Is there a simple, formulaic way to construct a modular exponentiation circuit?

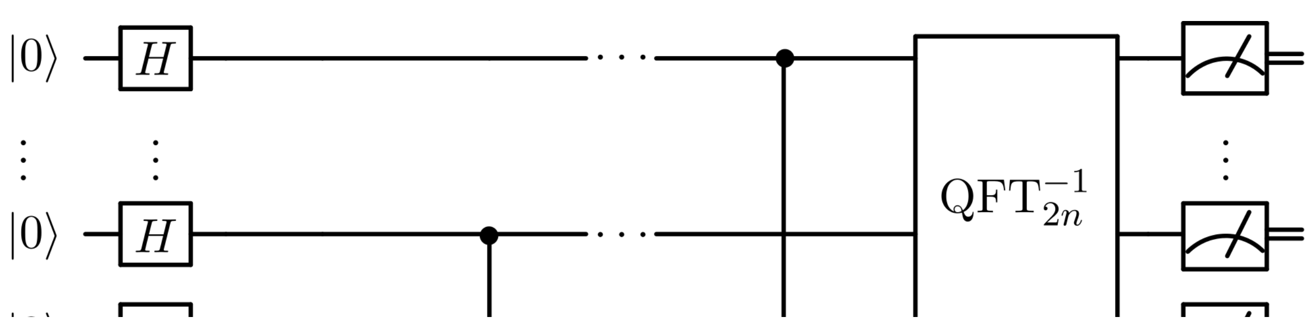
Asked 2 years, 4 months ago Active 1 year ago Viewed 2k times

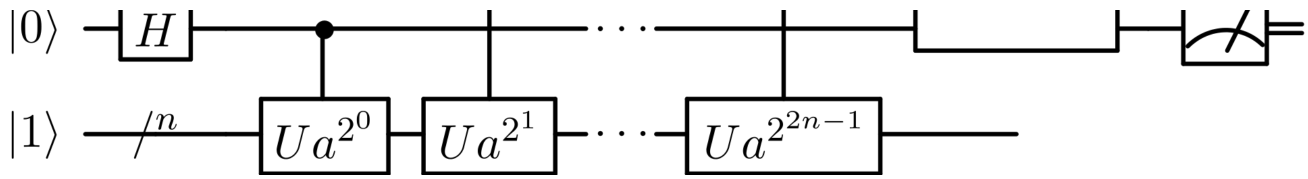
8

2

I'm a newcomer to quantum computing and circuit construction, and I've been struggling to understand how to make a modular exponentiation circuit. From what I know, there are several papers on the matter (like Pavlidis, van Meter, Markov and Saeedi, etc.) but they are all so complicated and involve a lot of efficiency and optimization scheme that make it impossible for me to understand. When I read it in Nielsen and Chuang, specifically in Box 5.2 the author wrote them without any example, as if it is very easy to make (it probably is, but not for me).

Anyway, I've learned about the algorithm to do modular exponentiation using binary representation (it's simple enough at least this thing), but I don't know how to make a circuit out of it. Here's the picture I believe describing the process:





So how do I build those U circuit? Can anyone, for example, tell me how do things changed when say I went from $11^x (\mod 15)$ to $7^x (\mod 21)$? I don't care if the circuit is not gonna be optimized and contain thousands of gates, but I want to at least understand the first step before going into more advanced stuffs like optimization.

Thank you very much!

circuit-construction shors-algorithm

Share Improve this question Follow

asked Jul 22 '19 at 17:15

Jack Nathan
424 2 13

1 This question and answer about an implementation of the order finding circuit and modular exponation circuit gives some insight.quantumcomputing.stackexchange.com/questions/4852/... – Bram Jul 22 '19 at 20:26

Thank you! I've also seen these modular multiplication circuits in Markov and Saeedi in the form of SWAP gate, and even though I did try the numbers in and it works, I still couldn't see how; it's like something we make when we've already know the answer, and just make a circuit that gives out the correct answer. Why do we use the SWAP gates, and in that order for C = 2 , 4 or 7, 11? How do things changed when say M = 21 for example? I couldn't understand those things... – Jack Nathan Jul 23 '19 at 3:56

