Problem 1.
Exercise Set 4.1

52. For all integers m, if $m > 2$ then $m^2 - 4$ is composite.
    False.
    Counterexample:
    Let $m = 3$.
    $3^2 - 4 = 5$
    5 is not composite.

53. For all integers n, $n^2 - n + 11$ is a prime number.
    False.
    Counterexample:
    Let $n = 11$.
    $11^2 - 11 + 11 = 121$
    121 isn't prime.

Problem 2.
Exercise Set 4.1

61. Suppose that integers m and n are perfect squares. Then $m+n+2\sqrt{mn}$ is also a perfect square. Why?
    Proof:
    Let $m = a^2$, where $a \in \mathbb{Z}$.
    Let $n = b^2$, where $b \in \mathbb{Z}$.
    $$
    \begin{aligned}
    m + n + 2\sqrt{mn} &= a^2 + b^2 + 2\sqrt{a^2b^2} \quad \text{substitution} \\
    &= a^2 + b^2 + 2ab \\
    &= (a+b)^2 \qquad\qquad \text{factoring} \\
    &= c^2 \qquad\qquad\quad\; \text{let } c = a + b
    \end{aligned}
    $$
    $c \in \mathbb{Z}$ because addition is closed under $\mathbb{Z}$
    $m + n + 2\sqrt{mn}$ is a perfect square by definition (square of some integer).

Problem 3.
Exercise Set 4.2

30. Prove that if one solution for a quadratic equation of the form $x^2 + bx + c = 0$ is rational (where b and c are rational), then the other solution is also rational.
    Proof:
    Let s and r be the roots of the equation.
    $$
    \begin{aligned}
    x^2 + bx + c &= (x-s)(x-r) \qquad \text{factored form} \\
    &= x^2 - xr - xs + rs \quad \text{distribute} \\
    &= x^2 - (r+s)x + rs \quad \text{factor}
    \end{aligned}
    $$
    $b = -(r+s)$ and $c = rs$
    $b, c \in \mathbb{Q}$, therefore $-(r+s), rs \in \mathbb{Q}$.

Case 1: $r, s \in \mathbb{Q}$
Proof done.
Case 2: $r, s \notin \mathbb{Q}$
Irrelevant.
Case 3: WLOG let $r \in \mathbb{Q}$ and $s \notin \mathbb{Q}$.
$-(r+s) \in \mathbb{Q} \Rightarrow r + s \in \mathbb{Q}$
By Theorem 4.5.3, $r + s \notin \mathbb{Q}$.
Contradiction.

The only valid, relevant case shows that both r and s must be rational.

Problem 4.
Exercise Set 4.2

38. The "proof" does not prove that $\dfrac{a}{b} + \dfrac{c}{d}$ is rational.

39. The "proof" assumes $r + s \in \mathbb{Q}$ to prove $r + s \in \mathbb{Q}$.

Problem 5.
Exercise Set 4.3

30. For all integers a and n, if $a \mid n^2$ and $a \leq n$ then $a \mid n$.
    False.
    Counterexample:
    Let $a = 4$ and $n = 6$.
    $4 \mid 36$
    $4 \nmid 6$

Problem 6.
Exercise Set 4.3

34. Is it possible to have 50 coins, made up of pennies, dimes, and quarters, that add up to $3? Explain.
    Not possible.
    Proof:
    Let p, d, $q \in \mathbb{Z}^+$ be the number of pennies, dimes, and quarters, respectively.
    Assume there such a configuration is possible.
    $$\begin{aligned} p+ \quad d+ \quad q &= 50 \\ .01p+ \quad .10d+ \quad .25q &= 3 \end{aligned}$$

    $$\begin{aligned} p+ \quad 10d+ \quad 25q &= 300 \\ -(p+ \quad d+ \quad q &= 50) \\ = \qquad\qquad 9d+ \quad 24q &= 250 \end{aligned}$$
    $3(3d + 8q) = 250$
    $3d + 8q = \dfrac{250}{3}$

    $3d + 8q \in \mathbb{Z}^+$ because addition and multiplication is closed on $\mathbb{Z}^+$, but $\dfrac{250}{3} \notin \mathbb{Z}^+$.
    Contradiction, therefore no configuration exists.

Problem 7.
Exercise Set 4.3

35. Two athletes run a circular track at a steady pace so that the first completes one round in 8 minutes and the second in 10 minutes. If they both start from the same spot at 4 P.M., when will be the first time they return to the start together?
    At 4:40 P.M.

