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Hipparchus vs. Ptolemy and the Antikythera Mechanism: Pin–Slot device models lunar motions

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Abstract

The ancient Greek astronomical calculator known as the Antikythera Mechanism has been analyzed using geometrical, calculus, trigonometric and complex variable methods. This analysis demonstrates that the Mechanism modeled the variations in the Moon's angular velocity as seen from the Earth, to better than 1 part in 200. A major implication of this analysis is that the Antikythera Mechanism of the 2nd century BCE modeled the anomalistic motion of the Moon more accurately than Ptolemy's account of Hipparchus's theory of the 2nd century CE. In the present work, mathematics, astronomy, history and methodology of the sciences combine in the study of a unique artifact, preserved for posterity in an ancient ship that sank in the Mediterranean 2100 years ago and recovered by Greek sponge divers at the dawn of the 20th century.

Keywords: Hipparchus; Lunar motions; Lunar Anomaly; Ptolemy; Antikythera Mechanism; Ancient mechanical universes

1. Introduction

The ancient astronomical calculator known as the "Antikythera Mechanism" has a long and controversial history. Greek sponge divers found it in 1900 in a Roman wreck dated circa 100 BCE off the island of Antikythera. There are many accounts of the discovery; one striking fact is that it was found on a well-known traditional ancient route along the geographical parallel joining Rhodes to Gibraltar via Tunis, approximately 36.5° N called "the diaphragm". It is easy to sail along a geographical parallel. All the mariner has to do is to observe the path that the tip of the shadow of a vertical stick traces out at around noon. No compass is needed. Two millennia later, the sponge divers were following the same route so it was perhaps inevitable that they would discover it. The artifact was found alongside other equally precious finds, such as a large number of exquisite bronze and marble statues. All are now on permanent display at the Athens Archaeological Museum. Bronze artifacts are hard to come by, as most were recycled. This particular one happened to travel through a space-time wormhole (the wreck) and reached the 20th century! Or, to put it another way, it was delayed but preserved in the post for two millennia and eventually reached the modern age (Figs. 1 and 2).

The AM consists of a set of 30 bronze gears with teeth approximating equilateral triangles. The sheer complexity of the arrangement is obvious even under simple visual inspection. X-rays determined the number of teeth. While it is possible that some model-builders were massaging the data to fit their model, rigorous statistical analysis of the data has now narrowed down the possibilities. One may refer to the "Standard Model" of the Mechanism as the one reported in Freeth et al. (2006). It has been established that the Mechanism incorporated the *Metonic cycle*, the equivalence of 235 synodic months to 19 solar years. It also incorporated the *Saros* eclipse cycle of 223 synodic months. There was a separate dial for the *exeligmos=turn of the wheel*, which is three *Saros* cycles. After one *exeligmos* the sequence of solar and lunar eclipses repeats itself. It is clear that the instrument was designed very carefully

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Fig. 1. The Antikythera Mechanism, 100 BCE. The Mechanism's most iconic image. Note the similarity with the wheels in Dondi's Astrarium.

