Problem 1.

(a) Exercise Set 5.2, Problem 11

$1^3 + 2^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$, for all integers $n \geq 1$.

Let property $P(n)$ be $\displaystyle\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Basis:

$P(1) : \left(\displaystyle\sum_{i=1}^{1} i^3 = 1\right) = \left(\left[\dfrac{n(n+1)}{2}\right]^2 = 1\right)$ is true.

Inductive hypothesis:

Assume $P(k) : \displaystyle\sum_{i=1}^{k} i^3 = \left[\frac{k(k+1)}{2}\right]^2$ for $k \geq 1 \in \mathbb{Z}$ is true.

Prove $P(k+1) : \displaystyle\sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$:

$$\sum_{i=1}^{k+1} i^3 \stackrel{?}{=} \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$\left(\sum_{i=1}^{k} i^3\right) + (k+1)^3 \stackrel{?}{=} \left[\frac{k^2+3k+2}{2}\right]^2$$

$$\left[\frac{k(k+1)}{2}\right]^2 + k^3 + 3k^2 + 3k + 1 \stackrel{?}{=} \left[\frac{k^2+3k+2}{2}\right]^2$$

$$\left[\frac{k^2+k}{2}\right]^2 + k^3 + 3k^2 + 3k + 1 \stackrel{?}{=} \left[\frac{k^2+3k+2}{2}\right]^2$$

$$\frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4} \stackrel{?}{=} \frac{k^4+3k^3+2k^2+3k^3+9k^2+6k+2k^2+6k+4}{4}$$

$$\frac{k^4+6k^3+13k^2+12k+4}{4} = \frac{k^4+6k^3+13k^2+12k+4}{4}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.2, Problem 14

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2, \text{ for all integers } n \geq 0.$$

Let property $P(n)$ be $\displaystyle\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$.

Basis:

$P(0) : \left(\displaystyle\sum_{i=1}^{1} i \cdot 2^i = 2\right) = \left(0 \cdot 2^2 + 2 = 2\right)$ is true.

Inductive hypothesis:

Assume $P(k) : \displaystyle\sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$ for $k \geq 0 \in \mathbb{Z}$ is true.

Prove $P(k+1) : \displaystyle\sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$:

$$\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^i &\stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\
\left(\sum_{i=1}^{k+1} i \cdot 2^i\right) + (k+2) \cdot 2^{k+2} &\stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\
k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} &\stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\
k \cdot 2^{k+2} + 2 + k \cdot 2^{k+2} + 2^{k+3} &\stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\
2k \cdot 2^{k+2} + 2 + 2^{k+3} &\stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\
k \cdot 2^{k+3} + 2 + 2^{k+3} &\stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\
(k+1) \cdot 2^{k+3} + 2 &= (k+1) \cdot 2^{k+3} + 2
\end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(c) Exercise Set 5.2, Problem 17

$\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2n+2)!}$, for all integers $n\geq 0$.

Let property $P(n)$ be $\prod_{i=0}^{n}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)$.

Basis:

$P(0):\left(\prod_{i=0}^{0}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{2}\right)=\left(\frac{1}{(0+2)!}=\frac{1}{2}\right)$ is true.

Inductive Hypothesis:

Assume $P(k):\prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2k+2)!}$ for $k\geq 0\in\mathbb{Z}$ is true.

Prove $P(k+1):\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)=\frac{1}{(2(k+1)+2)!}$:

$$\prod_{i=0}^{k+1}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)\stackrel{?}{=}\frac{1}{(2(k+1)+2)!}$$

$$\prod_{i=0}^{k}\left(\frac{1}{2i+1}\cdot\frac{1}{2i+2}\right)\left(\frac{1}{2(k+1)+1}\cdot\frac{1}{2(2k+1)+2}\right)\stackrel{?}{=}\frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!}\left(\frac{1}{2k+3}\cdot\frac{1}{2k+4}\right)\stackrel{?}{=}\frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!(2k+3)(2k+4)}\stackrel{?}{=}\frac{1}{(2k+4)(2k+3)!}$$

$$\frac{1}{(2k+4)(2k+3)(2k+2)!}=\frac{1}{(2k+4)(2k+3)(2k+2)!}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(d) Exercise Set 5.3, Problem 10

n^3-7n+3 is divisible by 3, for each integer $n\geq 0$.