Problem 8.
Exercise Set 4.3

42c. Without computing the value of $(20!)^2$ determine how many zeros are at the end of this number when it is written in decimal form. Justify your answer.
8 zeroes.
Proof:
Multiplying by 10 adds a trailing zero.
Factoring out all 10s from 20! gives the number of trailing zeros in 20!.
$20! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13 \times 14 \times 15 \times 16 \times 17 \times 18 \times 19 \times 20$
$20! = \underbrace{(10 \times 20)}_{\text{obvious 10s}} \times 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 11 \times 12 \times 13 \times 14 \times 15 \times 16 \times 17 \times 18 \times 19$

$20! = \underbrace{(10 \times 20)}_{\text{obvious 10s}} \overbrace{(2 \times 5)(4 \times 15)}^{\text{remaining 10s}} \times \underbrace{\begin{pmatrix} 1 \times 3 \times 4 \times 6 \times 7 \times 8 \times 9 \times 11 \times \\ 12 \times 13 \times 14 \times 16 \times 17 \times 18 \times 19 \end{pmatrix}}_{\text{remaining factors}}$

Remaining factors don't have any more 10s to factor out and no more 5s to factor out to pair with 2s to create 10s.

$20! = 10^4 \times 12 \times \underbrace{\overbrace{\begin{pmatrix} 1 \times 3 \times 4 \times 6 \times 7 \times 8 \times 9 \times 11 \times \\ 12 \times 13 \times 14 \times 16 \times 17 \times 18 \times 19 \end{pmatrix}}^{a}}_{\text{remaining factors}} = 10^4 a$

a, like the remaining factors, doesn't have any 10s nor 5 and 2 pairs to factor out.
10 can be factored out of 20! 4 times, therefore 20! has 4 leading zeros.
$(20!)^2 = (10^4 a)^2 = 10^8 a^2$
a^2 has no 10s to factor out for the same reason a doesn't.
10 can be factored out of $(20!)^2$ 8 times, therefore $(20!)^2$ has 8 trailing zeros.

Problem 9.
Exercise Set 4.3

43. In a certain town 2/3 of the adult men are married to 3/5 of the adult women. Assume that all marriages are monogamous (no one is married to more than one other person). Also assume that there are at least 100 adult men in the town. What is the least possible number of adult men in the town? of adult women in the town?
108 men and 120 women.
Proof:
Let $M \geq 100 \in \mathbb{Z}$ and $F \in \mathbb{N}$ be the number of men and women respectively.
Let $M_{married} = \dfrac{2}{3}M \in \mathbb{Z}$ and $F_{married} = \dfrac{3}{5}F \in \mathbb{Z}$
$M_{married} = F_{married} \Rightarrow \dfrac{2}{3}M = \dfrac{3}{5}F$
$F = \dfrac{10}{9}M$ and $M = \dfrac{9}{10}F$
No fractional number of people so $9 \mid M$ and $10 \mid F$
$F > M$ and we want to minimize adults, so find the minimal M.
$M = ((\text{Smallest multiple of } 9) \geq 100) = 108$
$F = \dfrac{10}{9}M = \dfrac{10}{9}108 = 120$

3

Problem 10.
Exercise Set 4.4

30a. Use the quotient-remainder theorem with $d = 3$ to prove that the product of any two consecutive integers has the form $3k$ or $3k + 2$ for some integer k.
Proof:
Let n be an arbitrary integer.
By the quotient remainder theorem, $n = 3q + r$ where $q, r \in \mathbb{Z}$ and $0 \leq r < 3$.

Case 1: $n = 3q + 0$	**Case 2:** $n = 3q + 1$	**Case 3:** $n = 3q + 2$
$n + 1 = 3q + 1$	$n + 1 = 3q + 2$	$n + 1 = 3q + 3$
$n(n+1) = 3q(3q+1)$	$n(n+1) = (3q+1)(3q+2)$	$n(n+1) = (3q+2)(3q+3)$
$= 3\underbrace{(9q^2 + q)}_{k}$	$= 9q^2 + 9q + 2$	$= (3q+2)(3(q+1))$
	$= 3\underbrace{(3q^2 + 3q)}_{k} + 2$	$= 3\underbrace{(3q+2)(q+1)}_{k}$
$n(n+1) = 3k$ and $k \in \mathbb{Z}$	$n(n+1) = 3k + 2$ and $k \in \mathbb{Z}$	$n(n+1) = 3k$ and $k \in \mathbb{Z}$

In all cases, the product of consecutive integers can be written in the form $3k$ or $3k + 2$ for some integer k, therefore proposition is true.

