University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2014/2015**

Term Test Solutions

1. (a) (i) Suppose $f : U \subset \mathbb{R}^n \to \mathbb{R}$. A point $\boldsymbol{a} \in U$ is called a **local (relative) minimum** of f if there is an open ball $B_r(\boldsymbol{a})$ such that $f(\boldsymbol{x}) \geq f(\boldsymbol{a})$ for all $\boldsymbol{x} \in B_r(\boldsymbol{a})$.

   (ii) Those points $\boldsymbol{a}$, in the domain of f, at which f is either not differentiable or $D\,f(\boldsymbol{a}) = O$ are called **critical points**.

   (b) From the lecture notes we have

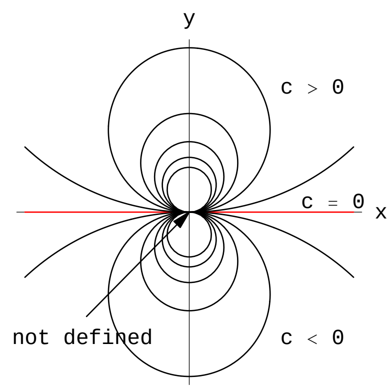
   Extreme Value Theorem. Let D be a compact set in $\mathbb{R}^n$ and let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Then f assumes both a (global) maximum and a (global) minimum on D.

2. (a) (i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 5xy + y^2}{x^2 + y^2}$. Evaluating along the line $y = 0$, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 5xy + y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$, but along the line $y = x$ we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 5xy + y^2}{x^2 + y^2} = \lim_{x\to 0} \frac{-3\,x^2}{2\,x^2} = \frac{-3}{2} \neq 1$. Hence this limit does not exist.

   (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(xy)\, - 1}{x^2y^2}$. Using another single variable technique, we have $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(xy)\, - 1}{x^2y^2} = \lim_{(x,y)\to(0,0)} \frac{(\cos xy - 1)\,(\cos xy + 1)}{x^2y^2\,(\cos xy + 1)} =$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{-\sin^2 xy}{x^2y^2\,(\cos xy + 1)} = -\lim_{(x,y)\to(0,0)} \left(\frac{\sin^2 xy}{(xy)^2}\right)\left(\frac{1}{\cos xy + 1}\right) = -\left(1\right)\left(\frac{1}{2}\right) =$ $-\dfrac{1}{2}$.

   (b) For f to be continuous at $(0,0)$, we need $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 2 = f(0,0)$. Now $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^4 + 2y^2 + 2x^2 - x^4}{x^2 + y^2} \overset{divide}{=} \lim_{(x,y)\to(0,0)} (2 - x^2 + y^2) =$ $2 = f(0,0)$. Hence we conclude that f is continuous at $(0,0)$.

3. $f(x,y) = \dfrac{y}{x^2+y^2}$.
Domain is $\{(x,y) \in \mathbb{R}^2 \mid (x,y) \neq (0,0)\}$.
Putting $f(x,y) = c$ we have $\dfrac{y}{x^2+y^2} = c$. For $c = 0$, the level curve is $y = 0$. For $c \neq 0$, we have $y = c\,(x^2+y^2) \iff x^2 + y^2 - \dfrac{y}{c} = 0 \iff x^2 + \left(y - \dfrac{1}{2c}\right)^2 = \left(\dfrac{1}{2c}\right)^2$. These are circles centered at $\left(0, \dfrac{1}{2c}\right)$ with radius $\dfrac{1}{2c}$.
(All circles pass through $(0,0)$.)



4. Recall, if we put $h(x,y,z) = f(x,y) - z$, the graph of f is a level surface of h, with normal $\nabla h(x,y,z) = (f_x(x,y),\, f_y(x,y),\, -1)$.
Hence the tangent plane to the graph of $f(x,y) = e^{x \sin y}$ at the point $(x,y,f(x,y))$ has normal $(f_x, f_y, -1) = ((\sin y)\, e^{x \sin y},\, (x \cos y)\, e^{x \sin y},\, -1)$. At the point $(1,0,1)$ the normal is $(0,1,-1)$.
Similarly, the tangent plane to the graph of $g(x,y) = 2x^2 + bxy + \dfrac{3}{2}\, y^2$ at the point $(x,y,g(x,y))$ has normal $(g_x, g_y, -1) = (4x + by,\, bx + 3y, -1)$. At the point $\left(1, -1, -b + \dfrac{7}{2}\right)$ the normal is $(4-b, b-3, -1)$.