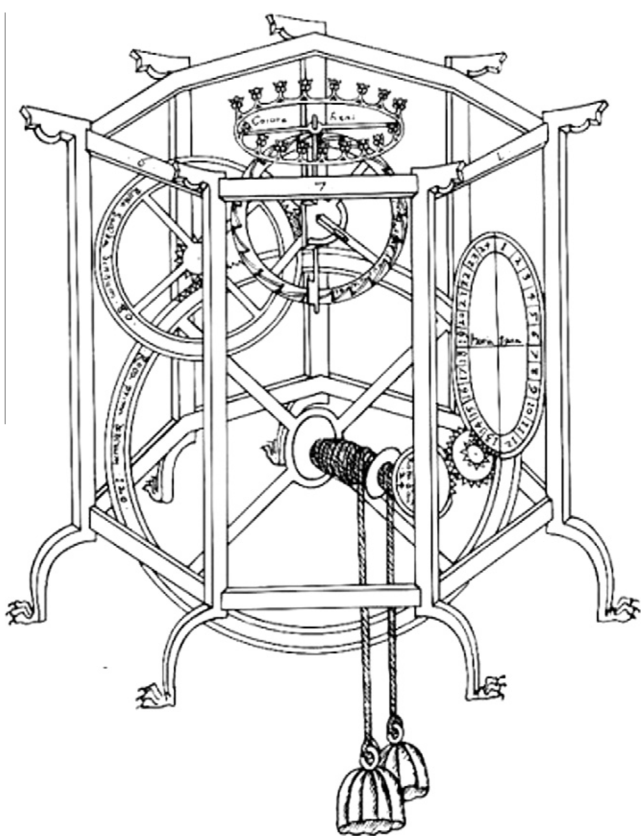

Fig. 2. Di Dondi's Astrarium, circa 1100 CE.

taking into account the latest astronomical knowledge. There are details such as a regular pentagon inscribed at the axle of a gear with 50 teeth. It is well known that a classical geometrical construction (using only rule and compasses) exists for the regular pentagon. Clichés such as these show that the study of geometry and astronomy had been formalized into a coherent body of knowledge, which, after Euclid, would continue almost unchanged until the time of Gauss and Riemann. Imperial edicts have been recovered from Byzantine times that prescribed the daily remuneration of a teacher of geometry. The great Christian Huygens (1673/1986) developed rigorous geometrical proofs involving infinitesimals. He used these methods to prove that the period of oscillation of a cycloidal pendulum is independent of the amplitude of oscillation. This paved the way to construction of "the perfect clock", the definition of the unit of length in terms of the unit of time and the solution of the longitude problem. Newton also used geometrical methods to prove results such as that the gravitational field of a planetoid of constant density is identical to the field of a point particle of equal mass situated at the centre of the planetoid. (This was a precursor of Birkhoff's theorem in the Theory of General Relativity, the fact that a pulsating spherically symmetric distribution of mass cannot emit gravitational waves).

Another feature of the Mechanism was a "Pin-and-Slot" device. Two identical gears were coupled. A fixed pin on the "driver" gear engaged into a slot in the "driven" gear. By adjusting the distance between the centres of the two gears and the distance between the centre of a gear and the pin, it became possible to mimic epicyclical motion. The exact analysis of this is the subject of our paper. It was found that the superimposed periodicity matched closely the variations in angular velocity of the moon as seen from the Earth. It is our contention that this was built into the design consciously and provides further evidence for the high level of sophistication in its construction. It also indicates that, three centuries before the time of Claudius Ptolemy, such *quantitative* models were commonplace. It is even possible to speculate that such advanced mathematicians/astronomers may have considered, if only as a theoretical possibility, a simple coordinate transformation from a geocentric to a heliocentric frame of reference, thus placing the Sun at the centre of the universe!

The conclusion is that this mechanical artifact has a level of sophistication that is comparable to the level of sophistication of other cultural activities, such as philosophy, theatre and the other arts, all flourishing at that particular time and place in the Ancient World. It is a microcosm of the heavens, and perhaps is tantamount to a philosophical statement: "*This is how things are, and this is our place in the great design. And it is possible to refine our model, or even discard it completely for an improved one, for the model may be wrong*".

In a recent paper on the Antikythera Mechanism, Freeth et al. (2006) investigated the extent that the "Pin–Slot" arrangement found in the Mechanism modeled the variation in the apparent angular velocity of the Moon as seen from the Earth (the Lunar Anomaly). Freeth et al. (2006) measured the distance between the centres of two wheels and the distance between the centre of the driving wheel and a pin. On the basis of a single calculation, the authors arrived at an angle, which did not agree with angles allegedly arising from Hipparchus's Lunar Theory.

