

Not so Discrete Math

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1 Sets

Opening Questions: What do we call a collection of things? does order or repetition matter? Can we contain collection of other collections? Is an empty collection considered a collection? How do we count members of a collection, and how do we define them? Can we combine collections, and if so, what are those operations?

1.1 Introduction to Sets

In discrete math we work with some group of ‘things,’ a thing or something we fancily call an **object**. A group or categorization of objects is called a set.

Definition 1.1: Set

Is a collection of objects.

For Example:

- S = The set of all students in a classroom.
- A = The set of all vowels in the English alphabet.
- \mathbb{Z} = The set of all integers.

Objects in a **set** are called **elements**.

Definition 1.2: Element

An object that is a member of a given set.

To expand on the previous example:

- $S = \{s_1, s_2, s_3\}$, where s_1, s_2, s_3 are students, elements of the set.
- $A = \{a, e, i, o, u\}$, where a, e, i, o, u are elements.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, elements of integer set.

Curly braces denote a set, commas separate elements, and the ‘...’ (ellipse) indicates an indefinite continuation, used only when the pattern is clear.

There is also notation to denote members of a set.

Definition 1.3: Membership

If x is an element of set A , $x \in A$. If x is not an element of set A , $x \notin A$.

For Example: Given $A = \{a, e, i, o, u\}$,
 $a \in A$, “ a is an element of A ,” and $b \notin A$, “ b is not an element of A .”

Order nor repetition matter:

- $A = \{1, 2, 3\} = \{3, 2, 1\} = \{1, 2, 3, 3, 3, 3, 3\}$.
- $B = \{a, b, c\} = \{a, b, c, a, b, c\}$.

Definition 1.4: Properties of a Set

- The order of elements do not matter.
- Duplicate elements are not counted.

A subset is a set contained within another set. If the set B is a subset of set A , then every element in B is also in A as shown in Figure 1:

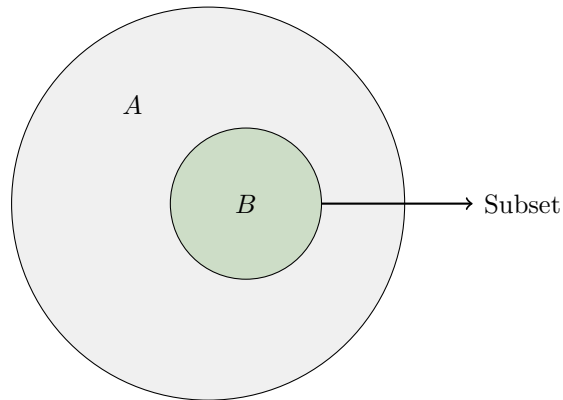


Figure 1: B is a subset of A .

Written $B \subseteq A$ or $A \supseteq B$, similar to the less than or equal to signs ‘ \leq ’ and ‘ \geq ’.

Definition 1.5: Subset

If every element in set B is also in set A , then B is a subset of A .
 Denoted: $B \subseteq A$ or $A \supseteq B$.

For Example:

- $\{-1, 0\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 1, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 0, 1, 2, 3\} \subseteq \{-1, 0, 1, 2, 3\}$
- $\{-1, 7\} \not\subseteq \{-1, 0, 1, 2, 3\}$

$\not\subseteq$ denotes ‘not a subset of.’

A set with no elements is called the empty set.

Definition 1.6: Empty Set

Commonly denoted by \emptyset or $\{\}$, refers to a collection with no objects.

Questions:

1. How many elements are in the set $\{\emptyset\}$?
2. True or False: $\emptyset \subseteq \{\emptyset\}$.
3. True or False: $\emptyset \in \{\emptyset\}$.
4. True or False: $\emptyset \subseteq \emptyset$.
5. True or False: $\emptyset \subseteq \mathbb{Z}$.
6. True or False: $\emptyset \in \mathbb{Z}$.

