

Problem 1.

(a) Exercise Set 5.2, Problem 11

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2, \text{ for all integers } n \geq 1.$$

$$\text{Let property } P(n) \text{ be } \sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2} \right]^2.$$

Basis:

$$P(1) : \left(\sum_{i=1}^1 i^3 = 1 \right) = \left(\left[\frac{1(1+1)}{2} \right]^2 = 1 \right) \text{ is true.}$$

Inductive hypothesis:

$$\text{Assume } P(k) : \sum_{i=1}^k i^3 = \left[\frac{k(k+1)}{2} \right]^2 \text{ for } k \geq 1 \in \mathbb{Z} \text{ is true.}$$

$$P(k+1) :$$

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \left[\frac{(k+1)(k+2)}{2} \right]^2 \\ \left(\sum_{i=1}^k i^3 \right) + (k+1)^3 &= \left[\frac{k^2 + 3k + 2}{2} \right]^2 \\ \left[\frac{k(k+1)}{2} \right]^2 + k^3 + 3k^2 + 3k + 1 &= \left[\frac{k^2 + 3k + 2}{2} \right]^2 \\ \left[\frac{k^2 + k}{2} \right]^2 + k^3 + 3k^2 + 3k + 1 &= \left[\frac{k^2 + 3k + 2}{2} \right]^2 \\ \frac{k^4 + 2k^3 + k^2}{4} + \frac{4k^3 + 12k^2 + 12k + 4}{4} &= \frac{k^4 + 3k^3 + 2k^2 + 3k^3 + 9k^2 + 6k + 2k^2 + 6k + 4}{4} \\ \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} &= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.2, Problem 14

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2, \text{ for all integers } n \geq 0.$$

Let property $P(n)$ be $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$.

Basis:

$$P(0) : \left(\sum_{i=1}^1 i \cdot 2^i = 2 \right) = (0 \cdot 2^2 + 2 = 2) \text{ is true.}$$

Inductive hypothesis:

Assume $P(k) : \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$ for $k \geq 0 \in \mathbb{Z}$ is true.

$P(k+1) :$

$$\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$$

$$\left(\sum_{i=1}^{k+1} i \cdot 2^i \right) + (k+2) \cdot 2^{k+2} = (k+1) \cdot 2^{k+3} + 2$$

$$k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} = (k+1) \cdot 2^{k+3} + 2$$

$$k \cdot 2^{k+2} + 2 + k \cdot 2^{k+2} + 2^{k+3} = (k+1) \cdot 2^{k+3} + 2$$

$$2k \cdot 2^{k+2} + 2 + 2^{k+3} = (k+1) \cdot 2^{k+3} + 2$$

$$k \cdot 2^{k+3} + 2 + 2^{k+3} = (k+1) \cdot 2^{k+3} + 2$$

$$(k+1) \cdot 2^{k+3} + 2 = (k+1) \cdot 2^{k+3} + 2$$

Basis and inductive hypothesis proven, therefore original statement is true.

(c) Exercise Set 5.2, Problem 17

$$\prod_{i=0}^n \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}, \text{ for all integers } n \geq 0.$$

Let property $P(n)$ be $\prod_{i=0}^n \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right)$.

Basis:

$$P(0) : \left(\prod_{i=0}^0 \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{2} \right) = \left(\frac{1}{(0+2)!} = \frac{1}{2} \right) \text{ is true.}$$

Inductive Hypothesis:

$$\text{Assume } P(k) : \prod_{i=0}^k \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2k+2)!} \text{ for } k \geq 0 \in \mathbb{Z} \text{ is true.}$$

$$P(k+1) :$$

$$\prod_{i=0}^{k+1} \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2(k+1)+2)!}$$

$$\prod_{i=0}^k \left(\frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) \left(\frac{1}{2(k+1)+1} \cdot \frac{1}{2(2k+1)+2} \right) = \frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!} \left(\frac{1}{2k+3} \cdot \frac{1}{2k+4} \right) = \frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!(2k+3)(2k+4)} = \frac{1}{(2k+4)(2k+3)!}$$

$$\frac{1}{(2k+4)(2k+3)(2k+2)!} = \frac{1}{(2k+4)(2k+3)(2k+2)!}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(d) Exercise Set 5.3, Problem 10

$$n^3 - 7n + 3 \text{ is divisible by 3, for each integer } n \geq 0.$$

$$\text{Let property } P(n) \text{ be } n^3 - 7n + 3 \text{ is divisible by 3.}$$

