Problem 1.

(a) Exercise Set 5.2, Problem 11

$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$, for all integers $n \geq 1$.

Let property $P(n)$ be $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Basis:

$P(1) : \left(\displaystyle\sum_{i=1}^{1} i^3 = 1\right) = \left(\left[\frac{n(n+1)}{2}\right]^2 = 1\right)$ is true.

Inductive hypothesis:

Assume $P(k) : \displaystyle\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$ for $k \geq 1 \in \mathbb{Z}$ is true.

$P(k+1)$:

$$\sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$\left(\sum_{i=1}^{k} i^3\right) + (k+1)^3 = \left[\frac{k^2+3k+2}{2}\right]^2$$

$$\left[\frac{k(k+1)}{2}\right]^2 + k^3 + 3k^2 + 3k + 1 = \left[\frac{k^2+3k+2}{2}\right]^2$$

$$\left[\frac{k^2+k}{2}\right]^2 + k^3 + 3k^2 + 3k + 1 = \left[\frac{k^2+3k+2}{2}\right]^2$$

$$\frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4} = \frac{k^4+3k^3+2k^2+3k^3+9k^2+6k+2k^2+6k+4}{4}$$

$$\frac{k^4+6k^3+13k^2+12k+4}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.2, Problem 14

$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$, for all integers $n \geq 0$.

Let property $P(n)$ be $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$.

Basis:

$P(0) : \left(\sum_{i=1}^{1} i \cdot 2^i = 2 \right) = \big(0 \cdot 2^2 + 2 = 2\big)$ is true.

Inductive hypothesis:

Assume $P(k) : \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$ for $k \geq 0 \in \mathbb{Z}$ is true.

$P(k+1)$:

$$\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^i &= (k+1) \cdot 2^{k+3} + 2 \\
\left(\sum_{i=1}^{k+1} i \cdot 2^i \right) + (k+2) \cdot 2^{k+2} &= (k+1) \cdot 2^{k+3} + 2 \\
k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} &= (k+1) \cdot 2^{k+3} + 2 \\
k \cdot 2^{k+2} + 2 + k \cdot 2^{k+2} + 2^{k+3} &= (k+1) \cdot 2^{k+3} + 2 \\
2k \cdot 2^{k+2} + 2 + 2^{k+3} &= (k+1) \cdot 2^{k+3} + 2 \\
k \cdot 2^{k+3} + 2 + 2^{k+3} &= (k+1) \cdot 2^{k+3} + 2 \\
(k+1) \cdot 2^{k+3} + 2 &= (k+1) \cdot 2^{k+3} + 2
\end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(c) Exercise Set 5.2, Problem 17

$\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!}$, for all integers $n \geq 0$.

Let property $P(n)$ be $\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)$.

Basis:

$P(0) : \left(\prod_{i=0}^{0}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{2}\right)=\left(\frac{1}{(0+2)!}=\frac{1}{2}\right)$ is true.

Inductive Hypothesis:

Assume $P(k) : \prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2k+2)!}$ for $k \geq 0 \in \mathbb{Z}$ is true.

$P(k+1)$:

$$\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2(k+1)+2)!}$$

$$\prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)\left(\frac{1}{2(k+1)+1}\cdot\frac{1}{2(2k+1)+2}\right)=\frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!}\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right)=\frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!(2k+3)(2k+4)}=\frac{1}{(2k+4)(2k+3)!}$$

$$\frac{1}{(2k+4)(2k+3)(2k+2)!}=\frac{1}{(2k+4)(2k+3)(2k+2)!}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(d) Exercise Set 5.3, Problem 10

$n^3 - 7n + 3$ is divisible by 3, for each integer $n \geq 0$.

Let property $P(n)$ be $n^3 - 7n + 3$ is divisible by 3.

Basis:

$P(0) : (0^3 - 7\cdot 0 + 3 = 0)$ is divisible by 3 is true.

Inductive Hypothesis:

Assume $P(k) : k^3 - 7k + 3$ is divisible by 3 where $k \geq 0 \in \mathbb{Z}$ is true.

$P(k+1) : (k+1)^3 - 7(k+1) + 3$

$k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3$

$\underbrace{(k^3 - 7k + 3)}_{3a} + 3k^2 + 3k + 1 - 7$

$P(k)$ true, therefore multiple of some integer a

$3a + 3k^2 + 3k - 6$

$3a + 3k^2 + 3k - 6$

$3\left(a + k^2 + k - 2\right)$

Expression is a multiple of 3, therefore $P(k+1)$ is divisible by 3.

Basis and inductive hypothesis proven, therefore original statement is true.