   (a) For π_1 and π_2 to be parallel, their normals must also be parallel. Hence we need $(4-b, b-3, -1) = k\,(0,1,-1)$, $k \neq 0$. This is only possible if $b = 4$.

   (b) In order for π_1 and π_2 to intersect in a line parallel to $(1,1,1)$, $(1,1,1)$ must be orthogonal to both normals. This holds because $(1,1,1)\boldsymbol{\cdot}(0,1,-1) = 0$ and $(1,1,1)\boldsymbol{\cdot}(4-b, b-3, -1) = 0$, all b. From part (a), π_1 and π_2 will intersect for all b except $b = 4$. Hence, they intersect in a line parallel to $(1,1,1)$ for all b except $b = 4$.

5. (a) To find the equation of the tangent plane to the surface $x^2z^3 + y^2x^3 + z^2y^3 = 1$ at $\boldsymbol{p} = (1,-1,1)$, we put $g(x,y,z) = x^2z^3 + y^2x^3 + z^2y^3 - 1$. A normal to the level surface $g(x,y,z) = 0$ is the gradient, $\nabla g(x,y,z) = (2xz^3 + 3y^2x^2,\, 2yx^3 + 3z^2y^2,\, 3x^2z^2 + 2zy^3)$. Hence a normal at $\boldsymbol{p} = (1,-1,1)$ is $\nabla g(1,-1,1) = (5,1,1)$. Therefore the tangent plane has normal $(5,1,1)$ and its equation is of the form $5x + y + z = d$. Since $(1,-1,1)$ is a point on the plane, we have $d = 5(1) + (-1) + (1) = 5$. Hence the equation of the tangent plane is $5x + y + z = 5$.

   (b) Let $\boldsymbol{p} = (1,2,0)$ be a position vector for the line ℓ. Then $\boldsymbol{v} = (0,1,2) - (1,2,0) = (-1,-1,2)$ is a direction vector for ℓ. Hence a parametric description of ℓ is $\boldsymbol{p} + t\,\boldsymbol{v} = (1,2,0) + t\,(-1,-1,2)$, $t \in \mathbb{R}$.
   The line ℓ will intersect the tangent plane from (a) if and only if $(1,2,0) + t\,(-1,-1,2) = (1-t, 2-t, 2t)$, a point on ℓ, satisfies $5x + y + z = 5$, the equation of the tangent plane. Now $5(1-t) + (2-t) + 2t = 5 \iff 7 - 4t = 5 \iff$

$-4t = -2 \iff t = \dfrac{1}{2}$. Hence ℓ meets the tangent plane when $t = \dfrac{1}{2}$. The point of intersection is $\left(1 - \dfrac{1}{2},\, 2 - \dfrac{1}{2},\, 2\left(\dfrac{1}{2}\right)\right) = \left(\dfrac{1}{2},\, \dfrac{3}{2},\, 1\right)$.

6. We know that $D_v\, f(\boldsymbol{a}) = \nabla f(\boldsymbol{a}) \cdot \dfrac{1}{\|\boldsymbol{v}\|}\boldsymbol{v} = \|\nabla f(\boldsymbol{a})\| \dfrac{\|\boldsymbol{v}\|}{\|\boldsymbol{v}\|} \cos\theta = \|\nabla f(\boldsymbol{a})\| \cos\theta$, where θ is the angle between ∇f and $\boldsymbol{v}$. Thus $D_v\, f(\boldsymbol{a})$ will be maximized when $\cos\theta$ is maximized. This occurs when $\cos\theta = 1 \implies \theta = 0$ (or an integer multiple multiple of 2π). Therefore $D_v\, f(\boldsymbol{a})$ is maximized when $\boldsymbol{v}$ is in the same direction as $\nabla f(\boldsymbol{a})$. Hence the required unit vector is $\boldsymbol{v} = \dfrac{1}{\|\nabla f(\boldsymbol{a})\|} \nabla f(\boldsymbol{a})$.

