

# Not so Discrete Math

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# 1 Sets

**Opening Questions:** What do we call a collection of things? does order or repetition matter? Can we contain collection of other collections? Is an empty collection considered a collection? How do we count members of a collection, and how do we define them? Can we combine collections, and if so, what are those operations?

## 1.1 Introduction to Sets

In discrete math we work with some group of ‘things,’ a thing or something we fancily call an **object**. A group or categorization of objects is called a set.

### Theorem 1.1: Set

Is a collection of objects.

**For Example:**

- $S$  = The set of all students in a classroom.
- $A$  = The set of all vowels in the English alphabet.
- $\mathbb{Z}$  = The set of all integers.

Objects in a **set** are called **elements**.

### Theorem 1.2: Element

An object that is a member of a given set.

To expand on the previous example:

- $S = \{s_1, s_2, s_3\}$ , where  $s_1, s_2, s_3$  are students, elements of the set.
- $A = \{a, e, i, o, u\}$ , where  $a, e, i, o, u$  are elements.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ , elements of integer set.

Curly braces denote a set, commas separate elements, and the ‘...’ (ellipse) indicates an indefinite continuation, used only when the pattern is clear.

There is also notation to denote members of a set.

**Theorem 1.3: Membership**

If  $x$  is an element of set  $A$ ,  $x \in A$ . If  $x$  is not an element of set  $A$ ,  $x \notin A$ .

**For Example:** Given  $A = \{a, e, i, o, u\}$ ,  
 $a \in A$ , “ $a$  is an element of  $A$ ,” and  $b \notin A$ , “ $b$  is not an element of  $A$ .”

**Order nor repetition matter:**

- $A = \{1, 2, 3\} = \{3, 2, 1\} = \{1, 2, 3, 3, 3, 3, 3\}$ .
- $B = \{a, b, c\} = \{a, b, c, a, b, c\}$ .

**Theorem 1.4: Properties of a Set**

- The order of elements do not matter.
- Duplicate elements are not counted.

A subset is a set contained within another set. If the set  $B$  is a subset of set  $A$ , then every element in  $B$  is also in  $A$  as shown in Figure 1:

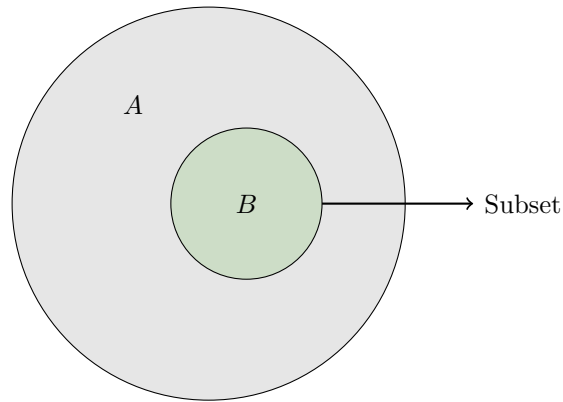


Figure 1

Written  $B \subseteq A$  or  $A \supseteq B$ , similar to the less than or equal to signs ‘ $\leq$ ’ and ‘ $\geq$ ’.

**Theorem 1.5: Subset**

If every element in set  $B$  is also in set  $A$ , then  $B$  is a subset of  $A$ .  
 Denoted:  $B \subseteq A$  or  $A \supseteq B$ .

**For Example:**

- $\{-1, 0\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 1, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 0, 1, 2, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 7\} \not\subseteq \{-1, 0, 1, 2, 3\}$

$\not\subseteq$  denotes ‘not a subset of.’

A set with no elements is called the empty set.

**Theorem 1.6: Empty Set**

Commonly denoted by  $\emptyset$  or  $\{\}$ , refers to a collection with no objects.

