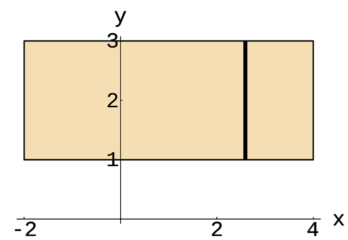
University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #9

1. (a) $\displaystyle\int_D \frac{x}{y}\,dA, \quad D=[-2,4]\times[1,3].$

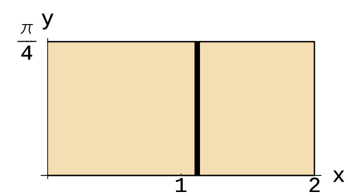
With this integral we could use either horizontal or vertical strips or we can separate the integral and integrate as two single variable integrals. We will use the latter.



$$\iint_D \frac{x}{y}\,dA = \int_{-2}^{4}\int_1^3 \frac{x}{y}\,dy\,dx = \int_{-2}^{4} x\,dx \int_1^3 \frac{dy}{y} = \left[\frac{x^2}{2}\right]_{-2}^{4}\left[\ln|y|\right]_1^3 = \frac{1}{2}(16-4)(\ln 3 - \ln 1) = 6\,\ln 3.$$

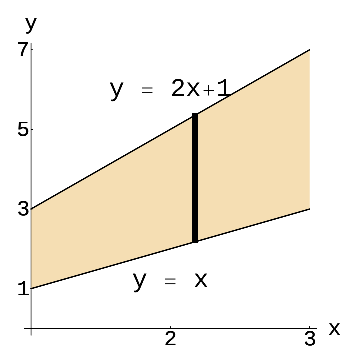
(b) $\displaystyle\int_D e^x \sin y\,dA, \quad D=[0,2]\times[0,\frac{\pi}{4}].$

The same options apply here as in part (a).



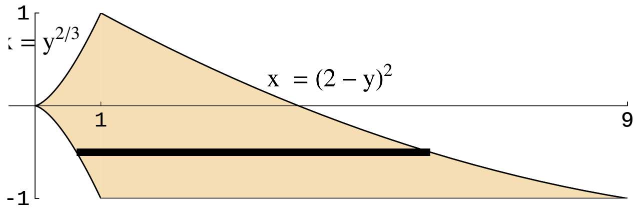
$$\int_D e^x \sin y\,dA = \int_0^2\int_0^{\frac{\pi}{4}} e^x \sin y\,dy\,dx = \int_0^2 e^x\Big[-\cos y\Big]_0^{\frac{\pi}{4}}dx = \int_0^2 e^x\Big(-\cos\frac{\pi}{4} + \cos 0\Big)\,dx = \Big(1-\frac{1}{\sqrt{2}}\Big)\int_0^2 e^x\,dx = \Big(1-\frac{1}{\sqrt{2}}\Big)\Big[e^x\Big]_0^2 = \Big(\frac{2-\sqrt{2}}{2}\Big)\big(e^2-1\big).$$

(c) $\displaystyle\int_D x^2 y\,dA,$ D is the region bounded by the lines $x=y$ and $y=2x+1$ between $x=1$ and $x=3$.



$$\int_D x^2 y\,dA = \int_1^3\int_x^{2x+1} x^2 y\,dx = \int_1^3 x^2\left[\frac{y^2}{2}\right]_x^{2x+1}dx = \int_1^3 \frac{x^2}{2}\big((2x+1)^2-x^2\big)\,dy = \int_1^3\Big(\frac{3\,x^4}{2}+2x^3+\frac{x^2}{2}\Big)\,dx = \left[\frac{3\,x^5}{10}+\frac{x^4}{2}+\frac{x^3}{6}\right]_1^3 = \frac{3^6}{10}+\frac{3^4}{2}+\frac{3^3}{6}-\Big(\frac{3}{10}+\frac{1}{2}+\frac{1}{6}\Big) = \frac{17545}{15}.$$