7. (a) $f(x,y,z) = x^3 + y^2 - z^2 - 6xy + 6x + 3y + 1$. Since f is continuous for all $(x,y,z) \in \mathbb{R}^3$, critical points can only occur when $\nabla f = \mathbf{0}$. Computing the partial derivatives we have $f_x = 3x^2 - 6y + 6$, $f_y = 2y - 6x + 3$ and $f_z = -2z$. Equating to 0, we have, from the third, $z = 0$, and from the second, $y = 3x - \dfrac{3}{2}$. Now the first becomes $3x^2 - 18x + 15 = 0$ or $0 = x^2 - 6x + 5 = (x-5)(x-1) \implies x = 5$ or $x = 1$. Hence there are two critical points: $\left(1,\, \dfrac{3}{2},\, 0\right)$ and $\left(5,\, \dfrac{27}{2},\, 0\right)$.

   (b) When moving from $(1,2,0)$ to $(3,0,1)$ you move in direction $\boldsymbol{v} = (3,0,1) - (1,2,0) = (2,-2,1)$. The rate of change in f is given by the directional derivative of f at $(1,2,0)$ in direction $\boldsymbol{v} = (2,-2,1)$. Now $\nabla f = (3x^2 - 6y + 6,\, 2y - 6x + 3,\, -2z)$ and $\nabla f(1,2,0) = (-3,1,0)$ so $D_v f(1,2,0) =$
   $\nabla f(1,2,0) \cdot \dfrac{1}{\|(2,-2,1)\|}(2,-2,1) = \dfrac{(-3,1,0)\cdot(2,-2,1)}{\sqrt{4+4+1}} = \dfrac{-8}{\sqrt{9}} = \dfrac{-8}{3}$.

   (c) For the most rapid increase you would go in the direction of the gradient; i.e., in direction $\nabla f(2,1,0) = (12,-7,0)$.
   The rate of maximum increase is $D_{\nabla f} f(2,1,0) = \|\nabla f(2,1,0)\| = \|(12,-7,0)\| = \sqrt{144 + 49 + 0} = \sqrt{193}$.

8. (a) From the lecture notes we have

   Chain Rule. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subset \mathbb{R}^m \to \mathbb{R}^k$ be given functions such that $f[U] \subset V$ so that $g \circ f$ is defined. Let $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} = f(\boldsymbol{a}) \in \mathbb{R}^m$. If f is differentiable at $\boldsymbol{a}$ and g is differentiable at $\boldsymbol{b}$, then $g \circ f$ is differentiable at $\boldsymbol{a}$ and

   $$D(g \circ f)(\boldsymbol{a}) = [Dg(\boldsymbol{b})]\,[Df(\boldsymbol{a})]\ .$$

   (b) $f : \mathbb{R}^4 \to \mathbb{R}^3$ is given by $f(x,y,z,w) = (y\,z\,w,\ x^2\,y,\ x\,z)$ so
   $$Df = \begin{pmatrix} 0 & zw & yw & yz \\ 2xy & x^2 & 0 & 0 \\ z & 0 & x & 0 \end{pmatrix}.$$
   $g : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $g(x,y,z) = (x\,y,\ y\,z)$ so $D\,g = \begin{pmatrix} y & x & 0 \\ 0 & z & y \end{pmatrix}$ and

$D\,g(f(x,y,z)) = \begin{pmatrix} x^2y & yzw & 0 \\ 0 & xz & x^2y \end{pmatrix}$.

Now $D(g\circ f)(x,y,z,w) = [D\,g(f(x,y,z,w))]\,[D\,f(x,y,z,w)]$

$= \begin{pmatrix} x^2y & yzw & 0 \\ 0 & xz & x^2y \end{pmatrix} \begin{pmatrix} 0 & zw & yw & yz \\ 2xy & x^2 & 0 & 0 \\ z & 0 & x & 0 \end{pmatrix}$

$= \begin{pmatrix} 2xy^2zw & 2x^2yzw & x^2y^2w & x^2y^2z \\ 3x^2yz & x^3z & x^3y & 0 \end{pmatrix}$.

