Not so Discrete Math

Christian Rudder

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1 Sets

Opening Question: What do we call a collection of things? does order or repetition matter? Can we contain collection of other collections? Is an empty collection considered a collection? How do we count members of a collection, and how do we define them?

1.1 Introduction to Sets

In discrete math we work with some group of 'things,' a thing or something we fancily call an **object**. A group or categorization of objects is called a set.

Theorem 1.1: Set

Is a collection of objects.

For Example:

- $S =$ The set of all students in a classroom.
- $A =$ The set of all vowels in the English alphabet.
- $\mathbb{Z} =$ The set of all integers.

Objects in a **set** are called **elements**.

Theorem 1.2: Element

An object that is a member of a given set.

To expand on the previous example:

- $S = \{s_1, s_2, s_3\}$, where s_1, s_2, s_3 are students, elements of the set.

- $A = \{a, e, i, o, u\}$, where a, e, i, o, u are elements.
- $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, elements of integer set.

Curly braces denote a set, commas to separate elements, The '...' or 'ellipse' indicate that the set continues indefinitely in that direction. When the set's pattern is clear to the reader, we use the dotted notation.

Symbols used to denote members of a set:

- '$\in$' = in the set.
- '$\notin$' = not in the set.

Theorem 1.3: Membership

If x is an element of set A, $x \in A$. If x is not an element of set A, $x \notin A$.

For Example: Given $A = \{a, e, i, o, u\}$,
$a \in A$, "a is an element of A," and $b \notin A$, "b is not an element of A."

Order nor repetition matter:

- $A = \{1, 2, 3\} = \{3, 2, 1\} = \{1, 2, 3, 3, 3, 3, 3\}$.
- $B = \{a, b, c\} = \{a, b, c, a, b, c\}$.

Theorem 1.4: Properties of a Set

- The order of elements does not matter.
- Duplicate elements are not counted.

A subset is a set contained within another set. If the set B is a subset of set A, then every element in B is also in A as shown in Figure 1:
Written $B \subseteq A$ or $A \supseteq B$, similar to the less than or equal to signs '$\leq$' and '$\geq$'.

Theorem 1.5: Subset

If every element in set B is also in set A, then B is a subset of A.
Denoted: $B \subseteq A$ or $A \supseteq B$.

For Example:

- $\{-1, 0\} \subseteq \{-1, 0, 1, 2, 3\}$

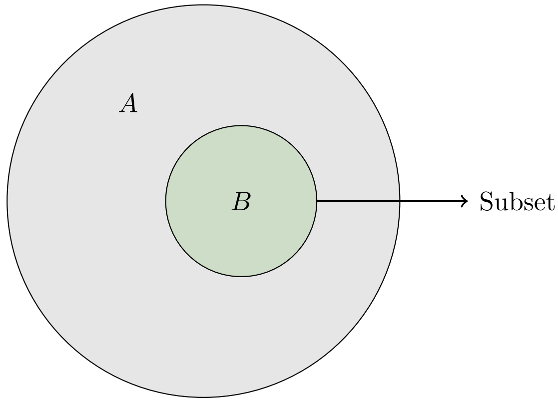

Figure 1

- $\{-1, 1, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 0, 1, 2, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 7\} \not\subseteq \{-1, 0, 1, 2, 3\}$

$\not\subseteq$ denotes 'not a subset of.'

A set with no elements is called the empty set.

Theorem 1.6: Empty Set

Commonly denoted by $\emptyset$ or $\{\}$, refer to a collection with no objects.

Questions:

1. How many elements are in the set $\{\emptyset\}$?
2. True or False: $\emptyset \subseteq \{\emptyset\}$.
3. True or False: $\emptyset \in \{\emptyset\}$.
4. True or False: $\emptyset \subseteq \emptyset$.
5. True or False: $\emptyset \subseteq \mathbb{Z}$.
6. True or False: $\emptyset \in \mathbb{Z}$.