Let property $P(n)$ be n^3-7n+3 is divisible by 3.

Basis:

$P(0):(0^3-7\cdot 0+3=0)$ is divisible by 3 is true.

Inductive Hypothesis:

Assume $P(k):k^3-7k+3$ is divisible by 3 where $k\geq 0\in\mathbb{Z}$ is true.

Prove $P(k+1):(k+1)^3-7(k+1)+3$ is divisible by 3:

$(k+1)^3-7(k+1)+3$

$=k^3+3k^2+3k+1-7k-7+3$

$=\underbrace{(k^3-7k+3)}_{m}+3k^2+3k+1-7$

$P(k)$ is true, therefore m is a multiple of 3 and $m=3a$ for some integer a.

$=3a+3k^2+3k-6$

$=3\left(a+k^2+k-2\right)$

Expression is a multiple of 3, therefore $(k+1)^3-7(k+1)+3$ is divisible by 3.

Basis and inductive hypothesis proven, therefore original statement is true.

(e) Exercise Set 5.3, Problem 17

$1 + 3n \leq 4^n$, for every integer $n \geq 0$.

Let property $P(n)$ be $1 + 3n \leq 4^n$.

Basis:

$P(0) : (1 + 3 \cdot 0 = 1) \leq (4^0 = 1)$ is true.

Inductive hypothesis:

Assume $P(k) : 1 + 3k \leq 4^k$ for $k \geq 0 \in \mathbb{Z}$ is true.

Prove $P(k+1) : 1 + 3(k+1) \leq 4^{k+1}$:

$$\begin{aligned}
1 + 3(k+1) &\overset{?}{\leq} 4^{k+1} \\
4 + 3k &\overset{?}{\leq} 4^{k+1} \\
(1 + 3k) + 3 &\overset{?}{\leq} 4^{k+1} \\
4^k + 3 &\overset{?}{\leq} 4^{k+1} && P(k) \text{ is true} \\
3 &\overset{?}{\leq} 4^{k+1} - 4^k \\
3 &\overset{?}{\leq} 4^k(4-1) \\
3 &\overset{?}{\leq} 3 \cdot 4^k \\
1 &\leq 4^k
\end{aligned}$$

Last inequality holds true for all $k \geq 0 \in \mathbb{Z}$.

Basis and inductive hypothesis proven, therefore original statement is true.

(f) Exercise Set 5.3, Problem 21

$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$, for all integers $n \geq 2$.

Let property $P(n)$ be $\sqrt{n} < \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$.

Basis:

$P(2) : (\sqrt{2}) < \left(\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} \right)$ is true.

Inductive hypothesis:

Assume $P(k) : \sqrt{k} < \sum_{i=1}^{k} \frac{1}{\sqrt{i}}$ for $k \geq 2 \in \mathbb{Z}$ is true.

Prove $P(k+1) : \sqrt{k+1} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$:

$$\sqrt{k+1} \overset{?}{<} \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$$

$$\sqrt{k+1} \overset{?}{<} \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}}$$

$$\sqrt{k+1} \overset{?}{<} \sqrt{k} + \frac{1}{\sqrt{k+1}} \qquad P(k) \text{ is true}$$

$$\frac{k+1}{\sqrt{k+1}} \overset{?}{<} \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$k+1 \overset{?}{<} \sqrt{k}\sqrt{k+1} + 1$$

$$\sqrt{k}\sqrt{k} \overset{?}{<} \sqrt{k}\sqrt{k+1}$$

$$k \overset{?}{<} k+1$$

$$0 < 1$$

Basis and inductive hypothesis proven, therefore original statement is true.

