University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

MAT B41 — Midterm Exam

Examiner: Adrian Butscher

Date: 25 October 2006
Duration: 170 minutes

FAMILY NAME: ______________________________

GIVEN NAME(S): ______________________________

STUDENT NUMBER: ______________________________

THE DATE AND TIME OF YOUR TUTORIAL: ______________________________

THE NAME OF YOUR TA: ______________________________

Paula Ehlers   Shay Fuchs

YOUR SIGNATURE: ______________________________

DO NOT OPEN THIS TEST UNTIL INSTRUCTED TO DO SO.

INSTRUCTIONS:

- Your signature above indicates that you have abided by the UTSC Code of Conduct while writing this test.
- All questions have equal value (20 points).
- This exam contains 17 numbered pages after this one. Ensure that no pages are missing.
- You may quote theorems from your textbook if you make an appropriate reference.
- No electronic devices of any kind (e.g. calculators, cell-phones) are allowed.

Question	Marks
1	
2	
3	
4	
5	
Total (100 points)	

1. Define the following vector subspaces $V, W \subseteq \mathbf{R}^4$

$$V = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \right\} \qquad W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \begin{array}{l} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{array} \right\}$$

(a) Find a basis for V. What is the dimension of V?

(b) Find a basis for W. What is the dimension of W?

(c) Find a basis for $V + W$. What is the dimension of $V + W$?

(d) Find a basis for $V \cap W$. What is the dimension of $V \cap W$?

2. Define the following vector subspace $W \subseteq \mathbf{R}^3$

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;:\; x_1 - x_2 + x_3 = 0 \right\}$$

   (a) Find an orthonormal basis for W.

(b) Find an orthonormal basis for $W^{\perp}$.

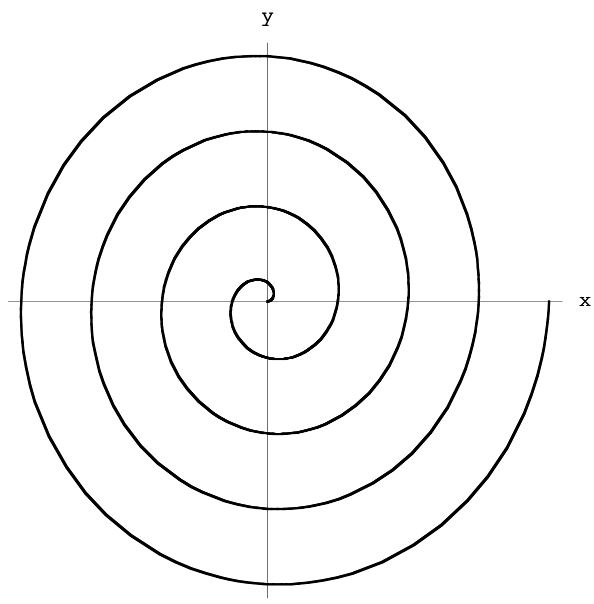
(c) According to the Orthogonal Decomposition Theorem, it is possible to write uniquely $\vec{x} = \vec{x}_1 + \vec{x}_2$ for every $\vec{x} \in \mathbf{R}^3$, where $\vec{x}_1 \in W$ and $\vec{x}_2 \in W^{\perp}$. The vector x_1 is called the *orthogonal projection of $\vec{x}$ onto W* and the vector $\vec{x}_2$ is called the *orthogonal projection of $\vec{x}$ onto $W^{\perp}$*. If

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

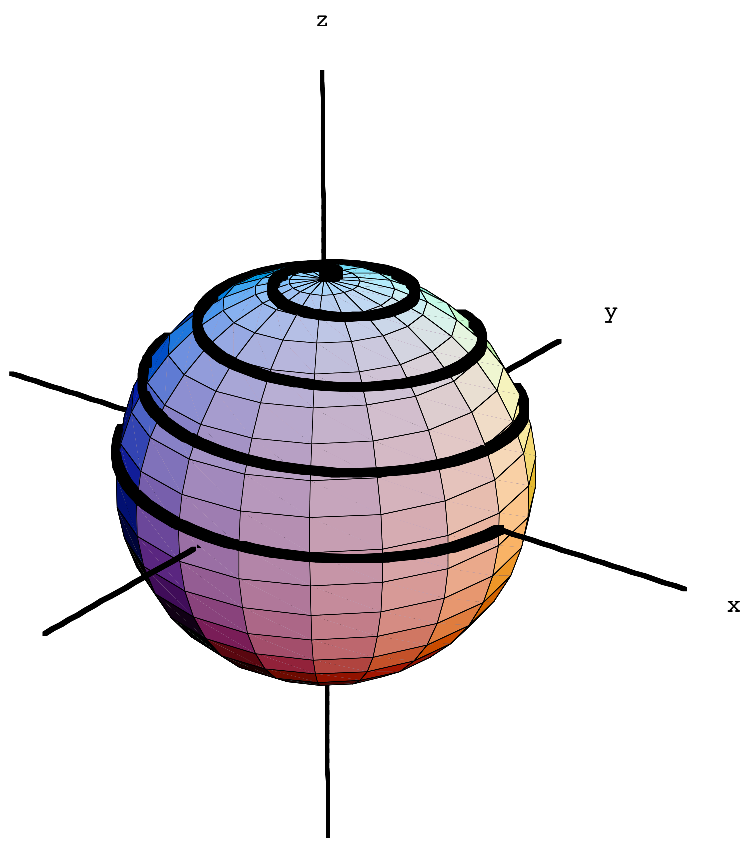
then what is its orthogonal projection onto W?

(d) What is the matrix of the linear transformation that takes an arbitrary vector $\vec{x} \in \mathbf{R}^3$ to its orthogonal projection onto W?