Problem 11.
Exercise Set 4.4

31a. Prove that for all integers m and n, $m + n$ and $m - n$ are either both odd or both even.
Proof:

Case 1: m and n are odd	
$m + n = (2j + 1) + (2k + 1)$	$m - n = (2j + 1) - (2k + 1)$
$\quad = 2j + 2k + 2$	$\quad = 2j - 2k$
$\quad = 2\underbrace{(j + k + 1)}_{a}$	$\quad = 2\underbrace{(j - k)}_{a}$
$\quad = 2a$ and $a \in \mathbb{Z}$ therefore even	$\quad = 2a$ and $a \in \mathbb{Z}$ therefore even

Both expressions are even.

Case 2: m and n are even	
$m + n = (2j) + (2k)$	$m - n = (2j) - (2k)$
$\quad = 2\underbrace{(j + k)}_{a}$	$\quad = 2\underbrace{(j - k)}_{a}$
$\quad = 2a$ and $a \in \mathbb{Z}$ therefore even	$\quad = 2a$ and $a \in \mathbb{Z}$ therefore even

Both expressions are even.

Case 3: WLOG m is odd and n is even	
$m + n = (2j + 1) + (2k)$	$m - n = (2j + 1) - (2k)$
$\quad = 2j + 2k + 1$	$\quad = 2j - 2k + 1$
$\quad = 2\underbrace{(j + k)}_{a} + 1$	$\quad = 2\underbrace{(j - k)}_{a} + 1$
$\quad = 2a + 1$ and $a \in \mathbb{Z}$ therefore odd	$\quad = 2a + 1$ and $a \in \mathbb{Z}$ therefore odd

Both expressions are odd.

In all cases, $m + n$ and $m - n$ have the same parity, therefore proposition is true.

Problem 12.
Exercise Set 4.4

40. For any integer n, $n(n^2 - 1)(n + 2)$ is divisible by 4.
    Proof:
    Let $E = n(n^2 - 1)(n + 2) = n(n - 1)(n + 1)(n + 2) = (n - 1)(n)(n + 1)(n + 2)$.
    Let $a = n - 1$.
    $E = a(a + 1)(a + 2)(a + 3)$

Case 1: $a \bmod 4 = 0$	**Case 2:** $a \bmod 4 = 1$
$a = 4q$ where $q \in \mathbb{Z}$	$a = 4q + 1$ where $q \in \mathbb{Z}$
$E = (4q)(4q + 1)(4q + 2)(4q + 3)$ $= 4\underbrace{q(4q + 1)(4q + 2)(4q + 3)}_{k}$	$E = (4q + 1)(4q + 2)(4q + 3)\overbrace{(4q + 4)}^{4(q+1)}$ $= 4\underbrace{(q + 1)(4q + 1)(4q + 2)(4q + 3)}_{k}$
$E = 4k$ and k $\in \mathbb{Z}$	$E = 4k$ and k $\in \mathbb{Z}$
Case 3: $a \bmod 4 = 2$	**Case 4:** $a \bmod 4 = 3$
$a = 4q$ where $q \in \mathbb{Z}$	$a = 4q + 1$ where $q \in \mathbb{Z}$
$E = (4q + 2)(4q + 3)\overbrace{(4q + 4)}^{4(q+1)}(4q + 5)$ $= 4\underbrace{(q + 1)(4q + 2)(4q + 3)(4q + 5)}_{k}$	$E = (4q + 3)\overbrace{(4q + 4)}^{4(q+1)}(4q + 5)(4q + 6)$ $= 4\underbrace{(q + 1)(4q + 3)(4q + 4)(4q + 6)}_{k}$
$E = 4k$ and k $\in \mathbb{Z}$	$E = 4k$ and k $\in \mathbb{Z}$

In all cases, $E = n(n^2 - 1)(n + 2)$ is divisible by 4, therefore proposition is true.

Problem 13.
Exercise Set 4.5

24. The reciprocal of any irrational number is irrational.
    Proof:
    Using $\mathbb{I}$ as the set of irrationals.

Theorem A:
If $a \neq 0 \in \mathbb{Q}$ and $b \in \mathbb{I}$ then $ab \in \mathbb{I}$.
Proof:
Let $c = ab$.
Assume opposite and $c \in \mathbb{Q}$.
$ab = c \Rightarrow \dfrac{j}{k}b = \dfrac{m}{n}$ where $j,\ k \neq 0,\ m,\ n \neq 0 \in \mathbb{Z}$
$b = \dfrac{mk}{nj}$
$mk,\ nj \in \mathbb{Z}$
b is a ratio of two integers, therefore $b \in \mathbb{Q}$.
Contradiction because $b \in \mathbb{I}$ and $b \in \mathbb{Q}$, therefore assumption is false and $ab \in \mathbb{I}$.

