University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #1

1. (a) If a finite limit exists, $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim_{x\to a} \dfrac{2x^2+x-3-(2a^2+a-3)}{x-a} = \lim_{x\to a} \dfrac{2(x^2-a^2)+(x-a)}{x-a} = \lim_{x\to a} \big(2(x-a)+1\big) = 2(a+a)+1 = 4a+1.$

   (b) Choose a regular partition of $[0,2]$, $x_0 = 0$, $x_i = \dfrac{2i}{n}$ and $\Delta x = \dfrac{2}{n}$. Now choose $w_i = x_i$, so $f(w_i) = f(x_i) = 2x_i^2 + x_i - 3 = 2\left(\dfrac{2i}{n}\right)^2 + \left(\dfrac{2x_i}{n}\right) - 3$.
$\displaystyle\sum_{i=1}^{n} f(w_i)\,\Delta x = \sum_{i=1}^{n}\left[2\left(\frac{2i}{n}\right)^2 + \left(\frac{2i}{n}\right) - 3\right]\frac{2}{n} = \frac{16}{n^3}\sum_{i=1}^{n} i^2 + \frac{4}{n^2}\sum_{i=1}^{n} i - \frac{6}{n}\sum_{i=1}^{n} 1 = \frac{16}{n^3}\left(\frac{n\,(n+1)\,(2n+1)}{6}\right) + \frac{4}{n^2}\left(\frac{n\,(n+1)}{2}\right) - 6 = \frac{8}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) + 2\left(1+\frac{1}{n}\right) - 6.$
Now $\displaystyle\int_0^2 f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(w_i)\,\Delta x = \lim_{n\to\infty}\left[\frac{8}{3}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) + 2\left(1+\frac{1}{n}\right) - 6\right] = \frac{16}{3} + 2 - 6 = \frac{4}{3}.$

   (c) $\displaystyle F(x) = \int_{\cos x}^{1-x^3} e^{t^2}\,dt = \int_{\cos x}^{a} e^{t^2}\,dt + \int_{a}^{1-x^3} e^{t^2}\,dt = \int_{a}^{1-x^3} e^{t^2}\,dt - \int_{a}^{\cos x} e^{t^2}\,dt.$
Hence $\dfrac{dF}{dx} = \left(e^{(1-x^3)^2}\right)\left(-3x^2\right) - \left(e^{\cos^2 x}\right)\left(-\sin x\right).$

2. (a) (i) $\displaystyle\int \frac{x^6+x^3}{1+x^2}\,dx \overset{divide}{=} \int\left(x^4 - x^2 + x + 1 + \frac{x-1}{1+x^2}\right)dx = \int\left(x^4 - x^2 + x + 1 + \frac{x}{1+x^2} - \frac{1}{1+x^2}\right)dx = \frac{x^5}{5} - \frac{x^3}{3} + \frac{x^2}{2} + x + \frac{1}{2}\ln(1+x^2) - \tan^{-1}x + C.$

   (ii) $\displaystyle\int \frac{(\ln w)^3}{w}\,dw \underset{z\,=\,\ln w}{\overset{substitute}{=}} \int z^3\,dz = \frac{1}{4}z^4 + C = \frac{1}{4}\Big(\ln w\Big)^4 + C.$

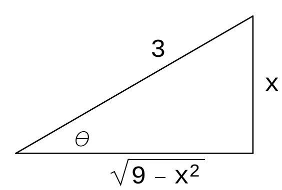
   (iii) $\displaystyle\int \sin^4 x\cos^3 x\,dx = \int \sin^4 x\left(1-\sin^2 x\right)\cos x\,dx = \int\left(\sin^4 x - \sin^6 x\right)\cos x\,dx \underset{u\,=\,\sin x}{\overset{substitute}{=}} \int\left(u^4 - u^6\right)du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$

   (iv) $\displaystyle\int z^2\cos z\,dz \underset{u\,=\,z^2,\,dv\,=\,\cos z\,dz}{\overset{parts\_with}{=}} z^2\,\sin z - 2\int z\,\sin z\,dz \underset{u\,=\,x,\,dv\,=\,\sin z\,dz}{\overset{parts\_with}{=}} z^2\,\sin z - 2\left(-z\,\cos z + \int \cos z\,dz\right) = z^2\,\sin z + 2\,z\,\cos z - 2\,\sin z + C.$

(v) $\displaystyle\int \sin(\ln x)\,dx \underset{u=\sin(\ln x)\,,\,dv=dx}{\overset{parts\ with}{=}} x\sin(\ln x) - \int \cos(\ln x)\,dx \underset{u=\cos(\ln x)\,,\,dv=dx}{\overset{parts\ with}{=}}$ $x\sin(\ln x) - x\cos(\ln x) - \displaystyle\int \sin(\ln x)\,dx$. Hence $2\displaystyle\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) + C'$, so $\displaystyle\int \sin(\ln x)\,dx = \frac{x}{2}\Big(\sin(\ln x) - \cos(\ln x)\Big) + C$.