3 Answers

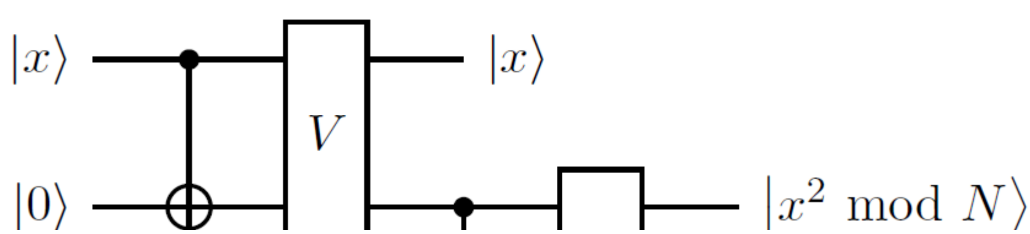
Active Oldest Votes

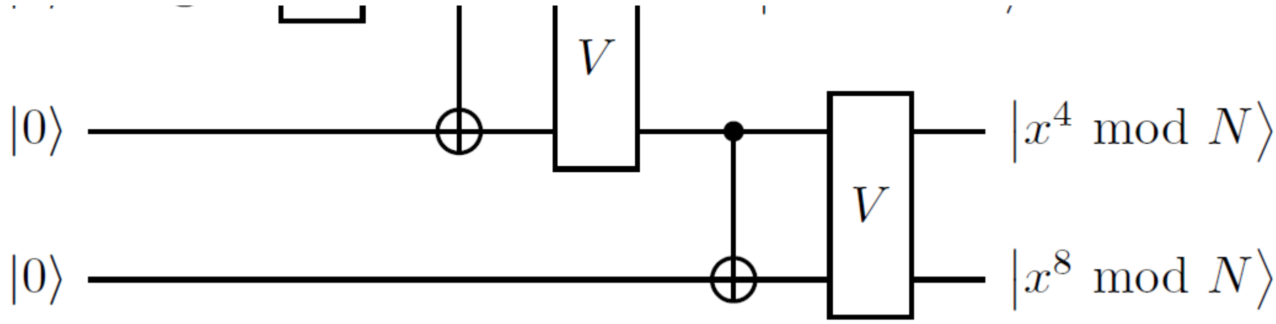
3

Here's, surely, a very non-optimal way of doing it. Imagine we have a unitary V which performs the operation

$$V|x\rangle|y\rangle = |x\rangle|xy \bmod N\rangle.$$

We can deal with how V might work separately, but if you have that, we want to see how we can use it to calculate any $|x^{2^i} \bmod N\rangle$. The trick is that if both inputs are $x^{2^j} \bmod N$, then the output is $x^{2^{j+1}} \bmod N$, so we only have to repeat this construction i times. For example, in the circuit below:

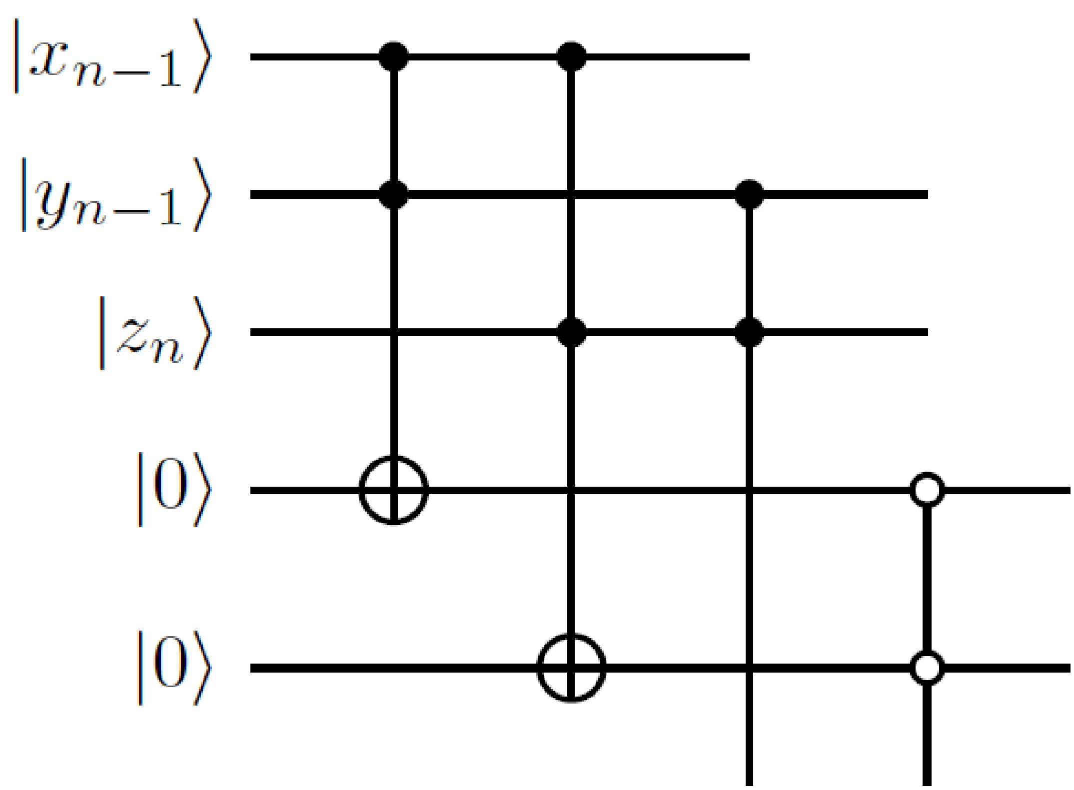


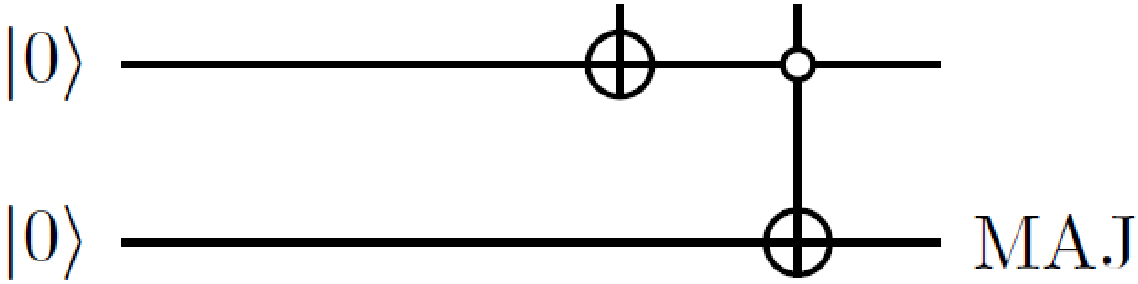


Here, I've used the control-not to denote transversal application to achieve copying of one (effectively classical) register to another. This lets you do any of the U operations that you need, assuming that you know how to implement V. Don't forget that, as part of a larger circuit, you have to 'uncompute' the data on any auxiliary registers.

So, how do we implement V? Let me give some of the components. Let $x = x_1x_2x_3\ldots x_n$ and $y = y_1y_2\ldots y_n$ be the binary representations of x and y. The product xy is easy to calculate by long multiplication. For example, x_iy_j is a bit value (so there are never any carries from the multiplication steps) that is equivalent to applying the Toffoli (controlled-controlled-not) with x_i and y_j as the two inputs. So you can calculate y_1x, y_2x, $y_3x\ldots$ on separate registers and then add them up.

Addition is another standard circuit. Imagine you want to add $x_1x_2\ldots x_n$ and $y_1y_2\ldots y_n$. We need to have additional two registers: one for the output, one for the carry bit. The least significant bit of output is $x_n \oplus y_n$, which can be calculated with controlled-nots. The carry bit has value $z_n = x_ny_n$. The next output is $x_{n-1} \oplus y_{n-1} \oplus z_n$, which we can again do with controlled nots. The carry bit is a majority vote - are two or more of x_{n-1}, y_{n-1}, z_n value 1? One way to implement this is:





You can keep repeating this process bit by bit to compute the sum. Then, again, don't forget to uncompute all the ancillas.

