1. The temperature in °C at points in the $xy$-plane is given by the function $T(x, y) = x^2 - 2y^2$ where $x$ and $y$ are in centimetres.

   (a) **[5 points]** Find the direction that a ladybug at the point $(2, -1)$ should crawl if she wishes to cool off as quickly as possible.

From the point $(2,-1)$ $T$ is decreased maximally by moving in the direction $-\nabla T(2,-1)$

$\nabla T(x,y) = (2x, -4y)$

$-\nabla T(2,-1) = -(4,4) = (-4,-4)$

$\therefore$ lady bug should crawl in the direction of $(-4,-4)$ from $(2,-1)$.

   (b) **[7 points]** Assume the ladybug's crawl speed is a constant 2 cm/s. Find the rate with respect to time at which the temperature changes if she crawls in a straight line from $(2, -1)$ towards the point $(-1, 1)$.

Let $P=(2,-1)$, $Q=(-1,1)$. Ladybug crawls in the direction $\underline{w} = \overrightarrow{PQ} = Q-P = (-3,2)$

$\underline{u} = \dfrac{\underline{w}}{\|\underline{w}\|} = \dfrac{(-3,2)}{\sqrt{13}} = \left(-\dfrac{3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}}\right)$

$D_{\underline{u}} T(2,-1)$ = rate of change of $T$ per unit of distance when crawling from $(2,-1)$ in $\underline{u}$-direction

$= \nabla T(2,-1) \cdot \underline{u} = (4,4)\cdot\left(-\dfrac{3}{\sqrt{13}}, \dfrac{2}{\sqrt{13}}\right)$

$= -\dfrac{4}{\sqrt{13}}$ (°C/m)

$\therefore \left(-\dfrac{4}{\sqrt{13}}\right)(2) = -\dfrac{8}{\sqrt{13}}$ °C/s gives the rate of temperature change with respect to time

(− sign means temperature is decreasing)

2. **[13 points]** Find and classify the critical points of $f(x,y,z) = x^3+xy^2+x^2+y^2+3z^2$.

We solve $\nabla f(x,y,z) = 0$ for critical points

① $0 = f_x = 3x^2+y^2+2x$

② $0 = f_y = 2xy+2y = 2y(x+1) \Rightarrow y=0$ or $x=-1$

③ $0 = f_z = 6z \Rightarrow z = 0$

Case 1 $y=0$ Sub-in ① to get $3x^2+2x=0$

$x(3x+2)=0$

$\therefore x=0$ or $x=-\frac{2}{3}$

$\therefore$ Critical points here are $(0,0,0)$ & $(-\frac{2}{3},0,0)$

Case 2 $x=-1$ Sub-in ① to get $3+y^2-2=0$

$y^2=-1$ which has no real solutions.

$\therefore$ the only critical points are $(0,0,0)$ & $(-\frac{2}{3},0,0)$ Now we classify these.

Hessian matrix $\mathcal{H}(x,y,z) = \begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 6x+2 & 2y & 0 \\ 2y & 2x+2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$

$\mathcal{H}(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ $\therefore D_1 = 2 \quad D_2 = 4 \quad D_3 = 24$

$\because D_i > 0$, $i=1,2,3$, $2^{nd}$ DT $\Rightarrow f$ has a local minimum @ $(0,0,0)$ of value $0$.

$\mathcal{H}(-\frac{2}{3},0,0) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ $\therefore D_1 = -2 \quad D_2 = -\frac{4}{3} \quad D_3 = -8$

$\because D_i < 0$, $i=1,2,3$, $2^{nd}$ DT $\Rightarrow f$ has a saddle point @ $(-\frac{2}{3},0,0)$ of value $\frac{4}{27}$

3. **[8 points]** Find the $6^{\text{th}}$–degree Taylor polynomial at $(0,0)$ for the function $f(x,y) = \dfrac{\sin(2xy)}{3+9x}$.

[You may use any facts and techniques about Taylor/Maclaurin series from 1–variable calculus.]

