Università degli Studi di Padova
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Introduction to Standard Model
Lecture Notes

Lectures by
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based on the lectures *Introduction to Standard Model* (2016/17) and *Theoretical Physics of Fundamental Interactions* (2017/18) given by Pierpaolo Mastrolia:
Suggested literature:

- F. Mandl, G. Shaw "Quantum Field Theory",
- M.D. Schwartz, "Quantum Field Theory and the Standard Model",
- W. Greiner, J. Reinhardt, "Quantum Electrodynamics",
- J.C. Romão, "Advanced Quantum Field Theory",
- M.E.Peskin, D.V.Schroeder, "An Introduction to Quantum Field Theory",
- L.H. Ryder, "Quantum Field Theory",
- F.Halzen, A.D.Martin, "Quarks and Leptons: An Introductory Course in Modern Particle Physics" ,
- D.J.Griffiths, "Introduction to Elementary Particles" .
- T. Gehrmann, and R. Wallny, "Phenomenology of Particle Physics 1" (lecture notes).

Quanto segue è una VERSIONE PROVVISORIA da ultimare e da correggere.

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Chapter 1

QED processes in lowest order

1.1 QED as a gauge theory

In non-relativistic quantum mechanics, the wave equation of an electron of charge $q = -e$ ($e > 0$) in the presence of an external electromagnetic field, described by the scalar potential $\phi(x)$ and the vector potential $\mathbf{A}(x)$, is obtained from Schrödinger equation with the substitutions

$$i\frac{\partial}{\partial t} \longrightarrow i\frac{\partial}{\partial t} - q\phi(x)\ , \qquad -i\boldsymbol{\nabla} \longrightarrow -i\boldsymbol{\nabla} - q\mathbf{A}(x). \tag{1.1}$$

where we have used natural units $\hbar = c = 1$

This scheme is usually referred to as the *minimal coupling.* In four-vector notation, it corresponds to the substitution of the ordinary derivative $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ with the so called covariant derivative,

$$\partial_\mu \longrightarrow D_\mu \equiv \partial_\mu + iqA_\mu(x), \tag{1.2}$$

where $A^\mu(x) \equiv (\phi, \mathbf{A})$ is the four-potential.

In the case of Dirac free electron Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x), \tag{1.3}$$

the substitution (1.2) gives

$$\mathcal{L}_0 \longrightarrow \mathcal{L}_0' = \mathcal{L}_0 + \mathcal{L}_I\ , \tag{1.4}$$

where the interaction term is

$$\mathcal{L}_I = e\bar{\psi}A\!\!\!/\psi. \tag{1.5}$$

Let us now consider the QED Lagrangian,

$$\begin{aligned} \mathcal{L}_{QED} &= \mathcal{L}_{em} + \mathcal{L}_0 + \mathcal{L}_I = \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x) + e\bar{\psi}A\!\!\!/\psi. \end{aligned} \tag{1.6}$$

As a consistency check, we observe that the interaction term can be written in the form of a conserved Noether current times the four potential

$$\mathcal{L}_I = -j_\mu A^\mu, \tag{1.7}$$

with $j^\mu \equiv -e\bar{\psi}\gamma^\mu\psi$ satisfying $\partial_\mu j^\mu = 0$ as a consequence of the invariance of Dirac Lagrangian under a global $U(1)$ transformation

$$\begin{cases} \psi' = e^{i\theta}\psi, \\ \bar{\psi}' = \bar{\psi}e^{-i\theta}. \end{cases} \tag{1.8}$$

where θ is a real constant. To prove that j^μ must be a conserved current, it suffices to differentiate the equations of motion for A^μ with respect to x^ν,

$$\partial_\mu F^{\mu\nu} = j^\nu. \tag{1.9}$$

As the four-potential appears in the electromagnetic Lagrangian $\mathcal{L}_{em}$ only through its field strength $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, this term of QED Lagrangian is invariant under a gauge transformation of the form

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu f(x), \tag{1.10}$$

where $f(x)$ is an arbitrary scalar function. However, the total Lagrangian is not invariant, since the interaction term transforms as

$$\mathcal{L}_I \longrightarrow \mathcal{L}'_I = \mathcal{L}_I + e\bar{\psi}\gamma^\mu\psi\partial_\mu f. \tag{1.11}$$

In order to restore gauge invariance, we can implement a local transformation of the abelian group $U(1)$ on the spinor fields, by preforming a multiplication by a locally-dependent phase factor

$$\begin{cases} \psi'(x) = e^{i\theta(x)}\psi(x) \equiv e^{ief(x)}\psi(x), \\ \bar{\psi}'(x) = \bar{\psi}(x)e^{-i\theta(x)} \equiv \bar{\psi}(x)e^{-ief(x)}. \end{cases} \tag{1.12}$$

The only term that doesn't transform trivially now is

$$\mathcal{L}_0 \longrightarrow \mathcal{L}'_0 = \mathcal{L}_0 - e\bar{\psi}\gamma^\mu\psi\partial_\mu f. \tag{1.13}$$

Considering the transformations 1.10 and 1.12 acting respectively on the gauge field $A_\mu(x)$ and matter fields $(\bar{\psi}(x), \psi(x))$, we can see that the invariance-breaking terms cancel out to give rise to an invariant total Lagrangian. Thus the coupled transformations

$$\begin{cases} A'_\mu(x) = A_\mu(x) + \partial_\mu f(x), \\ \psi'(x) = e^{ief(x)}\psi(x), \\ \bar{\psi}'(x) = \bar{\psi}(x)e^{-ief(x)}. \end{cases} \tag{1.14}$$

is a symmetry of the theory. It can also be seen as a consequence of the fact that the quantity

$$(D_\mu\psi)' = \partial_\mu\psi' - ieA'_\mu\psi' = e^{ief(x)}(D_\mu\psi) \tag{1.15}$$

transforms as a matter field, and therefore $\bar{\psi}(x)\not{D}\psi(x)$ is consequently invariant.

1.2 S-matrix and Feynman rules

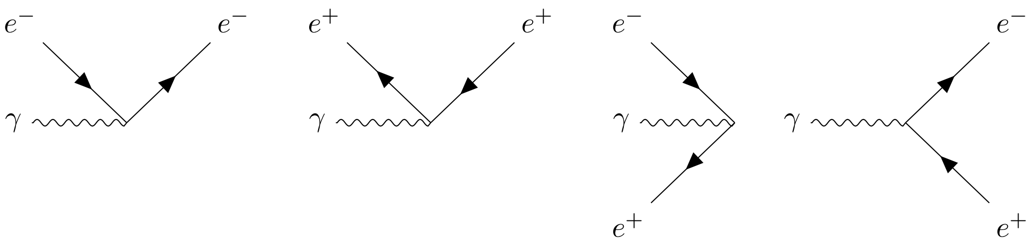
When studying scattering processes, one is interested in the probability density p_{fi} of finding the system, initially prepared in a state $|i\rangle$, in a certain final state $|f\rangle$. The operator encoding the information about the interaction is the so called S-matrix, which can be written as

$$S = \sum_{n=0}^{\infty} S^{(n)} = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \cdots d^4x_n T\Big\{\mathcal{H}_I(x_1)\cdots\mathcal{H}_I(x_n)\Big\}, \tag{1.16}$$

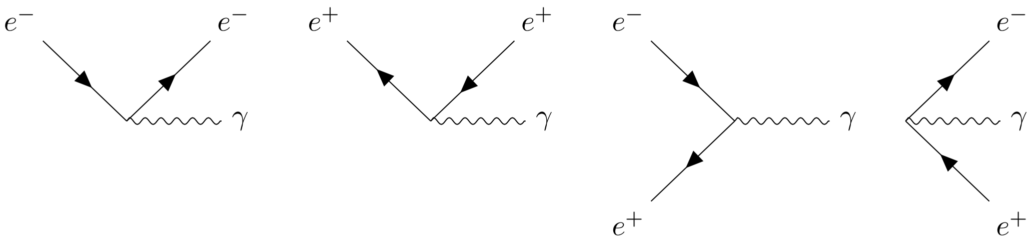
where $\mathcal{H}_\mathcal{I}$ is the interaction Hamiltonian. In the case of QED,

$$\begin{aligned} \mathcal{H}_I(x) = -\mathcal{L}_I(x) = -eN\Big\{\bar{\psi}(x)\not{A}(x)\psi(x)\Big\} = \\ = -eN\Big\{(\bar{\psi}^+ + \bar{\psi}^-)(\not{A}^+ + \not{A}^-)(\psi^+ + \psi^-)\Big\}_x. \end{aligned}$$

bear in mind that, this interaction Hamiltonian corresponds to the annihilation(creation) of electron-positron pair with the creation(annihilation) of a photon. This gives rise to 8 processes that can be diagrammatically represented as in Figure 1.1,



(a) γ absorption - $(\bar{\psi}^- A\!\!\!/^+ \psi^+), (\bar{\psi}^+ A\!\!\!/^+ \psi^-), (\bar{\psi}^+ A\!\!\!/^+ \psi^+), (\bar{\psi}^- A\!\!\!/^+ \psi^-)$



(b) γ emission - $(\bar{\psi}^- A\!\!\!/^- \psi^+), (\bar{\psi}^+ A\!\!\!/^- \psi^-), (\bar{\psi}^+ A\!\!\!/^- \psi^+), (\bar{\psi}^- A\!\!\!/^- \psi^-)$

Figure 1.1

The transition probabilities are proportional to the corresponding normalized S-matrix elements squared,

$$|S_{fi}|^2 = \frac{|\langle f|\, S\, |i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} \tag{1.17}$$

In the momentum space, these matrix elements are given by

$$S_{fi} = \delta_{fi} + \left[(2\pi)^4 \delta^{(4)}(p_f - p_i) \prod_{fermions} \left(\frac{m}{2VE} \right)^{\frac{1}{2}} \prod_{photons} \left(\frac{1}{2V\omega} \right)^{\frac{1}{2}} \right] \mathcal{M}, \tag{1.18}$$

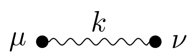
where products are normalization factors corresponding to fermions and photons of external lines and $\mathcal{M}$ is the Feynman scattering amplitude.

The calculation of this amplitude at a certain perturbative order can be performed directly from Feynman diagrams using the following rules:

1. the QED vertex is $ie\gamma^\mu$;

2. the photon propagator is

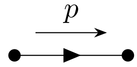
$$iD_{F\mu\nu}(k) \equiv i\frac{-g_{\mu\nu}}{k^2 + i\varepsilon} \tag{1.19}$$



;

3. the fermion propagator is

$$iS_F(p) \equiv i\frac{1}{p\!\!\!/ - m + i\varepsilon} \equiv i\frac{p\!\!\!/ + m}{p^2 - m^2 + i\varepsilon} \tag{1.20}$$



;

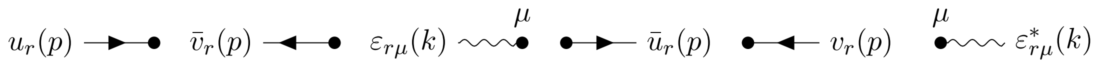
4. external lines are represented as in Figure 1.2, where $r = 1, 2$ labels spin and polarization states;

5. spinor factors (γ matrices, S_F propagators, spinors) must be written from left to right in the same order as following the fermion line in the opposite direction of its arrows;

6. a closed fermion loop corresponds to a trace since all spinorial indices are contracted and brings a factor (-1);

7. each loop integration brings a $\frac{d^4p}{(2\pi)^4}$ factor;

8. signs ± 1 appear for final states involving bosonic/fermionic exchanges.



(a) destroys e^- (ψ^+) (b) destroys e^+ ($\bar{\psi}^+$) (c) destroys γ (A^+) (d) creates e^- ($\bar{\psi}^-$) (e) creates e^+ (ψ^-) (f) creates γ (A^-)

Figure 1.2

1.3 Cross section

In order to introduce cross section[1] which is a natural quantity to measure experimentally, let us consider Rutherford's experiment: its aim was to investigate atomic structure and, in particular, Rutherford was interested in the size r of an atomic nucleus. By colliding $\alpha-$particles with gold foil and measuring how many $\alpha-$particles were scattered, he was able to determine the cross-sectional area, $\sigma = \pi r^2$, of the nucleus. Imagine there is just a single nucleus. Then the *cross-sectional area* is given by

$$\sigma = \frac{N}{\Big(\text{time}\Big) \times \left(\begin{array}{c}\text{number density of}\\ \text{particles in the beam}\end{array}\right) \times \Big(\text{relative velocity}\Big)} \equiv \frac{N}{T\Phi}, \tag{1.21}$$

where T is the time for the experiment and Φ= the incoming flux (number density $\times$ velocity of beam = Number of particles per unit area per unit time) and N is the number of particles scattered. Therefore, the number of particles scattered classically is proportional to the cross-sectional area of the scattering object.

In quantum mechanics, we generalize the notion of cross-sectional area to a *cross section*, which still has units of area, but has a more abstract meaning as a measure of the interaction strength. While classically an $\alpha-$particle either scatters off the nucleus or it does not scatter, quantum mechanically it has a probability for scattering. So the quantum mechanical differential cross section is naturally

$$d\sigma = \frac{1}{T}\frac{1}{\Phi}dP, \tag{1.22}$$

where Φ is the flux, now normalized as if the beam has just one particle, and P is the (quantum mechanical) probability of scattering. We can write the differential probability dP in terms of the transition probability and phase space $d\Pi$

$$dP = \frac{|\langle f| S |i\rangle|^2}{\langle f|f\rangle \langle i|i\rangle} d\Pi \tag{1.23}$$

The differential number of scattering events measured in a collider experiment is

$$dN = L d\sigma \tag{1.24}$$

where L is the *luminosity*, which is defined by this equation. Nevertheless we can provide a more "classical and pictorial" description of luminosity. Typical scattering experiment consists collision of two beams, described by initial wave packets. In all cases of practical interest, the initial state wave packets do not spread significantly and hence, we can define the luminosity L of the prepared system as (Figure 1.3)

$$L \equiv \lim_{d\to 0} \frac{P(d_{12} < d)}{\pi d^2}, \tag{1.25}$$

where $P(d_{12} < d)$ is the probability that the two colliding particles come to a relative distance[2]. $r = d_{12} < d$

[1] References for this section are: "*Quantum field theory and the standard model*" by M. D. Schwartz and "*Quantum field theory*" by F. Mandl and G. Shaw.

[2] More details about this alternative interpretation of luminosity are provided in "*An introduction to relativistic processes and the standard model of electroweak interactions*", C.M. Becchi, G. Ridolfi.

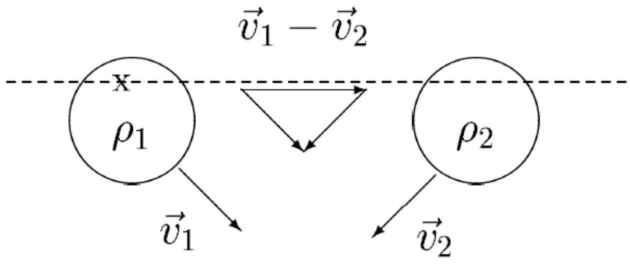


Figure 1.3: Schematic of a scattering event; ρ_1,ρ_2 are the probability densities of the initial state particles.

Now let us relate the formula for the differential cross section to S-matrix elements. From a practical point of view it is impossible to collide more than two particles at a time, so we consider a scattering process in which two particles, with four-momenta $p_i = (E_i, \mathbf{p}_i)$ and $i = 1, 2$, collide and produce N final particles with four-momenta $p'_f = (E'_f, \mathbf{p}'_f)$ and $f = 1, \ldots, N$:

$$p_1 + p_2 \longrightarrow \sum_{f=1}^{N} p'_f. \tag{1.26}$$

As a first step, it is meaningful to consider the normalized scattering matrix elements, since they appear in the differential probability term (1.23). A natural way to write them will be in the following form

$$S_{fi} \equiv \frac{\langle f|\, S\, |i\rangle}{\sqrt{\langle f|f\rangle}\sqrt{\langle i|i\rangle}} \tag{1.27}$$

the normalization factors can be written in terms of the initial and final states $|i\rangle$ and $|f\rangle$ which are

$$|i\rangle = |p_1\rangle\, |p_2\rangle \tag{1.28}$$

$$|f\rangle = \prod_{j}^{N} |p'_j\rangle \tag{1.29}$$

and therefore we get for each of the normalization factors squared

$$\langle i|i\rangle = (2E_1 V)(2E_2 V) \tag{1.30}$$

$$\langle f|f\rangle = \prod_{j}^{N} (2E_j V) \tag{1.31}$$

Finally, the normalized scattering matrix can be written as

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)} \left(\sum p'_f - \sum p_i\right) \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \prod_f \left(\frac{1}{2VE_f}\right)^{1/2} \prod_l (2m_l)^{1/2} \mathcal{M}, \tag{1.32}$$

where the index l runs over all external leptons in the process, but $2m_l$ factor is not relevant because of its presence also in Feynman amplitude's denominator. We'll show it later. In deriving the cross section, it is helpful to take T and V finite. For this purpose , it is useful to define the transition probability per unit time, called *rate*,

$$w = \frac{|S_{fi}|^2}{T} \qquad (f \neq i). \tag{1.33}$$

Equation (1.33) involves the square of the δ function which is a mathematically not well defined quantity. However, in order to obtain an explicit expression for the rate we can write *formally*

$$\left(\delta^{(4)}\left(\sum p\right)\right)^2 = \delta^{(4)}\left(\sum p\right)\delta^{(4)}(0) \tag{1.34}$$

and so we just need to compute $\delta^4(0)$. By definition

$$\delta^{(4)}(p) = \int \frac{d^4x}{(2\pi)^4}e^{ip\cdot x} \quad \Rightarrow \quad \delta(0) = \frac{VT}{(2\pi)^4}. \tag{1.35}$$

Therefore

$$w = V(2\pi)^4\delta^{(4)}\left(\sum p'_f - \sum p_i\right)\prod_i\left(\frac{1}{2VE_i}\right)\prod_f\left(\frac{1}{2VE_f}\right)\prod_l(2m_l)|\mathcal{M}|^2. \tag{1.36}$$

To obtain the transition rate to a final state with momenta in the intervals $[p'_f, p'_f + dp'_f]$ we must multiply w by the number of these states which is the *phase space*

$$\prod_f \frac{d^3\mathbf{p}'_f}{(2\pi)^3}V. \tag{1.37}$$

The differential cross section is the transition rate into this group of final states for one scattering center and unit incident flux. With our choice of normalization for the states, the volume V which we are considering contains one scattering center, and the incident flux is v_{rel}/V, where v_{rel} is the relative velocity of the colliding particles. Combining these results with equation (1.36), we obtain the required expression for the differential cross-section

$$\begin{aligned} d\sigma &= \frac{V}{v_{rel}}w\prod_f\frac{d^3\mathbf{p}'_f}{(2\pi)^3}V = \\ &= (2\pi)^4\delta^{(4)}\left(\sum p'_f - \sum p_i\right)\frac{1}{4E_1E_2v_{rel}}\left(\prod_f\frac{d^3\mathbf{p}'_f}{(2\pi)^3 2E'_f}\right)\left(\prod_l(2m_l)\right)|\mathcal{M}|^2. \end{aligned} \tag{1.38}$$

Note that there is no explicit dependence on the volume and time which were introduced for the normalization of the states. Furthermore, this expression holds in any Lorentz frame in which the colliding particles move collinearly, as we can check:

- $\delta^{(4)}\left(\sum p'_f - \sum p_i\right)$ is manifestly invariant;
- $\frac{d^3\mathbf{p}}{2E}$ can be rewritten in the form (manifestly invariant)

$$\frac{d^3\mathbf{p}}{2E} = \int d^4p\,\delta(p^2 - m^2)\Theta(p^0), \tag{1.39}$$

in fact

$$\begin{aligned} \int d^4p\,\delta(p^2-m^2)\Theta(p_0) &= \int_{-\infty}^{+\infty} dp_0\Theta(p_0)\delta(p^2-m^2)\int d^3p = \\ &= \int_{-\infty}^{+\infty} dp_0\Theta(p_0)\delta(p_0^2 - \mathbf{p}^2 - m^2)d^3p = \int_{-\infty}^{+\infty} dp_0\Theta(p_0)\delta(p_0^2 - E^2)\int d^3p \\ &= \int_0^{+\infty} dp_0\delta\left((p_0 - E)(p_0+E)\right)\int d^3p \end{aligned}$$

where $E^2 = \mathbf{p}^2 + m^2$, at this stage we can use a property of the δ-function, namely

$$\delta\left(f(x)\right) = \sum_k \frac{\delta(x - x_k)}{|f'(x_k)|} \qquad \text{with} \quad f(x_k) = 0 \tag{1.40}$$

in our case, $f'(p_0) = 2p_0$ and we have x_1 at $p_0 = E > 0$ and x_2 at $p_0 = -E < 0$. Given the integration boundary we can consider only x_1 and hence the integral becomes

$$= \int_{-\infty}^{+\infty} d^3p \int_0^{+\infty} dp_0 \left(\frac{\delta(p_0 - E)}{|2p_0|} \right) + \left(\frac{\delta(p_0 + E)}{|2p_0|} \right) = \frac{d^3\mathbf{p}}{2E}$$

- using the expression

$$|\mathbf{v}_1 - \mathbf{v}_2| = \sqrt{\mathbf{v}_1^2 + \mathbf{v}_2^2 - 2\mathbf{v}_1 \cdot \mathbf{v}_2} = \sqrt{\frac{\mathbf{p}_1^2}{E_1^2} + \frac{\mathbf{p}_2^2}{E_2^2} - \frac{2}{E_1 E_2}\mathbf{p_1} \cdot \mathbf{p_2}} \tag{1.41}$$

  we can express $E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2|$ in the form

$$E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2| = \sqrt{\mathbf{p}_1^2 E_2^2 + \mathbf{p}_2^2 E_1^2 - 2\mathbf{p}_1 \cdot \mathbf{p}_2 E_1 E_2}, \tag{1.42}$$

  with $\mathbf{p}_1 \cdot \mathbf{p}_2 = |\mathbf{p}_1||\mathbf{p}_1| \cos\theta$, for an appropriate θ. Consider now the quantity (which certainly is Lorentz invariant)

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = (E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2)^2 - m_1^2 m_2^2 = \tag{1.43}$$
$$= E_1^2 E_2^2 - 2E_1 E_2 \mathbf{p}_1 \cdot \mathbf{p}_2 + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 - m_1^2 m_2^2 \tag{1.44}$$

  but $E_i^2 = m_i^2 + \mathbf{p}_i^2$ and so

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \mathbf{p}_1^2 E_2^2 + m_1^2 \mathbf{p}_2^2 - 2E_1 E_2 \mathbf{p}_1 \cdot \mathbf{p}_2 + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 = \tag{1.45}$$
$$= \mathbf{p}_1^2 E_2^2 + \mathbf{p}_2^2 E_1^2 - \mathbf{p}_1^2 \mathbf{p}_2^2 - 2E_1 E_2 \mathbf{p}_1 \cdot \mathbf{p}_2 + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2 \tag{1.46}$$

  where, in the last passage, we have used $m_1^2 = E_1^2 - \mathbf{p}_1^2$ again. Our hope is to obtain (the square of) expression (1.42) and so we are annoyed by presence of term $-\mathbf{p}_1^2 \mathbf{p}_2^2 + (\mathbf{p}_1 \cdot \mathbf{p}_2)^2$. Nevertheless, we are considering situations in which $\theta = 0$ and hence

$$\mathbf{p}_1 \cdot \mathbf{p}_2 = |\mathbf{p}_1||\mathbf{p}_2| \cos\theta = |\mathbf{p}_1||\mathbf{p}_2|. \tag{1.47}$$

  Therefore we can write

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = \mathbf{p}_1^2 E_2^2 + \mathbf{p}_2^2 E_1^2 - 2E_1 E_2 \mathbf{p}_1 \cdot \mathbf{p}_2 \tag{1.48}$$

  and therefore we have the equivalence

$$E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2| = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \tag{1.49}$$

  At this level, we can conclude that the differential cross section is a Lorentz invariant quantity.

and so, going back to equation (1.38)

$$\frac{m_1 m_2}{E_1 E_2} \frac{1}{|\mathbf{v_2} - \mathbf{v_1}|} \xrightarrow[\theta=0]{} \frac{m_1 m_2}{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}. \tag{1.50}$$

Two different and important reference frames, in which the colliding particles move collinearly, are:

1. Center of mass (CoM) system, in which $\mathbf{p_1} = -\mathbf{p_2}$ and considering Eq.1.41, we get

$$E_1 E_2 |\mathbf{v_1} - \mathbf{v_2}|_{\theta=0} = |\mathbf{p_1}| \sqrt{(E_1 + E_2)^2} = |\mathbf{p_1}|(E_1 + E_2), \tag{1.51}$$

   whence

$$v_{rel} = |\mathbf{p_1}| \frac{E_1 + E_2}{E_1 E_2}; \tag{1.52}$$

2. Laboratory (Lab) system, in which $\mathbf{p_2} = \mathbf{0}$ and

$$v_{rel} = |\frac{\mathbf{p_1}}{E_1} - \frac{\mathbf{p_2}}{E_2}| = \frac{|\mathbf{p_1}|}{E_1}. \tag{1.53}$$

Let us now consider the frequently occurring case of a process leading to a two-body final state.

$$p_1 + p_2 \longrightarrow p'_1 + p'_2. \tag{1.54}$$

Equation (1.38) becomes

$$d\sigma = f(p'_1, p'_2)\delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)d^3\mathbf{p_1}'d^3\mathbf{p_2}' \tag{1.55}$$

with

$$f(p'_1, p'_2) \equiv \frac{1}{64\pi^2 v_{rel} E_1 E_2 E'_1 E'_2}\left(\prod_l (2m_l)\right)|\mathcal{M}|^2. \tag{1.56}$$

Integration of this equation with respect to $\mathbf{p_2}'$ gives

$$d\sigma = f(p'_1, p'_2)\delta(E'_1 + E'_2 - E_1 - E_2)\overbrace{|\mathbf{p_1}'|^2 d|\mathbf{p_1}'|d\Omega'_1}^{d^3\mathbf{p_1}'} \tag{1.57}$$

where $\mathbf{p_2}' = \mathbf{p_1} + \mathbf{p_2} - \mathbf{p_1}'$. In the CoM frame,

$$\boldsymbol{p}'_1 = -\boldsymbol{p}'_2 \quad \Rightarrow \quad |\boldsymbol{p}'_1| = |\boldsymbol{p}'_2| \quad \Rightarrow \quad E'_i = \sqrt{m'^2_i + |\boldsymbol{p}'_1|^2}, \quad i = 1, 2 \tag{1.58}$$

Integrating over $|\mathbf{p_1}'|$ and using the general relation

$$\int f(x)\delta[g(x)]dx = \sum_i \int f(x)\frac{1}{|g'(x)|}\delta(x - x_i) = \sum_i \left[\frac{f(x_i)}{g'(x_i)}\right], \tag{1.59}$$

using this one can find two roots for $|\mathbf{p_1}'|$, where we neglect the negative solution as physically unacceptable. [Note: $|\mathbf{p_1}'|$ is the only variable here. E'_2 and $|\mathbf{p_2}'|$ intrinsically depends on $|\mathbf{p_1}'|$]. So, we have

$$d\sigma = \sum_i f(p'_1, p'_2)|\mathbf{p_1}'|^2 d\Omega \left|\left(\frac{\partial(E'_1 + E'_2 - E_1 - E_2)}{\partial|\mathbf{p_1}'|}\right)^{-1}\right|_{E'_1+E'_2=E_1+E_2}. \tag{1.60}$$

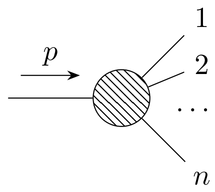
and the derivative gives

$$\left|\frac{\partial(E'_1 + E'_2 - E_1 - E_2)}{\partial|\mathbf{p}'_1|}\right|_{E'_1+E'_2=E_1+E_2} = \left|\frac{E'_1 + E'_2}{E'_1 E'_2}|\boldsymbol{p}'_1|\right|_{E'_1+E'_2=E_1+E_2} = \frac{E_1 + E_2}{E'_1 E'_2}|\boldsymbol{p}'_1|, \tag{1.61}$$

Thus, we finally obtain,

$$\left(\frac{d\sigma}{d\Omega_1}\right)_{CoM} = \frac{1}{64\pi^2}\frac{1}{(E_1 + E_2)^2}\frac{|\boldsymbol{p}'_1|}{|\boldsymbol{p}_1|}\left(\prod_l (2m_l)\right)|\mathcal{M}|^2. \tag{1.62}$$

As the transition amplitude may in general depend on the polar angles θ and ϕ, we need to know its explicit expression to integrate over the solid angle and obtain the total cross section.



1.4 Decay rate

Now we will deal with another similar kind of calculation, where we consider the decay of a single particle to N particles, given by,

$$p \longrightarrow \sum_{f=1}^{N} p'_f \tag{1.63}$$

The Scattering matrix for this process is,

$$S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)} \left(\sum p'_f - \sum p_i\right) \left(\frac{1}{2VE}\right)^{1/2} \prod_f \left(\frac{1}{2VE_f}\right)^{1/2} \prod_l (2m_l)^{1/2} \mathcal{M}, \tag{1.64}$$

and directly comparing with our previous equations, the decay rate is,

$$w = V(2\pi)^4 \delta^{(4)} \left(\sum p'_f - \sum p_i\right) \left(\frac{1}{2E}\right) \prod_f \left(\frac{1}{2VE_f}\right) \prod_l (2m_l) |\mathcal{M}|^2. \tag{1.65}$$

The differential cross section for the decay rate is defined as the product of the decay rate with the phase space of the final particles (Note here that there is no incident flux),

$$\begin{aligned} d\Gamma &= w \prod_f \frac{d^3\mathbf{p}'_f}{(2\pi)^3} V = \\ &= (2\pi)^4 \delta^{(4)} \left(\sum p'_f - p\right) \frac{1}{2E} \left(\prod_f \frac{d^3\mathbf{p}'_f}{(2\pi)^3 2E'_f}\right) \left(\prod_l (2m_l)\right) |\mathcal{M}|^2. \end{aligned} \tag{1.66}$$

From this expression, we can calculate the total decay rate by integration,

$$\Gamma = \int d\Gamma \tag{1.67}$$

The life time of the particle is just the inverse of the total decay rate $\tau = \frac{1}{\Gamma}$. If there are multiple decay modes present, where the same particle can decay into N particles in one mode and N' particles in other mode and etc. , then the total decay rate is just the sum of the decay rate for each mode.

$$\Gamma_{tot} = \sum_f \Gamma_{i \to f} \tag{1.68}$$

"f" sums over each decay mode. and again the life time is the inverse of the total decay rate. Thus, the total decay rate of an arbitrary decay process is given by

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \left(\prod_l (2m_l)\right) \int |\mathcal{M}_{fi}|^2 \prod_{j=1}^{n} \frac{d^3 p_j}{(2\pi)^3 (2E_j)} \delta^{(4)} \Big(p^\mu - \sum_{j=1}^{n} p_j^\mu\Big), \tag{1.69}$$

where "j" sums over all final particles. and $\prod_{j=1}^{n} \frac{d^3 p_j}{(2\pi)^3 (2E_j)} \delta^{(4)}(p^\mu - \sum_{j=1}^{n} p_j^\mu)$ is the Lorentz invariant phase space element.

Consider now a two-body decay: one initial particle decays into two particles. The kinematics of the system in the center of mass frame is summarized by:

$$
\begin{gathered}
p_i^\mu = (E_i, \mathbf{p_i}) = (m_i, \mathbf{0}), \\
p_1^\mu = (E_1, \mathbf{p_1}), \quad p_2^\mu = (E_2, \mathbf{p_2}), \quad \mathbf{p_1} + \mathbf{p_2} = \mathbf{0}, \\
E_1^2 = m_1^2 + |\mathbf{p_1}|^2, \quad E_2^2 = m_2^2 + |\mathbf{p_2}|^2 = m_2^2 + |\mathbf{p_1}|^2.
\end{gathered}
$$

We have:

$$
\begin{aligned}
\Gamma_{fi} &= \frac{(2\pi)^4}{2m_i} \left(\prod_l (2m_l) \right) \int |\mathcal{M}_{fi}|^2 \frac{d^3p_1}{(2\pi)^3(2E_1)} \frac{d^3p_2}{(2\pi)^3(2E_2)} \delta^{(3)}(\mathbf{p_1} + \mathbf{p_2}) \delta(m_i - E_1 - E_2) \\
&= \frac{1}{32\pi^2 m_i} \left(\prod_l (2m_l) \right) \int |\mathcal{M}_{fi}|^2 \frac{d^3p_1}{E_1 E_2} \delta(m_i - E_1 - E_2). \qquad (1.70)
\end{aligned}
$$

This computation can be easily handed in polar coordinates: $d^3p_1 = |\mathbf{p_1}|^2 \, d|\mathbf{p_1}| \, d\Omega$.

$$
\Gamma_{fi} = \frac{1}{32\pi^2 m_i} \left(\prod_l (2m_l) \right) \int |\mathcal{M}_{fi}|^2 g(|\mathbf{p_1}|) \delta(f(|\mathbf{p_1}|)) \, d|\mathbf{p_1}| \, d\Omega, \qquad (1.71)
$$

where

$$
g(|\mathbf{p_1}|) = \frac{\mathbf{p_1}^2}{E_1 E_2}, \quad f(|\mathbf{p_1}|) = m_i - \sqrt{m_1^2 + |\mathbf{p_1}|^2} - \sqrt{m_2^2 + |\mathbf{p_1}|^2}.
$$

There are two solution to $f(|\mathbf{p_1}|) = 0$,

$$
|\mathbf{p_1}|_{1,2} = \pm p* = \pm \frac{1}{2m_i} \sqrt{[m_i^2 - (m_1 + m_2)^2][m_i^2 - (m_1 - m_2)^2]},
$$

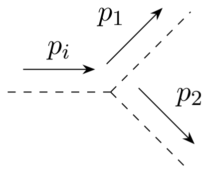
but only the positive one is physically acceptable.

$$
\int d|\mathbf{p_1}| \, g(|\mathbf{p_1}|) \delta(f(|\mathbf{p_1}|)) = \int d|\mathbf{p_1}| \, g(|\mathbf{p_1}|) \frac{\delta(|\mathbf{p_1}| - p*)}{|(f'(p*)|}
$$

and we have $f'(|\mathbf{p_1}|) = -|\mathbf{p_1}| \frac{E_1 + E_2}{E_1 E_2}$. Now we are able to integrate over the 3-momentum magnitude in the integral of Equation 1.71:

$$
\begin{aligned}
\Gamma_{fi} &= \frac{1}{32\pi^2 m_i} \left(\prod_l (2m_l) \right) \frac{E_1 E_2}{p*(E_1 + E_2)} \frac{(p*)^2}{E_1 E_2} \int |\mathcal{M}_{fi}|^2 \, d\Omega \\
&= \frac{p*}{32\pi^2 m_i^2} \left(\prod_l (2m_l) \right) \int |\mathcal{M}_{fi}|^2 \, d\Omega. \qquad (1.72)
\end{aligned}
$$

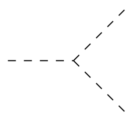
This is general. Now, we want to compute the total decay rate of a two-particle decay process in ϕ^3 theory.

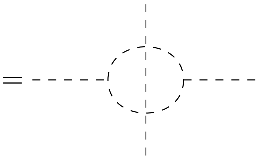
$i\mathcal{M}$ =  = $i\lambda$

Let's assume that the out-coming particles are on shell (i.e., their mass is m). The initial particle is off shell (condition needed to decay):

$$
m_i^2 = p_i^2 \neq m^2; \quad p_1^2 = p_2^2 = m^2; \quad p* = \frac{m_i}{2} \sqrt{1 - \frac{4m^2}{p_i^2}}.
$$

$$|\bar{\mathcal{M}}|^2 = \left| \right|^2 = \qquad (1.73)$$





so we have the following expression for the total decay rate of the process:

$$\Gamma_{\phi\to\phi\phi} = \frac{\lambda^2}{64\pi^2 m_i}\sqrt{1-\frac{4m^2}{p_i^2}}\left(\prod_l (2m_l)\right)\int d\Omega = \frac{\lambda^2}{16\pi m_i}\sqrt{1-\frac{4m^2}{p_i^2}}\left(\prod_l (2m_l)\right). \qquad (1.74)$$

From the above expression, it can be deduced that, there exists a minimum threshold for the process to happen i.e. $p_i^2 > 4m^2$ which is known as production threshold.

1.5 Spin sums

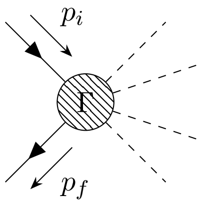
Formula (1.38) for the cross section holds when the polarization states of both initial and final particles are specified. An experimentally interesting case is when the colliding beams are unpolarized and the final polarization states are not measured. In this case, we must average over all initial polarization states and sum over all final polarization states, thus obtaining the unpolarized Feynman amplitude

$$\left|\bar{\mathcal{M}}\right|^2 = \Big(\prod_i \frac{1}{2j_i+1}\Big)\sum_{s_i}\sum_{s_f}|\mathcal{M}_{fi}|^2\,, \qquad (1.75)$$

where i and f label the initial and final particles respectively and $2j_i+1$ is the number of polarization states of the particle i. In this section we will study averages and sums over initial and final fermion spins.

Let us consider a Feynman amplitude of the form

$$\mathcal{M}_{fi} = \qquad = \bar{u}_{s_f}(p_f)\Gamma u_{s_i}(p_i), \qquad (1.76)$$

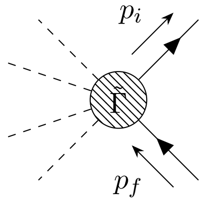


where Γ is a 4×4 matrix built up out of γ-matrices. Its complex conjugate can be explicitly calculated

$$\mathcal{M}^*_{fi} = (\mathcal{M}_{fi})^\dagger = u^\dagger_{s_i}\Gamma^\dagger\bar{u}^\dagger_{s_f} = u^\dagger_{s_i}\Gamma^\dagger(u^\dagger_{s_f}\gamma_0)^\dagger = u^\dagger_{s_i}\Gamma^\dagger\gamma_0 u_{s_f} = \underbrace{u^\dagger_{s_i}\gamma_0}_{\bar{u}_{s_i}}\underbrace{(\gamma_0\Gamma^\dagger\gamma_0)}_{\equiv\tilde{\Gamma}}u_{s_f}\,, \qquad (1.77)$$

and diagrammatically represented as

$$\mathcal{M}^*_{fi} = \qquad = \bar{u}_{s_i}(p_i)\tilde{\Gamma}u_{s_f}(p_f). \qquad (1.78)$$



This is the diagram related to the reversed process, in which initial and final states are interchanged. In the general case, there may also be a relative sign depending on the number of vertices and propagators of the diagram. We may then add to the set of Feynman rules the following one,

9. $\mathcal{M}^* = (\text{reversed process}) \times (-1)^{\#\text{vertices}+\#\text{propagators}}$, where the reversed process is obtained by interchanging initial and final states.

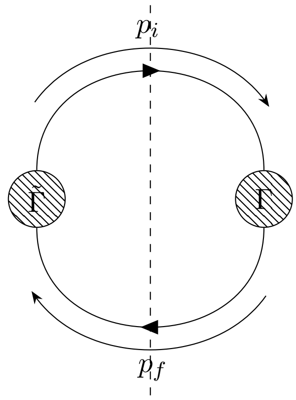
The complex conjugation of the Feynman amplitude may be seen as the action of the time reversal anti-linear operator T,

$$\mathcal{M}^* = T\Big(\mathcal{M}\Big), \tag{1.79}$$

which sends $t \to -t$, $i \to -i$ and so flips the momenta[3].

The unpolarized Feynman squared amplitude can be diagrammatically represented as

$$\left|\bar{\mathcal{M}}\right|^2 =$$



$$= \frac{1}{2}\sum_{s_i,s_f} \mathcal{M}^*_{fi}\mathcal{M}_{fi}$$

$$= \frac{1}{2}\sum_{s_i,s_f} \bar{u}_{s_f}(p_f)\Gamma u_{s_i}(p_i)\bar{u}_{s_i}(p_i)\tilde{\Gamma} u_{s_f}(p_f), \tag{1.80}$$

where the cut means that the two internal fermionic lines are not propagators, but represent *on shell* particles satisfying $p_{i,f}^2 = m_{i,f}^2$. Using the completeness relations for Dirac spinors,

$$\sum_s u_s(p)\bar{u}_s(p) = \frac{\not{p}+m}{2m} \quad \text{and} \quad \sum_s v_s(p)\bar{v}_s(p) = \frac{\not{p}-m}{2m}, \tag{1.81}$$

formula (1.80) becomes

$$\left|\bar{\mathcal{M}}\right|^2 = \frac{1}{2}\sum_{s_i,s_f} \bar{u}_{s_f}(p_f)\left(\Gamma\frac{\not{p}_i+m}{2m}\tilde{\Gamma}\right) u_{s_f}(p_f) = \tag{1.82}$$

$$= \frac{1}{2}\sum_{s_i,s_f} \Big(\bar{u}_{s_f}(p_f)\Big)_\alpha \left(\Gamma\frac{\not{p}_i+m}{2m}\tilde{\Gamma}\right)_{\alpha\beta} \Big(u_{s_f}(p_f)\Big)_\beta = \tag{1.83}$$

$$= \frac{1}{2}\left(\Gamma\frac{\not{p}_i+m}{2m}\tilde{\Gamma}\right)_{\alpha\beta} \left(\sum_{s_i,s_f} u_{s_f}(p_f)\bar{u}_{s_f}(p_f)\right)_{\beta\alpha} = \tag{1.84}$$

$$= \frac{1}{2}\left(\Gamma\frac{\not{p}_i+m}{2m}\tilde{\Gamma}\right)_{\alpha\beta} \left(\frac{\not{p}_f+m}{2m}\right)_{\beta\alpha} = \tag{1.85}$$

$$= \frac{1}{2}\operatorname{Tr}\left(\Gamma\frac{\not{p}_i+m}{2m}\tilde{\Gamma}\frac{\not{p}_f+m}{2m}\right). \tag{1.86}$$

Thus,

$$\left|\bar{\mathcal{M}}\right|^2 = \frac{1}{2}\operatorname{Tr}\left(\Gamma\frac{\not{p}_i+m}{2m}\tilde{\Gamma}\frac{\not{p}_f+m}{2m}\right). \tag{1.87}$$

Comparing with the 6^{th} Feynman rule, that associates each closed fermionic loop with a trace, we observe that the presence of a cut in the diagram ensure that only the numerator of the fermionic propagator (i.e. $\not{p}+m$ or $\not{p}-m$, according to the flow of the momenta) appears. As it had already been anticipated, we note that the factors $\frac{1}{2m}$ in equation (1.87) cancels out with those in the definition of the cross section.

[3] See M. D. Schwartz, "*Quantum field theory and the standard model*", chapter 11.6

γ matrices

We have shown that the unpolarized cross section can always be expressed in terms of traces of products of γ matrices. We shall now briefly recall some definitions and properties that may be useful to carry out this kind of calculations.

Dirac representation for γ matrices is

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{1.88}$$

The following properties hold:

- $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (Clifford algebra);
- $\gamma_0^\dagger = \gamma_0$ and $\gamma_i^\dagger = -\gamma_i$, which can be put together into $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$;
- $\gamma_0^2 = \mathbf{1}$ and $\gamma_i^2 = -\mathbf{1}$;
- $\{\gamma_5, \gamma^\mu\} = 0$.

Let us calculate the matrix $\tilde{\Gamma}$ in two simple cases:

- $\Gamma = \gamma^\mu \quad \Rightarrow \quad \tilde{\Gamma} = \gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$,
- $\Gamma = \gamma^{\mu_1}\cdots\gamma^{\mu_n} \quad \Rightarrow \quad \tilde{\Gamma} = \gamma^0(\gamma^{\mu_1}\cdots\gamma^{\mu_n})^\dagger\gamma^0 = \gamma^{\mu_n}\cdots\gamma^{\mu_1}$.

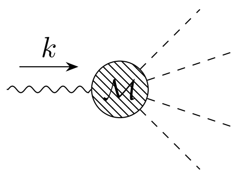
Some useful formulas in calculating traces of γ matrices are

- $\mathrm{Tr}\,(\mathbf{1}) = 4$;
- $\mathrm{Tr}\,(\gamma^{\mu_1}\cdots\gamma^{\mu_n}) = 0$ if n is odd;
- $\mathrm{Tr}\,(\gamma_5) = 0$;
- $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$;
- $\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$;
- $\mathrm{Tr}\,(\gamma_5\gamma^\mu\gamma_\nu) = 0$.

1.6 Photon polarization sums

Let us now consider a Feynman amplitude involving only an external photon,

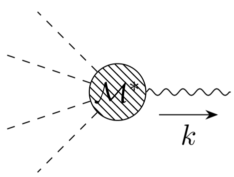
$\mathcal{M} =$



$$= \varepsilon_\mu(k)\mathcal{M}^\mu, \tag{1.89}$$

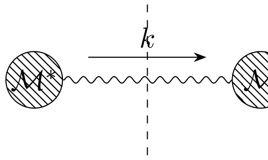
and its complex conjugate,

$\mathcal{M}^* =$



$$= \varepsilon^*_\mu(k)\mathcal{M}^{*\mu}. \tag{1.90}$$

The unpolarized squared amplitude can be diagrammatically represented as

$$\left|\bar{\mathcal{M}}\right|^2 =$$ 

$$= \sum_{pol} \mathcal{M}^* \mathcal{M} = \sum_{pol} \varepsilon_\mu(k)\varepsilon^*_\nu(k)\mathcal{M}^\mu \mathcal{M}^{*\nu} = -\mathcal{M}^\mu \mathcal{M}^*_\mu, \tag{1.91}$$

where in the last passage we have used the completeness relation for photon polarizations,

$$\sum_{pol} \varepsilon_\mu(k)\varepsilon^*_\nu(k) = -g_{\mu\nu}. \tag{1.92}$$

Ward identity

Gauge invariance of the theory implies the gauge invariance of the Feynman amplitudes. If we perform the gauge transformation

$$\begin{aligned} A^\mu &\longrightarrow A'^\mu = A^\mu + \partial^\mu f && \text{in the position space,} \\ \varepsilon^\mu(k) &\longrightarrow \varepsilon'^\mu(k) = \varepsilon^\mu(k) + \alpha k^\mu && \text{in the momentum space,} \end{aligned}$$

on the amplitude (1.89), we get

$$\mathcal{M} \longrightarrow \mathcal{M}' = \mathcal{M} + \alpha k^\mu \mathcal{M}_\mu. \tag{1.93}$$

Invariance under this transformation requires

$$k^\mu \mathcal{M}_\mu = 0, \tag{1.94}$$

i.e. when the external photon polarization vector is replaced by the corresponding four-momentum, the amplitude must vanish. This result is usually referred to as *Ward identity* and is valid at every order in perturbation theory. Be careful: a given process may receive contribution, at a certain order of perturbation theory, from several Feynman diagrams. The corresponding integrals are not required to be individually gauge-invariant, even though their sum is. The generalization to the case in which an arbitrary number of external photons appear in the amplitude,

$$\mathcal{M} = \varepsilon^{\mu_1} \cdots \varepsilon^{\mu_n} \mathcal{M}_{\mu_1 \cdots \mu_n}, \tag{1.95}$$

is straightforward, as there is a Ward identity holding for each photon.

In order to investigate the link between Lorentz invariance and gauge invariance, we note that a gauge transformation can be seen as a transformation of the Little Group, i.e. a Lorentz transformation Λ that preserves a given momentum k^μ,

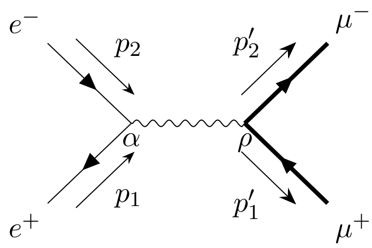
$$\Lambda^\mu_\nu k^\nu = k^\mu. \tag{1.96}$$

1.7 Muon pair production in $(e^- e^+)$ collision

Let us consider the scattering process

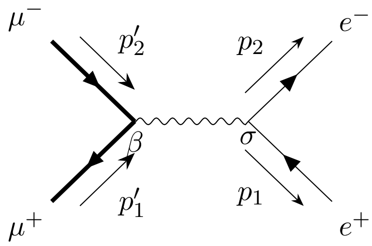
$$e^+(p_1, r_1) + e^-(p_2, r_2) \longrightarrow \mu^+(p'_1, s_1) + \mu^-(p'_2, s_2), \tag{1.97}$$

where r and s label spins. The Feynman amplitude for this process is given by

$$\mathcal{M} =$$ 

$$= (ie)^2 \bar{v}_{r_1}(p_1)\gamma_\alpha u_{r_2}(p_2) \frac{-ig^{\alpha\rho}}{(p_1+p_2)^2} \bar{u}_{s_2}(p'_2)\gamma_\rho v_{s_1}(p'_1),$$

and

$\mathcal{M}^* =$ 

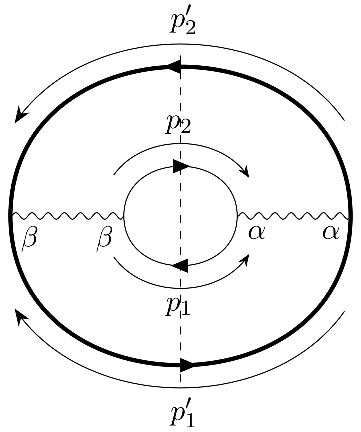
$\cdot (-1) = (ie)^2 \bar{v}_{s_1}(p_1')\gamma_\beta u_{s_2}(p_2') \frac{ig^{\beta\sigma}}{(p_1+p_2)^2} \bar{u}_{r_2}(p_2)\gamma_\sigma v_{r_1}(p_1).$

For the unpolarized amplitude

$$|\bar{\mathcal{M}}|^2 = \frac{1}{4} \sum_{r_1,r_2,s_1,s_2} |\mathcal{M}|^2 \tag{1.98}$$

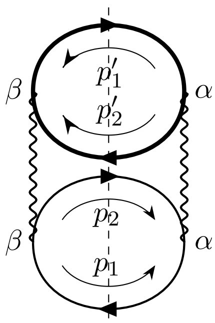
we use a "pictorial" representation

$$|\bar{\mathcal{M}}|^2 = \frac{1}{4} \sum \mathcal{M}^* \mathcal{M} = \tag{1.99}$$



This diagram can be "stretched" to obtain the equivalent one:

$$|\bar{\mathcal{M}}|^2 = \tag{1.100}$$



in which we can distinguish two different fermionic loops. Probably, at this level of complexity in scattering events, this stretching-trick" seems useless or not necessary. Nevertheless, it helps us to identify different loops and it will be very useful for more articulated diagrams, as we'll appreciate later on.

Using what we have learnt in the previous sections we can write

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{4\left((p_1+p_2)^2\right)^2} A_{\alpha\beta} B^{\alpha\beta} \tag{1.101}$$

where we have called $A_{\alpha\beta}$ the loop related to muons

$$A_{\alpha\beta} = \mathrm{Tr}\left(\frac{\not{p}_2' + m_\mu}{2m_\mu} \gamma_\alpha \frac{\not{p}_1' - m_\mu}{2m_\mu} \gamma_\beta \right) \tag{1.102}$$

and $B^{\alpha\beta}$ the loop related to electrons

$$B^{\alpha\beta} = \text{Tr}\left(\frac{\not{p}_1 - m_e}{2m_e}\gamma^\alpha \frac{\not{p}_2 + m_e}{2m_e}\gamma^\beta\right). \tag{1.103}$$

Computing these traces, we can take advantage of linearity and consider separately

- traces with four γ-matrices;
- traces with two γ-matrices;
- traces with an odd number of γ-matrices (but they don't contribute).

Therefore we obtain

$$A_{\alpha\beta} = \frac{1}{4m_\mu^2}\left(\text{Tr}(\not{p}'_2\gamma_\alpha \not{p}'_1\gamma_\beta) - m_\mu^2 \text{Tr}(\gamma_\alpha\gamma_\beta)\right) \tag{1.104}$$

and since

$$\text{Tr}(\gamma^\alpha\gamma^\beta) = 4g^{\alpha\beta},$$
$$\text{Tr}(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta) = 4(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}),$$

it follows that

$$A_{\alpha\beta} = \frac{4}{4m_\mu^2}\left(p'_{1\alpha}p'_{2\beta} + p'_{2\alpha}p'_{1\beta} - (m_\mu^2 + p'_1 \cdot p'_2)g_{\alpha\beta}\right). \tag{1.105}$$

Similarly one finds

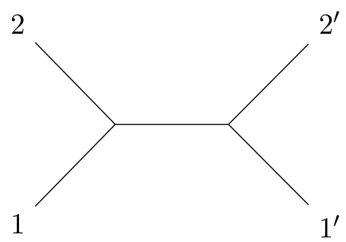
$$B^{\alpha\beta} = \frac{4}{4m_e^2}\left(p_1^\alpha p_2^\beta + p_2^\alpha p_1^\beta - (m_e^2 + p_1 \cdot p_2)g^{\alpha\beta}\right). \tag{1.106}$$

Substituting into equation (1.101)

$$\begin{aligned} |\bar{\mathcal{M}}|^2 = \frac{e^4}{2m_e^2 m_\mu^2\left((p_1+p_2)^2\right)^2} &\left((p_1\cdot p'_1)(p_2\cdot p'_2) + (p_1\cdot p'_2)(p_2\cdot p'_1) + \right. \\ &\underbrace{\left. + m_e^2(p'_1\cdot p'_2) + m_\mu^2(p_1\cdot p_2) + 2m_e^2 m_\mu^2\right)}_{\equiv X} \end{aligned} \tag{1.107}$$

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{2m_e^2 m_\mu^2\left((p_1+p_2)^2\right)^2}\cdot X \tag{1.108}$$

We observe that this result can be used as a "building block" for all processes which have the following shape



$$. \tag{1.109}$$

Indeed, one can write $|\bar{\mathcal{M}}|^2$ directly introducing into equation (1.107) momenta related to particles $1, 2, 2'$ and $1'$ instead of, respectively, p_1, p_2, p'_2 and p'_1.

So far we have worked in an arbitrary reference frame. We now specialize to the CoM frame, as showed in Figure (1.4):

$$e^+ \to (E, \mathbf{p}) = p_1 \qquad e^- \to (E, -\mathbf{p}) = p_2 \qquad \mu^+ \to (E, \mathbf{p}') = p'_1 \qquad \mu^- \to (E, -\mathbf{p}') = p'_2$$

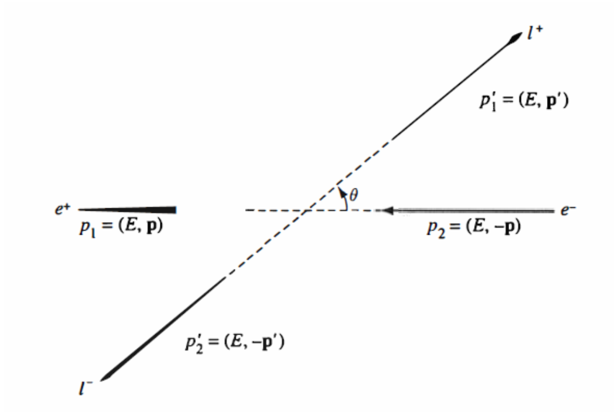


Figure 1.4: Kinematics for the process $e^+e^- \to \mu^+\mu^-$ in the CoM system ($l \equiv \mu$)

with on-shell conditions $m_e^2 = E^2 - |\mathbf{p}|^2$, $m_\mu^2 = E^2 - |\mathbf{p}'|^2$. In this case, the scattering is always planar and it's regulated by an angle θ. Energy conservation implies that the particles must scatter the same energy (two equal particles). The cross section formula yields the equation

$$\frac{d\sigma}{d\Omega}\Big|_{CoM} = \frac{1}{64\pi^2}\frac{1}{(E_1+E_2)^2}\frac{|\mathbf{p_1'}|}{|\mathbf{p_1}|}\prod_l (2m_l)|\bar{\mathcal{M}}|^2 \tag{1.110}$$

where $l = e^-, e^+, \mu^-, \mu^+$ and $(E_1+E_2)^2 = (2E)^2$ leads to

$$\frac{d\sigma}{d\Omega}\Big|_{CoM} = \frac{1}{16\pi^2}\frac{1}{E^2}\frac{p'}{p}m_e^2 m_\mu^2|\bar{\mathcal{M}}|^2 \tag{1.111}$$

with $|\mathbf{p}'| \equiv p'$ and $|\mathbf{p}| \equiv p$. Furthermore, since $E \geq m_\mu \approx 207 m_e$ it's a good approximation to take $p = |\mathbf{p}| = E$ and to neglect terms proportional to m_e^2. Substituting equation (1.108), we obtain

$$\frac{d\sigma}{d\Omega}\Big|_{CoM} \approx \frac{1}{16\pi^2}\frac{1}{E^2}\frac{p'}{E}m_e^2 m_\mu^2 \frac{e^4}{2m_e^2 m_\mu^2}\frac{1}{(2E)^4}X = \frac{e^4}{16\pi^2}\frac{1}{2E^2}\frac{p'}{E}\frac{1}{16E^4}X \tag{1.112}$$

and so we have to compute X. The kinematic factors occurring in equation (1.107) now take the form

$$p_1 \cdot p_1' = E^2 - pp'\cos\theta = p_2 \cdot p_2', \qquad p_1 \cdot p_2 = E^2 + p^2,$$
$$p_1 \cdot p_2' = E^2 + pp'\cos\theta = p_2 \cdot p_1', \qquad p_1' \cdot p_2' = E^2 + p'^2, \qquad (p_1+p_2)^2 = 4E^2,$$

hence (writing also terms proportional to m_e^2)

$$X = \Big((E^2 - pp'\cos\theta)^2 + (E^2 + pp'\cos\theta)^2 + m_e^2(E^2+p'^2) + m_\mu^2(E^2+p^2) + 2m_e^2 m_\mu^2\Big). \tag{1.113}$$

Taking the limit $m_e \to 0$ ($p \to E$), we find

$$X = \Big((E^2 - pp'\cos\theta)^2 + (E^2 + pp'\cos\theta)^2 + m_\mu^2(E^2+p^2)\Big)_{|p=E} = \tag{1.114}$$
$$= \Big(2E^4 + 2p^2p'^2\cos^2\theta + m_\mu^2(E^2+p^2)\Big)_{|p=E} = \tag{1.115}$$
$$= 2E^2(E^2 + p'^2\cos^2\theta + m_\mu^2) \tag{1.116}$$

and finally

$$\frac{d\sigma}{d\Omega}\Big|_{CoM} = \frac{e^4}{16\pi^2}\frac{1}{2E^2}\frac{p'}{E}\frac{1}{16E^4}2E^2(E^2 + p'^2\cos^2\theta + m_\mu^2) = \tag{1.117}$$
$$= \frac{\alpha^2}{16E^4}\left(\frac{p'}{E}\right)\left(E^2 + m_\mu^2 + p'^2\cos^2\theta\right) \tag{1.118}$$

with $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$.
If the energy of the beam is very high $E \gg m_\mu$ (ultra-relativistic limit), we can neglect also the muon mass and so

$$\frac{d\sigma}{d\Omega}\Big|_{CoM}^{UR} \approx \frac{\alpha^2}{16E^4} E^2 (1 + \cos^2\theta) = \frac{\alpha^2}{16E^2}(1 + \cos^2\theta). \tag{1.119}$$

In order to calculate the total cross section we have to integrate over Ω

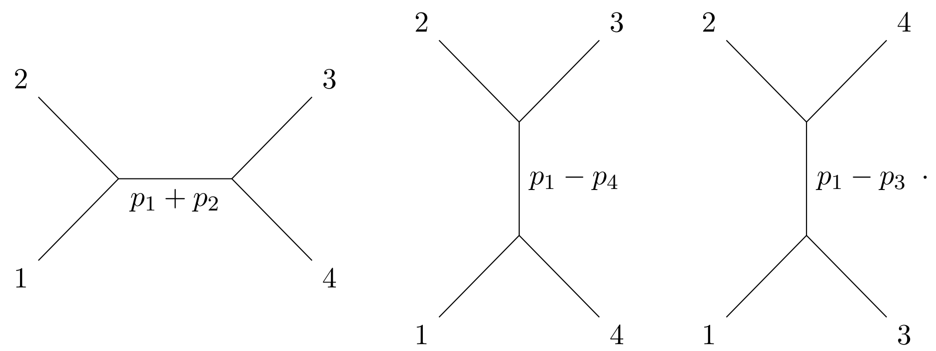
$$\sigma_{Tot} = \int d\Omega \frac{d\sigma}{d\Omega} = \int_{-1}^{1} d(\cos\theta) \int_0^{2\pi} d\phi \frac{d\sigma}{d\Omega} = \tag{1.120}$$

$$= 2\pi \frac{\alpha^2}{16E^2} \int_{-1}^{1} d(\cos\theta)(1 + \cos^2\theta) = \frac{\pi}{3}\frac{\alpha^2}{E^2}. \tag{1.121}$$

As we have already pointed out, formula (1.107) gives the unpolarized squared amplitude associated to the skeleton diagram of a two-to-two process where momenta are labeled as in Figure 1.109, irrespective of which kind of particles is involved. It is true in general that inequivalent Feynman diagrams associated to a scattering processes can be related to *Mandelstam variables*,

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2, \tag{1.122}$$
$$t \equiv (p_1 - p_4)^2 = (p_2 - p_3)^2, \tag{1.123}$$
$$u \equiv (p_1 - p_3)^2 = (p_2 - p_4)^2, \tag{1.124}$$

according to the momentum carried by the propagator. The three different channels are called *s channel*, *t channel* and *u channel* respectively:



. (1.125)

The following identity holds,

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2, \tag{1.126}$$

as can been easily proved by direct calculation,

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 = \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2(p_1 \cdot p_2) - 2(p_1 \cdot p_3) - 2(p_1 \cdot p_4) = \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2(-p_1^2 + p_1 \cdot p_3 + p_1 \cdot p_4) - 2(p_1 \cdot p_3) - 2(p_1 \cdot p_4), \end{aligned}$$

where in the last passage we have used momentum conservation.