(g) Exercise Set 5.3, Problem 22

$1 + nx \leq (1+x)^n$, for all real numbers $x > -1$ and integers $n \geq 2$.

Let property $P(n)$ be $1 + nx \leq (1+x)^n$ for all $x > -1 \in \mathbb{R}$.

Basis:

$P(2): \quad 1 + 2x \leq (1+x)^2 \quad$ is true.

$$\Rightarrow 1 + 2x \leq 1 + 2x + x^2$$

$$\Rightarrow 0 \leq x^2$$

Inductive hypothesis:

Assume $P(k): 1 + kx \leq (1+x)^k$ for $k \geq 2 \in \mathbb{Z}$ and $x > -1 \in \mathbb{R}$ is true.

Prove $P(k+1): 1 + (k+1)x \leq (1+x)^{k+1}$:

$$\begin{aligned} 1 + (k+1)x &\overset{?}{\leq} (1+x)^{k+1} \\ 1 + kx + x &\overset{?}{\leq} (1+x)^{k+1} \\ (1+x)^k + x &\overset{?}{\leq} (1+x)^{k+1} \\ x &\overset{?}{\leq} (1+x)^k((1+x) - 1) \\ x &\overset{?}{\leq} (1+x)^k x \end{aligned}$$

Case 1: $x = 0$

Prove $x \leq (1+x)^k x$:

$0 \leq (1+0)^k \cdot 0$

$0 \leq 0$

Proof done.

Case 2: $x > 0$

Prove $x \leq (1+x)^k x$:

$$\begin{array}{l|l} x > 0 & x \overset{?}{\leq} (1+x)^k x \\ 1 < 1 + x & 1 \overset{?}{\leq} (1+x)^k \\ 1^k < (1+x)^k & \\ 1 < (1+x)^k & \end{array}$$

$1 < (1+x)^k$

$\therefore 1 \leq (1+x)^k$

$\therefore x \leq (1+x)^k x$

Proof done.

Case 3: $-1 < x < 0$

Prove $x \leq (1+x)^k x$:

$$\begin{array}{rr|l} 0 > & x > -1 & x \overset{?}{\leq} (1+x)^k x \\ 1 > & 1 + x > 0 & 1 \overset{?}{\geq} (1+x)^k \\ 1^k > & (1+x)^k > 0^k & \\ 1 > & (1+x)^k > 0 & \end{array}$$

$1 > (1+x)^k$

$\therefore 1 \geq (1+x)^k$

$\therefore x \leq (1+x)^k x$

Proof done.

$P(k+1)$ is true in all cases.

Basis and inductive hypothesis proven, therefore original statement is true.

(h) Exercise Set 5.3, Problem 29

As each of a group of businesspeople arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if n people come to the meeting then $[n(n-1)]/2$ handshakes occur.

If the set of businesspeople has size n, then the number of handshakes is $\binom{n}{2}$.

For a set of 0 and 1 businesspeople, no handshakes occur.

$\binom{n}{2} = \dfrac{n!}{2!(n-2)!} = \dfrac{n!}{2(n-2)!}$

Let property $P(n)$ be $\begin{cases} 0 = \dfrac{n(n-1)}{2} & \text{if } n = 0,\ n = 1 \\ \dfrac{n!}{2(n-2)!} = \dfrac{n(n-1)}{2} & \text{if } n \geq 2 \end{cases}$.

Basis:

$P(0) : 0 = \left(\dfrac{0(0-1)}{2} = 0\right)$ is true.

$P(1) : 0 = \left(\dfrac{1(1-1)}{2} = 0\right)$ is true.

$P(2) : \left(\dfrac{2!}{2(2-2)!} = \dfrac{2}{2 \cdot 1} = 1\right) = \left(\dfrac{2(2-1)}{2} = 1\right)$ is true.