Tip: Mathematicians define things? So can you! Let’s define a collection that infinitely repeats the string “bees.” We will fancily call it “**Bioths Non-determinant Sequence**,” or a β_{seq} for short.

$$\beta_{seq} = \{\text{“bees”, “bees”, “bees”, “bees”, “bees”, “bees”, } \dots\}$$

Names are names, no matter how fancy, they were labeled by another human, like you. They thought,... “Damn, this would be a *kick-ass* name.” Never be intimidated, complex ideas are just groupings of basic concepts.

Answers:

1. 1 element, the empty set.
2. True, the empty set is a subset of $\{\emptyset\}$.
3. True, the empty set is an element of $\{\emptyset\}$.
4. True, the empty set is a subset of itself.
5. True, the empty set is a subset of all sets.
6. False, the empty set is not an element of the integers.

Why (1.): A collection is an object. The emptyset is a collection, a collection without objects. Likewise, a house is still a house without furniture.

Why (5.): Take sets $A = \{\}$ and $B = \{1, 2, 3\}$

By definition of a subset, every element in A must be in B . It's difficult to argue elements in A are indeed in B , but it's undeniable that elements in A are not in B . Since our statement cannot be denied, it's **Vacuously true**.

Say we have an empty box. How many objects do we have? **Zero**.
Put an empty box inside our original box. How many objects now? **One**!

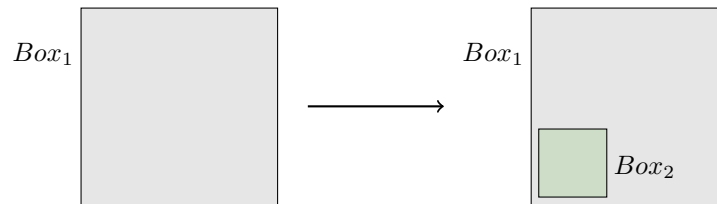


Figure 2: Box_1 contains 1 object, which is Box_2 , an empty box. Hence Box_1 represents $\{\{\}\}$ or $\{\emptyset\}$.

Counting the number of elements in a set is called the **cardinality** of the set.

Definition 1.7: Cardinality

The number of elements in a set.
Denoted over a set A as $|A|$.

For Example:

- $A = \{1, 2, 3\}$, $|A| = 3$.
- $B = \{a, e, i, o, u\}$, $|B| = 5$.
- \mathbb{Z} the set of all integers, $|\mathbb{Z}| = \infty$.

Questions:

What are the cardinalities of the following sets?

1. $|\{1, 2, 3\}|$
2. $|\emptyset|$
3. $|\{\}|$
4. $|\{\emptyset\}|$
5. $|\{1, \{2, 3\}\}|$
6. $|\{1, 2, 2, 3, 3, 3\}|$

Try to think about the answer before looking at the solution.

Things stick when you struggle.

Tip: Whenever you approach a problem, always break things down into simple components. “What defines a set? What defines an element? What defines a subset? What defines cardinality?”

Answers:

1. 3
2. 0
3. 0
4. 1
5. 2
6. 3

Explicitly defining a set, say $\{1, 2, 3, \dots\}$, is called **set-roster notation**. **set-builder notation** enables us to create more complex definitions.

Definition 1.8: Set-Builder Notation

General form: $\{x \mid P(x)\}$,

- x = defines some variable.
- “ \mid ” = is short hand for “such that.”
- $P(x)$ = describes the properties x must satisfy.

For Example: Lets Define the set of even integers

- $\{x \mid x \text{ is an even integer}\}$: “ x , such that, x is an even integer.”
- $\{x \in \mathbb{Z} \mid x \text{ is even}\}$: “ x in Integers, such that, x is an even.”
- $\{x \in \mathbb{Z} \mid x \text{ is not odd}\}$: “ x in Integers, such that, x is not odd.”

It’s important to Define exactly what variables are. In the above, x was stated directly as an integer. If not, x could be water-balloons or puppies.