(vi) We need to find A and B so that $\dfrac{1}{(x+1)(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{x-2}$; i.e., we need $1 = A(x-2) + B(x+1)$. Solving we get $A = -\dfrac{1}{3}$ and $B = \dfrac{1}{3}$. Hence $\displaystyle\int \frac{dx}{(x+1)(x-2)} = -\frac{1}{3}\int \frac{dx}{x+1} + \frac{1}{3}\int \frac{dx}{x-2} = -\frac{1}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + C = \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C$.

(vii) $\displaystyle\int x^2\sqrt{9-x^2}\,dx \underset{x=3\sin\theta}{\overset{substitute}{=}}$
$\displaystyle\int 9\sin^2\theta\sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta =$
$81\displaystyle\int \sin^2\theta\cos^2\theta\,d\theta =$
$81\displaystyle\int \left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{81}{4}\int \big(1 - \cos^2(2\theta)\big)\,d\theta = \frac{81}{4}\int\left(1 - \frac{1+\cos 4\theta}{2}\right)d\theta =$
$\dfrac{81}{8}\displaystyle\int \big(1 - \cos 4\theta\big)\,d\theta = \frac{81\,\theta}{8} - \frac{81}{32}\sin 4\theta + C = \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right) - \frac{81}{32}\big(2\sin 2\theta\,\cos 2\theta\big) + C = \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right) - \frac{81}{16}\big(2\sin\theta\,\cos\theta\big)\big(1 - 2\sin^2\theta\big) + C = \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right) - \frac{81}{8}\left(\frac{x\sqrt{9-x^2}}{9}\right)\left[1 - 2\left(\frac{x}{3}\right)^2\right] + C = \frac{81}{8}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\big(9-2x^2\big)\sqrt{9-x^2}}{8} + C$.



(b) (i) $\displaystyle\int_4^9 \frac{e^{\sqrt{y}}}{\sqrt{y}}\,dy \underset{w=\sqrt{y}}{\overset{substitute}{=}} 2\int_2^3 e^w\,dw = 2\Big[e^w\Big]_2^3 = 2\big(e^3 - e^2\big)$.

(ii) There is a singularity when $x = 4$, so the integral is improper. Now $\displaystyle\int_0^4 \frac{dx}{(x-4)^{\frac{2}{3}}} = \lim_{t\to 4^-} 3\big(x-4\big)^{\frac{1}{3}}\Big|_0^t = 3\lim_{t\to 4^-}\Big[\big(t-4\big)^{\frac{1}{3}} - \big(-4\big)^{\frac{1}{3}}\Big] = 3\,\big(4^{\frac{1}{3}}\big)$ and $\displaystyle\int_4^6 \frac{dx}{(x-4)^{\frac{2}{3}}} = \lim_{t\to 4^+} 3\big(x-4\big)^{\frac{1}{3}}\Big|_t^6 = 3\lim_{t\to 4^+}\Big[2^{\frac{1}{3}} - \big(t-4\big)^{\frac{1}{3}}\Big] = 3\big(2^{\frac{1}{3}}\big)$. Since both integrals converge, $\displaystyle\int_0^6 \frac{dx}{(x-4)^{\frac{2}{3}}}$ will also converge and $\displaystyle\int_0^6 \frac{dx}{(x-4)^{\frac{2}{3}}} =$

$$\int_0^4 \frac{dx}{(x-4)^{\frac{2}{3}}} + \int_4^6 \frac{dx}{(x-4)^{\frac{2}{3}}} = 3\left(4^{\frac{1}{3}}\right) + 3\left(2^{\frac{1}{3}}\right).$$

(iii) Since one of the limits of integration is infinite, the integral is improper. Now $\displaystyle\int_0^\infty \frac{x}{e^x}\,dx \underset{u=x\,,\,dv=e^{-x}\,dx}{\overset{parts\ with}{=}} \lim_{t\to\infty}\left[-x\,e^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx\right] =$ $\displaystyle\lim_{t\to\infty}\Big[(-1)\,(1+x)\,e^{-x}\Big]_0^t = 1 - \lim_{t\to\infty}(1+t)\,e^{-t} = 1 - 0 = 1.$ The integral converges to 1.