**Questions:**

1. How many elements are in the set  $\{\emptyset\}$ ?
2. True or False:  $\emptyset \subseteq \{\emptyset\}$ .
3. True or False:  $\emptyset \in \{\emptyset\}$ .
4. True or False:  $\emptyset \subseteq \emptyset$ .
5. True or False:  $\emptyset \subseteq \mathbb{Z}$ .
6. True or False:  $\emptyset \in \mathbb{Z}$ .

**Tip:** Mathematicians define things? So can you! Let’s define a collection that infinitely repeats the string “bees.” We will fancily call it “**Bioths Non-determinant Sequence**,” or a  $\beta_{seq}$  for short.

$$\beta_{seq} = \{\text{“bees”, “bees”, “bees”, “bees”, “bees”, “bees”, ...}\}$$

**Names are names**, no matter how fancy, they were labeled by another human, like you. They thought,... “Damn, this would be a *kick-ass* name.” Never be intimidated, complex ideas are just groupings of basic concepts.

**Answers:**

1. 1 element, the empty set.
2. True, the empty set is a subset of  $\{\emptyset\}$ .
3. True, the empty set is an element of  $\{\emptyset\}$ .
4. True, the empty set is a subset of itself.
5. True, the empty set is a subset of all sets.
6. False, the empty set is not an element of the integers.

**Why (1.): A collection is an object.** The empty set is a collection, a collection without objects. Likewise, a house is still a house without furniture.

**Why (5.):** Take sets  $A = \{\}$  and  $B = \{1, 2, 3\}$

By definition of a subset, every element in  $A$  must be in  $B$ . It's difficult to argue elements in  $A$  are indeed in  $B$ , but it's undeniable that elements in  $A$  are not in  $B$ . Since our statement cannot be denied, it's **Vacuously true**.

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Say we have an empty box. How many objects do we have? **Zero**.  
Put an empty box inside our original box. How many objects now? **One**!

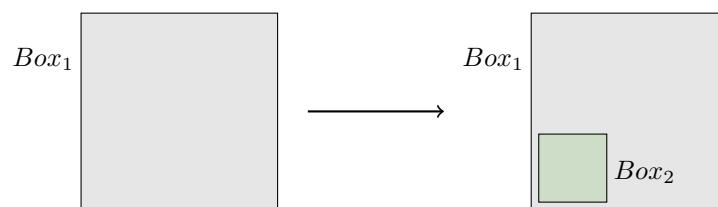


Figure 2:  $Box_1$  contains 1 object, which is  $Box_2$ , an empty box. Hence  $Box_1$  represents  $\{\{\}\}$  or  $\{\emptyset\}$ .

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Counting the number of elements in a set is called the **cardinality** of the set.

**Theorem 1.7: Cardinality**

The number of elements in a set.  
Denoted over a set  $A$  as  $|A|$ .

**For Example:**

- $A = \{1, 2, 3\}$ ,  $|A| = 3$ .
- $B = \{a, e, i, o, u\}$ ,  $|B| = 5$ .
- $\mathbb{Z}$  the set of all integers,  $|\mathbb{Z}| = \infty$ .

**Questions:**

What are the cardinalities of the following sets?

1.  $|\{1, 2, 3\}|$
2.  $|\emptyset|$
3.  $|\{\}|$
4.  $|\{\emptyset\}|$
5.  $|\{1, \{2, 3\}\}|$
6.  $|\{1, 2, 2, 3, 3, 3\}|$

**Try to think about the answer before looking at the solution.**

Things stick when you struggle.

**Tip:** Whenever you approach a problem, always break things down into simple components. “What defines a set? What defines an element? What defines a subset? What defines cardinality?”

**Answers:**

1. 3
2. 0
3. 0
4. 1
5. 2
6. 3

Explicitly defining a set, say  $\{1, 2, 3, \dots\}$ , is called **set-roster notation**. **set-builder notation** enables us to create more complex definitions.

### Theorem 1.8: Set-Builder Notation

General form:  $\{x \mid P(x)\}$ ,

- $x$  = defines some variable.
- “ $\mid$ ” = is short hand for “such that.”
- $P(x)$  = describes the properties  $x$  must satisfy.