9. We first note that $\dfrac{\partial f}{\partial v} \overset{Chain}{\underset{Rule}{=}} \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v} = \dfrac{\partial f}{\partial x}(u) + \dfrac{\partial f}{\partial y}(-3) = u\,\dfrac{\partial f}{\partial x} - 3\,\dfrac{\partial f}{\partial y}$. Now $\dfrac{\partial^2 f}{\partial u\,\partial v} = \dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial v}\right) = \dfrac{\partial}{\partial u}\left[u\,\dfrac{\partial f}{\partial x} - 3\,\dfrac{\partial f}{\partial y}\right] = \dfrac{\partial}{\partial u}\left[u\,\dfrac{\partial f}{\partial x}\right] - 3\,\dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial f}{\partial x} + u\,\dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial x}\right) - 3\,\dfrac{\partial}{\partial u}\left(\dfrac{\partial f}{\partial y}\right) \overset{Chain}{\underset{Rule}{=}} \dfrac{\partial f}{\partial x} + u\left[\dfrac{\partial^2 f}{\partial x^2}\dfrac{\partial x}{\partial u} + \dfrac{\partial^2 f}{\partial y\,\partial x}\dfrac{\partial y}{\partial u}\right] - 3\left[\dfrac{\partial^2 f}{\partial x\,\partial y}\dfrac{\partial x}{\partial u} + \dfrac{\partial^2 f}{\partial y^2}\dfrac{\partial y}{\partial u}\right] = \dfrac{\partial f}{\partial x} + u\left[\dfrac{\partial^2 f}{\partial x^2}(v) + \dfrac{\partial^2 f}{\partial y\,\partial x}(2)\right] - 3\left[\dfrac{\partial^2 f}{\partial x\,\partial y}(v) + \dfrac{\partial^2 f}{\partial y^2}(2)\right] = \dfrac{\partial f}{\partial x} + uv\,\dfrac{\partial^2 f}{\partial x^2} + 2u\,\dfrac{\partial^2 f}{\partial y\,\partial x} - 3v\,\dfrac{\partial^2 f}{\partial x\,\partial y} - 6\,\dfrac{\partial^2 f}{\partial y^2} \overset{f\ is\ of}{\underset{class\ C^2}{=}} \dfrac{\partial f}{\partial x} + uv\,\dfrac{\partial^2 f}{\partial x^2} + (2u-3v)\,\dfrac{\partial^2 f}{\partial y\,\partial x} - 6\,\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial f}{\partial x} + x\,\dfrac{\partial^2 f}{\partial x^2} + y\,\dfrac{\partial^2 f}{\partial y\,\partial x} - 6\,\dfrac{\partial^2 f}{\partial y^2}$.

10. Recall $e^t = \displaystyle\sum_{k=0}^{\infty}\frac{t^k}{k\,!}$, $|t|<\infty$, so $e^{x^2} = 1 + x^2 + \dfrac{x^4}{2\,!} + \cdots$, $|x| < \infty$ (by replacement). Also recall that $\cos t = \displaystyle\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)\,!}$, $|t|<\infty$ so $\cos(xy) = 1 - \dfrac{x^2y^2}{2\,!} + \dfrac{x^4y^4}{4\,!} - \cdots$, $|xy| < \infty$ (by replacement). We now obtain a Taylor series for $f(x,y) = e^{x^2}\cos(xy)$,

$$T = \left(1 + x^2 + \frac{x^4}{2\,!} + \cdots\right)\left(1 - \frac{x^2y^2}{2\,!} + \frac{x^4y^4}{4\,!} - \cdots\right),$$

by multiplication of series. Hence the 4th degree Taylor polynomial for f about the origin is

$$T_4 = 1 + x^2 + \frac{x^4}{2} - \frac{x^2y^2}{2} = 1 + x^2 + \frac{1}{2}\left(x^4 - x^2y^2\right).$$