Basis:

$$P(0) : (0^3 - 7 \cdot 0 + 3 = 3) \text{ is divisible by 3 is true.}$$

Inductive Hypothesis:

$$\text{Assume } P(k) : k^3 - 7k + 3 \text{ is divisible by 3 where } k \geq 0 \in \mathbb{Z} \text{ is true.}$$

$$P(k+1) : (k+1)^3 - 7(k+1) + 3$$

$$k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3$$

$$\underbrace{(k^3 - 7k + 3)}_{3a} + 3k^2 + 3k + 1 - 7$$

$$P(k) \text{ true, therefore multiple of some integer } a$$

$$3a + 3k^2 + 3k - 6$$

$$3a + 3k^2 + 3k - 6$$

$$3(a + k^2 + k - 2)$$

Expression is a multiple of 3, therefore $P(k+1)$ is divisible by 3.

Basis and inductive hypothesis proven, therefore original statement is true.

(e) Exercise Set 5.3, Problem 17

$1 + 3n \leq 4^n$, for every integer $n \geq 0$.

Let property $P(n)$ be $1 + 3n \leq 4^n$.

Basis:

$P(0) : (1 + 3 \cdot 0 = 1) \leq (4^0 = 1)$ is true.

Inductive hypothesis:

Assume $P(k) : 1 + 3k \leq 4^k$ for $k \geq 0 \in \mathbb{Z}$ is true.

$P(k + 1) :$

$$1 + 3(k + 1) \leq 4^{k+1}$$

$$4 + 3k \leq 4^{k+1}$$

$$1 + 3k + 3 \leq 4^{k+1}$$

$$P(k) + 3 \leq 4^{k+1}$$

$$4^k + 3 \leq 4^{k+1}$$

$$3 \leq 4^{k+1} - 4^k$$

$$3 \leq 4^k(4 - 1)$$

$$3 \leq 3 \cdot 4^k$$

$$1 \leq 4^k$$

Last inequality holds true for all $k \geq 0 \in \mathbb{Z}$.

Basis and inductive hypothesis proven, therefore original statement is true.

(f) Exercise Set 5.3, Problem 21

$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for all integers $n \geq 2$.

Let property $P(n)$ be $\sqrt{n} < \sum_{i=1}^n \frac{1}{\sqrt{i}}$.

Basis:

$P(2) : (\sqrt{2}) < \left(\sum_{i=1}^2 \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} \right)$ is true.

Inductive hypothesis:

Assume $P(k) : \sqrt{k} < \sum_{i=1}^k \frac{1}{\sqrt{i}}$ for $k \geq 2 \in \mathbb{Z}$ is true.

$P(k + 1) :$

$$\sqrt{k+1} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$$

$$\sqrt{k+1} < \sum_{i=1}^k \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}}$$

$$\sqrt{k+1} < \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

$$\frac{k+1}{\sqrt{k+1}} < \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$k+1 < \sqrt{k}\sqrt{k+1} + 1$$

$$\sqrt{k}\sqrt{k} < \sqrt{k}\sqrt{k+1}$$

$$k < k+1$$

Inequality holds true for all $k \geq 2$.

Basis and inductive hypothesis proven, therefore original statement is true.

(g) Exercise Set 5.3, Problem 22

$1 + nx \leq (1 + x)^n$, for all real numbers $x > -1$ and integers $n \geq 2$.

Let property $P(n, x)$ be $1 + nx \leq (1 + x)^n$.

Basis:

$P(2, x) : 1 + 2x \leq (1 + x)^2 \Rightarrow 1 + 2x \leq 1 + 2x + x^2 \Rightarrow 0 \leq x^2$ is true for all $x \in \mathbb{R}$.