3. We are given that $\dfrac{dv}{dt} = -20$, so integrating gives $v = v_0 - 20\,t$ where v_0 is the speed when the brakes were first applied. Integrating again for distance we have $s = s_0 + v_0\,t - 10\,t^2$. Since the skidding started when the brakes were applied, $s_0 = 0$. When $v = 0$ (car stopped) we have $t = \dfrac{v_0}{20}$ and $s = 200$, so $200 = \dfrac{v_0^2}{20} - 10\,\dfrac{v_0^2}{400}$. Hence $v_0 = \sqrt{(200)\,(40)} = 40\,\sqrt{5} \approx 89.4$ m/sec.

4. For $\boldsymbol{v} = (1,-1,1)$ and $\boldsymbol{w} = (0,1,-2)$ we have $\boldsymbol{v}\cdot\boldsymbol{w} = 0-1-2 = -3$, $\|\boldsymbol{v}\| = \sqrt{1+1+1} = \sqrt{3}$ and $\|\boldsymbol{w}\| = \sqrt{0+1+4} = \sqrt{5}$.

(a) If θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$, we have $\cos\theta = \dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\,\|\boldsymbol{w}\|} = \dfrac{-3}{\sqrt{3}\,\sqrt{5}} = -\dfrac{\sqrt{3}}{\sqrt{5}}$, so $\theta = \cos^{-1}\left(-\dfrac{\sqrt{3}}{\sqrt{5}}\right)$.

(b) Now $\|\boldsymbol{v}\|\,\|\boldsymbol{w}\| = \sqrt{3}\,\sqrt{5} > \sqrt{3}\,\sqrt{3} = 3 = |\,\boldsymbol{v}\cdot\boldsymbol{w}\,|$. Hence the Cauchy-Schwarz inequality holds for $\boldsymbol{v}$ and $\boldsymbol{w}$. Also $\boldsymbol{v}+\boldsymbol{w} = (1,0,-1)$ and $\|\boldsymbol{v}+\boldsymbol{w}\| = \sqrt{1+0+1} = \sqrt{2} < \sqrt{3} < \sqrt{3}+\sqrt{5} = \|\boldsymbol{v}\| + \|\boldsymbol{w}\|$. Hence the triangle inequality holds for $\boldsymbol{v}$ and $\boldsymbol{w}$.

(c) Let $\boldsymbol{u} = (a,b,c)$ be orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$; hence $(a,b,c)\cdot(1,-1,1) = a-b+c = 0$ and $(a,b,c)\cdot(0,1,-2) = b-2c = 0$. Solving these two equations we get $b = 2c$ and $a = c$. Hence all vectors in $\mathbb{R}^3$ which are orthogonal to both $\boldsymbol{v}$ and $\boldsymbol{w}$ are of the form $c\,(1,2,1)$, $c \in \mathbb{R}$. We also need $\|\boldsymbol{u}\| = 1$ or $c^2\,(1+4+1) = 1 \implies c^2 = \dfrac{1}{6}$ or $c = \pm\dfrac{1}{\sqrt{6}}$. The required vectors are $\left(\dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$ and $\left(\dfrac{-1}{\sqrt{6}}, -\dfrac{2}{\sqrt{6}}, -\dfrac{1}{\sqrt{6}}\right)$.

(d) (i) The projection of $\boldsymbol{v}$ onto $\boldsymbol{w}$ is a vector in direction $\dfrac{1}{\|\boldsymbol{w}\|}\boldsymbol{w}$ with length $\|\boldsymbol{v}\|\,\cos\theta = \left(\dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{w}\|}\right)$. Hence the projection is $\left(\dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{w}\|}\right)\left(\dfrac{1}{\|\boldsymbol{w}\|}\right)\boldsymbol{w} = \dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{w}\|^2}\,\boldsymbol{w}$ $= \dfrac{-3}{5}\Big(0,1,-2\Big) = \left(0, -\dfrac{3}{5}, \dfrac{6}{5}\right)$.

(ii) The projection of $\boldsymbol{w}$ onto $\boldsymbol{v}$ is $\dfrac{\boldsymbol{w}\cdot\boldsymbol{v}}{\|\boldsymbol{v}\|^2}\,\boldsymbol{v} = \dfrac{-3}{3}\,\big(1,-1,1\big) = \big(1,-1,1\big)$.

