P58、

3、~~解~~：证明：对于 x_{k+1}，有

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right) \geq \frac{1}{2} \cdot 2\sqrt{x_k \cdot \frac{a}{x_k}} = \sqrt{a}$$

当且仅当 $x_k^2 = a$ 时成立

∴对一切 $k = 1, 2, \cdots$，$x_k \geq \sqrt{a}$ 成立

对于 $\frac{x_{k+1}}{x_k}$，有：$\frac{x_{k+1}}{x_k} = \frac{\frac{1}{2}\left(x_k + \frac{a}{x_k}\right)}{x_k} = \frac{1}{2}\left(1 + \frac{a}{x_k^2}\right)$

∵ $x_k \geq \sqrt{a} \Rightarrow \frac{a}{x_k^2} \leq 1$

∴ $\frac{x_{k+1}}{x_k} = \frac{1}{2}\left(1 + \frac{a}{x_k^2}\right) \leq \frac{1}{2}(1+1) = 1$

∵不存在 $x_k^2 = a$ 与 $x_{k+1}^2 = a$ 同时成立

∴ $\frac{x_{k+1}}{x_k} < 1$　　∴ $x_1, x_2 \cdots$ 是递减的

4、解：由已知，得

$$x_{k+1} = x_k - \frac{x_k^3 - a}{3x_k^2} = \frac{2}{3}x_k + \frac{a}{3x_k^2}, \quad k = 0, 1 \cdots$$

令 $\varphi(x) = \frac{2}{3}x + \frac{a}{3x^2}$

则 $\varphi'(x) = \frac{2}{3} - \frac{2a}{3x^3}$　∴ $\varphi'(x^*) = 0$

$\varphi''(x) = \frac{2a}{x^4}$　　$\varphi''(x^*) = \frac{2}{\sqrt[3]{a}} \neq 0$

∴该迭代法 2 阶收敛

7、解：证明：不妨设 $f(x^*)=0$，$f'(x^*)\neq 0$

则将 $f(x)$ 在 x_n 处作 2 阶泰勒展开，并代入 x^*

$$0=f(x^*)=f(x_n)+f'(x_n)(x^*-x_n)+\frac{f''(\xi_n)}{2!}(x^*-x_n)^2$$

等式两边除以 $f'(x_n)$ ∴ $x^*=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{f''(x_n)}{2f'(x_n)}(x^*-x_n)^2$

由牛顿法，得 $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$

$$\therefore\ x^*=x_{n+1}-\frac{f''(\xi_n)}{2f'(x_n)}(x^*-x_n)^2$$

$$\therefore\ \frac{x_{n+1}-x^*}{(x^*-x_n)^2}=\frac{f''(\xi_n)}{\cancel{2f'(\xi_n)}\,2f'(x_n)}$$ 即 $\frac{e(x_{n+1})}{e(x_n)^2}=\frac{f''(\xi)}{2f'(\xi)}$

$$\therefore\ e(x_{n+1})=\frac{f''(\xi)}{2f'(\xi)}e(x_n)^2$$ 证毕

9、解：(1) 由已知，得

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}=\frac{2x_k^3+1}{3x_k^2-3}$$

将初始值代入公式，得下表

k	0	1	2	3	4
x_k	~~2.0000~~ ~~2.000~~	~~1.8888~~	~~1.8794~~	~~1.8793~~	~~1.8793~~
	2.000	~~1.888~~ 1.889	1.879	1.879	1.879

∴ 取 $x^*\approx 1.879$

(2) 由已知，得

$$x_{k+1}=x_k-\frac{f(x_k)}{f(x_k)-f(x_{k-1})}(x_k-x_{k-1})$$

将 $x_0=2$，$x_1=1.9$ 代入，得下表

k	1	2	3	4	5
x_k	2	1.881	1.879	1.879	1.879
x_{k-1}	1.9	2	1.881	1.879	1.879

∴ 取 $x^*\approx 1.879$