(e) Exercise Set 5.3, Problem 17
$1 + 3n \leq 4^n$, for every integer $n \geq 0$.
Let property $P(n)$ be $1 + 3n \leq 4^n$.
Basis:
$P(0) : (1 + 3 \cdot 0 = 1) \leq (4^0 = 1)$ is true.
Inductive hypothesis:
Assume $P(k) : 1 + 3k \leq 4^k$ for $k \geq 0 \in \mathbb{Z}$ is true.
$P(k+1)$:

$$\begin{aligned}
1 + 3(k+1) &\leq 4^{k+1} \\
4 + 3k &\leq 4^{k+1} \\
1 + 3k + 3 &\leq 4^{k+1} \\
P(k) + 3 &\leq 4^{k+1} \\
4^k + 3 &\leq 4^{k+1} \\
3 &\leq 4^{k+1} - 4^k \\
3 &\leq 4^k(4-1) \\
3 &\leq 3 \cdot 4^k \\
1 &\leq 4^k
\end{aligned}$$

Last inequality holds true for all $k \geq 0 \in \mathbb{Z}$.
Basis and inductive hypothesis proven, therefore original statement is true.

(f) Exercise Set 5.3, Problem 21
$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for all integers $n \geq 2$.
Let property $P(n)$ be $\sqrt{n} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$.
Basis:
$P(2) : (\sqrt{2}) < \left(\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} \right)$ is true.
Inductive hypothesis:
Assume $P(k) : \sqrt{k} < \sum_{i=1}^{k} \frac{1}{\sqrt{i}}$ for $k \geq 2 \in \mathbb{Z}$ is true.
$P(k+1)$:

$$\begin{aligned}
\sqrt{k+1} &< \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} \\
\sqrt{k+1} &< \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} \\
\sqrt{k+1} &< \sqrt{k} + \frac{1}{\sqrt{k+1}} \\
\frac{k+1}{\sqrt{k+1}} &< \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}} \\
k+1 &< \sqrt{k}\sqrt{k+1} + 1 \\
\sqrt{k}\sqrt{k} &< \sqrt{k}\sqrt{k+1} \\
k &< k+1
\end{aligned}$$

Inequality holds true for all $k \geq 2$.
Basis and inductive hypothesis proven, therefore original statement is true.

(g) Exercise Set 5.3, Problem 22

$1 + nx \leq (1+x)^n$, for all real numbers $x > -1$ and integers $n \geq 2$.

Let property $P(n, x)$ be $1 + nx \leq (1+x)^n$.

Basis:

$P(2, x) : 1 + 2x \leq (1+x)^2 \Rightarrow 1 + 2x \leq 1 + 2x + x^2 \Rightarrow 0 \leq x^2$ is true for all $x \in \mathbb{R}$.

Inductive hypothesis:

Assume $P(k, x) : 1 + kx \leq (1+x)^k$ for $k \geq 2 \in \mathbb{Z}$ and $x > -1 \in \mathbb{R}$ is true.

$P(k+1, x)$:

$$\begin{aligned}
1 + (k+1)x &\leq (1+x)^{k+1} \\
1 + kx + x &\leq (1+x)^{k+1} \\
(1+x)^k + x &\leq (1+x)^{k+1} \\
x &\leq (1+x)^k((1+x) - 1) \\
x &\leq (1+x)^k x
\end{aligned}$$

Case 1: $x = 0$

Prove: $0 \leq (1+0)^k \cdot 0$

$0 \leq (1+0)^k \cdot 0$

$0 \leq 0$

Proof done.

Case 2: $x > 0$

Prove: $x \leq (1+x)^k x$

$$\begin{array}{r|l}
x \leq (1+x)^k x & x > 0 \\
= 1 \leq (1+x)^k & 1 < 1 + x \\
& 1^k < (1+x)^k \\
& 1 < (1+x)^k
\end{array}$$

$1 < (1+x)^k \Rightarrow 1 \leq (1+x)^k$

Proof done.

Case 3: $-1 < x < 0$

Prove: $x \leq (1+x)^k x$

$$\begin{array}{r|rr}
x \leq (1+x)^k x & 0 > & x > -1 \\
= 1 \geq (1+x)^k & 1 > & 1 + x > 0 \\
& 1^k > & (1+x)^k > 0^k \\
& 1 > & (1+x)^k > 0
\end{array}$$

$1 > (1+x)^k > 0 \Rightarrow 1 \geq (1+x)^k$

Proof done.

$P(k+1, x)$ for $k \geq 2 \in \mathbb{Z}$ and $x > -1 \in \mathbb{R}$ is true in all cases.