1.8 Crossing symmetries

When two Feynman diagrams have an identical structure in terms of lines and vertices and can be mapped one into the other by an appropriate relabeling of the momenta, one speaks of a *crossing symmetry.* As an example, let us consider the two following processes: electron-muon elastic scattering,

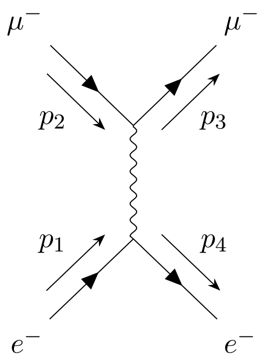
$$e^- + \mu^- \longrightarrow e^- + \mu^-, \tag{1.127}$$

and muon pair production,

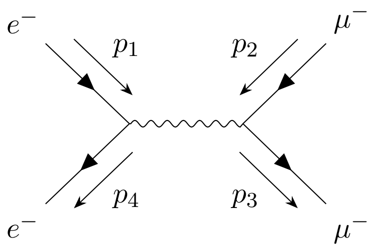
$$e^+ + e^- \longrightarrow \mu^+ + \mu^-. \tag{1.128}$$

Their Feynman amplitudes,

$$\mathcal{M}_{e^-+\mu^-\longrightarrow e^-+\mu^-} =$$



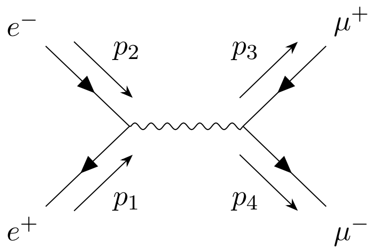
$$=$$



(1.129)

and

$$\mathcal{M}_{e^++e^-\longrightarrow \mu^++\mu^-} =$$



(1.130)

can be obtained one from the other upon the substitution

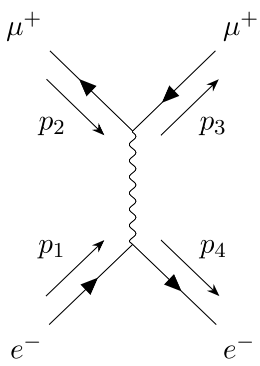
$$\mathcal{M}_{e^-+\mu^-\longrightarrow e^-+\mu^-} = \mathcal{M}_{e^++e^-\longrightarrow \mu^++\mu^-}(p_1, p_2, p_3, p_4) \to (-p_4, p_1, -p_2, p_3). \tag{1.131}$$

Another example is the electron-antimuon scattering,

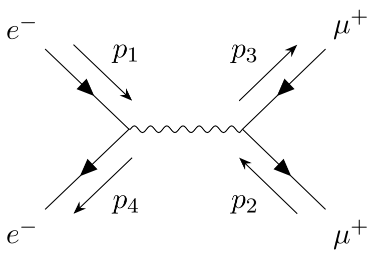
$$e^- + \mu^+ \longrightarrow e^- + \mu^+, \tag{1.132}$$

whose Feynman amplitude is

$$\mathcal{M}_{e^-+\mu^+\longrightarrow e^-+\mu^+} =$$



$$=$$



(1.133)

and can be obtained upon the substitution,

$$\mathcal{M}_{e^-+\mu^+\longrightarrow e^-+\mu^+} = \mathcal{M}_{e^++e^-\longrightarrow \mu^++\mu^-}(p_1, p_2, p_3, p_4) \to (-p_4, p_1, p_3, -p_2). \tag{1.134}$$

This redefinition of the momenta allow us to avoid explicit calculation and read the unpolarized squared amplitude for the process directly from the formula (1.107) we have previously derived for the muon pair production,

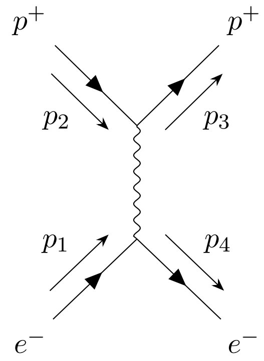
$$\begin{aligned} |\bar{\mathcal{M}}|^2 = \frac{e^4}{2m_e^2 m_\mu^2} \frac{1}{[(p_1 - p_4)^2]^2} \{ &(-p_3 \cdot p_4)(-p_1 \cdot p_2) + (p_2 \cdot p_4)(p_1 \cdot p_3) + \\ &+ m_e^2(-p_2 \cdot p_3) + m_\mu^2(-p_1 \cdot p_4) + 2m_e^2 m_\mu^2 \}. \end{aligned} \tag{1.135}$$

1.9 Electron-proton elastic scattering

Let us now promote the muon μ^+ to a proton p^+ of mass M and consider the electron-proton elastic scattering

$$e^-(p_1) + p^+(p_2) \longrightarrow p^+(p_3) + e^-(p_4). \tag{1.136}$$

At the lowest perturbative order, the Feynman diagram associated to this process is



. (1.137)

The Feynman amplitude squared in this case is

$$|\bar{\mathcal{M}}|^2 = \frac{e^4}{2m^2M^2}\frac{1}{[(p_1-p_4)^2]^2}((p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_3)(p_2\cdot p_4) \tag{1.138}$$

$$- m^2(p_2\cdot p_3) - M^2(p_1\cdot p_4) + 2m^2M^2) \tag{1.139}$$

In the frame of reference in which the proton is at rest, we have

$$\begin{aligned} p_1 &= (E, \mathbf{p}_i), \qquad p_4 = (E, \mathbf{p}_f),\\ p_2 &= (M, \mathbf{0}), \qquad p_3 = (M, \mathbf{0}). \end{aligned}$$

Now, since

$$|\mathbf{p}_i| = |\mathbf{p}_f| = p, \tag{1.140}$$

the scalar product between the initial and final 3-momentum of the electron is

$$\mathbf{p}_i\cdot\mathbf{p}_f = p^2\cos\theta = v^2E^2\cos\theta, \tag{1.141}$$

where we have defined

$$v \equiv \frac{p}{E} = \sqrt{1-\frac{m^2}{E^2}} \tag{1.142}$$

and

$$\begin{aligned} p_1\cdot p_2 = p_1\cdot p_3 &= p_3\cdot p_4 = p_2\cdot p_4 = EM,\\ p_1\cdot p_4 &= E^2 - \mathbf{p}_i\cdot\mathbf{p}_f = E^2 - p^2\cos\theta = E^2(1-v^2\cos\theta),\\ p_2\cdot p_3 &= M^2,\\ (p_1-p_4)^2 &= -(\mathbf{p}_i-\mathbf{p}_f)^2 = -(2p^2 - 2\mathbf{p}_i\cdot\mathbf{p}_f) = -2p^2(1-\cos\theta). \end{aligned}$$

Therefore,

$$\begin{aligned} |\bar{\mathcal{M}}|^2 &= \frac{e^4}{2m^2}\frac{1}{4p^4(1-\cos\theta)^2}\left(E^2+m^2+E^2v^2\cos\theta\right) =\\ &= \frac{e^4}{8m^2}\frac{E^2}{p^4(1-\cos\theta)^2}\left(2-v^2(1-\cos\theta)\right) =\\ &= \frac{e^4}{16m^2}\frac{E^2}{p^4\sin^4\frac{\theta}{2}}\left(1-v^2\sin^2\frac{\theta}{2}\right) =\\ &= \frac{e^4}{16m^2}\frac{1}{v^4E^2\sin^4\frac{\theta}{2}}\left(1-v^2\sin^2\frac{\theta}{2}\right) \end{aligned} ,$$

where we have used

$$m^2 = E^2 - p^2 = E^2(1 - v^2) \tag{1.143}$$

and the trigonometric identity

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}. \tag{1.144}$$

Since the center-of-mass frame is essentially the lab frame, we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{1}{(E+M)^2}4m^2 4M^2|\bar{\mathcal{M}}|^2. \tag{1.145}$$

Moreover, in the $E \ll M$ limit, we can approximate $E + M \approx M$, obtaining

$$\frac{d\sigma}{d\Omega} \approx \frac{1}{64\pi^2}16m^2|\bar{\mathcal{M}}|^2 = \frac{e^4}{64\pi^2}\frac{1}{v^4E^2\sin^4\frac{\theta}{2}}\left(1 - v^2\sin^2\frac{\theta}{2}\right), \tag{1.146}$$

which is called *Mott formula* since it was first derived by Mott in the context of Special Relativity.

In the non relativistic limit, in which $v \ll 1$ and $E \approx m$, the familiar *Rutherford formula* is recovered,

$$\frac{d\sigma}{d\Omega} \approx \frac{e^4}{64\pi^2}\frac{E^2}{p^4}\frac{1}{\sin^4\frac{\theta}{2}} \approx \frac{e^4}{64\pi^2}\frac{m^2}{p^4}\frac{1}{\sin^4\frac{\theta}{2}}. \tag{1.147}$$

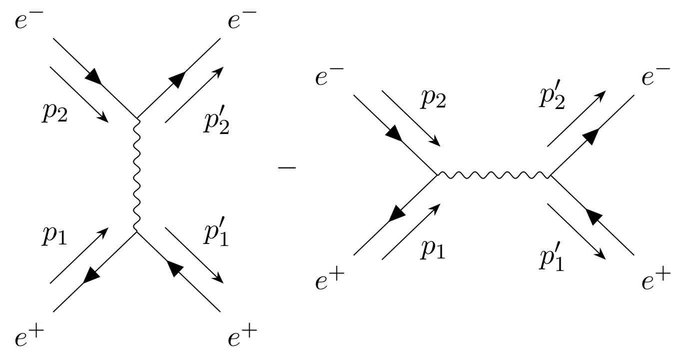
1.10 Bhabha scattering

The electron-positron elastic scattering,

$$e^+(p_1) + e^-(p_2) \longrightarrow e^+(p_1') + e^-(p_2'), \tag{1.148}$$

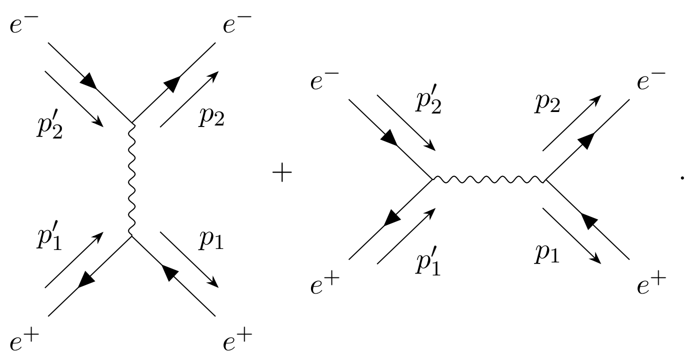
is a little more complicated than the electron-muon scattering since now there are two inequivalent Feynman diagrams (a t-channel and an s-channel) contributing to the scattering amplitude at the lowest perturbative order,

$$\mathcal{M} = \mathcal{M}_a - \mathcal{M}_b = \tag{1.149}$$

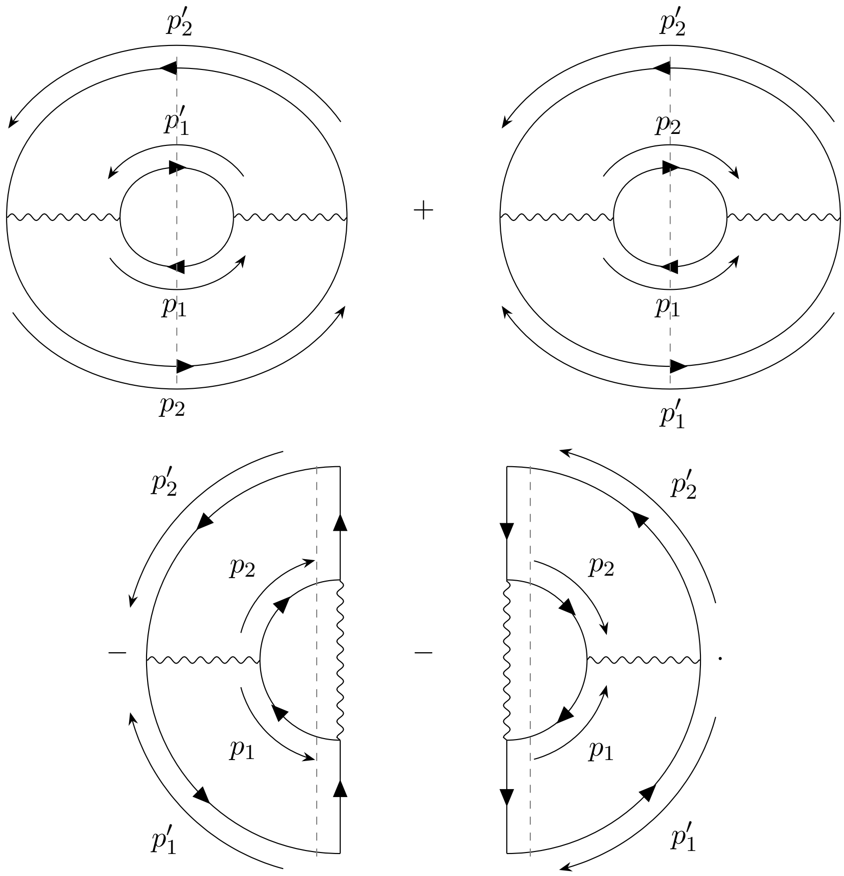


where the relative sign between the two diagrams is a *minus* because of the odd number of external fermionic exchanges. The conjugate amplitude is

$$\mathcal{M}^* = \mathcal{M}_a^* - \mathcal{M}_b^* = - \tag{1.150}$$



The unpolarized squared amplitude receives contribution from the four following cut diagrams,



We observe that the first two terms contain two closed fermionic loop, while the other two have only one loop. Thus we expect to have two traces and one single trace respectively. Hence, using the approximation $E \gg m$, we have

$$|\bar{\mathcal{M}}|^2 = \sum_{spins} \frac{1}{4} \left(|\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 - \mathcal{M}_a \mathcal{M}_b^* - \mathcal{M}_a^* \mathcal{M}_b \right) \tag{1.151}$$

with

$$\sum_{spins} \frac{1}{4} |\mathcal{M}_a|^2 = \frac{e^4}{2m^4} \frac{1}{[(p_1 - p_1')^2]^2} \left((p_1 \cdot p_2')(p_2 \cdot p_1') + (-p_1 \cdot p_2)(-p_1' \cdot p_2') + o(m^2) \right), \tag{1.152}$$

$$\sum_{spins} \frac{1}{4} |\mathcal{M}_b|^2 = \frac{e^4}{2m^4} \frac{1}{[(p_1 + p_2)^2]^2} \left((p_1 \cdot p_2')(p_2 \cdot p_1') + (p_1 \cdot p_1')(p_2 \cdot p_2') + o(m^2) \right) \tag{1.153}$$

and

$$\begin{aligned} \sum_{spins} \frac{1}{4} \mathcal{M}_a \mathcal{M}_b^* &= \frac{e^4}{4(p_1 - p_1')^2 (p_1 + p_2)^2} \mathrm{Tr} \left(\frac{\not{p}_2' + m}{2m} \gamma_\alpha \frac{\not{p}_2 + m}{2m} \gamma_\beta \frac{\not{p}_1 - m}{2m} \gamma^\alpha \frac{\not{p}_1' - m}{2m} \gamma^\beta \right) = \\ &= \frac{e^4}{64 m^4 (p_1 - p_1')^2 (p_1 + p_2)^2} \left(\mathrm{Tr}(\not{p}_2' \gamma_\alpha \not{p}_2 \gamma_\beta \not{p}_1 \gamma^\alpha \not{p}_1' \gamma^\beta) + o(m^2) \right). \end{aligned} \tag{1.154}$$

In order to compute the traces we remember the contractions:

$$\gamma_\lambda \gamma^\lambda = 4, \tag{1.155}$$

$$\gamma_\lambda \gamma_\alpha \gamma^\lambda = (-\gamma_\alpha \gamma_\lambda + 2 g_{\lambda\alpha}) \gamma^\lambda = -2\gamma_\alpha, \tag{1.156}$$

$$\gamma_\lambda \gamma_\alpha \gamma_\beta \gamma^\lambda = \cdots = 4 g_{\alpha\beta}, \tag{1.157}$$

$$\gamma_\lambda \gamma_\alpha \gamma_\beta \gamma_\gamma \gamma^\lambda = \cdots = -2 \gamma_\gamma \gamma_\beta \gamma_\alpha. \tag{1.158}$$

hence

$$
\begin{aligned}
\mathrm{Tr}(\not{p}_2'\gamma_\alpha\not{p}_2\gamma_\beta\not{p}_1\gamma^\alpha\not{p}_1'\gamma^\beta) &= \mathrm{Tr}(p_2'^\mu p_2^\nu p_1^\rho p_1'^\sigma\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta\gamma_\rho\gamma^\alpha\gamma_\sigma\gamma^\beta) = \\
&= p_2'^\mu p_2^\nu p_1^\rho p_1'^\sigma \mathrm{Tr}(\gamma_\mu(-2\gamma_\rho\gamma_\beta\gamma_\nu)\gamma_\sigma\gamma^\beta) = \\
&= -2p_2'^\mu p_2^\nu p_1^\rho p_1'^\sigma \mathrm{Tr}(\gamma_\mu\gamma_\rho(4g_{\nu\sigma})) = \\
&= -8(p_2\cdot p_1')p_2'^\mu p_1^\rho(4g_{\mu\rho}) = \\
&= -32(p_2\cdot p_1')(p_2'\cdot p_1).
\end{aligned}
$$

Finally, we obtain

$$
\sum_{spins}\frac{1}{4}\mathcal{M}_a\mathcal{M}_b^* = \frac{-e^4}{2m^4(p_1-p_1')^2(p_1+p_2)^2}\Big((p_1\cdot p_2')(p_2\cdot p_1') + o(m^2)\Big). \tag{1.159}
$$

We see from this equation that $\mathcal{M}_a\mathcal{M}_b^*$ is real, so $\mathcal{M}_a\mathcal{M}_b^* = \mathcal{M}_a^*\mathcal{M}_b$. In order to go on, we evaluate the process in the CoM system in which

$$
(p_1\cdot p_2')(p_2\cdot p_1') = (E^2(1+\cos\theta))^2 = 4E^4\cos^4\left(\frac{\theta}{2}\right), \tag{1.160}
$$

$$
(p_1-p_1')^2 = -4E^2\sin^2\left(\frac{\theta}{2}\right), \tag{1.161}
$$

$$
(p_1+p_2)^2 = 4E^2. \tag{1.162}
$$

Therefore

$$
\sum_{spins}\frac{1}{4}|\mathcal{M}_a|^2 = \frac{e^4}{8m^4\sin^4\left(\frac{\theta}{2}\right)}\left(1+\cos^4\left(\frac{\theta}{2}\right)+o(m^2)\right), \tag{1.163}
$$

$$
\sum_{spins}\frac{1}{4}|\mathcal{M}_b|^2 = \frac{e^4}{16m^4}\Big(1+\cos^2\theta + o(m^2)\Big), \tag{1.164}
$$

$$
\sum_{spins}\frac{1}{4}\mathcal{M}_a\mathcal{M}_b^* = \frac{e^4}{8m^4\sin^2\left(\frac{\theta}{2}\right)}\left(\cos^4\left(\frac{\theta}{2}\right)+o(m^2)\right). \tag{1.165}
$$

If we combine all these results in the cross section formula we find

$$
\frac{d\sigma}{d\Omega}|_{CoM} = \frac{m^4}{16\pi^2}\frac{1}{E^2}\frac{|p'|}{|p|}|\bar{\mathcal{M}}|^2 = \frac{\alpha^2}{8E^2}\left(\frac{1+\cos^4\left(\frac{\theta}{2}\right)}{\sin^4\left(\frac{\theta}{2}\right)} + \frac{1+\cos^2(\theta)}{2} - 2\frac{\cos^4\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\right) \tag{1.166}
$$

and we observe that the minus sign before the third term derives from the sign between "a" and "b" diagrams.

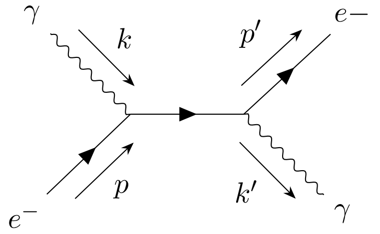
1.11 Compton scattering

In this process a photon is present in both the initial and final states, and we shall apply our earlier results to carry out the photon polarization sums, as well as the electron spin sums. In particular we consider

$$
e^-(p) + \gamma(k) \to e^-(p') + \gamma(k') \tag{1.167}
$$

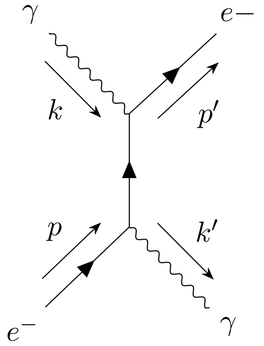
with $p = (E, \mathbf{p})$, $k = (\omega, \mathbf{k})$, $p' = (E', \mathbf{p}')$ and $k' = (\omega', \mathbf{k}')$. In lowest order, the Feynman amplitude $\mathcal{M}$ results from the two Feynman graphs

a) $\mathcal{M}_a =$



$$= -ie^2 \bar{u}' \not{\varepsilon}'^* \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} \not{\varepsilon} u \qquad (s\text{-channel}), \tag{1.168}$$

b) $\mathcal{M}_b =$



$$= -ie^2 \bar{u}' \not{\varepsilon} \frac{\not{p} - \not{k}' + m}{(p-k')^2 - m^2} \not{\varepsilon}'^* u \qquad (t\text{-channel}), \tag{1.169}$$

whence

$$\mathcal{M} = \mathcal{M}_a + \mathcal{M}_b = -ie^2 \bar{u}' \varepsilon_\nu'^* \varepsilon_\mu \left(\gamma^\nu \frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} \gamma^\mu + \gamma^\mu \frac{\not{p} - \not{k}' + m}{(p-k')^2 - m^2} \gamma^\nu \right) u. \tag{1.170}$$

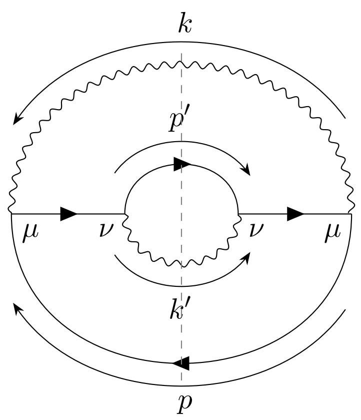
Let us concentrate on denominators. In general, the presence of masses guarantees non divergent objects but using on shell conditions $p^2 = p'^2 = m^2$, $k^2 = k'^2 = 0$ we find

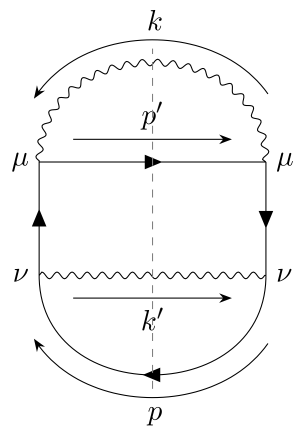
$$(p+k)^2 - m^2 = p^2 + 2p \cdot k + k^2 - m^2 = 2p \cdot k, \tag{1.171}$$

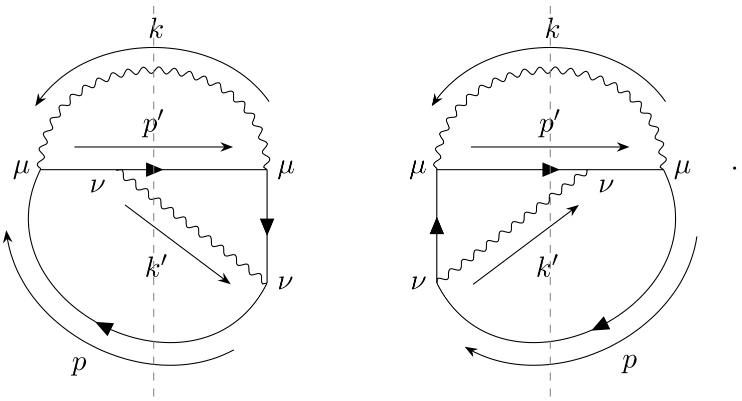
$$(p-k')^2 - m^2 = -2p \cdot k'. \tag{1.172}$$

Therefore singularities may appear when the previous scalar products vanish (i.e. we expect to have singularities when we have emission and absorption of massless quanta); classification of those singularities is very important in Quantum Field Theory.

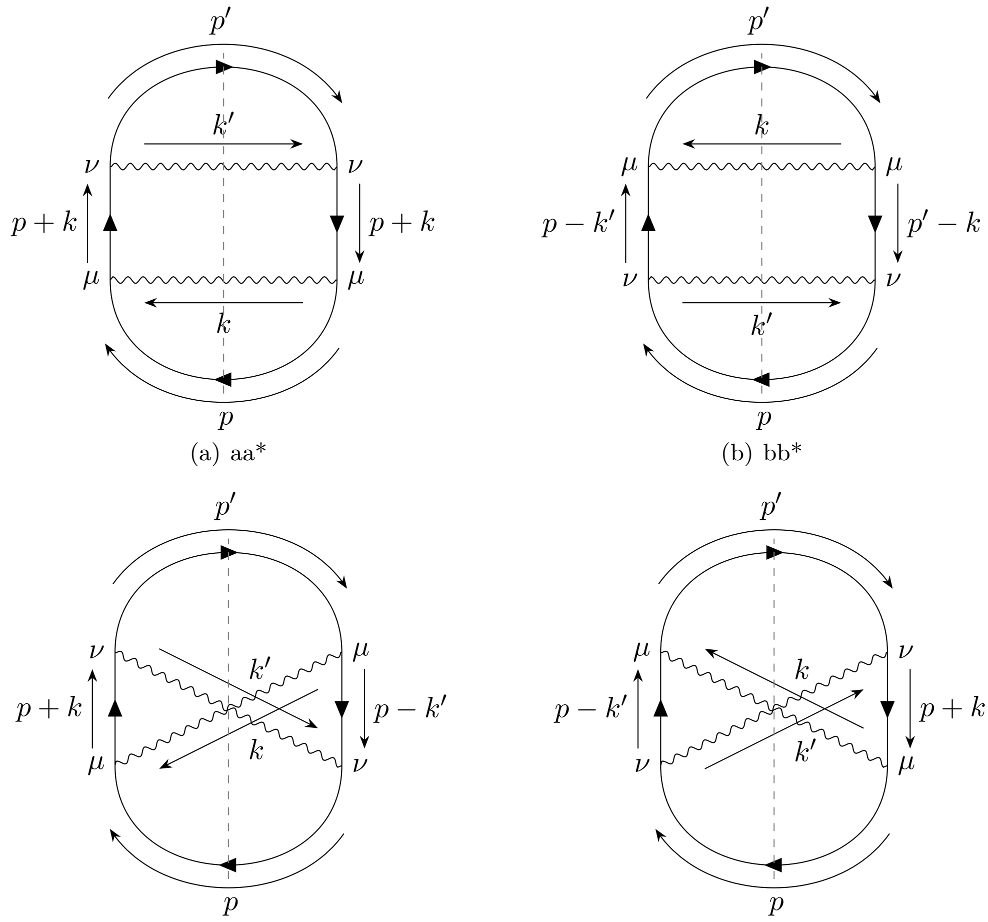
Contributions to the square of the amplitude are represented by







Before continuing, let us reshape these diagrams as shown in Figure 1.5. We note that



(a) aa* (b) bb* (c) ab* (d) ba*

Figure 1.5

- we have only fermion traces;
- we can obtain the second (fourth) diagram by relabeling k in the first (third).

Analogously to Bhabha scattering and taking into consideration the additional summation over the photon polarization, one finds

$$
\begin{aligned}
|\bar{\mathcal{M}}|^2 &= \frac{1}{4} \sum_{pol,spin} \Big(|\mathcal{M}_a|^2 + |\mathcal{M}_b|^2 + \mathcal{M}_a \mathcal{M}_b^* + \mathcal{M}_a^* \mathcal{M}_b \Big) = \\
&= \frac{e^4}{64m^2} \left(\frac{X_{aa}}{(p\cdot k)^2} + \frac{X_{bb}}{(p\cdot k')^2} - \frac{X_{ab}+X_{ba}}{(p\cdot k)(p\cdot k')} \right)
\end{aligned}
$$

with

$$
X_{aa} = \mathrm{Tr}\left(\gamma^\nu (\not{p} + \not{k}) + m)\gamma^\mu (\not{p} + m)\gamma_\mu ((\not{p} + \not{k} + m)\gamma_\nu (\not{p}' + m) \right), \tag{1.173}
$$

$$
X_{ab} = \mathrm{Tr}\left(\gamma^\nu (\not{p} + \not{k}) + m)\gamma^\mu (\not{p} + m)\gamma_\nu ((\not{p}' - \not{k} + m)\gamma_\mu (\not{p}' + m) \right) \tag{1.174}
$$

Note that the factor $\frac{e^4}{64m^2}$ can be written as ($\frac{1}{4} \cdot \frac{1}{2^2}$(coming from the fermion loop) . $\frac{1}{2m^2}$ (related to the fermion cuts)). and one can obtain X_{bb} and X_{ba} from X_{aa} and X_{ab} with the substitutions

$$
k \leftrightarrow -k', \qquad \varepsilon \leftrightarrow \varepsilon'^*.
$$

Traces calculation leads to

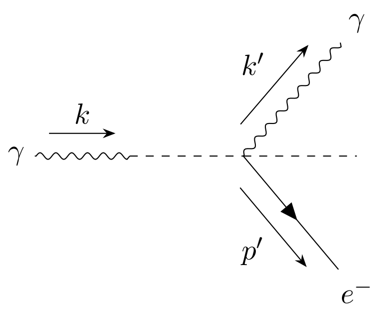
$$
X_{aa} = 32(m^4 + m^2(p\cdot k) + (p\cdot k)(p\cdot k')), \tag{1.175}
$$

$$
X_{ab} = 16m^2(2m^2 + (p\cdot k) - (p\cdot k')) \tag{1.176}
$$

hence

$$
|\bar{\mathcal{M}}|^2 = \frac{e^4}{2m^2} \left\{ \left(\frac{p\cdot k}{p\cdot k'} + \frac{p\cdot k'}{p\cdot k} \right) + 2m^2 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right) + m^4 \left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'} \right)^2 \right\}. \tag{1.177}
$$

This result refers to a general reference frame. We are interested in the case in which the photon is incident on a fixed electron (*laboratory frame*).



. (1.178)

In this case, we have $p = (m, \mathbf{0})$, $k = (\omega, \mathbf{k})$, $p' = (E', \mathbf{p}')$ and $k' = (\omega', \mathbf{k}')$, with $|\mathbf{k}| = \omega$ and $|\mathbf{k}'| = \omega'$ because the photon is on shell. The scalar products are

$$
p\cdot k = m\omega, \qquad p\cdot k' = m\omega'. \tag{1.179}
$$

Momentum conservation gives

$$
p + k = p' + k' \quad \Rightarrow \quad \begin{cases} m + \omega = E' + \omega' \\ \mathbf{p}' = \mathbf{k} - \mathbf{k}' \end{cases} . \tag{1.180}
$$

and

$$
\begin{aligned}
E'^2 &= m^2 + |\mathbf{p}'|^2 = m^2 + |\mathbf{k} - \mathbf{k}'|^2 = \\
&= m^2 + |\mathbf{k}|^2 + |\mathbf{k}'|^2 - 2|\mathbf{k}||\mathbf{k}|' \cos\theta = \\
&= m^2 + \omega^2 + \omega'^2 - 2\omega\omega' \cos\theta,
\end{aligned}
$$

hence

$$E' = \sqrt{m^2 + \omega^2 + \omega'^2 - 2\omega\omega'\cos\theta}, \tag{1.181}$$

$$\frac{\partial E'}{\partial \omega'} = \frac{\omega' - \omega\cos\theta}{E'} \overset{(1.180)}{=} -1 + \frac{m + \omega(1-\cos\theta)}{E'}. \tag{1.182}$$

Now, from $(p+k)^2 = (p'+k')^2$ and $p' = p + k - k'$, we have generally

$$p \cdot k = p' \cdot k' = p \cdot k' + k \cdot k' \tag{1.183}$$

and specializing to the lab frame

$$m\omega = m\omega' + \omega\omega' - |\mathbf{k}||\mathbf{k}'|\cos\theta = m\omega' + \omega\omega'(1-\cos\theta). \tag{1.184}$$

Therefore, isolating ω'

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos\theta)}, \tag{1.185}$$

which is the relation between the frequencies before and after the scattering, often written as

$$\frac{1}{\omega} - \frac{1}{\omega'} = \frac{\cos\theta - 1}{m}. \tag{1.186}$$

We can also obtain an equation in terms of the wavelength $\frac{1}{\omega} = \frac{\lambda}{2\pi c}$ reintroducing the appropriate powers of $\hbar$ and c:

$$\frac{1}{\omega} - \frac{1}{\omega'} = \frac{\hbar}{c^2}\left(\frac{\cos\theta - 1}{m}\right) \qquad \Rightarrow \qquad \lambda' - \lambda = \lambda_C(1-\cos\theta), \tag{1.187}$$

where $\lambda_C = \frac{h}{mc}$ is the Compton wavelength of the electron. From the general formula (1.60) for the cross section, we find

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{(2m_e)^2}{64\pi^2 v_{rel} E_1 E_2 E_1' E_2'} |\bar{\mathcal{M}}|^2 |\mathbf{k}'|^2 \left(\frac{\partial(E_1' + E_2')}{\partial|\mathbf{k}'|}\right)^{-1} = \\
&= \frac{m^2}{16\pi^2 v_{rel} E_1 E_1'} \frac{\omega'}{\omega} \left(\frac{\partial(E_1' + \omega')}{\partial\omega'}\right)^{-1} |\bar{\mathcal{M}}|^2 \overset{lab}{=} \\
&= \frac{m^2}{16\pi^2 \left(\frac{|\mathbf{k}|}{\omega}\right) m E_1'} \frac{\omega'}{\omega} \left(\frac{\partial(E_1' + \omega')}{\partial\omega'}\right)^{-1} |\bar{\mathcal{M}}|^2 \overset{(1.182)}{=} \\
&= \frac{1}{16\pi^2} \frac{\omega'}{\omega} \frac{1}{1 + \frac{\omega}{m}(1-\cos\theta)} |\bar{\mathcal{M}}|^2 \overset{(1.185)}{=} \\
&= \frac{1}{16\pi^2} \left(\frac{\omega'}{\omega}\right)^2 |\bar{\mathcal{M}}|^2.
\end{aligned}$$

Specializing (1.177) to the lab frame,

$$|\bar{\mathcal{M}}|^2_{lab} = \frac{e^4}{2m^2}\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right), \tag{1.188}$$

we obtain

$$\frac{d\sigma}{d\Omega}^{lab}_{spin,\ pol.\ sum} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right), \tag{1.189}$$

which is the *fully unpolarized Klein-Nishina* formula. If both initial and final photons are in state of definite polarization, summing and averaging must be carried out on electron spins only. Using the identities

$$\not{p}\not{\varepsilon}u = (-\not{\varepsilon}\not{p} + 2p\cdot\epsilon)u = -m\not{\varepsilon}u + 2p\cdot\varepsilon u,$$
$$\bar{u}'\not{\varepsilon}\not{p}' = \bar{u}'(-\not{p}'\not{\varepsilon} + 2(p'\cdot\varepsilon)) = \bar{u}'(-m\not{\varepsilon} + 2(p'\cdot\varepsilon)),$$

which follow from Dirac equation,

$$(\not{p} - m)u = 0, \qquad \bar{u}'(\not{p}' - m) = 0, \tag{1.190}$$

the amplitude (1.170) can be rewritten as follows,

$$\begin{aligned}\mathcal{M} &= -ie^2\bar{u}'\varepsilon'^*_\nu\varepsilon_\mu\left(\gamma^\nu\frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\gamma^\mu + \gamma^\mu\frac{\not{p}'-\not{k}+m}{(p'-k)^2-m^2}\gamma^\nu\right)u = \\ &= -ie^2\bar{u}'\left(\frac{\not{\varepsilon}'^*\not{k}\not{\varepsilon} + 2(p\cdot\varepsilon)\not{\varepsilon}'^*}{2p\cdot k} + \frac{-\not{\varepsilon}\not{k}\not{\varepsilon}'^* + 2(p'\cdot\varepsilon)\not{\varepsilon}'^*}{-2p'\cdot k}\right)u.\end{aligned} \tag{1.191}$$

Ward identity To verify that Ward identity is satisfied, we consider the transformation

$$\begin{cases}\varepsilon^\mu \to \varepsilon'^\mu = \varepsilon^\mu + \alpha k^\mu, \\ \mathcal{M} \to \mathcal{M}' = \mathcal{M} + \delta\mathcal{M}.\end{cases} \tag{1.192}$$

and, using $k^2 = 0$,

$$\begin{aligned}(\delta\mathcal{M})_{\alpha=1} &= -ie^2\bar{u}'\left(\frac{\not{\varepsilon}'^*\not{k}\not{k} + 2(p\cdot k)\not{\varepsilon}'^*}{2p\cdot k} + \frac{\not{k}\not{k}\not{\varepsilon}'^* - 2(p'\cdot k)\not{\varepsilon}'^*}{2p'\cdot k}\right)u = \\ &= -ie^2\bar{u}'\not{\varepsilon}^{*\prime}u(1-1) = 0.\end{aligned} \tag{1.193}$$

You may check also with ε'^*. Now we are sure that equation (1.191) is gauge invariant, so we can choose a convenient gauge.

Physical gauge A convenient gauge choice is the *physical gauge* (or radiation gauge), in which

$$\varepsilon^\mu = (0, \varepsilon), \qquad \varepsilon'^{*\mu} = (0, \varepsilon'^*), \tag{1.194}$$

with

$$\varepsilon\cdot k = -\varepsilon\cdot\mathbf{k} = 0 = -\varepsilon'^*\cdot\mathbf{k}' = \varepsilon'^*\cdot k'. \tag{1.195}$$

Moreover, in the lab frame we have also

$$p\cdot\varepsilon = p\cdot\varepsilon'^* = 0. \tag{1.196}$$

Thus, the amplitudes in (1.170) become

$$\mathcal{M}_a = -ie^2\bar{u}'\frac{\not{\varepsilon}'^*\not{k}\not{\varepsilon}}{2p\cdot k}u, \qquad \mathcal{M}_b = -ie^2\bar{u}'\frac{\not{\varepsilon}\not{k}'\not{\varepsilon}'^*}{2p'\cdot k}u, \tag{1.197}$$

and

$$\frac{1}{2}\sum_{spin}|\mathcal{M}|^2 = \frac{e^4}{32m^2}\left(\frac{Y_{aa}}{(p\cdot k)^2} + \frac{Y_{bb}}{(p\cdot k')^2} + \frac{Y_{ab}+Y_{ba}}{(p\cdot k)(p\cdot k')}\right), \tag{1.198}$$

where

$$
\begin{aligned}
Y_{aa} &= \mathrm{Tr}\left[\not{\varepsilon}'^{*}\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}\not{k}\not{\varepsilon}'^{*}(\not{p}'+m)\right] = 8(p\cdot k)(2(\varepsilon'^{*}\cdot k)^2+(p\cdot k')),\\
Y_{bb} &= Y_{aa}(k\to -k', \varepsilon\to\varepsilon'^{*}),\\
Y_{ab} &= \mathrm{Tr}\left[\not{\varepsilon}'^{*}\not{k}\not{\varepsilon}(\not{p}+m)\not{\varepsilon}'^{*}\not{k}'\not{\varepsilon}'^{*}(\not{p}'+m)\right]\\
&= 8(p\cdot k)(p\cdot k')\Big[2(\varepsilon\cdot\varepsilon'^{*})^2-1\Big] - 8(k\cdot\varepsilon'^{*})^2(p\cdot k') + 8(k'\cdot\varepsilon)^2(p\cdot k),\\
Y_{ba} &= Y_{ab}(k\to -k', \varepsilon\to\varepsilon'^{*}).
\end{aligned}
$$

Finally, the cross section in the lab frame is

$$
\frac{d\sigma}{d\Omega}_{lab} = \frac{\alpha^2}{4m^2}\left(\frac{\omega'}{\omega}\right)^2\Big(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}+4(\varepsilon\cdot\varepsilon'^{*})^2-2\Big), \tag{1.199}
$$

which is the *polarized photons Klein-Nishina* formula. From it the unpolarized cross section can be obtained averaging over initial and summing over final photon polarizations. In the physical gauge, we can assume that $\mathbf{k}, \boldsymbol{\varepsilon}_1(\mathbf{k}), \boldsymbol{\varepsilon}_2(\mathbf{k})$ and $\mathbf{k}', \boldsymbol{\varepsilon}_1(\mathbf{k}') \equiv \boldsymbol{\varepsilon}_1', \boldsymbol{\varepsilon}_2(\mathbf{k}') \equiv \boldsymbol{\varepsilon}_2'$ form two orthogonal sets respectively and that $\boldsymbol{\varepsilon}_2 \parallel \boldsymbol{\varepsilon}_2'$. With this choice,

$$
\boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_1' = \cos\theta, \qquad \boldsymbol{\varepsilon}_2\cdot\boldsymbol{\varepsilon}_2' = 1, \qquad \boldsymbol{\varepsilon}_1\cdot\boldsymbol{\varepsilon}_2' = \boldsymbol{\varepsilon}_1'\cdot\boldsymbol{\varepsilon}_2 = 0, \qquad \varepsilon\cdot\varepsilon' = -\boldsymbol{\varepsilon}\cdot\boldsymbol{\varepsilon}' \tag{1.200}
$$

and summing and averaging over the polarization term in (1.199),

$$
\frac{1}{2}\sum_{\lambda,\lambda'=1}^{2}|\varepsilon_\lambda\cdot\varepsilon'_{\lambda'}|^2 = \frac{1}{2}\left(\cos^2\theta+1\right), \tag{1.201}
$$

we recover equation (1.189).

In the $\omega \ll m$ limit, we have

$$
\begin{aligned}
\frac{\omega}{\omega'} &= 1+\frac{\omega}{m}x = 1+\delta,\\
\frac{\omega'}{\omega} &= \frac{1}{1+\frac{\omega}{m}x} \simeq 1-\frac{\omega}{m}x = 1-\delta,
\end{aligned}
$$

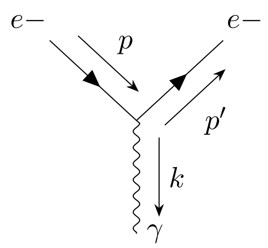
where $x \equiv 1-\cos\theta$ and $\delta = \frac{\omega}{m}x \lll 1$, and (1.189) becomes

$$
\frac{d\sigma}{d\Omega}^{lab}_{spin,\ pol\ sum} \simeq \frac{\alpha^2}{2m^2}(1-2\delta)(1+\delta+1-\delta-1+\cos^2\theta) \simeq \frac{\alpha^2}{2m^2}(1+\cos^2\theta+o(\delta)), \tag{1.202}
$$

which is the *Thomson formula* for the elastic electron-photon scattering in classical electrodynamics, derived in our case from Feynman diagrams.

1.12 Scattering by an external field

Let us consider the process related to the Feynman diagram



. (1.203)

This is a not physical process because quadrimomentum is not conserved[4]. Indeed, from $p = p' + k$ we find

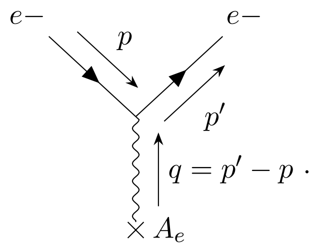
$$m^2 = m^2 + 2p' \cdot k \qquad \Rightarrow \qquad p' \cdot k = 0 = p'^0 k_0 - \mathbf{p}' \cdot \mathbf{k} \tag{1.204}$$

and in the rest frame of the final electron this condition reads

$$p'^0 k_0 - \mathbf{p}' \cdot \mathbf{k} = k^0 (p'^0 + |\mathbf{p}'|)) = 0, \tag{1.205}$$

where we have used the properties $\mathbf{k} = -\mathbf{p}'$ and $k^0 = |\mathbf{k}|$. Since $p'^0 + |\mathbf{p}'| > 0$, momentum conservation implies $k^0 = 0$, namely that there is no photon and the initial electron stays still.

Nevertheless in some problems, we can have meaningful diagrams with the same shape of the previous one. In particular, where the quantum fluctuations of the field are unimportant, it may be adequate to describe the field as a purely classical function of the space-time coordinates. An example would be the scattering of electrons or positrons by an applied external electromagnetic field $A_e^\mu(x)$, such as the Coulomb field. In this case, we can give a new meaning to the previous Feynman diagram. From a "graphical" point of view we introduce a new symbol, marking external source by a cross



(1.206)

Physically, instead, we are studying the scattering of an electron by a static external field defined by its Fourier transform

$$A_e^\mu(x) = A_e^\mu(\mathbf{x}) = \frac{1}{(2\pi)^3} \int d^3\mathbf{q} e^{i\mathbf{q}\cdot\mathbf{x}} A_e^\mu(\mathbf{p}). \tag{1.207}$$

We can consider as initial state an electron with momentum $p^\mu = (E, \mathbf{p})$ and as final state an electron with momentum $p'^\mu = (E', \mathbf{p}')$. Evaluation the matrix element $\langle f| S^{(1)} |i\rangle$ in the momentum space, we get

$$\langle f| S^{(1)} |i\rangle = (2\pi)\delta(E' - E) \left(\frac{m}{VE}\right)^{\frac{1}{2}} \left(\frac{m}{VE'}\right)^{\frac{1}{2}} \mathcal{M} \tag{1.208}$$

and

$$\mathcal{M} = ie\bar{u}'(\mathbf{p}') \not{A}_e(\mathbf{q} = \mathbf{p}' - \mathbf{p}) u(\mathbf{p}). \tag{1.209}$$

Unlike what we have met so far, equation (1.208) does not contain a momentum conserving δ-function, since we are ignoring the momentum of the source of the field (so, in this case, we have no problem of quadrimomentum conservation and Feynman diagram (1.206) is now meaningful). On the other hand, the δ-function in equation (1.208) leads to conservation of the electron' s energy, therefore

$$q = p' - p = (E' - E, \mathbf{p}' - \mathbf{p}) = (0, \mathbf{q}).$$

As usual, the transition probability per unit time yields the equation

$$w = \frac{|\langle f| S^{(1)} |i\rangle|^2}{T} = (2\pi)^2 \left(\delta(E' - E)\right)^2 \left(\frac{m}{VE}\right) \left(\frac{m}{VE'}\right) |\mathcal{M}|^2. \tag{1.210}$$

[4]Since 2004 a new interpretation to this diagram has emerged; in particular, it can be shown that using complex momenta this process become meaningful. Thanks to this new interpretation a completely new branch of Quantum Field Theory was born.

Once again, we have found the square of Dirac delta function which is a formal writing with the meaning introduced in Section 1.3:

$$\left(\delta(E'-E)\right)^2 = \delta(E'-E)\frac{T}{2\pi}. \tag{1.211}$$

Hence

$$w = (2\pi)\delta(E'-E)\left(\frac{m}{VE}\right)\left(\frac{m}{VE'}\right)|\mathcal{M}|^2. \tag{1.212}$$

In order to derive cross section we divide w by the incident electron flux $\Phi = v/V = |\mathbf{p}|/(VE)$ and we multiply by the density of final states

$$d\Pi = \frac{V}{(2\pi)^3}d^3\mathbf{p}' = \frac{V}{(2\pi)^3}|\mathbf{p}'|^2 d|\mathbf{p}'|d\Omega' = \frac{V}{(2\pi^3)}|\mathbf{p}'|E'dE'd\Omega' \tag{1.213}$$

where we have used, based on the on-shell condition,

$$m^2 = E'^2 - |\mathbf{p}'|^2 \quad \to \quad dm^2 = 0 = 2E'dE' - 2|\mathbf{p}'|d|\mathbf{p}'|. \tag{1.214}$$

We obtain

$$d\sigma = \frac{1}{(2\pi)^2}\delta(E'-E)\left(\frac{m}{E}\right)^2\frac{|\mathbf{p}|'}{|\mathbf{p}|}E'EdE'd\Omega'|\mathcal{M}|^2 \tag{1.215}$$

considering the energies of the electrons

$$E = \sqrt{m^2+|\mathbf{p}|^2}, \qquad E' = \sqrt{m^2+|\mathbf{p}'|^2}$$

$$E = E' \Leftrightarrow |\mathbf{p}| = |\mathbf{p}'| \tag{1.216}$$

and integrating over E' we find the differential cross section for an electron scattering into an element of solid angle $d\Omega'$.

$$\frac{d\sigma}{d\Omega'} = \left(\frac{m}{2\pi}\right)^2|\mathcal{M}|^2 \tag{1.217}$$

with $|\mathcal{M}|^2 = e^2|\bar{u}\not{A}_e u|^2$.

In the Coulomb gauge, the potential is given by

$$A_e^\mu(x) = \left(\frac{Ze}{4\pi|\mathbf{x}|}, 0, 0, 0\right) \tag{1.218}$$

using the Fourier transform[5],

$$\int d^3\mathbf{x}\frac{1}{|\mathbf{x}|}e^{-i\mathbf{q}\cdot\mathbf{x}} = \frac{-1}{|\mathbf{q}|^2}\int d^3\mathbf{x}\frac{1}{|\mathbf{x}|}\nabla^2\left(e^{-i\mathbf{q}\cdot\mathbf{x}}\right) = \tag{1.219}$$

$$= \frac{-1}{|\mathbf{q}|^2}\int d^3\mathbf{x}\left(\nabla^2\frac{1}{|\mathbf{x}|}\right)e^{-i\mathbf{q}\cdot\mathbf{x}} \tag{1.220}$$

$$= \frac{-1}{|\mathbf{q}|^2}\int d^3\mathbf{x}\left(-4\pi\delta^3(\mathbf{x})\right)e^{-i\mathbf{q}\cdot\mathbf{x}} = \tag{1.221}$$

$$= \frac{4\pi}{|\mathbf{q}|^2} \tag{1.222}$$

momentum space potential can be written as,

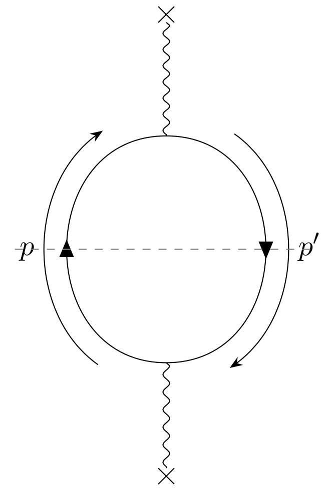
$$A_e^\mu(\mathbf{q}) = \left(\frac{Ze}{|\mathbf{q}|^2}, 0, 0, 0\right), \tag{1.223}$$

[5]For a detailed calculation of the Fourier transform of the field A^μ, see W. Greiner, J. Reinhardt "Quantum Electrodynamics", page 78

therefore summing and averaging over electron spins, one obtains the unpolarized cross-section for Coulomb scattering

$$\frac{d\sigma}{d\Omega'} = \frac{(2m\alpha Z)^2}{|\mathbf{q}|^4} \underbrace{\frac{1}{2}\sum_{\text{spin}} |\bar{u}\gamma^0 u|^2}_{|\bar{\mathcal{M}}|^2} . \tag{1.224}$$

The diagram related to $|\bar{\mathcal{M}}|^2$ is



which produces

$$\frac{d\sigma}{d\Omega'} = \frac{(\alpha Z)^2}{2|\bar{q}|^4} \text{Tr}\left((\not{p}' + m)\gamma^0(\not{p} + m)\gamma^0 \right) . \tag{1.225}$$

The trace which appears in previous equation can be divided into two terms

$$\text{Tr}\left((\not{p}' + m)\gamma^0(\not{p} + m)\gamma^0 \right) = \text{Tr}(\not{p}'\gamma^0\not{p}\gamma^0) + m^2\text{Tr}(\gamma^0\gamma^0)). \tag{1.226}$$

Since $\text{Tr}(\gamma^0\gamma^0) = 4$, we just need to compute the first term

$$\begin{aligned} \text{Tr}(\not{p}'\gamma^0\not{p}\gamma^0) &= p'_\alpha p_\gamma \left(4(g^{\alpha 0}g^{\gamma 0} - g^{\alpha\gamma}g^{00} + g^{\alpha 0}g^{0\gamma}) \right) = 4p'_\alpha p_\gamma \left(2g^{\alpha 0}g^{\gamma 0} - g^{\alpha\gamma} \right) = \\ &= 4(2p'^0 p^0 - p' \cdot p) = 4(2E'E - E'E + \mathbf{p}' \cdot \mathbf{p})_{|_{E'=E}} = \\ &= 4(E^2 + \mathbf{p}' \cdot \mathbf{p}) \end{aligned}$$

which gives

$$\frac{d\sigma}{d\Omega'} = \frac{2(\alpha Z)^2}{|\bar{q}|^4}(E^2 + \mathbf{p}' \cdot \mathbf{p} + m^2). \tag{1.227}$$

Introducing the scattering angle θ, we can write

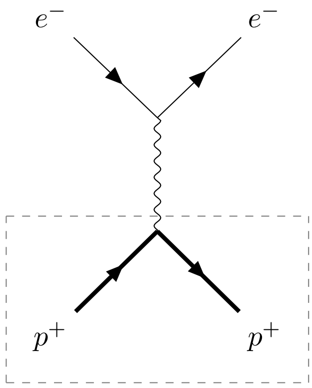
$$\mathbf{p}' \cdot \mathbf{p} = |\mathbf{p}||\mathbf{p}'| \cos\theta$$

$$\begin{aligned} |\mathbf{q}|^2 = |\mathbf{p} - \mathbf{p}'|^2 &= |\mathbf{p}|^2 + |\mathbf{p}'|^2 - 2|\mathbf{p}||\mathbf{p}'|\cos\theta = 2|\mathbf{p}|^2(1 - \cos\theta) = \\ &= 4|\mathbf{p}|^2 \sin^2\left(\frac{\theta}{2}\right) \end{aligned}$$

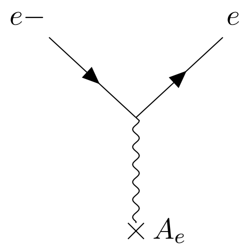
and remembering that $|\mathbf{p}| = Ev$, one finds the *Mott cross section*

$$\frac{d\sigma}{d\Omega'} = \frac{(\alpha Z)^2}{4E^2 v^4 \sin^4\left(\frac{\theta}{2}\right)} \left(1 - v^2 \sin^2\left(\frac{\theta}{2}\right) \right) . \tag{1.228}$$

Therefore, as result of our "semi-quantic" treatment, we have found a correspondence, for Z=1, between



and



. (1.229)

We can also point out that analogy writing

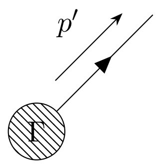
$$A_e^\mu = \left(\frac{Ze}{|\mathbf{q}|^2}, \mathbf{0} \right) \equiv \frac{-ig^{\mu\nu}}{q^2}(ie)j_\nu \tag{1.230}$$

with $j^\mu = Z(\bar{u}_r(\mathbf{0})\gamma^0 u_s(\mathbf{0}), \mathbf{0})$. In the following, we will use this diagram as a template, in order to study properties of Field Theory.

