P236. 19. 解：$R_3(x_j)=f(x_j)-P(x_j)=0$

$R_3'(x_j)=f'(x_j)-P'(x_j)=0 \quad j=k,k+1$

$\therefore x_k, x_{k+1}$ 均为 $R_3(x)$ 的二重零点，因而有形式

$$R_3(x)=K(x)(x-x_k)^2(x-x_{k+1})^2$$

设 $\varphi(t)=f(t)-p(t)-K(x)(t-x_k)^2(t-x_{k+1})^2$

则 $\varphi(x_k)=0$，$\varphi(x)=0$，$\varphi'(x_k)=0$ $\varphi(x_{k+1})=0$ $\varphi'(x_{k+1})=0$

由罗尔定理，得，存在 $\xi_1\in(x_k,x)$，$\xi_2\in(x,x_{k+1})$ 有：

$$\varphi'(\xi_1)=0 \quad \varphi'(\xi_2)=0$$

同理使用3次罗尔定理有：$\xi\in(\xi_1,\xi_2)\subset(x_1,x_2)$ 使

$\varphi^{(4)}(\xi)=0$，又 $\varphi^{(4)}(t)=f^{(4)}(t)-4!\,K(x)$

$\therefore K(x)=\dfrac{f^{(4)}(\xi)}{4!}$

$\therefore R_3(x)=\dfrac{f^{(4)}(\xi)(x-x_k)^2(x-x_{k+1})^2}{4!} \quad \xi\in(x_k,x_{k+1})$

20. 解：由 $P(0)=0$，$P(1)=1$，$P(2)=1$ 构造两次的插值多项式 $P_2(x)$，有

$P_2(x)=-\frac{1}{2}x^2+\frac{3}{2}x$

设 $P_4(x)=P_2(x)+(Ax+B)(x-0)(x-1)(x-2)$

$\because P_4'(0)=1 \quad P_4'(1)=1 \Rightarrow \begin{cases}\frac{3}{2}+2B=0\\ \frac{1}{2}-(A+B)=1\end{cases}$

$\therefore A=\frac{1}{4} \quad B=-\frac{3}{4}$

$\therefore P_4(x)=-\frac{1}{2}x^2+\frac{3}{2}x+(\frac{1}{4}x-\frac{3}{4})(x-0)(x-1)(x-2)$ 即为所求

23. 解：$f'(x)=4x^3$，则 $I_h(x)$ 在每个小区间 $[x_k,x_{k+1}]$ 上表示为

$$I_h(x)=\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)x_k^4+\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2\left(1+2\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)x_{k+1}^4$$
$$+4\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2(x-x_k)x_k^3+4\left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2(x-x_{k+1})x_{k+1}^3$$

$$|R(x)|=\left|\frac{f^{(4)}(\xi)(x-x_k)^2(x-x_{k+1})^2}{4!}\right|\le\frac{4!\cdot\left(\frac{x_{k+1}+x_k}{2}-x_k\right)^2\left(\frac{x_{k+1}+x_k}{2}-x_{k+1}\right)^2}{4!}=\frac{h^4}{16}$$

24. 解：由已知，得

$$\cancel{M_0 = S''(27.7) = 3 \quad M_1 = S''(30) = -4}$$

则 $S''(x) = \underset{M_j}{\cancel{3}} \left(\dfrac{x-x_{j+1}}{x_{j+1}-x_j}\right) + \underset{M_{j+1}}{\cancel{(-4)}} \left(\dfrac{x-x_j}{x_{j+1}-x_j}\right)$

$\therefore S(x) = -\dfrac{\cancel{3}M_j}{6h_j}(x-x_{j+1})^3 + \dfrac{\cancel{-4}M_{j+1}}{6h_j}(x-x_j)^3 + \cancel{a_j x}\, a_j x + b_j$ ①

$\cancel{= -\dfrac{1}{2h_j}(x-x_{j+1})^3 - \dfrac{2}{3h_j}(x-x_j)^3 + a_j x + b_j}$ 其中 $h_j = x_{j+1} - x_j$

由表可知，得 $\longrightarrow$ 其中 $a_j = \dfrac{f_{j+1}-f_j}{h_j} - \dfrac{M_{j+1}-M_j}{6} h_j$ ②

$b_j = \dfrac{f_j x_{j+1} - f_{j+1} x_j}{h_j} + \dfrac{M_{j+1} x_j - M_j x_{j+1}}{6} h_j$ ③

$x_0 = 27.7 \quad x_1 = 28 \quad x_2 = 29 \quad x_3 = 30$

$f_0 = 4.1 \quad f_1 = 4.3 \quad f_2 = 4.1 \quad f_3 = 3$

$h_0 = 0.3 \quad h_1 = 1 \quad h_2 = 1 \quad \cancel{h_3 = 1}$

$\mu_1 = \dfrac{3}{13} \quad \mu_2 = \dfrac{1}{2} \quad \lambda_1 = \dfrac{10}{13} \quad \lambda_2 = \dfrac{1}{2}$

$d_1 = 6\left[\dfrac{4.1}{0.3(0.3+1)} + \dfrac{4.1}{1(0.3+1)} - \dfrac{4.3}{0.3\times 1}\right]$ $\quad \cancel{d_3 = 6\left[\dfrac{4.1}{\ } + \right.}$

$d_2 = 6\left[\dfrac{4.3}{1(1+1)} + \dfrac{3}{1(1+1)} - \dfrac{4.1}{1\times 1}\right]$

其中，$\lambda_0 = \mu_3 = 1 \quad d_0 = \dfrac{40}{3} \quad d_3 = 6.6$

则 $\begin{bmatrix} 2 & \lambda_0 & & \\ \mu_1 & 2 & \lambda_1 & \\ & \mu_2 & 2 & \lambda_2 \\ & & \mu_3 & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix}$ 解该方程组解得 M_0, M_1, M_2, M_3 代入①②③ 即为所求