(d) $\displaystyle\int_{-1}^{1}\int_{y^{2/3}}^{(2-y)^2}\left(\frac{3}{2}\sqrt{x}-2y\right)dx\,dy.$

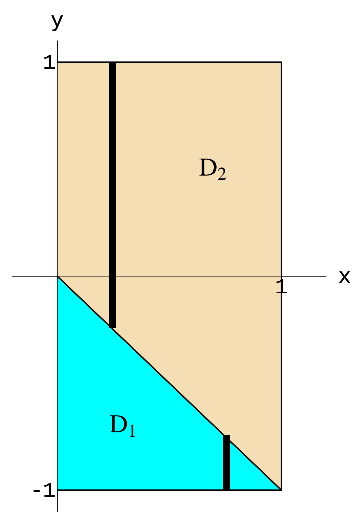


$$\int_{-1}^{1}\int_{y^{2/3}}^{(2-y)^2}\left(\frac{3}{2}\sqrt{x}-2y\right)dx\,dy = \int_{-1}^{1}\Big[x^{3/2}-2xy\Big]_{y^{2/3}}^{(2-y)^2}dy = \int_{-1}^{1}(2-y)^3-2(2-y)^2y-|\,y\,|+2\,y^{5/3}\,dy = -\left[\frac{(2-y)^4}{4}\right]_{-1}^{1} - 2\int_{-1}^{1}4y-4y^2+y^3\,dy - 2\int_0^1 y\,dy + 2\left[\left(\frac{3}{5}\right)y^{8/3}\right]_{-1}^{1} = 20-2\left[2y^2-\frac{4}{3}y^3+\frac{1}{4}y^4\right]_{-1}^{1}-\Big[y^2\Big]_0^1+0 = 20-2\left(-\frac{8}{3}\right)-1=\frac{73}{3}.$$

(e) $\displaystyle\int_D |\,x+y\,|\,dA$, where $D=[0,1]\times[-1,1]$.

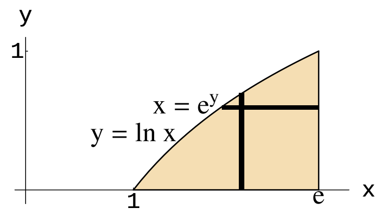
We first notice that $x+y\le 0$ below the line $y=-x$ which is the triangle at the bottom left of the rectangle. Now $x+y\ge 0$ in the rest of the rectangle. We label these regions as D_1 and D_2. Then

$$\int_D|\,x+y\,|\,dA=\int_{D_1}|\,x+y\,|\,dA+\int_{D_2}|\,x+y\,|\,dA = -\int_{D_1}(x+y)\,dA+\int_{D_2}(x+y)\,dA = -\int_0^1\int_{-1}^{-x}(x+y)\,dy\,dx+\int_0^1\int_{-x}^{1}(x+y)\,dy\,dx = -\int_0^1\left(x(1-x)+\frac{x^2}{2}-\frac{1}{2}\right)dx+\int_0^1\left(x(1+x)+\frac{1}{2}-\frac{x^2}{2}\right)dx = -\left[\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^3}{6}-\frac{x}{2}\right]_0^1+\left[\frac{x^2}{2}+\frac{x^3}{3}+\frac{x}{2}-\frac{x^3}{6}\right]_0^1 = -\left(-\frac{1}{6}\right)+\frac{7}{6}=\frac{4}{3}.$$



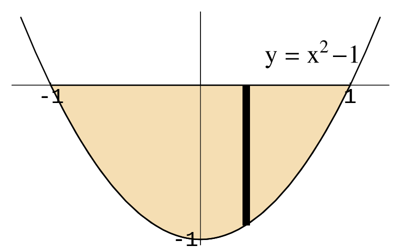
(f) $\displaystyle\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\, dx\, dy$.