1.13 Radiation from external legs

Let us consider a general process with one external leg

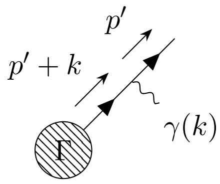
$\mathcal{M}_0(p') =$



$$= \bar{u}'\Gamma(p'). \tag{1.231}$$

We want to compare it with

$\mathcal{M} =$



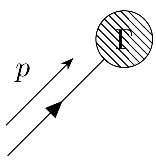
$$\begin{aligned} &= (ie)\bar{u}' \not{\varepsilon}^* i \frac{\not{p}' + \not{k} + m}{(p'+k)^2 - m^2} \Gamma(p'+k) \\ &= (-e)\bar{u}' \frac{2p' \cdot \varepsilon^* + \not{k}}{2p' \cdot k} \Gamma(p'+k), \end{aligned} \tag{1.232}$$

where we have used the relation $\not{\varepsilon}^* \not{p}' = -\not{p}' \not{\varepsilon}^* + 2(p' \cdot \varepsilon^*)$. In the *soft approximation* (i.e. photon in $k \to 0$ limit) we can write $\mathcal{M}$ in a factorized form:

$$\mathcal{M} \approx -e \frac{2(p' \cdot \varepsilon^*)}{2(p' \cdot k)} \bar{u}'\Gamma(p') = -e\mathcal{M}_0 \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} \right) \tag{1.233}$$

where $\mathcal{M}_0$ is a non-radiating term and $\left(\frac{p' \cdot \varepsilon^*}{p' \cdot k}\right)$ is a *soft factor.*
Analogously, we can consider

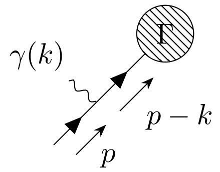
$\mathcal{M}_0(p) =$



$$= \Gamma(p)u, \tag{1.234}$$

and

$$\mathcal{M} = $$



$$= (ie)\Gamma(p-k)i\frac{\not{p}-\not{k}+m}{(p-k)^2-m^2}\not{\varepsilon}^* u. \tag{1.235}$$

In the soft approximation

$$\mathcal{M} \approx -e\Gamma(p)u\frac{2p\cdot\varepsilon^*}{-2p\cdot k} = e\mathcal{M}_0\frac{p\cdot\varepsilon^*}{p\cdot k}. \tag{1.236}$$

Therefore, we have found

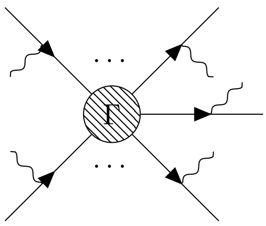
$$\mathcal{M} \approx -e\mathcal{M}_0\left(\frac{p'\cdot\varepsilon^*}{p'\cdot k}\right) \qquad \text{OUTGOING} \tag{1.237}$$

$$\mathcal{M} \approx e\mathcal{M}_0\left(\frac{p\cdot\varepsilon^*}{p\cdot k}\right) \qquad \text{INCOMING} \tag{1.238}$$

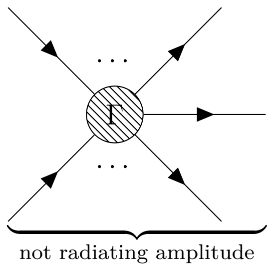
and we note that the previous expressions differ by a sign.

We can generalize to many legs

$$\mathcal{M} = $$



$$\overset{\text{soft}}{\approx}$$



$$\left(\sum_{i\in\text{inc.}} Q_i\frac{p_i\cdot\varepsilon_i^*}{p_i\cdot k_i} - \sum_{j\in\text{outg.}} Q_j\frac{p_j\cdot\varepsilon_j^*}{p_j\cdot k_j}\right) \tag{1.239}$$

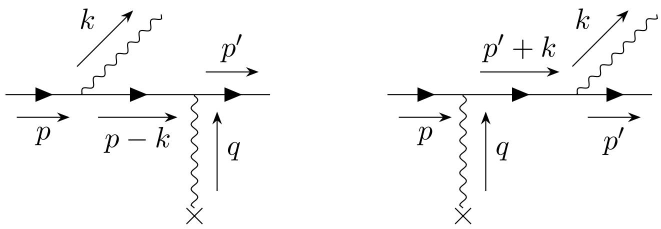
where Q is the electric charge. In order to require gauge invariance, we impose the Ward identity. Using the replacement $\varepsilon_i \to k_i$ we find

$$\sum_{\text{incoming}} Q_i - \sum_{\text{outgoing}} Q_j = 0. \tag{1.240}$$

This means that gauge invariance implies charge conservation between initial and final states. We note that this is not a perturbative result.

1.14 Bremsstrahlung

The scattering of any charged particle leads to the emission of radiation. Both this process and the radiation are referred to as bremsstrahlung (literally translated, braking radiation). In this section we shall consider Bremsstrahlung resulting from the scattering of electrons by the Coulomb field of a heavy nucleus. We have two different kinds of diagram



(1.241)

with $q = p' + k - p$. The amplitude for this process is

$$\mathcal{M} = -ie^2\bar{u}(\mathbf{p}')\left(\not{\varepsilon}^*\frac{\not{p}'+\not{k}+m}{2p'\cdot k}\not{A}_e(\mathbf{q}) + \not{A}_e(\mathbf{q})\frac{\not{p}-\not{k}+m}{-2p\cdot k}\not{\varepsilon}^*\right)u(\mathbf{p}). \tag{1.242}$$

The S-matrix element is given by

$$\langle f| S^{(1)} |i\rangle = (2\pi)\delta(E' + \omega - E) \left(\frac{m}{VE}\right)^{\frac{1}{2}} \left(\frac{m}{VE'}\right)^{\frac{1}{2}} \left(\frac{1}{2V\omega}\right)^{\frac{1}{2}} \mathcal{M}. \tag{1.243}$$

Multiplying the transition rate $|\langle f| S^{(1)} |i\rangle|^2/T$, the density of final states

$$V\frac{d^3\mathbf{p}'}{(2\pi)^3} V\frac{d^3\mathbf{k}}{(2\pi)^3} \tag{1.244}$$

and dividing by the incident electron flux $|p|/(VE)$ leads to

$$d\sigma = \frac{m^2}{(2\pi)^5 2\omega} \frac{|\mathbf{p}'|}{|\mathbf{p}|} |\mathcal{M}|^2 d^3\mathbf{k} d\Omega'. \tag{1.245}$$

In the soft approximation (i.e. $k \to 0, \omega \to 0$)

$$\mathcal{M} = -e\mathcal{M}_0 \left(\frac{p' \cdot \varepsilon^*}{p' \cdot k} - \frac{p \cdot \varepsilon^*}{p \cdot k}\right) \tag{1.246}$$

where $\mathcal{M}_0$ is the Feynman amplitude (1.209), hence

$$\left(\frac{d\sigma}{d\Omega'}\right)_B = \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\alpha}{(2\pi)^2} \left|\frac{p' \cdot \varepsilon^*}{p' \cdot k} - \frac{p \cdot \varepsilon^*}{p \cdot k}\right|^2 \frac{d^3\mathbf{k}}{\omega} \qquad (\omega \approx 0) \tag{1.247}$$

where $(d\sigma/d\Omega')_0$ is the cross section for elastic scattering without photon emission (i.e. equation (1.217))

$$\left(\frac{d\sigma}{d\Omega'}\right)_0 = \left(\frac{m}{2\pi}\right)^2 |\mathcal{M}_0|^2. \tag{1.248}$$

Now we want to sum over the polarizations of the emitted photons, assuming that this is not observed. We rewrite equation (1.247) as

$$\left(\frac{d\sigma}{d\Omega'}\right)_B = \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\alpha}{(2\pi)^2} \frac{d^3\mathbf{k}}{\omega} |V^\mu \varepsilon^*_\mu|^2 \qquad (\omega \approx 0) \tag{1.249}$$

where $V^\mu = \frac{p'^\mu}{p' \cdot k} - \frac{p^\mu}{p \cdot k}$ and so

$$\begin{aligned}
\sum_{pol} \left(\frac{d\sigma}{d\Omega'}\right)_B &= \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\alpha}{(2\pi)^2} \frac{d^3\mathbf{k}}{\omega} \sum_{pol} (V^\mu \varepsilon^*_\mu)(V^\nu \varepsilon_\nu) = \\
&= \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\alpha}{(2\pi)^2} \frac{d^3\mathbf{k}}{\omega} V^\mu V^\nu \sum_{pol} \varepsilon^*_\mu \varepsilon_\nu = \left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\alpha}{(2\pi)^2} \frac{d^3\mathbf{k}}{\omega} (-V^2) = \\
&= -\left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\alpha}{(2\pi)^2} \frac{d^3\mathbf{k}}{\omega} \left(\frac{p'^2}{(p' \cdot k)^2} + \frac{p^2}{(p \cdot k)^2} - \frac{2p' \cdot p}{(p' \cdot k)(p \cdot k)}\right) = \\
&= -\left(\frac{d\sigma}{d\Omega'}\right)_0 \frac{\alpha}{(2\pi)^2} d^4k \delta(k^2) \Theta(k_0) \left(\frac{m^2}{(p' \cdot k)^2} + \frac{m^2}{(p \cdot k)^2} - \frac{2p' \cdot p}{(p' \cdot k)(p \cdot k)}\right) \qquad (1.250) \\
&=\equiv \left(\frac{d\sigma}{d\Omega'}\right)_0 \alpha B \qquad (1.251)
\end{aligned}$$

where we defined B as

$$B \equiv \frac{-1}{(2\pi)^2} d^4k \delta(k^2) \Theta(k_0) \left(\frac{m^2}{(p' \cdot k)^2} + \frac{m^2}{(p \cdot k)^2} - \frac{2p' \cdot p}{(p' \cdot k)(p \cdot k)}\right) \tag{1.252}$$

If we take a look at 1.250 we can see that on one hand, the quantity B is ill defined since it contains infrared and collinear divergences, and on the other hand the presence of α makes the Bremsstrahlung- differential cross section one order higher in α with respect to to the elastic contribution. Therefore one needs to take into account the experimental set-up, in other words, to consider the case when the photon becomes "visible".

1.15 Radiative corrections

Let us consider an experiment of elastic electron scattering. In such an experiment, a photon may be emitted which is too soft to be detected, and it is the energy resolution ΔE of the apparatus which determines whether such a photon emission event is recorded as elastic or inelastic scattering. Therefore, the *experimental* cross section, i.e. the number that is given to us from the experiment, can be written as the sum of the elastic cross section and the soft Bremsstrahlung cross section, integrated over the range of photon energy $0 \leq |E| \leq \Delta E$,

$$\left(\frac{d\sigma}{d\Omega'}\right)_{exp} = \left(\frac{d\sigma}{d\Omega'}\right)_{el} + \left(\frac{d\sigma}{d\Omega'}\right)_{B}. \tag{1.253}$$

The bremsstrahlung term contains one more power of α with respect to the lowest order elastic cross section, that we denote with subscript 0,

$$\left(\frac{d\sigma}{d\Omega'}\right)_{B} = \left(\frac{d\sigma}{d\Omega'}\right)_{0} \alpha B, \tag{1.254}$$

with

$$B = -\frac{1}{(2\pi)^2}\int_{0\leq|E|\leq\Delta E} d^3k \frac{1}{\omega}\left(\frac{p'}{p'\cdot k} - \frac{p}{p\cdot k}\right)^2, \tag{1.255}$$

where we note that the integrand behaves like $1/\omega$ for small ω, so that the integral is logarithmically divergent at the lower limit of integration[6]. Therefore, for consistency, we must include corrections of order α to the elastic cross section,

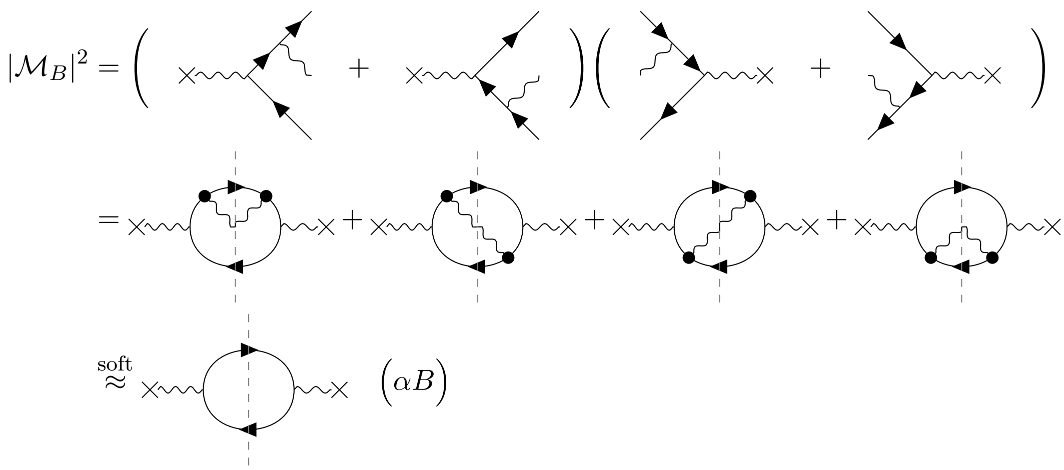
$$\left(\frac{d\sigma}{d\Omega'}\right)_{el} = \left(\frac{d\sigma}{d\Omega'}\right)_{0} (1+\alpha R). \tag{1.256}$$

Combining equations (1.256) and (1.255), we obtain

$$\left(\frac{d\sigma}{d\Omega'}\right)_{exp} = \left(\frac{d\sigma}{d\Omega'}\right)_{0} (1+\alpha(R+B)). \tag{1.257}$$

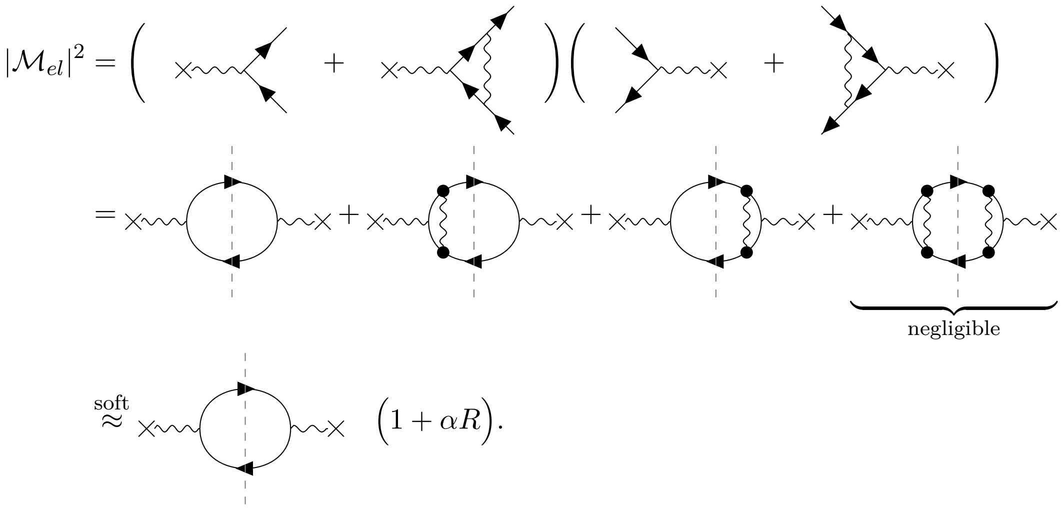
These correction of order α are called *radiative corrections*. What we expect is that the divergences in R and B exactly cancel so that the sum $R+B$ is well-defined and finite. In other words, we expect the infrared catastrophe to arise through treating soft bremsstrahlung and elastic scattering as separate processes in perturbation theory (this separation is artificial, as one always observes elastic scattering together with some soft Bremsstrahlung).

Let's consider the elastic and Bremsstrahlung contributions from a diagrammatic point of view.



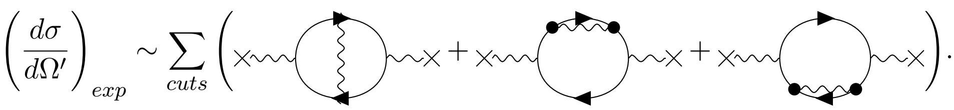
(1.258)

[6] This divergence is known as the *infrared catastrophe* and it is a consequence of the zero mass of the photon.



(1.259)

What we have discovered is that the experimental cross section is related to the cuts of the following diagrams,



(1.260)

Radiative and Bremsstrahlung corrections seemed to come independently, but now they turn out to be generated as cuts of the same underlying Green function. This is a hint to the fact that, the divergences that appear in these diagrams have something in common. In both cases, we have to integrate over the momentum of the photon. However, the phase space is different: in the Bremsstrahlung correction there are three final states and the emitted photon is real, while in the radiative correction there are only two final states and the photon is virtual. The infrared divergences cancellation is called

1. Bloch-Nordsieck theorem in QED;

2. Kinoshita-Lee-Nauenberg theorem in QCD.

This cancellation is crucial because infrared safe quantities are good candidates for physical observables.

Chapter 2

Radiative corrections

Up to now we have calculated QED processes in the lowest order of perturbation theory. On taking higher orders into account, we expect correction contributions from both real and virtual radiation. However, these corrections drag with them divergences in the corresponding integrals which make the results unphysical ones. In order to remove these inherent effects in the higher order perturbative approach, we can consider adopting the tools provided by regularization and renormalization theory based on the concept of modifying quantities to keep the finite and well-defined feature of the physical theory. The Feynman diagrams representing higher

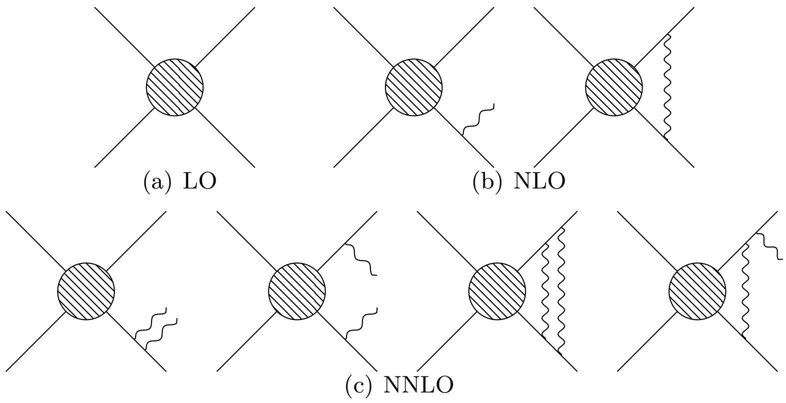
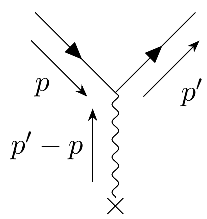


Figure 2.1: LO: leading order, NLO: next-to-leading order, NNLO: next-to-next-to leading order.

order corrections contain additional vertices, compared with those describing the process in the lowest order of perturbation theory. If we restrict ourselves to diagrams which contain two extra vertices, these corrections are of second order in the electronic charge, i.e. first order in the fine structure constant α. Let us consider the electronic scattering in an external potential. The lowest order diagram of the process is



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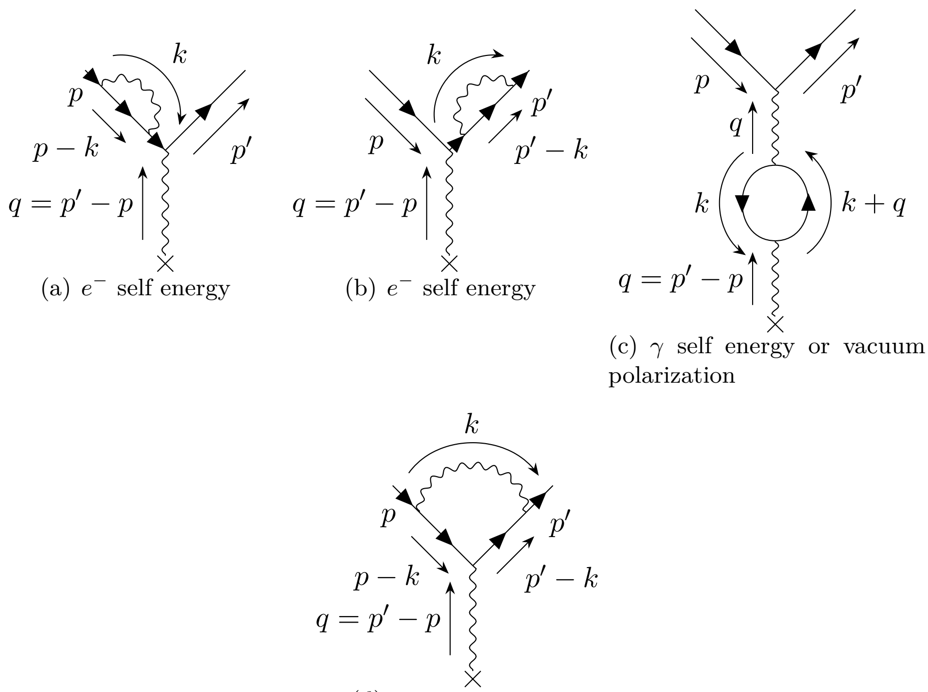
The four contributions to the second order corrections are represented in Figure 2.2.



(a) e^- self energy

(b) e^- self energy

(c) γ self energy or vacuum polarization

(d) vertex correction

Figure 2.2

The Feynman amplitudes for these diagrams follow from the Feynman rules where $q = p' - p$

$$
\begin{aligned}
\text{(LO)} \quad \mathcal{M}^{(0)} &= ie_0 \bar{u}(p') \not{A}_e(q) u(p), \\
\text{(NLO)} \quad \mathcal{M}_a^{(2)} &= ie_0 \bar{u}(p') \not{A}_e(q) iS_F(p) \Big[ie_0^2 \Sigma(p) \Big] u(p), \\
\mathcal{M}_b^{(2)} &= ie_0 \bar{u}(p') \Big[ie_0^2 \Sigma(p') \Big] iS_F(p') \not{A}_e(q) u(p), \\
\mathcal{M}_c^{(2)} &= ie_0 \bar{u}(p') \gamma^\mu u(p) iD_{F\mu\alpha}(q) \Big[ie_0^2 \Pi^{\alpha\beta}(q) \Big] A_{e\beta}(q), \\
\mathcal{M}_d^{(2)} &= ie_0 \bar{u}(p') \Big[e_0^2 \Lambda^\mu(p, p') \Big] u(p) A_{e\mu}(q).
\end{aligned}
$$

From dimensional arguments, all of these integrals are divergent as $k \longrightarrow \infty$ and they have singularities in the ultraviolet region. For instance, the divergence of the loop integral for the electron self energy is linear,

$$
ie_0^2 \Sigma(p) = (ie_0)^2 \int \frac{d^4k}{(2\pi)^4} iD_{F\alpha\beta}(k) \gamma^\alpha iS_F(p-k) \gamma^\beta \overset{\text{UV}}{\sim} k, \tag{2.1}
$$

that of the photon self energy is quadratic,

$$
ie_0^2 \Pi^{\mu\nu}(q) = (ie_0)^2 (-1) \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} \Big[\gamma^\nu iS_F(k) \gamma^\mu iS_F(k+q) \Big] \overset{\text{UV}}{\sim} k^2, \tag{2.2}
$$

and that of the vertex correction is logarithmic,

$$
e_0^2 \Lambda^\mu(p, p') = (ie_0)^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\alpha iS_F(p'-k) \gamma^\mu iS_F(p-k) \gamma^\beta iD_{F\alpha\beta}(k) \overset{\text{UV}}{\sim} \log k. \tag{2.3}
$$

In order to handle them, we can proceed in two steps:

- *regularization*, in order to identify and isolate divergences. A parameter is introduced so that the result of the calculation is well defined and finite. In the end, an appropriate limit is taken and the regulator dependence drops out of physical predictions;

- *renormalization*, in order to remove (subtract) UV divergences.

For simplicity, let us consider the following toy model in one dimension. The integral

$$I_n \equiv \int_0^\infty dx\ x^n \tag{2.4}$$

behaves differently according to the value of n:

- if $n \geq 0$,
$$I_n \sim x^{n+1} \begin{cases} \text{finite for } x \to 0, \\ \text{UV divergent for } x \to \infty; \end{cases} \tag{2.5}$$

- if $n = -1$,
$$I_n \sim \log x \begin{cases} \text{IR divergent for } x \to 0, \\ \text{UV divergence for } x \to \infty; \end{cases} \tag{2.6}$$

- if $n < -1$, and $m \equiv -n > 1$,
$$I_n \sim \frac{1}{x^{m-1}} \begin{cases} \text{IR divergent for } x \to 0, \\ \text{finite for } x \to \infty. \end{cases} \tag{2.7}$$

Let us briefly mention some regularization techniques:

- in order to handle a UV divergent integral, we can introduce a parameter Λ, that is called *cut off*,
$$\int^\infty \equiv \lim_{\Lambda \to \infty} \int^\Lambda \tag{2.8}$$
and take the limit after the calculation of the integral has been performed;

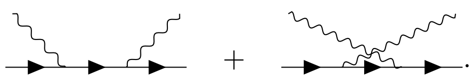
- a fictitious, *mass-like* parameter λ can be introduced to regularize an IR divergent integral,
$$\int_0 \frac{1}{x^m} \equiv \lim_{\lambda \to 0} \int \frac{1}{(x-\lambda)^m}. \tag{2.9}$$
Introducing a mass into a theory completely modifies it (gauge invariance is violated, an extra polarization state appears);

- another technique, developed by 't Hooft and Veltmann, that can be used to cure both UV and IR divergences, is dimensional regularization,
$$\int dx x^n \equiv \lim_{D \to 1} \int d^D x x^n. \tag{2.10}$$

Before performing explicit loop calculations, we can study their structure and Lorentz properties from a diagrammatic point of view.

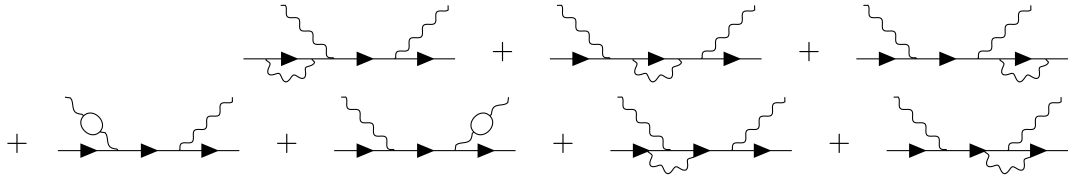
Compton scattering at second order

As an example, let us consider radiative corrections to the Compton scattering. In the lowest order, the process is described by the two following Feynman diagrams,



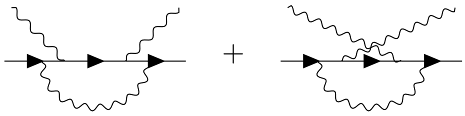
(2.11)

The second order radiative corrections arise from 14 Feynman diagrams obtained from the two tree-level ones with insertions of an electron self-energy, a photon self-energy or a vertex modification, which are responsible for 6, 4 and 4 contributions respectively, in such a way that each one contains four vertices and the correct number of external lines. The 7 terms related to the tree-level diagram of type (a) are shown in (2.12).

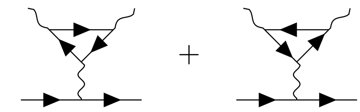


(2.12)

In addition, there are 4 fourth-order diagrams which cannot be obtained from the second-order diagrams by inserting a self-energy or a vertex correction and are shown in (2.13) and (2.14).



(2.13)



(2.14)

Diagrams of (2.13) are both finite and well defined, yielding radiative corrections of order α. Conversely, the two diagrams of (2.14) differ only in the direction of the arrows in the fermion loop: their contributions differ only in sign and thus cancel each other. This result is a particular case of *Furry's theorem*, which states that diagrams containing a fermion loop with an odd number of vertices don't contribute to the Feynman amplitude.

2.1 Self energies renormalization: basics

The idea of renormalization is based on a freedom that our Lagrangian has. Let's consider $\mathcal{L}_{QED}$ expliciting parameter dependences:

$$\mathcal{L}_{QED} = \mathcal{L}_{QED}(A^\mu, \psi, e, m).$$

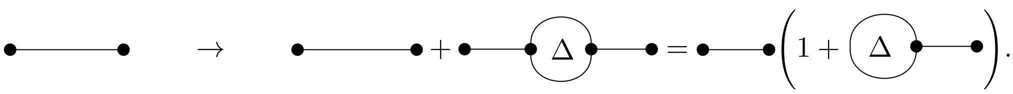
Even though at tree level we have used experimental results, we ignore what are the actual values of those parameters and in general we may have differences between Lagrangian parameters, called *bare quantities*, and physical measured ones. Therefore, adopting the renormalization technique we are allowed to reabsorb divergences into the new defined Lagrangian parameters.

2.1.1 A general model

In this section we'll introduce the renormalization idea studying a general model. We define

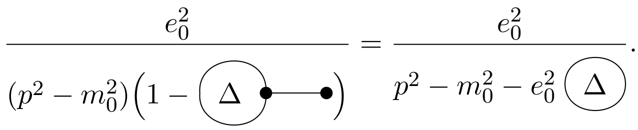
$$\equiv \frac{e_0^2}{p^2 - m_0^2} \tag{2.15}$$

in which we consider e_0 and m_0 as bare quantities. We now add a new contribution to this propagator



(2.16)

and introducing (2.15) we can rewrite (2.16) in the form



(2.17)

Using Taylor expansion and choosing $p^2 = m^2$

$$\Delta \equiv \Delta(p^2) = \underbrace{\Delta(p^2 = m^2)}_{\equiv\bar{\Delta}} + \underbrace{\Delta'(p^2 = m^2)}_{\equiv\bar{\Delta}'}(p^2 - m^2) + \Delta_C(p^2 - m^2) \tag{2.18}$$

in which m is the physical mass and Δ_C vanishes (at least) linearly when $p^2 \to m^2$. If we substitute (2.18) into equation (2.17) we obtain

$$\frac{e_0^2}{p^2 - m_0^2 - e_0^2\Delta} = \frac{e_0^2}{p^2 - m_0^2 - e_0^2\bar{\Delta} - (p^2 - m^2)e_0^2\bar{\Delta}' - (p^2 - m^2)e_0^2\Delta_C}. \tag{2.19}$$

We observe that considered so called radiative corrections shift the pole of the propagator to the physical mass and this has a strong meaning, since the propagator provides us with two crucial information, namely, the pole of this object corresponds to the mass whereas its residue is presented by the charge as we will see later. Therefore we define the (*mass renormalization condition*)

$$m^2 \equiv (m_0^2 + e_0^2\bar{\Delta}) \qquad \text{(physical mass)} \tag{2.20}$$

and (2.19) becomes

$$\frac{e_0^2}{(p^2 - m^2)(1 - e_0^2\bar{\Delta}' - e_0^2\Delta_C)} \approx \frac{e_0^2}{p^2 - m^2}(1 + e_0^2\bar{\Delta}') + \frac{e_0^2}{p^2 - m^2}(e_0^2\Delta_C). \tag{2.21}$$

Note that, the numerator of the left hand side of this equation represents the residue-shifting of the propagator, thus we can introduce the *renormalized charge*

$$e^2 \equiv e_0^2(1 + e_0^2\bar{\Delta}') \equiv e_0^2\bar{Z}, \tag{2.22}$$

with $\bar{Z}$ a renormalization constant, we obtain

$$\frac{e_0^2}{p^2 - m^2}(1 + e_0^2\bar{\Delta}') + \frac{e_0^2}{p^2 - m^2}(e_0^2\Delta_C) = \frac{e^2}{p^2 - m^2} + \frac{e_0^2}{p^2 - m^2}(e_0^2\Delta_C). \tag{2.23}$$

It follows from the definition (2.22) of the charge that

$$e = e_0\sqrt{1 + e_0^2\bar{\Delta}'} \approx e_0\left(1 + \frac{1}{2}e_0^2\bar{\Delta}'\right) \quad \Rightarrow \quad e = e_0(1 + o(e_0^2)). \tag{2.24}$$

We highlight the presence of 1/2 factor in the previous equation, because it will be important in the following. Finally, we have

$$\frac{e^2}{p^2 - m^2} + \frac{e_0^2}{p^2 - m^2}(e_0^2\Delta_C) \approx \frac{e^2}{p^2 - m^2} + \frac{e^2}{p^2 - m^2}(e^2\Delta_C) = \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet\left(1 + \Delta_C\right) \tag{2.25}$$

where the first line in the diagram represents the renormalized mass and the second circular symbol is the the renormalized charge. Therefore, by means of renormalization procedure $\bar{\Delta} \equiv \Delta(p^2 = m^2)$ and $\bar{\Delta}' = \Delta'(p^2 = m^2)$ disappear. We'll show later that UV divergences are contained in these two terms, whereas Δ_C is UV finite. This also suggests us that, to obtain a finite physical parameter, we must have divergences in bare quantities[1].

Essentially, from a "technical" point of view, radiative corrections shift the *pole* of propagators and we fix it choosing the physical mass. Moreover, radiative corrections also make *residue* of (leading term of) propagator proportional to the physical charge. Hence, renormalization procedure fix both *poles* and *residues* of propagators and introduce new definitions of physical parameters.

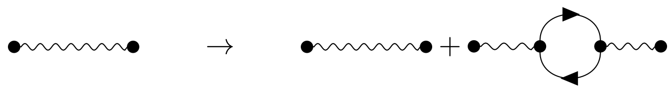
We conclude this section by showing a different way to write the renormalization condition, namely

$$m^2 = m_0^2\left(1 + e_0^2\frac{\bar{\Delta}}{m_0^2}\right) = m_0^2\bar{Z}_m. \tag{2.26}$$

[1]Don't be then confused by (2.24): we are stopping ourselves at the first order in the perturbation series

2.1.2 Photon self energy

We now apply the previous general treatment to the cases of our interest, starting from photon self energy. Let's consider (as in (2.16))



(2.27)

and equivalently (we are omitting the "i" in external vertices)

$$-i\frac{g_{\alpha\beta}}{k^2}e_0^2 \qquad \rightarrow \qquad -i\frac{g_{\alpha\beta}}{k^2}e_0^2 + e_0^2\frac{-ig_{\alpha\mu}}{k^2}\left(ie_0^2\Pi^{\mu\nu}(k)\right)\frac{-ig_{\nu\beta}}{k^2}. \tag{2.28}$$

In general, $\Pi^{\mu\nu}$ admit a Lorentz decomposition

$$\Pi^{\mu\nu}(k) = -g^{\mu\nu}A(k^2) + k^\mu k^\nu B(k^2) \tag{2.29}$$

but for the purpose of this discussion the second term does not contribute[2]. Therefore we rewrite equation (2.28) as

$$e_0^2\frac{-ig_{\alpha\beta}}{k^2}\left(1 - e_0^2 A(k^2)\frac{1}{k^2}\right) \approx e_0^2\frac{-ig_{\alpha\beta}}{k^2 + e_0^2 A(k^2)}. \tag{2.30}$$

Proceeding as previously, we require $A(k^2 = 0) = 0$ so that the physical photon stays massless (mass renormalization condition) and hence we can expand

$$A(k^2) = A(0) + A'(k^2 = 0)k^2 + \Pi_C k^2 = A'(k^2 = 0)k^2 + \Pi_C k^2, \tag{2.31}$$

where Π_C vanishes linearly when $k^2 \rightarrow 0$. Hence

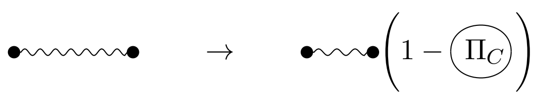
$$e_0^2\frac{-ig_{\alpha\beta}}{k^2}\left(1 - e_0^2 A(k^2)\frac{1}{k^2}\right) = e_0^2\frac{-ig_{\alpha\beta}}{k^2}(1 - e_0^2 A'(0)) + e_0^2\frac{ig_{\alpha\beta}}{k^2}e_0^2\Pi_C. \tag{2.32}$$

We define[3]

$$e^2 \equiv e_0^2(1 - e_0^2 A'(0)) \equiv e_0^2 \bar{Z}_3 \tag{2.33}$$

and then

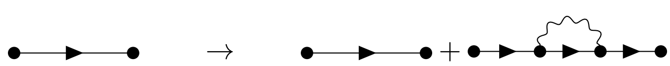
$$e_0^2\frac{-ig_{\alpha\beta}}{k^2}(1 - e_0^2 A'(0)) + e_0^2\frac{ig_{\alpha\beta}}{k^2}e_0^2\Pi_C \approx \frac{-ig_{\alpha\beta}}{k^2}e^2 + \frac{ig_{\alpha\beta}}{k^2}e^4\Pi_C \tag{2.34}$$



(2.35)

2.1.3 Electron self energy

Let's repeat the previous discussion in the case of the electron self energy:



(2.36)

$$e_0^2\frac{i}{\not p - m_0} \qquad \rightarrow \qquad e_0^2\frac{i}{\not p - m_0} + e_0^2\frac{i}{\not p - m_0}(ie_0^2\Sigma(\not p))\frac{i}{\not p - m_0}. \tag{2.37}$$

We can write

$$e_0^2\frac{i}{\not p - m_0} + e_0^2\frac{i}{\not p - m_0}(ie_0^2\Sigma(\not p))\frac{i}{\not p - m_0} \approx e_0^2\frac{i}{\not p - m_0 + e_0^2\Sigma(\not p)} \tag{2.38}$$

[2] This is due to the Ward identity, see for example F.Mandl, G. Shaw "Quantum Field Theory" second edition, pag 169.

[3] Equations (2.33) and (2.22) differ by a sign, but there is nothing deep in this difference. It comes from the "i" factor before $\Pi^{\mu\nu}$.

Using the Taylor expansion we obtain

$$\Sigma(\not{p}) = \Sigma(\not{p} = m) + \Sigma'(\not{p} = m)(\not{p} - m) + \Sigma_C(\not{p} - m). \tag{2.39}$$

and we define the physical mass (the renormalized electron mass)

$$m = m_0 + \delta m, \quad \delta m = -e_0^2 \Sigma(\not{p} = m). \tag{2.40}$$

Equivalently, we write

$$m = m_0 \left(1 - \frac{e_0^2 \Sigma(\not{p} = m)}{m_0}\right) = m_0 \bar{Z}_m. \tag{2.41}$$

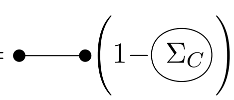
Therefore

$$e_0^2 \frac{i}{\not{p} - m_0 + e_0^2 \Sigma(\not{p})} \approx e_0^2 \frac{i}{\not{p} - m}(1 - e_0^2 \Sigma'(\not{p} = m)) + e_0^2 \frac{i}{\not{p} - m}(-e_0^2 \Sigma_C). \tag{2.42}$$

Defining the renormalized charge as

$$e^2 = e_0^2 \left(1 - e_0^2 \Sigma'(\not{p} = m)\right) \tag{2.43}$$

we can express equation (2.42) in the form

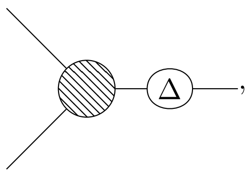
$$e_0^2 \frac{i}{\not{p} - m}(1 - e_0^2 \Sigma'(\not{p} = m)) + e_0^2 \frac{i}{\not{p} - m}(-e_0^2 \Sigma_C) \approx e^2 \left(\frac{i}{\not{p} - m} - e^2 \frac{i}{\not{p} - m} \Sigma_C\right) =$$  (2.44)

Charge renormalization condition can also be expressed as

$$e^2 \equiv e_0^2 \bar{Z}_2 \qquad \Rightarrow \qquad e = e_0 \bar{Z}_2^{\frac{1}{2}} \approx e_0 \left(1 - \frac{1}{2} e_0^2 \Sigma'(\not{p} = m)\right). \tag{2.45}$$

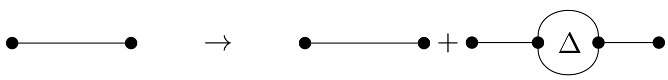
2.2 External legs (wave functions) renormalization: basics

This is another type of correction related to two-point functions. In this case we consider diagrams like



(2.46)

with blobs in external legs. In order to study them, let's recall propagator results presented in the general model. Starting from

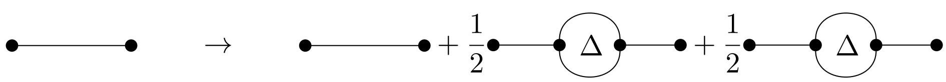


(2.47)

we arrived to the charge renormalization

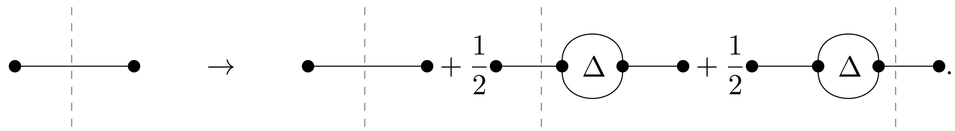
$$e^2 \equiv e_0^2 (1 + e_0^2 \bar{\Delta}') \quad \Rightarrow \quad e \approx e_0 \left(1 + \frac{1}{2} e_0^2 \bar{\Delta}'\right). \tag{2.48}$$

We can relate wave functions to propagators through the use of the completeness relation. Diagrammatically, we rewrite (2.47) in the form



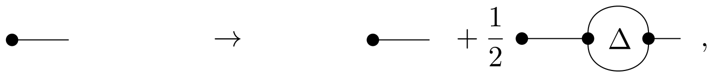
(2.49)

and we introduce cuts (sum over wave functions):

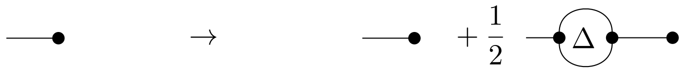


(2.50)

We can "factorize" (2.50) in



(2.51)

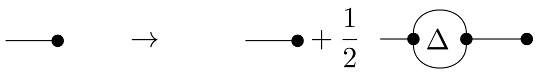


(2.52)

and we have

$$(2.51) \times (2.52) = (2.50) + \text{higher orders}.$$

Therefore, the renormalization of external legs is given by



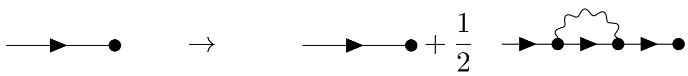
(2.53)

and proceeding as in the previous section we obtain the renormalization condition

$$e = e_0 \left(1 + \frac{1}{2} e_0^2 \bar{\Delta}' \right). \tag{2.54}$$

We note that the 1/2 factor is compatible with renormalization procedure done for propagator.

QED fermion: We can now apply what we have found to electron external lines. We have



(2.55)

$$u(p) \quad \to \quad u(p) + \frac{1}{2} \frac{i}{\not{p} - m} \Big(ie_0^2 \Sigma'(\not{p} = m)(\not{p} - m) + ie_0^2 \Sigma_C(\not{p} - m) \Big) u(p) \tag{2.56}$$

where m is the renormalized mass. The first term in parentheses can be simplified using the denominator (N.B. $\Sigma'(\not{p} = m)$ is a constant) whereas the second term vanishes thanks to Dirac equation. Therefore we can rewrite equation (2.56) as

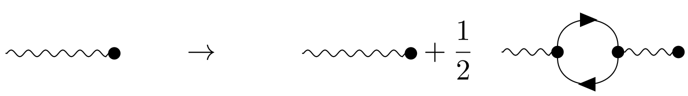
$$u(p) \left(1 + \frac{1}{2} (i)^2 e_0^2 \Sigma'(\not{p} = m) \right) = u(p) \underbrace{\left(1 - \frac{1}{2} e_0^2 \Sigma'(\not{p} = m) \right)}_{\bar{Z}_2^{\frac{1}{2}}} \tag{2.57}$$

and we obtain

1. for fermions
$$u(p) \to Z_2^{\frac{1}{2}} u(p). \tag{2.58}$$

2. Similar arguments applied to anti-fermion self-energy insertions in other external lines lead to the analogous result for
$$\bar{u}(p) \to Z_2^{\frac{1}{2}} \bar{u}(p). \tag{2.59}$$

3. Similarly, the photon self-energy insertion in an external photon line



(2.60)

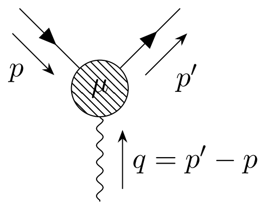
leads to
$$\varepsilon^\mu(k) \to Z_3^{\frac{1}{2}} \varepsilon^\mu(k). \tag{2.61}$$

2.3 Vertex renormalization

Before dealing with the vertex modification, we need to study the structure of the QED vertex from a general point of view and therefore in this section we will start with this consideration and then move to apply renormalization tools to the QED vertex.

2.3.1 Structure of the QED vertex

By Lorentz decomposition, we expect that our QED vertex to be an object carrying one Lorentz index μ that, in general, may have three contributions coming from two external fermionic momenta[4] as well from the γ^μ that characterizes QED interaction, and thus we can write the vertex in the following general form



$$\equiv V^\mu = \bar{u}'\Big(a_1\gamma^\mu + a_2 p^\mu + a_3 p'^\mu\Big)u, \tag{2.62}$$

where the coefficients a_1, a_2, a_3 are called *form factors.* The Ward identity applied in this case

$$\bar{u}'\Big(a_1\not{q} + a_2 q\cdot p + a_3 q\cdot p'\Big)u = 0, \tag{2.63}$$

and the Dirac equation is then

$$\bar{u}'\not{q}u = \bar{u}'\Big(\not{p}' - \not{p}\Big)u = \bar{u}'\Big(m - m\Big)u = 0, \tag{2.64}$$

imply that

$$a_2 q\cdot p + a_3 q\cdot p' = 0. \tag{2.65}$$

Moreover, if we calculate

$$\begin{aligned} q\cdot p &= (p'-p)\cdot p = p'\cdot p - p^2 = p'\cdot p - m^2, \\ q\cdot p' &= (p'-p)\cdot p' = p'^2 - p'\cdot p = m^2 - p'\cdot p = -q\cdot p, \end{aligned}$$

using (2.65) we discover that $a_2 = a_3$, i.e. there are only two independent form factors. Thus, (2.62) becomes

$$V^\mu = \bar{u}'\Big[a_1\gamma^\mu + a_2\Big(p+p'\Big)^\mu\Big]u. \tag{2.66}$$

In the following we will assume that the vertex form factors are to be evaluated for on-shell spinors and in order to get the limit $q = p' - p \longrightarrow 0$ it is useful to use Gordon decomposition which implies a decomposition into current and spin density part for the evaluated spinor.

Gordon decomposition Using the anti-commutation relation for gamma matrices,

$$\Big\{\gamma^\mu, \gamma^\nu\Big\} = 2g^{\mu\nu}, \tag{2.67}$$

the commutator of two gamma matrices can be written as

$$\Big[\gamma^\mu, \gamma^\nu\Big] = 2\gamma^\mu\gamma^\nu - \Big\{\gamma^\mu, \gamma^\nu\Big\} = 2\gamma^\mu\gamma^\nu - 2g^{\mu\nu}. \tag{2.68}$$

If we introduce

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\Big[\gamma^\mu, \gamma^\nu\Big], \tag{2.69}$$

[4]The presence of a term proportional to q^μ is automatically accounted for because of momentum conservation.

we can write

$$i\sigma^{\mu\nu} = -\frac{1}{2}\Big[\gamma^\mu, \gamma^\nu\Big] = g^{\mu\nu} - \gamma^\mu\gamma^\nu = \gamma^\nu\gamma^\mu - g^{\mu\nu}, \tag{2.70}$$

and thus

$$\begin{aligned}
\bar{u}' i\sigma^{\mu\nu}\Big(p'_\nu - p_\nu\Big)u &= \bar{u}'\Big[\Big(\gamma^\nu\gamma^\mu - g^{\mu\nu}\Big)p'_\nu - \Big(g^{\mu\nu} - \gamma^\mu\gamma^\nu\Big)p_\nu\Big]u = \\
&= \bar{u}'\Big[\not{p}'\gamma^\mu - \Big(p' + p\Big)^\mu + \gamma^\mu\not{p}\Big]u = \\
&= \bar{u}'\Big[2m\gamma^\mu - \Big(p' + p\Big)^\mu\Big]u,
\end{aligned} \tag{2.71}$$

whence

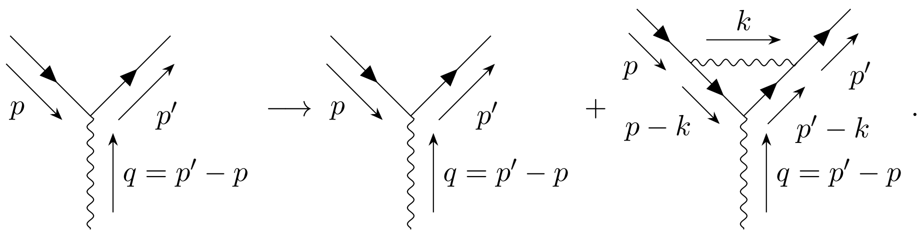
$$\bar{u}'\Big(p' + p\Big)^\mu u = \bar{u}'\Big[2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu\Big]u. \tag{2.72}$$

Finally, inserting (2.72) into equation (2.66) and redefining the form factors for the vertex in terms of F_1 and F_2 we find

$$V^\mu = \bar{u}'\left[F_1(q^2)\gamma^\mu + \frac{F_2(q^2)}{2m} i\sigma^{\mu\nu}q_\nu\right]u. \tag{2.73}$$

2.3.2 Vertex renormalization

Now that we have all the tools we need to take a look at the renormalization of the QED vertex, we can start by considering the one-loop correction to the tree-level vertex,



(2.74)

This corresponds to the replacement

$$ie_0\gamma^\mu \longrightarrow i\Gamma^\mu \equiv ie_0\left[\gamma^\mu + e_0^2\Lambda^\mu(p, p')\right], \tag{2.75}$$

Now, making use of the results derived from Gordon identity in equation (2.73) we can identify $\Lambda^\mu(p, p')$ with the right hand side in Eq. (2.73)

$$\bar{u}'\Lambda^\mu(p, p')u = \bar{u}'\left(f_1(q^2)\gamma^\mu + \frac{f_2(q^2)}{2m} i\sigma^{\mu\nu}q_\nu\right)u. \tag{2.76}$$

We now define

$$P^\mu = \frac{p'^\mu + p^\mu}{2}, \quad \not{P} = m \quad \text{on shell} \tag{2.77}$$

$$p^\mu \equiv P^\mu - \frac{1}{2}q^\mu, \quad p'^\mu \equiv P^\mu + \frac{1}{2}q^\mu. \tag{2.78}$$

In the $q \to 0$ limit, $p \to P$, $p' \to P$ and equation (2.76) becomes

$$\bar{u}'(P)\Lambda^\mu(P, P)u(P) = L\bar{u}'(P)\gamma^\mu u(P), \tag{2.79}$$

where the second term in 2.76 the scalar constant is equal to

$$L \equiv f_1(0) \tag{2.80}$$

For general four-vectors p, p' we can define $\Lambda_c^\mu(p,p')$ by

$$\Lambda^\mu(p,p') \equiv L\gamma^\mu + \Lambda_c^\mu(p,p'), \tag{2.81}$$

which satisfies

$$\bar{u}'(P)\Lambda_c^\mu(P,P)u(P) = 0. \tag{2.82}$$

Note that since our aim is to restore QED after renormalization, we find that L contains the UV divergence although $\Lambda_c^\mu(P',P)$ maintains a finite well-defined form.
Substituting equation (2.81) in (2.76), we obtain

$$i\Gamma^\mu = ie_0\Big[\gamma^\mu\Big(1+e_0^2L\Big) + e_0^2\Lambda_c^\mu(p',p)\Big]. \tag{2.83}$$

If we define the renormalized charge as

$$e \equiv e_0\Big(1+e_0^2L\Big) \equiv \frac{e_0}{\bar{Z}_1}, \tag{2.84}$$

the renormalized vertex correction can be written as

$$ie_0\gamma^\mu \to i\Gamma^\mu(p,p') \approx ie\Big[\gamma^\mu + e^2\Lambda_c^\mu(p,p')\Big]. \tag{2.85}$$

Let us have a closer look at $\Lambda^\mu(p,p')$,

$$\begin{aligned}\Lambda^\mu(p,p') &\equiv -\int \frac{d^4k}{(2\pi)^4}\gamma^\alpha iS_F(p'-k)\gamma^\mu iS_F(p-k)\gamma^\beta iD_F(k)_{\alpha\beta} = \\ &= -\frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2}\gamma^\alpha\frac{1}{\not{p}'-\not{k}-m}\gamma^\mu\frac{1}{\not{p}-\not{k}-m}\gamma_\alpha.\end{aligned} \tag{2.86}$$

If we introduce $\Delta \equiv P-k$, the auxiliary momentum, we have

$$\begin{aligned}p-k &= p-P+P-k = -\frac{1}{2}q+\Delta,\\ p'-k &= \frac{1}{2}q+\Delta,\end{aligned}$$

and equation the loop integral in Eq(2.86) can be rewritten as

$$\Lambda^\mu(p,p') = -\frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2}\gamma^\alpha\frac{1}{\not{\Delta}+\frac{1}{2}\not{q}-m}\gamma^\mu\frac{1}{\not{\Delta}-\frac{1}{2}\not{q}-m}\gamma_\alpha. \tag{2.87}$$

In the limit $q \to 0$, this becomes

$$\begin{aligned}\Lambda^\mu(p,p') \to \Lambda^\mu(P,P) &= \frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2}\gamma^\alpha\left(-\frac{1}{\not{\Delta}-m}\gamma^\mu\frac{1}{\not{\Delta}-m}\right)\gamma_\alpha\\ &= \frac{i}{(2\pi)^4}\int\frac{d^4k}{k^2}\gamma^\alpha\Big(-S_F(\Delta)\gamma^\mu S_F(\Delta)\Big)\gamma_\alpha.\end{aligned} \tag{2.88}$$

Obviously,

$$S_F(\Delta)S_F^{-1}(\Delta) = 1. \tag{2.89}$$

Taking the derivative with respect to P^μ,

$$\begin{aligned}\frac{\partial}{\partial P^\mu}\Big(S_F(\Delta)S_F^{-1}(\Delta)\Big) &= \left(\frac{\partial}{\partial P^\mu}S_F(\Delta)\right)S_F^{-1}(\Delta) + S_F(\Delta)\left(\frac{\partial}{\partial P^\mu}S_F^{-1}(\Delta)\right) = \\ &= \left(\frac{\partial}{\partial P^\mu}S_F(\Delta)\right)S_F^{-1}(\Delta) + S_F(\Delta)\gamma^\mu = 0,\end{aligned}$$

and multiplying by $S_F(\Delta)$ from the right, we discover that the term into round brackets in equation (2.88) is

$$-S_F(\Delta)\gamma^\mu S_F(\Delta) = \frac{\partial}{\partial P^\mu} S_F(\Delta), \tag{2.90}$$

and

$$\Lambda^\mu(P,P) = \frac{\partial}{\partial P^\mu}\left(\frac{i}{(2\pi)^4}\int \frac{d^4k}{k^2}\gamma^\alpha S_F(P-k)\gamma_\alpha\right). \tag{2.91}$$

Comparing with the formula for the electron self-energy,

$$\Sigma(\not{p}) = \frac{i}{(2\pi)^4}\int \frac{d^4k}{k^2}\gamma^\alpha \frac{1}{\not{p}-\not{k}-m}\gamma_\alpha, \tag{2.92}$$

we obtain

$$\Lambda^\mu(P,P) = \frac{\partial}{\partial P^\mu}\Sigma(\not{P}). \tag{2.93}$$

If we consider

$$\Sigma(\not{P}) = \underbrace{A}_{=\Sigma(m)} + (\not{P}-m)\underbrace{B}_{=\Sigma'(m)} + (\not{P}-m)\Sigma_c(\not{P}) \overset{\not{P}\to m}{\approx} A + (\not{P}-m)B, \tag{2.94}$$

and take the derivative with respect to P^μ, we find

$$\frac{\partial}{\partial P^\mu}\Sigma(\not{P}) = \frac{\partial \not{P}}{\partial P^\mu}\frac{\partial}{\partial \not{P}}\Sigma(\not{P}) = \gamma^\mu \Sigma'(\not{P}). \tag{2.95}$$

Since

$$\Lambda^\mu(P,P) = L\gamma^\mu, \tag{2.96}$$

we discover that

$$f_1(0) = \Sigma'(m) \Leftrightarrow L = B. \tag{2.97}$$

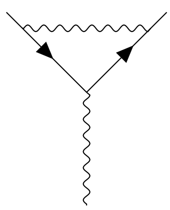
Since

$$\begin{aligned}\bar{Z}_1 &= \left(1 - e_0^2 L\right),\\ \bar{Z}_2 &= \left(1 - e_0^2 B\right),\end{aligned} \tag{2.98}$$

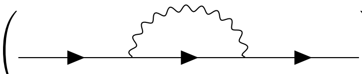
this relation can be expressed in term of the charge renormalization constants as

$$\bar{Z}_1 = \bar{Z}_2. \tag{2.99}$$

This result can be also diagrammatically represented as



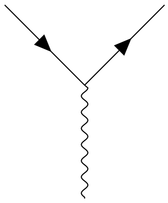
$$\overset{q\to 0}{=} \frac{\partial}{\partial P^\mu}\Bigg(\qquad \Bigg). \tag{2.100}$$



Combined charge renormalization We now combine the charge renormalization (2.84) resulting from the vertex modification with the charge renormalization of the photon and fermion self-energy effects,

$$\begin{cases}\psi_0 \to \sqrt{\bar{Z}_2}\psi,\\ A_0^\mu \to \sqrt{\bar{Z}_3}A^\mu.\end{cases} \tag{2.101}$$

Since each vertex has one photon line and two fermion lines attached, the net effect is the following replacement,



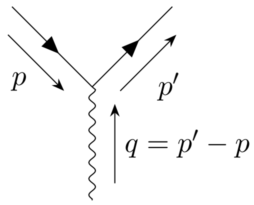
$$= e_0 \bar{\psi}_0 \not{A}_0 \psi_0 \longrightarrow e_0 \bar{Z}_2 \sqrt{\bar{Z}_3} \bar{\psi} \not{A} \psi = \bar{Z}_1 e \bar{\psi} \not{A} \psi. \tag{2.102}$$

The renormalized charge is then

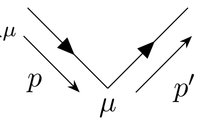
$$e = e_0 \sqrt{\bar{Z}_3} \frac{\bar{Z}_2}{\bar{Z}_1} \overset{(2.99)}{=} e_0 \sqrt{\bar{Z}_3}. \tag{2.103}$$

Thus, the charge renormalization does not depend on fermion self energy effects or vertex modifications, but is entirely due to photon self energy effects, i.e. to vacuum polarization. Vacuum is not empty, but can generate virtual e^--e^+ pairs.

Generalized Ward identity Is the equivalence (2.99) an accident or is it a consequence of some properties of the theory? Let's investigate what's behind it[5].

