Sampling $\mathbf{u} \in \mathcal{U}\left([0,1]^d\right)$

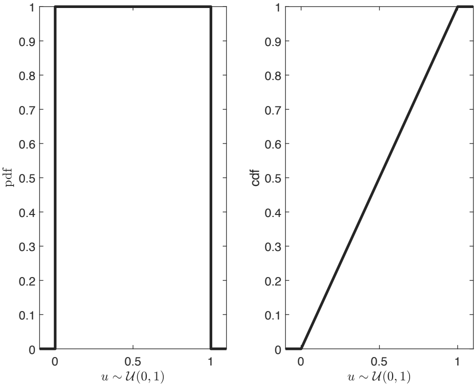
Definition: A random variable (RV) u uniformly distributed over $[0, 1]$ has a probability density function (pdf) defined as follows,

$$p_u(u) = \begin{cases} 1 & \text{if } u \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Its cumulative density function (cdf) is,

$$F_u(x) = \int_{-\infty}^{x} p_x(u)\mathrm{d}u = \int_{0}^{x} \mathrm{d}u = x$$

u is completely defined by p_u or F_u.



pdf
cdf
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
$u \sim \mathcal{U}(0,1)$
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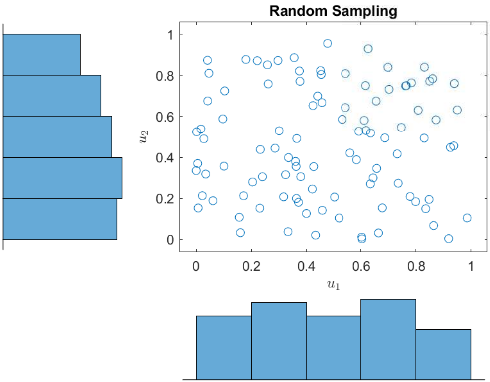
Most of the programming languages contain by default a pseudo-random generator for $x \sim \mathcal{U}([0,1])$. For example,
Matlab/Octave $->$ rand
R $->$ runif



Random Sampling
u_1
u_2
0
0.2
0.4
0.6
0.8
1

From $\mathbf{u} \in \mathcal{U}\left(]0,1[^d\right)$

to

$\mathbf{x} \sim p_{x_1} \times \cdots \times p_{x_d}$

The Integral Transform: Let $\mathbf{x} \sim p_{x_1} \times \cdots \times p_{x_d}$ be a random vector of **independent** RVs arbitrary distributed, and $F_{x_1}, \ldots, F_{x_d}$ be their associated cdfs.

Given a random vector $\mathbf{u} \in \mathcal{U}\left(]0,1[^d\right)$, it is straightforward to derive $\mathbf{x}$ with the integral transformation,

$$x_i = F_{x_i}^{-1}(u_i) \tag{1}$$

with $i = 1, \ldots, d$.

Given a sample $\mathbf{U}$ of $\mathbf{u}$ Eq.(1) allows to generate a sample $\mathbf{X}$ of $\mathbf{x}$.

Some analytical integral transforms,

- if $x \sim \mathcal{U}(x|a,b)$ then $x = u(b-a)+a$
- if $x \sim \mathcal{DU}(x|l_1,l_2)$, $l_j \in \mathbb{Z}$, then $x = \mathrm{E}[(l_2-l_1+1)u]+l_1$, where E is the integer part operator
- if $x \sim \mathcal{N}(x|\mu,\sigma^2) = (2\pi\sigma^2)^{-1/2}\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$, then $x = \sigma\sqrt{2}\mathrm{erf}^{-1}(2u-1)+\mu$, erf is the error function.
- if $x \sim \mathcal{T}(x|a,b,c)$, then
$$x = \begin{cases} a+\sqrt{u(b-a)(c-b)} & \text{if } 0<u<\frac{b-a}{c-a} \\ c-\sqrt{(1-u)(b-a)(c-b)} & \text{otherwise} \end{cases}.$$

Useful functions:
Matlab $->$ erfinv ($=\mathrm{erf}^{-1}$), gaminv (inverse cdf of Γ law),
R $->$ qnorm ($=\sqrt{2}\mathrm{erf}^{-1}(2u-1)$), qgamma (inverse cdf of Γ law),

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