Share Improve this answer Follow

answered Jul 24 '19 at 8:15
DaftWullie
42.1k 1 25 88

Thank you for your answer. I still have 2 problems though: 1. I'm still not sure how we can use multiplication and addition to make a circuit that calculate modulus. I'm a physics student and I don't really know about circuit. Can you elaborate for me? 2. This method looks like it's gonna take a massive number of qubits since we effectively just multiply everything out instead of doing memory-efficient modular arithmetic right? I estimated using 30 qubits just to store the x^j answer if we were to do $19^7 \mod (21) = 1$. So this is definitely unusable as of right now right? – Jack Nathan Jul 24 '19 at 12:47

1 @D.Tran You didn't ask for usable. You asked for a general method that could get you started. All the methods that have been used have involved a very deliberate choice of a that makes things easy, and some very specific improvements for those cases. – DaftWullie Jul 24 '19 at 12:59

Yes, the 2nd question isn't too important, but as in the 1st one I'm still not sure how to use it to do modulo... – Jack Nathan Jul 24 '19 at 13:18

1 @D.Tran Functionally, what is it you would do if you were asked to calculate $x \bmod N$, given x and N? – DaftWullie Jul 25 '19 at 7:17

If I were to do it by hand, I'll try to divide x to N, then multiply N by the rounded down result, then take x minus the multiplied result. – Jack Nathan Jul 25 '19 at 8:55

2

The method of transversal copying by a CNOT is solid and you can stack up the building blocks for the quantum part of shor algorithm. However ad-hoc circuit synthesis based at a pattern of the function truth table could be efficiënt in some cases as described in [arxiv 1310.6446v2.](https://arxiv.org/abs/1310.6446v2) First case is for factoring N=15 and base a = 2 with period r = 4. In exponential formulation we have

$$f(x) = a^x \bmod 15$$

With values

x	f(x)
0	1

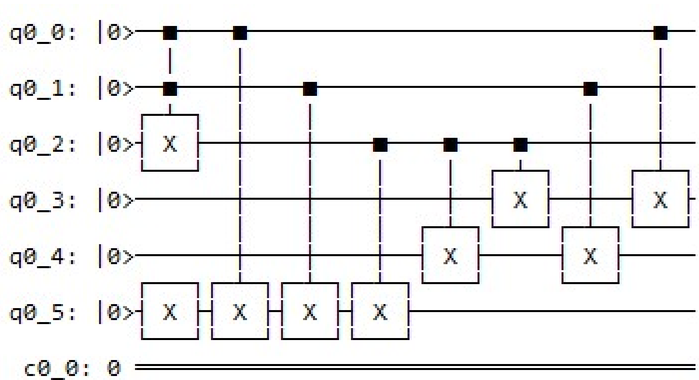
0	1
1	2
2	4
3	8

Setup a truth table for input x between 0 and 3. Input x is represented by 2 qubits x2 and x1. Output y is represented by 4 qubits y4,y3,y2,y1 For example if x = 2 then x2=1 and x1=0 then only y3 =1 so put a NOT on this line.

x_2	x_1	y_4	y_3	y_2	y_1
0	0	0	0	0	1
0	1	0	0	1	0
1	0	0	1	0	0
1	1	$\underline{1}$	$\underline{0}$	$\underline{0}$	$\underline{0}$

TABLE I: The binary truth table for $y = f_{2,15}(x)$.

Furthermore underlined entries in table 1 are the ones that are modified by a toffoli gate to get the right output in the circuit according the table 1. We can use this module in the overall algorithm according to https://arxiv.org/pdf/0705.1398.pdf



Share Improve this answer Follow edited Jul 27 '19 at 13:29 answered Jul 27 '19 at 10:56

Bram
479 3 8

2

Nielsen and Chuang Box 5.2 does indeed need more elaborate explanation.

I'm going to describe the architecture of efficient $O(n^3)$ modular exponentiation circuit from the paper 'Quantum Networks for Elementary Arithmetic Operations' – Vedral, Barenco, Ekert, 1995, for the case $n = 3$ using specific 3-bit numeric values in order to make general approach more illustrative. It seems to be exactly what you need, since illustrative. It seems to be exactly what you need, since

1. The circuit's efficiency is $O(n^3)$
2. The circuit is not subject to further lower-level optimization, but it is still efficient and replicates logic behind commonly used decomposition technique (you can find more elaborate comments on efficiency in the paper)