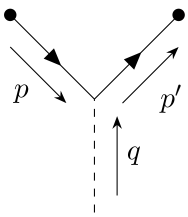


$$\equiv \varepsilon^\mu$$



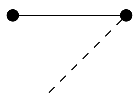
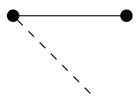
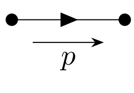
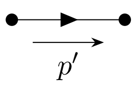
$$= \frac{i}{\not{p}' - m} \not{\varepsilon} \frac{i}{\not{p} - m}. \tag{2.104}$$

Having Ward identity in mind, we should be tempted to replace in this formula the photon polarization with its momentum. We can give a representation of the new object in terms of a different diagram, where the dashed line represents an arbitrary (i.e. not on shell) q insertion,



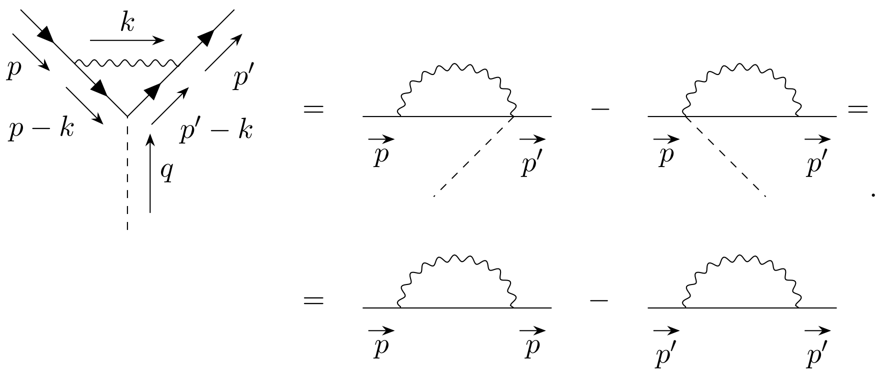
$$\equiv \frac{i}{\not{p}' - m} \not{q} \frac{i}{\not{p} - m} = \frac{i}{\not{p}' - m} (\not{p}' - \not{p}) \frac{i}{\not{p} - m} =$$

$$= \frac{i}{\not{p}' - m} \Big[(\not{p}' - m) - (\not{p} - m) \Big] \frac{i}{\not{p} - m} = \frac{i}{\not{p} - m} - \frac{i}{\not{p}' - m} = \qquad . \tag{2.105}$$

$\equiv$  $-$  $=$  $-$ 

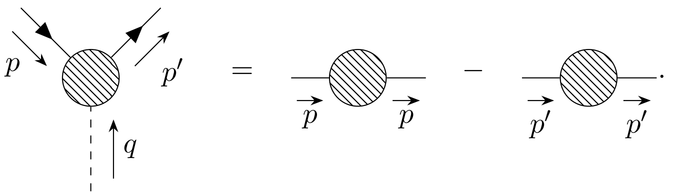
[5] For a more formal discussion see Section 7.4 of Ryder, Quantum Field Theory.

At one loop,



(2.106)

What we have found is a relation between a diagram with one dashed line and its two sub-diagrams without dashed lines. Every time that we replace a "photon" (remember that q is not on shell) with its momentum, we get the difference of two two-point functions obtained by removing the photon leg,



(2.107)

In the limit in which $q \to 0$, also $q^2 \to 0$, thus recovering an on shell photon. In this limit, the two functions become the same and their subtraction gives zero, as we expect from Ward identity. Moreover, as the two momenta p and p' differ by q, dividing this difference by q^μ we get by definition a derivative. The equivalence (2.99) between the two renormalization constants is a consequence of the generalized Ward identity.

2.4 Renormalization summary

We can summarize our results as follows.

- **Propagators**

$$\gamma: \qquad -i\frac{g^{\alpha\beta}}{k^2} \quad \to \quad -i\frac{g^{\alpha\beta}}{k^2}\Big(1 - e^2\Pi_C(k^2)\Big) + o(e^4), \tag{2.108}$$

$$e^-: \qquad \frac{i}{\not{p} - m_0} \quad \to \quad \frac{i}{\not{p} - m}\Big(1 - e^2\Sigma_C(\not{p})\Big) + o(e^4); \tag{2.109}$$

- **Vertex**

$$ie_0\gamma^\mu \quad \to \quad ie\Big(\gamma^\mu + e^2\Lambda^\mu_C(p', p)\Big) + o(e^4); \tag{2.110}$$

- **Parameters**

$$m_0 \to m = \bar{Z}_m m_0 \qquad e_0 \to e = e_0 \bar{Z}_3^{\frac{1}{2}}\frac{\bar{Z}_2}{\bar{Z}_1} \equiv \bar{Z}_e e_0. \tag{2.111}$$

  We define $\bar{Z}_i \equiv (1 + \bar{\delta}_i)$ with

$$\bar{\delta}_m = -e_0^2\frac{\Sigma(\not{p} = m)}{m_0}, \qquad \bar{\delta}_2 = -e_0^2\Sigma'(\not{p} = m), \qquad \bar{\delta}_3 = -e_0^2 A'(0). \tag{2.112}$$

If we recall

$$\frac{1}{\bar{Z}_1} \equiv (1 + e_0^2 L) \tag{2.113}$$

then

$$\bar{Z}_1 = (1 - e_0^2 L) \quad \Rightarrow \quad \bar{\delta}_1 = -e_0^2 L = -e_0^2 f_1(0). \tag{2.114}$$

By generalized Ward identity

$$\bar{Z}_1 = \bar{Z}_2 \Leftrightarrow \bar{\delta}_1 = \bar{\delta}_2; \tag{2.115}$$

- **Fields** (external legs)

$$\psi^0 \to \bar{Z}_2^{\frac{1}{2}} \psi, \qquad A_\mu^0 \to \bar{Z}_3^{\frac{1}{2}} A_\mu. \tag{2.116}$$

2.5 Diagrammatic renormalization

So far, we have introduced basics ideas of renormalization. We now start realizing in a quantitative and systematic way that procedure. Indeed, renormalization can be completely carried on diagrammatically. Let us write the Lagrangian of QED

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu}^0 F^{0\mu\nu} + \bar{\psi}^0 \left(i\partial\!\!\!/ - m_0 \right) \psi^0 + e_0 \bar{\psi}^0 A\!\!\!/^0 \psi^0 \tag{2.117}$$

where we have used superscript "0" to denote bare quantities. If we define

$$\psi^0 \equiv Z_2^{\frac{1}{2}} \psi, \qquad A_\mu^0 \equiv Z_3^{\frac{1}{2}} A_\mu, \qquad m_0 \equiv Z_m m, \qquad e_0 = Z_e e, \tag{2.118}$$

we can write

$$\mathcal{L}_{QED} = -\frac{1}{4} Z_3 F_{\mu\nu} F^{\mu\nu} + i Z_2 \bar{\psi} \partial\!\!\!/ \psi - Z_2 Z_m m \bar{\psi} \psi + e Z_e Z_2 Z_3^{\frac{1}{2}} \bar{\psi} A\!\!\!/ \psi. \tag{2.119}$$

Comparing with definitions given in the previous section we find

$$Z_1 = \bar{Z}_1, \qquad Z_2 = \bar{Z}_2, \qquad Z_3 = \bar{Z}_3, \qquad Z_m = \frac{1}{\bar{Z}_m}, \qquad Z_e \equiv \frac{1}{\bar{Z}_e} \tag{2.120}$$

hence

$$e = e_0 \bar{Z}_3^{\frac{1}{2}} \frac{\bar{Z}_2}{\bar{Z}_1} = e_0 Z_3^{\frac{1}{2}} \frac{Z_2}{Z_1} \quad \Rightarrow \quad Z_e = \frac{Z_1}{Z_2 Z_3^{\frac{1}{2}}}. \tag{2.121}$$

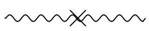
Therefore, defining $Z_i \equiv (1 + \delta_i)$ we obtain

$$\begin{aligned}
\mathcal{L}_{QED} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - m)\psi + e\bar{\psi} A\!\!\!/ \psi + \\
&\quad - \frac{1}{4}\delta_3 F_{\mu\nu} F^{\mu\nu} + \delta_2 \bar{\psi}(i\partial\!\!\!/ - m)\psi - m\delta_m \bar{\psi}\psi + \delta_1 e \bar{\psi} A\!\!\!/ \psi \equiv \\
&\equiv \mathcal{L}_{QED}^R + \mathcal{L}_{CT}
\end{aligned} \tag{2.122}$$

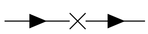
(CT=counterterms) and we have

$$\delta_1 = \bar{\delta}_1, \qquad \delta_2 = \bar{\delta}_2, \qquad \delta_3 = \bar{\delta}_3, \qquad \delta_m = -\bar{\delta}_m, \qquad \delta_e = \delta_1 - \delta_2 - \frac{1}{2}\delta_3 = -\frac{1}{2}\delta_3. \tag{2.123}$$

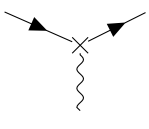
Nothing new derives from $\mathcal{L}_{QED}^R$ term. Instead, $\mathcal{L}_{CT}$ gives us new Feynman rules:



$$= i\delta_3 \left(-g^{\mu\nu} k^2 + k^\mu k^\nu \right), \tag{2.124}$$



$$= i \left(-m\delta_m + \delta_2 (p\!\!\!/ - m) \right), \tag{2.125}$$



$$= ie\delta_1 \gamma^\mu. \tag{2.126}$$

Hence, we have a structure similar to that obtained by the initial Lagrangian but now one loop counterterms are present. Let's see an example.

Example: electron self energy In the "old fashion" we arrived to (expansion (2.18))

$$e_0^2\Sigma(\not{p}) - e_0^2\Sigma(\not{p}=m) - e_0^2\Sigma'(\not{p}=m)(\not{p}-m) \simeq e^2\Sigma_C(\not{p})(\not{p}-m) \tag{2.127}$$

and

$$-e_0^2\Sigma(\not{p}=m) = m_0\bar{\delta}_m, \qquad -e_0^2\Sigma'(\not{p}=m) = \bar{\delta}_2. \tag{2.128}$$

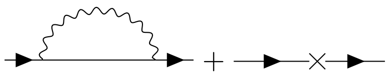
Since

$$m_0 = m + o(e^2), \qquad e_0 = e + o(e^2), \qquad \bar{\delta}_m = -\delta_m, \qquad \bar{\delta}_2 = \delta_2 \tag{2.129}$$

we obtain

$$e^2\Sigma_C(\not{p})(\not{p}-m) = e^2\Sigma(\not{p}) - m\delta_m + \delta_2(\not{p}-m). \tag{2.130}$$

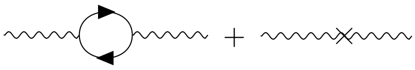
Using instead the Feynman rules of $\mathcal{L}^R_{QED} + \mathcal{L}_{CT}$, we have to add two graphs



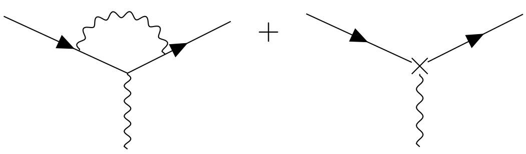
$$= ie^2\Sigma(\not{p}) + i\left(-m\delta_m + \delta_2(\not{p}-m)\right) = ie^2\Sigma_C(\not{p})(\not{p}-m). \tag{2.131}$$

Aside from "i" factor, we have obtained the same result.

Our procedure works also for vertex and photon propagators. In these cases we have to consider



$$= \text{UV finite}, \tag{2.132}$$



$$= \text{UV finite}. \tag{2.133}$$

2.6 Renormalizability

Renormalization in QED can be realized by using a finite number of counterterms (δ_j). In order to show this, we'll study the renormalizability of a theory.

Degree of superficial divergence of a diagram Let's consider a graph with

- n vertices,
- f_i = number of internal fermion lines,
- b_i = number of internal boson lines,
- f_e = number of external fermion legs,
- b_e = number of external boson legs,
- l = number of loops (independent internal momenta).

Then we define the superficial degree of divergence K as

$$K \equiv \underbrace{4l}_{\sim d^4k} - \underbrace{f_i}_{\sim\frac{1}{k}} - \underbrace{2b_i}_{\sim\frac{1}{k^2}} \begin{cases} \geq 0 & \text{potentially UV divergent,} \\ < 0 & \text{UV finite.} \end{cases} \tag{2.134}$$

In order to obtain results valid for any number of loops, we now try to express K in terms of external legs of diagram.

For a generic graph the number of loops l is given by

$$l = \underbrace{f_i + b_i}_{\text{internal lines}} - \overbrace{(n-1)}^{\text{ind. vertices}} \tag{2.135}$$

where we have considered momentum conservation to obtain the number of independent vertices $n-1$. In QED we can exploit two other relations. Naming the number of fermions at every vertex F_v we have, in general the total number of fermions

$$F_v \cdot n = f_e + 2f_i \tag{2.136}$$

and, since in QED $F_v = 2$, we obtain

$$2n = f_e + 2f_i. \tag{2.137}$$

Similarly, one finds for the total number of bosons

$$F_b \cdot n = b_e + 2b_i \tag{2.138}$$

and, since in QED $F_b = 1$

$$n = b_e + 2b_i. \tag{2.139}$$

Therefore

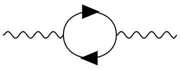
$$K = 4 - \frac{3}{2}f_e - b_e \geq 0. \tag{2.140}$$

Actually, we could have found a dependence on the number of loops l. Instead in QED K depends only on the number of external legs.

Let's now classify superficial degree in terms of f_e and b_e

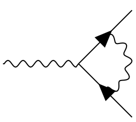
$$K \equiv D_{f_e,b_e} = 4 - \frac{3}{2}f_e - b_e. \tag{2.141}$$

We have

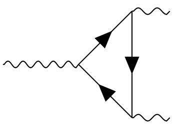


$$\int d^4k \frac{1}{\not{k}\not{k}} \sim k^2, \qquad D_{0,2} = 2 \qquad \text{UV divergent} \tag{2.142}$$

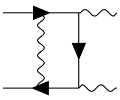
$$\int d^4k \frac{1}{k^2\not{k}} \sim k, \qquad D_{2,0} = 1 \qquad \text{UV divergent} \tag{2.143}$$



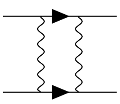
$$\int d^4k \frac{1}{k^2\not{k}\not{k}} \sim k^0, \qquad D_{2,1} = 0 \qquad \text{UV divergent} \tag{2.144}$$



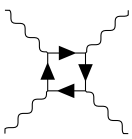
$$\int d^4k \frac{1}{\not{k}\not{k}\not{k}} \sim k, \qquad D_{0,3} = 1 \qquad \text{UV divergent} \tag{2.145}$$



$$\int d^4k \frac{1}{\not{k}\not{k}\not{k}k^2} \sim k^{-1}, \qquad D_{2,2} = -1 \qquad \text{Finite} \tag{2.146}$$

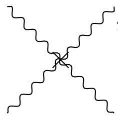


$$\int d^4k \frac{1}{\not{k}k^2\not{k}k^2} \sim k^{-2}, \qquad D_{4,0} = -2 \qquad \text{Finite} \tag{2.147}$$



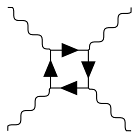
$$\int d^4k \frac{1}{\not{k}\not{k}\not{k}\not{k}} \sim k^0, \qquad D_{0,4} = 0 \qquad \text{Log divergent} \tag{2.148}$$

While divergence of diagram (2.145) can be removed by Furry's theorem [6], in diagram (2.148) we have 4 (an even number) photon legs, so that theorem cannot be used. We have to renormalize vertex (2.148). In this case the counter-term associated, δ_4, looks like



, (2.149)

but in $\mathcal{L}_{QED}$ there are no terms yielding δ_4 (or Z_4). Therefore renormalization process cannot be realized. There are six diagrams that contribute to M, and it is the combination of these that must be canceled.[7]

$\mathcal{M} \sim$  $= \varepsilon_1^\mu \varepsilon_2^\nu \varepsilon_3^\rho \varepsilon_4^\sigma \mathcal{M}_{\mu\nu\rho\sigma}$, (2.150)

where

$$\mathcal{M}_{\mu\nu\rho\sigma} \sim c \log(k^2)(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) + \text{finite}. \tag{2.151}$$

Using Ward identity

$$0 = p_1^\mu \mathcal{M}_{\mu\nu\rho\sigma} = c \log(k^2)(p_{1\nu}g_{\rho\sigma} + p_{1\rho}g_{\nu\sigma} + p_{1\sigma}g_{\nu\rho}) + p^\mu \text{finite}. \tag{2.152}$$

The previous equation can be verified if and only if $c = 0$, i.e. when there are not contribution by $\mathcal{M}$.

If now we consider diagram with a larger number of external legs the value, D_{f_e,b_e} can only become more negative. Therefore we are sure that divergences we have studied (and corresponding counterterms δ_1, δ_2, δ_3 and δ_m) are the only necessary to renormalize QED at all orders.

We define

- *Renormalizable theories*: finite number of counterterms;
- *Unrenormalizable theories*: infinite number of counterterms[8].

2.7 Dimensional regularization

At this level, we can distinguish two methods underlying a regularization of the theory, namely the cut-off method and the dimensional regularization one. It is obvious that cut-off regularization has the disadvantage of not being gauge invariant, but it offers a unique possibility to investigate theories at larger values of the coupling constant, because the natural perturbative expansion is often in terms of the inverse coupling constant. Whereas the dimensional regularization rely on the fact that the degree of divergence of Feynman integrals depends on the number of space-time dimensions and ensures automatically gauge invariance to all order small-coupling expansion of perturbation theory. For instance, the divergent loop integrals of Field Theory are four-dimensional integrals in energy-momentum space and dimensional regularization method consists in modifying the dimensionality of these integrals, $d \equiv 4 - \varepsilon$, so that they become finite. Generalizing QED from four to d dimensions, we obtain for the free action

$$S_0 = \int d^d x \mathcal{L}_0 = \int d^d x \Big(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} \, (i \partial\!\!\!/ - m) \, \psi \Big), \tag{2.153}$$

[6] See the section "Compton Scattering at second order".

[7] Three diagrams can be found by fixing for example one of the four external photons and exchanging the remaining three, and the last three by inverting the fermionic line.

[8] Don't be confused by names: in a "unrenormalizable theory" renormalization procedure can be still realized but order by order".

and for the interaction term

$$S_I = \int d^d x e \bar{\psi} \not{A} \psi. \tag{2.154}$$

For the action to be dimensionless, the various quantities must have the following natural units dimensions:

$$[A^\mu] = \frac{d-2}{2} = 1 - \frac{\varepsilon}{2}, \qquad [\psi] = \frac{d-1}{2} = \frac{3-\varepsilon}{2}, \qquad [e] = \frac{4-d}{2} = \frac{\varepsilon}{2}. \tag{2.155}$$

As it is more convenient to have a dimensionless coupling constant, a parameter μ with dimensions of a mass $[m] = 1$ can be introduced, while keeping e dimensionless,

$$\underset{(4\text{ dim})}{e} \to \underset{(d\text{ dim})}{e\mu^{\frac{4-d}{2}}} = \underset{(d\text{ dim})}{e\mu^{\frac{\varepsilon}{2}}}. \tag{2.156}$$

Before proceeding with exploring this regularization method, we will present some mathematical tools used to derive the underlying concept regularizing the dimensions of a loop integral.

2.7.1 Metric tensor and γ matrices

The algebra of γ matrices can be generalized to arbitrary dimensions, introducing the following rules:

- $g^{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$;
- $\left\{\gamma^\mu, \gamma^\nu\right\} = 2g^{\mu\nu}$;
- γ^0, γ^1, ..., γ^{d-1} are $f(d) \times f(d)$ matrices, where $f(d) \underset{d\to 4}{\longrightarrow} 4$ (for example, $f(d) = 2^{\frac{d}{2}}$);
- the identity matrix $\mathbf{1}$ is a $f(d) \times f(d)$ matrix, such that $\mathrm{Tr}\,\mathbf{1} = f(d)$;
- contractions

$$\begin{aligned} \gamma_\lambda \gamma^\lambda &= d\mathbf{1}, \\ \gamma_\lambda \gamma^\alpha \gamma^\lambda &= -(d-2)\gamma^\alpha, \\ \gamma_\lambda \gamma^\alpha \gamma^\beta \gamma^\lambda &= (d-4)\gamma^\alpha \gamma^\beta + 4g^{\alpha\beta}; \end{aligned}$$

- traces

$$\begin{aligned} \mathrm{Tr}\,\mathbf{1} &= f(d), \\ \mathrm{Tr}\left[\gamma^\alpha \gamma^\beta\right] &= f(d) g^{\alpha\beta}, \\ \mathrm{Tr}\left[\gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta\right] &= f(d)\left[g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma}\right], \\ \mathrm{Tr}\left[\gamma^{\alpha_1} \ldots \gamma^{\alpha_{2n+1}}\right] &= 0. \end{aligned}$$

2.7.2 Feynman integrals in d dimension

The most general form for a rank r tensor integral associated to the one-loop diagram of Figure 2.3 is

$$T_n^{\mu_1 \cdots \mu_r} = \int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \ldots k^{\mu_r}}{D_1 \ldots D_n}, \tag{2.157}$$

where

$$D_i = (k + r_i)^2 - m_i^2 + i\varepsilon, \tag{2.158}$$

and the momenta r_i are related with the external momenta (all taken to be incoming) through

$$r_i = \sum_{j=1}^{i} p_j, \quad r_n = \sum_{j=1}^{n} p_j = 0. \tag{2.159}$$

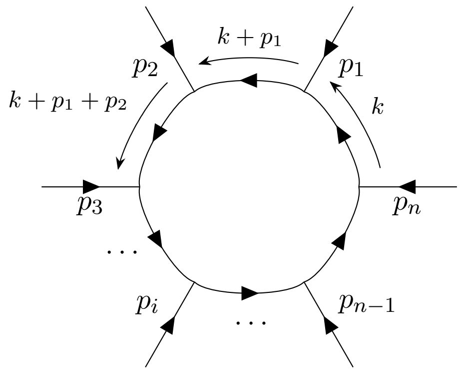


Figure 2.3

In the denominator of (2.157) there appear products of the denominators of the propagators of particles in the loop. It is convenient to combine these products in just one common denominator, introducing the so-called Feynman parameters.

Feynman parameters

$$\frac{1}{ab} = \int_0^1 \frac{dx}{\left[ax + b(1-x)\right]^2}, \tag{2.160}$$

$$\frac{1}{a^m b^n} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^1 dx \frac{x^{m-1}(1-x)^{n-1}}{\left[ax + b(1-x)\right]^{m+n}}, \tag{2.161}$$

$$\frac{1}{a_1 \dots a_n} = \Gamma(n) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\dots-x_{n-2}} \frac{dx_{n-1}}{\left[a_1 x_1 + a_2 x_2 + \dots + a_n(1 - x_1 - \dots - x_{n-1})\right]^n}. \tag{2.162}$$

Euler Γ function In the formulas above we have used the Γ function, which is defined by the integral

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}, \qquad \begin{aligned} \Gamma(z+1) &= z\Gamma(z), \\ \Gamma(n+1) = n! \quad &if \quad n \in \mathbb{N}. \end{aligned} \tag{2.163}$$

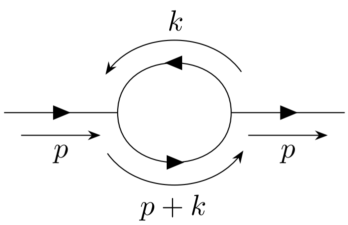
We will often need to expand the γ function around the pole $x = 0$

$$\Gamma\left(\frac{\varepsilon}{2}\right) = \frac{2}{\varepsilon} - \gamma + o(\varepsilon), \tag{2.164}$$

where $\gamma \simeq 0,577$ is the Euler constant. We can write a series expansion

$$\Gamma(1+\varepsilon) = 1 - \gamma\varepsilon + \left(\gamma^2 + \frac{\pi^2}{6}\right)\frac{\varepsilon^2}{2!} + o(\varepsilon^3), \tag{2.165}$$

where $\frac{\pi^2}{6}$ is the value of the Riemann zeta function in 2. As an example, we consider the following diagram, (numerator is one because we are considering a scalar theory, for the sake of simplicity)



$$\equiv I = \int \frac{d^d k}{(2\pi)^d} \frac{1}{D_1 D_2}, \tag{2.166}$$

where

$$D_1 = (k+p)^2 - m_1^2 + i\varepsilon,$$
$$D_2 = k^2 - m_2^2 + i\varepsilon.$$

Introducing a Feynman parameter, we have to compute

$$\begin{aligned} D_1 x + D_2(1-x) &= k^2 x + 2p\cdot kx + p^2 x - m_1^2 x + k^2 - k^2 x - m_2^2(1-x) + p^2x^2 - p^2x^2 = \\ &= (k+px)^2 + p^2 x(1-x) - m_1^2 x - m_2^2(1-x) = \\ &= (k+px)^2 - C(p^2, m_1^2, m_2^2, x), \end{aligned}$$

where $C = C(p^2, m_1^2, m_2^2, x)$ depends on the external momenta and the masses but not on the loop momentum. Then (2.166) becomes

$$I = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[(k+px)^2 - C + i\varepsilon\right]^2} \overset{(k_\mu \to k_\mu - p_\mu x)}{=} \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{\left[k^2 - C + i\varepsilon\right]^2}. \tag{2.167}$$

2.7.3 Scalar integrals in d dimensions

In general, in a dimensional regularization procedure we may need a scalar integral of the form

$$I_{r,m} = \int \frac{d^dk}{(2\pi)^d} \frac{(k^2)^r}{\left[k^2 - C + i\varepsilon\right]^m}. \tag{2.168}$$

In the previous example, we have met $\int_0^1 dx I_{0,2}$. All tensor integrals can be obtained from scalar integral of this form, for instance

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu}{\left[k^2 - C + i\varepsilon\right]^m} = 0 \quad \text{i.e. there is no contribution from single powers of } k, \tag{2.169}$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{\left[k^2 - C + i\varepsilon\right]^m} = \frac{g^{\mu\nu}}{d} \int \frac{d^dk}{(2\pi)^d} \frac{k^2}{\left[k^2 - C + i\varepsilon\right]^m} \tag{2.170}$$

and so on.

Wick rotation In order to compute $I_{r,m}$, we consider $C > 0$ (the case $C < 0$ can be obtained by analitical continuation) and use an integration in the complex plane of the variable k^0,

$$I_{r,m} = \int \frac{d^{d-1}k}{(2\pi)^d} \int dk_0 \frac{(k^2)^r}{\left[k_0^2 - |\mathbf{k}|^2 - C + i\varepsilon\right]^m}. \tag{2.171}$$

The function under the integral has two poles for

$$k_0 = \pm\left(\sqrt{|\mathbf{k}|^2 + C} - i\varepsilon\right), \tag{2.172}$$

as shown in Figure 2.4. By Cauchy's residue theorem, the integral along a closed loop which doesn't contain any pole is zero,

$$\int_{-\infty}^{+\infty} dk_0 + \int_{+i\infty}^{-i\infty} dk_0 + \int_{\gamma_1,\gamma_2} dk_0 = 0. \tag{2.173}$$

Now, the contribution of the two arcs at infinity vanishes[9] and we can change the integration along the real axis into an integration along the imaginary axis,

$$\int_{-\infty}^{+\infty} dk_0 = \int_{-i\infty}^{+i\infty} dk_0 \overset{(k_0 \equiv ik_E^0)}{=} i \int_{-\infty}^{+\infty} dk_E^0. \tag{2.174}$$

[9]This claim is actually problematic, see for example DeWitt *"The global approach to Quantum Field Theory"*, Claredon Press - Oxford, 2003, vol.2 pag 696

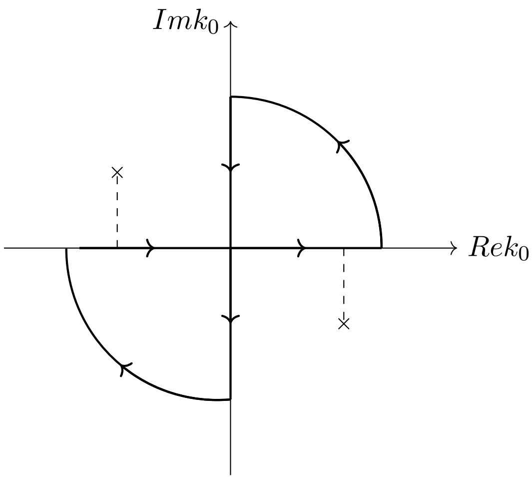


Figure 2.4

If we introduce the euclidean vector $k_E = (k_E^0, \mathbf{k})$, so that scalar products can be calculated using euclidean metric diag$(+, \dots, +)$ instead of Minkowsky,

$$k^2 = k_0^2 - |\mathbf{k}|^2 = -(k_E^0)^2 - |\mathbf{k}|^2 \equiv -k_E^2, \tag{2.175}$$

the integral (2.171) becomes

$$I_{r,m} = i(-1)^{r-m} \int \frac{d^d k_E}{(2\pi)^d} \frac{(k_E^2)^r}{\left[k_E^2 + C\right]^m}, \tag{2.176}$$

where the denominator is well defined because $C > 0$ and the ε prescription is no more necessary. This technique is usually called *Wick rotation*.

Generalized polar coordinates In order to find an explicit expression for $I_{r,m}$ we introduce the d dimensional generalization of polar coordinates,

$$\int d^d k_E = \int dk_E \; k_E^{d-1} d\Omega_{d-1}, \tag{2.177}$$

where $k_E = \sqrt{(k_E^0)^2 + |\mathbf{k}|^2}$ is the length of the euclidean vector k_E and the integration over the solid angle is defined as

$$\int d\Omega_{d-1} \equiv \int_0^\pi \sin\theta_{d-1}^{d-2} d\theta_{d-1} \cdots \int_0^\pi \sin\theta_2 d\theta_2 \int_0^{2\pi} d\theta_1. \tag{2.178}$$

For example,

$$\begin{aligned}
d = 2 \qquad & \Omega_1 = \int d\Omega_1 = \int_0^{2\pi} d\theta_1 = 2\pi; \\
d = 3 \qquad & \Omega_2 = \int d\Omega_2 = \int_0^\pi \sin\theta_2 d\theta_2 \int_0^{2\pi} d\theta_1 = 2 \cdot 2\pi = 4\pi; \\
d = 4 \qquad & \Omega_3 = \int d\Omega_3 = \int_0^\pi \sin^2\theta_3 d\theta_3 \int_0^\pi \sin\theta_2 d\theta_2 \int_0^{2\pi} d\theta_1 = \frac{\pi}{2} \cdot 2 \cdot 2\pi = 2\pi^2.
\end{aligned}$$

In order to evaluate (2.178) we can use the gaussian integral,

$$\pi^{\frac{d}{2}} = \left(\int_{-\infty}^{+\infty} dx \; e^{-x^2} \right)^d = \int d\Omega_{d-1} \int_0^\infty dr \; r^{d-1} e^{-r^2} = \frac{1}{2} \Omega_{d-1} \Gamma\left(\frac{d}{2}\right), \tag{2.179}$$

and obtain

$$\Omega_{d-1} = 2\frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{2.180}$$

Moreover, recalling the Euler's beta function β

$$\int_0^\infty dz \frac{z^{a-1}}{(1+z)^{a+b}} \equiv \beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{2.181}$$

we have

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2}\left(\frac{4\pi}{C}\right)^{\frac{\varepsilon}{2}} C^{2+r-m}\frac{\Gamma\left(2+r-\frac{\varepsilon}{2}\right)}{\Gamma\left(2-\frac{\varepsilon}{2}\right)}\frac{\Gamma\left(m-r-2+\frac{\varepsilon}{2}\right)}{\Gamma\left(m\right)}. \tag{2.182}$$

For example

$$I_{0,2} = \frac{i}{(4\pi)^2}\left(\frac{4\pi}{C}\right)^{\frac{\varepsilon}{2}}\frac{2\Gamma\left(1+\frac{\varepsilon}{2}\right)}{\varepsilon} = \frac{i}{16\pi^2}\left(\Delta_\varepsilon - \ln C + o(\varepsilon)\right) \tag{2.183}$$

with

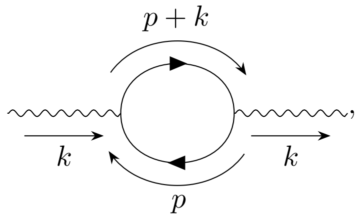
$$\Delta_\varepsilon = \frac{2}{\varepsilon} - \gamma + \ln(4\pi), \tag{2.184}$$

which contains an UV logarithmic divergence in 4−dim.

2.8 1-loop vacuum polarization

We can apply to the vacuum polarization what we have learnt. We have

$$ie_0^2\Pi_{\mu\nu}(k^2) = \tag{2.185}$$



with the Lorentz decomposition

$$\Pi^{\mu\nu}(k^2) = -g^{\mu\nu}A(k^2) + k^\mu k^\nu B(k^2). \tag{2.186}$$

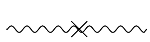
Using Ward identity one finds

$$0 = k_\mu k_\nu \Pi^{\mu\nu} = -k^2 A + k^4 B, \tag{2.187}$$

hence

$$\Pi^{\mu\nu}(k^2) = (-g^{\mu\nu}k^2 + k^\mu k^\nu)B(k^2) \equiv (-g^{\mu\nu}k^2 + k^\mu k^\nu)\Pi(k^2). \tag{2.188}$$

The previous equation has the same structure of equation (2.124)



$$= i\delta_3\left(-g^{\mu\nu}k^2 + k^\mu k^\nu\right), \tag{2.189}$$

and there is a reason for this. Indeed, when studying the renormalization of photon self energy we required $A(0) = 0$ but this is guaranteed by the relation $A = k^2 B$, assuming B regular. Moreover, we imposed a condition on $A'(k^2)$ but

$$A'(k^2) = \frac{d}{dk^2}A(k^2) = \Pi(k^2) \quad \Rightarrow \quad A'(0) = \Pi(0). \tag{2.190}$$

Therefore

$$\delta_3 = \bar{\delta_3} = -e_0^2 A'(0) = -e_0^2\Pi(0) \tag{2.191}$$

and

$$e_0^2\Pi_C(k^2) = e_0^2\left(\Pi(k^2) - \Pi(0)\right) = e_0^2\Pi(k^2) + \delta_3, \tag{2.192}$$

with

$$\Pi_C(0) = 0. \tag{2.193}$$

Thus, in order to obtain an expression for δ_3, we can compute $\Pi(0)$.

Dimensional regularization In *dimensional regularization*

$$ie_0^2\Pi_{\mu\nu} \equiv \left(ie_0\mu^{\frac{\varepsilon}{2}}\right)^2 \int \frac{d^dp}{(2\pi)^d}(-1)(i^2)\frac{\mathrm{Tr}\left(\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)\right)}{D_1D_2} \tag{2.194}$$

where $D_1 = p^2 - m^2 + i\varepsilon$ and $D_2 = (p+k)^2 - m^2 + i\varepsilon$. Using

$$\mathrm{Tr}\left(\gamma_\mu(\not{p}+m)\gamma_\nu(\not{p}+\not{k}+m)\right) = 4(2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)), \tag{2.195}$$

we find

$$i\Pi_{\mu\nu} = -4\mu^\epsilon \int \frac{d^dp}{(2\pi)^d}\frac{N_{\nu\mu}}{D_1D_2} \tag{2.196}$$

where $N_{\nu\mu}(p,k) \equiv 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)$.

Feynman parameterization We can now exploit Feynman parameterization. We write

$$D_2x + D_1(1-x) = p^2x + k^2x + 2p\cdot kx - m^2x + p^2 - m^2 - p^2x + m^2x \tag{2.197}$$

and, adding and subtracting k^2x^2,

$$D_2x + D_1(1-x) = k^2x + 2p\cdot kx + p^2 - m^2 + k^2x^2 - k^2x^2 = (p+kx)^2 - C \tag{2.198}$$

with

$$C = m^2 - k^2x(1-x) \equiv C(x, m^2, k^2). \tag{2.199}$$

Thus we have

$$i\Pi_{\mu\nu} = -4\mu^\varepsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p,k)}{\left((p+kx)^2 - C + i\varepsilon\right)^2}. \tag{2.200}$$

Redefining $p^\mu \to p^\mu - k^\mu x$, $d^dp \to d^dp$ we find

$$i\Pi_{\mu\nu} = -4\mu^\varepsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d}\frac{N_{\mu\nu}(p-kx,k)}{(p^2 - C + i\varepsilon)^2} \tag{2.201}$$

with

$$\begin{aligned} N_{\mu\nu}(p-kx,k) = 2(p_\mu - k_\mu x)(p_\nu - k_\nu x) + (p_\mu - k_\mu x)k_\nu + (p_\nu - k_\nu x)k_\mu + \\ - g_{\mu\nu}\left((p-kx)^2 + (p-kx)\cdot k - m^2\right). \end{aligned} \tag{2.202}$$

Since

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu}{(p^2 - C + i\varepsilon)^2} = 0, \tag{2.203}$$

we can drop some terms in equation (2.202) and consider

$$\begin{aligned} N_{\mu\nu}(p-kx,k)\Big|^{\text{no single}}_{\text{power}} &= 2p_\mu p_\nu + k_\mu k_\nu x^2 - k_\mu k_\nu x - k_\mu k_\nu x - g_{\mu\nu}(p^2 + k^2x^2 - k^2x - m^2) = \\ &= 2p_\mu p_\nu + 2x(x-1)k_\mu k_\nu - g_{\mu\nu}(p^2 - m^2 + k^2x(x-1)). \end{aligned} \tag{2.204}$$

Moreover, we recall that

$$\int \frac{d^dp}{(2\pi)^d}\frac{p^\mu p^\nu}{(p^2 - C + i\varepsilon)^2} = \frac{g^{\mu\nu}}{d}\int \frac{d^dp}{(2\pi)^d}\frac{p^2}{(p^2 - C + i\varepsilon)^2} \tag{2.205}$$

and so

$$\begin{aligned}
\mu^\varepsilon M_{\mu\nu} &\equiv \mu^\varepsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p-kx,k)}{(p^2-C+i\varepsilon)^2} = \\
&= \mu^\varepsilon \left(\frac{2}{d} g_{\mu\nu} I_{1,2} + 2x(x-1)k_\mu k_\nu I_{0,2} - g_{\mu\nu} I_{1,2} - g_{\mu\nu} \left(-m^2 + k^2(x-1)x \right) I_{0,2} \right) = \\
&= \frac{2-d}{d} g_{\mu\nu} \mu^\varepsilon I_{1,2} + \left(-2x(1-x)k_\mu k_\nu + \left(x(1-x)k^2 + m^2 \right) g_{\mu\nu} \right) \mu^\varepsilon I_{0,2} = \\
&= \frac{\varepsilon-2}{4-\varepsilon} g_{\mu\nu} \mu^\varepsilon I_{1,2} + \left(-2x(1-x)k_\mu k_\nu + \left(x(1-x)k^2 + m^2 \right) g_{\mu\nu} \right) \mu^\varepsilon I_{0,2}. \qquad (2.206)
\end{aligned}$$

In general, $I_{0,2}$ and $I_{1,2}$ admit a full analitical expression in ε but, for simplicity, we can consider expansions

$$\mu^\varepsilon I_{0,2} = \frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{C} \right)^{\frac{\varepsilon}{2}} \frac{\Gamma\left(\frac{\varepsilon}{2}\right)}{\Gamma(2)} \overset{\varepsilon \to 0}{\approx} \frac{i}{16\pi^2} \left(\Delta_\varepsilon - \ln \frac{C}{\mu^2} \right) + o(\varepsilon), \qquad (2.207)$$

$$\mu^\varepsilon I_{1,2} = -\frac{i}{16\pi^2} \left(\frac{4\pi\mu^2}{C} \right)^{\frac{\varepsilon}{2}} C \frac{\Gamma\left(3-\frac{\varepsilon}{2}\right)}{\Gamma\left(2-\frac{\varepsilon}{2}\right)} \frac{\Gamma\left(-1+\frac{\varepsilon}{2}\right)}{\Gamma(2)} \overset{\varepsilon \to 0}{\approx} \frac{i}{16\pi^2} C \left(1 + 2\Delta_\varepsilon - 2\ln \frac{C}{\mu^2} \right) + o(\varepsilon) \qquad (2.208)$$

(note that C is not dimensionless). Furthermore we have to expand

$$\frac{\varepsilon-2}{4-\varepsilon} = \frac{2}{4-\varepsilon} - 1 = -\frac{1}{2} + \frac{1}{8}\varepsilon + o(\varepsilon^2) \qquad (2.209)$$

and now we can insert equations (2.207), (2.208) and (2.209) into equation (2.206) obtaining

$$\mu^\varepsilon M_{\mu\nu} = i\frac{x(1-x)}{8\pi^2} \left(\ln \frac{C}{\mu^2} - \Delta_\varepsilon \right) (-g_{\mu\nu}k^2 + k_\mu k_\nu) + o(\varepsilon). \qquad (2.210)$$

The structure we have just isolated is the same of equation (2.188) and since

$$i\Pi_{\mu\nu} = -4 \int_0^1 dx \mu^\varepsilon M_{\mu\nu} \equiv i\Pi(k^2)(-g^{\mu\nu}k^2 + k^\mu k^\nu) \qquad (2.211)$$

we eventually extract the expression

$$\Pi(k^2) = \frac{1}{2\pi^2} \int_0^1 dx\ x(1-x) \left(\Delta_\varepsilon - \ln \frac{C}{\mu^2} \right) \qquad (2.212)$$

with $C \equiv m^2 - k^2x(1-x)$. We can rewrite

$$\begin{aligned}
\Pi(k^2) &= \frac{1}{2\pi^2} \left(\Delta_\varepsilon \underbrace{\int_0^1 dx\ x(1-x)}_{=\frac{1}{6}} - \int_0^1 dx\ x(1-x) \ln \frac{C}{\mu^2} \right) = \qquad (2.213) \\
&= \frac{1}{12\pi^2} \Delta_\varepsilon - \frac{1}{2\pi^2} \int_0^1 dx\ x(1-x) \ln \frac{C}{\mu^2}. \qquad (2.214)
\end{aligned}$$

At $k^2 = 0$, $C = m^2$ and so

$$\Pi(0) = \frac{1}{12\pi^2} \left(\Delta_\varepsilon - \ln \frac{m^2}{\mu^2} \right). \qquad (2.215)$$

Therefore

$$\delta_3 = -e_0^2 \Pi(0) = -\frac{1}{3} \left(\frac{\alpha}{\pi} \right) \left(\Delta_\varepsilon - \ln \frac{m^2}{\mu^2} \right) \qquad (2.216)$$

and, correctly, δ_3 is divergent.

In this section we have shown how to calculate δ_3 using dimensional regularization but the same result can be achieved using cut-off integrations.

2.8.1 Large momentum expansion

Given the importance of δ_3, which is related to the normalization of charge, we want to study it better. In particular we consider

$$\Pi_C(k^2) = \Pi(k^2) - \Pi(0) = -\frac{1}{2\pi}\int_0^1 dx\; x(1-x)\ln\frac{C}{\mu^2} + \frac{1}{12\pi^2}\ln\frac{m^2}{\mu^2} \tag{2.217}$$

in the limit of large momentum $k^2 \gg m^2$ we can use the approximation

$$C = m^2 - k^2x(1-x) \approx -k^2x(1-x) \tag{2.218}$$

we obtain

$$-\frac{1}{2\pi}\int_0^1 dxx(1-x)\ln\frac{C}{\mu^2} = \frac{5-3\ln\left(-\frac{k^2}{\mu^2}\right)}{36\pi^2} \tag{2.219}$$

hence

$$\Pi_C(k^2) = \frac{5-3\ln\left(-\frac{k^2}{\mu^2}\right)}{36\pi^2} + \frac{1}{12\pi^2}\ln\frac{m^2}{\mu^2} \overset{k^2\gg m^2}{\approx} -\frac{1}{12\pi^2}\ln\left(-\frac{k^2}{m^2}\right). \tag{2.220}$$

Where does Π_C is involved? We met it in the renormalization of the photon

$$-i\frac{g_{\alpha\beta}}{k^2+i\varepsilon}e_0^2 \to -\frac{ig_{\alpha\beta}}{k^2+i\varepsilon}e^2 + \frac{-ig_{\alpha\beta}e^2}{k^2+i\varepsilon}\left(-e^2\Pi_C(k^2)\right). \tag{2.221}$$

Now we can use expression (2.220)

$$-\frac{ig_{\alpha\beta}}{k^2+i\varepsilon}e^2 + \frac{-ig_{\alpha\beta}e^2}{k^2+i\varepsilon}\left(-e^2\Pi_C(k^2)\right) = -\frac{ig_{\alpha\beta}}{k^2+i\varepsilon}\frac{e^2}{1+e^2\Pi_C(k^2)} = \tag{2.222}$$

$$\approx -\frac{ig_{\alpha\beta}}{k^2+i\varepsilon}\frac{e^2}{1-\frac{e^2}{12\pi^2}\ln\left(-\frac{k^2}{m^2}\right)} = \tag{2.223}$$

$$\equiv -\frac{ig_{\alpha\beta}}{k^2+i\varepsilon}e_{eff}^2 \tag{2.224}$$

where we have used the approximation $k^2 \gg m^2$ and we have introduced an *effective charge*

$$e_{eff}^2 \equiv \frac{e^2}{1-\frac{e^2}{12\pi^2}\ln\left(-\frac{k^2}{m^2}\right)}. \tag{2.225}$$

that grows as the distance gets smaller and respectively a larger momentum. We can also define

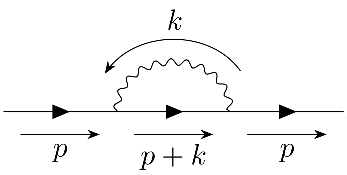
$$\alpha_{eff} \equiv \frac{\alpha}{1-\frac{1}{3}\left(\frac{\alpha}{\pi}\right)\ln\left(-\frac{k^2}{m^2}\right)}. \tag{2.226}$$

The minus sign of the coefficient of the ln function in the denominator implies that the effective charge gets larger at short distances. An other thing to notice is, the scale where perturbation theory breaks down happens for high energies $\sim 10^{286}$eV. Such scales where we have a pole and at which α_{eff} diverges is called **Landau Pole**.

2.9 1-loop electron self energy

Let's repeat the previous discussion for the electron self energy. We have

$$ie_0^2\Sigma(\not{p}) =$$



, (2.227)

and

$$
\begin{aligned}
e_0^2\Sigma(\not{p}) &= e_0^2\Sigma(\not{p}=m) + e_0^2\Sigma'(\not{p}=m)(\not{p}-m) + e_0^2\Sigma_C(\not{p})(\not{p}-m) = \\
&= -m\bar{\delta}_m - \bar{\delta}_2(\not{p}-m) + e_0^2\Sigma_C(\not{p})(\not{p}-m) = \\
&= m\delta_m - \delta_2(\not{p}-m) + e_0^2\Sigma_C(\not{p})(\not{p}-m). \qquad (2.228)
\end{aligned}
$$

Thus, we want to find two counter-terms, δ_m and δ_2 (even for the photon they were two, but one was trivial).

Dimensional regularization In dimensional regularization we have

$$
ie_0^2\Sigma(\not{p}) \equiv \left(ie_0\mu^{\frac{\varepsilon}{2}}\right)^2 \int \frac{d^dk}{(2\pi)^d}(i)^2(-1)\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma_\mu}{D_1D_2} \qquad (2.229)
$$

with $D_1 = k^2 + i\varepsilon$, $D_2 = (p+k)^2 - m^2 + i\varepsilon$. Proceeding as in the last section we write

$$
\Sigma(\not{p}) = i\mu^\varepsilon \int \frac{d^dk}{(2\pi)^d}\frac{N(p,k)}{D_1D_2} \qquad (2.230)
$$

where in this case the numerator has the form

$$
N(k,p) = -(d-2)(\not{k}+\not{p}) + dm. \qquad (2.231)
$$

Feynman parameterization Using a Feynman parameter, we consider

$$
D_2x + D_1(1-x) + p^2x^2 - p^2x^2 = (k+px)^2 - C \qquad (2.232)
$$

with

$$
C \equiv -p^2x(1-x) + m^2x. \qquad (2.233)
$$

Therefore

$$
\Sigma(\not{p}) = i\mu^\varepsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{N(k,p)}{\left((k+px)^2 - C + i\varepsilon\right)^2}. \qquad (2.234)
$$

Performing the shift $k \to k - px$ one finds

$$
\Sigma(\not{p}) = i\mu^\varepsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{N(k-px,p)}{(k^2 - C + i\varepsilon)^2}, \qquad (2.235)
$$

where

$$
N(p-kx,p) = -(d-2)\left(\not{k} + \not{p}(1-x)\right) + md. \qquad (2.236)
$$

Since we have shifted p, we can drop linear terms in equation (2.236)

$$
N_{\mu\nu}(p-kx,k)\big|^{\text{no single}}_{\text{power}} = -(d-2)\not{p}(1-x) + md. \qquad (2.237)
$$

Finally, we have

$$
\Sigma(\not{p}) \equiv a(p^2) + b(p^2)\not{p} \qquad (2.238)
$$

with

$$
a(p^2) = -im(\varepsilon - 4)\int_0^1 dx\mu^\varepsilon I_{0,2}, \qquad (2.239)
$$

$$
b(p^2) = i(\varepsilon - 2)\int_0^1 dx(1-x)\mu^\varepsilon I_{0,2}. \qquad (2.240)
$$

In order to evaluate δ_m, we consider the case $p^2 = m^2$ in which

$$\Sigma(m) = a(m^2) + b(m^2)m, \qquad C|_{p^2=m^2} = m^2x^2 \tag{2.241}$$

and

$$\mu^\varepsilon I_{0,2} = \frac{i}{16\pi^2}\left(\Delta_\varepsilon - \ln\frac{C}{\mu^2}\right) + o(\varepsilon). \tag{2.242}$$

Therefore

$$\Sigma(m) = -\frac{m}{16\pi^2}\left(3\Delta_\varepsilon + 3\ln\left(\frac{\mu^2}{m^2}\right) + 4\right) \tag{2.243}$$

and

$$\delta_m = e^2\frac{\Sigma(m)}{m} = -\frac{1}{4}\left(\frac{\alpha}{\pi}\right)\left(3\Delta_\varepsilon + 3\ln\left(\frac{\mu^2}{m^2}\right) + 4 + o(\varepsilon)\right). \tag{2.244}$$

We next consider δ_2, which is defined as

$$\delta_2 = -e_0^2\Sigma'(\not{p} = m). \tag{2.245}$$

Taking the derivative with respect to $\not{p}$ of equation (2.238), we find

$$\begin{aligned}
\Sigma'(\not{p}) &= \frac{\partial}{\partial\not{p}}a(p^2) + \frac{\partial}{\partial\not{p}}\left(b(p^2)\not{p}\right) = \frac{\partial}{\partial\not{p}}a(p^2) + b(p^2) + \not{p}\frac{\partial}{\partial\not{p}}b(p^2) = \\
&= \frac{\partial p^2}{\partial\not{p}}\frac{\partial}{\partial p^2}a(p^2) + b(p^2) + \not{p}\frac{\partial p^2}{\partial\not{p}}\frac{\partial}{\partial p^2}b(p^2) \underset{(p^2=\not{p}\not{p})}{=} \\
&= 2\not{p}a'(p^2) + b(p^2) + 2\underbrace{\not{p}\not{p}}_{p^2}b'(p^2).
\end{aligned} \tag{2.246}$$

Evaluating equation (2.246) at $\not{p} = m$, we find

$$\Sigma'(\not{p} = m) = 2ma'(m^2) + b(m^2) + 2m^2b'(m^2), \tag{2.247}$$

where

$$a'(p^2) = -im(\varepsilon - 4)\int_0^1 dx\mu^\varepsilon\frac{d}{dp^2}I_{0,2}, \tag{2.248}$$

$$b'(p^2) = i(\varepsilon - 2)\int_0^1 dx(1-x)\mu^\varepsilon\frac{d}{dp^2}I_{0,2}. \tag{2.249}$$

Since

$$\mu^\varepsilon\frac{d}{dp^2}I_{0,2} = \frac{i}{16\pi^2}\frac{(1-x)}{C} + \mathcal{O}(\varepsilon), \tag{2.250}$$

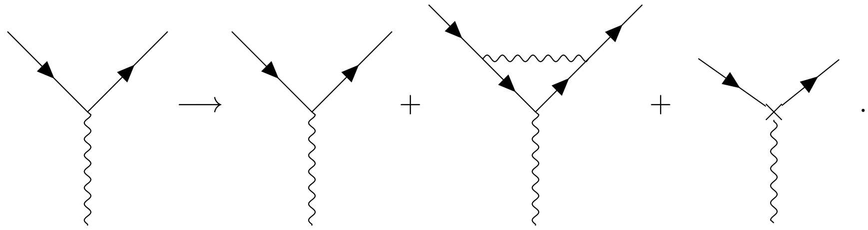
our final result is

$$\delta_2 = -e_0^2\Sigma'(\not{p} = m) = -\frac{1}{4}\left(\frac{\alpha}{\pi}\right)\left(\Delta_\varepsilon + \ln\left(\frac{\mu^2}{m^2}\right) + 4\right) + \frac{\alpha}{\pi}\int_0^1 dx\ \frac{1}{x} + \mathcal{O}(\varepsilon). \tag{2.251}$$

We note that the integral is IR divergent and compensates the bremsstrahlung contribution.

2.10 Vertex correction

Let us consider the one-loop correction to the tree-level vertex,



(2.252)

This corresponds to the replacement

$$ie_0\gamma^\mu \longrightarrow i\Gamma^\mu \equiv ie\left[\gamma^\mu + e^2\Lambda^\mu(p,p') + \delta_1\gamma^\mu\right], \tag{2.253}$$

where $\Lambda^\mu(p,p')$ is from equation (2.73) given by

$$\bar{u}'\Lambda^\mu(p,p')u = \bar{u}'\left(f_1(q^2)\gamma^\mu + \frac{f_2(q^2)}{2m}i\sigma^{\mu\nu}q_\nu\right)u. \tag{2.254}$$

Equation (2.73) also allows us to write

$$\bar{u}\Gamma^\mu u \equiv e\bar{u}'\left(F_1(q^2)\gamma^\mu + \frac{F_2(q^2)}{2m}\sigma^{\mu\nu}q_\nu\right)u \tag{2.255}$$

with

$$F_1(q^2) \equiv 1 + e^2 f_1(q^2) + \delta_1. \tag{2.256}$$

Our renormalization condition is then

$$F_1(0) = 1 \quad \Leftrightarrow \quad e^2 f_1(0) = -\delta_1, \tag{2.257}$$

which is equivalent to the condition

$$\bar{u}'(P)\Lambda_c^\mu(P,P)u(P) = 0 \tag{2.258}$$

we have imposed in Section 2.3. Indeed, from

$$i\Gamma^\mu(p,p') = ie\left(\gamma^\mu + e^2\Lambda_c^\mu(p,p')\right) \tag{2.259}$$

we have the identification

$$e^2\Lambda_c^\mu(p,p') = e^2\Lambda^\mu(p,p') + \delta_1\gamma^\mu \tag{2.260}$$

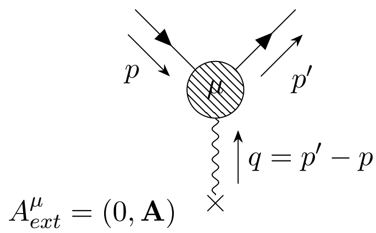
but when $q \to 0$

$$p' \to P, \quad p \to P \quad \Rightarrow \quad \Lambda^\mu(P,P) \to L\gamma^\mu \tag{2.261}$$

and we recover equation (2.257). We now can find an explicit expression for δ_1, but we omit this calculation.

2.11 Anomalous magnetic moment of the electron

As an application of dimensional regularization, we now derive the anomalous magnetic moment of the electron to order α in perturbation theory. The diagram describing the scattering of an electron by an external magnetic field, $\mathbf{B} = \mathbf{\nabla} \times \mathbf{A}$, is



While studying the vertex correction, we derived the following equation for the vertex function

$$\Gamma_\mu(p',p) = e\left(F_1(q^2)\gamma_\mu + \frac{F_2(q^2)}{2m} i\sigma_{\mu\nu}q^\nu\right) = e\left(\frac{F_1(q^2)}{2m}\left(p'+p\right)_\mu + \frac{F_1(q^2)+F_2(q^2)}{2m} i\sigma_{\mu\nu}q^\nu\right), \tag{2.262}$$

where in the second passage we have used Gordon identity. The electron transition current,

$$J_\mu = \bar{u}'\Gamma_\mu u, \tag{2.263}$$

coupled with the external electromagnetic field, gives the following interaction hamiltonian,

$$\begin{aligned}
\mathcal{H}_{int} &= -J_\mu A^\mu_{ext} = \\
&= -\frac{e}{2m}F_1(p'+p)_\mu A^\mu_{ext} - \frac{e}{2m}(F_1+F_2)i\sigma_{\mu\nu}q^\nu A^\mu_{ext} = \\
&= \frac{e}{2m}F_1(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}_{ext} + \frac{e}{2m}(F_1+F_2)i\sigma_{i\nu}q^\nu A^i_{ext} = \\
&= \frac{e}{2m}F_1(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}_{ext} + \frac{e}{2m}(F_1+F_2)i(\sigma_{i0}q^0 - \sigma_{ij}q^j)A^i_{ext} = \\
&= \frac{e}{2m}F_1(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}_{ext} + \frac{e}{2m}(F_1+F_2)i\sigma_{i0}q_0 - \frac{e}{2m}(F_1+F_2)i\sigma_{ij}q^j A^i_{ext}.
\end{aligned} \tag{2.264}$$

For elastic scattering, $q_0 = E' - E \simeq 0$ and the equation above becomes

$$\mathcal{H}_{int} = \underbrace{\frac{e}{2m}F_1(\mathbf{p}'+\mathbf{p})\cdot\mathbf{A}_{ext}}_{\mathcal{H}^E_{int}} - \underbrace{\frac{e}{2m}(F_1+F_2)i\sigma_{ij}q^j A^i_{ext}}_{\mathcal{H}^B_{int}}. \tag{2.265}$$

Since $q_j = -i\nabla_j$, the second term is

$$\begin{aligned}
\mathcal{H}^B_{int} &= -\frac{e}{2m}(F_1+F_2)\underbrace{\sigma_{ij}}_{\varepsilon_{ijk}\sigma_k}\nabla^j A^i_{ext} = \frac{e}{2m}(F_1+F_2)\varepsilon_{kji}\sigma_k\nabla^j A^i_{ext} = \\
&= \frac{e}{2m}(F_1+F_2)\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\times\mathbf{A} = \frac{e}{2m}(F_1+F_2)\underbrace{\boldsymbol{\sigma}}_{2\mathbf{S}}\cdot\mathbf{B},
\end{aligned} \tag{2.266}$$

and at $q = 0$ becomes

$$\begin{aligned}
\mathcal{H}^B_{int}(q^2=0) &= \frac{e}{2m}\left(F_1(0)+F_2(0)\right)2\mathbf{S}\cdot\mathbf{B} = \frac{e}{2m}\left[2\left(1+F_2(0)\right)\right]\mathbf{S}\cdot\mathbf{B} = \\
&= \frac{e}{2m}g_e\mathbf{S}\cdot\mathbf{B} = -\boldsymbol{\mu}_e\cdot\mathbf{B},
\end{aligned} \tag{2.267}$$

where

$$g_e = 2\left(1+F_2(0)\right) \tag{2.268}$$

is the Landé factor and

$$\boldsymbol{\mu}_e \equiv -\frac{e}{2m}g_e\mathbf{S} \tag{2.269}$$

is the electron magnetic moment. At tree level, $F_2(0) = 0$ and from equation (2.268) we see that the Landé factor is $g_e = 2$, as predicted from Dirac's theory. The deviation from this value is called *anomaly*,

$$a_e \equiv F_2(0) = \frac{1}{2}(g_e - 2). \tag{2.270}$$

Before proceeding with the calculation of this correction at one loop, let us introduce a class of projectors which can be used to extract information about form factors from a given diagram.