Inductive hypothesis:

Assume $P(k) : \dfrac{k!}{2(k-2)!} = \dfrac{k(k-1)}{2}$ for $k \geq 2 \in \mathbb{Z}$ to be true.

Prove $P(k+1) : \dfrac{(k+1)!}{2((k+1)-2)!} = \dfrac{(k+1)((k+1)-1)}{2}$:

$$\begin{aligned}
\frac{(k+1)!}{2((k+1)-2)!} &\stackrel{?}{=} \frac{(k+1)((k+1)-1)}{2} \\
\frac{(k+1)k!}{2(k-1)!} &\stackrel{?}{=} \frac{(k+1)k}{2} \\
\frac{(k+1)k(k-1)!}{2(k-1)!} &\stackrel{?}{=} \frac{(k+1)k}{2} \\
\frac{(k+1)k}{2} &= \frac{(k+1)k}{2}
\end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(i) Prove that in an n-sided regular polygon, where $n \geq 3$, the number of diagonals is $n(n-3)/2$.

The number of possible vertex pairs in an n-sided regular polygon is $\binom{n}{2}$, and n of these vertex pairs are the edges of the polygon. The number of diagonals is $\binom{n}{2} - n$.

$\binom{n}{2} - n = \dfrac{n!}{2!(n-2)!} - n = \dfrac{n!}{2(n-2)!} - n$

Let property $P(n)$ be $\dfrac{n!}{2(n-2)!} - n = \dfrac{n(n-3)}{2}$.

Basis:

$P(3) : \left(\dfrac{3!}{2(3-2)!} - 3 = \dfrac{6}{2} - 3 = 0\right) = \left(\dfrac{3(3-3)}{2} = 0\right)$ is true.

Inductive hypothesis:

Assume $P(k) : \dfrac{k!}{2(k-2)!} - k = \dfrac{k(k-3)}{2}$ for $k \geq 3 \in \mathbb{Z}$ to be true.

Prove $P(k+1) : \dfrac{(k+1)!}{2((k+1)-2)!} - (k+1) = \dfrac{(k+1)((k+1)-3)}{2}$:

$$\begin{aligned}
\frac{(k+1)!}{2((k+1)-2)!} - (k+1) &\stackrel{?}{=} \frac{(k+1)((k+1)-3)}{2} \\
\frac{(k+1)k!}{2(k-1)!} - k - 1 &\stackrel{?}{=} \frac{(k+1)(k-2)}{2} \\
\frac{(k+1)k(k-1)!}{2(k-1)!} - k - 1 &\stackrel{?}{=} \frac{(k+1)(k-2)}{2} \\
\frac{(k+1)k}{2} - \frac{2k+2}{2} &\stackrel{?}{=} \frac{(k+1)(k-2)}{2} \\
\frac{k^2+k-2k-2}{2} &\stackrel{?}{=} \frac{k^2-k-2}{2} \\
\frac{k^2-k-2}{2} &= \frac{k^2-k-2}{2}
\end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(j) Prove that the number of permutations of the set $\{1, 2, \ldots, n\}$ with n elements is $n!$, for natural number $n \geq 1$.

A set of $n = 1$ elements has $1! = 1$ permutation.

Let property $P(n)$ be $\{1, 2, \ldots, n\}$ has $n!$ permutations.

Basis:

$P(1) : \{1\}$ has $1! = 1$ permutation is true.

Inductive hypothesis:

Assume $P(k) : \{1, 2, \ldots, k\}$ has $k!$ permutations for $k \geq 1 \in \mathbb{Z}$ to be true.

In order to create a permuted set B_p of size $k+1$, one can insert $k+1$ into A_p, an arbitrary permutation of set A of k elements.

This action is equivalent to permuting a set of size $k+1$.

There are $k+1$ positions to insert such an element into A_p: k positions before each element and one position after the last element of A_p.