The region $e^y \le x \le e$, $0 \le y \le 1$ can also be written as $0 \le y \le \ln x$, $1 \le x \le e$. So, changing the order of integration we have $\displaystyle\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\, dx\, dy = \int_1^e \int_0^{\ln x} \frac{x}{\ln x}\, dy\, dx = \int_1^e \left[y\, \frac{x}{\ln x} \right]_0^{\ln x} dx = \int_1^e x\, dx = \left[\frac{x^2}{2} \right]_1^e = \frac{1}{2}\big(e^2 - 1\big)$.



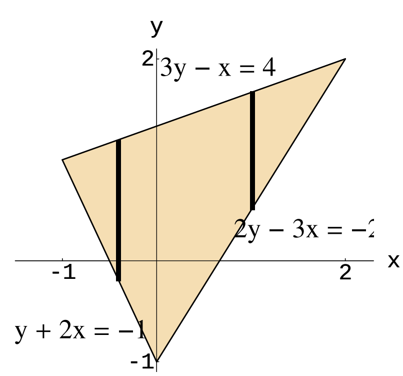
(g) $\displaystyle\int_D \|\nabla f\|^2\, dA$, where $f(x,y) = y - x^2 + 1$ and $D = \big\{(x,y) \mid f(x,y) \ge 0,\ y \le 0\big\}$.

Since $f(x,y) = y - x^2 + 1$, $\nabla f = \big(-2x,\, 1\big)$ and $\|\nabla f\|^2 = 4x^2 + 1$. The curve $f(x,y) = 0$ is the graph of $y = x^2 - 1$ so the region D is given by $x^2 - 1 \le y \le 0$, $-1 \le x \le 1$. Thus $\displaystyle\int_D \|\nabla f\|^2\, dA = \int_{-1}^1 \int_{x^2-1}^0 \big(4x^2 + 1\big)\, dy\, dx = \int_{-1}^1 \Big[4x^2 y + y \Big]_{x^2-1}^0 dx = \int_{-1}^1 \Big(-4x^2(x^2-1) - (x^2-1) \Big)\, dx = \left[-\frac{4\,x^5}{5} + \frac{4\,x^3}{3} - \frac{x^3}{3} + x \right]_{-1}^1 = \frac{12}{5}$.



(h) $\displaystyle\int_D e^x y\, dA$, where D is the interior of the triangle with vertices $(-1,1)$, $(2,2)$ and $(0,-1)$.

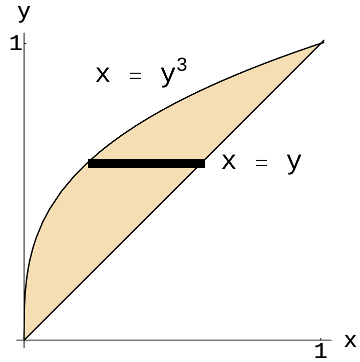
We will need to write $\displaystyle\int_D e^x y\, dA$ as the sum of two integrals: $\displaystyle\int_D e^x y\, dA = \int_{-1}^0 \int_{-2x-1}^{\frac{x+4}{3}} e^x y\, dy\, dx + \int_0^2 \int_{\frac{3x-2}{2}}^{\frac{x+4}{3}} e^x y\, dy\, dx = \int_{-1}^0 e^x \left[\frac{y^2}{2} \right]_{-2x-1}^{\frac{x+4}{3}} dx + \int_0^2 e^x \left[\frac{y^2}{2} \right]_{\frac{3x-2}{2}}^{\frac{x+4}{3}} dx = \int_{-1}^0 \frac{e^x}{2} \left(-\frac{35\,x^2}{9} - \frac{28\,x}{9} + \frac{7}{9} \right) dx + \int_0^2 \frac{e^x}{72} \big(-77x^2 + 140x + 28 \big)\, dx = -\frac{7}{18} \Big[5\, e^x \big(x^2 - 2x + 2\big) + 4\, e^x (x-1) - e^x \Big]_{-1}^0 + \frac{7}{72} \Big[e^x \big(-11\big(x^2 - 2x + 2\big) + 20\big(x-1\big) + 4\big) \Big]_0^2 = \frac{7\, e^2}{36} - \frac{203}{36} + \frac{56}{9\, e}$.