The Idea

Let's first revisit the idea which is used to construct the circuit of the interest. Using the property of modular multiplication $(A \times B) \mod N = (A \mod N \times B \mod N) \mod N$, we can see that modular exponentiation is a succession of modular multiplications:

$$y^x \mod N = (y^{x_0 2^0} \times y^{x_1 2^1} \times \ldots \times y^{x_{n-1} 2^{n-1}}) \mod N =$$

$$= (\ldots([(y^{x_0 2^0} \times y^{x_1 2^1}) \mod N] \times \ldots \times y^{x_{n-1} 2^{n-1}}) \mod N \ldots) \mod N,$$

where $x = x_0 2^0 + x_1 2^1 + \ldots + x_{n-1} 2^{n-1}$. Now, any modular multiplication operation can be represented by modular additions in the following way:

$$zm \mod N = (z_0 2^0 m + z_1 2^1 m + .. + z_{n-1} 2^{n-1} m) \mod N,$$

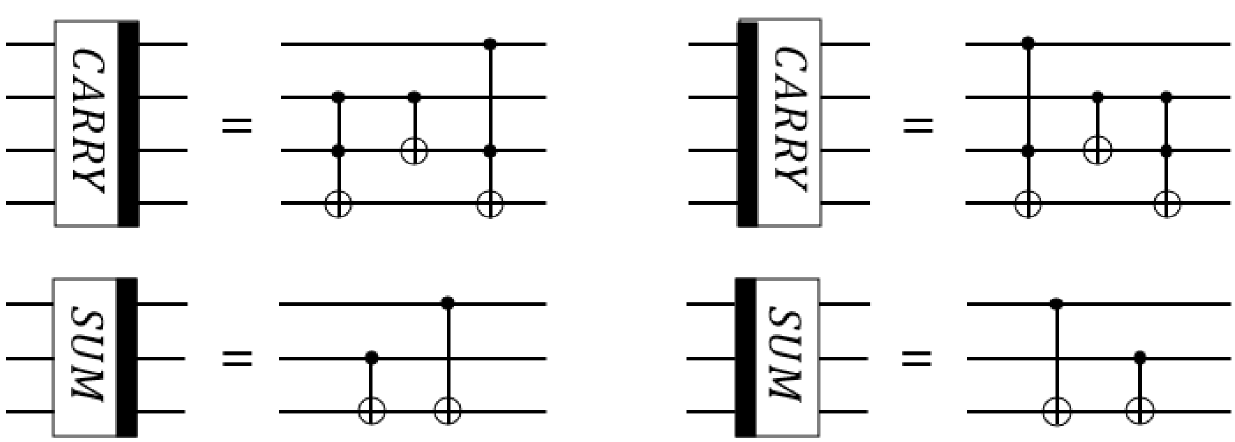
where $z = z_0 2^0 + z_1 2^1 + \ldots + z_{n-1} 2^{n-1}$. Finally, modular addition can be represented using addition and logical operations, as you will see later in the text.

The Circuit

Some comments on notations: wires marked in **blue** are auxiliary wires for lower-level operations. I decided to keep them in order for the reader not to lose track of what's going on. Values and circuit elements corresponding to known-in-advance classical information are marked **red**.

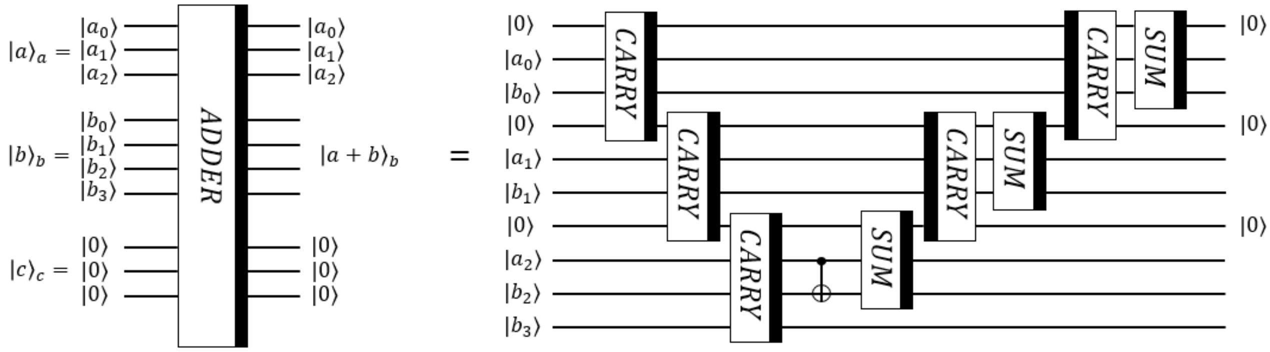
Let's go through the logic of building the circuit from the lowest level with elementary quantum operations to the highest level with modular multiplications

3-qubit addition circuit ADDER. We will use circuits CARRY and SUM which implement bitwise carry and sum operations.