3. (a) The following curve is a spiral starting at the origin and ending at $(0, 1)$. Suggest a possible parametrization of this curve. Try to capture as many details of the curve as you can.



(b) The following curve is a spiral lying on the surface of the unit sphere starting at $(0, 0, 1)$ and ending at $(1, 0, 0)$. Suggest a possible parametrization of this curve. Try to capture as many details of the curve as you can.



4. Do the following limits exist? Prove your assertions by appealing to theorems or techniques learned in class, the textbook or the problem sets.

   (a) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$

   (b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 6x^2y^2 + y^4}{x^4 + 6x^2y^2 + y^4}$

(d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2 + y^4}$

5. Consider the function $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{x} & \forall\ (x,y) \text{ with } x \neq 0 \\ 0 & \text{whenever } x = 0 \end{cases}$

   (a) Show that the level set $L_c(f)$ for $c \neq 0$ is a circle of radius $|c|/2$ centered at the point $(c/2, 0)$. What is $L_0(f)$?

   (b) Make a rough sketch of several level sets.

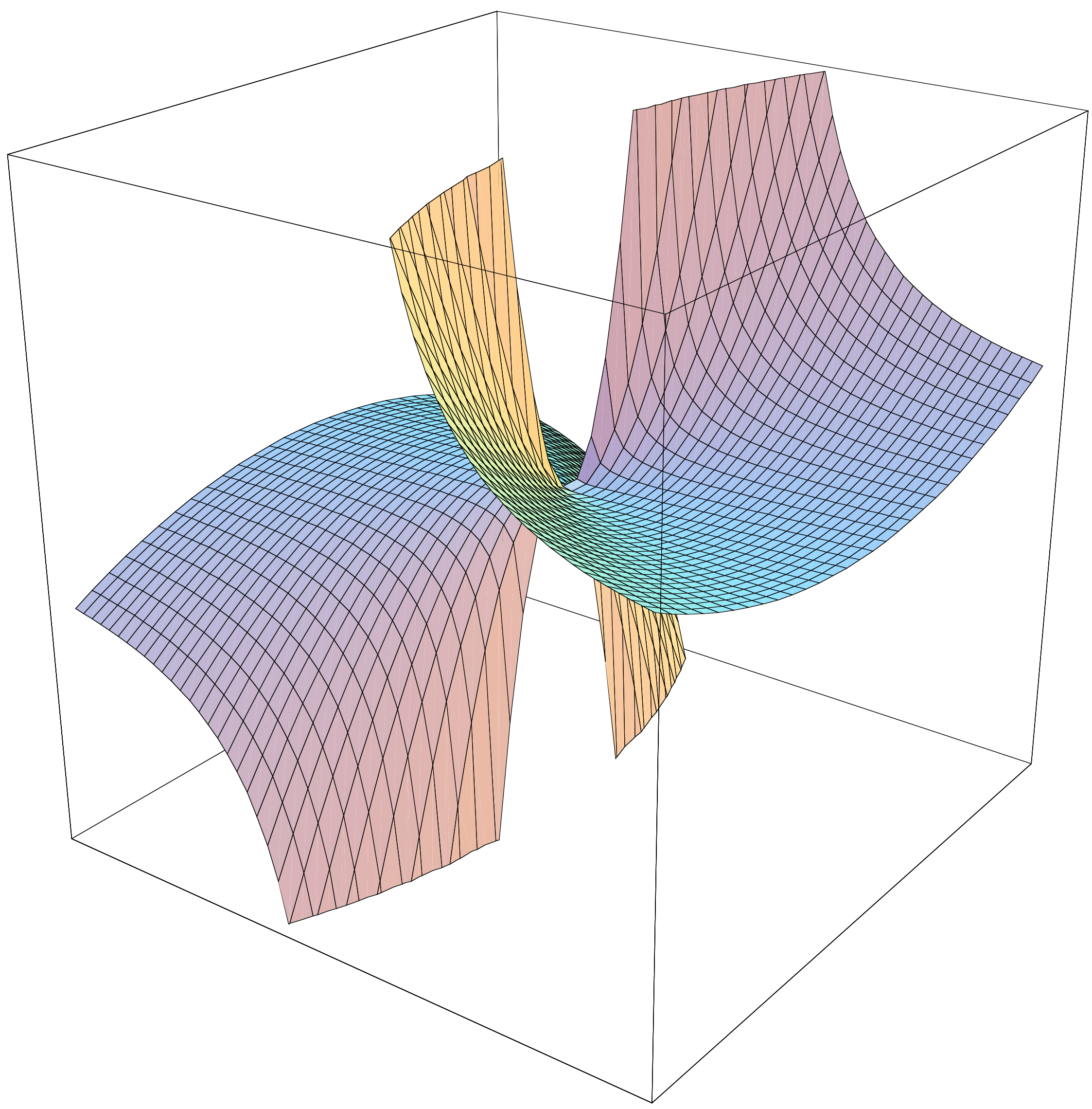
(c) For which values of (x, y) is f continuous? Prove your assertions by appealing to theorems or techniques learned in class, the textbook or the problem sets.

(d) Describe the $x = \mathit{constant}$ and $y = \mathit{constant}$ cross sections of the graph of f. Draw these for several values of the constant.

(e) Explain as fully as you can why the following picture is the graph of f.

Where are the axes? Show a variety of level sets, cross-sections and examples of discontinuous behaviour from the previous questions. Are there any other important features that you can explain mathematically?

Note that the part of the graph above a small sliver containing the x-axis (i.e. the part above the set of points $A = \{(x,y) : -0.01 < x < 0.01\}$) is missing because the computer is unable to draw it. Why is this so? Where is A in the picture? What would the part above of the graph above A look like?



(e) . . . Continued from the previous page.

This page is blank for you to use as scrap paper.