Let a be an arbitrary irrational number.
Assume opposite and $\dfrac{1}{a} \in \mathbb{Q}$.
$\dfrac{1}{a} = \dfrac{x}{y}$ where $x,\ y \neq 0 \in \mathbb{Z}$
$y = ax$
By Theorem A, $ax = y$ must be irrational.
Contradiction because $y \in \mathbb{Z}$ and $y \in \mathbb{I}$, therefore assumption is false and original proposition is true.

Problem 14.
Exercise Set 4.5

31b. For all integers $n > 1$, if n is not prime, then there exists a prime number p such that $p \leq \sqrt{n}$ and n is divisible by p.

Proof:

Theorem 4.3.1:
For all integers a and b, if a and b are positive and a divides b, then $a < b$.

Theorem 4.3.3:
For all integers a, b, and c, if a divides b and b divides c, then a divides c.

Theorem 4.3.4:
Any integer $n > 1$ is divisible by a prime number.

Theorem A:
For any $n = ab$ where $a, b \in \mathbb{Z}^+$, if $a \leq b$ then $a \leq \sqrt{n}$ and $b \geq \sqrt{n}$.
Proof:
$n \in \mathbb{Z}^+ \Rightarrow \sqrt{n} \geq 1$
Assume opposite.
There exists $n = ab$ where $a, b \in \mathbb{Z}^+$ such that $a \leq b$ and $(a > \sqrt{n}$ or $b < \sqrt{n})$.
By DeMorgan's laws, $(a \leq b$ and $a > \sqrt{n})$ or $(a \leq b$ and $b < \sqrt{n})$.

> **Case 1:** $a \leq b$ and $a > \sqrt{n}$
>
> $\sqrt{n} < a \leq b$
> $a > \sqrt{n}$ and $b > \sqrt{n}$
> $b, \sqrt{n} > 0$, therefore $ab > b\sqrt{n}$ and $b\sqrt{n} > n$.
> $ab > n$
> Contradiction because $ab > n$ and $ab = n$.
>
> **Case 2:** $a \leq b$ and $b < \sqrt{n}$
>
> $a \leq b < \sqrt{n}$
> $a < \sqrt{n}$ and $b < \sqrt{n}$
> $b, \sqrt{n} > 0$, therefore $ab < b\sqrt{n}$ and $b\sqrt{n} < n$.
> $ab < n$
> Contradiction because $ab < n$ and $ab = n$.
>
> **Case 3:** $(a \leq b$ and $a > \sqrt{n})$ and $(a \leq b$ and $b < \sqrt{n})$
>
> See previous cases.

Contradiction in all cases of the opposite, therefore original proposition is true.

By definition of composite number, $n = pq$ where $q \in \mathbb{Z}^+$ and $0 \leq q < n$.

Case 1: $p \leq q$
By Theorem A, $p \leq \sqrt{n}$ and proposition is true.

Case 2: $p > q$
By Theorem A, $q \leq \sqrt{n}$.
By Theorem 4.3.4, let p_2 be a prime factor of q.
By Theorem 4.3.1, $p_2 < q$.
By Theorem 4.3.3, because $p_2 \mid q$ and $q \mid n$, $p_2 \mid n$.
$p_2 < q \leq \sqrt{n} \Rightarrow p_2 \leq \sqrt{n}$ and proposition is true.

In all cases the proposition is true.

Problem 15.
Exercise Set 4.5

33. Use the sieve of Eratosthenes to find all prime numbers less than 100.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Problem 16.
Show that among any set of arbitrary (1 trillion + 1) natural numbers, one can find two numbers so that their difference is divisible by 1 trillion.

Proof:

Let $\mathbb{S} \subset \mathbb{N}$ be the set of $10^{12} + 1$ arbitrary natural numbers.

Map every element $e \in \mathbb{S}$ under e mod 10^{12} and call the set $\mathbb{T}$.

The maximum cardinality of $\mathbb{T}$ is 10^{12}. By quotient remainder theorem, remainders of division by 10^{12} are in the range $[0, 10^{12})$. The largest $\mathbb{T}$ possible contains all these remainders.

Because $|\mathbb{S}| = 10^{12} + 1 > \max(|\mathbb{T}|) = 10^{12}$, by the pigeonhole principle, there exists two elements $a, b \in \mathbb{S}$ that map to the same element $m \in \mathbb{T}$.

$m = a$ mod $10^{12} = b$ mod 10^{12}

By the quotient remainder theorem, $a = 10^{12}j + m$ and $b = 10^{12}k + m$ where $j, k \in \mathbb{Z}$.

$a - b = (10^{12}j + m) - (10^{12}k + m) = 10^{12}(j - k)$

The difference $a - b$ is a multiple of 10^{12} therefore proposition is true.