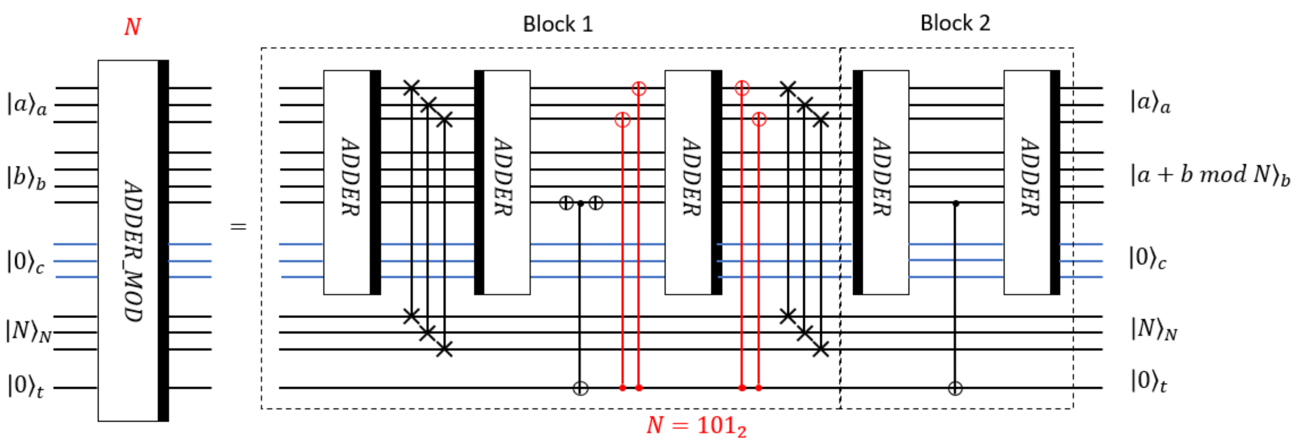
Note that thick black line on the right side of a block denotes operation itself, while thick black line on the left side of a block denotes reverse operation, i.e. operation with reverse order of all elementary operation for the block.

CARRY and SUM are used to construct 3-qubit addition transformation ADDER



Note that a is the number decoded with 3 qubits, b is the number decoded with 3 qubits, but the register $|b\rangle_b$ contains additional qubit to account for the possibility of 4-bit result of addition.

3-qubit modular addition circuit ADDER_MOD. Modular addition has two blocks: Block 1 and Block 2.



The logic of the Block 1 is the following: firstly, ADDER acts

$$|a\rangle_a|b\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t \rightarrow |a\rangle_a|a+b\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t$$

Then 3 SWAP gates swap the register a with the register N:

$$|a\rangle_a|a+b\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t \rightarrow |N\rangle_a|a+b\rangle_b|0\rangle_c|a\rangle_N|0\rangle_t$$

Then reverse ADDER extracts N:

$$|N\rangle_a|a+b\rangle_b|0\rangle_c|a\rangle_N|0\rangle_t \rightarrow |N\rangle_a|a+b-N\rangle_b|0\rangle_c|a\rangle_N|0\rangle_t$$

At that point we are interested in the sign of $a+b-N$. If it is greater than 0, we want to keep the result in the register b, but if it is less than 0, we want to make addition of N once again to get $a+b$ in the register b, and this is why see CNOTs, the third ADDER and SWAPS in the rest of Block 1

Block 1.

Note that CNOTs denoted by red color are there to make transformation $|N\rangle_a \to |0\rangle_a$ before ADDER if the value of register t is $|1\rangle_t$, and then undo this operation after ADDER. This is the first time when classically known N affects configuration of the circuit itself: in the case of $N = 5 = 101_2$ we need 2 CNOTs before ADDER and 2 CNOTS after ADDER, but if $N = 6 = 111_2$, we would have to use 3 red CNOTs before ADDER and 3 red CNOTS after ADDER.