In this paper a totally different approach is followed by deriving an analytical expression linking the angular velocities of the two wheels in terms of the numerical parameters discussed earlier. It is demonstrated that, using the values of the parameters measured by the CT scan and reported in the original paper, the resulting variation in angular velocity modeled the actual variation of the lunar angular velocity very accurately! The major implication of this is that the Antikythera Mechanism of the 2nd century BCE modeled the anomalistic motion of the Moon more accurately than Ptolemy's version of Hipparchus's theory of the 2nd century CE.

2. Derivation of the relationship between the angular velocities

In the diagram shown in Fig. 3, A is the centre of $k1$ (following the same nomenclature for the gears as in Freeth et al. (2006). B is the centre of $k2$, and P is the centre of the pin.

Let $AB = a =$ the distance between the centres of $k1$ and $k2 =$ constant, and $AP = a_1 =$ the distance between the pin and the centre of $k1 =$ constant.

Also let $\varepsilon \equiv \dfrac{a}{a_1} \ll 1$ be the eccentricity parameter. (1)

With reference to Fig. 3, α is the angle subtended by AB at P.

Clearly $\quad \varphi = \theta + \alpha \Rightarrow \alpha = \varphi - \theta.$ (2)

Using the Sine Rule, we have:

$$\frac{a}{\sin\alpha} = \frac{a_1}{\sin\varphi} \tag{3}$$

$\Rightarrow \sin\alpha = \left(\frac{a}{a_1}\right)\sin\varphi = \varepsilon\sin\varphi$, where, using Eq. (1)

$$\Rightarrow \sin(\varphi-\theta) = \varepsilon\sin\varphi \Rightarrow \sin\varphi\cos\theta - \cos\varphi\sin\theta = \varepsilon\sin\varphi$$

$$\Rightarrow \sin\varphi(\cos\theta-\varepsilon) = \cos\varphi\sin\theta$$

$$\Rightarrow \tan\varphi = \frac{\sin\theta}{\cos\theta-\varepsilon}. \tag{4}$$

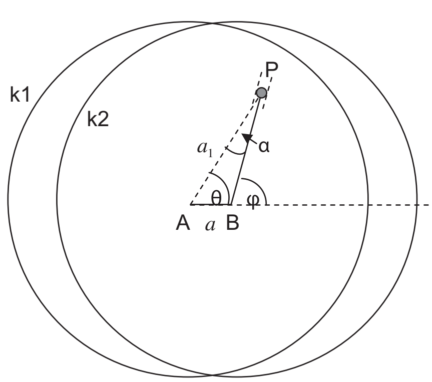

Fig. 3. The coupled gears in the "Pin-and-Slot" arrangement.

To relate the angular velocities $\dot\theta$ and $\dot\varphi$ of the two wheels, differentiate (4) with respect to the time, t, to obtain:

$$(\sec^2\varphi)\dot\varphi = \frac{(\cos\theta)\dot\theta(\cos\theta-\varepsilon) + (\sin^2\theta)\dot\theta}{(\cos\theta-\varepsilon)^2}$$

$$\Rightarrow \dot\varphi(1+\tan^2\varphi) = \frac{\dot\theta(\cos^2\theta - \varepsilon\cos\theta + \sin^2\theta)}{(\cos\theta-\varepsilon)^2}$$

$$\Rightarrow \dot\varphi = \frac{\dot\theta(1-\varepsilon\cos\theta)}{(\cos\theta-\varepsilon)^2\left\{1+\frac{\sin^2\theta}{(\cos\theta-\varepsilon)^2}\right\}}.$$

Hence,

$$\dot\varphi = \frac{\dot\theta(1-\varepsilon\cos\theta)}{(\cos\theta-\varepsilon)^2+\sin^2\theta} = \frac{\dot\theta(1-\varepsilon\cos\theta)}{1-2\varepsilon\cos\theta+\varepsilon^2}. \tag{5}$$

Now, wheel $k1$ rotates with

constant angular velocity ω, $\Rightarrow \dot\theta = \omega$ (6)

and thus $\quad \theta = \omega t.$ (7)

The angular velocity of $k2$ is given by

$$\dot\varphi = \frac{\omega(1-\varepsilon\cos\omega t)}{1-2\varepsilon\cos\omega t+\varepsilon^2} \tag{8}$$

where $\omega \equiv \dot\theta$ is the angular velocity of $k1$.