2.11.1 Projectors

e^- self energy Given

$$e_0^2\Sigma(\not{p}) = m\delta_m - \delta_2(\not{p}-m) + \mathcal{O}\left((\not{p}-m)^2\right),$$

we can define

$$P_{\delta_m} \equiv \frac{e_0^2}{4m^2}(\not{p}+m) \quad \Rightarrow \quad \delta_m = \mathrm{Tr}\left\{P_{\delta_m}\Sigma\right\}, \tag{2.271}$$

$$P_{\delta_2} \equiv -\frac{e_0^2}{4m^2}(\not{p}+m)\not{p}\frac{\partial}{\partial\not{p}} \quad \Rightarrow \quad \delta_2 = \mathrm{Tr}\left\{P_{\delta_2}\Sigma\right\}_{\not{p}=m}. \tag{2.272}$$

γ self energy As for

$$\Pi^{\mu\nu}(p^2) = \left(-g^{\mu\nu}p^2 + p^\mu p^\nu\right)\Pi(p^2), \tag{2.273}$$

we can define

$$P_\Pi^{\mu\nu} \equiv \frac{1}{(p^2)^2(d-1)}\left(-g^{\mu\nu}p^2 + p^\mu p^\nu\right) \quad \Rightarrow \quad \Pi(p^2) = P_\Pi^{\mu\nu}\Pi_{\mu\nu}. \tag{2.274}$$

Vertex correction Finally, given

$$\Lambda_\mu(p',p) = F_1(q^2)\gamma_\mu + F_2(q^2)\frac{1}{2m}i\sigma_{\mu\nu}q^\nu, \tag{2.275}$$

where $q^\nu = p'^\nu - p^\nu$, we can define

$$P_{F_i}^\mu \equiv (\not{p}+m)\left[c_{1i}\gamma^\mu + c_{2i}\frac{1}{2m}(p'+p)^\mu\right](\not{p}'+m) \quad \Rightarrow \quad F_i(q^2) = \mathrm{Tr}\left\{P_{F_i}^\mu\Lambda_\mu\right\}_{p^2=p'^2=m^2}, \tag{2.276}$$

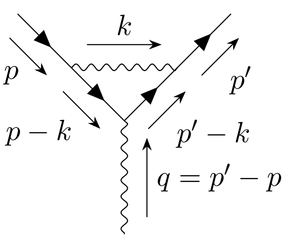
where

$$c_{11} \equiv \frac{1}{2(d-2)(q^2-4m^2)}, \quad c_{21} \equiv c_{11}\frac{4m^2}{(q^2-4m^2)}(d-1) \quad \Rightarrow \quad P_{F_1},$$

$$c_{12} \equiv -\frac{2m^2}{(d-2)q^2(q^2-4m^2)}, \quad c_{22} \equiv c_{12}\frac{(d-2)q^2+4m^2}{(q^2-4m^2)} \quad \Rightarrow \quad P_{F_2}.$$

2.11.2 $g-2$ at one loop

We are now ready to perform the explicit calculation of the anomaly at one loop. We consider the following diagram,



$$= e_0^2\Lambda^\mu(p',p). \tag{2.277}$$

By dimensional regularization,

$$\begin{aligned} e_0^2\Lambda^\mu(p',p) &\equiv \left(ie_0\mu^{\frac{\varepsilon}{2}}\right)^2\int\frac{d^dk}{(2\pi)^d}\gamma^\alpha iS_F(p'-k)\gamma^\mu iS_F(p-k)\gamma^\beta iD_{F\alpha\beta}(k) \\ &= -e_0^2(i)^5\mu^\varepsilon\int\frac{d^dk}{(2\pi)^d}\frac{N^\mu(p',p,k)}{D_1D_2D_3}, \end{aligned} \tag{2.278}$$

where

$$N^\mu(p',p,k) \equiv \gamma^\alpha(\not{p}'-\not{k}+m)\gamma^\mu(\not{p}-\not{k}+m)\gamma_\alpha,$$
$$D_1 = k^2+i\varepsilon, \quad D_2=(p'-k)^2-m^2+i\varepsilon, \quad D_3=(p-k)^2-m^2+i\varepsilon.$$

Using Feynman parametrization, we can write

$$\frac{1}{D_1D_2D_3} = 2\int_0^1 dx\int_0^{1-x}dy\frac{1}{[D_1+(D_2-D_1)x+(D_3-D_1)y]^3}. \tag{2.279}$$

The quantity inside square brackets is

$$k^2-2xp'\cdot k-2yp\cdot k = k^2-2k_\mu\underbrace{(xp'^\mu+yp^\mu)}_{\equiv a^\mu} = k^2-2k\cdot a = (k-a)^2-\underbrace{a^2}_{\equiv C}.$$

Hence,

$$\Lambda^\mu(p',p) = -i\mu^\varepsilon\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{2N^\mu(p',p,k)}{[(k-a)^2-C+i\varepsilon]^3}, \tag{2.280}$$

and shifting the integration variable $k^\mu\to k^\mu+a^\mu$,

$$\Lambda^\mu(p',p) = -i\mu^\varepsilon\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{2N^\mu(p',p,k+a)}{[k^2-C+i\varepsilon]^3}. \tag{2.281}$$

The numerator can be split in a k-independent, a linear and a quadratic contribution,

$$N^\mu(p',p,k+a) = \gamma^\alpha(\not{p}'-\not{a}+m)\gamma^\mu(\not{p}-\not{a}+m)\gamma_\alpha + (\text{linear terms in } k) + \gamma^\alpha\not{k}\gamma^\mu\not{k}\gamma_\alpha =$$
$$\equiv N_0^\mu(p',p)+N_1^\mu(p',p,k)+N_2^\mu(p',p,k).$$

As N_1^μ contains a single power of k, it vanishes upon integration and we are left with

$$\Lambda^\mu(p',p) = -i\mu^\varepsilon\int_0^1 dx\int_0^{1-x}dy\int\frac{d^dk}{(2\pi)^d}\frac{2(N_0^\mu+N_2^\mu)}{[k^2-C+i\varepsilon]^3} \equiv \Lambda_0^\mu+\Lambda_2^\mu. \tag{2.282}$$

We first consider Λ_0^μ and integrate in k,

$$\Lambda_0^\mu(p',p) = -i\mu^\varepsilon\int_0^1 dx\int_0^{1-x}dy 2N_0^\mu I_{0,3}. \tag{2.283}$$

This integral is UV finite in $d=4$,

$$I_{0,3} = \int\frac{d^4k}{(2\pi)^4}\frac{1}{[k^2-C+i\varepsilon]^3} \underset{\text{(Wick)}}{=} -i\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+C)^3} =$$
$$= -\frac{i}{32\pi^4}\underbrace{\Omega_3}_{2\pi^2}\underbrace{\int_0^\infty dk_E^2\frac{k_E^2}{[k_E^2+C]^3}}_{\frac{1}{2C}} = -\frac{i}{32\pi^2}\frac{1}{C},$$

where in the second line we have used polar coordinates,

$$d^4k_E = d\Omega_3k_E^3dk_E = \frac{1}{2}d\Omega_3k_E^2dk_E^2. \tag{2.284}$$

In $d=4$, equation (2.283) becomes

$$\Lambda_0^\mu = -\frac{1}{16\pi^2}\int_0^1 dx\int_0^{1-x}dy\frac{N_0^\mu}{C}, \tag{2.285}$$

where

$$C = C(q^2) = m^2(x^2 + y^2) + 2p \cdot p' xy = m^2(x^2 + y^2) + xy(2m^2 - q^2), \tag{2.286}$$

which in the elastic limit reduces to

$$C(q^2 = 0) = m^2(x + y)^2. \tag{2.287}$$

We can use the projector introduced in (2.276) to calculate the contribution of Λ_0^μ to the form factor F_2,

$$F_2(q^2) = -\frac{1}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{\mathrm{Tr}\,[P_{F_2\mu} N_0^\mu]}{C}. \tag{2.288}$$

In $d = 4$, the trace is given by

$$\mathrm{Tr}\,[P_{F_2\mu} N_0^\mu] = 4m^2(x + y - 1)(x + y). \tag{2.289}$$

Substituting in equation (2.288), we have

$$F_2(q^2) = \frac{m^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{(1 - x - y)(x + y)}{C}, \tag{2.290}$$

which in the elastic limit becomes

$$F_2(q^2 = 0) = \frac{1}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{1 - x - y}{(x + y)} = \frac{1}{8\pi^2}. \tag{2.291}$$

Accounting for e_0^2 (present in $e_0^2\Lambda^\mu$),

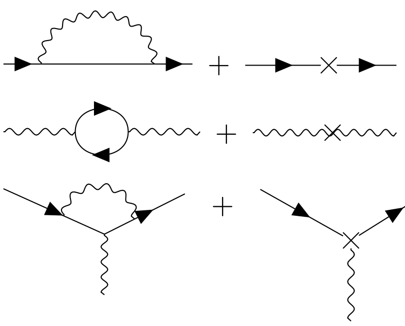
$$e_0^2 F_2(q^2 = 0) = \frac{e_0^2}{4\pi} \frac{1}{2\pi} = \frac{1}{2}\left(\frac{\alpha}{\pi}\right) \simeq 0.00116. \tag{2.292}$$

Moreover, it can be showed that Λ_2^μ doesn't give any contribution to the form factor and this concludes the calculation. This result was first obtained by J. Schwinger in 1948 and was in agreement with experiments.

2.12 Renormalization schemes

Let's sumarize our renormalization procedure.

On-shell renormalization scheme



$$= i\left(e^2\Sigma(\not{p}) - m\delta_m + \delta_2(\not{p} - m)\right) \equiv ie^2\Sigma_R(\not{p})$$

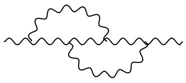
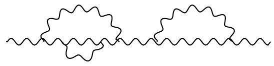
$$= i(-g^{\mu\nu}k^2 + k^\mu k^\nu)(e^2\Pi(k^2) + \delta_3) \equiv i(-g^{\mu\nu}k^2 + k^\mu k^\nu)\Pi_R(k^2)$$

$$= ie(e^2\Lambda^\mu(p, p') + \delta_1\gamma^\mu) \equiv ie\Lambda_R^\mu(p, p')$$

We have four renormalization conditions to fix four counterterms: δ_1, δ_2, δ_3 and δ_m. What we have performed is the so called *on-shell renormalization scheme*. This scheme is defined by setting the renormalized electron mass equivalent to the pole mass $m \equiv m_{pole}$ with conditions

$$\begin{cases} \Sigma_R(\not{p} = m) = 0 \ \rightarrow \ \delta_m, \\ \Sigma_R'(\not{p} = m) = 0 \ \rightarrow \ \delta_2, \\ \Gamma_R^\mu(q^2 = 0) = e\gamma^\mu \ \rightarrow \ \delta_1, \\ \Pi_R(k^2 = 0) = 0 \ \rightarrow \ \delta_3, \end{cases} \tag{2.293}$$

where we are considering renormalized functions and we have defined $q = p' - p$. Renormalized Green functions Σ_R, Π_R, Γ^{μ}_R contain one particle irreducible diagrams only (IP1), i.e. diagrams which don't fall into two pieces if you cut one internal line:

 is 1PI, while  is not.

We are not considering multiple loops because they are automatically included in the geometric series.

Minimal subtraction scheme (MS) What we have summarized in the previous paragraph is a possible renormalization scheme, but not the unique. In the minimal subtraction scheme, counterterms δ_i contain only the $\frac{1}{\varepsilon}$−pole part. For instance

$$\delta_3^{OS} = -\frac{1}{3}\left(\frac{\alpha}{\pi}\right)\left(\Delta_\varepsilon - \ln\frac{m^2}{\mu^2}\right) \quad \Rightarrow \quad \delta_3^{MS} = -\frac{1}{3}\left(\frac{\alpha}{\pi}\right)\frac{2}{\varepsilon} \tag{2.294}$$

where we have used $\Delta_\varepsilon = \frac{2}{\varepsilon} - \gamma + \ln(4\pi)$.

Since our aim is to subtract divergences, different schemes can only produce different finite parts. Therefore this new definition of counterterms is also valid. Differences which arises in finite parts disappear when computing physical quantities, if one takes into account both virtual diagrams and real radiations.

We'll limit our discussion of the minimal scheme to the following application.

2.12.1 The β function of QED

Bare quantities in general do not depend on μ^2, therefore we can establish a differential equation (valid at all orders in perturbation theory) considering, for example, the bare charge e_0

$$0 = \mu\frac{d}{d\mu}e_0 = \mu\frac{d}{d\mu}\left(\mu^{\frac{\varepsilon}{2}}eZ_e\right) = \mu^{\frac{\varepsilon}{2}}eZ_e\left(\frac{\varepsilon}{2} + \frac{\mu}{e}\frac{\partial e}{\partial\mu} + \frac{\mu}{Z_e}\frac{\partial Z_e}{\partial\mu}\right). \tag{2.295}$$

Since $Z_e \equiv (1 + \delta_e)$, we have

$$0 = \frac{\varepsilon}{2} + \frac{\mu}{e}\frac{\partial e}{\partial\mu} + \mu(1 - \delta_e)\frac{\partial}{\partial\mu}(1 + \delta_e). \tag{2.296}$$

We now read previous equation order by order. At the leading order

$$\mu\frac{\partial e}{\partial\mu} = -\frac{\varepsilon}{2}e. \tag{2.297}$$

To find the NLO term, instead, we consider (remember $\delta_e = -\delta_3/2$)

$$Z_e = 1 + \delta_e = 1 - \frac{1}{2}\delta_3 = 1 + \frac{1}{3}\frac{\alpha}{\pi}\frac{1}{\varepsilon} = 1 + \frac{1}{12}\frac{e^2}{\pi^2}\frac{1}{\varepsilon} \tag{2.298}$$

where we have used equation (2.294). Therefore

$$\mu\frac{\partial}{\partial\mu}(1 + \delta_e) = \frac{1}{\varepsilon}\frac{\mu}{12\pi^2}\frac{d}{d\mu}e^2 = \frac{1}{\varepsilon}\frac{\mu}{12\pi^2}2e\frac{de}{d\mu} = \tag{2.299}$$

$$= \frac{1}{6\varepsilon}\frac{e}{\pi^2}\mu\frac{de}{d\mu}. \tag{2.300}$$

Then, using the leading order result in equation (2.297), we write

$$\mu\frac{\partial}{\partial\mu}(1 + \delta_e) = -\frac{1}{12\pi^2}e^2 \tag{2.301}$$

and finally, at NLO

$$\mu \frac{\partial e}{\partial \mu} = -\frac{\varepsilon}{2} e + \frac{1}{12\pi^2} e^3. \tag{2.302}$$

Considering the limit $\varepsilon \to 0$ we find the relation

$$12\pi^2 \frac{\partial e}{e^3} = \frac{\partial \mu}{\mu} \quad \Rightarrow \quad 12\pi^2 \int_{e(\mu_0)}^{e(\mu)} \frac{de}{e^3} = \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} \tag{2.303}$$

which gives

$$12\pi^2 \frac{1}{2} \left(\frac{1}{e^2(\mu_0)} - \frac{1}{e^2(\mu)} \right) = \ln \left(\frac{\mu}{\mu_0} \right) = \frac{1}{2} \ln \left(\frac{\mu^2}{\mu_0^2} \right) \tag{2.304}$$

hence

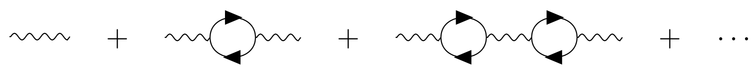
$$e^2(\mu^2) = \frac{e^2(\mu_0)}{1 - \beta_0 e^2(\mu_0^2) \ln \left(\frac{\mu^2}{\mu_0^2} \right)} \tag{2.305}$$

with $\beta_0 \equiv \frac{1}{12\pi^2}$. What we have just found is how the electric charge depends on the choice of μ scale. When we define a renormalization prescription we are choosing a particular scale, a particular μ_0. The on-shell renormalization prescription choose μ_0 equal to the electron mass.

Equation (2.305) has to be compared with the effective charge

$$e^2_{eff} = \frac{e^2}{1 - \frac{e^2}{12\pi} \ln \left(-\frac{k^2}{m^2} \right)} \tag{2.306}$$

which comes from the vacuum polarization resummation, namely when we consider all the blobs (considering the geometric series)



and we stop at the first order. So what we found using a mathematical construction has a very deep meaning and is related to the evolution of the coupling constant. The dependence $\beta(\mu)$

$$\beta(e) = \mu \frac{de}{d\mu} \tag{2.307}$$

is referred to as *running of the coupling constant*. We remember the importance of the sign of $\beta_0 \sim$ Landau pole, which is the first order of the expansion. In order to study the behavior of the theory it's sufficient to study UV behavior of counter-terms (*spiegare meglio*).

2.12.2 Renormalization group equation

In general we can extend the differential equation approach that we have introduced just for a charge to the complete Green function. Let's consider the bare Green function with n photons and m electrons

$$G^0_{n,m} \equiv \langle \Omega | \, T\{ A^0_{\mu_1} \dots A^0_{\mu_n} \psi^0_1 \dots \psi^0_m \} \, | \Omega \rangle \tag{2.308}$$

Applying our renormalization procedure we have

$$G^0_{n,m} = Z_3^{\frac{n}{2}} Z_2^{\frac{m}{2}} G_{n,m} \tag{2.309}$$

where $G_{n,m}$ is the renormalized Green function. In general $G_{n,m} \equiv G_{n,m}(\mu, e, m_e)$ is finite and depends on μ both explicitly and implicitly, so

$$0 = \mu \frac{d}{d\mu} G^0_{n,m} = Z_3^{\frac{n}{2}} Z_2^{\frac{m}{2}} \left(\mu \frac{\partial}{\partial \mu} + \frac{n}{2} \frac{\mu}{Z_3} \frac{\partial Z_3}{\partial \mu} + \frac{m}{2} \frac{\mu}{Z_2} \frac{\partial Z_2}{\partial \mu} + \mu \frac{\partial e}{\partial \mu} \frac{\partial}{\partial e} + \mu \frac{\partial m_e}{\partial \mu} \frac{\partial}{\partial m_e} \right) G_{n,m}. \tag{2.310}$$

Defining the *anomalous dimensions*

$$\gamma_3 \equiv \frac{\mu}{Z_3}\frac{\partial Z_3}{\partial \mu}, \qquad \gamma_2 \equiv \frac{\mu}{Z_2}\frac{\partial Z_2}{\partial \mu}, \qquad \gamma_{m_e} \equiv \frac{\mu}{m_e}\frac{\partial m_e}{\partial \mu}, \tag{2.311}$$

we find

$$\left(\mu\frac{\partial}{\partial \mu} + \frac{n}{2}\gamma_3 + \frac{m}{2}\gamma_2 + \beta\frac{\partial}{\partial e} + \gamma_{m_e}\frac{\partial}{\partial m_e}\right) G_{n,m} = 0. \tag{2.312}$$

Equation (2.312) is known as *Callan Symanzik equation* (for Green functions) or *renormalization group equation*.

Chapter 3

Weak Interactions

As light witnesses, the electromagnetic interaction has an infinite range which correspond to a massless exchanged particle, the photon. The weak interaction is instead associated with the exchange of elementary spin-1 bosons between quarks and/or leptons and these "force carriers" are very massive particles. There are three such "intermediate vector bosons": the charged bosons W^+ and W^- and the neutral Z^0 boson, with masses

$$M_W \simeq 80 \text{ GeV}, \qquad M_Z \simeq 90 \text{ GeV}.$$

To begin with, we develop and apply the intermediate vector boson (IVB) theory which takes account of $W^\pm$ exchange only.

3.1 IVB theory

We conventionally divide weak interaction processes into three categories:

1. *purely leptonic* processes (e.g. $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$),
2. *semileptonic* processes (e.g. $\beta-$decay: $n \to p + e^- + \bar{\nu}_e$),
3. *purely hadronic* processes (e.g. $\Lambda-$decay: $\Lambda \to p + \pi^-$).

Nevertheless, a complete and correct discussion of processes involving hadrons should use QCD and so we'll restrict ourselves to purely leptonic processes (or semileptonic in an approximated form).

Our treatment starts from an analogy with the electromagnetic current in the interactive Hamiltonian

$$H_{QED} = -e\bar{\psi}(x)\gamma^\alpha \psi(x) A_\alpha. \tag{3.1}$$

Indeed, the weak interaction Hamiltonian density responsible for leptonic processes is similarly constructed from bilinear forms of the lepton field operators. The experimental data are consistent with the assumption that the leptons fields enter the interaction only in the combinations

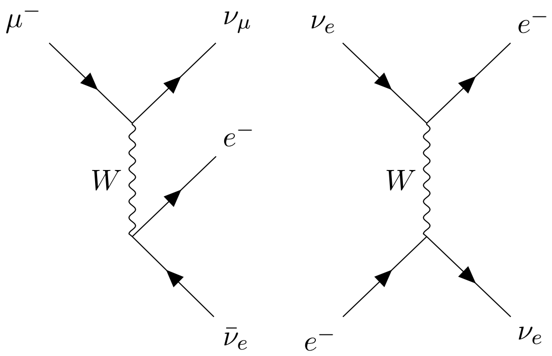
$$J_\alpha(x) \equiv \sum_l \bar{\psi}_l(x)\gamma_\alpha(1-\gamma_5)\psi_{\nu_l}(x), \tag{3.2}$$

$$J_\alpha^\dagger(x) \equiv \sum_l \bar{\psi}_{\nu_l}(x)\gamma_\alpha(1-\gamma_5)\psi_l(x). \tag{3.3}$$

where l labels the various charged lepton fields, $l = e, \mu, \ldots$ and ν_l the corresponding neutrino fields. Since electric charge of leptons is ± 1 (while neutrinos are neutral) we can distinguish between two different charged bosons. Moreover we stress the presence of γ_5 and we note that

$$(J_\alpha)^\dagger = \left(\bar{\psi}_l \gamma_\alpha (1-\gamma_5)\psi_{\nu_l}\right)^\dagger = \left(\psi_l^\dagger \gamma_0 \gamma_\alpha (1-\gamma_5)\psi_{\nu_l}\right)^\dagger = \tag{3.4}$$

$$= \psi_{\nu_l}^\dagger (1-\gamma_5)\gamma_\alpha^\dagger \gamma_0^\dagger \psi_l. \tag{3.5}$$



(a) Muon decay (b) $\nu_e + e^- \to \nu_e + e^-$

Figure 3.1

Since

$$(\gamma_\alpha)^\dagger = \gamma_0\gamma_\alpha\gamma_0, \qquad \gamma_5^\dagger = \gamma_5, \qquad \gamma_0^\dagger = \gamma_0, \qquad \{\gamma_\alpha, \gamma_5\} = 0, \tag{3.6}$$

we have

$$(J_\alpha)^\dagger = \psi_{\nu_l}^\dagger(1-\gamma_5)(\gamma_0\gamma_\alpha \underbrace{\gamma_0)\gamma_0}_{\mathbf{1}} \psi_l = \psi_{\nu_l}^\dagger \gamma_0\gamma_\alpha(1-\gamma_5)\psi_l = J_\alpha^\dagger. \tag{3.7}$$

Inspired by equation (3.1) we can then write

$$H_{IVB} = g_W J^{\alpha\dagger}(x) W_\alpha(x) + g_W J^\alpha(x) W_\alpha^\dagger(x) \tag{3.8}$$

where g_W is a dimensionless coupling constant and the field $W_\alpha(x)$ describes the W particles. With this interaction, we can describe processes like those represented in figure 3.1.

We have seen that QED interaction conserve lepton numbers such as

$$N(e) \equiv N(e^-) - N(e^+), \tag{3.9}$$

whereas they are not conserved by interaction (3.8) (for example, look at the muon decay). Nevertheless, if we modify the definition of lepton numbers to

$$N(e) \equiv N(e^-) - N(e^+) + N(\nu_e) - N(\bar{\nu}_e), \tag{3.10}$$
$$N(\mu) \equiv N(\mu^-) - N(\mu^+) + N(\nu_\mu) - N(\bar{\nu}_\mu), \tag{3.11}$$
$$N(\tau) \equiv N(\tau^-) - N(\tau^+) + N(\nu_\tau) - N(\bar{\nu}_\tau), \tag{3.12}$$

then the currents J_α and $J_\alpha^\dagger$ do conserve lepton numbers. For example, in bombarding nuclei with muon neutrinos, the processes

$$\nu_\mu + n \to p + \mu^- \quad \Leftrightarrow \quad \nu_\mu + (Z, A) \to (Z+1, A) + \mu^- \qquad \text{are ALLOWED,} \tag{3.13}$$
$$\nu_\mu + n \to \bar{p} + \mu^+ \quad \Leftrightarrow \quad \nu_\mu + (Z, A) \to (Z-1, A) + \mu^+ \qquad \text{are FORBIDDEN.} \tag{3.14}$$

The interaction (3.8) is called "V-A" interaction since the current $J^\alpha(x)$ can be written as

$$J^\alpha(x) = J_V^\alpha(x) - J_A^\alpha(x) \tag{3.15}$$

where we have introduced the vectorial current

$$J_V^\alpha \equiv \sum_l \bar{\psi}_l \gamma^\alpha \psi_{\nu_l} \tag{3.16}$$

and the axial vector current

$$J_A^\alpha \equiv \sum_l \bar{\psi}_l \gamma^\alpha \gamma_5 \psi_{\nu_l}. \tag{3.17}$$

Under the parity transformation $(\mathbf{x}, t) \to (-\mathbf{x}, t)$, the spatial components of $J_V^\alpha(x)$ changes sign, while $J_A^\alpha(x)$ does not. Hence, the interaction (3.8) is not invariant under spatial inversion and parity is not conserved.

We remember the chiral projection operators

$$P_L = \frac{1}{2}(1-\gamma_5), \qquad P_R = \frac{1}{2}(1+\gamma_5) \tag{3.18}$$

which satisfy

$$P_L^2 = P_L, \qquad P_R^2 = P_R \qquad P_L P_R = 0 = P_R P_L. \tag{3.19}$$

Therefore the operator

$$\psi_{\nu_l}^L(x) \equiv \frac{1}{2}(1-\gamma_5)\psi_{\nu_l}(x) \tag{3.20}$$

which occurs in equations (3.2) and (3.3) can annihilate only negative helicity neutrinos and create only positive helicity antineutrinos. Thus, the presence of axial current (which doesn't appear in QED theory) create a clear distinction between left and right particles. We can rewrite the leptonic current in the form

$$J^\alpha \equiv 2\sum_l \bar{\psi}_l^L \gamma^\alpha \psi_{\nu_l}^L. \tag{3.21}$$

The free vector boson field The simplest equation for a vector field $W^\alpha(x)$, describing particles of mass m_W and spin 1 (3 polarizations, since the requirement that the Lagrangian must have a positive definite energy density is equivalent to removing the 0-spin particle out of the picture) is the Proca equation [1]

$$\begin{cases} \Box W^\alpha + m_W^2 W^\alpha = 0 \\ \partial_\alpha W^\alpha = 0 \end{cases} \tag{3.22}$$

where $\partial_\alpha W^\alpha = 0$ is not a gauge invariance condition (one automatically finds it by taking the divergence of the general equation i.e Lorentz-invariance condition).

We now define

$$F_W^{\alpha\beta} = \partial^\beta W^\alpha - \partial^\alpha W^\beta, \tag{3.23}$$

and we expand as usual in a complete set of plane waves

$$W^\alpha = W^{\alpha+} + W^{\alpha-}, \tag{3.24}$$

with

$$W^{\alpha+} = \sum_{\mathbf{k}}\sum_r \left(\frac{1}{2V\omega_{\mathbf{k}}}\right)^{\frac{1}{2}} \varepsilon_r^\alpha(\mathbf{k}) a_r(\mathbf{k}) e^{-i\mathbf{kx}} \tag{3.25}$$

$$W^{\alpha-} = \sum_{\mathbf{k}}\sum_r \left(\frac{1}{2V\omega_{\mathbf{k}}}\right)^{\frac{1}{2}} \varepsilon_r^\alpha(\mathbf{k}) b_r^\dagger(\mathbf{k}) e^{i\mathbf{kx}} \tag{3.26}$$

and

$$\omega_{\mathbf{k}} = \left(m_W^2 + \mathbf{k}^2\right)^{\frac{1}{2}}. \tag{3.27}$$

The vectors $\varepsilon_r^\alpha(\mathbf{k})$, $r = 1, 2, 3$ are a complete set of real orthonormal polarization vectors, i.e.

$$\varepsilon_r^\alpha(\mathbf{k})\varepsilon_s^\alpha(\mathbf{k}) = -\delta_{rs} \tag{3.28}$$

[1] For a more detailed derivation of the Lagrangian and why we are not considering the 0-spin particle, see M.D. Schwartz, "Quantum Field Theory and the Standard Model" page 115

and the condition $\partial_\alpha W^\alpha = 0$ implies

$$k_\alpha \varepsilon_r(\mathbf{k}) = 0. \tag{3.29}$$

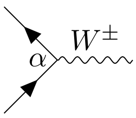
In the frame in which $k = (\omega_{\mathbf{k}}, 0, 0, |\mathbf{k}|)$ a suitable choice of polarization vectors is

$$\begin{cases} \varepsilon_1(\mathbf{k}) = (0,1,0,0) \\ \varepsilon_2(\mathbf{k}) = (0,0,1,0) \\ \varepsilon_3(\mathbf{k}) = (|\mathbf{k}|, 0, 0, \omega_{\mathbf{k}})/m_W \end{cases} \tag{3.30}$$

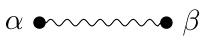
hence the completeness relation

$$\sum_{r=1}^{3} \varepsilon_r^\alpha(\mathbf{k}) \varepsilon_r^\beta(\mathbf{k}) = -g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}. \tag{3.31}$$

The Feynman rules We now write down the Feynman rules for treating leptonic processes in perturbation theory.



$$= -i g_W \gamma^\alpha (1 - \gamma_5), \tag{3.32}$$



$$= \frac{i\left(-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}\right)}{k^2 - m_W^2 + i\varepsilon} \equiv i D_F^{\alpha\beta}(k, m_W), \tag{3.33}$$

$$= \varepsilon_i^\alpha, \qquad (i = 1,2,3). \tag{3.34}$$

Note that, in order to obtain Feynman rules in IVB theory for all leptons, we only have to relabel the Feynman rules for electron propagators and external lines, replace the respective masses to the corresponding lepton one and substitute the external line factor with factors appropriate to negative leptons and neutrinos as well as for the corresponding antiparticles.

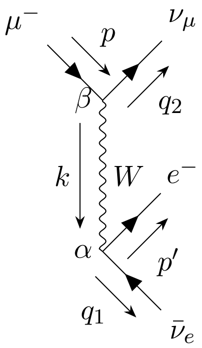
3.2 Muon decay

We now study the muon decay,

$$\mu^-(p) \to e^-(p') + \bar{\nu}_e(q_1) + \nu_\mu(q_2), \tag{3.35}$$

where $p^\mu = (E, \mathbf{p})$, $p'^\mu = (E', \mathbf{p}')$, $q_i^\mu = (E_i, \mathbf{q}_i)$ and $p = p' + q_1 + q_2$ by momentum conservation. In lowest order, the Feynman amplitude for this process is

$\mathcal{M} =$



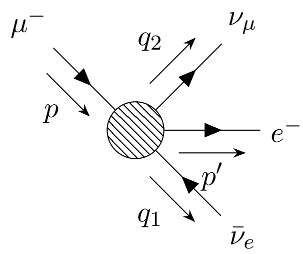
$$= -g_W^2 \left[\bar{u}(p')\gamma^\alpha (1-\gamma_5) v(q_1)\right] \frac{i\left(-g_{\alpha\beta} + \frac{k_\alpha k_\beta}{m_W^2}\right)}{k^2 - m_W^2 + i\varepsilon} \left[\bar{u}(q_2)\gamma^\beta (1-\gamma_5) u(p)\right]. \tag{3.36}$$

bearing in mind that $k = p - q_2 = p' + q_1$. Since $m_W \gg m_\mu \gg m_e$, we can consider take $m_W \to \infty$. In this limit, the propagator shrinks to a contact term,

$$\equiv \frac{i g_{\alpha\beta}}{m_W^2}, \tag{3.37}$$

and our amplitude becomes

$$\mathcal{M} =$$



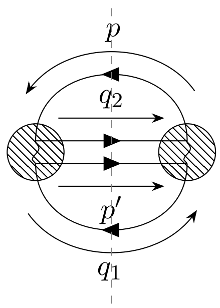
$$= -\frac{iG}{\sqrt{2}} \left[\bar{u}(p')\gamma^\alpha \left(1-\gamma_5\right) v(q_1)\right] \left[\bar{u}(q_2)\gamma_\alpha \left(1-\gamma_5\right) u(p)\right], \tag{3.38}$$

where we have introduced the Fermi constant,

$$\frac{G}{\sqrt{2}} \equiv \left(\frac{g_W}{m_W}\right)^2. \tag{3.39}$$

The unpolarized squared amplitude is

$$|\bar{\mathcal{M}}|^2 = \frac{1}{2}\sum_{\text{spin}} |\mathcal{M}|^2 =$$



$$= \frac{G^2}{4} \operatorname{Tr}\left(\frac{\not{p}' + m_e}{2m_e}\gamma^\alpha(1-\gamma_5)\frac{\not{q}_1 - m_{\nu_e}}{2m_{\nu_e}}\gamma^\beta(1-\gamma_5)\right) \times$$

$$\times \operatorname{Tr}\left(\frac{\not{q}_2 + m_{\nu_\mu}}{2m_{\nu_\mu}}\gamma_\alpha(1-\gamma_5)\frac{\not{p} + m_\mu}{2m_\mu}\gamma_\beta(1-\gamma_5)\right). \tag{3.40}$$

and in the $m_{\nu_e} = m_{\nu_\mu} = 0$ approximation the above amplitude becomes

$$|\bar{\mathcal{M}}|^2 = \frac{G^2}{4} \operatorname{Tr}\left(\frac{\not{p}' + m_e}{2m_e}\gamma^\alpha(1-\gamma_5)\frac{\not{q}_1}{2m_{\nu_e}}\gamma^\beta(1-\gamma_5)\right) \operatorname{Tr}\left(\frac{\not{q}_2}{2m_{\nu_\mu}}\gamma_\alpha(1-\gamma_5)\frac{\not{p} + m_\mu}{2m_\mu}\gamma_\beta(1-\gamma_5)\right). \tag{3.41}$$

Trace calculation leads to

$$\begin{aligned}
\operatorname{Tr}\left[\left(\not{p}' + m_e\right)\gamma^\alpha \left(1-\gamma_5\right)\not{q}_1\gamma^\beta \left(1-\gamma_5\right)\right] &= 2\operatorname{Tr}\left[\left(\not{p}' + m_e\right)\gamma^\alpha \left(1-\gamma_5\right)\not{q}_1\gamma^\beta\right] \\
&= 2\operatorname{Tr}\left[\not{p}'\gamma^\alpha \left(1-\gamma_5\right)\not{q}_1\gamma^\beta\right] \\
&= 2p'^\mu q_1^\nu \operatorname{Tr}\left[\gamma_\mu\gamma^\alpha \left(1-\gamma_5\right)\gamma_\nu\gamma^\beta\right] \\
&= 2p'^\mu q_1^\nu \operatorname{Tr}\left[\gamma_\mu\gamma^\alpha\gamma_\nu\gamma^\beta \left(1-\gamma_5\right)\right],
\end{aligned}$$

and analogously

$$\operatorname{Tr}\left[\not{q}_2\gamma_\alpha \left(1-\gamma_5\right)\left(\not{p} + m_\mu\right)\gamma_\beta \left(1-\gamma_5\right)\right] = 2q_2^\mu p^\nu \operatorname{Tr}\left[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta \left(1-\gamma_5\right)\right]. \tag{3.42}$$

Since

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] = 4\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}\right), \tag{3.43}$$

$$\mathrm{Tr}\left[\gamma_5\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right] = -4i\varepsilon^{\mu\alpha\nu\beta}, \tag{3.44}$$

our amplitude becomes

$$|\bar{\mathcal{M}}|^2 = \frac{G^2 p'_\mu q_{1\nu} x^{\mu\alpha\nu\beta} q_2^\sigma p^\tau x_{\sigma\alpha\tau\beta}}{m_\mu m_e m_{\nu_e} m_{\nu_\mu}}, \tag{3.45}$$

where x is defined as

$$x^{\mu\alpha\nu\beta} \equiv g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha} + i\varepsilon^{\mu\alpha\nu\beta}. \tag{3.46}$$

After computing the contraction,

$$\begin{aligned} x^{\mu\alpha\nu\beta}x_{\sigma\alpha\tau\beta} &= g^\mu_\sigma g^\nu_\tau - g^{\mu\nu}g_{\sigma\tau} + g^\mu_\tau g^\nu_\sigma + i\varepsilon_\sigma{}^\mu{}_\tau{}^\nu - g^{\mu\nu}g_{\sigma\tau} + 4g^{\mu\nu}g_{\sigma\tau} - g^{\mu\nu}g_{\sigma\tau} = \\ &+ g^\mu_\tau g^\nu_\sigma - g^{\mu\nu}g_{\sigma\tau} + g^\mu_\sigma g^\nu_\tau + i\varepsilon_\sigma{}^\nu{}_\tau{}^\mu + i\varepsilon^\mu{}_\sigma{}^\nu{}_\tau + i\varepsilon^\mu{}_\tau{}^\nu{}_\sigma - \varepsilon^{\mu\alpha\nu\beta}\varepsilon_{\sigma\alpha\tau\beta} = \\ &= 2g^\mu_\sigma g^\nu_\tau + 2g^\mu_\tau g^\nu_\sigma - \varepsilon^{\alpha\beta\mu\nu}\varepsilon_{\alpha\beta\sigma\tau} = \\ &= 2g^\mu_\sigma g^\nu_\tau + 2g^\mu_\tau g^\nu_\sigma + 2\left(g^\mu_\sigma g^\nu_\tau - g^\mu_\tau g^\nu_\sigma\right) = \\ &= 4g^\mu_\sigma g^\nu_\tau, \end{aligned}$$

our final result is

$$|\bar{\mathcal{M}}|^2 = \frac{4G^2\left(p\cdot q_1\right)\left(p'\cdot q_2\right)}{m_\mu m_e m_{\nu_e} m_{\nu_\mu}}. \tag{3.47}$$

While studying scattering processes, we used Feynman amplitudes to compute cross sections and the decay rate for decay processes. We will apply the steps highlighted in chapter 1 to derive the decay rate for the muon.

3.2.1 Muon decay rate

We are now ready to go back to the muon decay. $d\Gamma$ reads in our case,

$$d\Gamma = (2\pi)^4\delta^{(4)}\left(p' + q_1 + q_2 - p\right)\frac{m_\mu m_e m_{\nu_e} m_{\nu_\mu}}{E}\frac{1}{(2\pi)^9}\frac{d^3p'}{E'}\frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}|\mathcal{M}|^2, \tag{3.48}$$

and, combining with equation (3.47), we obtain

$$d\Gamma = \frac{4G^2}{(2\pi)^5}\frac{1}{E}\left(p\cdot q_1\right)\left(p'\cdot q_2\right)\delta^{(4)}\left(p' + q_1 + q_2 - p\right)\frac{d^3p'}{E'}\frac{d^3q_1}{E_1}\frac{d^3q_2}{E_2}. \tag{3.49}$$

The next step is the integration over neutrino's momenta, which can be performed borrowing some techniques from loop calculation. The integral we are interested in is

$$I^{\mu\nu}(q) \equiv \int d^3\mathbf{q}_1 d^3\mathbf{q}_2 \frac{q_1^\mu q_2^\nu}{E_1E_2}\delta^{(4)}\left(q_1 + q_2 - q\right), \tag{3.50}$$

where we have defined $q \equiv p - p'$, thus $q^\mu - \equiv (q_1 + q_2)^\mu$. By Lorentz decomposition,

$$I^{\mu\nu}(q) = Ag^{\mu\nu} + Bq^\mu q^\nu. \tag{3.51}$$

If we define

$$I_1 \equiv g_{\mu\nu}I^{\mu\nu} = 4A + Bq^2, \qquad I_2 \equiv q_\mu q_\nu I^{\mu\nu} = Aq^2 + B\left(q^2\right)^2, \tag{3.52}$$

the two coefficients of the decomposition are given by

$$A = \frac{I_1 - \frac{I_2}{q^2}}{3}, \qquad B = -\frac{1}{3q^2}\left(I_1 - \frac{4}{q^2}I_2\right). \tag{3.53}$$

In the $m_{\nu_e} = m_{\nu_\mu} = 0$ approximation,

$$q_1^2 = q_2^2 = 0 \quad \Rightarrow \quad q^2 = (q_1 + q_2)^2 = 2q_1 \cdot q_2. \tag{3.54}$$

Since I_1 and I_2 are scalar quantities, they can be evaluated in any frame of reference. A convenient choice is the center-of-momentum system of the two neutrinos,

$$q_1^\mu \equiv (\omega, \mathbf{q}_1), \quad q_2^\mu \equiv (\omega, -\mathbf{q}_1), q^\mu = (q_0, 0) \Rightarrow q^2 = q_0^2 \Rightarrow q \cdot q_1 = q_0 \cdot \omega = q \cdot q_2 \tag{3.55}$$

where the energy is given by the on shell condition,

$$\omega = E_1 = |\mathbf{q}_1| = E_2 = |\mathbf{q}_2|. \tag{3.56}$$

In this system, we have

$$I_1 = \frac{q^2}{2} \int \frac{d^3\mathbf{q}_1}{\omega} \frac{d^3\mathbf{q}_2}{\omega} \delta^{(3)}(\mathbf{q}_1 + \mathbf{q}_2 - \mathbf{q})\, \delta(2\omega - q_0) = \frac{q^2}{2} \underbrace{\int \frac{d^3 q_1}{\omega^2} \delta(2\omega - q_0)}_{\equiv I(q^2)}. \tag{3.57}$$

Using spherical coordinates,

$$d^3\mathbf{q}_1 = |\mathbf{q}_1|^2 d|\mathbf{q}_1| d\Omega = \omega^2 d\omega d\Omega, \tag{3.58}$$

this integral can be easily computed,

$$I(q^2) = \int \omega^2 d\omega d\Omega \frac{1}{2} \frac{\delta\left(\omega - \frac{q_0}{2}\right)}{\omega^2} = \frac{1}{2} \cdot 4\pi = 2\pi, \tag{3.59}$$

and we finally obtain

$$I_1 = g_{\mu\nu} I^{\mu\nu} = \pi q^2. \tag{3.60}$$

Similarly, one finds

$$\begin{aligned} I_2 = q_\mu q_\nu I^{\mu\nu} &= \int d^3\mathbf{q}_1 d^3\mathbf{q}_2 \,(q \cdot q_1)\,(q \cdot q_2)\, \frac{\delta^{(4)}(q_1 + q_2 - q)}{E_1 E_2} = \\ &= \int d\omega \,(q_0\omega)^2 \frac{1}{2} \delta\left(\omega - \frac{q_0}{2}\right) d\Omega = \frac{\pi}{2}\left(q_0^2\right)^2 = \frac{\pi}{2}\left(q^2\right)^2. \end{aligned} \tag{3.61}$$

Hence, equation (3.51) becomes

$$I^{\mu\nu}(q) = \frac{\pi}{6}\left(g^{\mu\nu} q^2 + 2q^\mu q^\nu\right) \tag{3.62}$$

and the muon decay rate for emission of an electron with momentum in the range $d^3\mathbf{p}$ at $\mathbf{p}'$ is

$$d\Gamma = \frac{2\pi}{3} \frac{G^2}{(2\pi)^5 E} \frac{d^3 p'}{E'} \left[(p \cdot p')\, q^2 + 2\,(p \cdot q)\,(p' \cdot q)\right]. \tag{3.63}$$

In order to integrate over $\mathbf{p}'$ we note that, in the rest frame of the muon

$$p^\mu = (M, 0), \qquad p'^\mu = (E', \mathbf{p}'), \qquad q^\mu = (M - E', -\mathbf{p}') \tag{3.64}$$

with $M \equiv m_\mu$. If we consider $m_e \simeq 0$ then

$$E' = |\mathbf{p}'|, \qquad p \cdot p' = ME', \tag{3.65}$$

$$q^2 = (M - E')^2 - |\mathbf{p}'|^2 = M^2 - 2ME', \qquad p \cdot q = M^2 - ME' \tag{3.66}$$

$$p' \cdot q = ME' - E'^2 - (-\mathbf{p} \cdot \mathbf{p}') = ME' - E'^2 + E'^2 = ME' \tag{3.67}$$

and

$$(p \cdot p')\, q^2 + 2\,(p \cdot q)\,(p' \cdot q) = ME'(3M^2 - 4ME'). \tag{3.68}$$

Using

$$\frac{d^3\mathbf{p}'}{EE'} = \frac{|\mathbf{p}'|E'dE'}{EE'}d\Omega = \frac{E'dE'}{M}d\Omega \tag{3.69}$$

we can finally write

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{3}\frac{G^2}{(2\pi)^5}\int_{E_{e,min}}^{E_{e,max}} E'^2(3M^2 - 4ME')dE'. \tag{3.70}$$

We now need to compute the integration extremes:

- the minimum value of E_e is the mass m_e hence

  $$E_{e,min} = m_e \simeq 0 \tag{3.71}$$

  and correspond to

  $$\underset{\bar{\nu}_e}{\longleftarrow\!\!-\!\!-\!\!-}\; e^- \;\underset{\nu_\mu}{-\!\!-\!\!-\!\!\longrightarrow}\;; \tag{3.72}$$

- we have the maximum in the energy of the electron when

  $$\overset{e^-}{\longleftarrow\!\!-\!\!-\!\!-}\overset{\bar{\nu}_e}{\underset{\nu_\mu}{=\!\!=\!\!=\!\!\Longrightarrow}} \tag{3.73}$$

  with the conditions

  $$\mathbf{p}' = \mathbf{q}_1 + \mathbf{q}_2 \tag{3.74}$$

  hence

  $$|\mathbf{p}'|^2 = E'^2 = |\mathbf{q}_1|^2 + |\mathbf{q}_2|^2 + 2\mathbf{q}_1\mathbf{q}_2 = (E_1 + E_2)^2 \quad \Rightarrow \quad E' = E_1 + E_2 \tag{3.75}$$

  and substituting into $M = E' + E_1 + E_2$ we have

  $$E_{e,max} = \frac{M}{2}. \tag{3.76}$$

Finally

$$\Gamma = \frac{G^2M^5}{192\pi^3} \tag{3.77}$$

and the $\mu-$life time is

$$\tau = \frac{1}{\Gamma} = \frac{192\pi^3}{G^2M^5}. \tag{3.78}$$

Therefore, in the Fermi approximation of the IVB theory we can compute the muon lifetime which can be also measured experimentally

$$\tau_{exp} \simeq 2 \cdot 10^{-6}\ \mathrm{s} \tag{3.79}$$

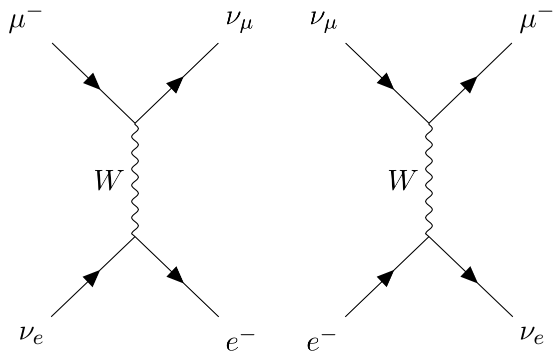
and since $M \equiv M_{\mu,exp} \simeq 100\ \mathrm{MeV/c^2}$ we find

$$G \simeq 1,1 \cdot 10^{-5}\ \mathrm{GeV^{-2}}. \tag{3.80}$$

Remembering that

$$\alpha_W \equiv \frac{G}{4\pi\sqrt{2}}m_W^2 \tag{3.81}$$

we can obtain an estimation of α_W and thus of the weak interaction strength, which is smaller with respect to the QED fine constant.



(a) $\mu^- + \nu_e \to \nu_\mu + e^-$ (b) $\nu_\mu + e^- \to \mu^- + \nu_e$

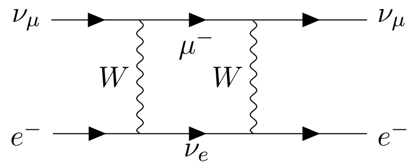
Figure 3.2

3.3 Problems within IVB theory

The calculation of the muon lifetime is a very successful result. Nonetheless, we find some problems with IVB theory already at tree level. Let's consider the reactions in Figure 3.2. Using what we have learnt by muon decay, we can easily compute the squared matrix elements and the cross section of these processes, which are perfectly allowed by IVB theory. Nevertheless, the "unexpected" result is that, experimentally, the elastic reaction

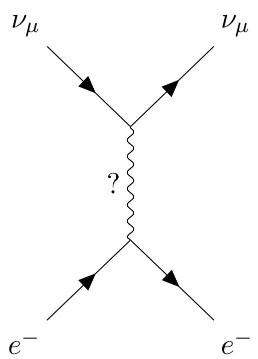
$$\nu_\mu + e^- \to \nu_\mu + e^- \tag{3.82}$$

has the same cross section of the reactions we have just considered in Figure 3.2. The strangeness of this result is clear when one tries to write down the Feynman diagram associated to reaction (3.82). A first attempt could be



. (3.83)

Unfortunately, (3.83) is an higher order diagram (compared to diagrams in Figure 3.2) and, actually, its contribution to the cross section is suppressed by the higher power of g_W. Therefore, we would like to have a process like



(3.84)

but in this case the intermediate vector boson must be electrically neutral. Therefore this experimental result suggests the existence of a third neutral boson.

Another problem concerns degrees of UV divergence. Let's recall that

$$iD_F^{\alpha\beta} = i\frac{-g^{\alpha\beta} + \frac{k^\alpha k^\beta}{m_W^2}}{k^2 - m_W^2 + i\varepsilon} \quad \overset{k\to\infty}{\to} \quad \frac{1}{m_W^2} \sim k^0 \tag{3.85}$$

and so, in this case, the degree of superficial divergence becomes

$$K_{IVB} \equiv \underbrace{4l}_{\sim d^4k} - \underbrace{f_i}_{\sim \frac{1}{k}} - 0 \cdot b_i. \tag{3.86}$$

Like in QED we have the relations (n = number of vertices)

$$l = f_i + b_i - (n-1), \qquad 2n = f_e + 2f_i, \qquad n = b_e + 2b_i \tag{3.87}$$

and substituting into equation (3.86) we obtain

$$K_{IVB} = 4 - \frac{3}{2}f_e - 2b_e + n. \tag{3.88}$$

Comparing the previous expression with the K found for QED, we note an additional term n and the fact that it grows with respect to it. Hence, as we consider a given process in higher orders of perturbation theory, new, progressively more severe divergences arise. To cancel these, we would have to introduce additional renormalization constants at each stage, ending up with infinitely many such constants. In other words, IVB theory is non-renormalizable.

In the view outlined above, we now want to enlarge our theory without losing results obtained by IVB approach.

3.4 Towards a gauge theory of weak interactions

We now attempt to formulate a gauge theory of weak interactions in analogy to QED for electromagnetic interactions. In the first lecture we followed three steps:

1. we introduced a global symmetry group, obtaining conserved currents;
2. we generalized global transformations to local phase (i.e. gauge) transformations;
3. we introduced a gauge field to restore invariance at the Lagrangian level (covariant derivative).

We'll build our theory using successive approximation. To begin with, we shall assume that all leptons are massless (because mass terms would break the local symmetry). The free lepton Lagrangian density is then given by

$$\mathcal{L}_0 = i\left(\bar{\psi}_l \partial\!\!\!/ \psi_l + \bar{\psi}_{\nu_l} \partial\!\!\!/ \psi_{\nu_l}\right) \qquad (m_l = 0 = m_{\nu_l}). \tag{3.89}$$

Previously, we found that the leptonic currents, and consequently the leptonic interaction, involve only the left-handed lepton fields. The left polarization is obtained as

$$\psi^{L,R} = P_{L,R}\psi, \qquad \psi = \psi^L + \psi^R \tag{3.90}$$

with

$$P_{L,R} = \frac{1}{2}(1 \pm \gamma_5), \qquad P_L P_R = 0. \tag{3.91}$$

Therefore we can write[2]

$$\mathcal{L}_0 = i\left(\bar{\psi}_l^L \partial\!\!\!/ \psi_l^L + \bar{\psi}_{\nu_l}^L \partial\!\!\!/ \psi_{\nu_l}^L + \bar{\psi}_l^R \partial\!\!\!/ \psi_l^R + \bar{\psi}_{\nu_l}^R \partial\!\!\!/ \psi_{\nu_l}^R\right). \tag{3.92}$$

For conciseness, in the following we'll put $\nu_l \equiv \nu$.

[2]Note that, in our notation, $\bar{\psi}^L = (\overline{P_L\psi}) = \bar{\psi}P_R$.

We now combine the fields ψ_l^L and ψ_ν^L into a two component field (called a weak isospinor)

$$\Psi_l^L \equiv \begin{pmatrix} \psi_\nu^L \\ \psi_l^L \end{pmatrix} \tag{3.93}$$

and, correspondingly

$$\bar{\Psi}_l^L \equiv \left(\bar{\psi}_\nu^L, \bar{\psi}_l^L \right) \tag{3.94}$$

hence

$$\mathcal{L}_0 = i \left(\bar{\Psi}_l^L \partial\!\!\!/ \Psi_l^L + \bar{\psi}_l^R \partial\!\!\!/ \psi_l^R + \bar{\psi}_\nu^R \partial\!\!\!/ \psi_\nu^R \right). \tag{3.95}$$

Note that we have not introduced two-component right-handed fields. We shall see that the left-right asymmetry of weak interactions can be described in terms of different transformation properties of the left- and right-handed fields.

For the two-component left-handed fields, the term $i\bar{\Psi}_l^L(x)\partial\!\!\!/\Psi_l^L(x)$ in $\mathcal{L}_0$ is left invariant by the $SU(2)$ transformations

$$\Psi_l^L \to \Psi_l^{L'} = U(\alpha)\Psi_l^L, \qquad U(\alpha) \equiv \exp\left(i\alpha_j \frac{\tau_j}{2} \right), \tag{3.96}$$

$$\bar{\Psi}_l^L \to \bar{\Psi}_l^{L'} = \bar{\Psi}_l^L U^\dagger(\alpha), \qquad U^\dagger(\alpha) \equiv \exp\left(-i\alpha_j \frac{\tau_j}{2} \right) \tag{3.97}$$

for any three real numbers $\alpha \equiv (\alpha_1, \alpha_2, \alpha_3)$. We have introduced the usual Pauli spin matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.98}$$

with

$$[\tau_i, \tau_j] = 2i\varepsilon_{ijk}\tau_k. \tag{3.99}$$

We shall now define each right handed lepton field to be a weak isoscalar, i.e. to be invariant under any $SU(2)$ transformation:

$$\psi_l^R(x) \to \psi_l^{R'}(x) = \psi_l^R(x), \qquad \psi_\nu^R(x) \to \psi_\nu^{R'}(x) = \psi_\nu^R(x), \tag{3.100}$$

$$\bar{\psi}_l^R(x) \to \bar{\psi}_l^{R'}(x) = \bar{\psi}_l^R(x), \qquad \bar{\psi}_\nu^R(x) \to \bar{\psi}_\nu^{R'}(x) = \bar{\psi}_\nu^R(x). \tag{3.101}$$

The Lagrangian density $\mathcal{L}_0$ is left invariant under (3.96), (3.97), (3.100) and (3.101). From this invariance the conservation of the leptonic currents $J_\alpha(x)$ and $J_\alpha^\dagger(x)$ follows. For infinitesimal α_j we have

$$\Psi_l^{L'}(x) = \left(\mathbf{1} + i\alpha_j \frac{\tau_j}{2} \right) \Psi_l^L, \qquad \bar{\Psi}_l^{L'} = \bar{\Psi}_l^L \left(\mathbf{1} - i\alpha_j \frac{\tau_j}{2} \right). \tag{3.102}$$

Noether theorem leads to three conserved currents (one for each generator):

$$J_i^\alpha = \frac{1}{2} \bar{\Psi}_l^L \gamma^\alpha \tau_i \Psi_l^L \qquad \text{with } i = 1, 2, 3 \tag{3.103}$$

which are called *weak isospin currents.* The corresponding conserved quantities

$$I_i^W = \int d^3x J_i^0(x) \tag{3.104}$$

are called *weak isospin charges.* Reshuffling J_i^α, we define

$$J^\alpha \equiv 2(J_1^\alpha - iJ_2^\alpha) = 2\left(\bar{\Psi}^L \gamma^\alpha \left(\frac{\tau_1}{2} - i\frac{\tau_2}{2} \right) \Psi^L \right) \tag{3.105}$$

but

$$\frac{\tau_1}{2} - i\frac{\tau_2}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.106}$$

hence

$$J^\alpha = 2\left(\bar{\psi}_\nu^L, \bar{\psi}_l^L\right)\gamma^\alpha \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_\nu^L \\ \psi_l^L \end{pmatrix} = 2\left(\bar{\psi}_\nu^L, \bar{\psi}_l^L\right)\gamma^\alpha \begin{pmatrix} 0 \\ \psi_\nu^L \end{pmatrix} = \tag{3.107}$$

$$= 2\bar{\psi}_l^L \gamma^\alpha \psi_\nu^L = \bar{\psi}_l \gamma^\alpha (1-\gamma_5)\psi_\nu = J_{IVB}^\alpha. \tag{3.108}$$

Analogously, using

$$\frac{\tau_1}{2} + i\frac{\tau_2}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3.109}$$

one finds

$$J^{\alpha\dagger} \equiv 2\left(J_1^\alpha + iJ_2^\alpha\right) = J_{IVB}^{\dagger\alpha}. \tag{3.110}$$

Most remarkably, the above development necessarily led to the conservation of a third current, namely the weak isospin current

$$J_3^\alpha = \frac{1}{2}\bar{\Psi}^L\gamma^\alpha\tau_3\Psi^L = \frac{1}{2}\left[\bar{\psi}_\nu^L\gamma^\alpha\psi_\nu^L - \bar{\psi}_l^L\gamma^\alpha\psi_l^L\right], \tag{3.111}$$

which couples either electrically neutral leptons or electrically charged leptons and can be rewritten as

$$J_3^\alpha = \frac{1}{2}\Big(\bar{\psi}_\nu \underbrace{P_R\gamma^\alpha P_L}_{\gamma^\alpha P_L^2 = \gamma^\alpha P_L} \psi_\nu - \bar{\psi}_l \underbrace{P_R\gamma^\alpha P_L}_{\gamma^\alpha P_L^2 = \gamma^\alpha P_L} \psi_l\Big) = \tag{3.112}$$

$$= \frac{1}{4}\Big(\bar{\psi}_\nu\gamma^\alpha(1-\gamma_5)\psi_\nu - \bar{\psi}_l\gamma^\alpha(1-\gamma_5)\psi_l\Big). \tag{3.113}$$

It is a neutral current, as the electromagnetic one,

$$S^\alpha = -e\bar{\psi}_l\gamma^\alpha\psi_l. \tag{3.114}$$

Combining these two currents together, we can define the weak hypercharge current as

$$\begin{aligned} J_Y^\alpha \equiv \frac{S^\alpha}{e} - J_3^\alpha &= -\bar{\psi}_l\gamma^\alpha\psi_l - \frac{1}{2}\bar{\psi}_\nu^L\gamma^\alpha\psi_\nu^L + \frac{1}{2}\bar{\psi}_l^L\gamma^\alpha\psi_l^L = \\ &= -\frac{1}{2}\bar{\psi}_\nu^L\gamma^\alpha\psi_\nu^L - \frac{1}{2}\bar{\psi}_l^L\gamma^\alpha\psi_l^L - \bar{\psi}_l^R\gamma^\alpha\psi_l^R = \qquad (3.115) \\ &= -\frac{1}{2}\bar{\Psi}^L\gamma^\alpha\Psi^L - \bar{\psi}_l^R\gamma^\alpha\psi_l^R, \qquad (3.116) \end{aligned}$$

and the corresponding charge, called weak hypercharge,

$$Y \equiv \int d^3x J_Y^0, \tag{3.117}$$

which is related to the electric charge Q and the weak isocharge I_3^W by

$$Y = \frac{Q}{e} - I_3^W. \tag{3.118}$$

The conservation of Q and I_3^W implies conservation of Y. Its values for the different leptons are summarized in Table 3.1. The conservation of the weak hypercharge, which we deduced from equation (3.118), directly follows from the invariance of the Lagrangian (3.95) under the $U(1)_Y$ global transformation

$$\begin{cases} \Psi'^L = e^{iY\beta}\Psi^L, & Y = -\frac{1}{2}, \\ \psi_l'^R = e^{iY\beta}\psi_l^R, & Y = -1, \\ \psi_\nu'^R = e^{iY\beta}\psi_\nu^R, & Y = 0, \end{cases} \tag{3.119}$$

where β is an arbitrary real number. In conclusion, the Lagrangian (3.95) is globally invariant under $SU(2)_L \otimes U(1)_Y$.