1.2 Set Operations

Combining the two sets, $\{1, 2, 3\}$ and $\{a, b, c\}$, produce the set $\{1, 2, 3, a, b, c\}$, which is called the **union**.

Definition 1.9: Union

The set of elements that appear in either set A or set B is the union.
Denoted: $A \cup B$.

This is also known as a **disjunction**, which is a fancy term for the word “OR”.

For Example:

- $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$.
- $\{1, 2\} \cup \emptyset = \{1, 2\}$. There is nothing to add.
- $\{1\} \cup \{\emptyset\} = \{1, \emptyset\}$. The \emptyset is an element in this case.

The common elements of the two sets, $\{1, 2, 3\}$ and $\{2, 3, 4\}$, produce the set $\{2, 3\}$, the **intersection**.

Definition 1.10: Intersection

The set of elements that appear in both sets A and B is the intersection.
Denoted: $A \cap B$.

This is also known as a **conjunction**, which is a fancy term for the word “AND”.

For Example:

- $\{1, 2\} \cap \{2, 3\} = \{2\}$.
- $\{1\} \cap \{2\} = \emptyset$. There is nothing in common.
- $\{1\} \cap \emptyset = \emptyset$. There is nothing to compare.

Tip: To lessen the confusion between \cup and \cap , think, “ \cap ” for “AND”, since \cap looks like a curved “A” without the line.

The combination of $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ in order pairs are:

$(1, a), (1, b), (1, c),$

$(2, a), (2, b), (2, c),$

$(3, a), (3, b), (3, c)$

Putting the above objects in a set yields the **cartesian product** of A and B .

Definition 1.11: Cartesian Product

The set of all possible order pairs of elements from sets A and B .
Denoted: $A \times B$.

For Example:

- $\{1, 2\} \times \{a, b\} = \{(1, a), (1, b), (2, a), (2, b)\}$.
- $\{1, 2\} \times \emptyset = \emptyset$. There is nothing to pair.
- $\{1\} \times \{\emptyset\} = \{(1, \emptyset)\}$. The \emptyset is an element in this case.

Note: Visit ‘**Figure 2**’ in the previous section if \emptyset causes confusion.

We have sets $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, to remove the common elements A has with B , i.e., take all in A that is not in B , yields the set $\{1\}$, the **difference**.

Definition 1.12: Difference

The set of all elements that are in set A but not in set B .
Denoted: $A - B$.

For Example:

- $\{1, 2\} - \{2, 3\} = \{1\}$.
- $\{1\} - \{1\} = \emptyset$.
- $\{1, 2\} - \emptyset = \{1, 2\}$. There is nothing to remove.

Summary: A **set** is a collection of ‘things’ or objects. An object that is a member of a set is an **element** $4 \in \mathbb{Z}$. In a set **order and repetition do not matter**. Sets can contain other sets, these subsections/slices/portions are called **subsets**. An **empty set** is denoted \emptyset or $\{\}$. **Cardinality** is the element count of a set, it does not count sets beyond the top layer. $\{1, 2, 3\}$ has a cardinality of 2, not 3. **Set-roster** notation is the explicit listing of a set like $\{\dots, 1, 2, 3, \dots\}$. **Set-builder** notation is the description of a set $\{x \in \mathbb{Z} | x \text{ is even}\}$. Combining two sets is called a **union**, the **intersection** is the common elements between two sets. The **cartesian product** of two sets is the set of all possible ordered pairs. The **difference** between sets A and B are elements in A that are not in B . There are many more operations that we could discuss, but we will just stick to these for now.

Comments: In or out of math, we talk about sets. In conversations, “yeah that thing!” “sorry, what thing?” Those people: are we talking about the group or a sub-group of those people. Will we make generalizations? Will we be specific? To draw connections, $\text{Humans} \in \text{Earth} \in \text{Universe}$. Organizing files, sub-folder to sub-folder. Everything is a set of something.

2 Functions

2.1 Introduction to Functions

To talk about functions, is to talk about **relationships**. Take the ' $<$ ' sign, this is a relationship. $x < y$ means x relates to y , such that x is less than y .