There are $k!$ possible A_p made from A.

$(k+1$ ways to insert into $A_p) \times (k!$ possible $A_p) = (k+1)k! = (k+1)!$ ways to create B_p.

$P(k+1)$ is true.

Basis and inductive hypothesis proven, therefore original statement is true.

Problem 2.

(a) Exercise Set 5.4, Problem 8
Suppose that $h_0, h_1, h_2, \ldots$ is a sequence defined as follows:
$h_0 = 1$, $h_1 = 2$, $h_2 = 3$,
$h_k = h_{k-1} + h_{k-2} + h_{k-3}$ for all integers $k \geq 3$.

(a) Prove that $h_n \leq 3^n$ for all integers $n \geq 0$.
Let property $P(n)$ be $h_n \leq 3^n$.
Basis:
$P(0) : (h_0 = 1) \leq (3^0 = 1)$ is true.
$P(1) : (h_1 = 2) \leq (3^1 = 3)$ is true.
$P(2) : (h_2 = 3) \leq (3^2 = 9)$ is true.
Inductive hypothesis:
Let $k \geq 2$.
Assume $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq 3^i$ for $0 \leq i \leq k$ and $i \in \mathbb{Z}$ is true.
Prove $P(k+1) : h_{k+1} = h_k + h_{k-1} + h_{k-2} \leq 3^{k+1}$:

$$\begin{aligned} h_{k+1} &= h_k + h_{k-1} + h_{k-2} \\ &\leq 3^k + 3^{k-1} + 3^{k-2} && P(i) \text{ is true} \\ &\leq 3^{k-2}\left(3^2 + 3 + 1\right) \\ &\leq 13 \cdot 3^{k-2} && \leq \left(3^3 \cdot 3^{k-2} = 3^{k+1}\right) \\ &\leq 3^{k+1} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Suppose that s is any real number such that $s^3 \geq s^2 + s + 1$. (This implies that $2 > s > 1.83$.)
Prove that $h_n \leq s^n$ for all $n \geq 2$.
Let property $P(n)$ be $h_n \leq s^n$.
Basis:
$P(2) : (h_2 = 2) \leq (3.34 < s^2 < 4)$ is true.
$P(3) : (h_3 = 6) \leq (6.12 < s^3 < 8)$ is true.
$P(4) : (h_4 = 11) \leq (11.21 < s^4 < 16)$ is true.
Inductive hypothesis:
Let $k \geq 4$.
Assume $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq s^i$ for $2 \leq i \leq k$ and $i \in \mathbb{Z}$.
Prove $P(k+1) : h_{k+1} = h_k + h_{k-1} + h_{k-2} \leq s^{k+1}$:

$$\begin{aligned} h_{k+1} &= h_k + h_{k-1} + h_{k-2} \\ &\leq s^k + s^{k-1} + s^{k-2} \\ &\leq s^{k-2}\left(s^2 + s + 1\right) \\ &\leq s^{k-2} s^3 && s^2 + s + 1 \leq s^3 \\ &\leq s^{k+1} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.4, Problem 9

Define a sequence $a_1, a_2, a_3, \ldots$ as follows: $a_1 = 1$, $a_2 = 3$, and $a_k = a_{k-1} + a_{k-2}$ for all integers $k \geq 3$. Use strong mathematical induction to prove that $a_n \leq \left(\frac{7}{4}\right)^n$ for all integers $n \geq 1$.

Let property $P(n)$ be $a_n \leq \left(\frac{7}{4}\right)^n$.

Basis:

$P(1) : 1 \leq \frac{7}{4}$ is true.

$P(2) : 3 \leq \frac{49}{16}$ is true.

Inductive hypothesis:

Let $k \geq 2$.

Assume $P(i) : a_i = a_{i-1} + a_{i-2} \leq \left(\frac{7}{4}\right)^i$ for $1 \leq i \leq k$ and $i \in \mathbb{Z}$ is true.

