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MAT B41H **2009/2010**

Term Test Solutions

1. (a) For a function $f : \mathbb{R}^2 \to \mathbb{R}$, the equation of the tangent plane at (a, b) is $z = f(a, b) + \dfrac{\partial f}{\partial x}(a, b)\,(x - a) + \dfrac{\partial f}{\partial y}(a, b)\,(y - b)$. Here we have $f_x = 3x^2 - 6y$, $f_x(1, 2) = -9$; $f_y = -6x + 3y^2$, $f_y(1, 2) = 6$ and $f(1, 2) = -3$. Hence the equation of the tangent plane is $z = -3 + (-9)(x - 1) + (6)(y - 2) = -3 - 9x + 9 + 6y - 12 = -9x + 6y - 6$, which can be rewritten as $9x - 6y + z = -6$.

   (b) Since (0.99, 2.01) is "near" $(1, 2)$ we can use our work from part (a). Hence the linear approximation is given by $T_1(x, y) = -3 + (-9)(x - 1) + (6)(y - 2)$. Evaluating at (0.99, 2.01) we have $T_1(0.99,\ 2.01) = -3 + (-9)(0.99 - 1) + (6)(2.01 - 2) = -3 + (-9)(-0.01) + (6)(0.01) = -3 + 0.09 + 0.06 = -2.85$.

2. (a) (i) $\displaystyle\lim_{(x,y)\to(2,0)} \frac{(x-2)^2}{(x-2)^2 + y^2}$. If we restrict to the line $x = 2$, the limit reduces to $\displaystyle\lim_{y\to 0} \left(\frac{0}{y^2}\right) = 0$. On the other hand, if we restrict to the line $y = x - 2$, the limit reduces to $\displaystyle\lim_{x\to 2} \left(\frac{(x-2)^2}{(x-2)^2 + (x-2)^2}\right) = \lim_{x\to 2} \left(\frac{1}{2}\right) = \frac{1}{2}$. Hence this limit does not exist.

   (ii) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\|(x, y)\|} = \lim_{(x,y)\to(0,0)} \frac{\sin(x-y)}{\sqrt{x^2 + y^2}}$. If we restrict to the line $y = x$, the limit reduces to $\displaystyle\lim_{x\to 0} \frac{\sin 0}{\sqrt{2x^2}} = 0$. On the other hand, if we restrict to the line $y = -x$, the limit reduces to $\displaystyle\lim_{x\to 0^+} \frac{\sin(2x)}{\sqrt{2x^2}} = \lim_{x\to 0^+} \frac{\sqrt{2}\,\sin(2x)}{2\,x} = \sqrt{2}$. Hence this limit does not exist.

   (b) $f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & , \text{ if } (x, y) \neq (0, 0) \\ 0 & , \text{ if } (x, y) = (0, 0) \end{cases}$. For f to be continuous at $(0, 0)$, we need $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y) = 0 = f(0, 0)$. Evaluating the limit along the line $y = x$, we have $\displaystyle\lim_{x\to 0} \left(\frac{x^2}{2\,x^2}\right) = \frac{1}{2}$. Since $f(0, 0) = 0 \neq \frac{1}{2}$, we can conclude that f is not continuous at $(0, 0)$.

3. $f(x,y) = \dfrac{x^2}{x+y+1}$.

   Domain is $\{(x,y) \in \mathbb{R}^2 \mid y \neq -x-1\}$.

   Putting $f(x,y) = c$ we have $\dfrac{x^2}{x+y+1} = c \iff x^2 = c\,x + c\,y + c$.

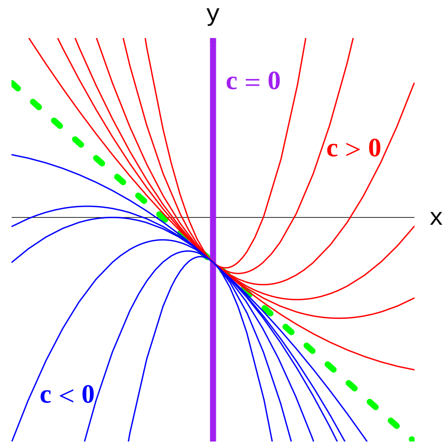
   For $c = 0$, we have $x^2 = 0$ or $x = 0$. The level curve is the y–axis (purple)

   For $c \neq 0$, we have $c\,y = x^2 - c\,x - c \iff y = \dfrac{1}{c}x^2 - x - 1 \overset{\textit{complete}}{\underset{\textit{the square}}{=}} \dfrac{1}{c}\left(x - \dfrac{c}{2}\right)^2 - \dfrac{4+c}{4}$.

   For $c > 0$, the level curves are parabolas opening upward (red).

   For $c < 0$, the level curves are parabolas opening downward (blue).

   All parabolas have y–intercept $(0,-1)$.

   (For $c > 0$ and $c \leq -4$, the parabolas have x–intercepts $\left(\dfrac{c \pm \sqrt{c^2+4c}}{2}, 0\right)$. For $c > -4$ and $c < 0$, the parabolas have vertices below the x–axis and, therefore, have no x–intercepts.)