Basis and inductive hypothesis proven, therefore original statement is true.

(h) Exercise Set 5.3, Problem 29

As each of a group of businesspeople arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if n people come to the meeting then $[n(n-1)]/2$ handshakes occur.

If the set of businesspeople has size n, then the number of handshakes is $\binom{n}{2}$.

For a set of 0 and 1 businesspeople, no handshakes occur.

$\binom{n}{2} = \dfrac{n!}{2!(n-2)!} = \dfrac{n!}{2(n-2)!}$

Let property $P(n)$ be $\begin{cases} 0 = \dfrac{n(n-1)}{2} & n = 0,\ n = 1 \\ \dfrac{n!}{2(n-2)!} = \dfrac{n(n-1)}{2} & n \geq 2 \end{cases}$ for non negative integers n.

Basis:

$P(0) : 0 = \left(\dfrac{0(0-1)}{2} = 0\right)$ is true.

$P(1) : 0 = \left(\dfrac{1(1-1)}{2} = 0\right)$ is true.

$P(2) : \left(\dfrac{2!}{2(2-2)!} = \dfrac{2}{2 \cdot 1} = 1\right) = \left(\dfrac{2(2-1)}{2} = 1\right)$ is true.

Inductive hypothesis:

Assume $P(k) : \dfrac{k!}{2(k-2)!} = \dfrac{k(k-1)}{2}$ for $k \geq 0 \in \mathbb{Z}$ to be true.

$P(k+1)$:

$$\frac{(k+1)!}{2((k+1)-2)!} = \frac{(k+1)((k+1)-1)}{2}$$

$$\frac{(k+1)k!}{2(k-1)!} = \frac{(k+1)k}{2}$$

$$\frac{(k+1)k(k-1)!}{2(k-1)!} = \frac{(k+1)k}{2}$$

$$\frac{(k+1)k}{2} = \frac{(k+1)k}{2}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(i) Prove that in an n-sided regular polygon, where $n \geq 3$, the number of diagonals is $n(n-3)/2$.

The number of possible vertex pairs in an n-sided regular polygon is $\binom{n}{2}$, and n of these vertex pairs are the edges of the polygon. The number of diagonals is $\binom{n}{2} - n$.

$$\binom{n}{2} - n = \frac{n!}{2!(n-2)!} - n = \frac{n!}{2(n-2)!} - n$$

Let property $P(n)$ be $\frac{n!}{2(n-2)!} - n = \frac{n(n-3)}{2}$.

Basis:

$P(3) : \left(\frac{3!}{2(3-2)!} - 3 = \frac{6}{2} - 3 = 0\right) = \left(\frac{3(3-3)}{2} = 0\right)$ is true.

Inductive hypothesis:

Assume $P(k) : \frac{k!}{2(k-2)!} - k = \frac{k(k-3)}{2}$ for $k \geq 3$ to be true.

$P(k+1)$:

$$\frac{(k+1)!}{2((k+1)-2)!} - (k+1) = \frac{(k+1)((k+1)-3)}{2}$$
$$\frac{(k+1)k!}{2(k-1)!} - k - 1 = \frac{(k+1)(k-2)}{2}$$
$$\frac{(k+1)k(k-1)!}{2(k-1)!} - k - 1 = \frac{(k+1)(k-2)}{2}$$
$$\frac{(k+1)k}{2} - \frac{2k+2}{2} = \frac{(k+1)(k-2)}{2}$$
$$\frac{k^2+k-2k-2}{2} = \frac{k^2-k-2}{2}$$
$$\frac{k^2-k-2}{2} = \frac{k^2-k-2}{2}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(j) Prove that the number of permutations of the set $\{1, 2, \ldots, n\}$ with n elements is $n!$, for natural number $n \geq 1$.

A set of $n = 1$ elements has $1! = 1$ permutation.

Let property $P(n)$ be $\{1, 2, \ldots, n\}$ has $n!$ permutations.

Basis:

$P(1) : \{1\}$ has $1! = 1$ permutation is true.

Inductive hypothesis:

Assume $P(k) : \{1, 2, \ldots, k\}$ has $k!$ permutations for $k \geq 1$ to be true.

In order to create a permuted set B_p of size $k+1$, one can insert $k+1$ into A_p, an arbitrary permutation of set A of k elements.

This action is equivalent to permuting a set of size $k+1$.

There are $k+1$ positions to insert such an element into A_p: k positions before each element and one position after the last element of A_p.

There are $k!$ possible A_p made from A.

($k+1$ ways to insert into A_p) $\times$ ($k!$ possible A_p) $= (k+1)k! = (k+1)!$ ways to create B_p.