Prove $P(k+1) : a_k + a_{k-1} \leq \left(\frac{7}{4}\right)^i$:

$$
\begin{aligned}
a_{k+1} &= a_k + a_{k-1} \\
&\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \\
&\leq \left(\frac{7}{4}\right)^{k-1} \left(1 + \frac{7}{4}\right) \\
&\leq \left(\frac{11}{4}\left(\frac{7}{4}\right)^{k-1} = \frac{44}{16}\left(\frac{7}{4}\right)^{k-1}\right) \leq \left(\frac{49}{16}\left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^2 \left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^{k+1}\right) \\
&\leq \left(\frac{7}{4}\right)^{k+1}
\end{aligned}
$$

Basis and inductive hypothesis proven, therefore original statement is true.

(c) Exercise Set 5.4, Problem 25(b)

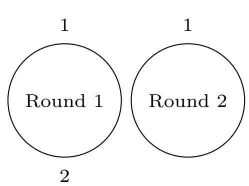
Use mathematical induction to prove that for all integers $n \geq 1$, given any set of 2^n people arranged in a circle and numbered consecutively 1 through 2^n, if one starts from person #1 and goes repeatedly around the circle successively eliminating every second person, eventually only person #1 will remain.

Let property $P(n)$ be 2^n people eliminated as given above will eventually leave only person #1.

Eliminating halves the number of people, so after one round there are $2^n/2 = 2^{n-1}$ people.

Basis:

$P(1)$:



is true.

Inductive hypothesis:

Assume $P(k) : 2^k$ people eliminated as given above will eventually leave only person #1 for $k \geq 1$ is true.

Prove $P(k+1) : 2^{k+1}$ people eliminated as given above will eventually leave only person #1:

After round 1 there are 2^k people.

Because $P(k)$ is true, by round 2, when there are 2^k people left, carrying out eliminations as given above must leave only person #1.

Basis and inductive hypothesis proven, therefore original statement is true.

(d) Exercise Set 5.4, Problem 30

It is a fact that every integer $n \geq 1$ can be written in the form
$c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0$,
where $c_r = 1$ or 2 and $c_i = 0, 1,$ or 2 for all integers $i = 0, 1, 2, \ldots, r-1$. Sketch a proof of this fact.
Let property $P(n)$ be the the property given above.
$P(n)$ says that any positive integer can be written as the sum of multiples of powers of 3.

Basis:

$P(1) : (c_{r=0} = 1) \cdot 3^{r=0} = 1$ is true.
$P(2) : (c_{r=0} = 1) \cdot 3^{r=0} + (c_0 = 1) = 2$ is true.
$P(3) : (c_{r=1} = 1) \cdot 3^{r=1} = 3$ is true.
$P(4) : (c_{r=1} = 1) \cdot 3^{r=1} + (c_{r=0} = 1) \cdot 3^{r=0} = 4$ is true.

Inductive hypothesis:

Let $k \geq 4$.
Assume $P(i)$ for $1 \leq i \leq k$ and $i \in \mathbb{Z}$ to be true.
Prove $P(k+1)$ can be written as the sum of multiples of powers of 3:

Case 1: $k + 1 \bmod 3 = 0$

$1 \leq (k+1)/3 \leq k$

$2 \leq k \leq 3k - 1$

$k \geq 2$ is true.

$\dfrac{k+1}{3} = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0$

$k + 1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3$

$k + 1$ is the sum of multiples of powers of 3.

Proof done.

Case 2: $k + 1 \bmod 3 = 1$

$k \bmod 3 = 0$

$1 \leq k/3 \leq k$

$3 \leq k \leq 3k$

$k \geq 3$ is true.

$\dfrac{k}{3} = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0$

$k = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3$

$k + 1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3 + (1 \cdot 3^0 = 1)$

$k + 1$ is the sum of multiples of powers of 3.

