\underset{\textit{column } 2}{=}} -(-2)\det\begin{pmatrix} -2 & 0 \\ 3 & -4-\lambda \end{pmatrix} +$

$(-\lambda)\det\begin{pmatrix} 1-\lambda & 3 \\ 3 & -4-\lambda \end{pmatrix} = 2\big(8+2\lambda\big) - \lambda\big(\lambda^2+3\lambda-13\big) = -\lambda^3-3\lambda^2+17\lambda+16.$ To solve we need Newton's Method. Let $f(\lambda) = -\lambda^3-3\lambda^2+17\lambda+16$. Now $f'(\lambda) = -3\lambda^2-6\lambda+17 = 0$ if $\lambda = \dfrac{6\pm\sqrt{36+204}}{-6} = -1 \pm \dfrac{2\sqrt{15}}{3}$. $f''(x) = -6\lambda-6 = 0$ if $\lambda = -1$. $f'(\lambda) > 0$ if $\lambda \in \left(-1-\dfrac{2\sqrt{15}}{3}, -1+\dfrac{2\sqrt{15}}{3}\right)$ and $f'(\lambda) < 0$ if $\lambda \in \left(-\infty, -1-\dfrac{2\sqrt{15}}{3}\right) \cup \left(-1+\dfrac{2\sqrt{15}}{3}, \infty\right)$.



$f''(x) > 0$ if $\lambda \in \big(-\infty, -1\big)$ and $f''(x) < 0$ if $\lambda \in \big(-1, \infty\big)$. Hence we get the graph shown and we see that there are three roots. Using the Newton algorithm, $x_i = x_{i-1} - \dfrac{f(x_{i-1})}{f'(x_{i-1})}$, we get $\lambda_1 \approx -5.54532927$, $\lambda_2 \approx -0.84983068$ and $\lambda_3 \approx 3.39515995$.

4. Since $\det A = \det\begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{pmatrix} \underset{r_3 -> r_3 - r_1}{\overset{r_2 -> r_2 - 2r_1}{=}} \det\begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 6 \\ 0 & 1 & -2 \end{pmatrix} \underset{on\ c_1}{\overset{expand}{=}}$ $(1)\det\begin{pmatrix} -2 & 6 \\ 1 & -2 \end{pmatrix} = (1)(4-6) = -2 \neq 0$, A has an inverse. The cofactor matrix C is

$$C = \begin{pmatrix} +\det\begin{pmatrix} 2 & 4 \\ 3 & -3 \end{pmatrix} & -\det\begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix} & +\det\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \\ -\det\begin{pmatrix} 2 & -1 \\ 3 & -3 \end{pmatrix} & +\det\begin{pmatrix} 1 & -1 \\ 1 & -3 \end{pmatrix} & -\det\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \\ +\det\begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} & -\det\begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} & +\det\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{pmatrix}.$$

Now the (classical) adjoint of A, $\operatorname{adj}(A) = C^t = \begin{pmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{pmatrix}$, and

$$A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \left(-\frac{1}{2}\right)\begin{pmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{pmatrix}.$$

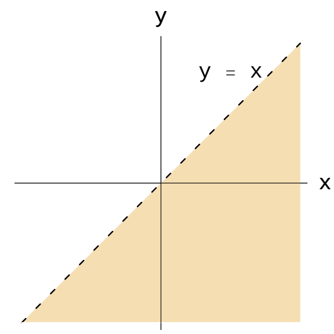
5. (a) $f(x,y) = 5x^2+2y^2-3$.

(i) The domain of $f(x,y)$ is $\big\{(x,y) \in \mathbb{R}^2\big\}$ and its range is $\big\{z \in \mathbb{R} \mid z \geq -3\big\}$.

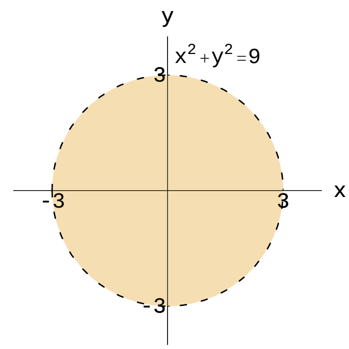
(ii) One possible domain on which $f(x, y)$ is one-to-one is $\{(x, x) \mid x \geq 0\}$. There many other possibilities.

(iii) To ensure that $f(x, y)$ is onto, we put the codomain of f to be equal to the range of f.

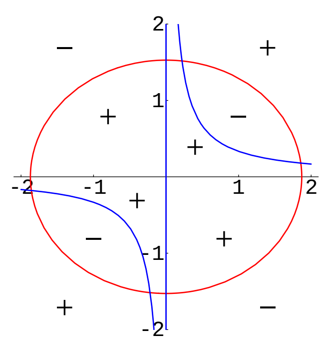
(b) (i) $f(x, y) = \log_2(x - y)$. The domain of $f(x, y)$ is $\{(x, y) \in \mathbb{R}^2 \mid x - y > 0\}$. (This is the part of $\mathbb{R}^2$ to the right of the line $y = x$).
The range of $f(x, y)$ is $\mathbb{R}$.