	$\frac{Q}{e}$	I_3^W	Y
ψ_ν^L	0	$\frac{1}{2}$	$-\frac{1}{2}$
ψ_l^L	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
ψ_ν^R	0	0	0
ψ_l^R	-1	0	-1

Table 3.1

3.5 Gauge invariant EW theory

The next step is to localize the global $SU(2)_L \times U(1)_Y$ simmetry.

3.5.1 Local $SU(2)$

We start considering the local $SU(2)_L$ transformation,

$$\begin{cases} \Psi^L & \longrightarrow & \Psi'^L & = & e^{ig\frac{\tau_j}{2}\omega_j(x)}\Psi^L, \\ \psi'^R_{l,\nu} & \longrightarrow & \psi^R_{l,\nu} & = & \psi^R_{l,\nu}, \end{cases} \tag{3.120}$$

where $\omega_j(x)$ $(j = 1, 2, 3)$ are three arbitrary real functions and g is a real constant, which we will later interpeter as a coupling constant. The Lagrangian (3.95) transforms as

$$\mathcal{L}_0 \longrightarrow \mathcal{L}'_0 = \mathcal{L}_0 + \delta\mathcal{L}_0, \tag{3.121}$$

where the second term,

$$\delta\mathcal{L}_0 = -\frac{1}{2}g\bar{\Psi}^L\tau_j\left(\partial\!\!\!/\omega_j\right)\Psi^L, \tag{3.122}$$

explicitly breaks the $SU(2)_L$ symmetry. In order to restore the invariance, we can introduce a covariant derivative, defined as

$$\partial_\mu\Psi^L \longrightarrow D_\mu\Psi^L = \left[\partial_\mu + ig\frac{\tau_j}{2}W_{j\mu}\right]\Psi^L, \tag{3.123}$$

where W_j $(j = 1, 2, 3)$ are massless gauge bosons. Hence, differently from the QED, we have *three* (not one) gauge fields associated to *three* different generators. Nevertheless, this is not the unique difference, since $SU(2)$ is a non abelian group, whereas $U(1)$ is abelian.

W boson gauge transformation (Yang-Mills theory for non abelian gauge groups)

Let $\mathcal{G}$ be the $SU(N)$ Lie group with generators

$$t_a \equiv \frac{\tau_a}{2}, \qquad \left[t_a, t_b\right] = i\varepsilon_{abc}t_c. \tag{3.124}$$

Given $\omega(x) \equiv \omega_a t_a$, by exponentiation we obtain an element of the group,

$$U(x) \equiv e^{igt_a\omega_a(x)} \equiv e^{ig\omega(x)}. \tag{3.125}$$

We suppose that matter fields transform as

$$\begin{cases} \psi(x) & \longrightarrow & \psi'(x) & = & U(x)\psi(x), \\ \partial_\mu\psi(x) & \longrightarrow & (\partial_\mu\psi(x))' & = & U(x)\partial_\mu\psi(x) + (\partial_\mu U)\,\psi(x), \end{cases} \tag{3.126}$$

where the last term in second line is the gauge invariance breaking term. We can introduce a covariant derivative,

$$D_\mu\psi(x) \equiv (\partial_\mu + igW_\mu)\,\psi(x) \longrightarrow (D_\mu\psi(x))' \equiv U(x)D_\mu\psi, \tag{3.127}$$

where $W_\mu \equiv t_a W_{a\mu}$, so that the derivative of the field transforms like the field itself. Since

$$\begin{aligned} ig\left(W_\mu\psi\right)' &= \left(D_\mu\psi\right)' - \partial_\mu\psi' = U\left(D_\mu\psi\right) - \left(\partial_\mu U\right)\psi - U\partial_\mu\psi \\ &= ig\left[UW_\mu U^{-1} + \frac{i}{g}\left(\partial_\mu U\right)U^{-1}\right]\psi', \end{aligned} \tag{3.128}$$

W_μ transforms as

$$W_\mu \longrightarrow W'_\mu \equiv UW_\mu U^{-1} + \frac{i}{g}\left(\partial_\mu U\right)U^{-1} = U\left(W_\mu + \frac{i}{g}U^{-1}\partial_\mu U\right)U^{-1}. \tag{3.129}$$

Since

$$U \simeq \left(\mathbf{1} + ig\omega_b t_b\right), \qquad U^{-1} \simeq \left(\mathbf{1} - ig\omega_b t_b\right), \tag{3.130}$$

$$UW_\mu U^{-1} = \left(\mathbf{1} + ig\omega_b t_b\right)W_\mu\left(\mathbf{1} - ig\omega_b t_b\right) \simeq W_\mu + ig\omega_b\Big[t_b, W_\mu\Big] = W^a_\mu t_a - g\omega_b W^c_\mu \varepsilon_{bca} t_a, \tag{3.131}$$

$$\left(\partial_\mu U\right)U^{-1} \simeq \left(\partial_\mu\left(\mathbf{1} + ig\omega_b t_b\right)\right)\left(\mathbf{1} - ig\omega_b t_b\right) \simeq ig\left(\partial_\mu\omega_a\right)t_a, \tag{3.132}$$

the infinitesimal form of equation (3.129) is

$$W'^a_\mu = \underbrace{W^a_\mu - \partial_\mu\omega_a}_{\text{abelian} \sim QED} - \overbrace{g\omega_b W^c_\mu\varepsilon_{bca}}^{\text{non abelian}} \equiv W^a_\mu + \delta W^a_\mu, \tag{3.133}$$

where the infinitesimal variation of the gauge field is the sum of an abelian and a non abelian term. We can rewrite the gauge field trasformation with the use of the covariant derivative, namely:

$$W'_\mu = W_\mu + D_\mu\omega \tag{3.134}$$

where ω is a generic function charged under the $SU(N)$ gruop, and the action of the covariant derivative on ω is :

$$D_\mu\omega = \partial_\mu\omega + ig\left[W_\mu, \omega\right] \tag{3.135}$$

note that the commutator is non trivial for the non abelian properties of the $SU(N)$ gruop.

3.5.2 Local $U(1)$

In order to localize $U(1)_Y$, we proceed as in QED. The matter field transformation is

$$\psi(x) \longrightarrow \psi'(x) = e^{ig'Yf(x)}\psi(x). \tag{3.136}$$

We introduce a covariant derivative,

$$\partial_\mu\psi \longrightarrow D_\mu\psi \equiv \left[\partial_\mu + ig'YB_\mu\right]\psi, \tag{3.137}$$

where B_μ is a massless gauge boson transforming as

$$B^\mu \longrightarrow B^\mu - \partial^\mu f(x). \tag{3.138}$$

Summarizing, we have constructed the Lagrangian

$$\mathcal{L} \equiv i\left(\bar{\Psi}^L \not{D}\Psi^L + \bar{\psi}^R_l \not{D}\psi^R_l + \bar{\psi}^R_\nu \not{D}\psi^R_\nu\right), \tag{3.139}$$

where covariant derivatives are defined as

$$\left\{\begin{array}{lcl} D^\mu\Psi^L &=& \left(\partial^\mu + ig\frac{\tau_i}{2}W^\mu_i - ig'\frac{1}{2}B^\mu\right)\Psi^L, \\ D^\mu\psi^R_l &=& \left(\partial^\mu - ig'B^\mu\right)\psi^R_l, \\ D^\mu\psi^R_\nu &=& \partial^\mu\psi^R_\nu, \end{array}\right. \tag{3.140}$$

which is invariant under a local $SU(2)_L \times U(1)_Y$ symmetry group. Our aim is now to extract from this Lagrangian a description of both the electromagnetic and the weak interactions of leptons.

3.5.3 Gauge bosons mixing

We can write the Lagrangian (3.139) in the form

$$\mathcal{L} \equiv \mathcal{L}_0 + \mathcal{L}_I, \tag{3.141}$$

where

$$\mathcal{L}_I \equiv -g J_i^\mu(x) W_{i\mu}(x) - g' J_Y^\mu(x) B_\mu(x), \tag{3.142}$$

describes the interaction of the weak isospin currents J_i^μ ($i = 1, \ldots, 3$) and the weak hypercharge current J_Y^μ with the gauge fields $W_{i\mu}$ and B_μ. If we introduce the charged vector bosons

$$W_\mu \equiv \frac{1}{\sqrt{2}} \left[W_{1\mu} - iW_{2\mu}\right], \qquad W_\mu^\dagger \equiv \frac{1}{\sqrt{2}} \left[W_{1\mu} + iW_{2\mu}\right], \tag{3.143}$$

we can write the first two terms of $\mathcal{L}_I$ in the form

$$\mathcal{L}_{I12} \equiv -g \sum_{i=1}^{2} J_i^\mu W_{i\mu} = -\frac{g}{2\sqrt{2}} \left[J^{\dagger\mu} W_\mu + J^\mu W_\mu^\dagger\right], \tag{3.144}$$

so that IVB theory is recovered. We are left with the two gauge fields $W_{3\mu}$ and B_μ, which we write as linear combinations of two gauge fields Z_μ and A_ν, defined by

$$\begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}, \qquad \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} W_{3\mu} \\ B_\mu \end{pmatrix}, \tag{3.145}$$

so that the remaining two terms of $\mathcal{L}_I$ can be rewritten as

$$\begin{aligned} \mathcal{L}_{I34} &\equiv -g J_3^\mu W_{3\mu} - g' J_Y^\mu B_\mu \\ &= -\frac{g'}{e} S^\mu \left(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu\right) + \\ &\quad - J_3^\mu \left[g\left(\cos\theta_W Z_\mu + \sin\theta_W A_\mu\right) - g'\left(-\sin\theta_W Z_\mu + \cos\theta_W A_\mu\right)\right]. \end{aligned} \tag{3.146}$$

The requirement that A_μ couples only to the QED current S^μ leads to the condition

$$e \equiv g' \cos\theta_W \equiv g \sin\theta_W. \tag{3.147}$$

Inserting (3.147) in (3.142), we obtain

$$\mathcal{L}_I = -S^\mu A_\mu - \frac{g}{2\sqrt{2}} \left[J^{\dagger\mu} W_\mu + J^\mu W_\mu^\dagger\right] - \frac{g}{\cos\theta_W} \left[J_3^\mu - \sin^2\theta_W \frac{S^\mu}{e}\right] Z_\mu. \tag{3.148}$$

The angle θ_W is called *Weinberg* or *weak mixing* angle and has an experimental value of

$$\sin^2\theta_{W,exp} \simeq 0.22. \tag{3.149}$$

The electroweak coupling constant,

$$g_W \equiv \frac{g}{2\sqrt{2}}, \tag{3.150}$$

is related to the Fermi constant by

$$\left(\frac{g_W}{m_W}\right)^2 = \frac{G_F}{\sqrt{2}}. \tag{3.151}$$

The third term in equation (3.148) represents a neutral current coupled to a real vector field $Z_\mu(x)$ and it can be rewritten in the form

$$J_3^\mu - \sin^2\theta_W \frac{S^\mu}{e} = \frac{1}{4}\bar{\psi}_\nu \gamma^\mu (1 - \gamma_5)\psi_\nu - \frac{1}{4}\bar{\psi}_l \gamma^\mu \left((1 - 4\sin^2\theta_W) - \gamma_5\right)\psi_l \tag{3.152}$$

which is sometimes used in literature.

3.5.4 Properties of gauge bosons

To complete the construction of our theory we now need to introduce in the Lagrangian kinematic terms for gauge bosons. These terms must also be $SU(2) \times U(1)$ gauge invariant. In analogy to the electromagnetic case, a $U(1)$ gauge invariant Lagrangian density for the $B^\mu(x)$ field is given by

$$\mathcal{L}_B = -\frac{1}{4} B^{\mu\nu} B_{\mu\nu} \tag{3.153}$$

with

$$B^{\mu\nu} \equiv \partial^\nu B^\mu - \partial^\mu B^\nu. \tag{3.154}$$

In the case of $SU(2)_W$, instead, we have a non abelian theory and therefore we need to be more careful. It can be shown that the previous solution doesn't work in this case and that it's necessary to introduce additional terms[3]. Nevertheless, we'll follow a more general construction (a general construction of the field strength tensor).

The starting point is the covariant derivative. Let us consider the commutator

$$\begin{aligned} [D_\mu, D_\nu] = [\partial_\mu + igW_\mu, \partial_\nu + igW_\nu] &= -igF_{\mu\nu} - g^2[W_\mu, W_\nu] &(3.155)\\ &= -ig\left(F_{\mu\nu} - ig[W_\mu, W_\nu]\right). &(3.156)\end{aligned}$$

We prove the second equality explicitly:

$$\begin{aligned} D_\mu(D_\nu\psi) &= (\partial_\mu + igW_\mu)(\partial_\nu\psi + igW_\nu\psi) = &(3.157)\\ &= \partial_\mu\partial_\nu\psi + ig\partial_\mu(W_\nu\psi) + igW_\mu\partial_\nu\psi - g^2 W_\mu W_\nu\psi = &(3.158)\\ &= \partial_\mu\partial_\nu\psi + ig(\partial_\mu W_\nu)\psi + igW_\nu(\partial_\mu\psi) + igW_\mu\partial_\nu\psi - g^2 W_\mu W_\nu\psi. &(3.159)\end{aligned}$$

and similarly

$$D_\nu(D_\mu\psi) = \partial_\nu\partial_\mu\psi + ig(\partial_\nu W_\mu)\psi + igW_\mu(\partial_\nu\psi) + igW_\nu\partial_\mu\psi - g^2 W_\nu W_\mu\psi \tag{3.160}$$

hence

$$\begin{aligned} [D_\mu, D_\nu]\psi = D_\mu(D_\nu\psi) - D_\nu(D_\mu\psi) &= 0 + ig(\partial_\mu W_\nu - \partial_\nu W_\mu)\psi - g^2[W_\mu, W_\nu]\psi = &(3.161)\\ &= (-igF_{\mu\nu} - g^2[W_\mu, W_\nu])\psi. &(3.162)\end{aligned}$$

Since $W_\mu = W_{\mu j} t_j$, we can rewrite

$$-ig[W_\mu, W_\nu] = -igi\varepsilon_{ijk} W_{\mu j} W_{\nu k} t_i = g\varepsilon_{ijk} W_{\mu j} W_{\nu k} t_i \tag{3.163}$$

and then

$$\frac{i}{g}[D_\mu, D_\nu] = F_{\mu\nu} + g\varepsilon_{ijk} W_{\mu j} W_{\nu k} t_i \equiv G_{\mu\nu}. \tag{3.164}$$

From the previous equation we can define the *field strength* $G_{\mu\nu}$ which can be written as

$$G_{\mu\nu} = F_{\mu\nu} - ig[W_\mu, W_\nu]. \tag{3.165}$$

The fact that the group is non abelian implies the presence of the (non vanishing) commutator.

[3] See F. Mandl, G. Shaw "Quantum Field Theory", 17.4

Properties of the field strength tensor Since a covariant derivative satisfies the Jacobi identity

$$[D_\mu,[D_\nu,D_\rho]]+[D_\nu,[D_\rho,D_\mu]]+[D_\rho,[D_\mu,D_\nu]]=0, \tag{3.166}$$

we obtain the Bianchi identity

$$D_\mu G_{\nu\rho}+D_\nu G_{\rho\mu}+D_\rho G_{\mu\nu}=0. \tag{3.167}$$

Previous equation can be easily found from

$$[D_\mu,[D_\nu,D_\rho]]\psi = D_\mu\left([D_\nu,D_\rho]\psi\right)-[D_\nu,D_\rho]D_\mu\psi = \tag{3.168}$$

$$= \left(D_\mu[D_\nu,D_\rho]\right)\psi+[D_\nu,D_\rho](D_\mu\psi)-[D_\nu,D_\rho](D_\mu\psi)= \tag{3.169}$$

$$= -igD_\mu G_{\nu\rho}\psi. \tag{3.170}$$

Moreover

$$D_\mu G_{\nu\rho}=\partial_\mu G_{\nu\rho}+ig[W_\mu,G_{\nu\rho}] \tag{3.171}$$

which we can expand in components,

$$D_\mu G^i_{\nu\rho}=\partial_\mu G^i_{\nu\rho}-g\varepsilon_{ijk}W^j_\mu G^k_{\nu\rho}. \tag{3.172}$$

It's interesting to see differences with respect to the action of D_μ on the matter field

$$D_\mu\psi=(\partial_\mu+igW_\mu)\psi; \tag{3.173}$$

whereas ψ transform under the fundamental representation, $G_{\mu\nu}$ transforms under the adjoint representation. We now show that $G_{\mu\nu}$ has the right properties.

$G_{\mu\nu}$ under $SU(2)$ transformations Matter fields generically transform as

$$\psi'=U\psi, \tag{3.174}$$

$$(D_\mu\psi)'=UD_\mu\psi=UD_\mu U^{-1}\psi'. \tag{3.175}$$

Therefore, if we call

$$(D_\mu\psi)'\equiv D'_\mu\psi' \tag{3.176}$$

we find

$$D'_\mu=UD_\mu U^{-1}. \tag{3.177}$$

Knowing how the covariant derivative transforms, we can see how $G_{\mu\nu}$ transforms:

$$G_{\mu\nu}=\frac{i}{g}(D_\mu D_\nu-D_\nu D_\mu) \qquad\to\qquad G'_{\mu\nu}=\frac{i}{g}(D'_\mu D'_\nu-D'_\nu D'_\mu)=UG_{\mu\nu}U^{-1} \tag{3.178}$$

For the infinitesimal transformation, we have ($\omega\equiv\omega_i t^i$)

$$UG_{\mu\nu}U^{-1}=(1+ig\omega)G_{\mu\nu}(1-ig\omega)=G_{\mu\nu}+\underbrace{ig[\omega,G_{\mu\nu}]}_{\equiv\delta G_{\mu\nu}} \tag{3.179}$$

$$G_{\mu\nu}\qquad\to\qquad G'_{\mu\nu}=G_{\mu\nu}+\delta G_{\mu\nu}. \tag{3.180}$$

In order to build a Lagrangian which is invariant under $SU(2)$ we consider

$$\mathcal{L}_{Gauge}=-\frac{1}{4}G_{i\mu\nu}G_i^{\mu\nu}\equiv-\frac{1}{4}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) \tag{3.181}$$

where the trace is over the $SU(2)-$algebra index; indeed we have

$$\mathcal{L}_G\to\mathcal{L}'_G=-\frac{1}{4}\mathrm{Tr}\left(UG_{\mu\nu}U^{-1}UG^{\mu\nu}U^{-1}\right)=-\frac{1}{4}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}U^{-1}U\right)=\mathcal{L}_G. \tag{3.182}$$

We note that:

- for an *abelian* theory we have that

$$G_{\mu\nu} \ \rightarrow \ G'_{\mu\nu} = G_{\mu\nu} \ \Rightarrow \ \delta G_{\mu\nu} = 0 \tag{3.183}$$

  and

$$\partial^\nu F_{\mu\nu} = J_\mu \ \Rightarrow \ \partial^\mu J_\mu = 0. \tag{3.184}$$

  where we have used the contraction between symmetric and antisymmetric indices;

- for a *non abelian* theory (we don't prove the first equality)

$$D^\mu G_{\mu\nu} = J_\nu \ \Rightarrow \ D^\mu J_\mu = 0 \ \Leftrightarrow \ \partial^\mu J_\mu = -ig[W^\mu, J_\mu] \tag{3.185}$$

  so the current is not conserved. In this case $D^\mu J_\mu = 0$ derives from

$$[D_\mu, [D_\nu, D_\rho]] = -igD_\mu G_{\nu\rho} \ \Leftrightarrow \ [D_\mu, G_{\nu\rho}] = D_\mu G_{\nu\rho} \tag{3.186}$$

  from a formal point of view this equation tells us that the covariant derivative acting on "something" which transforms under the adjoint representation is obtained by the commutator. Therefore

$$D_\mu J^\mu = D_\mu D_\nu G^{\mu\nu} = \frac{1}{2}[D_\mu, D_\nu]G^{\mu\nu} \equiv \frac{1}{2}\left[[D_\mu, D_\nu], G^{\mu\nu}\right] = 0. \tag{3.187}$$

3.5.5 $SU(2) \times U(1)$ gauge-invariant Lagrangian density for the gauge bosons

Combining what we have obtained so far, we find the complete $SU(2) \times U(1)$ gauge-invariant Lagrangian density for the gauge bosons

$$\mathcal{L}^B \equiv -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}G^i_{\mu\nu}G^{\mu\nu}_i \tag{3.188}$$

with $G^{\mu\nu}_i = F^{\mu\nu}_i + g\varepsilon_{ijk}W^\mu_j W^\nu_k$. Since

$$G^{\mu\nu}_i G^i_{\mu\nu} = (F^{\mu\nu}_i + g\varepsilon_{ijk}W^\mu_j W^\nu_k)(F^i_{\mu\nu} + g\varepsilon_{imn}W^\mu_m W^\nu_n) \tag{3.189}$$

$$= F^{\mu\nu}_i F^i_{\mu\nu} + g\varepsilon_{imn}F^{\mu\nu}_i W^\mu_m W^\nu_n + g\varepsilon_{ijk}F^i_{\mu\nu}W^\mu_j W^\nu_k + g^2\varepsilon_{ijk}\varepsilon_{imn}W^\mu_j W^\nu_k W^\mu_m W^\nu_n = \tag{3.190}$$

$$= F^{\mu\nu}_i F^i_{\mu\nu} + 2g\varepsilon_{ijk}F^{\mu\nu}_i W^j_\mu W^k_\nu + g^2\varepsilon_{ijk}\varepsilon_{imn}W^\mu_j W^\nu_k W^\mu_m W^\nu_n \tag{3.191}$$

and

$$\varepsilon_{ijk}F^{\mu\nu}_i W^j_\mu W^k_\nu = \varepsilon_{ijk}(\partial^\nu W^\mu_i)W^j_\mu W^k_\nu - \varepsilon_{ijk}(\partial^\mu W^\nu_i)W^j_\mu W^k_\nu = \tag{3.192}$$

$$= \varepsilon_{ijk}W^k \cdot \partial W_i \cdot W_j - \varepsilon_{ijk}W^j \cdot \partial W_i \cdot W^k = (\text{renaming } k \leftrightarrow j) \tag{3.193}$$

$$= -2\varepsilon_{ijk}W^j \cdot \partial W_i \cdot W^k \stackrel{i \leftrightarrow k}{=} -2\varepsilon_{kji}W^j \cdot \partial W_k \cdot W^i \stackrel{i \leftrightarrow j}{=} \tag{3.194}$$

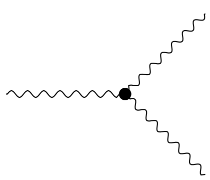
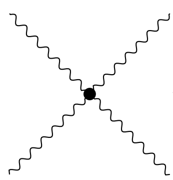
$$= -2\varepsilon_{kij}W^i \cdot \partial W_k \cdot W^j = -2\varepsilon_{ijk}W_{i\mu}\partial^\mu W^\nu_k W_{j\nu} = \tag{3.195}$$

$$= -2\varepsilon_{ijk}W_{i\mu}W_{j\nu}\partial^\mu W^\nu_k. \tag{3.196}$$

Therefore

$$\mathcal{L}^B = \underbrace{-\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}F^{\mu\nu}_i F^i_{\mu\nu}}_{\text{Free field term}} \overbrace{+g\varepsilon_{ijk}W_{i\mu}W_{j\nu}\partial^\mu W^\nu_k}^{\text{Triple self int.}} \underbrace{-\frac{1}{4}g^2\varepsilon_{ijk}\varepsilon_{ilm}W^\mu_j W^\mu_k W_{l\mu}W_{m\nu}}_{\text{Quartic self int.}} \tag{3.197}$$

and we would expect vertices like

 and  . (3.198)

3.6 Introduction to the Higgs mechanism

The Higgs mechanism is based on the assumption that it exists a spin-zero field, the *Higgs* field, which is a doublet under $SU(2)$, has a non-zero $U(1)$ hypercharge and is a singlet in color space. Gauge bosons and fermions can interact with this field, thus acquiring their mass.

A key role is played by the spontaneous symmetry breaking of the $SU(2)_W \otimes U(1)_Y$ symmetry of the Lagrangian (3.148) by the ground state choice. Before illustrating its application to weak interactions, we shall treat the topic of symmetry breaking from a general point of view.

3.6.1 Spontaneous symmetry breaking

We begin with some examples.

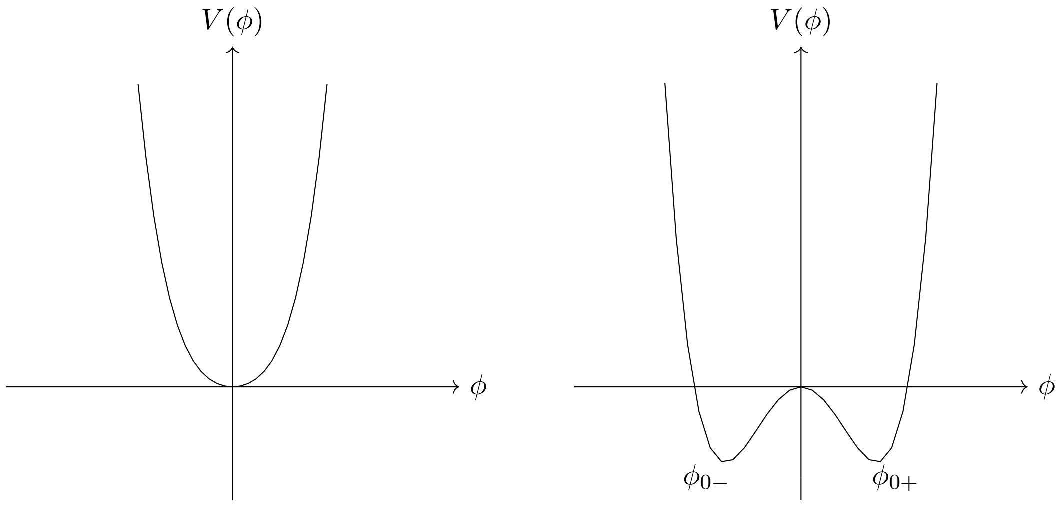
Discrete $\mathbb{Z}_2$ symmetry We consider the following Lagrangian,

$$\mathcal{L} = T - V(\phi) = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \left(\frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4\right), \tag{3.199}$$

where ϕ is a real scalar field, $\lambda > 0$ is a self-coupling constant and μ is not regarded as a mass, but as a parameter only. This Lagrangian is invariant under the transformation

$$\phi \longrightarrow -\phi. \tag{3.200}$$

In order to find the spectrum of the theory, we have to identify the ground state $\phi(x) = \phi_0$, with ϕ_0 chosen to minimize the potential V, and then make an expansion of the field near this minimum to find out its excitations. Depending on the sign of μ^2, there are two possible cases:



(a) (b)

1. if $\mu^2 > 0$, there is only a minimum at $\phi_0 = 0$ and μ represents the mass of the field;
2. if $\mu^2 < 0$, imposing the stationarity condition,

$$\frac{\partial V}{\partial \phi} = \phi\left(\mu^2 + \lambda\phi^2\right) = 0, \tag{3.201}$$

we find two minima at

$$|\phi_0| = \sqrt{-\frac{\mu^2}{\lambda}} \equiv v, \tag{3.202}$$

where the constant v is called the *vacuum expectation value* of ϕ.

The presence of degenerate vacua makes the second case a very interesting situation, as we are free to choose which vacuum is the physical one. We choose the vacuum $\phi_0 = v$ and expand the field around it,

$$\phi(x) = v + h(x), \tag{3.203}$$

where $h(x)$ represents excitations. Plugging this expansion in the Lagrangian (3.199), we find (remember that $-\mu^2 = \lambda v^2$)

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) \underbrace{+\mu^2 h^2}_{\text{mass term}} \overbrace{-\lambda v h^3 - \frac{1}{4}\lambda h^4}^{\text{self-interaction}} - \frac{\lambda}{4} v^4, \tag{3.204}$$

which describes a scalar field of mass

$$m_h^2 \equiv -2\mu^2, \tag{3.205}$$

with cubic and quartic self-interactions. This mass can be interpreted as the energy that is spent in excitations around the vacuum. Finally, the reflection symmetry (3.200) is no longer apparent.

Summarizing, the points we have to keep in mind are the following:

- the theory written in terms of ϕ or h must be equivalent;
- the scalar particle with $\mu^2 < 0$ is a real scalar and its mass is obtained by its self-interactions (because $v \neq 0$);
- the $\mathbb{Z}_2$ symmetry disappears from the new Lagrangian. We say that the symmetry is broken by the choice of the vacuum.

More interesting theories arise when the broken symmetry is continuous, rather than discrete. The generalization of what we have seen for the discrete $\mathbb{Z}_2$ symmetry is the Goldstone mechanism for global abelian and non abelian symmetry groups. Localizing these symmetries, the Higgs mechanism is obtained.

Global $U(1)$ symmetry We consider the Lagrangian

$$\mathcal{L} = \left(\partial_\mu \phi\right)^* \left(\partial^\mu \phi\right) - \mu^2 \phi^* \phi - \lambda \left(\phi^* \phi\right)^2, \tag{3.206}$$

where ϕ is a complex scalar field. The $U(1)$ global phase transformation

$$\phi \longrightarrow \phi' = e^{i\chi}\phi \tag{3.207}$$

is a symmetry of the theory. Decomposing the complex field in its real and imaginary parts,

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \tag{3.208}$$

where ϕ_1 and ϕ_2 are real fields, the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1\right)\left(\partial^\mu \phi_1\right) + \frac{1}{2}\left(\partial_\mu \phi_2\right)\left(\partial^\mu \phi_2\right) - \frac{1}{2}\mu^2\left(\phi_1^2 + \phi_2^2\right) - \frac{1}{4}\lambda\left(\phi_1^2 + \phi_2^2\right)^2. \tag{3.209}$$

According to the sign of μ^2, there are two possible cases (Figure 3.3):

1. if $\mu^2 > 0$, the minimum of the potential is $\phi_0 = 0$ and μ is the mass of ϕ;

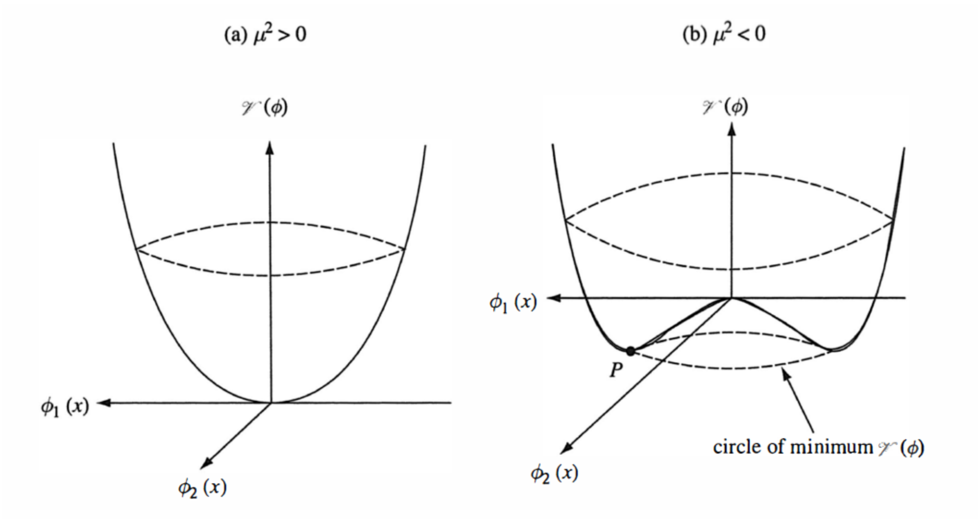


Figure 3.3

2. if $\mu^2 < 0$, we have a "Mexican hat" potential whose minima lie along a circle of radius,

$$\frac{\partial V}{\partial \phi_i} = \phi_i \left[\mu^2 + \lambda \left(\phi_1^2 + \phi_2^2\right)\right] = 0 \quad \Rightarrow \quad |\phi_0| \equiv \sqrt{\frac{\phi_{0,1}^2 + \phi_{0,2}^2}{2}} = \sqrt{-\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}, \tag{3.210}$$

and form a set of infinitely degenerate vacua related to each other by rotation. Excitations are realized by performing perturbations around one of these vacua.

If we choose the vacuum $\phi_{0,1} = v$, $\phi_{0,2} = 0$, we have the expansion

$$\phi(x) = \frac{v + h(x) + i\rho(x)}{\sqrt{2}}, \tag{3.211}$$

where h and ρ parametrize excitations along the ϕ_1 and the ϕ_2 axis respectively. Plugging in the Lagrangian, we find

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \rho)(\partial^\mu \rho) + \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \overbrace{\mu^2 h^2}^{\text{mass term}} \underbrace{-\lambda v \left(h\rho^2 + h^3\right) - \frac{\lambda}{2} h^2 \rho^2 - \frac{\lambda}{4} h^4 - \frac{\lambda}{4} \rho^4}_{\text{interactions}}, \tag{3.212}$$

where we have a mass term for h, with mass defined by

$$m_h^2 = -2\mu^2, \tag{3.213}$$

while there is no mass term for ρ, so

$$m_\rho = 0. \tag{3.214}$$

As a result of spontaneous symmetry breaking, what would otherwise be two massive real fields (ϕ_1 and ϕ_2), become one massive and one massless field (h and ρ). This result may be interpreted as a consequence of vacuum degeneracy: it clearly costs energy to ascend a slope of the potential, whereas displacements along the circular valley of minima are costless excitations. The ρ particle is called a *Goldstone boson*. The existence of such a massless, spin-zero boson is a general phenomenon, occurring whenever a continuous global symmetry is spontaneously broken, as stated by the Goldstone theorem.

Global $SO(3)$ symmetry We now turn to a non abelian theory, described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_i\right)\left(\partial^\mu \phi_i\right) - \frac{1}{2}\mu^2 \phi_i \phi_i - \frac{1}{4}\lambda \left(\phi_i \phi_i\right)^2, \tag{3.215}$$

where summation over $i = 1, 2, 3$ is understood. This Lagrangian is invariant under $SO(3)$ rotations,

$$\phi_i \longrightarrow \phi_i' = \left(e^{-iT_a \omega_a}\right)_{ij} \phi_j, \tag{3.216}$$

where T_a are the generators of the group,

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.217}$$

with

$$[T_i, T_j] = i\varepsilon_{ijk} T_k. \tag{3.218}$$

We look at the minima of the potential, distinguishing between the two cases:

1. if $\mu^2 > 0$, the minimum occurs at $\phi_i = 0$;
2. if $\mu^2 < 0$, minima lie on the sphere of radius

$$\frac{\partial V}{\partial \phi_i} = \phi_i \left[\mu^2 + \lambda\left(\phi_1^2 + \phi_2^2 + \phi_3^2\right)\right] = 0 \quad \Rightarrow \quad |\phi_0| \equiv \sqrt{\phi_{0,1}^2 + \phi_{0,2}^2 + \phi_{0,3}^2} = \sqrt{-\frac{\mu^2}{\lambda}} \equiv v. \tag{3.219}$$

The second case is again characterized by degenerate vacua, among which we are free to choose which one is the physical one. We choose

$$\phi_0 = v\hat{\mathbf{e}}_\mathbf{3} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{3.220}$$

which is a vector pointing in the 3 direction. We clearly don't expect it to be invariant under the full $SO(3)$, which is the symmetry group of the potential, but only under the subgroup of rotations about the 3 axis. As a general rule, we can check invariance by applying generators: if a generator kills the field it's acting on, it's said to be conserved, contrariwise it's said to be broken. In this case, we have

$$\begin{array}{lllc} T_i \phi_0 \neq 0 & \text{for } i = 1, 2 & \Rightarrow & \text{BROKEN} \quad ; \\ T_3 \phi_0 = 0 & & \Rightarrow & \text{CONSERVED} \quad . \end{array} \tag{3.221}$$

Thus, our vacuum is only invariant under the subgroup generated by T_3. How many Goldstone bosons are there? Expanding around the vacuum,

$$\phi_1(x) = \rho_1(x), \quad \phi_2(x) = \rho_2(x), \quad \phi_3(x) = v + h(x), \tag{3.222}$$

the potential becomes

$$V = \frac{1}{2}\mu^2 \left[\rho_1^2 + \rho_2^2 + (v + h)^2\right] + \frac{1}{4}\lambda \left[\rho_1^2 + \rho_2^2 + (v + h)^2\right]^2 = -\mu^2 h^2 + \dots . \tag{3.223}$$

Only the field h has a quadratic term, and therefore a mass

$$m_h^2 \equiv -2\mu^2, \tag{3.224}$$

whereas the other two fields are massless,

$$m_{\rho_1} = m_{\rho_2} = 0, \tag{3.225}$$

and so, after spontaneous symmetry breaking, we have two Goldstone bosons and one massive scalar field, corresponding to the two broken generators and the conserved generator respectively.

3.6.2 Goldstone theorem

We are now ready to prove Goldstone theorem. Let $\mathcal{G}$ be the symmetry group of the potential V, $\mathcal{H}$ the subgroup of $\mathcal{G}$ which leaves the vacuum invariant, dim$\mathcal{G}$ and dim$\mathcal{H}$ their dimensions. Expanding the potential about its minimum, since

$$\left(\frac{\partial V}{\partial \phi_i}\right)_{\phi=\phi_0} = 0, \tag{3.226}$$

we have

$$V(\phi) = V(\phi_0) + \frac{1}{2}\left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right)_{\phi=\phi_0} \chi_i \chi_j + \mathcal{O}\left(\chi^3\right), \tag{3.227}$$

where $\chi(x) \equiv \phi(x) - \phi_0$. Since ϕ_0 is a minimum, the mass matrix,

$$M_{ij} \equiv \left(\frac{\partial^2 V}{\partial \phi_i \partial \phi_j}\right)_{\phi=\phi_0}, \tag{3.228}$$

is positive semi-definite. Imposing the invariance of the potential under a group transformation,

$$V(\phi_0) = V(\phi_0') = V(\phi_0) + \frac{1}{2} M_{ij} \delta\phi_i \delta\phi_j + \text{higher orders}, \tag{3.229}$$

where $\delta\phi_i$ is the variation of the field under the group transformation, we find the condition

$$M_{ij} \delta\phi_i \delta\phi_j = 0. \tag{3.230}$$

In order for the above expression to vanish, or $\delta\phi_i$=0 either $\delta\phi_i \neq 0$ and $M_{ij} = 0$. In the first case, the group transformation belongs to the subgroup $\mathcal{H}$, the corresponding generators are conserved and massive bosons are admitted. The other possibility arises when the group transformation doesn't belong to $\mathcal{H}$, generators are broken and entail massless Goldstone bosons. Thus, the number of Goldstone bosons is a matter of group theory: dim$\mathcal{H}$ is the number of massive modes, instead the dimension of the coset $\mathcal{G}/\mathcal{H}$, is the number of massless modes. In our $SO(3)$ example,

$$\begin{array}{ll} \mathcal{G} = SO(3) & \dim\mathcal{G} = 3, \\ \mathcal{H} = SO(2) & \dim\mathcal{H} = 1, \end{array} \quad \Rightarrow \quad \begin{cases} \dim\mathcal{G}/\mathcal{H} = 2 \text{ massless modes}, \\ \dim\mathcal{H} = 1 \text{ massive modes}. \end{cases} \tag{3.231}$$

3.7 Spontaneous symmetry breaking of gauge symmetry

We now repeat the same steps we did for the Goldstone model in the case of a local symmetry.

$U(1)$ **(abelian)** Let us consider the transformation

$$\phi \quad \rightarrow \quad \phi' = \phi e^{i\Lambda(x)}. \tag{3.232}$$

We know that local gauge invariance is associated to the existence of a massless gauge boson, which is equivalent to the photon. As usual, we replace the ordinary derivative by the covariant derivative,

$$\partial_\mu \quad \rightarrow \quad D_\mu \equiv \partial_\mu + ieA_\mu, \tag{3.233}$$

and add the free Lagrangian of the gauge field, obtaining

$$\begin{aligned} \mathcal{L} &= \left(D_\mu \phi\right)^* \left(D^\mu \phi\right) - \mu^2 \phi^* \phi - \lambda (\phi^* \phi)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \\ &= \frac{1}{2}\left(\partial_\mu \phi_1\right)^2 + \frac{1}{2}\left(\partial_\mu \phi_2\right)^2 + \frac{1}{2}\left(e^2 A_\mu A^\mu - \mu^2\right)\left(\phi_1^2 + \phi_2^2\right) + \\ &\quad + eA^\mu \left(\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1\right) - \frac{1}{4}\lambda\left(\phi_1^2 + \phi_2^2\right)^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \end{aligned} \tag{3.234}$$

which is invariant under $U(1)$ gauge transformations. If we assume $\mu^2 < 0$, the ground state is

$$|\phi_0| = \sqrt{-\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}. \tag{3.235}$$

As in the Goldstone model, we consider the expansion

$$\phi(x) = \frac{v + h(x) + i\rho(x)}{\sqrt{2}}, \tag{3.236}$$

hence

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\left(\partial_\mu h\right)^2 + \frac{1}{2}\left(\partial_\mu \rho\right)^2 + \frac{1}{2}\left(e^2 A_\mu A^\mu - \mu^2\right)\left(v^2 + h^2 + 2vh + \rho^2\right) - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \\ &\quad + eA^\mu\left[\left(v+h\right)\partial_\mu\rho - \rho\partial_\mu h\right] - \frac{1}{4}\lambda\left(v^4 + h^4 + 6v^2h^2 + \rho^4 + 4v^3h + 2v^2\rho^2 + 4h^3v + 2h^2\rho^2 + 4vh\rho^2\right) \\ &= \frac{1}{2}\left(\partial_\mu h\right)^2 + \frac{1}{2}\left(\partial_\mu \rho\right)^2 + \frac{1}{2}e^2v^2A_\mu A^\mu + evA^\mu\partial_\mu\rho + \mu^2h^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \ldots,\end{aligned} \tag{3.237}$$

We now have a Lagrangian in terms of the fields h and ρ and we will try to recognize its spectrum. We note that

- μ^2h^2 is a mass term for h (excitation around the vacuum),
- ρ is massless (like a Goldstone boson),
- $\frac{1}{2}e^2v^2A_\mu A^\mu$ looks like a mass term for A_μ, but at this level we cannot conclude it's a really mass term since we have to study also $evA^\mu\partial_\mu\rho$.

In order to understand the role of ρ, we collect the terms:

$$\frac{1}{2}\left(\partial_\mu\rho\right)^2 + \frac{1}{2}e^2v^2A_\mu A^\mu + evA^\mu\partial_\mu\rho = \frac{1}{2}e^2v^2\left(A_\mu + \frac{1}{ev}\partial_\mu\rho\right)^2. \tag{3.238}$$

Therefore we can remove ρ from the Lagrangian using a gauge transformation

$$A_\mu \;\rightarrow\; A'_\mu = A_\mu + \partial_\mu f(x), \qquad f(x) \equiv \frac{\rho}{ev} \tag{3.239}$$

with the corresponding matter fields transformations

$$\phi \;\rightarrow\; \phi' = \phi e^{-ief(x)}, \tag{3.240}$$

$$\phi^* \;\rightarrow\; \phi^{*'} = \phi^* e^{ief(x)}. \tag{3.241}$$

Since

$$\begin{aligned}\phi' &= e^{-i\frac{\rho}{v}}\phi = e^{-i\frac{\rho}{v}}\frac{(v+h+i\rho)}{\sqrt{2}} \approx \left(1 - i\frac{\rho}{v}\right)\frac{(v+h+i\rho)}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\left(v + h + i\rho - i\rho + o\left(h\rho, \rho^2\right)\right) = \frac{v+h}{\sqrt{2}},\end{aligned} \tag{3.242}$$

we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}e^2v^2A_\mu A^\mu + \frac{1}{2}\left(\partial_\mu h\right)^2 + \mu^2h^2 + \text{couplings} \tag{3.243}$$

where ρ is disappeared and we can now (safely) claim that

$$m_A^2 = e^2v^2, \qquad m_h^2 = -2\mu^2. \tag{3.244}$$

Actually, after the gauge transformation we discover that the additional degree of freedom related to ρ (massless) becomes an additional degree of freedom characterizing a massive vector boson. This particular gauge in which no Goldstone bosons are present is usually referred to as *unitary gauge*.

In the Lagrangian (3.243), A^μ is the massive vector boson and h is the massive Higgs boson. In this case ($U(1)$), the Higgs boson "gives mass" to the vector boson because the mass of A^μ is proportional to v. Summarizing what we have learnt:

- *Goldstone mechanism* (SSB of global $U(1)$ symmetry)

2 massive scalars (ϕ_1, ϕ_2)	$\rightarrow$	1 massive scalar (h)
		1 massless (Goldstone) scalar (ρ)

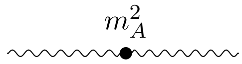
- *Higgs mechanism* (SSB of local $U(1)$ symmetry)

2 massive scalars (ϕ_1, ϕ_2)	$\rightarrow$	1 massive scalar (h)
1 photon (massless)		1 massive vector
(4 d.o.f)		*(4 d.o.f)*

A diagrammatic interpretation We now want to interpret[4] in a diagrammatic way the mixing term $evA_\mu\partial^\mu\rho$. From the Lagrangian, we saw that A^μ acquire a mass

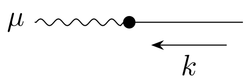
$$m_A = ev. \tag{3.245}$$

If we treat the mass term $\frac{1}{2}m_A^2 A_\mu A^\mu$ as a graphic rule



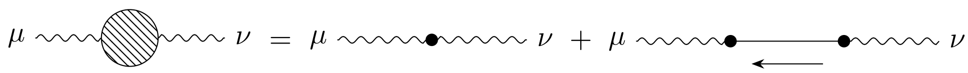
(3.246)

and we represent the coupling between the gauge boson A^μ and the Goldstone boson ρ as



$\equiv iev(-ik^\mu) \equiv m_A k^\mu.$ (3.247)

we can define a leading order vacuum polarization



(3.248)

$$= im_A^2 g^{\mu\nu} + (m_A k^\mu)\frac{i}{k^2}(-m_A k^\nu) = \tag{3.249}$$

$$= im_A^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) \tag{3.250}$$

which has the same structure of $\Pi^{\mu\nu}$.

SO(3) (non abelian) We can now apply the same idea to $SO(3)$. We have a Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(D_\mu\phi_i\right)\left(D^\mu\phi_i\right) - \frac{1}{2}\mu^2\phi_i\phi_i - \frac{1}{4}\lambda(\phi_i\phi_i)^2 - \frac{1}{4}G^i_{\mu\nu}G^{\mu\nu}_i, \qquad i = 1, 2, 3 \tag{3.251}$$

and

$$D_\mu\phi_i = \left(\partial_\mu\phi_i + g\varepsilon_{ijk}W^j_\mu\phi_k\right) \tag{3.252}$$

where W^j_μ $(j = 1, 2, 3)$ are three massless gauge bosons. If $\mu^2 < 0$ the minimum of V (following the discussion on the Goldstone model) lies on the sphere of radius

$$|\phi_0| = \sqrt{-\frac{\mu^2}{\lambda}} \equiv v. \tag{3.253}$$

As in the last section, we choose the vacuum

$$\phi_0 \equiv v\hat{\mathbf{e}}_3 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}, \tag{3.254}$$

[4]This discussion can be found in M.E.Peskin, D.V.Schroeder, "An Introduction to Quantum Field Theory", chapter 20.1

and make the expansion

$$\phi_1(x) = \rho_1(x), \qquad \phi_2(x) = \rho_2(x), \qquad \phi_3(x) = v + h(x). \tag{3.255}$$

Our Lagrangian is then

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\rho_1)^2 + (\partial_\mu\rho_2)^2 + (\partial_\mu h)^2\right] + vg\left[(\partial_\mu\rho_1)\, W_2^\mu - (\partial_\mu\rho_2)\, W_1^\mu\right] + \\ + \frac{v^2g^2}{2}\left[\left(W_\mu^1\right)^2 + \left(W_\mu^2\right)^2\right] - \frac{1}{4}G_{\mu\nu}^i G_i^{\mu\nu} + \mu^2h^2 + \text{couplings}.\end{aligned} \tag{3.256}$$

So we see that ρ_1 and ρ_2 are massless (Goldstone bosons), h is massive (Higgs boson), with

$$m_h^2 = -2\mu^2, \tag{3.257}$$

and we have again some coupling terms involving W_1^μ and W_2^μ, while W_3^μ is massless. As seen previously in the unitary gauge, the Higgs field around the vacuum can be expanded as follows:

$$\phi \equiv \begin{pmatrix} 0 \\ 0 \\ v + h(x) \end{pmatrix}, \tag{3.258}$$

where the Goldstone boson does not appear for the choice of the unitary gauge, and we refer to the appendix for the demonstration of how to go from (3.256) to (3.259), so finally we find:

$$\mathcal{L} \underset{U.G.}{=} -\frac{1}{4}G_{\mu\nu}^i G^{i\mu\nu} - \frac{1}{2}v^2g^2\left[\left(W_\mu^1\right)^2 + \left(W_\mu^2\right)^2\right] + \frac{1}{2}(\partial_\mu h)^2 - v^2\lambda h^2 + \text{couplings}. \tag{3.259}$$

Summarizing:

- *Goldstone mechanism* $SO(3)$

3 massive scalars (ϕ_1, ϕ_2, ϕ_3)	$\rightarrow$	1 massive scalar (h)
		2 massless (Goldstone) scalar (ρ_1, ρ_2)

- *Higgs mechanism* $SO(3)$ local

3 massive scalars (ϕ_1, ϕ_2, ϕ_3)	$\rightarrow$	1 massive scalar (h)
3 massless vectors		2 massive vectors (W_μ^1, W_μ^2)
		1 massless vector (W_μ^3)
(9 d.o.f)		*(9 d.o.f)*

Note that T_3 is the only generator which is conserved and W_3 is the only massless vector. A different choice of the vacuum would have been produce a different situation.

We can now summarize what we have learnt. Let $\mathcal{G}$ be the symmetry group of the potential V, $\mathcal{H}$ the subgroup of $\mathcal{G}$ which leaves the vacuum invariant, dim$\mathcal{G}$ and dim$\mathcal{H}$ their dimensions; we have

	Goldstone mechanism	*Higgs mechanism*
# unbroken gener. = dim$\mathcal{H}$	# massive scalars	# massless gauge bosons
# broken gener. = dim$\mathcal{G}/\mathcal{H}$	# massless Gold. bosons	# massive gauge bosons

3.8 The standard model of electroweak interactions

We start from the $SU(2)_L \times U(1)_Y$ invariant Lagrangian,

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_B, \tag{3.260}$$

where $\mathcal{L}_L$ is the Lagrangian of leptons and $\mathcal{L}_B$ is the Lagrangian of massless gauge bosons. We introduce a Higgs field,

$$\Phi(x) = \begin{pmatrix} \phi_a(x) \\ \phi_b(x) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_{a,1}(x) + i\phi_{a,2}(x) \\ \phi_{b,1}(x) + i\phi_{b,2}(x) \end{pmatrix}, \tag{3.261}$$

which transforms both under $SU(2)_L$ and $U(1)$,

$$SU(2): \quad \begin{cases} \Phi \longrightarrow \Phi' = e^{ig\frac{\tau^i}{2}\omega_i(x)}\Phi, \\ \Phi^\dagger \longrightarrow \Phi'^\dagger = \Phi^\dagger e^{-ig\frac{\tau^i}{2}\omega_i(x)}, \end{cases} \qquad U(1): \quad \begin{cases} \Phi \longrightarrow \Phi' = e^{ig'Yf(x)}\Phi, \\ \Phi^\dagger \longrightarrow \Phi'^\dagger = \Phi^\dagger e^{-ig'Yf(x)}, \end{cases} \tag{3.262}$$

and add a new term to the Lagrangian,

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_B + \mathcal{L}_H, \tag{3.263}$$

where

$$\mathcal{L}_H = (D^\mu\Phi)^\dagger (D_\mu\Phi) - \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2, \tag{3.264}$$

with $\lambda > 0$, $\mu^2 < 0$. The covariant derivative of the Higgs field is defined as

$$D^\mu\Phi = \left(\partial^\mu + ig\frac{\tau_j}{2}W^{j\mu} + ig'YB^\mu\right)\Phi. \tag{3.265}$$

We have a minimum at

$$\Phi_0 = \begin{pmatrix} \phi_a^0(x) \\ \phi_b^0(x) \end{pmatrix}, \tag{3.266}$$

which satisfies

$$\sqrt{\Phi_0^\dagger\Phi_0} = \sqrt{|\phi_a^0|^2 + |\phi_b^0|^2} = \sqrt{-\frac{\mu^2}{2\lambda}} \equiv \frac{v}{\sqrt{2}}. \tag{3.267}$$

We choose the vacuum

$$\Phi_0 \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{3.268}$$

and make an expansion of the field around this vacuum,

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} \rho_1(x) + i\rho_2(x) \\ v + h(x) + i\rho_3(x) \end{pmatrix}, \tag{3.269}$$

where ρ_1, ρ_2 and ρ_3 play the role of Goldstone bosons and can be reabsorbed by an unitary transformation. Thus, in the unitary gauge the field becomes

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}. \tag{3.270}$$

Generators of

$$SU(2)_L: \quad \begin{aligned} \tau_1\Phi_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} &&= \tfrac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} &&\neq 0, \\ \tau_2\Phi_0 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} &&= -i\tfrac{1}{\sqrt{2}}\begin{pmatrix} v \\ 0 \end{pmatrix} &&\neq 0, \\ \tau_3\Phi_0 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} &&= -\tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} &&\neq 0 \end{aligned} \tag{3.271}$$

and

$$U(1)_Y : \quad Y\Phi_0 \neq 0, \tag{3.272}$$

are all broken, while the generator of

$$\begin{aligned} U(1)_{QED}: \quad Q\Phi_0 &= \left(\frac{1}{2}\tau_3 + Y\mathbf{1}\right)\Phi_0 \overset{Y=\frac{1}{2}}{\equiv} \frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\Phi_0 = \\ &= \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} = 0, \end{aligned} \tag{3.273}$$

is conserved. Note that we have chosen $Y = \frac{1}{2}$ to obtain this generator conserved and hence the photon massless. In the unitary gauge, the Higgs Lagrangian is

$$\mathcal{L}_H = (D^\mu\Phi)^\dagger (D_\mu\Phi) - V, \tag{3.274}$$

where

$$D_\mu\Phi = \left(\partial_\mu + ig\frac{1}{2}\tau^j W_\mu^j + ig' Y B_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v + h(x)\end{pmatrix}, \qquad \left(Y = \frac{1}{2}y\right) \tag{3.275}$$

and generates masses for gauge bosons plus interactions. We consider $Y = \frac{1}{2}y$ to show the role of Y inside the mass matrix and have a more general discussion. We focus only on the v-dependent terms,

$$\begin{aligned} D_\mu\Phi &\overset{(h=0)}{=} \frac{1}{\sqrt{2}}\left[ig\frac{1}{2}\tau_j W_\mu^j + ig'\frac{1}{2}yB_\mu\right]\begin{pmatrix}0\\ v\end{pmatrix} = \\ &= \frac{i}{2\sqrt{2}}\left[g\left(\tau_1 W_\mu^1 + \tau_2 W_\mu^2 + \tau_3 W_\mu^3\right) + g'yB_\mu\right]\begin{pmatrix}0\\ v\end{pmatrix} = \\ &= \frac{i}{2\sqrt{2}}\left[g\left(\begin{pmatrix}0 & W_\mu^1\\ W_\mu^1 & 0\end{pmatrix} + \begin{pmatrix}0 & -iW_\mu^2\\ iW_\mu^2 & 0\end{pmatrix} + \begin{pmatrix}W_\mu^3 & 0\\ 0 & -W_\mu^3\end{pmatrix}\right) + g'\begin{pmatrix}yB_\mu & 0\\ 0 & yB_\mu\end{pmatrix}\right]\begin{pmatrix}0\\ v\end{pmatrix} = \\ &= \frac{i}{2\sqrt{2}}\begin{pmatrix}gW_\mu^3 + g'yB_\mu & g\left(W_\mu^1 - iW_\mu^2\right)\\ g\left(W_\mu^1 + iW_\mu^2\right) & -gW_\mu^3 + g'yB_\mu\end{pmatrix}\begin{pmatrix}0\\ v\end{pmatrix} = \\ &= \frac{iv}{2\sqrt{2}}\begin{pmatrix}g\left(W_\mu^1 - iW_\mu^2\right)\\ -gW_\mu^3 + g'yB_\mu\end{pmatrix} \end{aligned} \tag{3.276}$$

and

$$(D_\mu\Phi)^\dagger \overset{(h=0)}{=} -\frac{iv}{2\sqrt{2}}\left(g\left(W_\mu^1 + iW_\mu^2\right), \quad -gW_\mu^3 + g'yB_\mu\right). \tag{3.277}$$

Hence

$$(D^\mu\Phi)^\dagger (D_\mu\Phi) \overset{(h=0)}{=} \frac{v^2}{8}\left[\underbrace{g^2\left(W_1^2 + W_2^2\right)}_{(*)} + \underbrace{\left(-gW_\mu^3 + g'yB_\mu\right)^2}_{(**)}\right], \tag{3.278}$$

where

$$\begin{aligned} (*) &= g^2\left(W_\mu^\dagger W^\mu\right), \\ (**) &= \begin{pmatrix}W_\mu^3 & B_\mu\end{pmatrix}\underbrace{\begin{pmatrix}g^2 & -gg'y\\ -gg'y & g'^2y^2\end{pmatrix}}_{M}\begin{pmatrix}W_\mu^3\\ B_\mu\end{pmatrix}. \end{aligned}$$

If $y = 1$, the matrix M has two eigenvalues, with normalized eigenvectors

$$\lambda = 0 \quad \longleftrightarrow \quad \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g' \\ g \end{pmatrix},$$
$$\lambda = g^2 + g'^2 \quad \longleftrightarrow \quad \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g \\ -g' \end{pmatrix},$$

and is diagonalized by the orthogonal matrix R,

$$R \equiv \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g' & g \\ g & -g' \end{pmatrix}, \qquad RMR^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & g^2 + g'^2 \end{pmatrix}. \tag{3.279}$$

Thus, we can rewrite

$$(**) = \begin{pmatrix} W^3_\mu & B_\mu \end{pmatrix} R^{-1} RMR^{-1} R \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} \equiv \begin{pmatrix} A_\mu & Z_\mu \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & g^2 + g'^2 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix}, \tag{3.280}$$

where we have defined

$$\begin{pmatrix} A_\mu \\ Z_\mu \end{pmatrix} \equiv R \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix} = \frac{1}{\sqrt{g^2 + g'^2}} \begin{pmatrix} g' W^3_\mu + g B_\mu \\ g W^3_\mu - g' B_\mu \end{pmatrix}. \tag{3.281}$$

Note that this definition matches with the one we had already given in terms of the Weinberg angle in (3.145). Now we can rewrite equation (3.278) in terms of the physical fields,

$$\left(D^\mu \Phi\right)^\dagger \left(D_\mu \Phi\right) \overset{(h=0)}{=} \frac{v^2}{8} \left[g^2 \left(W^\dagger_\mu W^\mu \right) + \left(g'^2 + g^2 \right) \left(Z_\mu \right)^2 + 0 \cdot \left(A_\mu \right)^2 \right], \tag{3.282}$$

which tells us that

$$m_{W^+} = m_{W^-} = \frac{vg}{2}, \qquad m_Z = \frac{v}{2} \sqrt{g^2 + g'^2}, \qquad m_\gamma = 0, \tag{3.283}$$

and moreover, since $e = g \sin\theta_W = g' \cos\theta_W$,

$$\frac{m_W}{m_Z} = \frac{g}{\sqrt{g^2 + g'^2}} = \cos\theta_W. \tag{3.284}$$

It is useful to introduce the parameter

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1, \tag{3.285}$$

which was a first successful prediction of the model. The mass of the Higgs boson,

$$m_h = \sqrt{2\lambda v^2}, \tag{3.286}$$

is given in terms of v, which is experimentally known, for example from the muon decay,

$$\begin{cases} \frac{G_F}{\sqrt{2}} = \frac{g^2}{8 m_W^2}, \\ m_W = \frac{vg}{2}, \end{cases} \qquad \Rightarrow \quad v = \frac{1}{\sqrt{2} G_F} \simeq 250 \text{ GeV (electroweak scale)}, \tag{3.287}$$

and λ, which instead is a free parameter of the theory.