(i) $\displaystyle\int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y}\, dy\, dx.$

We can not integrate $e^{x/y}$ w.r.t. y by exact means, so we will reverse the order of integration.

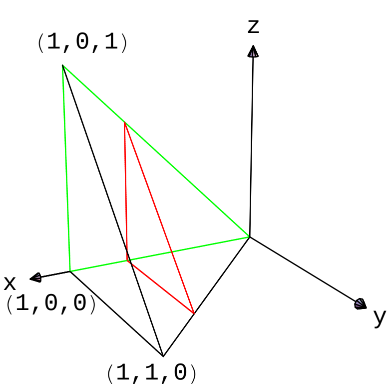
$$\int_0^1 \int_x^{\sqrt[3]{x}} e^{x/y}\, dy\, dx = \int_0^1 \int_{y^3}^{y} e^{x/y}\, dx\, dy = \int_0^1 \left(y e^{x/y} \Big|_{y^3}^{y} \right) dy = \int_0^1 (ye - y e^{y^2})\, dy = \left[\frac{y^2 e}{2} - \frac{e^{y^2}}{2} \right]_0^1 = \frac{1}{2}.$$

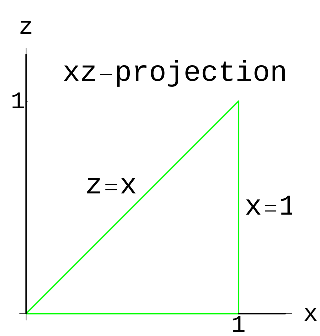


2. The function $f(x, y) = x^2 + y^2 + 1$ has the value $r^2 + 1$ where r is the distance from the origin. Hence, on the disk of radius 2, we have $1 = 0 + 1 \le f(x, y) \le 4 + 1 = 5$. We also know that the area of D, the disk of radius 2, is 4π. Now, integrating this inequality over D, and using the properties of the integral, we have

$$\begin{aligned}
& \int_D 1\, dx\, dy \le \int_D \left(x^2 + y^2 + 1\right) dx\, dy \le \int_D 5\, dx\, dy \\
\Longrightarrow\ & (1) \int_D dx\, dy \le \int_D \left(x^2 + y^2 + 1\right) dx\, dy \le (5) \int_D dx\, dy \\
\Longrightarrow\ & (1)\,(4\pi) \le \int_D \left(x^2 + y^2 + 1\right) dx\, dy \le (5)\,(4\pi) \\
\Longrightarrow\ & 4\pi \le \int_D \left(x^2 + y^2 + 1\right) dx\, dy \le 20\pi\, .
\end{aligned}$$

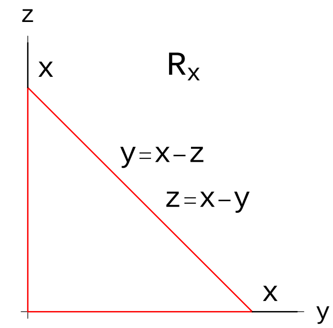
3. We are given the integral $\displaystyle\int_0^1 \int_z^1 \int_0^{x-z} f(x, y, z)\, dy\, dx\, dz$ which we will first regard as an integral where x and z are fixed and the y is integrated out first. The projection into the xz–plane is given as $\left\{ \begin{matrix} z & \le & x & \le & 1 \\ 0 & \le & z & \le & 1 \end{matrix} \right\}$. We can also describe this projection as $\left\{ \begin{matrix} 0 & \le & z & \le & x \\ 0 & \le & x & \le & 1 \end{matrix} \right\}$ and rewrite the integral as $\displaystyle\int_0^1 \int_0^x \int_0^{x-z} f(x, y, z) dy dz dx.$



