Section 3.2.

3.

$$\text{(d)}\quad \begin{array}{c} (a_1) \\ (b_1) \\ (c_1) \end{array} \left[\begin{array}{rrr|r} 2 & 4 & -2 & 4 \\ 1 & -2 & -4 & -1 \\ -2 & -1 & -3 & -4 \end{array}\right]$$

$$\begin{array}{c} (b_2) \\ (b_3) \\ (c_2) \\ (b_4) \\ (c_3) \\ (c_4) \end{array} \left[\begin{array}{lrr|r} 2 & -4 & -8 & -2 \\ 0 & -5 & -11 & -6 \\ 0 & 3 & -5 & 0 \\ 0 & -15 & -33 & -18 \\ 0 & 15 & -25 & 0 \\ 0 & 0 & -58 & -18 \end{array}\right] \begin{array}{l} 2(b_1) \\ (b_2)+(c_1) \\ (a_1)+(c_1) \\ 3(b_3) \\ 5(c_2) \\ (b_4)+(c_3) \end{array}$$

$$\left[\begin{array}{lrr|r} 2 & 4 & -2 & 4 \\ 0 & 5 & 11 & 6 \\ 0 & 0 & 58 & 18 \end{array}\right]$$

$$58x_3 = 18 \qquad 5x_2 + 11x_3 = 6 \qquad 2x_1 + 4x_2 - 2x_3 = 4$$

$$x_3 = \frac{9}{29} \qquad 5x_2 + \frac{99}{29} = 6 \qquad 2x_1 + 4\frac{15}{29} - 2\frac{9}{29} = 4$$

$$5x_2 = \frac{75}{29} \qquad 2x_1 = \frac{74}{29}$$

$$x_2 = \frac{15}{29} \qquad x_1 = \frac{37}{29}$$

5.

$$\text{(a)}\quad \begin{array}{c} (a_1) \\ (b_1) \\ (c_1) \end{array} \left[\begin{array}{rrr|r} 2 & 4 & -2 & 4 \\ 1 & -2 & -4 & -1 \\ -2 & -1 & -3 & -4 \end{array}\right]$$

$$\begin{array}{c} (a_2) \\ (b_2) \\ (c_2) \\ (c_3) \end{array} \left[\begin{array}{lrr|r} 1 & 2 & -1 & 2 \\ 0 & -4 & -3 & -3 \\ 0 & 3 & -5 & 0 \\ 0 & 0 & -\frac{29}{4} & -\frac{9}{4} \end{array}\right] \begin{array}{l} \frac{1}{2}(a_1) \\ (b_1)-(a_2) \\ (c_1)+(a_1) \\ (c_1)+(a_1)+\frac{3}{4}(b_2) \end{array}$$

$$\mathbf{U} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & -4 & -3 \\ 0 & 0 & -\frac{29}{4} \end{bmatrix}, \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -1 & -\frac{3}{4} & 1 \end{bmatrix}$$

$$\text{(b)}\quad \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -1 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & -4 & -3 \\ 0 & 0 & -\frac{29}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & -4 & -3 \\ 0 & 0 & -\frac{29}{4} \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & -4 & -3 \\ 0 & 0 & -\frac{29}{4} \end{bmatrix} = \begin{bmatrix} 1 & -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & -4 & -3 \\ 0 & 0 & -\frac{29}{4} \end{bmatrix} = \begin{bmatrix} -2 & -1 & -3 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 \\ 1 & -2 & -4 \\ -2 & -1 & -3 \end{bmatrix}$$

6. $\det(\mathbf{A}) = \mathbf{U}_{11} \cdot \mathbf{U}_{22} \cdot \mathbf{U}_{33} = 58$

Section 3.3.

5.

(a) $\mathbf{A} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$

$$\det(\mathbf{A}) = \frac{3}{4} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$$

$$\mathbf{A}^{-1} = 4 \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

7.