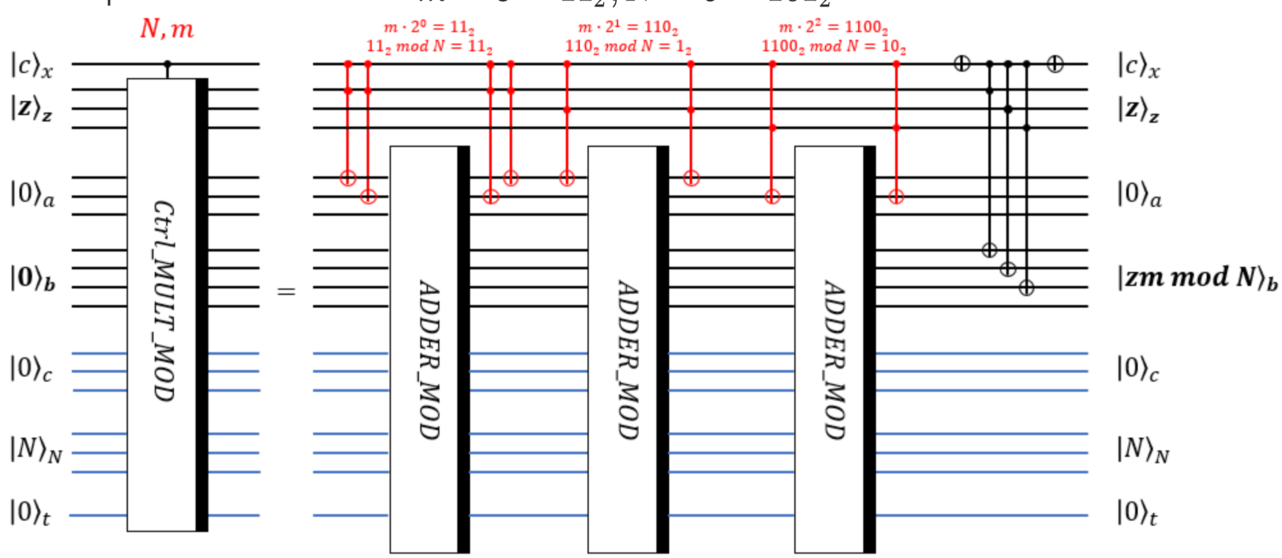
The role of the Block 2 is to uncompute the value $|1\rangle_t$ to $|0\rangle_t$, if it appears.

3-qubit controlled modular multiplication circuit Ctrl_MULT_MOD. Block Ctrl_MULT_MOD implements the following transformation:

$$|c\rangle_x|z\rangle_z|0\rangle_a|0\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t \to |c\rangle_x|z\rangle_z|0\rangle_a|zm \mod N\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t, \text{ if } c = 1$$

$$|c\rangle_x|z\rangle_z|0\rangle_a|0\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t \to |c\rangle_x|z\rangle_z|0\rangle_a|z\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t, \text{ if } c = 0$$

For this particular block we use $m = 3 = 11_2, N = 5 = 101_2$



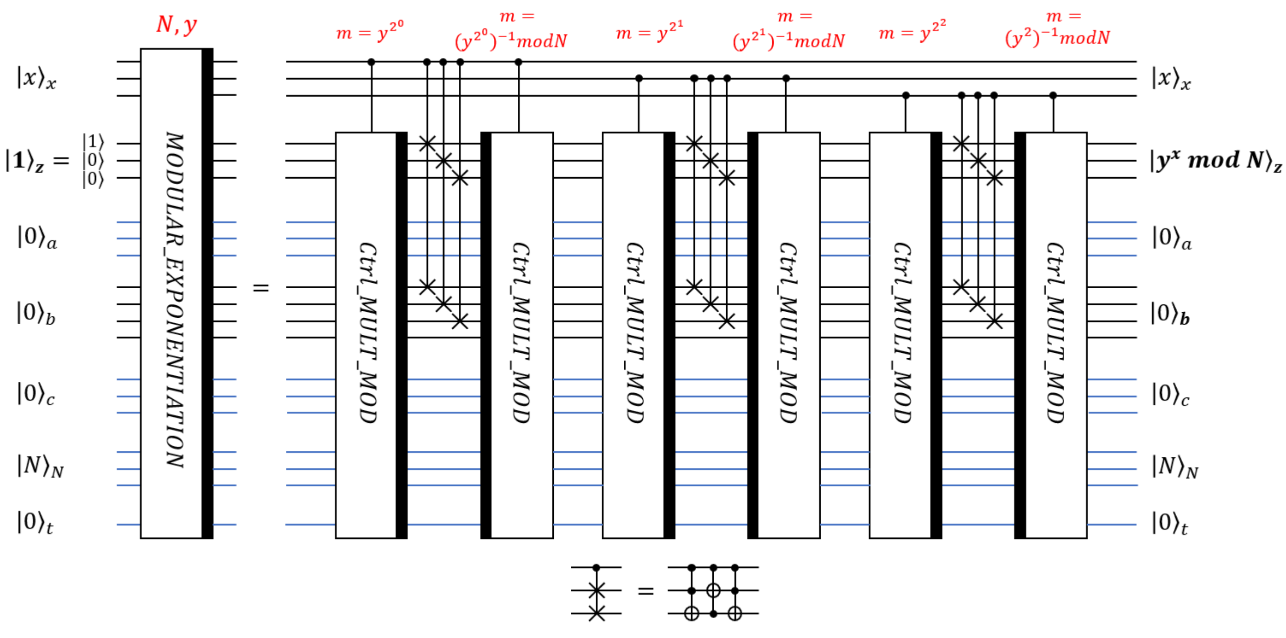
The role of red Toffoli gates is to replace zeros in the register $|0\rangle_a$ with the state $|m \times z_i 2^i \mod N\rangle_a$ to further add up all this numbers to get $|z \times m \mod N\rangle_b$. Red Toffoli gates put values $m \times 2^i \mod N$ in the register a conditionally on values in registers x and z. Note that numbers $m \times 2^i \mod N$ can be classically and efficiently computed. Also note that this is the second time when classically known information affects configuration of the circuit itself.