	ψ_ν^L	ψ_l^L	ψ_l^R	ψ_ν^R	ϕ^a	ϕ^b
I_3^W	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$
Y	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	0	$\frac{1}{2}$	$\frac{1}{2}$
Q	0	-1	-1	0	1	0

Table 3.2

3.9 Yukawa interactions

So far, we have given mass to gauge bosons. We are now going to give mass to fermions using the Yukawa interaction. Let's consider the Lagrangian

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_B + \mathcal{L}_H + \mathcal{L}_{HL} \tag{3.288}$$

(L: massless leptons, B: gauge bosons, H: Higgs, HL: Higgs-leptons) where we have introduced the *Yukawa term*

$$\mathcal{L}_{HL} = -g_l \left(\bar{\Psi}^L \psi_l^R \Phi + \Phi^\dagger \bar{\psi}_l^R \Psi^L\right) - g_{\nu_l} \left(\bar{\Psi}^L \psi_{\nu_l}^R \tilde{\Phi} + \tilde{\Phi}^\dagger \bar{\psi}_{\nu_l}^R \Psi^L\right), \tag{3.289}$$

and

$$\Phi(x) \equiv \begin{pmatrix} \phi^a(x) \\ \phi^b(x) \end{pmatrix} \tag{3.290}$$

$$\tilde{\Phi}(x) \equiv -i\left(\Phi^\dagger \tau_2\right)^T = \begin{pmatrix} \phi_b^*(x) \\ -\phi_a^*(x) \end{pmatrix} \tag{3.291}$$

Let's firstly motivate why this is a plausible term, starting by checking its invariance. Using Table 3.2, we recall that

- under $SU(2)_L$

$$\Psi^{L'} = e^{ig\omega_j t_j} \Psi^L \qquad t_j \equiv \frac{\tau_j}{2}, \tag{3.292}$$

$$\psi_l^{R'} = \psi_l^R, \qquad \psi_\nu^{R'} = \psi_\nu^R, \tag{3.293}$$

$$\Phi' = e^{ig\omega_j t_j} \Phi \ ; \tag{3.294}$$

- under $U(1)_Y$

$$\Psi^{L'} = e^{-\frac{1}{2}g' f(x)} \Psi^L, \tag{3.295}$$

$$\psi_l^{R'} = e^{-1ig' f(x)}, \qquad \psi_\nu^{R'} = \psi^R, \tag{3.296}$$

$$\Phi' = e^{\frac{1}{2}ig' f(x)} \Phi. \tag{3.297}$$

Therefore we can check what happen to the Yukawa term, studying separately the addends in equation (3.289). Considering $\bar{\Psi}^L \psi_l^R \Phi$ we have that

- under $SU(2)_L$
$$\bar{\Psi}^{L'} \psi_l^{R'} \Phi' = \bar{\Psi}^L e^{-ig\omega_j t_j} \psi_l^R e^{ig\omega_j t_j} \Phi = \bar{\Psi}^L \psi_l^R \Phi; \tag{3.298}$$

- under $U(1)_Y$
$$\bar{\Psi}^{L'} \psi_l^{R'} \Phi' = e^{ig'\left(\frac{1}{2}-1+\frac{1}{2}\right) f(x)} \bar{\Psi}^L \psi_l^L \Phi = \bar{\Psi}^L \psi_l^R \Phi \tag{3.299}$$

hence $\bar{\Psi}^L \psi_l^R \Psi$ is $SU(2) \times U(1)$ invariant.
Let's now check $\bar{\Psi}^L \psi_\nu^R \tilde{\Phi}$

- under $SU(2)_L$

$$\bar{\Psi}^{L'}\psi_\nu^{R'}\tilde{\Phi}' = \bar{\Psi}^L e^{-ig\omega_j t_j}\psi_\nu^R\tilde{\Phi}' \tag{3.300}$$

  which is invariant iff

$$\tilde{\Phi}' = e^{ig\omega_j t_j}\tilde{\Phi} \tag{3.301}$$

  i.e. iff $\tilde{\Phi}$ transforms like Φ;

- under $U(1)_Y$

$$\bar{\Psi}^{L'}\psi_\nu^{R'}\tilde{\Phi}' = e^{ig'\left(\frac{1}{2}+0-\frac{1}{2}\right)f(x)}\bar{\Psi}^L\psi_\nu^R\tilde{\Phi} = \bar{\Psi}^L\psi_\nu^R\tilde{\Phi}. \tag{3.302}$$

Therefore we want to show that, under $SU(2)$

$$\delta\tilde{\Phi} = \delta\Phi. \tag{3.303}$$

Remember that, under $SU(2)$

$$\Phi(x) \to \Phi'(x) = e^{ig\omega_j(x)\frac{\tau_j}{2}}\Phi(x) \tag{3.304}$$

hence

$$\Phi(x) \to \Phi(x) + \delta\Phi(x) = \left(1 + ig\omega_j(x)\frac{\tau_j}{2} + o(g^2)\right)\Phi(x) \tag{3.305}$$

and

$$\delta\Phi(x) = i\frac{1}{2}g\omega_j\tau_j\Phi(x) \tag{3.306}$$

$$\delta\Phi^\dagger(x) = -i\frac{1}{2}g\omega_j\Phi^\dagger\tau_j. \tag{3.307}$$

Therefore

$$\delta\tilde{\Phi} = -i\left(\delta\Phi^\dagger\tau_2\right)^T = -i\left(-i\frac{1}{2}g\omega_j\Phi^\dagger\tau_j\tau_2\right)^T \tag{3.308}$$

and using the property of the Pauli matrices, $\tau_j\tau_2 = -\tau_2\tau_j^T$ we find

$$\delta\tilde{\Phi} = -i\left(i\frac{1}{2}g\omega_j\Phi^\dagger\tau_2\tau_j^T\right)^T = i\frac{1}{2}g\omega_j\left(-i\Phi^\dagger\tau_2\tau_j^T\right)^T = \tag{3.309}$$

$$= i\frac{1}{2}g\omega_j\tau_j\left(-i\Phi^\dagger\tau_2\right)^T = i\frac{1}{2}g\omega_j\tau_j\tilde{\Phi}, \tag{3.310}$$

so $\tilde{\Phi}$ transforms, under $SU(2)$, like Φ. This result is somehow expected since the difference between Φ and $\tilde{\Phi}$ is just in the change of the role of up and down components in the isospinor.

Explicitly we have

$$\bar{\Psi}\psi_l^R\Phi \overset{UG}{=} \frac{1}{\sqrt{2}}\left(\bar{\psi}_\nu^L, \bar{\psi}_l^L\right)\psi_l^R\begin{pmatrix}0\\ v+h(x)\end{pmatrix} = \underbrace{\frac{v}{\sqrt{2}}\bar{\psi}_l^L\psi_l^R}_{(*)} + \underbrace{\frac{1}{\sqrt{2}}\bar{\psi}_l^L\psi_l^R h}_{(**)} \tag{3.311}$$

and similarly

$$\bar{\Psi}\psi_\nu^R\tilde{\Phi} \overset{UG}{=} \frac{2}{\sqrt{2}}\left(\bar{\psi}_\nu^L, \bar{\psi}_l^L\right)\psi_\nu^R\begin{pmatrix}v+h(x)\\ 0\end{pmatrix} = \underbrace{\frac{v}{\sqrt{2}}\bar{\psi}_\nu^L\psi_\nu^R}_{(*)} + \underbrace{\frac{1}{\sqrt{2}}\bar{\psi}_\nu^L\psi_\nu^R h}_{(**)}. \tag{3.312}$$

Then we can define the lepton mass,

$$m_l \equiv \frac{v}{\sqrt{2}}g_l \tag{3.313}$$

the neutrino mass

$$m_\nu \equiv \frac{v}{\sqrt{2}} g_\nu \tag{3.314}$$

and plugging everything into the Lagrangian we discover that

$$(*) \;\to\; m_i \bar{\psi}_i \psi_i \qquad \textit{Dirac mass} \tag{3.315}$$

$$(**) \;\to\; \frac{m_i}{v} \bar{\psi}_i \psi_i h \qquad \textit{Yukawa interaction} \tag{3.316}$$

where i means leptons or neutrinos.

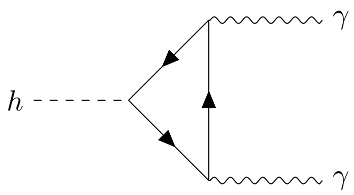
At this point, putting together all the pieces we have found, we have obtained the complete Lagrangian of electroweak theory,

$$\begin{aligned}\mathcal{L} = {} & i\left(\bar{\Psi}^L \not{D} \Psi^L + \bar{\psi}_l^R \not{D} \psi_l^R + \bar{\psi}_\nu^R \not{D} \psi_\nu^R\right) + \left(-\frac{1}{4} B_{\mu\nu} B^{\mu\nu} - \frac{1}{4} G_{i\mu\nu} G_i^{\mu\nu}\right) + \\ & + \left[(D^\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda \left(\Phi^\dagger \Phi\right)^2\right] + \\ & - g_l \left(\bar{\Psi}^L \psi_l^R \Phi + \Phi^\dagger \bar{\psi}_l^R \Psi^L\right) - g_{\nu_l} \left(\bar{\Psi}^L \psi_{\nu_l}^R \tilde{\Phi} + \tilde{\Phi}^\dagger \bar{\psi}_{\nu_l}^R \Psi^L\right).\end{aligned}$$

Actually we found an approximated form, since we didn't speak about quarks and we used a "diagonal Lagrangian", i.e. we treated neutrinos (of muon, electron and tau) like separated particles, while it's not true (oscillation of neutrinos).

3.10 The (missed) discovery channel

The discovery of Higgs boson at LHC was achieved using the channel

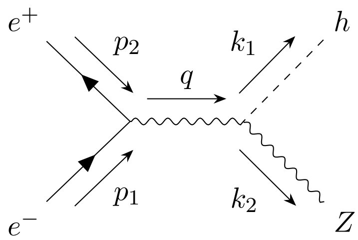


. (3.317)

Previous huntings instead, by means of LEP, used electrons. Actually, the Higgs boson is also produced in the process

$$e^-(p_1) + e^+(p_2) \to h(k_1) + Z(k_2) \tag{3.318}$$

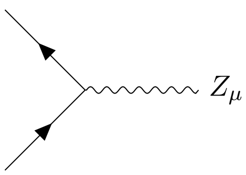
which can be described by



(3.319)

with $q = p_1 + p_2$. We need some interactions vertices that can be derived from the Lagrangian:

- $Zf\bar{f}$-vertex,



$$\to \quad \frac{-ig}{4\cos\theta_W} \gamma_\mu (V - A\gamma_5) \tag{3.320}$$

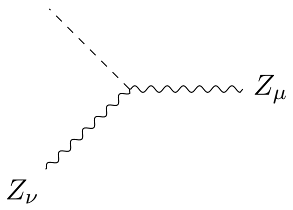
where

$$V = 2I_W^3 - 4Q\sin^2\theta_W, \qquad A = 2I_W^3 \tag{3.321}$$

and for the electron

$$V = -1 + 4\sin^2\theta_W, \qquad A = -1; \tag{3.322}$$

- hZZ-vertex,



$$\rightarrow \qquad ig\frac{m_Z}{\cos\theta_W}g_{\mu\nu} = i2\frac{m_Z^2}{v}g_{\mu\nu}. \tag{3.323}$$

In this way, one finds the amplitude

$$\mathcal{M} = i\frac{2m_Z^2}{v}\varepsilon_\nu^*(k_2)\frac{i\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_Z^2}\right)}{q^2 - m_Z^2}\frac{-ig}{4\cos\theta_W}\bar{v}(p_2)\gamma_\mu(V - A\gamma_5)u(p_1) = \tag{3.324}$$

$$= ig\frac{m_Z^2}{2v\cos\theta_W}\varepsilon_\nu^*(k_2)\frac{\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_Z^2}\right)}{q^2 - m_Z^2}\bar{v}(p_2)\gamma_\mu(V - A\gamma_5)u(p_1)\ . \tag{3.325}$$

We note that

$$\frac{gm_Z}{2v\cos\theta_W} = \frac{g}{m_W}\frac{m_Z^2}{2v}\cdot\frac{g}{g} = \frac{g^2m_Z^2}{4m_W^2} = 2m_Z^2\frac{G_F}{\sqrt{2}} = \sqrt{2}G_Fm_Z^2\ , \tag{3.326}$$

where we have used

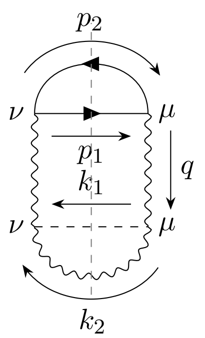
$$\frac{m_W}{m_Z} = \cos\theta_W \qquad \text{and} \qquad \frac{g^2}{8m_W^2} = \frac{G_F}{\sqrt{2}}. \tag{3.327}$$

Therefore

$$\mathcal{M} = i\sqrt{2}m_Z^3G_F\varepsilon_\nu^*(k_2)\frac{\left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{m_Z^2}\right)}{q^2 - m_Z^2}\bar{v}(p_2)\gamma_\mu(V - A\gamma_5)u(p_1)\ . \tag{3.328}$$

Since $m_Z^2 \geq m_e^2$, we can neglect $q^\mu q^\nu-$term and we consider $m_e \to 0$. The value of m_e enter when we want to compute

$$|\bar{\mathcal{M}}|^2 = \sum_{\text{spin, pol}}|\mathcal{M}^2| = \tag{3.329}$$



Explicitly, defining $s \equiv q^2 = (p_1 + p_2)^2$

$$|\bar{\mathcal{M}}|^2 = \frac{2G_F^2m_Z^6}{(s - m_z^2)^2}\mathrm{Tr}\left(\frac{\not{p}_2}{2m_e}\gamma_\mu(V - A\gamma_5)\frac{\not{p}_1}{2m_e}\gamma_\nu(V - A\gamma_5)\right)\left(-g^{\mu\nu} + \frac{k_2^\mu k_2^\nu}{m_Z^2}\right). \tag{3.330}$$

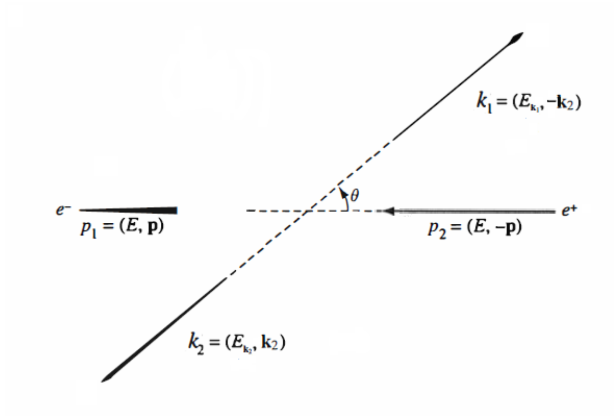


Figure 3.4

In order to evaluate the trace we observe that terms proportional to γ_5 produce zero or $\varepsilon^{\mu\nu\rho\sigma}p_{1\rho}p_{2\sigma}$, which is antisymmetric in $\mu \leftrightarrow \nu$. Since $g^{\mu\nu}$ and $k_2^\mu k_2^\nu$ are symmetric in $\mu \leftrightarrow \nu$, we can drop these terms:

$$(2m_e)^2|\bar{\mathcal{M}}|^2 = \frac{2G_F^2 m_Z^6}{(s-m_z^2)^2}(V^2+A^2)\left(-g^{\mu\nu}+\frac{k_2^\mu k_2^\nu}{m_Z^2}\right)\mathrm{Tr}(\not{p}_2\gamma_\mu\not{p}_1\gamma_\nu). \tag{3.331}$$

Expanding we obtain

$$(2m_e)^2|\bar{\mathcal{M}}|^2 = \frac{4G_F^2 m_Z^6}{(s-m_z^2)^2}(V^2+A^2)\left[s+\frac{(t-m_Z^2)(u-m_Z^2)}{m_Z^2}\right] \tag{3.332}$$

with

$$s \equiv (p_1+p_2)^2, \qquad t \equiv (p_1-k_2)^2, \qquad u \equiv (p_2-k_2)^2, \qquad s+t+u = m_Z^2+m_h^2\ . \tag{3.333}$$

In the center of mass system

$$p_1^2 = 0 = p_2^2 \ \Leftrightarrow\ E = |\mathbf{p}| = \frac{\sqrt{s}}{2} \tag{3.334}$$
$$k_1^2 = m_h^2 = E_{k_1}^2 - |\mathbf{k}_2|^2 \tag{3.335}$$
$$k_2^2 = m_Z^2 = E_{k_2}^2 - |\mathbf{k}_2|^2 \tag{3.336}$$

and we can compute

$$t = p_1^2+k_2^2-2p_1\cdot k_2 = m_Z^2 - \sqrt{s}\,(E_{k_2}-|\mathbf{k}_2|\cos\theta)\ , \tag{3.337}$$
$$u = p_2^2+k_2^2-2p_2\cdot k_2 = m_Z^2 - \sqrt{s}\,(E_{k_2}+|\mathbf{k}_2|\cos\theta)\ . \tag{3.338}$$

The energy conservation leads to

$$\frac{\sqrt{s}}{2}+\frac{\sqrt{s}}{2} = \sqrt{s} = \sqrt{m_Z^2+|\mathbf{k}_2|^2}+\sqrt{m_h^2+|\mathbf{k}_2|^2} \tag{3.339}$$

and using (3.336) e (3.339) we find

$$\begin{cases} |\mathbf{k}_2|^2 = \frac{\lambda(s,m_Z^2,m_h^2)}{4s} \\ E_{k_2} = \frac{s+m_Z^2-m_h^2}{2\sqrt{s}} \end{cases} \tag{3.340}$$

where

$$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz \tag{3.341}$$

is the *Challen function*. Therefore in the C.o.M.

$$(2m_e)^2|\bar{\mathcal{M}}|^2 = \frac{4G_F^2 m_Z^4 (V^2 + A^2)}{(s - m_Z^2)^2} s(m_Z^2 + E_{k_2}^2 - |\mathbf{k}_2|^2 \cos^2\theta)\ . \tag{3.342}$$

Averaging over initial states (namely multiplying by $\frac{1}{4}$)

$$(2m_e)^2|\bar{\mathcal{M}}|^2 = \frac{G_F^2 m_Z^4 (V^2 + A^2)}{(s - m_Z^2)^2} s(m_Z^2 + E_{k_2}^2 - |\mathbf{k}_2|^2 \cos^2\theta) \tag{3.343}$$

and recalling the differential cross section in the C.o.M.

$$\left(\frac{d\sigma}{d\Omega}\right)_{C.o.M} = \frac{1}{64\pi^2} \frac{1}{(E_1 + E_2)^2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \prod_l (2m_l)|\bar{\mathcal{M}}|^2 \tag{3.344}$$

we can write

$$\left(\frac{d\sigma}{d\Omega}\right)_{C.o.M} = \frac{1}{32\pi^2} \frac{1}{s} \frac{|\mathbf{k}_2|}{\sqrt{s}} (2m_l)^2 |\bar{\mathcal{M}}|^2 = \tag{3.345}$$

$$= \frac{V^2 + A^2}{16\pi} \frac{G_F^2 m_Z^4}{(s - m_Z^2)^2} \frac{|\mathbf{k}_2|}{\sqrt{s}} (m_Z^2 + E_{k_2}^2 - |\mathbf{k}_2|^2 \cos^2\theta). \tag{3.346}$$

Integrating

$$\sigma = \int d\sigma = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \left(\frac{d\sigma}{d\Omega}\right)_{C.o.M} = 2\pi \int_{-1}^{1} d\cos\theta \left(\frac{d\sigma}{d\Omega}\right)_{C.o.M} = \tag{3.347}$$

$$= \frac{V^2 + A^2}{16\pi} \frac{G_F^2 m_Z^4}{(s - m_Z^2)^2} \frac{|\mathbf{k}_2|}{\sqrt{s}} \int_{-1}^{1} d\cos\theta (m_Z^2 + E_{k_2}^2 - |\mathbf{k}_2|^2 \cos^2\theta) = \tag{3.348}$$

$$= \frac{V^2 + A^2}{12\pi} G_F^2 m_z^4 \frac{|\mathbf{k}_2|}{\sqrt{s}} \frac{3m_Z^2 + |\mathbf{k}_2|^2}{(s - m_Z^2)^2}. \tag{3.349}$$

We note that the total cross section depends on

$$|\mathbf{k}_2| = \sqrt{E_{|\mathbf{k}_2|}^2 - m_Z^2} = \sqrt{\left(\frac{s + m_Z^2 - m_h^2}{2\sqrt{s}}\right)^2 - m_Z^2} \tag{3.350}$$

which decreases with increasing m_h.

Chapter 4

Introduction to QCD

4.1 Quarks and color

Matter in the universe is composed of atoms which consist of a nucleus and electrons. The nucleus is further composed of protons and neutrons, generally called nucleons. They are bound together through the interaction of the pion which was first found in cosmic ray experiments. Since particle accelerators were introduced, many new particles, stable and unstable, have been created in scattering experiments of nucleons, pions, photons and electrons. The number of kinds of these particles now adds up to more than several hundreds. They are classified in the following way:

- Particles with *strong interactions*

$$Hadrons \begin{cases} Baryons(fermions), & \text{e.g., nucleons} \\ Mesons(bosons), & \text{e.g., pions} \end{cases} \tag{4.1}$$

- Particles with *no strong interactions*
  Leptons (fermions), e.g., electrons, muons, neutrinos.

- Particles which mediate electromagnetic and weak interactions
  Gauge bosons (bosons), e.g., photons, W-bosons, Z-bosons.

There are a number of different kinds of hadrons while the number of the kinds of leptons and gauge bosons is rather limited. It is then quite hard to think that all of these hadrons are ultimate building blocks of matter. Rather it is natural to expect that hadrons are made of a fewer number of fundamental particles, *the quarks.* On the hand there are not many kinds of leptons and gauge bosons, and they may still be considered as elementary and point-like. In fact there is so far no experimental evidence of the internal structure of leptons and gauge bosons. Hadrons, on the contrary, exhibit the extended structure in electron-scattering experiments. This structure may be considered as a manifestation of the fundamental entities within hadrons. Thus we are naturally led to an attempt of classifying all hadrons in term of their fundamental building blocks, the quarks. According to the quark model, all the hadrons found in the particle data tables may be classified in a consistent manner if one assumes that the baryons are made of three quarks (qqq) and the mesons are made of a quark-antiquark pair ($q\bar{q}$).
In order to account for the conservation of the isotopic-spin, strangeness, charm, bottom and top quantum numbers in hadronic reactions, one assumes the existence of different kinds of quarks corresponding to these conserved quantum numbers in strong interactions. They are called the up(u), down(d), strange(s), charm(c), bottom(b) and top(t) quarks respectively and the quantum numbers are generically called *flavor.*
The leptons are considered to be elementary just as the quarks are. According to their properties

in weak interactions it is commonly accepted that they form sequential doublets:

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix} \quad , \quad \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix} \quad , \quad \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}$$

where e^-, μ^- and τ^- are the electron, muon and tau lepton respectively and ν_e, ν_μ and ν_τ are the corresponding neutrinos. By studying the weak interaction of hadrons, one derives information on the property of quarks in weak processes and finds that quarks exhibit the same regularity as the sequential property of the leptons where t is the new quark (top quark) observed recently.

flavors	Q_{el}	I_3^W
$\begin{pmatrix} u \\ d \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$
$\begin{pmatrix} c \\ s \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$
$\begin{pmatrix} t \\ b \end{pmatrix}$	$\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$

Table 4.1: Electric charge Q_{el} and weak isospin I_3^W of quarks

The common sequential structure of the leptons and quarks is soften called *the family structure*. In addition to the flavor quantum number, quarks have a hidden quantum number, *color*. We now explain several facts to require the necessity of the color quantum number.

Low-lying baryon states In the naive quark model there is a difficulty in constructing the low-lying baryon states. As one of the simplest examples, we consider the pion-nucleon resonance Δ^{++} which is of spin $\frac{3}{2}$. On the basis of its charge, isospin and strangeness, we see that Δ^{++} is made of three up quarks. If we consider the third component $J_3 = \frac{3}{2}$ of the total angular momentum for the Δ^{++} system, we find that all three u-quarks must have spins aligned up since all relative orbital angular momenta are required to vanish for the lowest state in the three-quark systems. Thus the $J_3 = \frac{3}{2}$ Δ^{++} state is given by

$$|\Delta^{++}, J_3 = \frac{3}{2}\rangle = |u\uparrow, u\uparrow, u\uparrow\rangle \tag{4.2}$$

But this assignment is not acceptable because the quarks are assumed to be fermions and hence the state has to be antisymmetric with respect to the exchange of the quarks. Moreover the quarks cannot occupy the same state according to Pauli exclusion principle. Then we are forced to assume the existence of a hidden degrees of freedom for quarks, *color* in order to distinguish three quarks which are otherwise identical. We need at least three different colors to discriminate these three quarks. It is then easy to construct totally antisymmetric state for Δ^{++} in place of Eq (4.2)

$$|\Delta^{++}, J_3 = \frac{3}{2}\rangle = \varepsilon_{ijk}\, |u^i\uparrow, u^j\uparrow, u^k\uparrow\rangle \tag{4.3}$$

where the indices i, j, k imply the quark colors (we assume exactly three colors, i.e., $i = 1, 2, 3$, the advantage of which will be seen later on) and ε_{ijk} is the totally antisymmetric tensor (the repeated indices are summed). The same argument is applied to other baryon states and the

difficulty is now circumvented with the introduction of the extra color degree of freedom for quarks.
Since we do not observe the color degree of freedom directly, we may assume that the hadronic phenomena be unaltered under the exchange of colors. We choose the corresponding symmetry group from the Lie groups and adopt $SU(3)$, the special unitary transformation in three dimensions. A single quark state is assigned to the fundamental triplet, **3**, of $SU(3)$. The state (4.3) is then a singlet, **1**, of $SU(3)$ because $\varepsilon_{ijk}\,|u^j\uparrow, u^k\uparrow\rangle$ constitutes the complex conjugate representation, $\mathbf{3}^*$, and gives rise to the singlet (4.3) when contracted with $|u^i\uparrow\rangle$ which belongs to **3**.
Note that quarks have a Dirac structure, by introducing the color degree of freedom, we are intuitively constructing a wave function out of products in different spaces and therefore we have to bear in mind in which space we are building our theory. In our case we are working in the color space. But as a first step to build a gauge theory for the quarks, is to take a look at the properties of the $SU(3)$ group.

4.1.1 SU(3) Lie algebra

The Lie algebra of $SU(3)$ is

$$\left[T^a, T^b\right] = i f^{abc} T^c \tag{4.4}$$

Where f^{abc} denotes the structure constants and where there are 8 generators T^a (recall that the number of generators o is obtained from $o = N^2 - 1 = 8$) out of which $r = N - 1 = 2$ are diagonal. We can introduce two representations for the generators, namely:

- **The fundamental representation**: which is given by the 3×3 matrices $T^a = \frac{\lambda^a}{2}$ with the *Gell-Mann* matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \tag{4.5}$$

One can notice that these matrices are hermitian and traceless,

$$\lambda_a^\dagger = \lambda_a \qquad Tr\lambda^a = 0 \tag{4.6}$$

Moreover, only λ_3 and λ_8 are diagonal and the block in the matrix denoted by r is the $SU(2)$ group matrices , since every $SU(N)$ group contains $SU(N-1)$ in its structure which is inherent in the generator T^a. Furthermore, one can show that

$$Tr(\lambda^a \lambda^b) = 2\delta^{ab} \tag{4.7}$$
$$\lambda^a_{ij}\lambda^a_{kl} = 2\left(\delta_{il}\delta_{kj} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) \qquad \text{(Fierz identity)} \tag{4.8}$$

the structure constants of $SU(3)$ are given by

$$f_{abc} = \frac{1}{4i} Tr\left(\left[\lambda_a, \lambda_b\right]\lambda_c\right) \tag{4.9}$$

which are antisymmetric in a, b and c. The numerical values are

$$\begin{cases} f_{123} = 1 \\ f_{458} = f_{678} = \frac{\sqrt{3}}{2} \\ f_{147} = f_{156} = f_{246} = f_{257} = f_{345} = f_{367} = \frac{1}{2} \\ f_{abc} = 0, \qquad \text{else} \end{cases} \tag{4.10}$$

- **Adjoint representation**: As in the case of $SU(2)$, the adjoint representation is given by the structure constants which, in this case, are 8×8 matrices:

$$(t^a)_{bc} = -if_{abc} \tag{4.11}$$

The multiplets (again built out of the fundamental representation) is given by the direct sums

$$|q\bar{q}\rangle : \qquad \mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8} \tag{4.12}$$

for the mesons, where the bar denotes antiparticle states, and for baryons

$$|qqq\rangle : \qquad \mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \tag{4.13}$$

Only in the $|q\bar{q}\rangle$ case we can have a singlet that, by definition, have no color. however, the other multiplets are colored and can thus not be observed and therefore the physical states present in nature have to come in *singlet* state. Working out the $SU(3)$ potential structure, one finds that an attractive QCD potential exists only for the singlet states, while the potential is repulsive for all other multiplets.
The development of QCD outlined so far can be summarized as follows: Starting from the observation that the nucleons have similar properties, we considered isospin and $SU(2)$ symmetry. To satisfy the Pauli exclusion principle, we had to introduce a new quantum number and with it a new $SU(3)$ symmetry of the Lagrangian. This in turn led us to multiplet structures where the colors singlet states correspond to mesons and baryons.

4.1.2 QCD as SU(3) gauge theory

Now we take a closer look at this $SU(3)$ transformation of a color triplet

$$|q\rangle = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \to |q'\rangle = \begin{pmatrix} q'_1 \\ q'_2 \\ q'_3 \end{pmatrix} = e^{ig_s\alpha_a T^a} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = U\,|q\rangle \tag{4.14}$$

where $g_s \in \mathbb{R}$ is used as rescaling of the group parameter α introduced previously and is called the strong coupling constant, it will also be used for the perturbative expansion. The reason of introducing it becomes clear in the context of gauge theories. In analogy to QED current,

$$j^\mu_{QED} = -e\bar{\psi}\gamma^\mu\psi \tag{4.15}$$

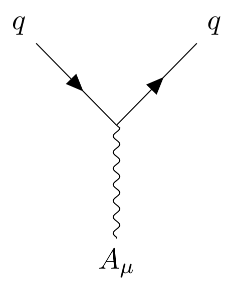
we introduce the **color** current, which is the conserved current associated with the $SU(3)$ symmetry,

$$j^\mu_a = g_s\bar{q}_i\gamma^\mu T^a_{ij}q_j \qquad a = 1\cdots 8 \tag{4.16}$$

In the same spirit, by looking at the QED interaction

$$\mathcal{L}^{int}_{QED} = -j^\mu_{QED}A_\mu = e\bar{\psi}\gamma^\mu\psi A_\mu \tag{4.17}$$

yielding the vertex

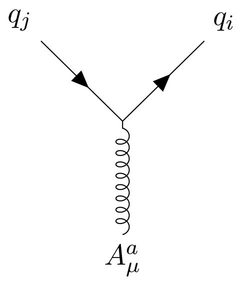


(4.18)

where we can see the photon, the electrically uncharged $U(1)$ gauge boson of QED, we postulate an interaction part if the QCD

$$\mathcal{L}_{QCD}^{int} = j^{\mu}_{QED} A_\mu = g_s \bar{q}_i \gamma^\mu T^a_{ij} q_j A^a_\mu \tag{4.19}$$

which translates in the vertex (that is not the only one of QCD as we shall see),



(4.20)

Now there are 8 $SU(3)$ gauge bosons A^a_μ for QCD, one for each possible value of a. They are called *gluons* and are themselves colored. Continuing with our analogy, we define *the covariant derivative of QCD*[1],

$$D_\mu = \partial_\mu \mathbb{1} + i g_s T^a A^a_\mu \tag{4.21}$$

and state that the QCD Lagrangian should have a term of the form

$$\tilde{\mathcal{L}}_{QCD} = \bar{q}\left(i \not{D}_\mu - m\right) q \tag{4.22}$$

Up to now, both QED and QCD look nearly identical. Their differences become crucial when we look at local gauge symmetries. Such a transformation can be written,

$$|q(x)\rangle \to |q'(x)\rangle = e^{i g_s \alpha_a(x) T^a} |q(x)\rangle \tag{4.23}$$

and we impose as before that the Lagrangian must be invariant under any such transformation. This is equivalent to imposing:

$$D'_\mu |q'(x)\rangle \overset{!}{=} e^{i g_s \alpha_a(x) T^a} D_\mu |q(x)\rangle \Leftrightarrow \langle \bar{q}'(x)| i \not{D} |q'(x)\rangle = \langle \bar{q}(x)| i \not{D} |q(x)\rangle \tag{4.24}$$

For $\alpha_a(x) \ll 1$, we cab expand the exponential map and keep only the first order term,

$$\begin{aligned} D'_\mu |q'(x)\rangle &= \left(\partial_\mu + i g_s T^c A'^c_\mu\right)\left(\mathbb{1} + i g_s \alpha_a T^a\right)|q(x)\rangle &(4.25)\\ &\overset{!}{=} \left(\mathbb{1} + i g_s \alpha_a T^a\right)\underbrace{\left(\partial_\mu + i g_s T^c A^c_\mu\right)}_{D\mu}|q(x)\rangle &(4.26)\end{aligned}$$

Making the Ansatz $A'^c_\mu = A^c_\mu + \delta A^c_\mu$ where $\left|\delta A^c_\mu\right| \ll \left|A^c_\mu\right|$ and expanding the former equation to first order in δA^c_μ (the term proportional to $\alpha_a(x)\delta A^c_\mu$ has also been ignored), we get,

$$\begin{aligned} & i g_s T^c \delta A^c_\mu + i g_s \left(\partial_\mu \alpha_a(x)\right) T^a + i^2 g_s^2 T^c A^c_\mu \alpha_a(x) T^a \overset{!}{=} i^2 g_s^2 \alpha_a(x) T^a T^c A^c_\mu \qquad (4.27)\\ & \Rightarrow T^c \delta A^c_\mu \overset{!}{=} -\left(\partial_\mu \alpha_a(x)\right) T^a + i g_s \left[T^a, T^c\right] \alpha_a(x) A^c_\mu \end{aligned}$$

or renaming the dummy indices and using the Lie algebra $\mathfrak{su}(3)$

$$\begin{aligned} T^a \delta A^a_\mu &= -\left(\partial_\mu \alpha_a(x)\right) T^a - i g_s f_{abc} T^a \alpha_b(x) A^c_\mu \qquad (4.28)\\ \Rightarrow A'^a_\mu &= A^a_\mu - \underbrace{\partial_\mu \alpha_a(x)}_{\text{like in QED}} \underbrace{- i g_s f_{abc} \alpha_b(x) A^c_\mu}_{\text{non-abelian part}} \end{aligned}$$

[1]Note that D_μ acts on color triplet and gives back a color triplet, ∂_μ does not mix the colors, whereas the other summand does (T^a is a 3×3 matrix)

Eq(4.28) describes the (infinitesimal) gauge transformation of the gluons field. In order for the gluon field to become physical, we need to introduce a kinematical term (depending on the derivatives of the field). Remember the photon term of QED

$$\mathcal{L}_{QED}^{Photon} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad\qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A\mu \tag{4.29}$$

where the last is gauge invariant. As we might expect from Eq(4.28) the non-abelian part will get us into trouble. Let's look at

$$\delta\left(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c\right) = -\partial_\mu\partial_\nu\alpha_a + \partial_\nu\partial_\mu\alpha_a - g_s f_{abc}\alpha_b\left(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c\right) \tag{4.30}$$

$$- g_s f_{abc}\left((\partial_\mu\alpha_b)A_\nu^c - (\partial_\nu\alpha_b)A_\mu^c\right) \tag{4.31}$$

We remark that the two first summands cancel each other and that the third looks like the $SU(3)$ transformation under the adjoint representation. We recall that

$$q_i \to q_i' = \left(\delta_{ij} + ig_s\alpha_a T_{ij}^a\right)q_j \qquad \text{(fundamental representation)} \tag{4.32}$$

$$B_a \to B_a' = \left(\delta_{ac} + ig_s\alpha_a t_{ac}^b\right)B_c \qquad \text{(adjoint representation)} \tag{4.33}$$

respectively, where

$$t_{ac}^b = if_{abc} = -if_{bac} \tag{4.34}$$

Hence if $F_{\mu\nu}^a$ transforms in the adjoint representation of $SU(3)$, we should have,

$$\delta F_{\mu\nu}^a \overset{!}{=} -g_s f_{abc}\alpha_b F_{\mu\nu}^c \tag{4.35}$$

We now make the Ansatz

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g_s f_{abc}A_\mu^b A_\nu^c \tag{4.36}$$

and prove it fulfills the above constraints. Finally, we get the full QCD Lagrangian

$$\mathcal{L}_{QCD} = -\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} + \bar{q}\left(i\not{D} - m_q\right)q, \tag{4.37}$$

The Lagrangian is by construction invariant under local $SU(3)$ gauge transformations. It is an other example of a non-abelian gauge theory, a so called **Yang-Mills theory**. Furthermore, if we consider the transformation of the Mesons and Baryons under $SU(3)$:

$$q \overset{SU(3)}{\overbrace{\to}} q' = Uq \Leftrightarrow q_i' = U_{ij}q_j \tag{4.38}$$

$$\bar{q} \overset{SU(3)}{\overbrace{\to}} \bar{q}' = \bar{q}U^\dagger \Leftrightarrow \bar{q}_i' = \bar{q}_k(U^\dagger)_{ki} \tag{4.39}$$

Applying the above transformations respectively to :

1. *Mesons*

$$|q\bar{q}\rangle = \sum_i q_i\bar{q}_i \to \sum_i q_i'\bar{q}_i' = \sum_{ijk}\underbrace{U_{ki}^\dagger U_{ij}}_{\delta_{kj}} q_j\bar{q}_k = \sum_j q_j\bar{q}_j = |q\bar{q}\rangle \tag{4.40}$$

2. *Baryons*

$$|qqq\rangle = \sum_{ijk}\epsilon_{ijk}q_iq_jq_k \to \sum_{i'j'k'}\underbrace{\epsilon_{ijk}U_{ii'}U_{jj'}U_{kk'}}_{\epsilon_{i'j'k'}det(U)} q_{i'}q_{j'}q_{k'} = |qqq\rangle \tag{4.41}$$

where in Eq(4.41) we have used $det(U) = 1$. As one can see that the above combination of quarks, and quark-antiquark, which are respectively dependent on δ_{ij} and ϵ_{ijk}, are invariant under the action of $SU(3)$.

4.1.3 QCD Feynman rules

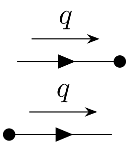
From the definition Eq(4.36) of $F^a_{\mu\nu}$, we see that

$$F^a_{\mu\nu}F^{\mu\nu}_a = \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a - g_s f_{ade} A^\mu_d A^\nu_e\right) \tag{4.42}$$

will have a much richer structure than in the case of QED[2].

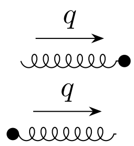
1. The external lines for a quark with momentum p, spin s and color c are denoted by

$$\begin{cases} Incoming : u^s(p)c \\ Outgoing : \bar{u}^s(p)c^\dagger \end{cases} \tag{4.43}$$

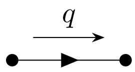


2. For an external gluon of Momentum p, polarization ϵ and a color a

$$\begin{cases} Incoming : \epsilon_\mu(q)a^\alpha \\ Outgoing : \epsilon^*_\mu(q)a^{\alpha *} \end{cases} \tag{4.44}$$



3. The propagator of the quark is represented by



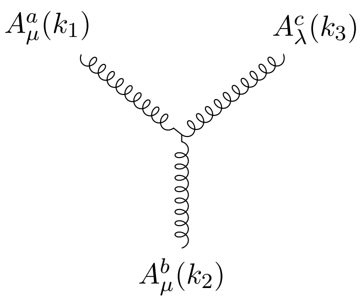
$$= \frac{i\left(\not{q} + m\right)}{q^2 - m^2} \tag{4.45}$$

4. Moreover we have -as in QED- a 2-gluon term $(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$ corresponding to *the gluon propagator*

$\mu\ a$  $\nu\ b$

$$= -\frac{g^{\mu\nu}}{k^2}\delta^{ab} \tag{4.46}$$

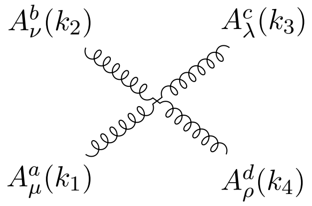
5. then we have a 3-gluon term $(-g_s f_{abc} A^b_\mu A^c_\nu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$ yielding a *3-gluon vertex*:



$$= g_s f_{abc}\left[g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu\right] \tag{4.47}$$

[2]Keep in mind that for antiparticles, the Feynman rules applies with e reversed momentum flow with respect the the particle- momentum flow

6. Finally we have also a 4-gluon term $(-g_s f_{abc} A_\mu^b A_\nu^c)(-g_s f_{ade} A_d^\mu A_e^\nu)$ yielding the *4-gluon vertex*



$$= -\,ig_s^2\Big[f_{abe}f_{cde}\left(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}\right) + f_{ade}f_{bce}\left(g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho}\right) + f_{ace}f_{bde}\left(g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\rho\lambda}\right)\Big]$$

Unlike in QED, gluons are bale to interact with themselves. This comes from the fact that the theory is non-abelian. As a consequence, there is no superposition principle for QCD: the field of a system of strongly interacting particles is not the sum of the individual fields. Thence, there is no plane wave solution to QCD problems, and we cannot make use of the usual machinery of Green's function and Fourier decomposition. Up to now there is no known solution.

4.1.4 Color factor

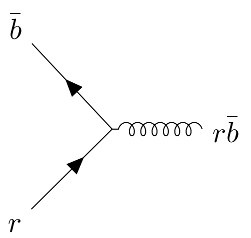
The next step is to investigate the interactions of colored particles. Electromagnetic interactions are mediated by photons, chromodynamic interactions by *gluons*.The strength of the chromodynamic force is set by the strong coupling constant

$$g_s = \sqrt{4\pi\alpha_s} \tag{4.48}$$

which may be thought of as the fundamental unit of color. Quarks come in three colors, "red" (r), "blue" (b), "green" (g). Thus the specification of a quark state in QCD requires not only the Dirac spinor $u^s(p)$, giving its momentum and spin, but also a three-element column vector c, giving its color

$$c = \begin{pmatrix}1\\0\\0\end{pmatrix} \text{ for red,} \qquad b = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ for blue,} \qquad g = \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ for green}$$

Typically, the quark color changes at a quark-gluon vertex, and the difference is carried off by the gluon. For example:



(4.49)

(In this diagram a red quark turned into a blue quark, emitting a red-antiblue gluon). Each gluon carries one unit of color and one of anticolor. It would appear, then, that there should be nine species of gluons $(r\bar{r}, r\bar{b}, r\bar{g}, b\bar{r}, b\bar{b}, b\bar{g}, g\bar{r}, g\bar{b}, g\bar{g})$. Such a nine-gluon theory is perfectly possible in principle, but it would describe a world very different from our own. In terms of color $SU(3)$ symmetry, these nine states constitute a "color octet"

$$|1\rangle = \frac{(r\bar{b} + b\bar{r})}{\sqrt{2}} \qquad |5\rangle = \frac{-i(r\bar{g} - g\bar{r})}{\sqrt{2}} \tag{4.50}$$

$$|2\rangle = \frac{-i(r\bar{b} - b\bar{r})}{\sqrt{2}} \qquad |6\rangle = \frac{(b\bar{g} + g\bar{b})}{\sqrt{2}} \tag{4.51}$$

$$|3\rangle = \frac{(r\bar{r} - b\bar{b})}{\sqrt{2}} \qquad |7\rangle = \frac{-i(b\bar{g} - g\bar{b})}{\sqrt{2}} \tag{4.52}$$

$$|4\rangle = \frac{(r\bar{g} + g\bar{r})}{\sqrt{2}} \qquad |8\rangle = \frac{(r\bar{r} + b\bar{b} - 2g\bar{g})}{\sqrt{6}} \tag{4.53}$$

and the *color singlet*:

$$|9\rangle = \frac{(r\bar{r} + b\bar{b} + g\bar{g})}{\sqrt{3}} \tag{4.54}$$

We are not concerned with isotopic spin, here, and we have used different linear combinations of states within the octet, which will simplify the notation later. If the singlet gluon existed, it would be as common and conspicuous as the photon? **Confinement** requires that all naturally occurring particles be color singlet, and this explain why the octet gluons never appear as free particles. But $|9\rangle$ is a color singlet, and if it exists as a mediator it should also occur as free particle. Moreover, it could be exchanged between two colors singlets (a proton and a neutron, say), giving rise to a long-range force with strong coupling[3] whereas in fact we know that the strong force is of very short range. In our world, then, there are evidently only eight kinds of gluons.
Like photons, gluons are massless particles of spin 1; they are represented by a polarization vector ϵ^μ, which is orthogonal to the gluons momentum p:

$$\epsilon^\mu \; p_\mu = 0 \qquad \text{Lorentz condition} \tag{4.55}$$

As before we adapt the Coulomb gauge:

$$\epsilon^0 = 0 \tag{4.56}$$

This spoils manifest Lorentz covariance, but it cannot be helped. To describe the color state of the gluon, we need in addition an eight element column vector as

$$a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{for } |1\rangle\,, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{for } |7\rangle \qquad \text{and so on}$$

[3]because the gluons are massless, they mediate a force of infinite range (same as in electrodynamics). In this sense the force between two quarks is actually **long** range. However confinement and the absence of a singlet gluon, conceals this from us. A singlet state (such as proton) can only emit and absorb a singlet (like a pion), so individual gluons cannot be exchanged between a proton and neutron. That's why the force we observe is of **short** range.

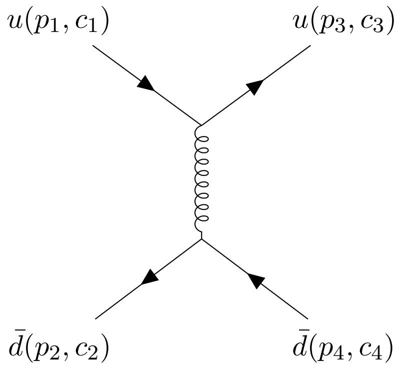
4.2 Quark-quark interactions

In this section we consider the interaction between two quarks in the lowest-order QCD. Of course we cannot observe quark-quark scattering directly in the laboratory (although a hadron-hadron is an indirect manifestation), so we won't be looking for cross section here. Instead we concentrate on the effective potentials between quarks -the QCD analog of the Coulomb potential in electrodynamics. Bear in mind that this is a perturbation theory calculation, valid only insofar as the coupling constant α_s is small. We cannot hope to get the confining term in the potential by this route- we are implicitly relying on asymptotic freedom, and all we are going to find is the short-range behavior. Nevertheless, we will obtain a very suggestive result: Quarks attract one another most strongly when they are in the color singlet *configuration* (indeed, in other arrangement they generally repel). At very short range, then, the color singlet is the "maximally attractive channel".

1. **Quark and antiquark**: we shall assume that they have different flavors, so the only diagram in lowest order is the following one representing for instance

$$u + \bar{d} \to u + \bar{d} \tag{4.57}$$

$\mathcal{M} =$



$$= \bar{u}(3) c_3^\dagger \left[-i \frac{g_s}{2} \lambda^\alpha \gamma^\mu \right] u(1) c_1 \left[\frac{-i g_{\mu\nu} \delta^{\alpha\beta}}{q^2} \right] \tag{4.58}$$

$$\times \bar{v}(2) c_2^\dagger \left[-i \frac{g_s}{2} \lambda^\beta \gamma^\nu \right] v(4) c_4$$

Thus

$$\mathcal{M} = \frac{-g_s^2}{4} \frac{1}{q^2} \left[\bar{u}(3) \gamma^\mu u(1) \right] \left[\bar{v}(2) \gamma_\mu v(4) \right] \left(c_3^\dagger \lambda^\alpha c_1 \right) \left(c_2^\dagger \lambda^\alpha c_4 \right) \tag{4.59}$$

This is exactly what we had for the electron-positron scattering, except that g_e is replaced by g_s and we have in addition *the color factor* :

$$f = \frac{1}{4} \left(c_3^\dagger \lambda^\alpha c_1 \right) \left(c_2^\dagger \lambda^\alpha c_4 \right) \tag{4.60}$$

$$= \frac{1}{4} \langle 3 | \lambda^\alpha | 1 \rangle \langle 2 | \lambda^\alpha | 4 \rangle \tag{4.61}$$

$$= \frac{1}{4} \sum_\alpha \left(\lambda^\alpha \right)_{c_3 c_1} \left(\lambda^\alpha \right)_{c_2 c_4} \tag{4.62}$$

The potential describing the $q\bar{q}$ interaction is, therefore, the same as that acting in electrodynamics between two opposite charges, only with a replaced by $f\alpha_s$:

$$V_{q\bar{q}}(r) = -f \frac{\alpha_s \hbar c}{r} \tag{4.63}$$

Now the color factor itself depends on the color state of the interacting quarks. From quark and antiquark we can make a color singlet (4.54) and a color (4.53) (all members of which yields the same f).

- **Color factor for octet configuration**
  A typical octet state (4.53) is $r\bar{b}$ (any of the others would do just as well). Here the incoming quark is red, and the incoming antiquark is antiblue. Because color is conserved, the outgoing quark must also be red and the antiquark antiblue. Thus

$$c_1 = c_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad c_2 = c_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

  and hence

$$f = \frac{1}{4}\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\right] = \frac{1}{4}\lambda^\alpha_{11}\lambda^\alpha_{22}$$

  A glance at the λ matrices reveals that the only ones with entries in the 11 and 22 positions are λ^3 and λ^8. So

$$f = \frac{1}{4}\left(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22}\right) = \frac{1}{4}\left[(1)(-1) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right] = -\frac{1}{6} \tag{4.64}$$

- **Color factor for singlet configuration**
  The color singlet color is (4.54)

$$\frac{(r\bar{r} + b\bar{b} + g\bar{g})}{\sqrt{3}}$$

  If the incoming quarks are in the singlet state (as they would be for a meson, let's say) the color factor is a sum of three terms:

$$\begin{aligned} f = \frac{1}{4}\frac{1}{\sqrt{3}}&\left\{\left[c_3^\dagger \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha c_4\right] + \left[c_3^\dagger \lambda^\alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \lambda^\alpha c_4\right]\right\} \\ + \frac{1}{4}\frac{1}{\sqrt{3}}&\left\{\left[c_3^\dagger \lambda^\alpha \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \lambda^\alpha c_4\right]\right\} \end{aligned} \tag{4.65}$$

  The outgoing quarks are necessary also in the singlet state, and we get nine terms in all, which can be written compactly as follows:

$$f = \frac{1}{4}\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}\lambda^a_{ij}\lambda^a_{ji} = \frac{1}{12}Tr(\lambda^\alpha\lambda^\alpha) \tag{4.66}$$

  (summing over i and j from 1 to 3, implied in the second expression). Now

$$Tr(\lambda^\alpha\lambda^\alpha) = 2\delta^{\alpha\beta} \tag{4.67}$$

  so with the summation over α

$$Tr(\lambda^\alpha\lambda^\alpha) = 16 \tag{4.68}$$

Evidently then for the color singlet

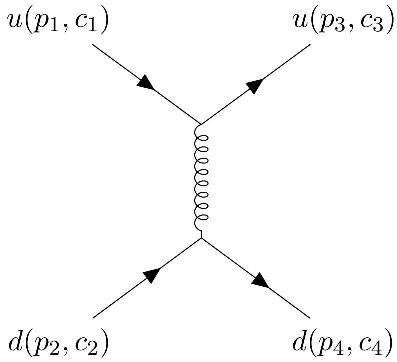
$$f = \frac{4}{3} \tag{4.69}$$

Putting equations 4.64 and 4.69 into equation 4.63 we conclude that the quark-antiquark potentials are

$$\begin{cases} V_{q\bar{q}}(r) = -\frac{4}{3}\frac{\alpha_s \hbar c}{r} & \text{color singlet} \\ V_{q\bar{q}}(r) = \frac{1}{6}\frac{\alpha_s \hbar c}{r} & \text{color octet} \end{cases} \tag{4.70}$$

From the signs we see that the force is attractive in the color singlet but repulsive for the octet. This helps to explain why quarks-antiquarks binding (mesons) occurs in the singlet configuration but not in the octet (which would have produced colored mesons).

2. **The quark-quark interaction**



(4.71)

We turn now to the interaction of two quarks. Again we shall assume that they have different flavors, so the only diagram (in lowest order is the one indicated in figure), representing for example

$$u + d \to u + d \tag{4.72}$$

the amplitude is

$$\mathcal{M} = \frac{-g_s^2}{4}\frac{1}{q^2}\left[\bar{u}\left(3\right)\gamma^\mu u\left(1\right)\right]\left[\bar{u}\left(4\right)\gamma_\mu u\left(2\right)\right]\left(c_3^\dagger \lambda^\alpha c_1\right)\left(c_4^\dagger \lambda^\alpha c_2\right) \tag{4.73}$$

This is the same as for the electron-muon scattering, except that g_e is replaced by g_s and there is a color factor

$$f = \frac{1}{4}\left(c_3^\dagger \lambda^\alpha c_1\right)\left(c_4^\dagger \lambda^\alpha c_2\right) \tag{4.74}$$

$$= \frac{1}{4}\left\langle 3\right|\lambda^\alpha\left|1\right\rangle\left\langle 4\right|\lambda^\alpha\left|2\right\rangle \tag{4.75}$$

$$= \frac{1}{4}\sum_\alpha \left(\lambda^\alpha\right)_{c_3 c_1}\left(\lambda^\alpha\right)_{c_4 c_2} \tag{4.76}$$

The potential, therefore, takes the same form as that for like charges in electrodynamics:

$$V_{qq}(r) = f\,\frac{(\alpha_s \hbar c)}{r} \tag{4.77}$$

Again the color factor depends on the configuration of the quarks. From two quarks, however, you can't make a singlet and an octet (as for $q\bar{q}$), we obtain a triplet (the antisymmetric combinations):

$$\begin{cases} \frac{(rb-br)}{\sqrt{2}} \\ \frac{(bg-gb)}{\sqrt{2}} \\ \frac{(gr-rg)}{\sqrt{2}} \end{cases} \qquad \text{triplet}$$

and a sextet (the antisymmetric combinations, since in group theoretical language: $3\otimes\bar{3} = 1 \oplus 8, \text{but} 3 \otimes 3 = \bar{3} \oplus 6$)

$$\begin{cases} rr, & bb, & gg, \\ \frac{(rb+br)}{\sqrt{2}}, & \frac{(bg+gb)}{\sqrt{2}}, & \frac{(gr+rg)}{\sqrt{2}} \end{cases}$$

- **Color factor for sextet configuration** A typical sextet state is rr (note that we will get the same result even when we work on any of the others). In this case

$$c_1 = c_2 = c_3 = c_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and hence

$$f = \frac{1}{4}\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] = \frac{1}{4}\left(\lambda^\alpha_{11}\lambda^\alpha_{22}\right)$$

$$\begin{aligned} &= \frac{1}{4}\left[\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11}\right] = \frac{1}{4}\left[(1)(1) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\right] \\ &= \frac{1}{3} \end{aligned} \tag{4.78}$$

- **Color factor for triplet configuration**
A typical triplet state is $(rb - br)/\sqrt{2}$, so

$$\begin{aligned} f = \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} &\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \\ &- \left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right] \\ &- \left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] \\ &- \left[\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right]\left[\begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \lambda^\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{8}\left\{\lambda^\alpha_{11}\lambda^\alpha_{22} - \lambda^\alpha_{21}\lambda^\alpha_{12} - \lambda^\alpha_{12}\lambda^\alpha_{21} + \lambda^\alpha_{22}\lambda^\alpha_{11}\right\} \\
&= \frac{1}{4}\left\{\lambda^\alpha_{11}\lambda^\alpha_{22} - \lambda^\alpha_{12}\lambda^\alpha_{21}\right\} \\
&= \frac{1}{4}\left(\lambda^3_{11}\lambda^3_{22} + \lambda^8_{11}\lambda^8_{22} - \lambda^1_{12}\lambda^1_{21} + \lambda^2_{12}\lambda^2_{21}\right) \\
&= \frac{1}{4}\left(-1 + \frac{1}{3} - 1 - 1\right) = -\frac{2}{3} \qquad (4.79)
\end{aligned}
$$

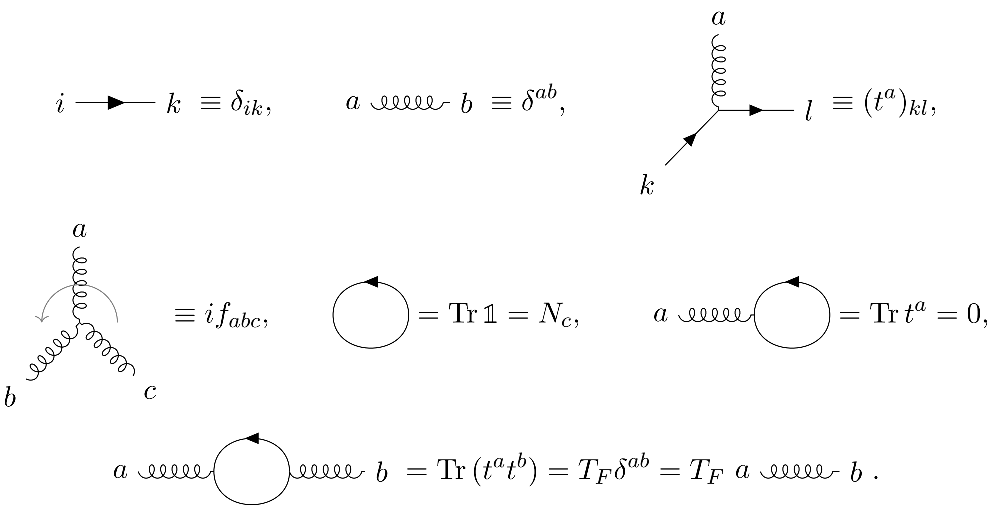
Putting equations (4.78) and (4.79) into equation (4.77), we conclude that the quark-quark potentials are

$$
\begin{cases}
V_{qq}(r) = -\frac{2}{3}\frac{(\alpha_s \hbar c)}{r} & \text{color triplet} \\
V_{qq}(r) = \frac{1}{3}\frac{(\alpha_s \hbar c)}{r} & \text{color sextet}
\end{cases}
$$

In particular, the signs indicate that the force is attractive for the triplet and repulsive for the sextet. Of course, that's not too helpful as it stands, because neither combination occurs in nature. However, it does have interesting implications for the binding of three quarks. This time we can make a singlet (completely antisymmetric), a decuplet (completely symmetric), and two octets (of mixed symmetry). Since the singlet is completely antisymmetric, every pair of quarks is in the antisymmetric triplet state- the attractive channel. In the decuplet, every pair is in the symmetric sextet state-they repel. As for two octets, some pairs are triplet and some are sextet: we expect some attraction, then, some repulsion. Only in the singlet configuration, though, do we get complete mutual attraction of the three quarks. Again, this is a comforting result: as in the case of mesons, the potential is most favorable for binding when the quarks are in the color singlet configuration.