5. (a) $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{w}\cdot\boldsymbol{v}$ means $\boldsymbol{u}\cdot\boldsymbol{v} - \boldsymbol{w}\cdot\boldsymbol{v} = 0$ or $\big(\boldsymbol{u} - \boldsymbol{w}\big)\cdot\boldsymbol{v} = 0$ for all v. To be true for all $\boldsymbol{v}$, it must be true for $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}$ which gives $\big(\boldsymbol{u} - \boldsymbol{w}\big)\cdot\big(\boldsymbol{u} - \boldsymbol{w}\big) = \big\|\boldsymbol{u} - \boldsymbol{w}\big\|^2 = 0 \implies \boldsymbol{u} - \boldsymbol{w} = \boldsymbol{0} \implies \boldsymbol{u} = \boldsymbol{w}$.

   (b) This time we have $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{w}\cdot\boldsymbol{v}$ or $\big(\boldsymbol{u} - \boldsymbol{w}\big)\cdot\boldsymbol{v} = 0$, but only for some nonzero $\boldsymbol{v}$. Recall $\boldsymbol{x}\cdot\boldsymbol{y} = 0 \implies \boldsymbol{x} \perp \boldsymbol{y}$, so here we have $\big(\boldsymbol{u} - \boldsymbol{w}\big)\cdot\boldsymbol{v}$. Choosing $\boldsymbol{u} = (0,3)$ and $\boldsymbol{w} = (0,2)$ we note that, with $\boldsymbol{v} = (1,0) \neq \boldsymbol{0}$, we have $\big(\boldsymbol{u} - \boldsymbol{w}\big)\cdot\boldsymbol{v} = \big((0,3) - (0,2)\big)\cdot(1,0) = (0,1)\cdot(1,0) = 0$, but $\boldsymbol{u} \neq \boldsymbol{w}$.

6. (a) To show that $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ form an orthonormal basis for $\mathbb{R}^3$ we must show that $\boldsymbol{b}_i\cdot\boldsymbol{b}_j = \delta_{ij}$, for $1 \leq i,j \leq 3$. Now

$$\boldsymbol{b}_1\cdot\boldsymbol{b}_1 = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)\cdot\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right) = \frac{1}{5} + \frac{4}{5} = 1,$$

$$\boldsymbol{b}_1\cdot\boldsymbol{b}_2 = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)\cdot\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{2}{\sqrt{30}} + 0 + \frac{2}{\sqrt{30}} = 0,$$

$$\boldsymbol{b}_1\cdot\boldsymbol{b}_3 = \left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)\cdot\left(-\frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right) = -\frac{2}{\sqrt{150}} + \frac{2}{\sqrt{150}} = 0,$$

$$\boldsymbol{b}_2\cdot\boldsymbol{b}_2 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\cdot\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \frac{4}{6} + \frac{1}{6} + \frac{1}{6} = 1,$$

$$\boldsymbol{b}_2\cdot\boldsymbol{b}_3 = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\cdot\left(-\frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right) = \frac{4}{\sqrt{180}} - \frac{5}{\sqrt{180}} + \frac{1}{\sqrt{180}} = 0,$$

and $\boldsymbol{b}_3\cdot\boldsymbol{b}_3 = \left(-\frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right)\cdot\left(-\frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right) = \frac{4}{30} + \frac{25}{30} + \frac{1}{30} = 1.$

Hence $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, $\boldsymbol{b}_3$ form an orthonormal basis for $\mathbb{R}^3$.

   (b) We want $\boldsymbol{v} = (1,0,1) = a_1\boldsymbol{b}_1 + a_2\boldsymbol{b}_2 + a_3\boldsymbol{b}_3$. The a_i are the projections of $\boldsymbol{v}$ onto the $\boldsymbol{b}_i$, $i = 1,2,3$. Hence

$$a_1 = \boldsymbol{v}\cdot\boldsymbol{b}_1 = (1,0,1)\cdot\left(\frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}} = \frac{3}{\sqrt{5}},$$

$$a_2 = \boldsymbol{v}\cdot\boldsymbol{b}_2 = (1,0,1)\cdot\left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} = -\frac{1}{\sqrt{6}}\ ,$$

and $a_3 = \boldsymbol{v}\cdot\boldsymbol{b}_3 = (1,0,1)\cdot\left(-\frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}\right) = -\frac{2}{\sqrt{30}} + \frac{1}{\sqrt{30}} = -\frac{1}{\sqrt{30}}.$

Hence $\boldsymbol{v} = \left(\frac{3}{\sqrt{5}}\right)\boldsymbol{b}_1 + \left(-\frac{1}{\sqrt{6}}\right)\boldsymbol{b}_2 + \left(-\frac{1}{\sqrt{30}}\right)\boldsymbol{b}_3.$

7. We have $A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ \frac{7}{3} & -\frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$, and $C = \begin{pmatrix} -1 & 3 & 2 \\ 4 & -1 & 1 \\ 3 & 2 & 3 \end{pmatrix}$.

