1、解：由已知，得 $y=\frac{1}{a+bx} \Rightarrow \frac{1}{y}=\cancel{\frac{1}{}}\ a+bx$

取 $y'=\frac{1}{y}$，则有如下：

x	1.0	1.4	1.8	2.2	2.6
y	0.931	0.473	0.297	0.224	0.168
y'	1.074	2.114	3.367	4.464	5.952

∴需要拟合的函数形式为 $y'=a+bx$

则有 $A=\begin{bmatrix}1 & 1.0\\ 1 & 1.4\\ 1 & 1.8\\ 1 & 2.2\\ 1 & 2.6\end{bmatrix}$ $\vec{f}=\begin{bmatrix}1.074\\ 2.114\\ 3.367\\ 4.464\\ 5.952\end{bmatrix}$

列法方程 $A^TA\vec{x}=A^T\vec{f}$ 为：

$$\begin{bmatrix}5 & 9\\ 9 & \cancel{17}\ 17.8\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}16.971\\ 35.3902\end{bmatrix}$$

解得 $a=\cancel{2.1643}$ $b=2.1643$

$=-0.5016$

$\therefore y=\frac{1}{2.1643x-0.5016}$ 为数据的拟合曲线

二、解：作二次多项式 $N(x)$，满足

$N(a)=f(a)$ $N'(a)=f'(a)$ $N''(a)=f''(a)$

$N(x)=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2$

设 $H(x)=N(x)+h(x) \Rightarrow h(x)=H(x)-N(x)$

易知 $h(x)$ 为一个三次多项式，且有

$h(a)=0$ $h'(a)=0$ $h''(a)=0$

$\therefore h(x)=A(x-a)^3$

$\therefore H(x)=N(x)+A(x-a)^3=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+A(x-a)^3$

求二阶导数，得 $H''(x)=f''(a)+6A(x-a)$

$\because H''(b)=f''(b)$ $\therefore H''(b)=f''(a)+6A(b-a)=f''(b)$

$\therefore A=\dfrac{f''(b)-f''(a)}{6(b-a)}$

$\therefore H(x)=f(a)+f'(a)(x-a)+\frac{1}{2}f''(a)(x-a)^2+\dfrac{f''(b)-f''(a)}{6(b-a)}(x-a)^3$ 即为所求

三、解：(1) 由已知，得 $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}=\begin{bmatrix} 15 & -3 & 2 \\ 1 & -1 & 8 \\ 2 & -3 & 20 \end{bmatrix}$ $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}=\begin{bmatrix} 4 \\ 1 \\ -7 \end{bmatrix}$

$\therefore$ 雅可比迭代格式为：

$$\begin{cases} x_1^{(k+1)}=-\frac{1}{15}(-3x_2^{(k)}+2x_3^{(k)})+\frac{4}{15} \\ x_2^{(k+1)}=-\frac{1}{-1}(x_1^{(k)}+8x_3^{(k)})+\frac{1}{-1} \\ x_3^{(k+1)}=-\frac{1}{20}(2x_1^{(k)}-3x_2^{(k)})+\frac{-7}{20} \end{cases}$$

$A=D-L-U$

其中，$D=\begin{bmatrix} 15 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 20 \end{bmatrix}$ $L=\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & 3 & 0 \end{bmatrix}$ $U=\begin{bmatrix} 0 & 3 & -2 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}$

$$x^{(k+1)}=D^{-1}(L+U)x^{(k)}+D^{-1}b$$

Gauss-Seidel 格式为：

$$\begin{cases} x_1^{(k+1)}=-\frac{1}{15}(-3x_2^{(k)}+2x_3^{(k)})+\frac{4}{15} \\ x_2^{(k+1)}=-\frac{1}{-1}(x_1^{(k+1)}+8x_3^{(k)})+\frac{1}{-1} \\ x_3^{(k+1)}=-\frac{1}{20}(2x_1^{(k+1)}-3x_2^{(k+1)})+\frac{-7}{20} \end{cases}$$

$$x^{(k+1)}=(D-L)^{-1}Ux^{(k)}+(D-L)^{-1}b$$

(2) 由(1)知，$D=\begin{bmatrix} 15 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 20 \end{bmatrix}$ $L=\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & 3 & 0 \end{bmatrix}$ $U=\begin{bmatrix} 0 & 3 & -2 \\ 0 & 0 & -8 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore D-L=\begin{bmatrix} 15 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -3 & 20 \end{bmatrix}$ $(D-L)^{-1}=\begin{bmatrix} \frac{1}{15} & 0 & 0 \\ \frac{1}{15} & -1 & 0 \\ \frac{1}{300} & -\frac{3}{20} & \frac{1}{20} \end{bmatrix}$

$\therefore B=(D-L)^{-1}U=\begin{bmatrix} 0 & \frac{1}{5} & -\frac{2}{15} \\ 0 & \frac{1}{5} & \frac{118}{15} \\ 0 & \frac{1}{100} & \frac{179}{150} \end{bmatrix}$

令 $\det(\lambda I - B) = 0$，则

$$\begin{vmatrix} \lambda & -\frac{1}{5} & \frac{2}{15} \\ 0 & \lambda-\frac{1}{5} & -\frac{118}{75} \\ 0 & -\frac{1}{100} & \lambda-\frac{179}{150} \end{vmatrix} = 0 \Rightarrow \lambda\left[(\lambda-\frac{1}{5})(\lambda-\frac{179}{150}) - (-\frac{1}{100})(-\frac{118}{150})\right] = 0$$

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$$\Leftrightarrow \lambda(\lambda^2 - 209\lambda + 34.62) = 0$$

解得 $\lambda_1 = 0 \quad \lambda_2 \approx 1.2 \quad \lambda_3 \approx 0.193$

$\therefore \rho(B) > 1$

$\therefore$ Gauss-Seidel 迭代格式不收敛

4、解：(1) 由已知，得

$$\int_0^1 f(x)\,dx = \int_0^1 x^4\,dx = \frac{1}{5}x^5\Big|_0^1 = \frac{1}{5}$$

$$\therefore [1][x_1] = \left[\frac{1}{5}\right] \Rightarrow x_1 = \frac{1}{5}$$

$\therefore f(x)$ 的 0 次最佳平方逼近多项式为 $S_0^*(x) = 0.2$

(2) 同 (1)，有

$$\int_0^1 x f(x)\,dx = \int_0^1 x^5\,dx = \frac{1}{6}$$

$$\therefore \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{6} \end{bmatrix} \Rightarrow x_1 = -\frac{1}{5} \qquad x_2 = \frac{4}{5}$$

$\therefore S_1^*(x) = -0.2 + 0.8x$ 即为所求

(3) 同 (1)、(2)，有

$$\int_0^1 x^2 f(x)\,dx = \int_0^1 x^7\,dx = \frac{1}{7}$$

$$\therefore \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{1}{6} \\ \frac{1}{7} \end{bmatrix} \Rightarrow x_1 = \frac{8}{35} \quad x_2 = -\frac{32}{35} \quad x_3 = \frac{12}{7}$$

$\therefore S_2^*(x) = \frac{8}{35} - \frac{32}{35}x + \frac{12}{7}x^2$ 即为所求