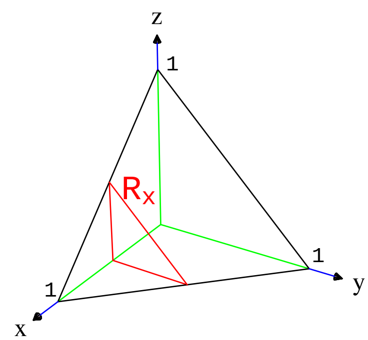


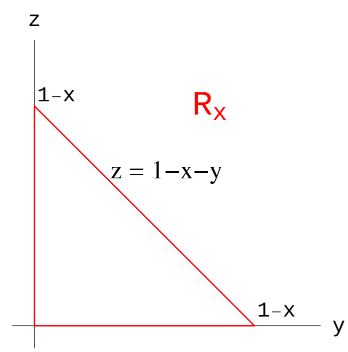
We now regard this integral as one where x is fixed and we first integrate over R_x, with

R_x described by $\left\{\begin{array}{ccccc} 0 & \le & y & \le & x-z \\ 0 & \le & z & \le & x \end{array}\right\}$. We can also describe R_x as $\left\{\begin{array}{ccccc} 0 & \le & z & \le & x-y \\ 0 & \le & y & \le & x \end{array}\right\}$. The integral can now be given by $\displaystyle\int_0^1\int_0^x\int_0^{x-y} f(x,y)\,dz\,dy\,dx$. (To see other ways of rewriting this integral you could take this integral and change the order of integration in the xy–plane. You could also take the original integral and change the order in the cross-section R_z.)



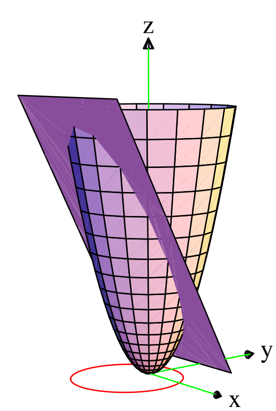
4. To evaluate this integral you can use any one of the six possible orders of integration.





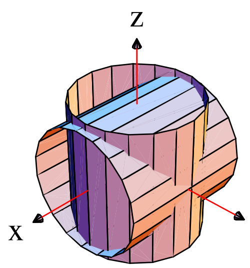
I'll fix x, $0 \le x \le 1$ and first integrate over R_x , shown above on the right. We can describe R_x by $\left\{\begin{array}{ccccc} 0 & \le & z & \le & 1-x-y \\ 0 & \le & y & \le & 1-x \end{array}\right\}$. Hence $\displaystyle\iiint_B y\,dV = \int_0^1\left(\iint_{R_x} y\,dA\right)dx = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y dz\,dy\,dx = \int_0^1\int_0^{1-x} y(1-x-y)\,dy\,dx = \int_0^1\left[\frac{(1-x)\,y^2}{2}-\frac{y^3}{3}\right]_0^{1-x}dx = \frac{1}{6}\int_0^1(1-x)^3\,dx = -\frac{1}{24}\,(1-x)^4\Big|_0^1 = \frac{1}{24}$.

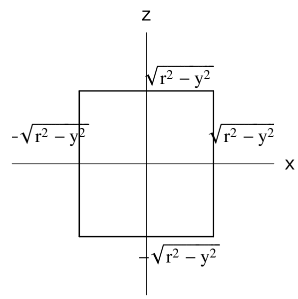
5. Fixing x and y we have $x^2+y^2 \le z \le 3-2y$ and the projection into the xy–plane is $\{(x,y) \mid x^2+y^2 \le 3-2y\} = \{(x,y) \mid x^2+(y+1)^2 \le 4\}$ which is a circular disk of radius 2 centered at $(0,-1)$. Now the volume is $\displaystyle\int_B 1\,dV = \iint_{proj}\left(\int_{x^2+y^2}^{3-2y} 1\,dz\right)dA \overset{symmetric\ about}{\underset{yz-plane}{=}}$
$\displaystyle 2\int_{-3}^1\int_0^{\sqrt{3-2y-y^2}}\int_{x^2+y^2}^{3-2y} 1\,dz\,dx\,dy =$
$\displaystyle 2\int_{-3}^1\int_0^{\sqrt{3-2y-y^2}}(3-2y-x^2-y^2)\,dx\,dy =$
$\displaystyle 2\int_{-3}^1\left[(3-2y-y^2)\,x-\frac{x^3}{3}\right]_0^{\sqrt{3-2y-y^2}}dy =$