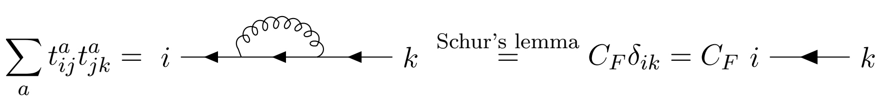
4.2.1 Chromatica

SU(3) color rules for color factors Color degrees of freedom factorize from kinematics. We will use the following diagrammatic rules for the color algebra:

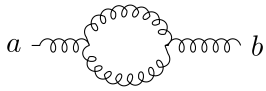
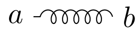


We can use these rules to compute the Casimir:

- in the fundamental representation:



- in the adjoint representation, defined $(F^a)_{bc} = -if_{abc}$:

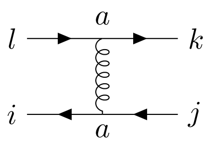
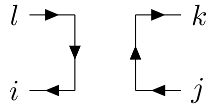
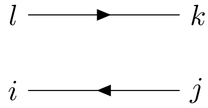
$\mathrm{Tr}\,(F^aF^b) =$  $\overset{\text{Schur's lemma}}{=} C_A\delta^{ab} = C_A$ 

This set of rules gives rise to several remarkable identities.

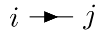
Example 1: fundamental identity. Let's write

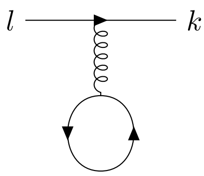
$$(t^a)^i_j(t^b)^k_l = A(\delta^i_l\delta^k_j + B\delta^i_j\delta^k_l),$$

since this is the only possible object I can form with these indeces. Diagrammatically:

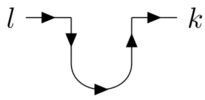
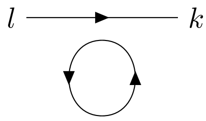
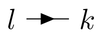
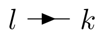
 $= A\Big($  $+ B$  $\Big).$ (4.80)

We determine A and B by projection:

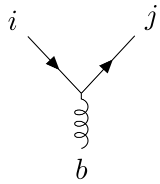
- we multiply (4.80) with δ^j_i, identified by the chromatica rules with the diagram . The left hand side becomes:

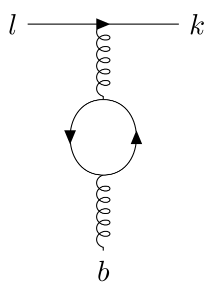
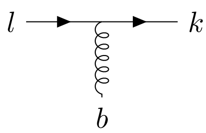
   $= 0,$

  while the right hand side becomes:

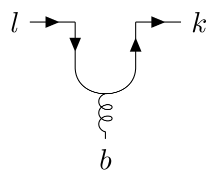
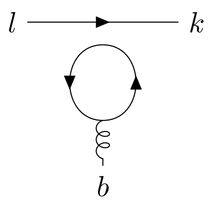
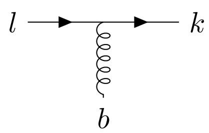
  $A\Big($  $+ B$  $\Big) = A($  $+ N_cB$  $),$

  from which we get $A(1 + N_cB) = 0$;

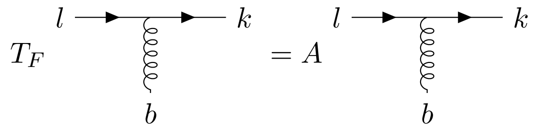
- we multiply (4.80) with $(t^b)^i_j$, identified with the diagram . The left hand side becomes:

   $= T_F$ 

  while the right hand side becomes:

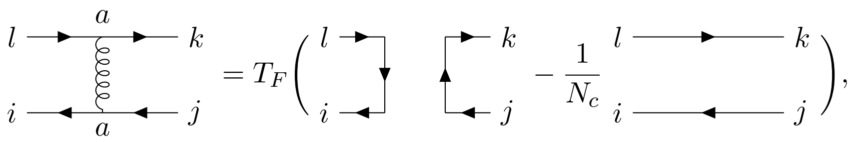
  $A\Big($  $+ B$  $\Big) = A$ 

from which we get



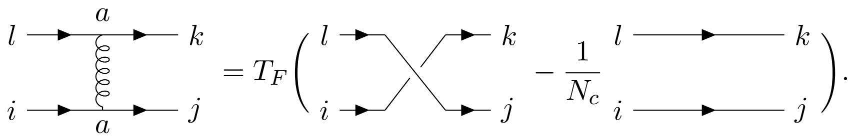
.

Hence, we have $A = T_F$ and $B = -\frac{1}{N_c}$, and so we have the *fundamental identity*



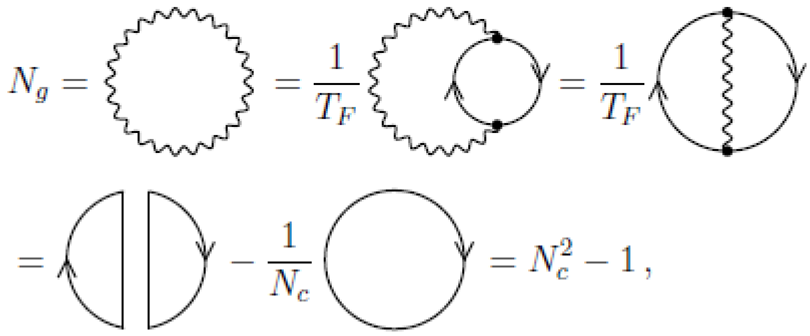
(4.81)

which allow us to eliminate gluons from diagrams.
An analogous identity is obtained when i and j are exchanged:

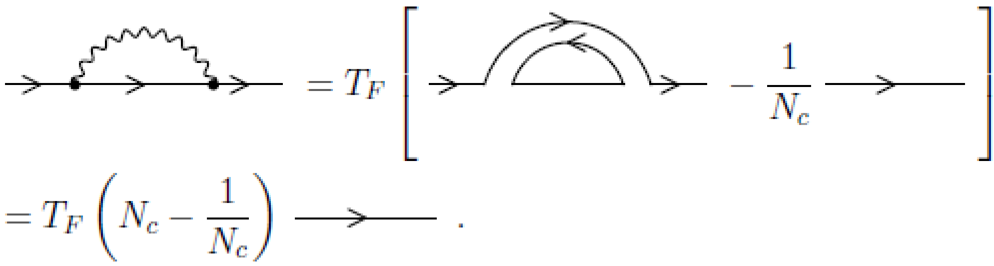


(4.82)

Example 2: relation between N_g and N_c. An example of the application of the fundamental identity: (taken from A.Grozin, "Lectures on QED and QCD")



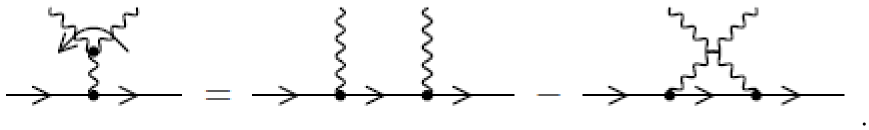
Example 3: the Casimir C_F. Another important example (from A.Grozin, "Lectures on QED and QCD") is



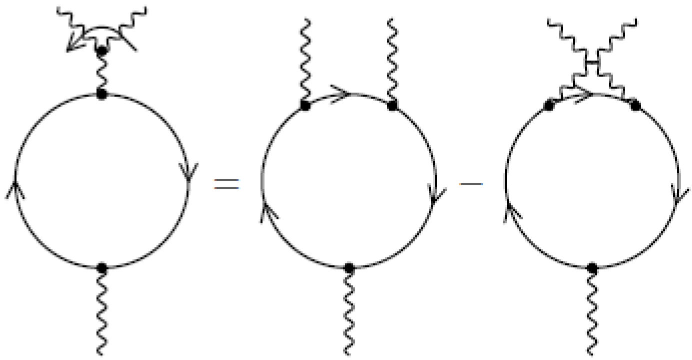
from which follows:

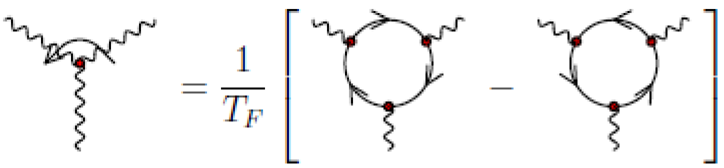
$$C_F = T_F\Big(N_c - \frac{1}{N_c}\Big). \tag{4.83}$$

Example 4: second fundamental identity Let's start from $\left[t^a, t^b\right] = if^{abc}t^c$. Using the chromatica rules, we write the commutator definition $\left[t^a, t^b\right] = t^a t^b - t^b t^a$ in the diagrammatic form (from A. Grozin, "Lectures on QED and QCD"):



Closing the quark line onto a gluon (from A. Grozin, "Lectures on QED and QCD"):



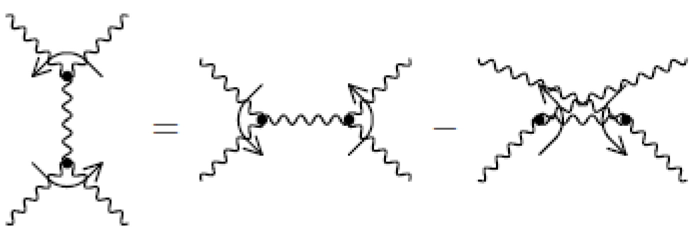


that is equivalent to:

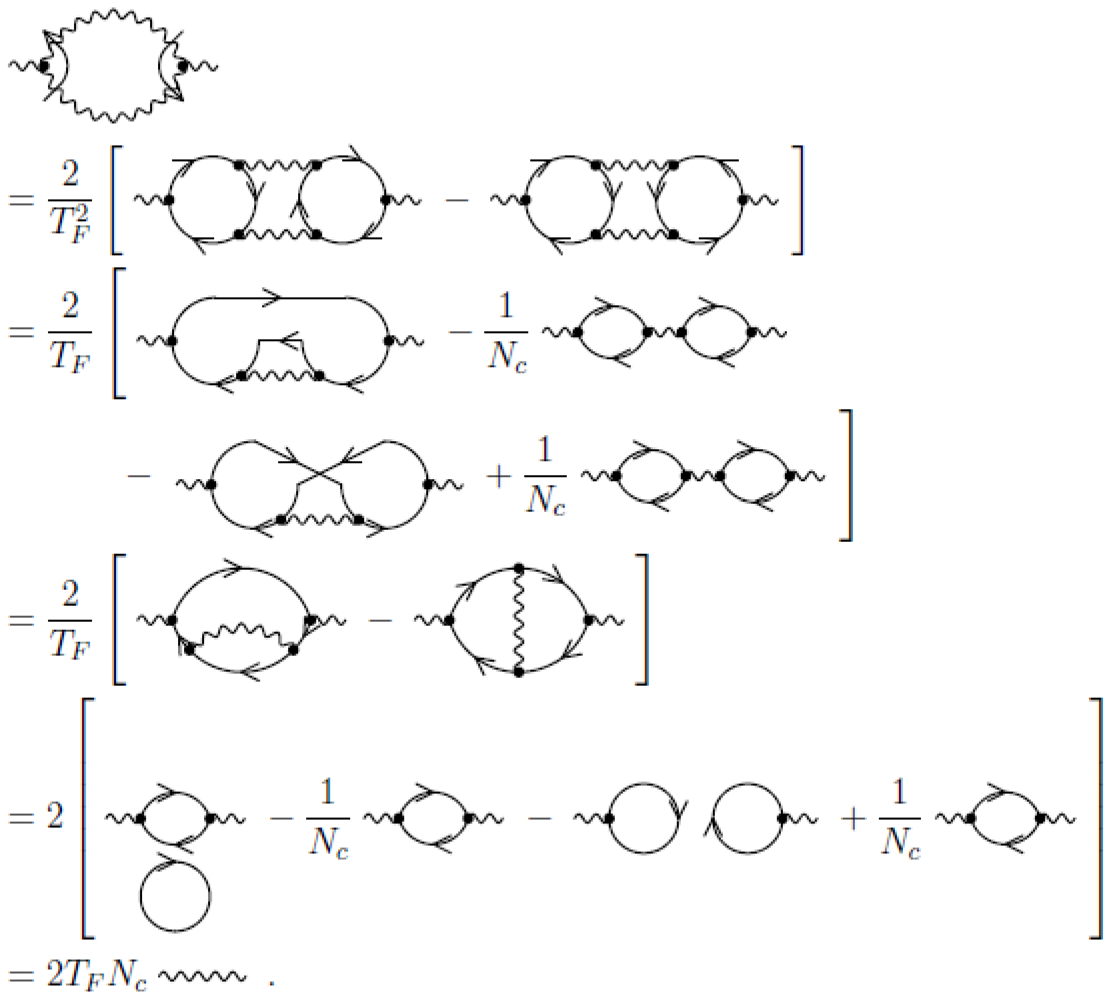
$$if^{abc} = \frac{1}{T_F}(\mathrm{Tr}\,(t^a t^b t^c) - \mathrm{Tr}\,(t^a t^c t^b)) \tag{4.84}$$

Two other important examples, both from A. Grozin, "Lectures on QED and QCD", are:

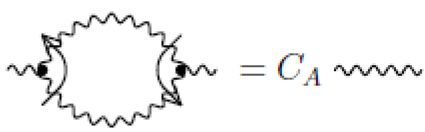
Example 5: Jacobi identity. The following identity is a diagrammatic representation of the Jacobi identity:



Example 6: the Casimir C_A.



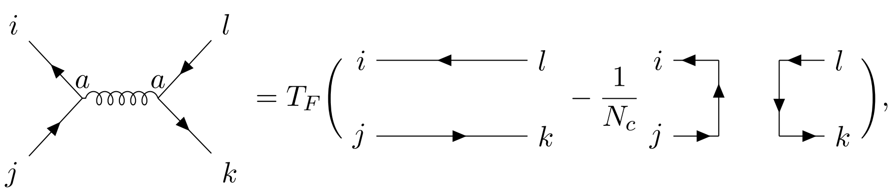
Remembering the Casimir relation:



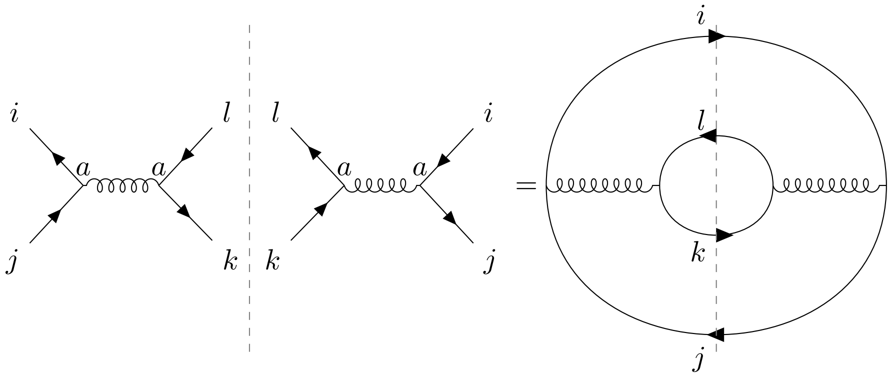
we conclude

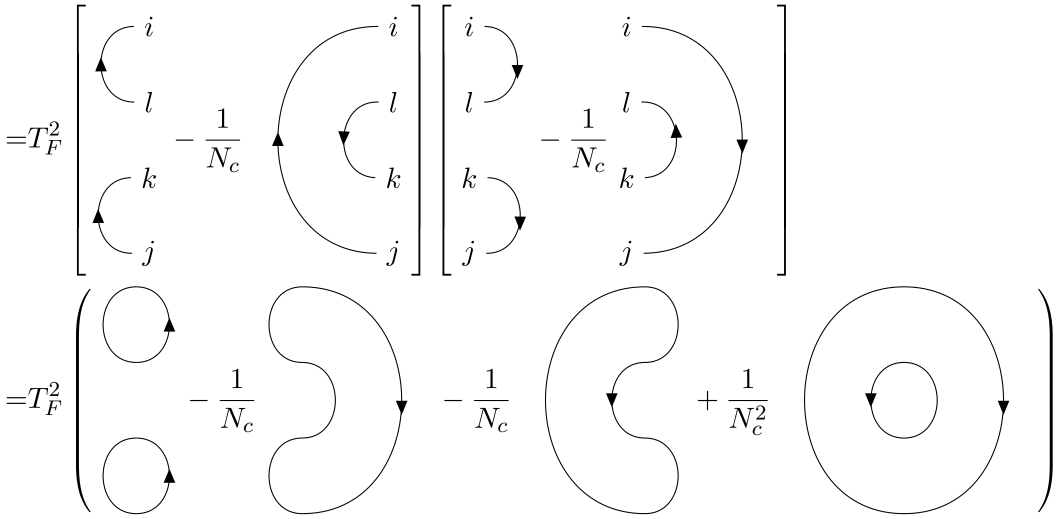
$$C_A = 2T_F N_c. \tag{4.85}$$

Example 7: color coefficients for $q\bar{q} \to q\bar{q}$ An interesting example to work out in detail is the color coefficient in $q\bar{q} \to q\bar{q}$. One of the involved diagram is the one we have considered in the fundamental identity:



so, among other diagrams (square and interference terms), in $\mathcal{M}$ there is a contribution

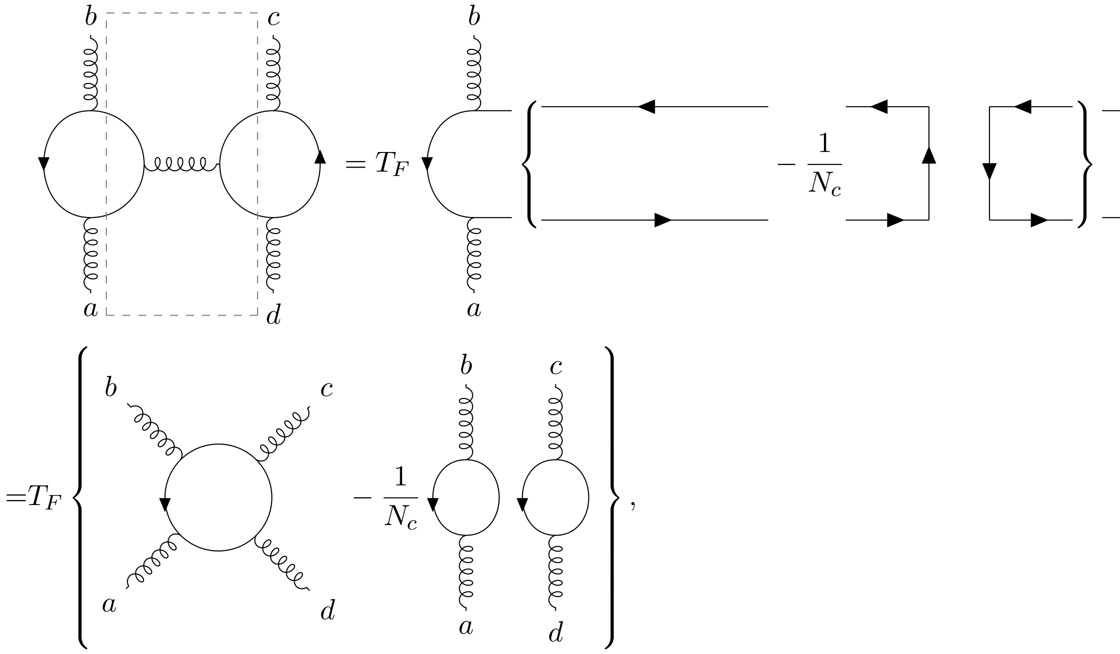




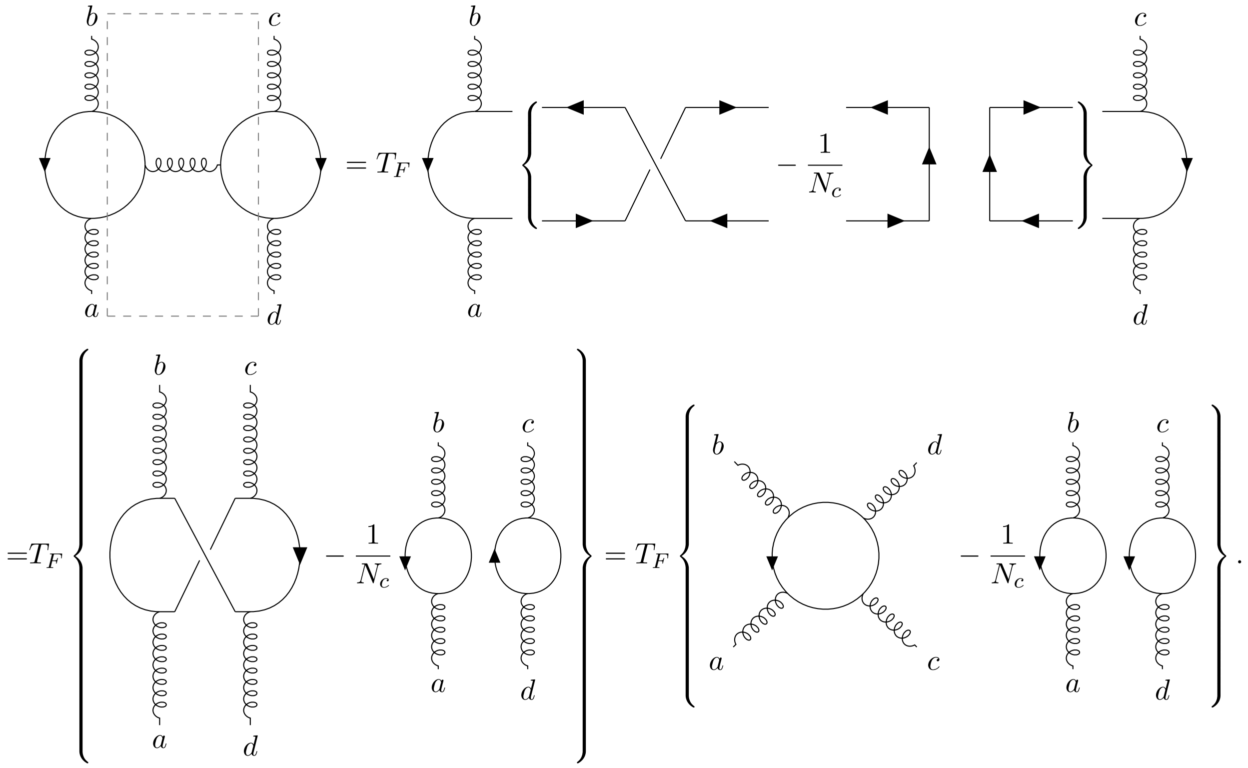
$$=T_F^2\Big(N_c^2 - \frac{1}{N_c}N_c - \frac{1}{N_c}N_c + \frac{1}{N_c^2}N_c^2\Big) = T_F^2(N_c^2 - 1),$$

and in the QCD case $T_F = \frac{1}{2}$, $N_c = 3$.
This is a good example of the computational power of the chromatica rules.

Example 8: application to the four-gluons scattering. Our last example is a couple of decomposition rule that will be useful in the study of the four-gluons scattering:



(4.86)



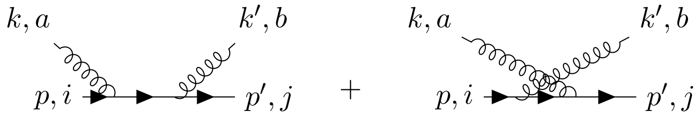
(4.87)

4.3 Gluon Compton scattering

As an example of typical QCD calculations, we sketch the calculation of the gluon Compton scattering:

$$g(k) + q(p) \to g(k') + q(p') \tag{4.88}$$

There are at first sight two Feynman diagrams coming into the calculations



(4.89)

which yields the following scattering matrix elements,

$$-i\mathcal{M}_{fi} = -ig_s^2 \left[\bar{u}(p')\not{\epsilon}^*(k')\frac{1}{\not{p}+\not{k}-m}\not{\epsilon}(k)u(p)T^b_{jl}T^a_{li} + \bar{u}(p')\not{\epsilon}(k)\frac{1}{\not{p}-\not{k}'-m}\not{\epsilon}^*(k')u(p)T^b_{jl'}T^a_{l'i} \right] \tag{4.90}$$

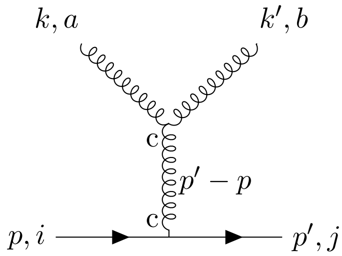
We start by checking the gauge invariance ($\mathcal{M}_{fi}$ must vanish under the substitution $\epsilon_\mu(k) \to k_\mu$)

$$-i\mathcal{M}_{fi} = -ig_s^2\bar{u}(p')\not{\epsilon}^*(k')u(p)\left(T^b_{ji}T^a_{li} - T^a_{jl'}T^b_{l'i}\right) \tag{4.91}$$

where

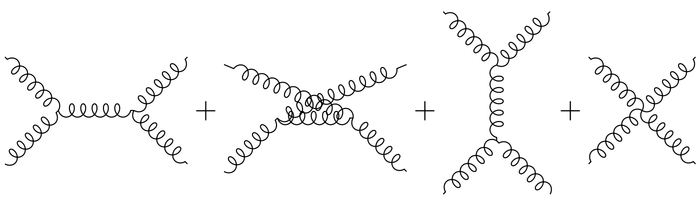
$$T^b_{ji}T^a_{li} - T^a_{jl'}T^b_{l'i} = \left[T^a, T^b\right] = if_{bac}T^c_{ji} \neq 0 \tag{4.92}$$

So we need another term, which turns out to be the one corresponding to the Feynman diagram,



(4.93)

whose color factor c is the same but with an other opposite contribution. The calculation of the gluon-gluon scattering goes analogously. We need to consider the graphs,

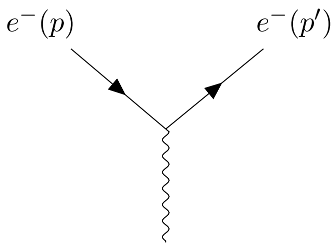


At this stage we note two features specific to the strong interaction, which we are going to handle in more details in a moment:

- **Confinement**: At low energies (large distances), the coupling becomes very large so that the perturbative treatment is no longer valid, and the process of hadronization becomes important. This is the reason why we cannot observe color directly.
- **Asymptotic freedom**: At high energy (small distances) the coupling becomes negligible, and the quarks and gluons can move almost freely.

4.4 QCD coupling constant

To leading order, a typical QED scattering process takes the form

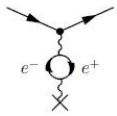


(4.94)

with $q^2 = (p' - p)^2 \leq 0$ In the Coulomb limit (long distance, low momentum transfer), the potential takes the form

$$V(R) = -\frac{\alpha}{R} \qquad\qquad R \sim \frac{1}{m_e} \approx 10^{-11}\,[cm] \tag{4.95}$$

When $R \leq m_e^{-1}$, quantum effects become important (loop interactions, also known as vacuum polarization), since the next leading order (NLO) diagram,



starts to play a significant (measurable role). This results in a change of the potential to,

$$V(R) = -\frac{\alpha}{R}\left[1 - \frac{2\alpha}{3\pi}\ln(\frac{1}{m_e R}) + \mathcal{O}(\alpha^2)\right] = -\frac{\bar{\alpha}(R)}{R} \tag{4.96}$$

where $\bar{\alpha}(R)$ is called the effective coupling.
We can understand the effective coupling in analogy to a solid state physics example: in an insulator an excess of charge gets screened by the polarization of the nearby atoms. Here we create e^+e^- pairs out of the vacuum, hence the name *vacuum polarization*.
As we can see from Eq (4.96), the smaller the distance $R \leq m_e^{-1}$, the bigger the observed "charge" $\bar{\alpha}(R)$. What we call the electric charge e (or the fine structure α) is the limiting value for the very large distances or low momentum transfer.

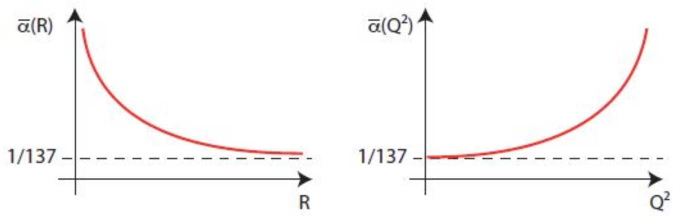
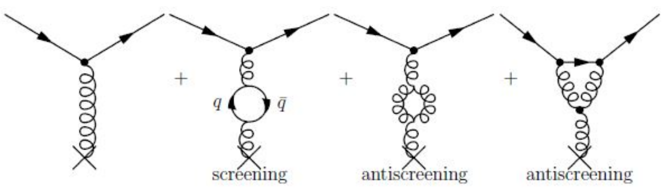


Figure 4.1: Evolution of the effective electromagnetic coupling with distance and energy ($Q^2 = -q^2$).

For example the measurements done at LEP show that, $\bar{\alpha}(Q^2 = m_Z^2) \approx \frac{1}{128} > \alpha$. In the case of QCD, we have at NLO, the following diagrams



We can picture the screening/antiscreening phenomenon as follows,

For QCD, the smaller the distance R (or the bigger the energy Q^2), the smaller the observed coupling $\bar{\alpha}_s(R)$. At large distances, $\bar{\alpha}_s(R)$ becomes comparable with unity, and the perturbative approach breaks down. The region concerning confinement and asymptotic freedom are also shown

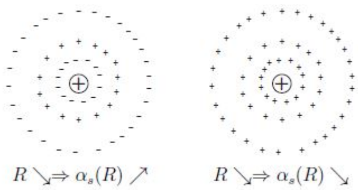


Figure 4.2: Screening and antiscreening.

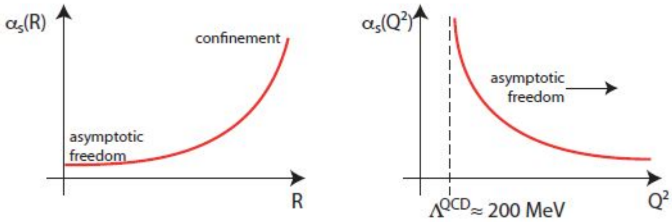


Figure 4.3: Evolution of the effective strong coupling with distance and energy ($Q^2 = -q^2$).

4.5 QCD in e^+e^-annihilations

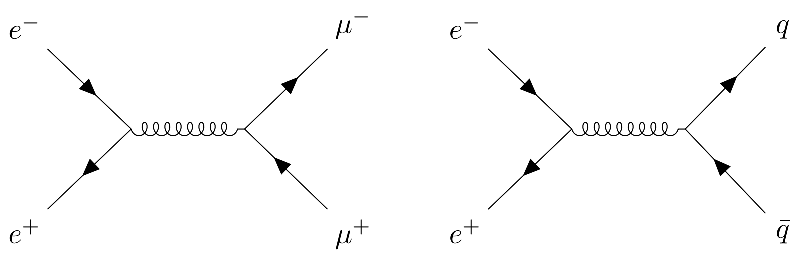
Now that we introduced QCD as a SU(3) gauge theory, we can continue this discussion and consider QCD processes following e^+e^- annihilations. The main focus is on the definition and application of observables linking theoretical predictions with measurable quantities: Jets and event shapes are discussed; the applications include measurements of the parton spins, the strong coupling constant, and the QCD color factors.

4.5.1 The basic Process: $e^+e^- \to q\bar{q}$

In chapter 1 1.120 we have calculated the cross section for $e^+e^- \to \mu^+\mu^-$ and found

$$\sigma^{e^+e^- \to \mu^+\mu^-} = \frac{4\pi\alpha_{em}^2}{3s} = \frac{86.9nbGeV^2}{s} \tag{4.97}$$

where the finite electron and muon masses have been neglected and s is the incoming electron energy $s = \frac{E}{2}$. Here, we consider the basic process $e^+e^- \to q\bar{q}$. In principle the same Feynman diagrams contributes:



(4.98)

The only differences are the fractional electric charges of the quarks and the fact that the quarks appears in $N_c = 3$ different colors which cannot be distinguished by measurement. Therefore, the cross section is increased by a factor N_c. For the quark-antiquark case one thus finds (for $m_q = 0$)

$$\sigma^{e^+e^- \to q\bar{q}} = \frac{4\pi\alpha_{em}^2}{3s} e_q^2 N_c = \frac{86.9nbGeV^{E}2}{s} e_q^2 N_c \tag{4.99}$$

We assume

$$\sum_q \sigma^{e^+e^-\to q\bar{q}} = \sigma^{e^+e^-\to hadrons} \tag{4.100}$$

in other words, the produced quark-antiquark will always hadronize. With Eq(4.99) and Eq(4.97, neglecting mass effects and gluon as well as photon radiation, we find the following ratio:

$$R = \frac{\sigma^{e^+e^-\to hadrons}}{\sigma^{e^+e^-\to\mu^+\mu^-}} \tag{4.101}$$

The sum over all flavors that can be produced at the available energy. For E_{CM} below the Z peaks and above the Υ resonance, we expect

$$R = N_c \sum_q e_q^2 = N_c \left[\underbrace{\left(\frac{2}{3}\right)^2}_{u} + \underbrace{\left(-\frac{1}{3}\right)^2}_{d} + \underbrace{\left(-\frac{1}{3}\right)^2}_{s} + \underbrace{\left(\frac{2}{3}\right)^2}_{c} + \underbrace{\left(-\frac{1}{3}\right)^2}_{b} \right] = N_c \frac{11}{9} \tag{4.102}$$

This is in good agreement with the data for $N_c = 3$ which confirms that there are three colors. At the Z peak one also has to include coupling to the Z boson which can be created from the e^+e^- pair instead of a photon. The small remaining difference visible is because of QCD corrections for gluon radiation.

Appendix A

The geometry of gauge fields

The gauge transformations that we have been considering have deliberately been chosen to be different at different points in space, so it is natural that we should try to cast our work into an explicitly geometrical language, which we shall do in this section.

A.0.1 Parallel transport and covariant derivative

To begin with, let us rewrite the formula for rotation of an isovector through an angle θ in isospace:

$$\phi \to \phi' = \phi - \theta \times \phi \tag{A.1}$$

This is the infinitesimal form of

$$\phi \to \phi' = e^{iI\theta}\phi \tag{A.2}$$

where the matrix generators I are given by

$$I_1 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 0 & i & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and as may easily be seen, have the matrix elements

$$(I_i)_{mn} = -i\varepsilon_{imn} \tag{A.3}$$

where ε_{imn} is the usual Levi-Civita symbol. Expanding (A.2) to first order θ, gives (with summation over repeated indices)

$$\begin{aligned} \phi' &= (1 + iI_i\theta_i)_{mn}\phi_n \\ &= (\delta_{mn} + \varepsilon_{imn}\theta_i)\,\phi_n \\ &= \phi_m - \varepsilon_{min}\theta_i\phi_n \end{aligned} \tag{A.4}$$

Now let θ depend on x^μ and write (A.2) as

$$\phi \to \phi' = e^{iI\theta(x)}\phi \tag{A.5}$$

The matrices I are representations of the generators of SO(3) (or SU(2)) and hence obey:

$$[I_i, I_j] = i\varepsilon_{ijk}I_k = C_{ijk}I_k \tag{A.6}$$

This equation identifies $i\varepsilon_{ijk}$ as the structure constants C_{ijk} of the group SU(2). Since the generators M_t of any group obey the Jacobi identity:

$$[[M_i, M_j], M_k] + [[M_j, M_k], M_i] + [[M_k, M_i], M_j] = 0 \tag{A.7}$$

The structure constants C_{ijk} which are totally antisymmetric in i, j, k obey the condition

$$C_{lim}C_{mjk} + C_{ljm}C_{mki} + C_{lkm}C_{mij} = 0 \tag{A.8}$$

Returning to SU(2), the explicit representation (A.3) with matrix elements

$$(I_i)_{mn} = C_{imn} \tag{A.9}$$

is called the adjoint representation of the group. From our treatment of the spin, we know that an isospinor ψ transforms like

$$\psi \to \psi' = e^{\frac{i}{2}\tau\theta(x)}\psi = S(x)\psi \tag{A.10}$$

where S(x) is a 2×2 matrix, and the Pauli matrices τ_i or more precisely $\frac{\tau_i}{2}$ obey the relation (A.6) as required for a representation of the group. Let us now write the general n-dimensional case as

$$\psi \to \psi' = e^{iM^a\theta^a(x)}\psi = S(x)\psi \tag{A.11}$$

where ψ is an n-component vector and M^a is an $n \times n$ matrices representing the generators and having the commutation relation (A.6). It is clear that $\partial_\mu\psi$ does not transform covariantly:

$$\partial_\mu\psi' = S(\partial_\mu\psi) + (\partial_\mu S)\psi \tag{A.12}$$

The problem is, we are preforming a different "isorotation" at each point in space, which we may be expressed by saying that the "axes" in isospace are oriented differently at each point. The reason $\frac{\partial\psi}{\partial x^\mu}$ is not covariant is that $\psi(x)$ and $\psi(x + dx) = \psi(x) + d\psi$ are measured in different coordinate systems. This is sketched in Fig(A.1); ψ is a field and so has different values at different points, but $\psi(x)$ and $\psi(x + dx) = \psi(x) + d\psi$ are measured with respect to different axes. The quantity $d\psi$, then, carries information about the variation of the field ψ itself with distance, but also about the rotation of the axes in isospace on moving from x to $x + dx$. To form a properly covariant derivative, we should compare $\psi(x + dx)$ not with $\psi(x)$ but with the value $\psi(x)$ would have if it were carried from x to $x + dx$ keeping the axes in isospace fixed- this we may call *parallel transport in isospace*, and it is illustrated in Fig(A.2). The resulting vector is denoted $\psi + \delta\psi$. Note that $\delta\psi$ is not zero, because $\psi + \delta\psi$ is not the vector which, when measured in the local iso-coordinate system at $x + dx$, is equal ("parallel") to the vector ψ measured in the local iso-coordinate system at x. these coordinate system are not the same, so neither are the vectors.

What is $\delta\psi$? It is sensible to assume that it is proportional to ψ itself, and also to dx^μ, so the distance over which the vector is carried, so we put

$$\delta\psi = igM^aA^a_\mu dx^\mu\psi \tag{A.13}$$

where g is a number put in to get the dimension right and A^a_μ is an additional field or potential- Feynman calls it a "universal influence"- which tells us to what extent the axes in isospace differ from point to point.

We now have two vectors at the point $x + dx$; $\psi + d\psi$ and $\psi + \delta\psi$. The true derivative of ψ is given by the difference between these vectors

$$\begin{aligned} D\psi &= (\psi + d\psi) - (\psi + \delta\psi) \\ &= d\psi - \delta\psi \\ &= d\psi - igM^aA^a_\mu dx^\mu\psi \\ \frac{D\psi}{dx^\mu} &= D_\mu\psi = \partial_\mu\psi - igM^aA^a_\mu dx^\mu\psi \end{aligned} \tag{A.14}$$

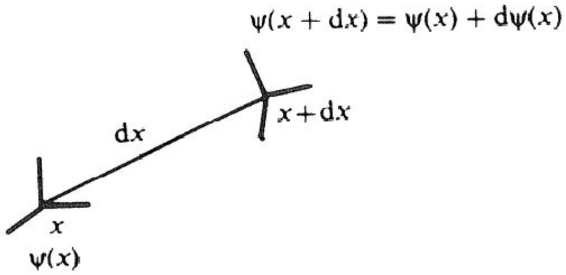


Figure A.1: $d\psi$ carries information about the variation in ψ as well as the change in coordinate axes between x and dx

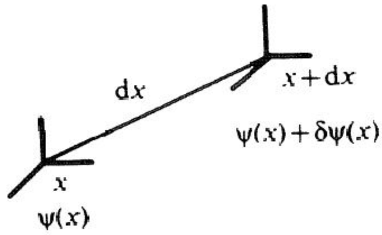


Figure A.2: $\delta\psi$ is defined by parallel transport, see text

The above equation defines the covariant derivative of an arbitrary field ψ transforming under an arbitrary group whose generators ares represented by the matrices M^a appropriate to the representation of ψ. The "derivation" of covariant derivatives for an internal gauge group given above is modeled as the derivation of the covariant derivative of a vector in general relativity, where in the case of a curved space-time, it is the spaces-time axes themselves which vary from point to point. In the case of a (contravariant) vector V^μ, its covariant derivative is:

$$D_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\lambda\nu} V^\lambda \tag{A.15}$$

The quantities $\Gamma^\mu_{\lambda\nu}$ are called *connection coefficients*, clearly play a similar roles to the vector potentials A^a_μ. They are called connection coefficients because they *connect* the components of a vector at one point with its components at nearby point, the vector being transported between the points by "parallel transport", as explained above. Because of this similarity, some physicists refer to A^a_μ as the connection.
Now we know how the generic vector ψ transforms when we do a rotation in isospace; $\psi \to S\psi$. Since $D_\mu\psi$ is the covariant derivative of ψ, it transforms in the same way, so we have

$$D_\mu\psi \to D'_\mu\psi = SD_\mu\psi \tag{A.16}$$

For simplicity, let us define the matrix

$$A_\mu = M^a A^a_\mu \tag{A.17}$$

so that (A.14) reads:

$$D_\mu\psi = (\partial_\mu - igA_\mu)\,\psi \tag{A.18}$$

Transforming to new isoframe gives, in view of (A.16),

$$\left(\partial_\mu - igA'_\mu\right)\psi' = S\left(\partial_\mu - igA_\mu\right)\psi \tag{A.19}$$

Putting $\psi' = S\psi$ is this equation then gives

$$A'_\mu = SA_\mu S^{-1} - \frac{i}{g}\left(\partial_\mu S\right)S^{-1} \tag{A.20}$$

This is the rule for the gauge transformation of the potential. Note the characteristic inhomogeneous term on the right. For the group U(1), $S = e^{-i\theta}$, $\partial_\mu S = -i\,(\partial_\mu \theta)\, e^{-i\theta}$ and (A.20) gives ($g \to e$ and $M = -1$)

$$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu \theta \tag{A.21}$$

In the case of SU(2), we have

$$S = \exp\left(\frac{i}{2}\tau\theta\right) \tag{A.22}$$

$$\partial_\mu S = \frac{i}{2}\tau \cdot \partial_\mu \theta \cdot S \tag{A.23}$$

After a little algebra, (A.20) gives (for an infinitesimal θ)

$$A'_\mu = A_\mu - \theta \times A_\mu + \frac{1}{g}\partial_\mu \theta \tag{A.24}$$

It may be noted that in passing that the connection coefficients in general relativity also have an inhomogeneous term in their transformation law. The formula is :

$$\Gamma'^{K}_{\lambda\mu} = \frac{\partial x'^K}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\lambda}\frac{\partial x^\gamma}{\partial x'^\mu}\Gamma^\alpha_{\beta\lambda} + \frac{\partial^2 x^\alpha}{\partial x'^\lambda x'^\mu}\frac{\partial x'^\gamma}{\partial x^\alpha} \tag{A.25}$$

If it were not for the second, inhomogeneous term, Γ would transforms like a tensor.

A.0.2 Gauge field tensor and Riemann-Christoffel curvature

Now the question presents itself: since A_μ transforms inhomogeneously, how do we know whether it may be transformed to zero at every point, and therefore have no physical effect? To test this, we preform a series of four infinitesimal displacements round the closed path ABCDA, as in Fig(A.3). We start at A with a vector ψ_A, denoted $\psi_{A,0}$ and transport the vector around the closed path using the role for the parallel transport, involving the covariant derivative, then compare the final value of the vector at A, $\psi_{A,1}$ with its initial value $\psi_{A,0}$. If they differ, we take this as a signal that the potential A_μ *does* have a physical effect.
transporting $\psi_{A,0}$ to B will give, working to second order in Δx and δx

$$\begin{aligned}\psi_B &= \psi_{A,0} + D_\mu \psi_{A,0}\Delta x^\mu + \frac{1}{2}D_\mu D_\nu \psi_{A,0}\Delta x^\mu \Delta x^\nu \\ &= \left(1 + \Delta x^\mu D_\mu + \frac{1}{2}\Delta x^\mu \Delta x^\nu D_\mu D_\nu\right)\psi_{A,0}\end{aligned} \tag{A.26}$$

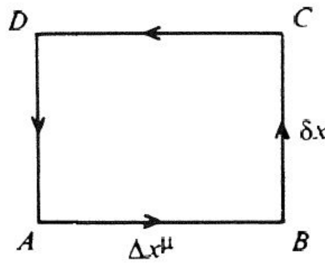


Figure A.3: A round trip by parallel transport

thence to C

$$\begin{aligned}\psi_C &= \left(1 + \delta x^\mu D_\mu + \frac{1}{2}\delta x^\mu \delta x^\nu D_\mu D_\nu\right)\psi_B \\ &= \left[1 + (\delta x^\mu + \Delta x^\mu)\, D_\mu + \left(\frac{1}{2}\delta x^\mu \Delta x^\nu + \delta x^\mu \Delta x^\nu + \frac{1}{2}\Delta x^\mu \Delta x^\nu D\right) D_\mu D_\nu\right]\psi_{A,0}\end{aligned} \tag{A.27}$$

Thence to D

$$
\begin{aligned}
\psi_D &= \left(1 - \Delta x^\mu D_\mu + \frac{1}{2}\Delta x^\mu \Delta x^\nu D_\mu D_\nu\right)\psi_C \\
&= \left[1 + \delta x^\mu D_\mu + \frac{1}{2}\delta x^\mu \delta x^\nu D_\mu D_\nu + \left(\delta x^\mu \Delta x^\nu - \Delta x^\mu \delta x^\nu D\right) D_\mu D_\nu\right]\psi_{A,0} \qquad \text{(A.28)}
\end{aligned}
$$

and finally back to A

$$
\begin{aligned}
\psi_{A,1} &= \left(1 - \delta x^\mu D_\mu + \frac{1}{2}\delta x^\mu \delta x^\nu D_\mu D_\nu\right)\psi_D \\
&= \left(1 + \delta x^\mu \Delta x^\nu + [D_\mu, D_\nu]\right)\psi_{A,0} \qquad \text{(A.29)}
\end{aligned}
$$

Note that what has appeared is the *commutator* of the covariant derivative operator. From (A.18) we find

$$
\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu - igA_\mu, \partial_\nu - igA_\nu] \qquad \text{(A.30)} \\
&= -ig\left(\partial_\mu - A_\nu - \partial_\nu - A_\mu - ig\left[A_\mu, A_\nu\right]\right) \qquad \text{(A.31)}
\end{aligned}
$$

If we define the gauge field as

$$
G_{\mu\nu} = \partial_\mu - A_\nu - \partial_\mu - A_\mu - ig\left[A_\mu, A_\nu\right] \qquad \text{(A.32)}
$$

then we have

$$
[D_\mu, D_\nu] = -igG_{\mu\nu} \qquad \text{(A.33)}
$$

In (A.29), $\delta x^\mu \Delta x^\mu$ represents the area of the rectangle $\Delta S^{\mu\nu}$, so we rewrite this equation (to second order) as

$$
\begin{aligned}
\psi_{A,1} &= \left(1 - ig\Delta S^{\mu\nu} G_{\mu\nu}\right)\psi_{A,0} \qquad \text{(A.34)} \\
\psi_{A,1} - \psi_{A,0} &= \Delta\psi = ig\Delta S^{\mu\nu} G_{\mu\nu} \qquad \text{(A.35)}
\end{aligned}
$$

and we see that, *if the gauge field is non-zero, a journey round a closed path has a physical effect*; the vector ψ is rotated in isospace.
The case of U(1), an abelian group, the commutator term is zero, so putting $G \to F$ (A.32), the gauge field is

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu - A_\mu \qquad \text{(A.36)}
$$

which is of course, just the electromagnetic field.
In the case of SU(2), the matrices M^a obey the commutation relations (A.6)so

$$
G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu - A_\mu + gA_\mu \times A_\nu \qquad \text{(A.37)}
$$

It may easily been seen that, since preforming a rotation in isospace gives

$$
\begin{aligned}
\psi_{A,0} &\to \psi'_{A,0} = S\psi_{A,0} \qquad \text{(A.38)} \\
\psi_{A,1} &\to \psi'_{A,1} = S\psi_{A,1} \qquad \text{(A.39)}
\end{aligned}
$$

with the same factor S, the field $G_{\mu\nu}$ transforms covariantly

$$
G'_{\mu\nu} = SG_{\mu\nu}S^{-1} \qquad \text{(A.40)}
$$

It follow that $G_{\mu\nu}$ cannot be transformed away to zero by a gauge transformation: it it is zero in one gauge, it is zero in all.

Returning one more to the general relativistic analogy, the quantity which is analogous to the field tensor $G^a_{\mu\nu}$ is the Riemann-Christoffel curvature tensor $R^{\mu}_{\rho\sigma\lambda}$ defined by

$$R^K_{\lambda\mu\nu} = \partial_\nu \Gamma^K_{\lambda\mu} - \partial_\mu \Gamma^K_{\lambda\nu} + \Gamma^\rho_{\lambda\mu}\Gamma^K_{\rho\nu} - \Gamma^\rho_{\lambda\nu}\Gamma^K_{\rho\mu} \tag{A.41}$$

comparing this with (A.32) and (A.37), we see the structural similarity: the first two terms are derivatives of the connection coefficients (potential) antisymmetric in μ and ν, and the last terms are the products of the connection coefficients, also antisymmetric in μ and ν. The way the curvature tensor is introduced is also analogous: on taking a vector V^μ on a round trip by parallel transport, the difference between the initial and final components of the vector is

$$\Delta V^\mu = \frac{1}{2} R^\mu_{\rho\sigma\lambda} V^\mu \Delta S^{\sigma\lambda} \tag{A.42}$$

where $\Delta S^{\sigma\lambda}$ is the area enclosed by the path. This equation parallels (A.34). ΔV^μ is non-zero only if the space is intrinsically curved: for example, on the surface of a sphere (a 2-dimensional space) a vector will point in a different direction after a round trip, but on a flat surface it will not. The curvature tensor, being a tensor, has the property that if it is non-zero (has any non-zero components) in one coordinate system, it is non-zero in all, and indicates that the space is curved. In general relativity, this means that there is a gravitational field present.

Bianchi Identity for the gauge field Finally, we derive an interesting identity satisfied by the gauge field. Consider the 3-dimensional closed path shown in Fig A.4(). Start with the vector $\psi_{A,0}$ at A and transport it round the path ABCDA, after which its value (from A.34) will be

$$\psi_{A,1} = (1 - ig\Delta S^\mu \nu G_{\mu\nu})\,\psi_{A,0} \tag{A.43}$$

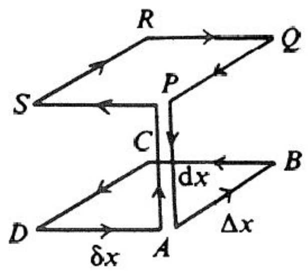


Figure A.4: A round trip used to derive Bianchi identity

where $\Delta S^\mu \nu = \delta x^\mu \Delta x^\nu$ is the area of ABCD. Now, transporting the vector ψ to P along the line AP of length dx, it will have the value

$$\psi_{P,0} = (1 + dx^\rho D_\rho)\,\psi_{A,1} \tag{A.44}$$

Thence, transporting it round the circuit PSRQP, its value will change to

$$\psi_{P,1} = (1 + ig\Delta S^\mu \nu G_{\mu\nu})\,\psi_{P,0} \tag{A.45}$$

where the plus sign arises because the rout is in the opposite direction. Finally, taking it down to A again, results in the final value

$$\psi_{A,2} = (1 + dx^\rho D_\rho)\,\psi_{P,1} \tag{A.46}$$

$$(1 + dx^\rho D_\rho)\,(1 + ig\Delta S^\mu \nu G_{\mu\nu})\,(1 + dx^\rho D_\rho)\,(1 - ig\Delta S^\mu \nu G_{\mu\nu})\,\psi_{A,0} \tag{A.47}$$

$$= \{1 - igV^{\rho\mu\nu}\,[D_\rho, G_{\mu\nu}]\} \tag{A.48}$$

where $\Delta V^{\rho\mu\nu} = dx^{\rho}\delta x^{\mu}\Delta x^{\nu}$ is th volume of the box. Taking into account the fact that the differential operator also acts on $\psi_{A,0}$, we may replace the commutator by the product $D_{\rho}G_{\mu\nu}$ giving

$$\psi_{A,2} = \left(1 - ig\Delta V^{\rho\mu\nu} D_{\rho}G_{\mu\nu}\right)\psi_{A,0} \tag{A.49}$$

The closed route we considered in Fig() consists of circuits round the top and bottom faces of the box. There are clearly two similar such circuits enclosing the two other pairs of faces. All six faces are circuited by the path

$$(ABCDAPSRQPA) + (ADSPABQRCBA) + (APQBADCRSDA) \tag{A.50}$$

which would result in a factor

$$1 - ig\Delta V^{\rho\mu\nu}\left(D_{\rho}G_{\mu\nu} + D_{\mu}G_{\nu\rho} + D_{\nu}G_{\rho\mu}\right) \tag{A.51}$$

operating on $\psi_{A,0}$ to give the new vector at A; call it $\psi_{A,3}$. However, as may be easily checked, the path (A.50) traverses each side of the box as many times in one direction as in the opposite one, so the path is equivalent to the reverse of itself, and consequently the vector ψ cannot change, $\psi_{A,3} = \psi_{A,0}$ and hence

$$D_{\rho}G_{\mu\nu} + D_{\mu}G_{\nu\rho} + D_{\nu}G_{\rho\mu} = 0 \tag{A.52}$$

This is the identity we wanted to proof. The covariant derivative $D_{\rho}G_{\mu\nu}$ is given by

$$D_{\rho}G_{\mu\nu} = \partial_{\rho}G_{\mu\nu} - ig\left[A_{\rho}, G_{\mu\nu}\right] \tag{A.53}$$

Recall that A_{μ} and $G_{\mu\nu}$ are both matrices: the commutator in (A.53) is a matrix commutator. Under the gauge transformation (A.20) and (A.40) it is straightforward to show that

$$D'_{\rho}G'_{\mu\nu} = S\left(D_{\rho}G_{\mu\nu}\right)S^{-1} \tag{A.54}$$

appropriate for the a covariant derivative.
In view of equation (A.33) the identity (A.52) is equivalent to

$$\left[D_{\rho}\left[D_{\mu}, D_{\nu}\right]\right] + \left[D_{\mu}\left[D_{\nu}, D_{\rho}\right]\right] + \left[D_{\nu}\left[D_{\rho}, D_{\mu}\right]\right] = 0 \tag{A.55}$$

which is in fact the Jacobi identity. It is a condition which is identically satisfied by the field tensor. I the abelian case U(1), it takes the form

$$\partial_{\rho}F_{\mu\nu} + \partial_{\mu}F_{\nu\rho} + \partial_{\nu}F_{\rho\mu} = 0 \tag{A.56}$$

This was shown to be equivalent to the homogeneous Maxwell equations. The identity (A.52) has an analogue in general relativity, known as the *Bianchi identity*, which takes the form

$$D_{\rho}R^{K}_{\lambda\mu\nu} + D_{\mu}R^{K}_{\lambda\nu\rho} + D_{\nu}R^{K}_{\lambda\rho\mu} \tag{A.57}$$

We may similarly refer to (A.52) as the Bianchi identity. The parallels which we have discussed between gauge theories and general relativity are summarized in Table (A.1).

Table A.1: Parallels between gauge theory and general relativity

Gauge theory	General relativity
Gauge transformation	coordinate transformation
Gauge group	Group of all coordinate transformation
gauge potential A_μ	connection coefficients $\Gamma^K_{\mu\nu}$
Field strength $G_{\mu\nu}$	Curvature tensor $R^K_{\lambda\mu\nu}$
Bianchi identity:	Bianchi identity
$\sum_{\rho\mu\nu,cyc} D_\rho G_{\mu\nu}=0$	$\sum_{\rho\mu\nu,cyc} D_\rho R^K_{\lambda\mu\nu}=0$