Let $a = \{1, 2, 3, 4\}$, $b = \{0, 1, 2, 3\}$. Let R be the relation ' $<$ ', aRb yields:

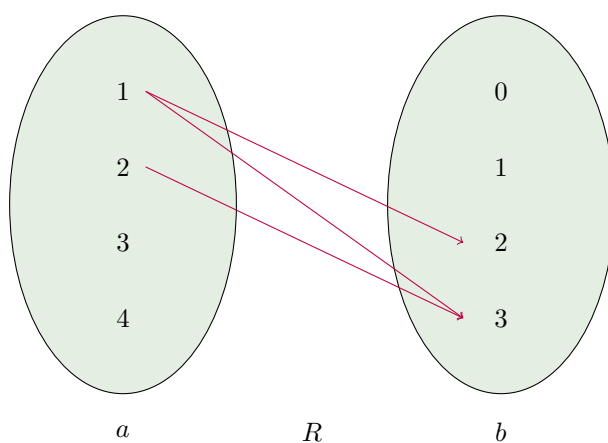


Figure 3: R produces ordered pairs $\{(1, 2), (1, 3), (2, 3)\}$, as $1 < 2$, $1 < 3$, $2 < 3$

Definition 2.1: Relation

A relation R on sets a and b , aRb , is a subset of $A \times B$.

Since $A \times B$ is the set of all possible ordered pairs. R 's pairings must contain some or all pairing of $A \times B$. This includes no pairing at all, the emptyset.

The arrows in **Figure 5** are often referred to as mappings. 1 maps to 2, 1 maps to 3, and 2 maps to 3.

Tip: When looking at Definitions, come up with examples on your own. The ability to explain a concept to someone else proves understanding.

Functions are a type of relation, where each input has exactly one output. We can visualize this as a machine:

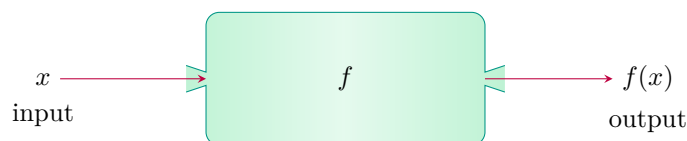


Figure 4: The machine f takes an input x and produces an output $f(x)$.

In our previous example we used ' $<$ ' as a relation. This is not a function, as 1 relates to 2 and 3. Instead let's use the absolute function, $f(x) = |x|$.

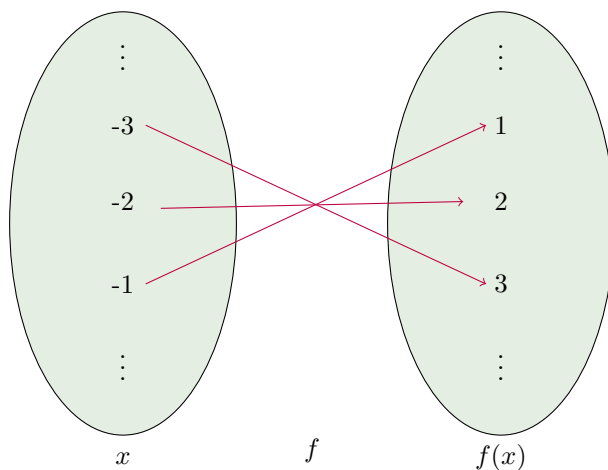


Figure 5: Pairs $\{(-3, 3), (-2, 2), (-1, 1)\}$, as $f(-3) = 3$, $f(-2) = 2$, $f(-1) = 1$

Note: $f(x)=|x|$ is where x hasn't been chosen yet, choose 3, $f(-3)=|-3|=3$.

In **Figure 5**, assume inputs are integers. The absolute function maps to integers, i.e., if I put in an integer, I get out an integer. In the figure we represent this by the two green ovals; The left (inputs), the right (outputs).

