University of Toronto Scarborough
Department of Computer & Mathematical Sciences

Midterm Test

MATB41H – Techniques of the Calculus of Several Variables I

Examiner: E. Moore

Date: October 31, 2014
Duration: 110 minutes

1. **[8 points]**

   (a) Given $f : U \subset \mathbb{R}^n \to \mathbb{R}$, define the following terms

   i. **local minimum**

   ii. **critical points**

   (b) Carefully state the **Extreme Value** (Min-Max) Theorem for real-valued functions of several variables.

2. **[15 points]**

   (a) Calculate the following limits, showing all your steps, or show that the limit does not exist.

   i. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - 5xy + y^2}{x^2 + y^2}$.

   ii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos(xy) - 1}{x^2 y^2}$.

   (b) Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{y^4 + 2y^2 + 2x^2 - x^4}{x^2 + y^2} & , \text{ if } (x,y) \neq (0,0) \\ 2 & , \text{ if } (x,y) = (0,0) \ . \end{cases}$$

   Is f continuous at $(0,0)$? (Explain your answer.)

3. **[12 points]** Characterize and sketch several level curves of the function

$$f(x,y) = \frac{y}{x^2 + y^2} \ .$$

Carefully indicate where f is zero, positive, negative and not defined.

4. **[10 points]** Let $f(x,y) = e^{x \sin y}$ and $g(x,y) = 2x^2 + bxy + \dfrac{3}{2}y^2$. Let π_1 be the plane tangent to the graph of f at $(1,0,1)$ and let π_2 be the plane tangent to the graph of g at $\left(1, -1, -b + \dfrac{7}{2}\right)$. Find all values of b such that

   (a) π_1 and π_2 are parallel

   (b) π_1 and π_2 intersect in a line parallel to the vector $(1,1,1)$.

5. **[11 points]**

   (a) Give the equation of the tangent plane to the surface $x^2z^3 + y^2x^3 + z^2y^3 = 1$ at the point $\boldsymbol{p} = (1,-1,1)$.

   (b) Let ℓ be the line through $(1,2,0)$ and $(0,1,2)$. Give a parametric description of ℓ and determine where ℓ meets the tangent plane from part (a), or show that ℓ does not intersect the tangent plane.

6. **[6 points]** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $\boldsymbol{a}$ and assume that $\nabla f(\boldsymbol{a})$ is nonzero. Find the unit vector $\boldsymbol{v} \in \mathbb{R}^2$ such that $D_{\boldsymbol{v}}\, f(\boldsymbol{a})$ is maximized. (Justify your answer).

7. **[15 points]** Let $f(x,y,z) = x^3 + y^2 - z^2 - 6xy + 6x + 3y + 1$.

   (a) Find the critical points of f.

   (b) What is the rate of change in f as you move from $(1,2,0)$ towards $(3,0,1)$.

   (c) In what direction from $(2,1,0)$ must you go for the most rapid increase in f? What is the rate of this increase.

8. **[16 points]**

   (a) Carefully state the **Chain Rule** for functions of more than one variable.

   (b) Let $f : \mathbb{R}^4 \to \mathbb{R}^3$ be given by $f(x,y,z,w) = (yzw, x^2y, xz)$ and let
   $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $g(x,y,z) = (xy, yz)$.
   `USE THE CHAIN RULE` to compute $D(g \circ f)(x,y,z,w)$.
   (**NOTE:** You must use the Chain Rule and show all your steps.)

9. **[6 points]** Let $f(x,y)$ be of class C^2. Putting $x = uv$ and $y = 2u - 3v$ makes f into a function of u and v. Compute a formula for $\dfrac{\partial^2 f}{\partial u\, \partial v}$ in terms of the partial derivatives of f with respect to x and y.

10. **[6 points]** Give the 4^{th} degree Taylor polynomial about the origin of
    $f(x,y) = e^{x^2} \cos(xy)$.