Eq. (8) determines the perturbed angular velocity $\dot\varphi$ of $k2$ as a function of the time and the angular velocity of $k1$, parameterized by the eccentricity parameter $\varepsilon \equiv \frac{a}{a_1}$, as defined by Eq. (1).

3. The linear approximation

Ignoring ε^2 and higher powers of ε, Eq. (8) becomes:

$$\dot\varphi \approx \frac{\omega(1-\varepsilon\cos\omega t)}{1-2\varepsilon\cos\omega t} = \omega(1-\varepsilon\cos\omega t)(1-2\varepsilon\cos\omega t)^{-1}.$$

Expanding by the Binomial Theorem, we obtain:

$$\dot\varphi \approx \omega(1-\varepsilon\cos\omega t)(1+2\varepsilon\cos\omega t)$$
$$= \omega\{1+2\varepsilon\cos\omega t - \varepsilon\cos\omega t + O(\varepsilon^2)\}$$

and hence

$$\dot\varphi = \omega(1+\varepsilon\cos\omega t) \tag{9}$$

or

$$\dot\varphi = \omega + A\cos\omega t \tag{10}$$

where $A\cos\omega t$ is the perturbing term, which is added to the angular velocity ω of $k1$ and where

$$A = \omega\varepsilon \tag{11}$$

is the perturbing amplitude.

Integrating Eq. (9) with respect to t, we obtain:

$$\varphi = \omega t + \varepsilon\sin\omega t \tag{12}$$

which provides the explicit dependence of the angle of rotation of $k2$ as a function of the time, the angular velocity of $k1$ and the eccentricity parameter, ε.

From Eq. (10), the graph of $\dot\varphi$ versus t is shown in Fig. 4.

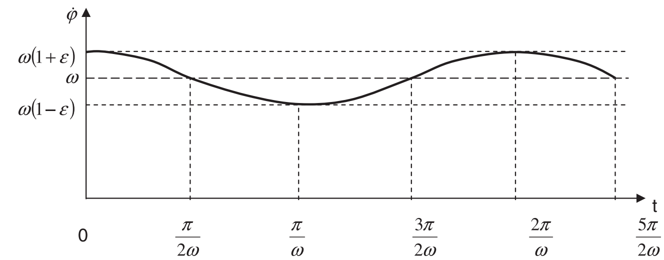

Fig. 4. The variation of the angular velocity of the driven gear $k2$ with time.

4. Comparison with actual lunar orbits

The longitudinal angular speed of the Moon is computed as a function of the time, taking position values of the Moon (in geocentric ecliptic coordinates) from a web-based spreadsheet (Burnett, 2000).

A graph of the results calculated for Birmingham, UK is displayed in Fig. 5.

This compares very well with the expected graph of the angular velocity, as shown in Fig. 4.

Taking $t = 0$ on day 8, we fitted to the data above, the formula:

$$\dot{\varphi} = 13.065 + 1.4917 \cos 13.065t \quad (13)$$

using the Length of Anomalistic Month = 27.555 days. (*Note:* This has hardly changed in the last 2100 years.) Based on this value, we obtain

$$\omega = \frac{360^\circ}{27.555} = 13.065^\circ/\text{day}. \quad (14)$$

Comparing this with Eq. (10), and making the identifications:

$$\omega = 13.065^\circ/\text{day}$$

$$\text{and} \quad A = \omega\varepsilon = 1.4917^\circ/\text{day} \quad (15)$$

the value of ε, as given by the model described herein is:

$$\varepsilon = \frac{1.4917}{13.065} \approx 0.1142. \quad (16)$$

In Freeth et al. (2006) the authors report measured values from the CT scan for the parameters a and a_1 which appear in Eq. (1). Using their values of $a = 1.1$ mm and $a_1 = 9.6$ mm, the value for the eccentricity parameter, ε is

$$\varepsilon = \frac{1.1}{9.6} = 0.1146. \quad (17)$$

The agreement between the value of ε using the model described in this paper and the value obtained from the CT scan is remarkable!

