Not so Discrete Math

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1 Sets

Opening Questions: What do we call a collection of things? does order or repetition matter? Can we contain collection of other collections? Is an empty collection considered a collection? How do we count members of a collection, and how do we define them? Can we combine collections, and if so, what are those operations?

1.1 Introduction to Sets

In discrete math we work with some group of 'things,' a thing or something we fancily call an **object**. A group or categorization of objects is called a set.

Theorem 1.1: Set

Is a collection of objects.

For Example:

- $S =$ The set of all students in a classroom.
- $A =$ The set of all vowels in the English alphabet.
- $\mathbb{Z} =$ The set of all integers.

Objects in a **set** are called **elements**.

Theorem 1.2: Element

An object that is a member of a given set.

To expand on the previous example:

- $S = \{s_1, s_2, s_3\}$, where s_1, s_2, s_3 are students, elements of the set.
- $A = \{a, e, i, o, u\}$, where a, e, i, o, u are elements.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, elements of integer set.

Curly braces denote a set, commas separate elements, and the '...' or 'ellipse' indicate that the set continues indefinitely in that direction. When the set's pattern **is clear** to the reader, we use the dotted notation.

Symbols used to denote members of a set:

- '$\in$' = in the set.
- '$\notin$' = not in the set.

Theorem 1.3: Membership

If x is an element of set A, $x \in A$. If x is not an element of set A, $x \notin A$.

For Example: Given $A = \{a, e, i, o, u\}$,
$a \in A$, "a is an element of A," and $b \notin A$, "b is not an element of A."

Order nor repetition matter:

- $A = \{1, 2, 3\} = \{3, 2, 1\} = \{1, 2, 3, 3, 3, 3, 3\}$.
- $B = \{a, b, c\} = \{a, b, c, a, b, c\}$.

Theorem 1.4: Properties of a Set

- The order of elements do not matter.
- Duplicate elements are not counted.

Tip: In math we just define things, and then they are that thing. You can define things too! You could define a collection that infinitely has the word "bees." We will now fancily call it "**Bioths Non-determinant Sequence**," or a
BND-Sequence for short.

$$\text{BND-Sequence} = \{\text{"bees"}, \text{"bees"}, \text{"bees"}, \text{"bees"}, \text{"bees"}, \text{"bees"}, \ldots\}$$

Names are just names, no matter how fancy a word may sound, they were labeled by another human like you who thought, "this would be a *kick-ass* name." So never be intimidated, complex ideas are just groupings of basic concepts.

A subset is a set contained within another set. If the set B is a subset of set A, then every element in B is also in A as shown in Figure 1:

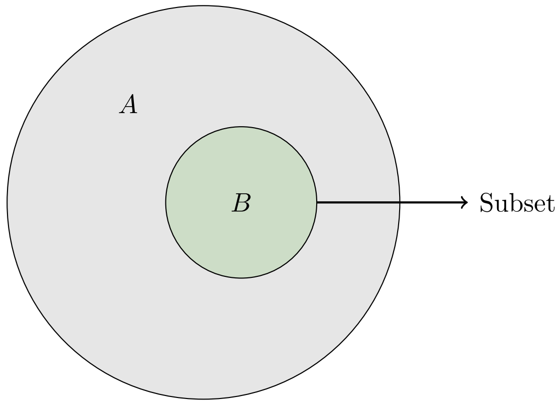

Figure 1

Written $B \subseteq A$ or $A \supseteq B$, similar to the less than or equal to signs '$\leq$' and '$\geq$'.

Theorem 1.5: Subset

If every element in set B is also in set A, then B is a subset of A.
Denoted: $B \subseteq A$ or $A \supseteq B$.

For Example:

- $\{-1, 0\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 1, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 0, 1, 2, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 7\} \nsubseteq \{-1, 0, 1, 2, 3\}$

$\nsubseteq$ denotes 'not a subset of.'

A set with no elements is called the empty set.

Theorem 1.6: Empty Set

Commonly denoted by $\emptyset$ or $\{\}$, refers to a collection with no objects.

Questions:

1. How many elements are in the set $\{\emptyset\}$?
2. True or False: $\emptyset \subseteq \{\emptyset\}$.
3. True or False: $\emptyset \in \{\emptyset\}$.
4. True or False: $\emptyset \subseteq \emptyset$.
5. True or False: $\emptyset \subseteq \mathbb{Z}$.
6. True or False: $\emptyset \in \mathbb{Z}$.

Answers:

1. 1 element, the empty set.
2. True, the empty set is a subset of $\{\emptyset\}$.
3. True, the empty set is an element of $\{\emptyset\}$.
4. True, the empty set is a subset of itself.
5. True, the empty set is a subset of all sets.
6. False, the empty set is not an element of the integers.

Why 1? $\emptyset = \{\}$, **A collection is an object**. The emptyset is a collection, a collection without objects. Likewise, a house is still a house without furniture.

Say we have an empty box. How many objects do we have? **Zero**.
Put an empty box inside our original box. How many objects now? **One**!