(b) $\mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -4 \\ 1 & -2 & 3 \end{bmatrix}$

$$\begin{array}{c} (a_1) \\ (b_1) \\ (c_1) \end{array} \left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 2 & -4 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} (a_2) \\ (b_2) \\ (c_2) \end{array} \left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -3 & 4 & 1 & 0 & 1 \\ 0 & 0 & -2 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} (a_1) \\ (c_1) + 2(a_1) \\ (b_1) + 2(a_1) \end{array}$$

$$\begin{array}{c} (a_3) \\ (a_3) \\ (b_3) \\ (c_3) \end{array} \left[\begin{array}{ccc|ccc} -1 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & -\frac{1}{3} \\ -1 & 0 & 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{3} \\ 0 & -3 & 0 & 5 & 2 & 1 \\ 0 & 0 & -2 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} (a_1) - \frac{1}{3}(b_2) \\ (a_1) - \frac{1}{3}(b_2) - \frac{1}{6}(c_2) \\ (b_2) + 2(c_2) \\ (c_2) \end{array}$$

$$\begin{array}{c} (a_4) \\ (b_4) \\ (c_4) \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & -1 & -\frac{1}{2} & 0 \end{array}\right] \begin{array}{l} -(a_3) \\ -\frac{1}{3}(b_3) \\ -\frac{1}{2}(c_2) \end{array}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ -\frac{5}{3} & -\frac{2}{3} & -\frac{1}{3} \\ -1 & -\frac{1}{2} & 0 \end{bmatrix}$$

18. Why must a matrix be square if it has an inverse?
Let $\mathbf{A}$ be a matrix of size $p \times q$ and $\mathbf{A}^{-1}$ of size $q \times r$.
$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} = \mathbf{A}^{-1}\mathbf{A}$
$\underbrace{\mathbf{A}\mathbf{A}^{-1}}_{p \times r} = \mathbf{A}^{-1}\mathbf{A}$
In order for $\mathbf{A}^{-1}\mathbf{A}$ to be valid $p = r$.
$\underbrace{\mathbf{A}\mathbf{A}^{-1}}_{r \times r} = \underbrace{\mathbf{A}^{-1}\mathbf{A}}_{q \times q}$
$p = r = q$ therefore $\mathbf{A}$ must be square.

33.

(a) $\mathbf{A} = \begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}$

$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 0 \\ 2 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 0 \cdot 2$

$\lambda_1 = 4,\ \lambda_2 = 2$

$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix} \qquad \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

$2u_1 - 2u_2 = 0 \qquad 2v_1 - 0v_2 = 0$

$u_1 = u_2 \qquad v_1 = 0$

$\mathbf{u} = [1, 1] \qquad \mathbf{v} = [0, 1]$

$\mathbf{U} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ \mathbf{U}^{-1} = \frac{1}{\det(\mathbf{U})} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$\mathbf{A} = \mathbf{U}\mathbf{D}_\lambda\mathbf{U}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 1 \cdot 2$

$6 - 5\lambda + \lambda^2 - 2 = \lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1)$

$\lambda_1 = 4,\ \lambda_2 = 1$

$\mathbf{A} - 4\mathbf{I} = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix} \qquad \mathbf{A} - 1\mathbf{I} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$2u_1 - u_2 = 0 \qquad v_1 + v_2 = 0$

$2u_1 = u_2 \qquad v_1 = -v_2$

$\mathbf{u} = [1, 2] \qquad \mathbf{v} = [1, -1]$

$\mathbf{U} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix},\ \mathbf{U}^{-1} = \frac{1}{\det(\mathbf{U})} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$

$\mathbf{A} = \mathbf{U}\mathbf{D}_\lambda\mathbf{U}^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$

34. $\mathbf{p} = [10, 10]$

$\mathbf{A}^{10}\mathbf{p} = \mathbf{U}\mathbf{D}_\lambda^{10}\mathbf{U}^{-1}$

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{10} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4^{10} & 0 \\ 4^{10} & 2^{10} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4^{10} & 0 \\ 4^{10} - 2^{10} & 2^{10} \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}^{10} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 4^{10} & 1 \\ 2^{10} & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{4^{10}}{3} + \frac{2}{3} & \frac{4^{10}}{3} - \frac{1}{3} \\ \frac{2^{10}}{3} - \frac{2}{3} & \frac{2^{10}}{3} + \frac{1}{3} \end{bmatrix}$

Section 3.4.

8.