Inductive hypothesis:

Assume $P(k, x) : 1 + kx \leq (1 + x)^k$ for $k \geq 2 \in \mathbb{Z}$ and $x > -1 \in \mathbb{R}$ is true.

$P(k + 1, x) :$

$$1 + (k + 1)x \leq (1 + x)^{k+1}$$

$$1 + kx + x \leq (1 + x)^{k+1}$$

$$(1 + x)^k + x \leq (1 + x)^{k+1}$$

$$x \leq (1 + x)^k((1 + x) - 1)$$

$$x \leq (1 + x)^k x$$

Case 1: $x = 0$
Prove: $0 \leq (1 + 0)^k \cdot 0$
$0 \leq (1 + 0)^k \cdot 0$
$0 \leq 0$
Proof done.
Case 2: $x > 0$
Prove: $x \leq (1 + x)^k x$
$\begin{array}{l l} x \leq (1 + x)^k x & x > 0 \\ = 1 \leq (1 + x)^k & 1 < 1 + x \\ & 1^k < (1 + x)^k \\ & 1 < (1 + x)^k \end{array}$
$1 < (1 + x)^k \Rightarrow 1 \leq (1 + x)^k$
Proof done.

Case 3: $-1 < x < 0$
Prove: $x \leq (1 + x)^k x$
$\begin{array}{l l} x \leq (1 + x)^k x & 0 > \quad x > -1 \\ = 1 \geq (1 + x)^k & 1 > \quad 1 + x > 0 \\ & 1^k > \quad (1 + x)^k > 0^k \\ & 1 > \quad (1 + x)^k > 0 \end{array}$
$1 > (1 + x)^k > 0 \Rightarrow 1 \geq (1 + x)^k$
Proof done.

$P(k + 1, x)$ for $k \geq 2 \in \mathbb{Z}$ and $x > -1 \in \mathbb{R}$ is true in all cases.

Basis and inductive hypothesis proven, therefore original statement is true.

(h) Exercise Set 5.3, Problem 29

As each of a group of businesspeople arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if n people come to the meeting then $[n(n-1)]/2$ handshakes occur.

If the set of businesspeople has size n , then the number of handshakes is $\binom{n}{2}$.

For a set of 0 and 1 businesspeople, no handshakes occur.

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n!}{2(n-2)!}$$

Let property $P(n)$ be
$$\begin{cases} 0 = \frac{n(n-1)}{2} & n = 0, n = 1 \\ \frac{n!}{2(n-2)!} = \frac{n(n-1)}{2} & n \geq 2 \end{cases}$$
 for non negative integers n .

Basis:

$$P(0) : 0 = \left(\frac{0(0-1)}{2} = 0 \right) \text{ is true.}$$

$$P(1) : 0 = \left(\frac{1(1-1)}{2} = 0 \right) \text{ is true.}$$

$$P(2) : \left(\frac{2!}{2(2-2)!} = \frac{2}{2 \cdot 1} = 1 \right) = \left(\frac{2(2-1)}{2} = 1 \right) \text{ is true.}$$

Inductive hypothesis:

Assume $P(k) : \frac{k!}{2(k-2)!} = \frac{k(k-1)}{2}$ for $k \geq 0 \in \mathbb{Z}$ to be true.

$P(k+1) :$

$$\frac{(k+1)!}{2((k+1)-2)!} = \frac{(k+1)((k+1)-1)}{2}$$

$$\frac{(k+1)k!}{2(k-1)!} = \frac{(k+1)k}{2}$$

$$\frac{(k+1)k(k-1)!}{2(k-1)!} = \frac{(k+1)k}{2}$$

$$\frac{(k+1)k}{2} = \frac{(k+1)k}{2}$$

Basis and inductive hypothesis proven, therefore original statement is true.

- (i) Prove that in an n -sided regular polygon, where $n \geq 3$, the number of diagonals is $n(n-3)/2$.

