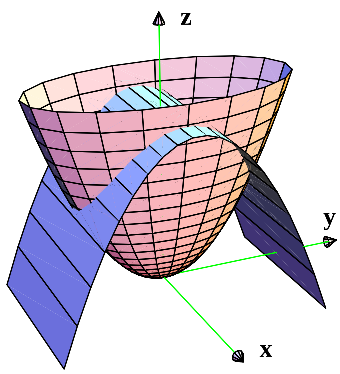
University of Toronto Scarborough
Department of Computer & Mathematical Sciences

MAT B41H **2013/2014**

Solutions #10

1. Fixing x and y, we have $4x^2 + y^2 \le z \le 2 - y^2$ (see the picture). The projection into the xy–plane is given by $\{\,(x, y) \mid 4x^2+y^2 \le 2-y^2\} = \{\,(x, y) \mid 4x^2 + 2y^2 \le 2\}$, an ellipitical disk. This question is easier if we use a change of variables. Put $x = \dfrac{r\cos\theta}{\sqrt{2}}$, $y = r\sin\theta$ and $z = z$. Now $Dg = \begin{pmatrix} \dfrac{\cos\theta}{\sqrt{2}} & \dfrac{-r\sin\theta}{\sqrt{2}} & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $|Jg| = |\det Dg| = \left|\dfrac{r}{\sqrt{2}}\cos^2\theta + \dfrac{r}{\sqrt{2}}\sin^2\theta\right| = \dfrac{r}{\sqrt{2}}$. The volume is $\iiint_B dV = \int_0^1\int_0^{2\pi}\int_{r^2(2\cos^2\theta+\sin^2\theta)}^{2-r^2\sin^2\theta} 1\,|Jg|\,dz\,d\theta\,dr =$



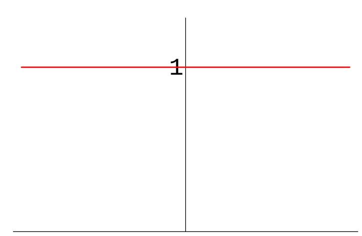
$$\frac{1}{\sqrt{2}}\int_0^1\int_0^{2\pi}\left(z\Big|_{r^2(2\cos^2\theta+\sin^2\theta)}^{2-r^2\sin^2\theta}\right) r\,d\theta\,dr = \frac{1}{\sqrt{2}}\int_0^1\int_0^{2\pi}(2-2r^2)\,r\,d\theta\,dr = \frac{2\pi}{\sqrt{2}}\left[r^2 - \frac{1}{2}r^4\right]_0^1$$
$$= \frac{2\pi}{\sqrt{2}}\left[1 - \frac{1}{2}\right] = \frac{\pi}{\sqrt{2}}.$$

(The volume can also be found directly with $V = \int_{-1}^1 \int_{-\sqrt{\frac{1-y^2}{2}}}^{\sqrt{\frac{1-y^2}{2}}} \int_{4x^2+y^2}^{2-y^2} 1\,dz\,dx\,dy$.)

2. (a) $r = \csc\theta$

$$\begin{aligned} &\Longrightarrow\quad r = \frac{1}{\sin\theta} \\ &\Longrightarrow\quad r\sin\theta = 1 \\ &\Longrightarrow\quad y = 1 \end{aligned}$$

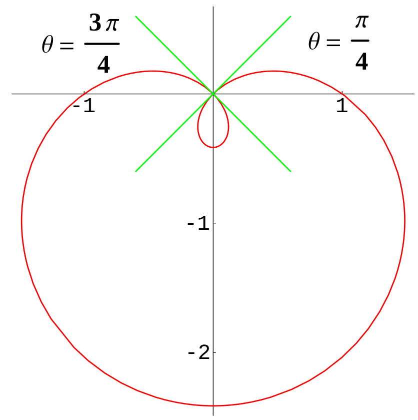
This is the line $y = 1$.