(a) $\mathbf{A} = \begin{bmatrix} 0.7 & -0.2 \\ -0.4 & 0.8 \end{bmatrix}$

$\mathbf{I} - \mathbf{D} = \mathbf{A}$

$\mathbf{D} = \mathbf{I} - \mathbf{A}$

$\mathbf{D} = \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.2 \end{bmatrix}$

$(\mathbf{I} - \mathbf{D})^{-1} = \sum_{i=0}^{\infty} \mathbf{D}^k \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.3 & 0.2 \\ 0.4 & 0.2 \end{bmatrix} + \begin{bmatrix} 0.09 + 0.08 & 0.06 + 0.04 \\ 0.12 + 0.08 & 0.08 + 0.04 \end{bmatrix}$

$\mathbf{A}^{-1} \approx \begin{bmatrix} 1.47 & 0.3 \\ 0.6 & 1.32 \end{bmatrix}$

$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.7 \end{bmatrix} = \frac{1}{0.56 - 0.08} \begin{bmatrix} 0.8 & 0.2 \\ 0.4 & 0.7 \end{bmatrix} = \begin{bmatrix} 1.66 & 0.42 \\ 0.83 & 1.46 \end{bmatrix}$

11.

$$
\begin{aligned}
7x_1 + x_2 + 2x_3 &= 30 \\
x_1 + 5x_2 + 3x_3 &= 10 \\
2x_1 + 3x_2 + 8x_3 &= 12
\end{aligned}
$$

(a) $$\underbrace{\begin{bmatrix} 7 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 8 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 30 \\ 10 \\ 12 \end{bmatrix}}_{\mathbf{b}}$$

$x_1' = 7x_1,\ x_2' = 5x_2,\ x_3' = 8x_3$

$$\underbrace{\begin{bmatrix} 1 & \frac{1}{5} & \frac{1}{4} \\ \frac{1}{7} & 1 & \frac{3}{8} \\ \frac{2}{7} & \frac{3}{5} & 1 \end{bmatrix}}_{\mathbf{A}'} \underbrace{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}}_{\mathbf{x}'} = \underbrace{\begin{bmatrix} 30 \\ 10 \\ 12 \end{bmatrix}}_{\mathbf{b}}$$

$(\mathbf{I} - \mathbf{D})\mathbf{x}' = b$

$\mathbf{x}' = \mathbf{D}\mathbf{x}' + \mathbf{b}$

$$\mathbf{x}' = \begin{bmatrix} 0 & -\frac{1}{5} & -\frac{1}{4} \\ -\frac{1}{7} & 0 & -\frac{3}{8} \\ -\frac{2}{7} & -\frac{3}{5} & 0 \end{bmatrix} \mathbf{x}' + \mathbf{b}$$

(b) $\mathbf{x}^{(0)} = [0, 0, 0]$
$\mathbf{x}^{(1)} = [30, 10, 12]$
$\mathbf{x}^{(2)} \approx [25, 1, -3]$
$\mathbf{x}^{(3)} \approx [30, 7, 4]$
$\mathbf{x}^{(4)} \approx [27, 4, -1]$
$\mathbf{x}^{(5)} \approx [29, 6, 2]$
$\mathbf{x}^{(6)} \approx [28, 5, 0]$
$\mathbf{x}^{(7)} \approx [29, 6, 1]$
$\mathbf{x}^{(8)} \approx [29, 6, 0]$
$\mathbf{x}^{(9)} \approx [29, 6, 1]$
$\mathbf{x}^{(10)} \approx [29, 6, 0]$
$\vdots$
$\mathbf{x}^{(100)} \approx [28.7662, 5.7711, 0.3184]$

(c) $\mathbf{x}^{(0)} = [100, 100, 100]$
$\mathbf{x}^{(1)} \approx [-15, -42, -77]$
$\mathbf{x}^{(2)} \approx [58, 41, 41]$
$\mathbf{x}^{(3)} \approx [11, -14, -29]$
$\mathbf{x}^{(4)} \approx [40, 19, 17]$
$\mathbf{x}^{(5)} \approx [22, -2, -11]$
$\mathbf{x}^{(6)} \approx [33, 11, 7]$
$\mathbf{x}^{(7)} \approx [26, 3, -4]$
$\mathbf{x}^{(8)} \approx [30, 8, 3]$
$\mathbf{x}^{(9)} \approx [28, 5, -1]$
$\mathbf{x}^{(10)} \approx [29, 7, 1]$
$\vdots$
$\mathbf{x}^{(100)} \approx [28.7662, 5.7711, 0.3184]$