1、(1) 证明：由已知，得

$$y_{n+1} = y_n + h\left(\frac{1}{4}k_1 + \frac{3}{4}k_2\right) = y_n + \frac{1}{4}h\lambda y_n + \frac{3}{4}h\lambda\left(y_n + \frac{2}{3}h\lambda y_n\right)$$

这里 $f(x,y) = \lambda y$

基于 $y_n = y(x_n)$ 的假设，有：

$$y_{n+1} = y(x_n)\left[1 + \left(\frac{1}{4}+\frac{3}{4}\right)h\lambda + \frac{3}{4}\times\frac{2}{3}(h\lambda)^2\right]$$

$$= y(x_n)\left[1 + h\lambda + \frac{1}{2}(h\lambda)^2\right] \quad ①$$

由泰勒展开式，$y(x_{n+1})$ 在 $x = x_n$ 处展开，有：

$$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{3!}y'''(x_n)\cdots$$

~~②-①，得~~ $= y(x_n)\left[1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{3!} + \cdots\right] \quad ②$

②-①，得局部截断误差，有：

$$l_{n+1} = y(x_{n+1}) - y_{n+1} = y(x_n)\left[\frac{(h\lambda)^3}{3!} + \frac{(h\lambda)^4}{4!} + \cdots\right]$$

即 $l_{n+1} = O(h^3)$ ∴这里一个2阶方法

(2) 由已知，得，

$$\begin{cases} y_{n+1} = y_n + \frac{h}{4}(k_1 + 3k_2) \\ k_1 = x_n^2 + y_n^2 \\ k_2 = \left(x_n + \frac{2}{3}h\right)^2 + \left(y_n + \frac{2}{3}hk_1\right)^2 = \left(x_n + \frac{2}{30}\right)^2 + \left(y_n + \frac{2}{30}k_1\right)^2 \end{cases}$$

~~又~~ 且有 $y_0 = 0$，$x_0 = 0$，$h = 0.1$

则 $k_1 = 0$，$k_2 = \cancel{\frac{2}{225}}\ \frac{1}{225}$

$$y_1 = y_0 + \frac{0.1}{4}\left(0 + 3\times\frac{1}{225}\right) = \frac{1}{3000} \approx 3.33\times 10^{-4}$$

此即所求

2.解：(1) $\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f[\frac{1}{2}(a+b)+\frac{1}{2}(\cancel{a-b}\ b-a)t]dt$ ①

其中，$x=\frac{1}{2}(a+b)+\frac{1}{2}(b-a)t$

∴由①得

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{n} A_i f(\frac{b-a}{2}t_i+\frac{a+b}{2})$$

∴ $n=2$ 时，有

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\sum_{i=1}^{2} A_i f(\frac{b-a}{2}t_i+\frac{a+b}{2})$$

其中，$A_i=\int_{-1}^{1}\frac{P_{n+1}(x)}{(x-x_i)P'_{n+1}(x_i)}dx = \frac{2}{(1-x_i^2)[P'_{n+1}(x_i)]^2}$

$$P_{n+1}(x)=\frac{1}{2^{n+1}(n+1)!}\frac{d^{n+1}}{dx^{n+1}}(x^2-1)^{n+1}$$

(2) 由已知，

$$I=\int_3^6 e^{-x}dx$$

作变量替换，$x=\frac{1}{2}(3+6)+\frac{1}{2}(6-3)t=\frac{3}{2}t+\frac{9}{2}$，则有

$$I=\int_3^6 e^{-x}dx=\int_{-1}^{1} e^{-(\frac{3}{2}t+\frac{9}{2})}dt$$

令 $f(t)=e^{-(\frac{3}{2}t+\frac{9}{2})}$，对 $n=2$，有

$$I\approx\frac{5}{9}f(-\sqrt{\tfrac{3}{5}})+\frac{8}{9}f(0)+\frac{5}{9}f(\sqrt{\tfrac{3}{5}})\approx 0.0315303$$

3.解：平面旋转变换：

① 先消去 $x_2=3$ 的旋转变换

$\cos\theta=\frac{x_1}{\sqrt{x_1^2+x_2^2}}=\frac{2}{\sqrt{13}}$ $\sin\theta=\frac{x_2}{\sqrt{x_1^2+x_2^2}}=\frac{3}{\sqrt{13}}$ ∴ $P_{12}=\begin{bmatrix}\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 & 0\\ -\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$

∴ $P_{12}x=(\sqrt{13},0,0,5)^T$，消去5的旋转变换

$\cos\theta=\frac{\sqrt{13}}{\sqrt{38}}$ $\sin\theta=\frac{5}{\sqrt{38}}$ $P_{14}=\begin{bmatrix}\frac{\sqrt{13}}{\sqrt{38}} & 0 & 0 & \frac{5}{\sqrt{38}}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ -\frac{5}{\sqrt{38}} & 0 & 0 & \frac{\sqrt{13}}{\sqrt{38}}\end{bmatrix}$

∴ $P_{14}P_{12}x=\sqrt{38}e_1$

所用变换为

$$T=P_{14}P_{12}=\begin{bmatrix}\frac{2}{\sqrt{38}} & \frac{3}{\sqrt{38}} & 0 & \frac{5}{\sqrt{38}}\\ -\frac{3}{\sqrt{38}} & \frac{2}{\sqrt{13}} & 0 & 0\\ 0 & 0 & 1 & 0\\ -\frac{10}{\sqrt{494}} & -\frac{15}{\sqrt{494}} & 0 & \frac{13}{\sqrt{494}}\end{bmatrix}$$

②用反射变换，由公式，得

$$\sigma = \mathrm{sign}(x_1)\,\|x\|_2 = \sqrt{38}$$

$$\mu = x + \sigma e_1 = (2+\sqrt{38}, 3, 0, 5)^T$$

$$\beta = \frac{1}{2}\|\mu\|_2^2 = 38 + 2\sqrt{38}$$