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Assignment 4

Due: Wednesday, November 1st, 2017, upload before 11:59pm

1) (50 pts.) Solve the recurrences:

a) $a_n = 4a_{n-1}$

$a_0=1$

$\forall n > 1$

$a_n = 4^1 \, a_{n-1}$

Substitute $= 4^1 \, [4 \, a_{n-2}]$

Simplify $= 4^2 \cdot a_{n-2}$

Substitute $= 4^2 \, [4 \, a_{n-3}]$

Simplify $= 4^3 \cdot a_{n-3}$

Etc…

$= 4^k \cdot a_{n-k}$

$k = n$

$4^n \cdot a_{n-n} = 4^n \cdot a_0 = 4^n \cdot 1$

$a_n = 4^n$ $\qquad\qquad$ $\forall n > 1$

Prove true for n = 1

$a_1 = 4^1 = 4$ $\qquad\qquad$ $a_1 = 4a_{1-1} = 4a_0 = 4(1) = 4$

True for n = 1

Assume true for n = k

$a_k = 4^k$

Consider n = k+1

$a_{k+1} = 4a_k$

$= 4(4^k)$

$= 4^{k+1}$

If true for n = k, then also true for n = k+1

As true for n = 1, true for all positive numbers

b) $a_n = ca_{n-1}$
$a_0=1$
$\forall n > 1$
c is some constant greater than 1

$a_n = c^1\, a_{n-1}$
Substitute $= c^1\, [c\, a_{n-2}]$
Simplify $= c^2 \cdot a_{n-2}$
Substitute $= c^2\, [c\, a_{n-3}]$
Simplify $= c^3 \cdot a_{n-3}$
Etc…
$= c^k \cdot a_{n-k}$
$k = n$
$c^n \cdot a_{n-n} = c^n \cdot a_0 = c^n \cdot 1$

$a_n = c^n$ $\qquad \forall n > 1$

Prove true for n = 1
$a_1 = c^1 = c$ $\qquad a_1 = ca_{1-1} = ca_0 = c(1) = c$
True for n = 1

Assume true for n = k
$a_k = c^k$

Consider n = k+1
$a_{k+1} = ca_k$
$= c(c^k)$
$= c^{k+1}$

If true for n = k, then also true for n = k+1
As true for n = 1, true for all positive numbers

c) $b_n = b_{n-1} + n/2$
$b_0=0$
$\forall n > 1$

Substitute $= [b_{n-2} + (n-1)/2] + n/2$
Simplify $= b_{n-2} + (n-1)/2 + n/2$
Substitute $= [b_{n-3} + (n-2)/2] + (n-1)/2 + n/2$
Simplify $= b_{n-3} + (n-2)/2 + (n-1)/2 + n/2$
ETC…
$= [b_{n-(k+1)} + (n-k)/2] + \ldots + (n-2)/2 + (n-1)/2 + n/2$
$= b_{n-(k+1)} + (n-k)/2 + \ldots + (n-2)/2 + (n-1)/2 + n/2$

Let $n-(k+1) = 0$
$n-k-1 = 0$
$n = k+1$

So $n-(k+1) = 0$
$n-k = 1$
$n-(k-1) = 2$ etc…

So we get
$b_0 + 1 + 2 + 3 + \ldots. (n-1) + n$

$= b_0 + n(n+1)/4$
$= 0 + n(n+1)/4$
$= n(n+1)/4$
$= (n^2+n)/4 \qquad \forall n > 1$

Prove true for n = 1
$b_1 = (1^2+1)/4 = (2)/4 = 1/2 \qquad b_1 = b_{1-1} + ½ = b_0 + ½ = 0 + ½ = 1/2$
True for n = 1

Assume true for n = k
$b_k = (k^2+k)/4$

Consider n = k+1
$b_{k+1} = b_k + (k+1)/2$
$= (k^2+k)/4 + (k+1)/2$
$= (k^2+3k+2)/4$

If true for n = k, then also true for n = k+1
As true for n = 1, true for all positive numbers

d) $c_n = 4c_{n-1} + n$
$c_0=0$
$\forall n > 1$

Substitute $= 4[4c_{n-2} + (n-1)] + n$
Simplify $= 4^2c_{n-2} + 4(n-1) + n$
Substitute $= 4^2[4c_{n-3} + (n-2)] + 4(n-1) + n$
Simplify $= 4^3c_{n-3} + 4(n-2) + 4(n-1) + n$
ETC…

