# University of Toronto Scarborough
# Department of Computer & Mathematical Sciences

## FINAL EXAMINATION

## MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: December 12, 2009
Duration: 3 hours

1. **[9 points]** Carefully state the following theorems. Make sure that you define your terms.

   (a) The Chain Rule for functions of more than one variable.

   (b) The Extreme Value (Min-Max) Theorem for real valued functions of several variables.

   (c) The Change of Variables Theorem for multiple integrals.

2. **[5 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \dfrac{2x^2 + xy + 2y^2}{x^2 + y^2} & , \text{ if } (x,y) \neq (0,0) \\ \quad 2 & , \text{ if } (x,y) = (0,0) \ . \end{cases}$$

   Determine the values of $(x,y)$ for which $f(x,y)$ is continuous.

3. **[5 points]** Give the $4^{\text{th}}$ degree Taylor polynomial about the origin of $f(x,y) = \big(\cos x\big)\big(\ln(1+xy)\big)$.

4. **[9 points]** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathrm{C}^3$.

   (a) Suppose that $n = 2$ and that the second degree Taylor polynomial of $f$ about the origin is $T_2 f = 2x^2 + 2xy + 3y^2$. Show that $(0,0)$ is a critical point of $f$ and classify it.

   (b) Suppose that $n = 2$ and that $\boldsymbol{a}$ is a critical point of $f$. Explain why $\begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$ can not be the Hessian matrix for $f$ at $\boldsymbol{a}$.

   (c) Suppose that $n = 4$ and that the Hessian matrix for $f$ at a critical point $\boldsymbol{a}$ is $\begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$. Classify the critical point.

5. **[9 points]** Let $f(x,y) = 2y^3 - 2xy + x^2$ and let $\boldsymbol{a} = (2,1)$ be a point in $\mathbb{R}^2$.

   (a) Find an equation for the tangent plane to the graph of $f$ at the point $\big(\boldsymbol{a},\, f(\boldsymbol{a})\big)$.

   (b) Determine the direction and magnitude of the maximal increase in $f$ at $\big(\boldsymbol{a},\, f(\boldsymbol{a})\big)$.

   (c) Compute the directional derivative of $f$ at $\boldsymbol{a}$ in the direction $(1,1)$.

6. **[8 points]** Let $f(x,y) = 2y^3 - 2xy + x^2$. Find and classify the critical points of $f$.

7. **[9 points]** Find the maximum and minimum values of $f(x,y,z) = x + 2y - z$ on the solid ellipsoid $2x^2 + y^2 + 2z^2 \leq 5$.

   Justify your answers including an explanation of why global extrema do exist.

8. **[15 points]**

   (a) Evaluate $\displaystyle\int_0^2 \int_{x/4}^{1/2} \sin(\pi\, y^2)\, dy\, dx$.

   (b) Let $D$ be the region of $\mathbb{R}^2$ bounded by the $x$–axis and $y = \sqrt{4 - x^2}$. Evaluate $\displaystyle\int_D e^{x^2+y^2}\, dA$.

   (c) Let $D$ be the triangular region of the $xy$–plane with vertices $(-1,0)$, $(2,0)$ and $(0,1)$. Evaluate $\displaystyle\int_D xy\, dA$.

9. **[5 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x,y) = (x^2y,\, xy,\, y^3)$ and let $g : \mathbb{R}^3 \to \mathbb{R}$ be given by $g(x,y,z) = xy + yz$.
   Use the Chain Rule to compute $\nabla(g \circ f)(x,y)$.

10. **[8 points]** Find the volume of the piece of the region between the planes $x+y+z = 1$ and $2x + 2y + z = 2$ which lies in the first octant.

11. **[8 points]** Evaluate $\displaystyle\int_B e^{x+y+z}\, dV$ where $B$ is the region in $\mathbb{R}^3$ bounded by the planes $y = 1$, $y = -x$, $z = -x$ and, the coordinate planes, $x = 0$ and $z = 0$.

12. **[10 points]** Use a triple integral to find the volume of the solid bounded below by $z = x^2+y^2$ and bounded above by $z = 8 - 2\sqrt{x^2 + y^2}$.