**For Example:** Lets define the set of even integers

- $\{x \mid x \text{ is an even integer}\}$ : “ $x$ , such that,  $x$  is an even integer.”
- $\{x \in \mathbb{Z} \mid x \text{ is even}\}$ : “ $x$  in Integers, such that,  $x$  is an even.”
- $\{x \in \mathbb{Z} \mid x \text{ is not odd}\}$ : “ $x$  in Integers, such that,  $x$  is not odd.”

**It’s important to define exactly what variables are.** In the above,  $x$  was stated directly as an integer. If not,  $x$  could be water-balloons or puppies.

## 1.2 Set Operations

Combining the two sets,  $\{1, 2, 3\}$  and  $\{a, b, c\}$ , produce the set  $\{1, 2, 3, a, b, c\}$ , which is called the **union**.

### Theorem 1.9: Union

The set of elements that appear in either set  $A$  or set  $B$  is the union.  
Denoted:  $A \cup B$ .

This is also known as a **disjunction**, which is a fancy term for the word “OR”.

**For Example:**

- $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$ .
- $\{1, 2\} \cup \emptyset = \{1, 2\}$ . There is nothing to add.
- $\{1\} \cup \{\emptyset\} = \{1, \emptyset\}$ . The  $\emptyset$  is an element in this case.



The common elements of the two sets,  $\{1, 2, 3\}$  and  $\{2, 3, 4\}$ , produce the set  $\{2, 3\}$ , the **intersection**.

**Theorem 1.10: Intersection**

The set of elements that appear in both sets  $A$  and  $B$  is the intersection.  
Denoted:  $A \cap B$ .

This is also known as a **conjunction**, which is a fancy term for the word “AND”.

**For Example:**

- $\{1, 2\} \cap \{2, 3\} = \{2\}$ .
- $\{1\} \cap \{2\} = \emptyset$ . There is nothing in common.
- $\{1\} \cap \emptyset = \emptyset$ . There is nothing to compare.

**Tip:** To lessen the confusion between  $\cup$  and  $\cap$ , think, “ $\cap$ ” for “AND”, since  $\cap$  looks like a curved “A” without the line.

The combination of  $A = \{1, 2, 3\}$  and  $B = \{a, b, c\}$  in order pairs are:

$(1, a), (1, b), (1, c),$

$(2, a), (2, b), (2, c),$

$(3, a), (3, b), (3, c)$

Putting the above objects in a set yields the **cartesian product** of  $A$  and  $B$ .

**Theorem 1.11: Cartesian Product**

The set of all possible order pairs of elements from sets  $A$  and  $B$ .  
Denoted:  $A \times B$ .

**For Example:**

- $\{1, 2\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b)\}$ .
- $\{1, 2\} \times \emptyset = \emptyset$ . There is nothing to pair.
- $\{1\} \times \{\emptyset\} = \{(1, \emptyset)\}$ . The  $\emptyset$  is an element in this case.

**Note:** Visit ‘**Figure 2**’ in the previous section if  $\emptyset$  causes confusion.

We have sets  $A = \{1, 2, 3\}$  and  $B = \{2, 3, 4\}$ , to remove the common elements  $A$  has with  $B$ , i.e., take all in  $A$  that is not in  $B$ , yields the set  $\{1\}$ , the **difference**.

**Theorem 1.12: Difference**

The set of all elements that are in set  $A$  but not in set  $B$ .  
Denoted:  $A - B$ .

**For Example:**

- $\{1, 2\} - \{2, 3\} = \{1\}$ .
- $\{1\} - \{1\} = \emptyset$ .
- $\{1, 2\} - \emptyset = \{1, 2\}$ . There is nothing to remove.
- $\{1\} - \{\emptyset\} = \{1\}$ . There is nothing to remove.