Since we want the Taylor polynomial of degree 6 about $(0,0)$, it is efficient to manipulate the appropriate Maclaurin series. For degree 6, we want terms with $m+n \leq 6$, $x^m y^n$

Write $f(x,y) = \frac{1}{3}\left[\dfrac{\sin(2xy)}{1+3x}\right]$

For $u \in \mathbb{R}$, $\sin(u) = \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^{m+1} u^{2m+1}}{(2m+1)!}$ Put $u = 2xy$

to get $\sin(2xy) \sim 2xy - \dfrac{(2xy)^3}{6} = 2xy - \frac{4}{3}x^3y^3$

For $v \in (-1,1)$, $\dfrac{1}{1-v} = \displaystyle\sum_{n=0}^{\infty} v^n$ Put $v = -3x$

to get $\dfrac{1}{1+3x} \sim 1 - 3x + 9x^2 - 27x^3 + 81x^4 - 243x^5$

$\therefore$ We examine, then appropriately truncate after the degree $= 6$ terms, the product

$$\frac{1}{3}\left[2xy - \frac{4}{3}x^3y^3\right]\left[1 - 3x + 9x^2 - 27x^3 + 81x^4 - 243x^5\right]$$

$$= \frac{1}{3}\left[2xy - 6x^2y + 18x^3y - 54x^4y + 162x^5y - \frac{4}{3}x^3y^3\right]$$

$$+ [\text{terms with degree} \geq 7]$$

$\therefore$ $6^{\text{th}}$–degree Taylor polynomial is

$$P_6(x,y) = \frac{2}{3}xy - 2x^2y + 6x^3y - 18x^4y + 54x^5y - \frac{4}{9}x^3y^3$$

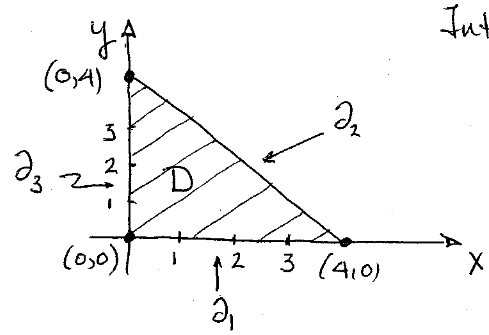
4. Throughout this question let $f(x,y) = x^2ye^{-(x+y)}$ and consider the triangular region

$$D = \{(x,y) \mid x \geq 0 \text{ and } y \geq 0 \text{ and } x + y \leq 4\} .$$

(a) **[3 points]** Draw the region $D$.

Use the line $x+y=4$

Intercepts $(0,4)$ + $(4,0)$




(b) **[4 points]** Justify briefly why $f$ has absolute extrema on $D$.

$f$ is continuous on $\mathbb{R}^2$, hence on D. $f$ is continuous because it is a product and exponentiation of polynomials. D is compact (because it is clearly bounded, and it is closed because it contains its boundary). It now follows by EVT that $f$ has absolute extrema on D.

(c) **[15 points]** Find the absolute extrema of $f$ on $D$.

We look for critical points of $f$ in $D^\circ$ using $\nabla f = 0$

$$0 = f_x = 2xye^{-(x+y)} + x^2ye^{-(x+y)}(-1)$$
$$= xye^{-(x+y)}(2-x)$$

$\nabla f = 0$ only @ the point $(2,1) \in D^\circ$

$$0 = f_y = x^2e^{-(x+y)} + x^2ye^{-(x+y)}(-1)$$
$$= x^2e^{-(x+y)}(1-y)$$

$f(2,1) = 4e^{-3} \sim .199$

Now we analyze $f$ on the boundary $\partial D = \partial_1 \cup \partial_2 \cup \partial_3$