The number of possible vertex pairs in an n -sided regular polygon is $\binom{n}{2}$, and n of these vertex pairs are the edges of the polygon. The number of diagonals is $\binom{n}{2} - n$.

$$\binom{n}{2} - n = \frac{n!}{2!(n-2)!} - n = \frac{n!}{2(n-2)!} - n$$

$$\text{Let property } P(n) \text{ be } \frac{n!}{2(n-2)!} - n = \frac{n(n-3)}{2}.$$

Basis:

$$P(3) : \left(\frac{3!}{2(3-2)!} - 3 = \frac{6}{2} - 3 = 0 \right) = \left(\frac{3(3-3)}{2} = 0 \right) \text{ is true.}$$

Inductive hypothesis:

$$\text{Assume } P(k) : \frac{k!}{2(k-2)!} - k = \frac{k(k-3)}{2} \text{ for } k \geq 3 \text{ to be true.}$$

$$P(k+1) :$$

$$\frac{(k+1)!}{2((k+1)-2)!} - (k+1) = \frac{(k+1)((k+1)-3)}{2}$$

$$\frac{(k+1)k!}{2(k-1)!} - k - 1 = \frac{(k+1)(k-2)}{2}$$

$$\frac{(k+1)k(k-1)!}{2(k-1)!} - k - 1 = \frac{(k+1)(k-2)}{2}$$

$$\frac{(k+1)k}{2} - \frac{2k+2}{2} = \frac{(k+1)(k-2)}{2}$$

$$\frac{k^2 + k - 2k - 2}{2} = \frac{k^2 - k - 2}{2}$$

$$\frac{k^2 - k - 2}{2} = \frac{k^2 - k - 2}{2}$$

Basis and inductive hypothesis proven, therefore original statement is true.

- (j) Prove that the number of permutations of the set $\{1, 2, \dots, n\}$ with n elements is $n!$, for natural number $n \geq 1$.

A set of $n = 1$ elements has $1! = 1$ permutation.

Let property $P(n)$ be $\{1, 2, \dots, n\}$ has $n!$ permutations.

Basis:

$$P(1) : \{1\} \text{ has } 1! = 1 \text{ permutation is true.}$$

Inductive hypothesis:

$$\text{Assume } P(k) : \{1, 2, \dots, k\} \text{ has } k! \text{ permutations for } k \geq 1 \text{ to be true.}$$

In order to create a permuted set B_p of size $k+1$, one can insert $k+1$ into A_p , an arbitrary permutation of set A of k elements.

This action is equivalent to permuting a set of size $k+1$.

There are $k+1$ positions to insert such an element into A_p : k positions before each element and one position after the last element of A_p .

There are $k!$ possible A_p made from A .

$$(k+1 \text{ ways to insert into } A_p) \times (k! \text{ possible } A_p) = (k+1)k! = (k+1)! \text{ ways to create } B_p.$$

$$P(k+1) \text{ is true.}$$

Basis and inductive hypothesis proven, therefore original statement is true.

Problem 2.

(a) Exercise Set 5.4, Problem 8

Suppose that h_0, h_1, h_2, \dots is a sequence defined as follows:

$$h_0 = 1, h_1 = 2, h_2 = 3,$$

$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \text{ for all integers } k \geq 3.$$

(a) Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.

Let property $P(n)$ be $h_n \leq 3^n$ for all integers $n \geq 0$.

Basis:

$$P(0) : (h_0 = 1) \leq (3^0 = 1) \text{ is true.}$$

$$P(1) : (h_1 = 2) \leq (3^1 = 3) \text{ is true.}$$

$$P(2) : (h_2 = 3) \leq (3^2 = 9) \text{ is true.}$$

Inductive hypothesis:

Let $k \geq 2$.

Assume $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq 3^i$ for $0 \leq i \leq k$ and $i \in \mathbb{Z}$.