4. (a) We put $\boldsymbol{p} = (0,1,-1)$. Two direction vectors for π are $(-2,-1,2) - (0,1,-1) = (-2,-2,3)$ and $(1,-1,-1) - (0,1,-1) = (1,-2,0)$. A normal vector for π is $\boldsymbol{n} = (-2,-2,3) \times (1,-2,0) = (6,3,6)$. An equation for π is of the form $6x+3y+6z = d$. Since $\boldsymbol{p} = (0,1,-1)$ is a point in π, $d = 6(0) + 3(1) + 6(-1) = -3$. An equation for π is $6x + 3y + 6z = -3$ or, equivalently, $2x + y + 2z = -1$.

   (b) We note that the tangent plane will be parallel to the plane $2x + y + 2z = -1$ when their normals are parallel. The normal to the hyperboloid $g(x,y,z) = x^2 - y^2 + 4z^2 = 4$ is given by the gradient $\nabla g = (2x, -2y, 8z)$. We need $(2x,-2y,8z) = \lambda\,(2,1,2)$, where $(2,1,2)$ is a normal vector to the plane $2x+y+2z = -1$. Equating components we have $x = \lambda$, $y = -\dfrac{1}{2}\lambda$ and $z = \dfrac{1}{4}\lambda$. To solve for λ we substitute into the equation of the hyperboloid, giving $x^2 - y^2 + 4z^2 = \lambda^2 - \dfrac{\lambda^2}{4} + \dfrac{\lambda^2}{4} = 4 \implies \lambda = \pm 2$. Hence the required points are $\left(2, -1, \dfrac{1}{2}\right)$ and $\left(-2, 1, -\dfrac{1}{2}\right)$.

5. (a) Since $\boldsymbol{p}$ is on the level surface, we have $c = g(1,1,0) = (1)(1) + (1)(0) + (0)(1) = 1$. The level surface is $g(x,y,z) = x\,y + y\,z + z\,x = 1$.

   (b) We have $\nabla g = (y+z,\ x+z,\ y+x)$, so a normal vector to the level surface at $\boldsymbol{p}$ is $\nabla g(\boldsymbol{p}) = \nabla g(1,1,0) = (1,1,2)$. The equation of the tangent plane is given by $0 = \nabla g(1,1,0) \cdot \big((x,y,z) - (1,1,0)\big) = (1)(x-1) + (1)(y-1) + (2)(z) = x + y + 2z - 2$, which we can rewrite as $x + y + 2z = 2$.

   (c) A direction vector for the normal line is $\nabla g(\boldsymbol{p}) = (1,1,2)$. Hence a parametric description of the normal line is $\boldsymbol{p} + t\,\nabla g(\boldsymbol{p}) = (1,1,0) + t\,(1,1,2)$, $t \in \mathbb{R}$.

6. (a) The direction of the maximum rate of increase is the direction of the gradient of f at $\boldsymbol{p}$. Now $\nabla f = (2xy^3z^2,\ 3x^2y^2z^2,\ 2x^2y^3z)$, so the direction of the maximum rate of increase is $\nabla f(\boldsymbol{p}) = \nabla f(2,1,-1) = (4,12,-8)$. $\big($ $(1,3,-2)$ would also be aceptable here. $\big)$

   The maximum rate is the magnitude of the gradient. Hence the maximum rate is $\|\nabla f(\boldsymbol{p})\| = \|\,(4,12,-8)\,\| = 4\,\|\,(1,3,-2)\,\| = 4\sqrt{1+9+4} = 4\sqrt{14}$.

   (b) We first note that the normal line for the plane $x+3y+2z=-2$ has direection vector $\boldsymbol{v} = (1,3,2)$. Now the directional derivative is $D_{\boldsymbol{v}} f(\boldsymbol{p}) = \nabla f(\boldsymbol{p}) \cdot \dfrac{\boldsymbol{v}}{\|\,\boldsymbol{v}\,\|} =$
   $(4,12,-8) \cdot \dfrac{(1,3,2)}{\|\,(1,3,2)\,\|} = \dfrac{4+36-16}{\sqrt{14}} = \dfrac{24}{\sqrt{14}}$.

7. From the lecture notes we have

   Extreme Value Theorem. Let D be a compact set in $\mathbb{R}^n$ and let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Then f assumes both a (global) maximum and a (global) minimum on D.

8. (a) From the lecture notes we have

   Chain Rule. Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g : V \subset \mathbb{R}^m \to \mathbb{R}^k$ be given functions such that $f[U] \subset V$ so that $g \circ f$ is defined. Let $\boldsymbol{a} \in \mathbb{R}^n$ and $\boldsymbol{b} = f(\boldsymbol{a}) \in \mathbb{R}^m$. If f is differentiable at $\boldsymbol{a}$ and g is differentiable at $\boldsymbol{b}$, then $g \circ f$ is differentiable at $\boldsymbol{a}$ and
   $$D(g \circ f)\,(\boldsymbol{a}) = [Dg(\boldsymbol{b})]\,[Df(\boldsymbol{a})]\ .$$

   (b) $f : \mathbb{R}^3 \to \mathbb{R}^4$ is given by $f(x,y,z) = (x^2y,\ y^2z^2,\ xyz^2,\ xy)$ so $Df = \begin{pmatrix} 2xy & x^2 & 0 \\ 0 & 2yz^2 & 2y^2z \\ yz^2 & xz^2 & 2xyz \\ y & x & 0 \end{pmatrix}$.