$= 4^k[4c_{n-(k+1)} + (n-k)] + \ldots + 4(n-2) + 4(n-1) + n$
$= 4^{k+1}c_{n-(k+1)} + (n-k) + \ldots + 4(n-2) + 4(n-1) + n$

Let $n-(k+1) = 0$
$n-k-1 = 0$
$n = k+1$

So $n-(k+1) = 0$
$n-k = 1$
$n-(k-1) = 2$ etc…

So we get
$c_0 + 1 + 2+ 3 + \ldots. (n-1) + n$
$= (4^{n+1} - 3n - 4)/9$

Prove true for n = 1
$c_1 = (4^{1+1} -3(1) -4)/9 = (16 – 3 – 4)/9 = 9/9 = 1$ $c_1 = 4c_{1-1} + 1 = 4c_0 + 1 = 4(0) + 1 = 1$
True for n = 1

Assume true for n = k
$c_k = (4^{k+1} – 3k - 4)/9$

Consider n = k+1
$c_{k+1} = 4c_k + (k+1)$
$= 4[(4^{k+1} – 3k – 4)/9] + (k+1)$
$= (4^{k+2} – 12k -16)/9 + (9k+9)/9$
$= (4^{k+2} – 3k -7)/9$

If true for n = k, then also true for n = k+1
As true for n = 1, true for all positive numbers

e) $t_n = 10t_{n-1} - 21t_{n-2}$
$t_0=0$
$t_1=1$
$\forall n \geq 2$

$t_n - 10t_{n-1} + 21t_{n-2} = 0$
$r^2 - 10r + 21 = 0$
$(r-3)(r-7) \qquad r = 3, 7$
$t_n = \alpha(3^n) + \beta(7^n)$
$t_0=0 = \alpha(3^0) + \beta(7^0) = \alpha+\beta$
$t_1=1 = \alpha(3^1) + \beta(7^1) = 3\alpha+7\beta$
$0 = -3\alpha-3\beta$
$1 = 3\alpha+7\beta$
+
$1 = 4\beta$
$\beta = 1/4$
$1 = 3\alpha + 7(1/4)$
$\alpha = -1/4$
$t_n = (-1/4)(3^n) + (1/4)(7^n)$

$= (7^n - 3^n)/4$

f) $t_n = -5t_{n-1} + 14t_{n-2} + 3^n$
$t_0 = 0$
$t_1 = 1$
$\forall n \geq 2$

$t_n + 5t_{n-1} - 14t_{n-2} = 0$
$r^2 - 5r - 14 = 0$
$(r+7)(r-2) = 0 \qquad r = -7, 2$

$t_n + 5t_{n-1} - 14t_{n-2} = 3^n$
$A3^n + 5A3^{n-1} - 14A3^{n-2} = 3^n$
$A + 5A3^{-1} - 14A3^{-2} = 1=$
$A + 5/3A - 14/9A = 1$
$9/9A + 15/9A - 14/9A = 1$
$10A = 1$
$A = 1/10$
Guess: $1/10(3^n)$

$\alpha(-7)^n + \beta(2)^n + 1/10(3^n)$
$t_0 = 0 = \alpha(-7)^0 + \beta(2)^0 + 1/10(3^0) = \alpha + \beta + 1/10$
$-1/10 = \alpha + \beta = 2/10 = -2\alpha - 2\beta$
$t_1 = 1 = \alpha(-7)^1 + \beta(2)^1 + 1/10(3^1) = -7\alpha + 2\beta + 3/10$
$7/10 = -7\alpha + 2\beta$

$9/10 = -9\alpha$
$9/10 = 9\alpha$
$\alpha = -1/10$
$-1/10 = -1/10 + \beta$
$\beta = 0$
$t_n = -1/10(-7)^n - 0(2)^n + 1/10(3^n)$
$= -(3^n - (-7)^n)/10$