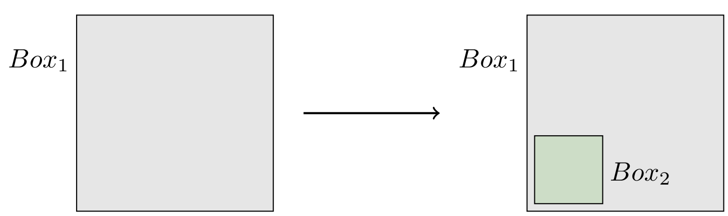

Figure 2: Box_1 contains 1 object, which is Box_2 (an empty box). Hence Box_1 represents $\{\{\}\}$ or $\{\emptyset\}$.

$\emptyset$ is a subset of every set? Take sets $A = \{\}$ and $B = \{1, 2, 3\}$
By definition of a subset, every element in A must be in B. It's difficult to argue elements in A are indeed in B, but it's undeniable that elements in A are not in B. Since our statement cannot be denied, it's **Vacuously true**.

Counting the number of elements in a set is called the **cardinality** of the set.

Theorem 1.7: Cardinality

The number of elements in a set.
Denoted over a set A as $|A|$.

For Example:

- $A = \{1, 2, 3\}$, $|A| = 3$.
- $B = \{a, e, i, o, u\}$, $|B| = 5$.
- $\mathbb{Z}$ the set of all integers, $|\mathbb{Z}| = \infty$.

Questions:
What are the cardinalities of the following sets?

1. $|\{1, 2, 3\}|$
2. $|\emptyset|$
3. $|\{\}|$
4. $|\{\emptyset\}|$
5. $|\{1, \{2, 3\}\}|$
6. $|\{1, 2, 2, 3, 3, 3\}|$

Answers:

1. 3
2. 0
3. 0
4. 1
5. 2
6. 3

Tip: Whenever you approach a problem, always break things down into simple components. "What defines a set? What defines an element? What defines a subset? What defines cardinality?"

Explicitly defining a set, say $\{1, 2, 3, ...\}$, is called **set-roster notation**. **Set-builder notation** enables more complex definitions of a set.

Theorem 1.8: Set-Builder Notation

General form: $\{x \mid P(x)\}$,

- x = defines some variable.
- " | " = is short hand for "such that."
- $P(x)$ = describes the properties x must satisfy.

For Example: Lets define the set of even integers

- $\{x \mid x \text{ is an even integer}\}$: "$x$, such that, x is an even integer."
- $\{x \in \mathbb{Z} \mid x \text{ is even}\}$: "$x$ in Integers, such that, x is an even."
- $\{x \in \mathbb{Z} \mid x \text{ is not odd}\}$: "$x$ in Integers, such that, x is not odd."

It's important to define exactly what variables are. In the above, x was stated directly as an integer. If not, x could be **water-balloons** or **puppies**.

1.2 Set Operations

Combining the two sets, $\{1, 2, 3\}$ and $\{a, b, c\}$, produce the set $\{1, 2, 3, a, b, c\}$, which is is called the **union** of A and B.

Theorem 1.9: Union

The union of two sets A and B is the set of all elements that are in both sets.

Denoted: $A \cup B$.

This is also known as a **disjunction**, which is a fancy term for the word "OR".

For Example:

- $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.
- $\{1, 2\} \cup \emptyset = \{1, 2\}$. There is nothing to add.
- $\{1\} \cup \{\emptyset\} = \{1, \emptyset\}$. The empty set is an object.

The common elements of the two sets, $\{1, 2, 3\}$ and $\{2, 3, 4\}$, produce the set $\{2, 3\}$, the **intersection**.

Theorem 1.10: Intersection

The intersection of two sets A and B is the set of all elements that are in both sets.

Denoted: $A \cap B$.

This is also known as a **conjunction**, which is a fancy term for the word "AND".

For Example:

- $\{1, 2\} \cap \{2, 3\} = \{2\}$.
- $\{1, 2\} \cap \emptyset = \emptyset$. There is nothing to compare.
- $\{1\} \cap \{\emptyset\} = \emptyset$. The empty set is not an object.

Tip: To lessen the confusion between $\cup$ and $\cap$, think, "$\cap$" for "AND", since $\cap$ looks like an "A" without the line.

The combination of $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ in order pairs are:

$$(1, a), (1, b), (1, c),$$
$$(2, a), (2, b), (2, c),$$
$$(3, a), (3, b), (3, c)$$

Putting the above objects in a set yields the **Cartesian Product** of A and B.

Theorem 1.11: Cartesian Product

The Cartesian product of two sets A and B is the set of all possible order pairs of elements from A and B.

Denoted: $A \times B$.

For Example:

- $\{1, 2\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b)\}$.
- $\{1, 2\} \times \emptyset = \emptyset$. B is empty, there is nothing to pair.
- $\{1\} \times \{\emptyset\} = \{(1, \emptyset)\}$. B contains 1 object to pair.

Note Figure 2 from the previous section if the above caused confusion.

We have sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, if we remove the common elements A has with B, i.e., take all in A that is not in B, we get the set $\{1\}$, the **difference**.

Theorem 1.12: Difference

The difference of two sets A and B is the set of all elements that are in A but not in B.

Denoted: $A - B$.

For Example:

- $\{1, 2\} - \{2, 3\} = \{1\}$.
- $\{1, 2\} - \emptyset = \{1, 2\}$. There is nothing to remove.
- $\{1\} - \{\emptyset\} = \{1\}$. The empty set is not an object.