   $g : \mathbb{R}^4 \to \mathbb{R}^2$ is given by $g(x,y,z,w) = (y\,e^z,\ xzw)$ so $Dg = \begin{pmatrix} 0 & e^z & ye^z & 0 \\ zw & 0 & xw & xz \end{pmatrix}$

   and $D\,g(f(x,y,z)) = \begin{pmatrix} 0 & e^{xyz^2} & y^2z^2e^{xyz^2} & 0 \\ x^2y^2z^2 & 0 & x^3y^2 & x^3y^2z^2 \end{pmatrix}$.

   Now $D(g \circ f)(x,y,z) = [D\,g(f(x,y,z))]\,[D\,f(x,y,z)]$
   $$= \begin{pmatrix} 0 & e^{xyz^2} & y^2z^2e^{xyz^2} & 0 \\ x^2y^2z^2 & 0 & x^3y^2 & x^3y^2z^2 \end{pmatrix} \begin{pmatrix} 2xy & x^2 & 0 \\ 0 & 2yz^2 & 2y^2z \\ yz^2 & xz^2 & 2xyz \\ y & x & 0 \end{pmatrix}$$
   $$= \begin{pmatrix} y^3z^4e^{xyz^2} & 2yz^2e^{xyz^2} + xy^2z^4e^{xyz^2} & 2y^2ze^{xyz^2} + 2xy^3z^3e^{xyz^2} \\ 2x^3y^3z^2 + x^3y^3z^2 + x^3y^3z^2 & x^4y^2z^2 + x^4y^2z^2 + x^4y^2z^2 & 2x^4y^3z \end{pmatrix}$$
   $$= \begin{pmatrix} y^3z^4e^{xyz^2} & yz^2e^{xyz^2}(2+xyz^2) & 2y^2ze^{xyz^2}(1+xyz^2) \\ 4x^3y^3z^2 & 3x^4y^2z^2 & 2x^4y^3z \end{pmatrix}.$$

9. $$\frac{\partial^2 f}{\partial v\,\partial u} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right) \overset{Chain}{\underset{Rule}{=}} \frac{\partial}{\partial v}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}\right] = \frac{\partial}{\partial v}\left[\left(\frac{\partial f}{\partial x}\right)(2) + \left(\frac{\partial f}{\partial y}\right)(4)\right] =$$

$$2\,\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right) + 4\,\frac{\partial}{\partial v}\left(\frac{\partial f}{\partial y}\right) \overset{Chain}{\underset{Rule}{=}} 2\left(\frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\left(\frac{\partial f}{\partial x}\right)}{\partial y}\frac{\partial y}{\partial v}\right)$$

$$+\,4\left(\frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial\left(\frac{\partial f}{\partial y}\right)}{\partial y}\frac{\partial y}{\partial v}\right) = 2\left(\left(\frac{\partial^2 f}{\partial x^2}\right)(-3) + \left(\frac{\partial^2 f}{\partial y\,\partial x}\right)(5)\right)$$

$$+\,4\left(\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)(-3) + \left(\frac{\partial^2 f}{\partial y^2}\right)(5)\right) = -6\,\frac{\partial^2 f}{\partial x^2} + 10\,\frac{\partial^2 f}{\partial y\,\partial x} - 12\,\frac{\partial^2 f}{\partial x\,\partial y} + 20\,\frac{\partial^2 f}{\partial y^2}$$

$$\overset{f\ is\ of}{\underset{class\ C^2}{=}} -6\,\frac{\partial^2 f}{\partial x^2} - 2\,\frac{\partial^2 f}{\partial x\,\partial y} + 20\,\frac{\partial^2 f}{\partial y^2}.$$

10. Recall $e^t = \displaystyle\sum_{k=0}^{\infty}\frac{t^k}{k!}$, $|t| < \infty$, so $e^{-xy} = 1 - xy + \dfrac{x^2y^2}{2!} - \dfrac{x^3y^3}{3!} + \cdots$, $|xy| < \infty$ (by replacement). We also recall that $\arctan y = y - \dfrac{y^3}{3} + \dfrac{y^5}{5} - \cdots$, $|y| < 1$. We obtain a Taylor series for $f(x,y) = e^{-xy}\arctan y$,

$$T = \left(1 - xy + \frac{x^2y^2}{2!} - \cdots\right)\left(y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots\right),$$

using multiplication of series. Hence the 5th degree Taylor polynomial for f about $(0,0)$ is

$$\begin{aligned} T_5 &= y - \frac{y^3}{3} + \frac{y^5}{5} - xy^2 + \frac{xy^4}{3} + \frac{x^2y^3}{2} \\ &= y - xy^2 - \frac{y^3}{3} + \frac{x^2y^3}{2} + \frac{xy^4}{3} + \frac{y^5}{5}. \end{aligned}$$