The last block of CNOTs is used to put value z in the register $|0\rangle_b$ if control $|c\rangle_x$ is $|0\rangle_x$

3-qubit modular exponentiation circuit MODULAR_EXPONENTIATION. Finally, using an array of controlled modular multiplications, we can implement modular exponentiation using known of controlled modular multiplications, we can implement modular exponentiation using known classical information for every step. It should be a succession of controlled modular multiplications with controls set on wires of the register x. But every Ctrl_MULT_MOD should be accompanied by SWAPs and reverse Ctrl_MULT_MOD to reset one of the registers to zero and free it for the next controlled modular multiplication (see the original paper for more details). Notation $(\dots)^{-1} \mod N$ is for modular inverse, which can be efficiently classically precomputed using Euclid's algorithm.

To sum up, this Ctrl_MULT_MOD blocks implement the following chain of transformations which lead to the desired result:

$$|x\rangle_x|1\rangle_z|0\rangle_a|0\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t \rightarrow |x\rangle_x|1 \times y^{x_0 2^0} \mod N\rangle_z|0\rangle_a|0\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t \rightarrow$$
$$\rightarrow |x\rangle_x|1 \times y^{x_0 2^0} \times y^{x_1 2^1} \mod N\rangle_z|0\rangle_a|0\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t \rightarrow \dots \rightarrow$$
$$\rightarrow |x\rangle_x|y^x \mod N\rangle_z|0\rangle_a|0\rangle_b|0\rangle_c|N\rangle_N|0\rangle_t,$$



The last thing I want to mention is that if the size of the register $|N\rangle_N$ is n, then the size of the register $|x\rangle_x$ should be $2n$ to make MODULAR_EXPONENTIATION circuit usable in Shor's algorithm. As one can see from the last picture, going to $2n = 6$ qubits in $|x\rangle_x$ for this particular case requires just additional 3 wires for $|x\rangle$ and additional 3 blocks of [Ctrl_MULT_MOD - SWAPs - inverse Ctrl_MULT_MOD].

Regarding your question about changes which occur when we go from $11^x \mod 15$ to $7^x \mod 21$: for $N = 15$ we need 4 bits to encode this number, so the current architecture requires 8 or less qubits for the register x, 4 qubits for the register z, 4 qubits for the register a, 4+1 qubits for the register b, 4 qubits for the register c, 4 qubits for the register N and 1 qubit for control t. If we use $N = 21$, then it will be 10 or less qubits for the register x, 5 qubits for the register z, 5 qubits for the register a, 5+1 qubits for the register b, 5 qubits for the register c, 5 qubits for the register N and 1 qubit for control t. So one can see that number of qubits grows as $O(n)$, which is acceptable according to the original paper

Share Improve this answer Follow edited Nov 24 '20 at 1:18 answered Nov 22 '20 at 1:26

Anatoly
96 6