5. Higher approximations

Eq. (8) can be written as

$$\dot{\varphi} = \omega(1 - \varepsilon\cos\omega t)(1 - 2\varepsilon\cos\omega t + \varepsilon^2)^{-1}. \quad (18)$$

We expand Eq. (18) into a series of ascending powers of ε up to and including ε^4.

From (18), we obtain:

$$\frac{\dot{\varphi}}{\omega} = (1 - \varepsilon\cos\omega t)(1 - 2\varepsilon\cos\omega t + \varepsilon^2)^{-1}. \quad (19)$$

When expanding the second bracket to ε^4 we find

$$\begin{aligned}(1 - 2\varepsilon\cos\omega t + \varepsilon^2)^{-1} &= \{1 - (2\varepsilon\cos\omega t - \varepsilon^2)\}^{-1} \\ &= 1 + 2\varepsilon\cos\omega t - \varepsilon^2 + (2\varepsilon\cos\omega t - \varepsilon^2)^2 \\ &\quad + (2\varepsilon\cos\omega t - \varepsilon^2)^3 + (2\varepsilon\cos\omega t - \varepsilon^2)^4 \\ &= 1 + 2\varepsilon\cos\omega t - \varepsilon^2 \\ &\quad + (4\varepsilon^2\cos^2\omega t - 4\varepsilon^3\cos\omega t + \varepsilon^4) \\ &\quad + 8\varepsilon^3\cos^3\omega t - 12\varepsilon^4\cos^2\omega t + 16\varepsilon^4\cos^4\omega t.\end{aligned}$$

Multiplying this by $(1 - \varepsilon\cos\omega t)$, we obtain, in a self-consistent approximation:

$$\begin{aligned}\frac{\dot{\varphi}}{\omega} &= 1 + \varepsilon\cos\omega t + \varepsilon^2\cos 2\omega t + \varepsilon^3(4\cos^3\omega t - 3\cos\omega t) \\ &\quad + \varepsilon^4(8\cos^4\omega t - 8\cos^2\omega t). \quad (20)\end{aligned}$$

Using $z = e^{i\theta}$ and the identities $z^n + z^{-n} = 2\cos n\theta$, we derive trigonometric identities for the last two terms of (20), i.e.

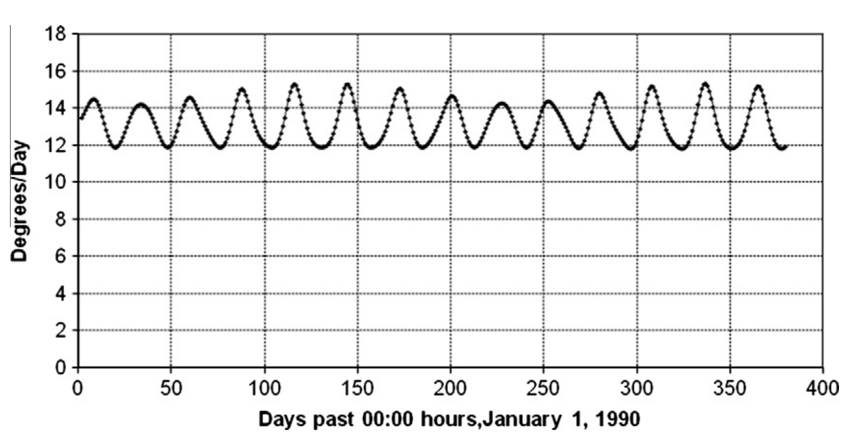

Fig. 5. The variation of the longitudinal angular velocity of the Moon with time. The values shown were determined for Birmingham, UK.