$\underline{\partial_1}$ Parametrize as $(x,0)$ where $0 \leq x \leq 4$

$$f|_{\partial_1}(x,y) = f(x,0) = 0$$

$\underline{\partial_3}$ Parametrize as $(0,y)$ where $0 \leq y \leq 4$

$$f|_{\partial_3}(x,y) = f(0,y) = 0$$

**Question 4 continued**

$\partial_2$ Parametrize as $y=-x+4$ where $0 \le x \le 4$

$f|_{\partial_2}(x,y) = f(x,-x+4) = x^2(4-x)e^{-4}$

Put $h(x) = x^2(4-x) = 4x^2 - x^3$ Checking $f|_{\partial_2}(x,y)$ is equivalent to analyzing $h(x)$ on $[0,4]$

$h(0) = 0 = h(4)$ (endpoints)

$h'(x) = 8x - 3x^2 = 0 \iff x(8-3x) = 0$

so $x=0$ or $x = \frac{8}{3}$

Only $x = \frac{8}{3} \in [0,4]^{\circ} = (0,4)$

$h\left(\frac{8}{3}\right) = \left(\frac{64}{9}\right)\left(\frac{4}{3}\right) = \frac{256}{27}$ so $f|_{\partial_2}\left(\frac{8}{3}, \frac{4}{3}\right) = \frac{256}{27}e^{-4}$

$\sim .174$

Conclusion: Absolute maximum of $f$ on $D$ is $\frac{4}{e^3}$

@ the point $(2,1)$

Absolute minimum of $f$ on $D$ is $0$

@ any point on the boundaries $\partial_1$ or $\partial_3$.

5. (a) **[5 points]** Find the volume of the region above $R = [1,3] \times [0,2]$ and under the graph of the function $f(x,y) = 3x^3 + 3x^2y$.

$$\text{Volume} = \iint_R f(x,y)\,dA = \int_1^3 \int_0^2 (3x^3 + 3x^2y)\,dy\,dx$$

$$= \int_1^3 \left[3x^3y + \frac{3x^2y^2}{2}\right]_{y=0}^{y=2} dx$$

$$= \int_1^3 (6x^3 + 6x^2)\,dx = \left[\frac{3x^4}{2} + 2x^3\right]_{x=1}^{x=3}$$

$$= \left(\frac{243}{2} + 54\right) - \left(\frac{3}{2} + 2\right) = 120 + 52$$

$$= 172$$

(b) **[5 points]** Find $\iint_R xe^{xy}dA$ where $R = [0,2] \times [0,1]$.

$$\iint_R xe^{xy}\,dA = \int_0^2 \int_0^1 xe^{xy}\,dy\,dx = \int_0^2 \left[e^{xy}\right]_{y=0}^{y=1} dx$$

$$= \int_0^2 (e^x - 1)\,dx = \left[e^x - x\right]_{x=0}^{x=2}$$

$$= (e^2 - 2) - (1 - 0) = e^2 - 3$$

6. **[13 points]** Use the Lagrange multiplier technique to find the absolute extrema of $f(x, y) = x^2y$ subject to the constraint $x^2 + 2y^2 = 6$.

Let $g(x,y) = x^2+2y^2$ Lagrange eq$^{\underline{ns}}$ to be solved are

$\nabla f = \lambda \nabla g$ and $g = 6$

We get ① $2xy = \lambda 2x \Rightarrow xy - \lambda x = 0$

② $x^2 = \lambda 4y \qquad x(y-\lambda) = 0$

③ $x^2 + 2y^2 = 6 \qquad \therefore x = 0$ or $y = \lambda$

Case 1 $x = 0$ Sub-in ③ to get $y = \pm\sqrt{3}$

(Sub-in ② to also get $\lambda = 0$ which is irrelevant here)

Constrained critical points (CCP) are $(0, \pm\sqrt{3})$

Case 2 $y = \lambda$ Sub-in ② to get $x^2 = 4y^2$ Sub this in ③ to get $4y^2 + 2y^2 = 6$ so $y = \pm 1$ We have $x = \pm 2$ so CCP are $(\pm 2, \pm 1)$ (all four are valid)

Evaluations: $f(\pm 2, 1) = 4 \quad f(\pm 2, -1) = -4$

Conclusion is $f$ has absolute maximum of 4 at the points $(\pm 2, 1)$ and absolute minimum of $-4$ at the points $(\pm 2, -1)$.