$P(k+1) :$

$$h_{k+1} = h_k + h_{k-1} + h_{k-2}$$

$$\leq 3^k + 3^{k-1} + 3^{k-2}$$

$$\leq 3^{k-2} (3^2 + 3 + 1)$$

$$\leq 13 \cdot 3^{k-2} \leq (3^3 \cdot 3^{k-2} = 3^{k+1})$$

$$P(k+1) : h_{k+1} \leq 3^{k+1} \text{ is true.}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Suppose that s is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $2 > s > 1.83$.) Prove that $h_n \leq s^n$ for all $n \geq 2$.

Let property $P(n)$ be $h_n \leq s^n$ for $n \geq 2 \in \mathbb{Z}$.

Basis:

$$P(2) : (h_2 = 2) \leq (3.34 < s^2 < 4) \text{ is true.}$$

$$P(3) : (h_3 = 6) \leq (6.12 < s^3 < 8) \text{ is true.}$$

$$P(4) : (h_4 = 11) \leq (11.21 < s^4 < 16) \text{ is true.}$$

Inductive hypothesis:

Let $k \geq 4$.

Assume $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq s^i$ for $2 \leq i \leq k$ and $i \in \mathbb{Z}$.

$P(k+1) :$

$$h_{k+1} = h_k + h_{k-1} + h_{k-2}$$

$$\leq s^k + s^{k-1} + s^{k-2}$$

$$\leq s^{k-2} (s^2 + s + 1)$$

$$\leq s^{k-2} s^3$$

$$\leq s^{k+1}$$

$$P(k+1) : h_{k+1} \leq s^{k+1} \text{ is true.}$$

Basis and inductive hypothesis proven, therefore original statement is true.

- (b) Exercise Set 5.4, Problem 9

Define a sequence a_1, a_2, a_3, \dots as follows: $a_1 = 1$, $a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$. Use strong mathematical induction to prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

Let property $P(n)$ be $a_n \leq \left(\frac{7}{4}\right)^n$.

Basis:

$P(1)$: $1 \leq \frac{7}{4}$ is true.

$P(2)$: $3 \leq \frac{49}{16}$ is true.

Inductive hypothesis:

Let $k \geq 2$.

Assume $P(i) : a_i = a_{i-1} + a_{i-2} \leq \left(\frac{7}{4}\right)^i$ for $1 \leq i \leq k$ and $i \in \mathbb{Z}$.

$P(k+1)$:

$$a_{k+1} = a_k + a_{k-1}$$

$$\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}$$

$$\leq \left(\frac{7}{4}\right)^{k-1} \left(1 + \frac{7}{4}\right)$$

$$\leq \left(\frac{11}{4}\right) \left(\frac{7}{4}\right)^{k-1} = \frac{44}{16} \left(\frac{7}{4}\right)^{k-1} \leq \left(\frac{49}{16}\right) \left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^2 \left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^{k+1}$$

$P(k+1) : a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$ is true.

Basis and inductive hypothesis proven, therefore original statement is true.

- (c) Exercise Set 5.4, Problem 25(b)

Use mathematical induction to prove that for all integers $n \geq 1$, given any set of 2^n people arranged in a circle and numbered consecutively 1 through 2^n , if one starts from person #1 and goes repeatedly around the circle successively eliminating every second person, eventually only person #1 will remain.

- (d) Exercise Set 5.4, Problem 30

It is a fact that every integer $n > 1$ can be written in the form

$$c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \dots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0,$$

where $c_r = 1$ or 2 and $c_i = 0, 1$, or 2 for all integers $i = 0, 1, 2, \dots, r-1$. Sketch a proof of this fact.

- (e) Exercise Set 5.5, Problem 30

F_0, F_1, F_2, \dots is the Fibonacci sequence.

Use mathematical induction to prove that for all integers $n \geq 0$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$

- (f) Let f be a function on whole numbers satisfying

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 4f(n/2) & \text{if } n > 0 \text{ and even} \\ f(n-1) + 2n - 1 & \text{if } n > 0 \text{ and odd} \end{cases}$$

Prove that $f(n) = n^2$ for all $n \geq 0$

Problem 3.

- (a) Exercise Set 5.5, Problems 28, 29
- (b) Exercise Set 5.6, Problem 2(b,d)
- (c) Exercise Set 5.6, Problems 9, 14, 15