Answers:

1. 1 element, the empty set.
2. True, the empty set is a subset of $\{\emptyset\}$.

3. True, the empty set is an element of $\{\emptyset\}$.
4. True, the empty set is a subset of itself.
5. True, the empty set is a subset of all sets.
6. False, the empty set is not an element of the integers.

Why 1? $\emptyset = \{\}$, A collection is an object, but this one just contains no objects.

Say we have an empty box. How many objects do we have? **Zero**.
Put an empty box inside our original box. How many objects now? **One**!

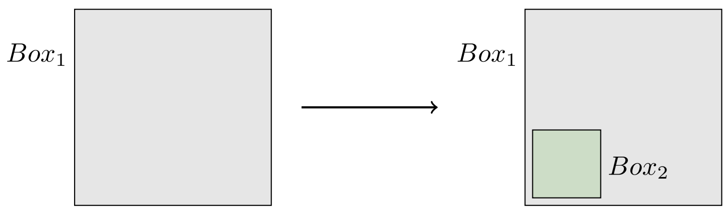

Figure 2: Box_1 contains **1** object, which is Box_2 (an empty box). Hence Box_1 represents $\{\{\}\}$ or $\{\emptyset\}$.

$\emptyset$ is a subset of every set? Take sets $A = \{\}$ and $B = \{1, 2, 3\}$
By definition of a subset, every element in A must be in B. It's difficult to argue elements in A are indeed in B, but it's undeniable that elements in A are not in B. Since our statement cannot be denied, it's **Vacuously true**.

Counting the number of elements in a set is called the **cardinality** of the set.

Theorem 1.7: Cardinality

The number of elements in a set.
Denoted over a set A as $|A|$.

For Example:

- $A = \{1, 2, 3\}$, $|A| = 3$.
- $B = \{a, e, i, o, u\}$, $|B| = 5$.
- $\mathbb{Z}$ the set of all integers, $|\mathbb{Z}| = \infty$.

Questions:
What are the cardinalities of the following sets?

1. $|\{1, 2, 3\}|$
2. $|\emptyset|$

3. $|\{\}|$

4. $|\{\emptyset\}|$

5. $|\{1, 2, 3\}|$

Answers:

1. 3

2. 0

3. 0

4. 1

5. 2

Explicitly defining a set, say $\{1, 2, 3, ...\}$, is called **set-roster notation**. **Set-builder notation** enables more complex definitions of a set.

Theorem 1.8: Set-Builder Notation

General form: $\{x \mid P(x)\}$,

- $x =$ defines some variable.
- " | " = is short hand for "such that."
- $P(x) =$ describes the properties x must satisfy.

For Example: Lets define the set of even integers

- $\{x \mid x \text{ is an even integer}\}$: "$x$, such that, x is an even integer."
- $\{x \in \mathbb{Z} \mid x \text{ is even}\}$: "$x$ in Integers, such that, x is an even."
- $\{x \in \mathbb{Z} \mid x \text{ is not odd}\}$: "$x$ in Integers, such that, x is not odd."

It's important to define exactly what variables are. In the above, x was stated directly as an integer. If not, x could be **water-balloons** or **puppies**.

1.2 Set Operations

The combination of $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ in order pairs are:

$$(1, a), (1, b), (1, c),$$
$$(2, a), (2, b), (2, c),$$
$$(3, a), (3, b), (3, c)$$

Putting the above objects in a set yields the **Cartesian** product of A and B.

Theorem 1.9: Cartesian

The Cartesian product of two sets A and B is the set of all possible order pairs of elements from A and B.

Denoted: $A \times B$.

For Example:

- $\{1, 2\} \times \{a, b\}$, then $A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$.
- $A = \{1, 2\}, B = \emptyset$, then $A \times B = \emptyset$. B is empty, there is nothing to pair.
- $A = \{1\}, B = \{\emptyset\}$, then $A \times B = \{(1, \emptyset)\}$. B contains 1 object to pair.

Note Figure 2 from the previous section if the above caused confusion.