   (a) (i) $\det A = (1)\det\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix} + 0 + (3)\det\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = (1)(-9) + (3)(1) = -6.$

   (ii) $\det B = (\frac{3}{2})\det\begin{pmatrix} -\frac{5}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix} - (-1)\det\begin{pmatrix} \frac{7}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{6} \end{pmatrix} +$

$(-\frac{1}{2})\det\begin{pmatrix} \frac{7}{3} & -\frac{5}{3} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix} = (\frac{3}{2})(-\frac{1}{6}) + (1)(\frac{1}{3}) + (-\frac{1}{2})(\frac{1}{2}) = -\frac{1}{6}.$

(iii) $\det C = (-1)\det\begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix} - (3)\det\begin{pmatrix} 4 & 1 \\ 3 & 3 \end{pmatrix} + (2)\det\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix} =$
$(-1)(-5) - (3)(9) + (2)(11) = 5 - 27 + 22 = 0.$

(iv) $\det(A\,B) = \det A\,\det B = (-6)(-\frac{1}{6}) = 1.$

(v) $\det(A+B) = \det\left[\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ \frac{7}{3} & -\frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}\right] =$

$\det\begin{pmatrix} \frac{5}{2} & -1 & \frac{5}{2} \\ \frac{13}{3} & -\frac{8}{3} & \frac{11}{3} \\ -\frac{19}{6} & \frac{7}{3} & \frac{7}{6} \end{pmatrix} = (\frac{5}{2})\det\begin{pmatrix} -\frac{8}{3} & \frac{11}{3} \\ \frac{7}{3} & \frac{7}{6} \end{pmatrix} - (-1)\det\begin{pmatrix} \frac{13}{3} & \frac{11}{3} \\ -\frac{19}{6} & \frac{7}{6} \end{pmatrix} +$

$(\frac{5}{2})\det\begin{pmatrix} \frac{13}{3} & -\frac{8}{3} \\ -\frac{19}{6} & \frac{7}{3} \end{pmatrix} = (\frac{5}{2})(-\frac{35}{3}) + (1)(\frac{50}{3}) + (\frac{5}{2})(\frac{5}{3}) = -\frac{25}{3}.$

(b) To show that A and B are inverse matrices we must show that $A\,B = I$ and $B\,A = I$.

$A\,B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix}\begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ \frac{7}{3} & -\frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix} =$

$\begin{pmatrix} \frac{3}{2} - \frac{1}{2} & -1 + 1 & -\frac{1}{2} + \frac{1}{2} \\ 3 - \frac{7}{3} - \frac{2}{3} & -2 + \frac{5}{3} + \frac{4}{3} & -1 + \frac{1}{3} + \frac{2}{3} \\ -\frac{9}{2} + \frac{14}{3} - \frac{1}{6} & 3 - \frac{10}{3} + \frac{1}{3} & \frac{3}{2} - \frac{2}{3} + \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$

$b\,A = \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ \frac{7}{3} & -\frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 1 & 0 & 3 \\ 2 & -1 & 4 \\ -3 & 2 & 1 \end{pmatrix} =$

$\begin{pmatrix} \frac{3}{2} - 2 + \frac{3}{2} & 1 - 1 & \frac{9}{2} - 4 - \frac{1}{2} \\ \frac{7}{3} - \frac{10}{3} + 1 & \frac{5}{3} - \frac{2}{3} & 7 - \frac{20}{3} - \frac{1}{3} \\ -\frac{1}{6} + \frac{2}{3} - \frac{1}{2} & -\frac{1}{3} + \frac{1}{3} & -\frac{1}{2} + \frac{4}{3} + \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$

(i) The system of equations $\begin{aligned} x \quad\;\; + 3z &= 1 \\ 2x - \;\, y + 4z &= 2 \\ -3x + 2y + \;\, z &= 3 \end{aligned}$ can be written as

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ so the solution can be given by $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} =$

$B\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 & -\frac{1}{2} \\ \frac{7}{3} & -\frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}.$

(ii) We now need to solve $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = B \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(c) If $\boldsymbol{v} = \begin{pmatrix} -5 \\ -9 \\ 11 \end{pmatrix}$, we have $C\boldsymbol{v} = \begin{pmatrix} -1 & 3 & 2 \\ 4 & -1 & 1 \\ 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} -5 \\ -9 \\ 11 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, so there are nonzero vectors such that $C\boldsymbol{v} = \mathbf{0}$.

The argument used in (b)(ii) requires the matrix to have an inverse (i.e. $\det C \neq 0$). Here we know that $\det C = 0$ and, consequently, that C is not invertible.