University of Toronto Scarborough
Department of Computer & Mathematical Sciences

Midterm Test

MATB41H – Techniques of the Calculus of Several Variables I

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Date: October 22, 2011
Duration: 110 minutes

1. **[8 points]**

   (a) Carefully complete the following definition:

   Let $f : U \subset \mathbb{R}^n \to \mathbb{R}^k$ be a given function. We say that f is *differentiable at* $\boldsymbol{a} \in U$ if $\cdots$

   (b) Carefully state the Chain Rule for functions of more than one variable.

2. **[6 points]** Choose a suitable function $f(x, y)$ and use a linear approximation about $(4, 3)$ to estimate $\sqrt{(4.01)^2 + (2.98)^2}$.

3. **[15 points]**

   (a) Calculate the following limits, showing all your steps, or show that the limit does not exist.

   i. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{1 - \cos 2x + \sin 2y}{x^2 + 2y}$

   ii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3\,x\,y^2}{x^2 + y^2}$

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

   $$f(x, y) = \begin{cases} \dfrac{xy + 2x}{x^2 + (y+2)^2} & , \text{ if } (x, y) \neq (0, -2) \\ 0 & , \text{ if } (x, y) = (0, -2) \ . \end{cases}$$

   Is f continuous at $(0, -2)$? (Explain your answer.)

4. **[12 points]** Let $A = \begin{pmatrix} 1 & 0 & x \\ y & 1 & 2 \\ 0 & 2 & z \end{pmatrix}$.

   (a) Find $\det A$.

   (b) Put $f(x, y, z) = \det A$ and compute ∇f, the gradient of f.

   (c) Let ℓ_1 be the line through $(2, 1, 0)$ with direction vector $\nabla f\,(2, 1, 0)$. If ℓ_2 is the line through $(2, 1, 0)$ and $(0, 1, 2)$ find the angle between ℓ_1 and ℓ_2.

5. **[15 points]**

   (a) Find the equation of the tangent plane to the graph of $f(x,y) = \cos(2\,x + y)$ at the point $\left(\frac{\pi}{2}, \frac{\pi}{4}, f\left(\frac{\pi}{2}, \frac{\pi}{4}\right)\right)$.

   (b) Find the equation of the tangent plane at the point $(2,1,1)$ to the graph of the function $z = f(x,y)$ defined implicitly by
   $$x\,z + 2\,x^2\,y + y^2\,z^3 = 11\,.$$

   (c) Show that every tangent plane to the surface $z^2 = x^2 + y^2$ passes through the origin.

6. **[15 points]** Let $T : \mathbb{R}^3 \to \mathbb{R}$, given by $T(x,y,z) = x^2 + y\,z + x\,z^2$, represent temperature in space. A bug has been living at $(1,1,1)$.

   (a) Describe the region of $\mathbb{R}^3$ where the bug can move and stay at the same temperature.

   (b) If the bug should feel too cold at $(1,1,1)$, in what direction should it move to increase temperature at the fastest rate? What is the maximum rate of increase?

   (c) The bug now learns that there is food at $(3,-2,1)$. What rate of change in temperature would it experience if it headed for the food?

7. **[5 points]** Determine if $f : \mathbb{R}^3 \to \mathbb{R}$, given by $f(x,y,z) = x^2 - 3y^2 + 2z^2$, is harmonic.

8. **[11 points]** Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ be given by $f(x,y,z,w) = (xw,\, yz,\, xy,\, zw)$ and let $g : \mathbb{R}^4 \to \mathbb{R}^2$ be given by $g(x,y,z,w) = (wx^2,\, wyz)$).
   `USE THE CHAIN RULE` to compute $D(g \circ f)(x,y,z,w)$.

   (**NOTE:** You must use the Chain Rule and show all your steps.)

9. **[7 points]** Let $z = f(x,y)$ be of class C^2. Putting $x = u - v$ and $y = 2v - 3u$ makes z into a function of u and v. Compute a formula for $\dfrac{\partial^2 z}{\partial v\,\partial u}$ in terms of the partial derivatives of z with respect to x and y.

10. **[6 points]** Give the 4^{th} degree Taylor polynomial about the origin of $f(x,y) = e^{2\,x}\,\ln(1 + x\,y)$.