$$4\cos^3\theta - 3\cos\theta \equiv \cos 3\theta \tag{21}$$

and

$$8(\cos^4\theta - \cos^2\theta) \equiv \cos 4\theta - 1. \tag{22}$$

Hence, Eq. (19) can be written in the form:

$$\frac{\dot{\varphi}}{\omega} = 1 - \varepsilon^4 + \varepsilon\cos\omega t + \varepsilon^2\cos 2\omega t + \varepsilon^3\cos 3\omega t + \varepsilon^4\cos 4\omega t. \tag{23}$$

When continuing the expansion of the second bracket of (19), up to and including ε^5, the relevant additional term is:

$$\varepsilon^5(5\cos\omega t - 20\cos^3\omega t + 16\cos^5\omega t).$$

Using the usual complex identities $z^n + z^{-n} = 2\cos n\theta$, the above expression can be transformed to

$$\varepsilon^5(\cos 5\omega t + 5\cos 3\omega t + 10\cos\omega t).$$

From this Eq. (19) can be approximated by:

$$\frac{\dot{\varphi}}{\omega} = 1 - \varepsilon^4 + (\varepsilon + 10\varepsilon^5)\cos\omega t + \varepsilon^2\cos 2\omega t + (\varepsilon^3 + 5\varepsilon^5)\cos 3\omega t + \varepsilon^4\cos 4\omega t + \varepsilon^5\cos 5\omega t. \tag{24}$$

This has the form of a Fourier Series up to and including the fifth harmonic!

Note that the expression for $\dot{\varphi}$ involves only even functions of t; hence it is invariant under time-reversal $t \to -t$.

Integrating Eq. (10) with respect to t, we obtain:

$$\varphi = \omega t(1 - \varepsilon^4) + \varepsilon(1 + 10\varepsilon^4)\sin\omega t + \frac{1}{2}\varepsilon^2\sin 2\omega t + \frac{1}{3}\varepsilon^3(1 + 5\varepsilon^2)\sin 3\omega t + \frac{1}{4}\varepsilon^4\sin 4\omega t + \frac{1}{5}\varepsilon^5\sin 5\omega t. \tag{25}$$

In the linear approximation, this becomes:

$\varphi = \omega t + \varepsilon\sin\omega t$ i.e. Eq. (12).

6. Concluding remarks

The methodology developed in this paper has resulted in a meaningful comparison of numerical parameters, which link quantitatively the Antikythera Mechanism to the Lunar Anomaly. The Epicyclical Theory realized in the Mechanism, passed the test with flying colors! One of the major implications of this analysis is that at around the time that the Greek astronomer Hipparchus was active (2nd century BC), a highly specialized device had been constructed. That device incorporated a fully developed theory of epicyclical motion that predicted very accurately the motion of the Moon, almost three centuries before Ptolemy. There are very few reliable sources on Hipparchus; perhaps one cannot avoid the temptation to associate the device with his theories, under the adage «εξ όνυχος τον λέοντα» ("A Lion can be recognized by his paw").

By adding more epicyclical gears to the Antikythera Mechanism, an orbit can be modeled more and more accurately. Schiaparelli (1926) had first suggested the equivalence of modeling planetary motions by epicycles to the determination of successive Fourier coefficients in a suitable approximation scheme of Newtonian orbital theory.

It is conceivable that there were further epicyclical gears and Pin–Slot devices in the Mechanism, modeling the motions of Mercury (ΕΡΜΗΣ) (eccentricity 0.2056) and Venus (ΑΦΡΟΔΙΤΗ), names mentioned in the Inscriptions found within the device (Freeth et al., 2006).

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