(b) $r = 1 - \sqrt{2}\sin\theta$

Now $r = 0 \implies \sqrt{2}\sin\theta = 1 \implies \sin\theta = \dfrac{1}{\sqrt{2}} \implies \theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$. Hence the graph will be tangent to $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{3\pi}{4}$ at the origin. Checking values we have

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$3\pi/2$	2π
r	1	0	$1-\sqrt{2}$	0	1	$1+\sqrt{2}$	1

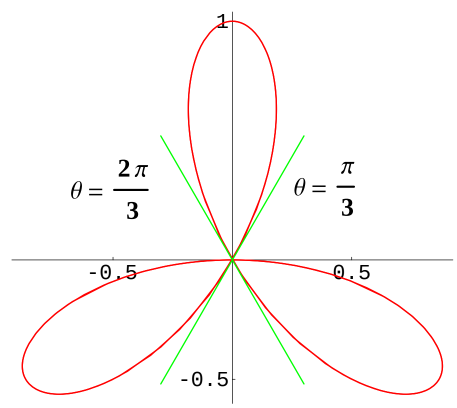


We start with $\theta = 0$. On $(0, \dfrac{\pi}{4})$, there is a positive loop where r decreases from 1 to 0. On $(\dfrac{\pi}{4}, \dfrac{3\pi}{4})$, there is a negative loop where r increases from 0 to $\sqrt{2}-1$ and then decreases to 0. On $(\dfrac{3\pi}{4}, 2\pi)$, there is a positive loop where r increases from 0 to $1+\sqrt{2}$ and then decreases to 1.

(c) $r = -\sin 3\theta$.

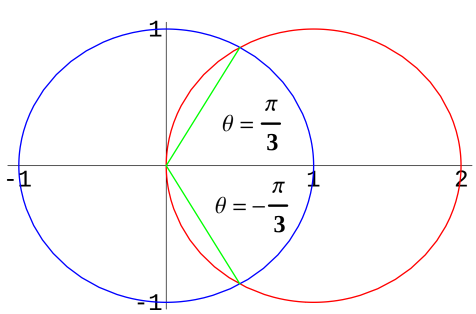
Now $r = 0 \implies -\sin 3\theta = 0 \implies 3\theta = k\pi, k \in \mathbb{Z} \implies \theta = 0\,, \dfrac{\pi}{3}\,, \dfrac{2\pi}{3}\,, \pi\,, \cdots$. Hence the graph will be tangent to $\theta = 0$, $\theta = \dfrac{\pi}{3}$ and $\theta = \dfrac{2\pi}{3}$ at the origin. Checking values we have

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π
r	0	-1	0	1	0	-1	0



(There is no need to continue as the loops of the graph will repeat). We start with $\theta = 0$. There are negative loops on $(0, \dfrac{\pi}{3})$ and on $(\dfrac{2\pi}{3}, \pi)$ and a positive loop on $(\dfrac{\pi}{3}, \dfrac{2\pi}{3})$. In each loop r increases from 0 to 1 and then decreases back to 0.

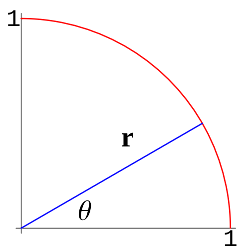
3. We know that $x^2 + y^2 = 1$ can be described in polars as $r = 1$. To describe $(x-1)^2 + y^2 = 1$ in polars, we put $x = r\cos\theta$ and $y = r\sin\theta$ giving $r^2\cos^2\theta - 2r\cos\theta + r^2\sin^2\theta = 0$ or $r^2 = 2\,r\,\cos\theta$. If $r \neq 0$, we have $r = 2\cos\theta$. The curves meet when $r = 1 = 2\cos\theta \implies \cos\theta = \dfrac{1}{2} \implies \theta = \dfrac{\pi}{3}\,, -\dfrac{\pi}{3}$. We describe the region by



$$\begin{array}{rcl} 1 & \leq\ r\ \leq & 2\cos\theta \\ -\dfrac{\pi}{3} & \leq\ \theta\ \leq & \dfrac{\pi}{3}\,. \end{array}$$

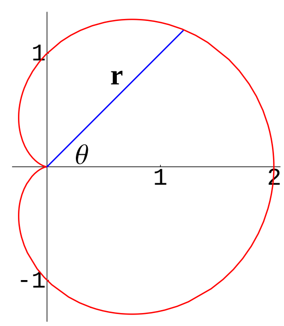
Now area is given by $\iint_D dA = \iint_D dx\,dy = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \int_1^{2\cos\theta} r\,dr\,d\theta =$ $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(2\cos^2\theta - \frac{1}{2}\right) d\theta = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(\frac{1}{2} + \cos 2\theta\right) d\theta = \left[\frac{\theta}{2} + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$.