(Since $f$ is continuous on the compact set $g(x,y) = 6$, EVT assures that $f$ has global extrema on this set. The Lagrange technique locates these extrema points)

7. **[12 points]** Let $S$ be the surface in $\mathbb{R}^3$ defined by the equation $x^2 + y^2 + 4z^2 = 16$. Find all points $(a, b, c)$ on $S$ for which the tangent plane to $S$ at $(a, b, c)$ is parallel to the plane $\underbrace{x + y + 2\sqrt{2}\, z = 97}$.

Call this plane $\Pi_2$

Let $\Pi$ denote the tangent plane to $S$ @ $(a,b,c)$

$\because$ $\Pi \parallel \Pi_2$ their normals are parallel, hence multiples of each other

$\therefore$ $(2a, 2b, 8c) = k(1, 1, 2\sqrt{2})$ for some $k \in \mathbb{R}$

This gives
$$2a = k \implies a = \frac{k}{2}$$
$$2b = k \implies b = \frac{k}{2}$$
$$8c = 2\sqrt{2} \implies c = \frac{\sqrt{2}}{4} k$$

$\because$ $(a,b,c) \in S$ we have
$$16 = a^2 + b^2 + 4c^2$$
$$= \left(\frac{k}{2}\right)^2 + \left(\frac{k}{2}\right)^2 + 4\left(\frac{\sqrt{2}\,k}{4}\right)^2$$
$$= k^2$$

$\therefore$ $k = \pm 4$ so the required points are $(2, 2, \sqrt{2})$ and $(-2, -2, -\sqrt{2})$.

8. **[10 points]** Let $\boldsymbol{a} = (2,1)$ and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $C^2$–function such that

$$f_x(\boldsymbol{a}) = 3 \qquad f_y(\boldsymbol{a}) = -2 \qquad f_{xx}(\boldsymbol{a}) = 0$$
$$f_{xy}(\boldsymbol{a}) = f_{yx}(\boldsymbol{a}) = 1 \qquad f_{yy}(\boldsymbol{a}) = 2$$

Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $g(u,v) = (u+v, uv)$. If $w = f \circ g$, find $\dfrac{\partial^2 w}{\partial v\, \partial u}$ at the point $(1,1)$.

We use the chain rule repeatedly

$x = u+v$
$y = uv$

When $u=v=1$, $x=2, y=1$

$$\frac{\partial^2 w}{\partial v\, \partial u} = \frac{\partial}{\partial v}\left(\frac{\partial w}{\partial u}\right) = \frac{\partial}{\partial v}\left(f_x(x,y)\frac{\partial x}{\partial u} + f_y(x,y)\frac{\partial y}{\partial u}\right)$$

$$= \frac{\partial}{\partial v}\left(f_x(x,y) + f_y(x,y)v\right)$$

$$= \left(f_{xx}(x,y)\frac{\partial x}{\partial v} + f_{xy}(x,y)\frac{\partial y}{\partial v}\right) + \left(f_{yx}(x,y)\frac{\partial x}{\partial v} + f_{yy}(x,y)\frac{\partial y}{\partial v}\right)v + f_y(x,y)$$

$$= f_{xx}(x,y) + f_{xy}(x,y)u + \left(f_{yx}(x,y) + f_{yy}(x,y)u\right)v + f_y(x,y)$$

$$\therefore \left.\frac{\partial^2 w}{\partial v\, \partial u}\right|_{\substack{u=1\\v=1}} = f_{xx}(\underline{a}) + f_{xy}(\underline{a}) + \left(f_{yx}(\underline{a}) + f_{yy}(\underline{a})\right) + f_y(\underline{a})$$

$$= 0 + 1 + (1+2) - 2 = 2$$