Our inputs are called the **domain**, the outputs the **codomain**, and all possible mappings the **range**.

Definition 2.2: Function

A function f is a relation between two sets, A (the domain) and B (the codomain), such that each element in A is mapped to exactly one element in B . The set of all possible outputs of f is called the range of f .

Say we create a function $f(x)$, which tells us if someone prefers cats or dogs.

Let $P = \{\text{"Lois", "Pete", "Stuart"}\}$, $Q = \{\text{"Cats", "Dogs"}\}$.

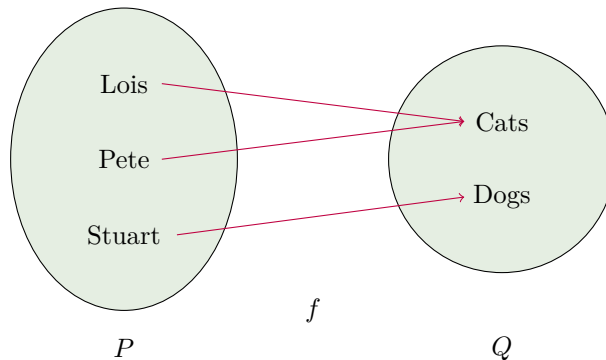


Figure 6: Produced ordered pairs $\{(\text{Lois, Cats}), (\text{Pete, Cats}), (\text{Stuart, Dogs})\}$.

In function f , all elements in our domain map to an element in our codomain. This makes our function **onto**, or **surjective**.

Definition 2.3: Onto (Surjective)

A function f is onto if every element in the codomain is mapped to by some element in the domain.

Comments: Unrelated, take a break, stretch, get water. We sit down way too much. Taking a break doesn't mean consuming media, it means sitting with yourself, taking a walk. Work smart not hard, that means taking care of yourself. Having dedicated study time for x hours of your day with frequent breaks is great. Dedicated the entire day or all your free-time, not so much, you'll burnout in that very session or the next. Consistent small chunks always outpaces the cram, and it's a lot less stressful too.

Now say we have a function $g(x)$, which tells us a persons student ID.

Let $T = \{\text{"Raven"}, \text{"Stella"}, \text{"Robbert"}\}$, $U = \{\text{"U001"}, \text{"U002"}, \text{U003}, \dots\}$.

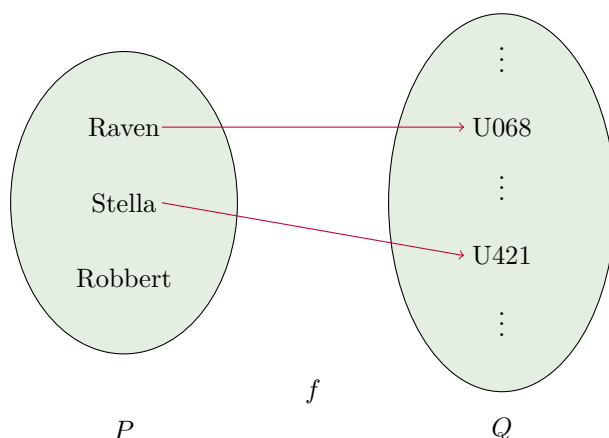


Figure 7: Produced ordered pairs $\{(\text{Raven}, \text{U068}), (\text{Stella}, \text{U421})\}$. not all elements have a mapping, meaning it is **not onto**.

Each element in the domain maps to exactly one element in the codomain.
This makes our function **one-to-one**, or **injective**.

In our example this suggests that "Robbert" is not a registered student.

Definition 2.4: One-to-One (Injective)

A function f is one-to-one if each element in the domain is mapped to exactly one element in the codomain.

If we were to change the function g , restricting the domain to only include registered students, then our function would be **onto** and **one-to-one**. Since each student has a unique ID, and each ID belongs to a student.

When a function is both onto and one-to-one, it is called a **bijection**.

Definition 2.5: Bijection

A function f is a bijection if it is both onto and one-to-one.