4. To evaluate this integral we will first change variables to polars giving



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)^{\frac{5}{2}}\,dy\,dx \overset{\text{change to}}{\underset{\text{polars}}{=}} \int_0^{\frac{\pi}{2}} \int_0^1 (r^5)\,r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^6\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\frac{r^7}{7}\right]_0^1 d\theta = \frac{1}{7}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{7}\left(\frac{\pi}{2}\right) = \frac{\pi}{14}.$$

5. The area of the region bounded by the polar graph $r = 1 + \cos\theta$ is given by $\iint_D dA =$

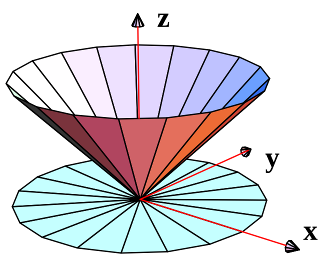


$$\int_0^{2\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\,d\theta =$$
$$\frac{1}{2}\int_0^{2\pi} \left(\frac{3}{2} + 2\cos\theta + \frac{1}{2}\cos 2\theta\right) d\theta =$$
$$\frac{1}{2}\left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{3\pi}{2}.$$

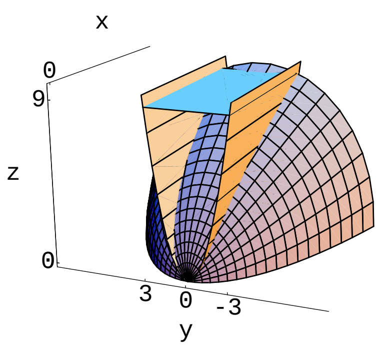
6. We will set this question up using the spherical polar coordinates: (ρ, θ, ϕ). The sphere $x^2 + y^2 + z^2 = a^2$ is $\rho = a$ and the sphere $x^2 + y^2 + z^2 = b^2$ is $\rho = b$. If we fix ρ, for $a \le \rho \le b$, R_ρ is the rectangle $\begin{cases} 0 \le \theta < 2\pi \\ 0 \le \phi \le \pi \end{cases}$. Now $\int_S \frac{dx\,dy\,dz}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} =$

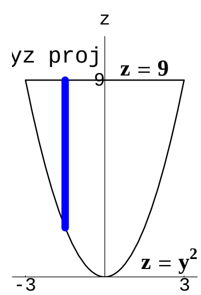
$$\int_a^b \int_0^{2\pi} \int_0^{\pi} \frac{\rho^2 \sin\phi}{\rho^3}\,d\phi\,d\theta\,d\rho = \int_a^b \frac{1}{\rho} \int_0^{2\pi} \int_0^{\pi} \sin\phi\,d\phi\,d\theta\,d\rho = \int_a^b \frac{1}{\rho} \int_0^{2\pi} \left(-\cos\phi\,\big|_0^{\pi}\right) d\theta\,d\rho = 2\int_a^b \frac{1}{\rho}\int_0^{2\pi} d\theta\,d\rho = 4\pi \int_a^b \frac{d\rho}{\rho} = 4\pi \ln\rho\,\big|_a^b = 4\pi \ln\left(\frac{b}{a}\right).$$

7. If we fix x and y, we have $0 \le z \le \sqrt{x^2+y^2}$ and the projection of the region B into the xy–plane is the unit disk, $x^2 + y^2 \le 1$. Hence $\int_B z\,dV = \iint_{unit\,disk} \int_0^{\sqrt{x^2+y^2}} z\,dz$. After switching to cylindrical polar coordinates, (r, θ, z), we have $\int_B z\,dV = \int_0^{2\pi}\int_0^1\int_0^r (z) r\,dz\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 \left(z^2\big|_0^r\right) r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = \frac{1}{8}\int_0^{2\pi}\left(r^4\big|_0^1\right) d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{1}{8}(2\pi) = \frac{\pi}{4}$.