$P(k+1)$ is true.

Basis and inductive hypothesis proven, therefore original statement is true.

Problem 2.

(a) Exercise Set 5.4, Problem 8
Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:
$h_0 = 1$, $h_1 = 2$, $h_2 = 3$,
$h_k = h_{k-1} + h_{k-2} + h_{k-3}$ for all integers $k \geq 3$.

(a) Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.
Let property $P(n)$ be $h_n \leq 3^n$ for all integers $n \geq 0$.
Basis:
$P(0) : (h_0 = 1) \leq (3^0 = 1)$ is true.
$P(1) : (h = 1 = 2) \leq (3^1 = 3)$ is true.
$P(2) : (h_2 = 3) \leq (3^2 = 9)$ is true.
Inductive hypothesis:
Let $k \geq 2$.
Assume $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq 3^i$ for $0 \leq i \leq k$ and $i \in \mathbb{Z}$.
$P(k+1) :$

$$
\begin{aligned}
h_{k+1} =& h_k + h_{k-1} + h_{k-2} \\
\leq& 3^k + 3^{k-1} + 3^{k-2} \\
\leq& 3^{k-2}\left(3^2 + 3 + 1\right) \\
\leq& 13 \cdot 3^{k-2} \leq \left(3^3 \cdot 3^{k-2} = 3^{k+1}\right)
\end{aligned}
$$

$P(k+1) : h_{k+1} \leq 3^{k+1}$ is true.
Basis and inductive hypothesis proven, therefore original statement is true.

(b) Suppose that s is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $2 > s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$.
Let property $P(n)$ be $h_n \leq s^n$ for $n \geq 2 \in \mathbb{Z}$.
Basis:
$P(2) : (h_2 = 2) \leq (3.34 < s^2 < 4)$ is true.
$P(3) : (h_3 = 6) \leq (6.12 < s^3 < 8)$ is true.
$P(4) : (h_4 = 11) \leq (11.21 < s^4 < 16)$ is true.
Inductive hypothesis:
Let $k \geq 4$.
Assume $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq s^i$ for $2 \leq i \leq k$ and $i \in \mathbb{Z}$.
$P(k+1) :$

$$
\begin{aligned}
h_{k+1} =& h_k + h_{k-1} + h_{k-2} \\
\leq& s^k + s^{k-1} + s^{k-2} \\
\leq& s^{k-2}\left(s^2 + s + 1\right) \\
\leq& s^{k-2} s^3 \\
\leq& s^{k+1}
\end{aligned}
$$

$P(k+1) : h_{k+1} \leq s^{k+1}$ is true.
Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.4, Problem 9

Define a sequence $a_1, a_2, a_3, \ldots$ as follows: $a_1 = 1$, $a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$. Use strong mathematical induction to prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

Let property $P(n)$ be $a_n \leq \left(\frac{7}{4}\right)^n$.

Basis:

$P(1) : 1 \leq \frac{7}{4}$ is true.

$P(2) : 3 \leq \frac{49}{16}$ is true.

Inductive hypothesis:

Let $k \geq 2$.

Assume $P(i) : a_i = a_{i-1} + a_{i-2} \leq \left(\frac{7}{4}\right)^i$ for $1 \leq i \leq k$ and $i \in \mathbb{Z}$.

$P(k+1)$:

$$
\begin{aligned}
a_{k+1} &= a_k + a_{k-1} \\
&\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \\
&\leq \left(\frac{7}{4}\right)^{k-1} \left(1 + \frac{7}{4}\right) \\
&\leq \left(\frac{11}{4}\left(\frac{7}{4}\right)^{k-1} = \frac{44}{16}\left(\frac{7}{4}\right)^{k-1}\right) \leq \left(\frac{49}{16}\left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^2\left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^{k+1}\right)
\end{aligned}
$$

$P(k+1) : a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$ is true.

Basis and inductive hypothesis proven, therefore original statement is true.

(c) Exercise Set 5.4, Problem 25(b)

(d) Exercise Set 5.4, Problem 30

(e) Exercise Set 5.5, Problem 30

(f) Let f be a function on whole numbers satisfying

$$
f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 4f(n/2) & \text{if } n > 0 \text{ and even} \\ f(n-1) + 2n - 1 & \text{if } n > 0 \text{ and odd} \end{cases}
$$

Prove that $f(n) = n^2$ for all $n \geq 0$

Problem 3.

(a) Exercise Set 5.5, Problems 28, 29

(b) Exercise Set 5.6, Problem 2(b,d)

(c) Exercise Set 5.6, Problems 9, 14, 15