$$\frac{4}{3}\int_{-3}^{1}\left(3-2y-y^2\right)^{3/2}dy = \frac{4}{3}\int_{-3}^{1}\left(4-(y+1)^2\right)^{3/2}dy \underset{y+1=2\sin\theta}{\overset{substitute}{=}} \frac{64}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\theta\,d\theta = \frac{128}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{32}{3}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta = 8\pi.$$

6. If we fix y for $-r \le y \le r$, the cross section R_y is a square which can be described by



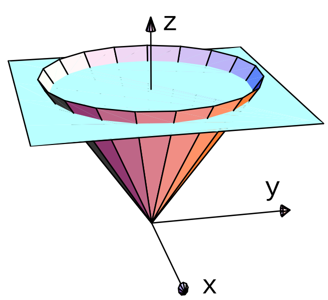


$$-\sqrt{r^2-y^2} \le x \le \sqrt{r^2-y^2},$$
$$-\sqrt{r^2-y^2} \le z \le \sqrt{r^2-y^2}.$$

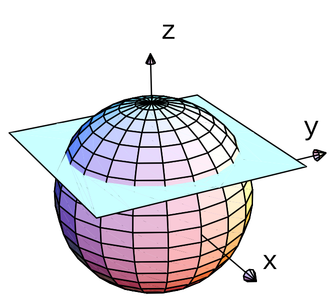
Hence the volume is $\int_B 1\,dV = \int_{-r}^{r}\left(\iint_{R_y} 1\,dA\right)dy = \int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\,dz\,dy$

$$\overset{symmetry}{=} 8\int_0^r\int_0^{\sqrt{r^2-y^2}}\int_0^{\sqrt{r^2-y^2}} dx\,dz\,dy = 8\int_0^r\int_0^{\sqrt{r^2-y^2}}\sqrt{r^2-y^2}\,dz\,dy = 8\int_0^r (r^2-y^2)\,dy = 8\left[r^2y-\frac{1}{3}y^3\right]_0^r = 8\left(\frac{2r^3}{3}\right) = \frac{16}{3}r^3.$$

7. (a) $W = (x,y,z) \mid \sqrt{x^2+y^2} \le z \le 1$ is the region inside the cone $z = \sqrt{x^2+y^2}$ below the plane $z = 1$. For $\sqrt{x^2+y^2} \le z \le 1$ we have $0 \le \sqrt{x^2+y^2} \le 1$ or $x^2+y^2 \le 1$ so the projection into the xy–plane is just the disk $x^2+y^2 \le 1$ (of radius 1 and centered at $(0,0)$). Now $\int_W f\,dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} f(x,y,z)\,dz\,dy\,dx.$



(b) We first note that $x^2+y^2+z^2=1$ describes the unit sphere. Since $\frac{1}{2} \le z \le 1$, W is the interior of the unit sphere above the plane $z = \frac{1}{2}$. Substituting $z = \frac{1}{2}$ into the equation of the unit sphere gives $x^2+y^2+\frac{1}{4} = 1$ or $x^2+y^2 = \frac{3}{4}$. Hence we have $\frac{1}{2} \le z \le \sqrt{1-(x^2+y^2)}$ and the projection into the xy–plane is the disk $x^2+y^2 \le \frac{3}{4}$ (of radius $\frac{\sqrt{3}}{2}$ and centered at $(0,0)$).



Now $\int_W f\,dV = \int_{\frac{-\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}\int_{-\sqrt{\frac{3}{4}-x^2}}^{\sqrt{\frac{3}{4}-x^2}}\int_{\frac{1}{2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$