8. A representation of the region is shown on the left with the face in the yz–plane removed. Fixing y and z, we have $0 \le x \le y^2 + z^2$. Then the projection into the yz–plane is the region bounded by the parabola $z = y^2$ and the line $z = 9$. Since $z \le 9$, $y^2 \le 9$, so $-3 \le y \le 3$. The volume is $V = \int_B 1\,dV = \int_{-3}^{3}\int_{y^2}^{9}\int_0^{y^2+z^2} 1\,dx\,dz\,dy = \int_{-3}^{3}\int_{y^2}^{9}(y^2 + z^2)\,dz\,dy = \int_{-3}^{3}\left[y^2 z + \frac{z^3}{3}\right]_{z=y^2}^{z=9} dy = \int_{-3}^{3}\left(9y^2 + 243 - y^4 - \frac{y^6}{3}\right) dy$

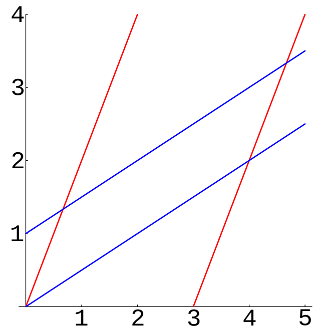




$$\overset{even\,function}{\underset{in\;\,y}{=}} 2\int_0^3\left(243 + 9y^2 - y^4 - \frac{y^6}{3}\right) dy = 2\left[243y + 3y^3 - \frac{y^5}{5} - \frac{y^7}{21}\right]_0^3 = \frac{46008}{35}.$$

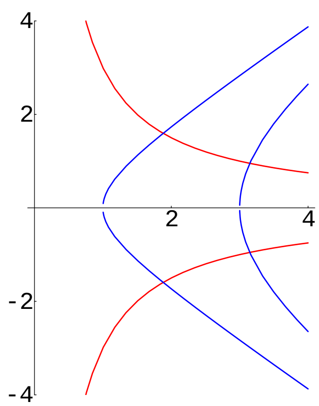
(You could also have fixed y and considered the cross section R_y. This would have given the same integral as we had above.)

9. The region D is the parallelogram bounded by $2x-y=0$, $2x-y=6$, $x-2y=0$ and $x-2y=-2$. We will use a change of variable. Put $u=2x-y$ and $v=x-2y$ so that



D^* is the rectangle $\left\{\begin{array}{ccccc} 0 & \le & u & \le & 6 \\ -2 & \le & v & \le & 0 \end{array}\right.$. Solving for x and y we get $x = g_1(u,v) = \dfrac{2}{3}u - \dfrac{1}{3}v$ and $y = g_2(u,v) = \dfrac{1}{3}u - \dfrac{2}{3}v$. Differenting we have $Dg = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$ so $Jg = \det(Dg) = \det\begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix} = -\dfrac{1}{3}$. Now $\displaystyle\iint_D e^{3x}\,dx\,dy = \iint_{D^*} e^{2u-v}\,|Jg|\,dv\,du = \frac{1}{3}\int_0^6\int_{-2}^0 e^{2u}\,e^{-v}\,dv\,du = -\frac{1}{3}\int_0^6 e^{2u}\left(e^{-v}\Big|_{-2}^0\right)du - \frac{1}{3}(1-e^2)\int_0^6 e^{2u}\,du = -\frac{1}{6}(1-e^2)\left(e^{2u}\Big|_0^6\right) = \frac{1}{6}(e^2-1)(e^{12}-1)$.