$= ((-7)^n - (3)^n)/10$

g) $t_n = 3t_{n-1} + n + 3^n$
$t_0=0$
$\forall n \geq 1$

$t_n - 3t_{n-1} = 0$
$r - 3 = 0$
$r = 3$

$t_n - 3t_{n-1} = n + 3^n$
$(x-3)(x-1)^2(x-3) = 0$
$r = 1,1,3,3$

$t_n = \alpha_1 1^n + \alpha_2 n1^n + \beta_1 3^n + \beta_2 n3^n$

$t_0 = \alpha_1 1^0 + \alpha_2(0)1^0 + \beta_1 3^0 + \beta_2(0)3^0 \rightarrow \alpha_1 + \beta_1 = 0$
$t_1 = \alpha_1 1^1 + \alpha_2(1)1^1 + \beta_1 3^1 + \beta_2(1)3^1 \rightarrow \alpha_1 + \alpha_2 + 3\beta_1 + 3\beta_2 = 4$
$t_2 = \alpha_1 1^2 + \alpha_2(2)1^2 + \beta_1 3^2 + \beta_2(2)3^2 \rightarrow \alpha_1 + 2\alpha_2 + 9\beta_1 + 18\beta_2 = 23$
$t_3 = \alpha_1 1^3 + \alpha_2(3)1^3 + \beta_1 3^3 + \beta_2(3)3^3 \rightarrow \alpha_1 + 3\alpha_2 + 27\beta_1 + 81\beta_2 = 99$

Solve the system of equations by substituting variables and get:
$\alpha_1 = -3/4$, $\alpha_2 = -1/2$, $\beta_1 = 3/4$, $\beta_2 = 1$

$t_n = (-3/4)1^n + (-1/2)n1^n + (3/4)3^n + (1)n3^n$

$= (-3/4) + (-1/2)n + (3/4)3^n + n3^n$

2) (20 pts.) Express the following solutions in Big-O notation

a) $T(n) = 4T(\ n/2) + n$
$\forall n > 1$

$a = 4,\ b = 2,\ d = 1$
$\log_b a = 2$
$2 > 1$

$\Theta(n^{\log_2 4})$

b) $T(n)= 5T(\ n/2) + n^2$
$\forall n > 1$

$a = 5,\ b = 2.\ d = 2$
$\log_b a = 2.32193$
$2.32193 > 2$
$\Theta(n^{\log_2 5})$

The following are bonus questions each worth 10 points.
c) $T(n) =4T^2(n-1)$
$T(0)=1$
$\forall n \geq 1$

d) $T(n) =nT^2(n/3)$
$T(1)=6$
$\forall n > 1$

Note: Please remember to complete your solution using proof by induction whenever your solution involves a guess.

Programming Questions: the following questions (3 and 4) are programming questions.

3) (20 pts.) Solve the recurrence relation for the "**Tower of Hanoi**" problem. The problem statement is as follows:

Given a stack of n disks arranged from largest on the bottom to smallest on top placed on a rod, together with two empty rods (totally 3 rods), the towers of Hanoi puzzle asks for the minimum number of moves required to move the stack from one rod to another, where moves are allowed only if they place smaller disks on top of larger disks.

Solve the relation and Plot a graph for the various values of n disks. Give appropriate justifications for your relation.

The first step is to move the (n-1)-disk tower to the spare peg; this takes T(n-1) moves. Then move the nth disk, taking 1 move. And finally move the (n-1)-disk tower again, this time on top of the nth disk, taking T(n-1) moves. This gives us our recurrence relation,

$T(1) = 1$
$T(n) = 2T(n-1) + 1$
Substitute: $2[2T(n-2) +1] + 1$
Simplify: $2^2T(n-2) + 2 + 1$
Substitute: $2^2[2T(n-3) +1] + 2 + 1$
Simplify: $2^3T(n-3) + 4 + 2 + 1$
ETC…
$T(n) = 2^kT(n-k) + 2^k – 1$

Since n is finite
lim T(n) $k \to n = 2^n – 1$

Runtime is: $T(n) = O(2^n)$

4) (20 pts.) Derive a recurrence relation on Binary Search.

Solve the relation and Plot a graph for the various values of n. Give appropriate justifications for your relation.

$B(1) = 1$
$B(n) = B(n/2) + 1$
$B(n/2) = B(n/4) + 1$
$B(n/4) = B(n/8) + 1$
….
$B(4) = B(2) + 1$
$B(2) = B(1) + 1$

$B(n) + B(n/2) + B(n/4) + \ldots + B(2) = B(n/2) + B(n/4) + B(n/8) + \ldots B(2) + B(1) + (1+1+\ldots+1)$

$B(n) = B(1) + Log(n)$
$B(n) = 1 + Log(n)$

Run Time is: $B(n) = O(Log(n))$

Note: The run time should be calculated for Problems 3 and 4. The graph should be plotted with time on the X axis and value of n on the Y axis.