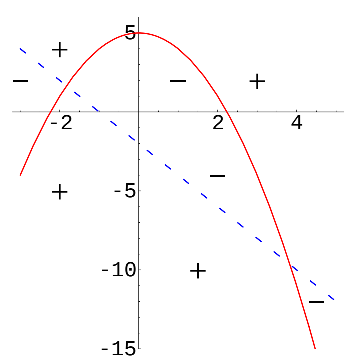
(ii) $f(x, y) = \dfrac{1}{\sqrt{9 - x^2 - y^2}}$. The domain of $f(x, y)$ is $\{\boldsymbol{x} \in \mathbb{R}^2 \mid \|\boldsymbol{x}\| < 3\}$. (This is the region of $\mathbb{R}^2$ which lies inside the circle $x^2 + y^2 = 9$ excluding the circle itself.)
The range of $f(x, y)$ is $\{z \in \mathbb{R} \mid z \geq \frac{1}{3}\}$.



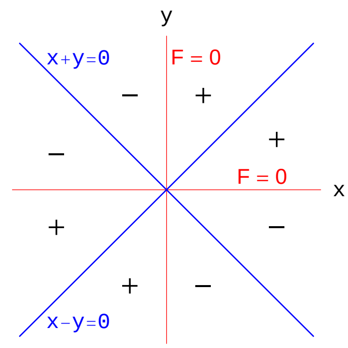
6. (a) $F(x, y) = (2x^2 + 3y^2 - 7)(3xy - 1)$ is defined for all $(x, y) \in \mathbb{R}^2$. $(2x^2 + 3y^2 - 7)$ is 0 on the ellipse $2x^2 + 3y^2 = 7$ (in red) and $(3xy - 1)$ is 0 on the hyperbola $3xy = 1$ (in blue); hence, $F(x, y)$ is 0 on both.



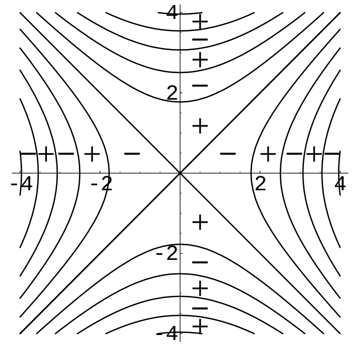
(b) $F(x, y) = \dfrac{y + x^2 - 5}{y + 2x + 2}$ is not defined on the dashed line $y = -2x - 2$ (in blue). $(y + x^2 - 5)$, and consequently, $F(x, y)$ is 0 on the parabola $y = 5 - x^2$ (in red).



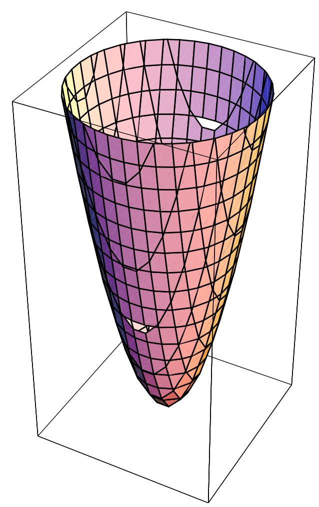
(c) $F(x,y) = |x+y| - |x-y|$ is defined for all $(x,y) \in \mathbb{R}^2$. We will consider 4 cases:
$x+y>0,\ x-y>0$: $F(x,y) = x+y-x+y = 2y$
$x+y>0,\ x-y<0$: $F(x,y) = x+y+x-y = 2x$
$x+y<0,\ x-y>0$: $F(x,y) = -x-y-x+y = -2x$
$x+y<0,\ x-y<0$: $F(x,y) = -x-y+x-y = -2y$.
$F(x,y) = 0$ when $x=0$ or $y=0$ (in red).



(d) $F(x,y) = \sin(y^2 - x^2)$ is defined for all $(x,y) \in \mathbb{R}^2$. $F(x,y) = 0$ if $y^2 - x^2 = k\pi$, $k \in \mathbb{Z}$.



7. We first complete the square. $z = 3x^2 + 3y^2 - 6x + 12y + 15 = 3(x^2 - 2x) + 3(y^2 + 4y) + 15 = 3(x^2 - 2x + 1) + 3(y^2 + 4y + 4) + 15 - (3)(1) - (3)(4) = 3(x-1)^2 + 3(y+2)^2$. This a paraboloid opening upward with vertex at $(1,-2,0)$. Note that the cross sections perpendicular to the x–axis and the y–axis are parabolas opening upward, while the cross sections perpendicular to the z–axis (contours at height z) are circles centered at $(1,-2,z)$ with radius $\sqrt{\dfrac{z}{3}}$.



8. The matchups are A $\longleftrightarrow$ II, B $\longleftrightarrow$ IV, C $\longleftrightarrow$ I and D $\longleftrightarrow$ III.