10. The region D is the region to the right of the y–axis bounded by $y=\dfrac{3}{x}$, $y=-\dfrac{3}{x}$, $x^2-y^2=1$ and $x^2-y^2=9$. We will use a change of variable. Put $u=xy$ and

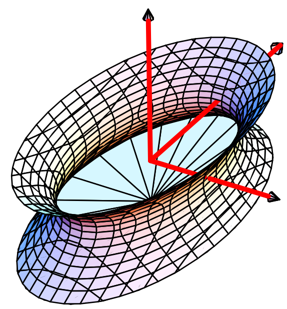


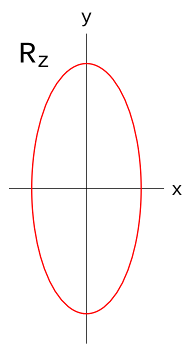
$v = x^2-y^2$ so that D^* is the rectangle $\left\{\begin{array}{ccccc} -3 & \le & u & \le & 3 \\ 1 & \le & v & \le & 9 \end{array}\right.$. Since we do not need to solve for x and y, it is easier here to compute $\dfrac{\partial(u,v)}{\partial(x,y)} = \det\begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} = -2\,(x^2+y^2)$ giving Jg, the Jacobian we want, as $Jg = \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{-1}{2(x^2+y^2)}$. Now $\displaystyle\int_D (x^2+y^2)\,\cos(xy)\,dx\,dy = \int_{D^*}(x^2+y^2)\,\cos(xy)\,|Jg|\,dv\,du = \frac{1}{2}\int_{-3}^3\int_1^9 \cos u\,dv\,du = 4\int_{-3}^3 \cos u\,du = 4\left(\sin u\,\Big|_{-3}^3\right) = 8\,\sin 3$.

11. The region B is bounded by $0 \le x+y+z \le 9$, $1 \le x+2y \le 4$ and $2 \le y-3z \le 6$. We will use a change of variable. Put $u=x+y+z$, $v=x+2y$ and $w=y-3z$ so that B^* is the rectangular box $\left\{\begin{array}{ccccc} 0 & \le & u & \le & 9 \\ 1 & \le & v & \le & 4 \\ 2 & \le & w & \le & 6 \end{array}\right.$. Since $u=x+y+z$ we do not need to solve for x, y and z. The substitutions are linear so it is sufficient to compute $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{pmatrix} = -2$. Hence $Jg = \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = -\dfrac{1}{2}$. Now $\displaystyle\int_B \sqrt{x+y+z}\,dV =$

$$\int_{B^*} \sqrt{u}\,|Jg|\,du\,dv\,dw = \frac{1}{2}\int_0^9\int_1^4\int_2^6 \sqrt{u}\,dw\,dv\,du = 6\int_0^9 \sqrt{u}\,du = 4\left(u^{3/2}\Big|_0^9\right) = 108.$$

12. A representation of the region is shown on the left with the ellipitical top face at $z = 1$ removed. If we fix z, $-1 \le z \le 1$, then R_z is the





ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{8} = z^2 + 1$. We will use a change of variable. Put $x = \sqrt{2}\,r\,\cos\theta$, $y = \sqrt{8}\,r\,\sin\theta$, and $z = z$. Now $Dg = \begin{pmatrix} \sqrt{2}\,\cos\theta & \sqrt{2}\,r\,\sin\theta & 0 \\ \sqrt{8}\,\sin\theta & \sqrt{8}\,r\,\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $Jg = \det Dg = 4r\cos^2\theta + 4r\sin^2\theta = 4r$. Now the volume is $V = \displaystyle\int_B 1\,dV = \int_{B^*} |Jg|\,dr\,d\theta\,dz = 4\int_{-1}^1\int_0^{2\pi}\int_0^{\sqrt{z^2+1}} r\,dr\,d\theta\,dz = 2\int_{-1}^1\int_0^{2\pi}(z^2+1)\,d\theta\,dz =$ $4\pi\displaystyle\int_{-1}^1 (z^2+1)\,dz = 4\pi\left[\frac{z^3}{3} + z\right]_{-1}^1 = (4\,\pi)\,(\frac{8}{3}) = \frac{32\,\pi}{3}$.

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This is the last solution set for MATB41H3 F for this year!

Good luck with your exams, enjoy the break and all the best for the New Year.
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