Proof done.

Case 3: $k + 1 \bmod 3 = 2$

$k - 1 \bmod 3 = 0$

$1 \leq (k-1)/3 \leq k$

$4 \leq k \leq 3k + 1$

$k \geq 4$ is true.

$\dfrac{k-1}{3} = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0$

$k - 1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3$

$k + 1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3 + (2 \cdot 3^0 = 2)$

$k + 1$ is the sum of multiples of powers of 3.

Proof done.

$P(k+1)$ is true in all cases.
Basis and inductive hypothesis proven, therefore original statement is true.

(e) Exercise Set 5.5, Problem 30

$F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.

Use mathematical induction to prove that for all integers $n \geq 0$, $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$

Let property $P(n)$ be $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$.

Basis:

$F_0 = 1$, $F_1 = 1$, $F_2 = 2$

$P(0) : (F_2F_0 - F_1^2 = 2 \cdot 1 - 1^2 = 1) = ((-1)^0 = 1)$ is true.

Inductive hypothesis:

Assume $P(k) : F_{k+2}F_k - F_{k+1}^2 = (-1)^k$ for $k \geq 0 \in \mathbb{Z}$ to be true.

Prove $P(k+1) : F_{k+3}F_{k+1} - F_{k+2}^2 = (-1)^{k+1}$:

$$\begin{aligned}
F_{k+3}F_{k+1} - F_{k+2}^2 &= (F_{k+2} + F_{k+1})F_{k+1} - (F_{k+1} + F_k)^2 \\
&= (2F_{k+1} + F_k)F_{k+1} - (F_{k+1}^2 + 2F_{k+1}F_k + F_k^2) \\
&= 2F_{k+1}^2 + F_kF_{k+1} - (F_{k+1}^2 + 2F_{k+1}F_k + F_k^2) \\
&= F_{k+1}^2 - F_kF_{k+1} - F_k^2 \\
&= F_{k+1}^2 - F_k(F_{k+1} + F_k) \\
&= F_{k+1}^2 - F_kF_{k+2} = -(F_{k+2}F_k - F_{k+1}^2) = -(-1)^k = (-1)^{k+1}
\end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(f) Let f be a function on whole numbers satisfying

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 4f(n/2) & \text{if } n > 0 \text{ and even} \\ f(n-1) + 2n - 1 & \text{if } n > 0 \text{ and odd} \end{cases}$$

Prove that $f(n) = n^2$ for all $n \geq 0$.

Let property $P(n)$ be $f(n) = n^2$.

Basis:

$P(0) : (f(0) = 0) = (0^2 = 0)$ is true.

$P(1) : (f(0) + 2 \cdot 1 - 1 = 1) = (1^2 = 1)$ is true.

Inductive hypothesis:

Let $k \geq 1$.

Assume $P(i) : f(i) = i^2$ for $0 \leq i \leq k$ and $i \in \mathbb{Z}$ to be true.

Prove $P(k+1) : f(k+1) = (k+1)^2$:

<table>
<tr><th>Case 1: $k+1$ is even</th><th>Case 2: $k+1$ is odd</th></tr>
<tr><td>$0 \leq (k+1)/2 \leq k$
$0 \leq k+1 \leq 2k$
$-k \leq 1 \leq k$
$k \geq 1$ is true.
$$\begin{aligned} f(k+1) &= 4f((k+1)/2) \\ &= 4\left(\frac{k+1}{2}\right)^2 && P(i) \text{ is true} \\ &= (k+1)^2 \end{aligned}$$
Proof done.</td><td>$0 \leq k/2 \leq k$
$0 \leq k \leq 2k$
$-k \leq 0 \leq k$
$k \geq 0$ is true.
$$\begin{aligned} f(k+1) &= f((k+1)-1) + 2(k+1) - 1 \\ &= f(k) + 2k + 1 \\ &= k^2 + 2k + 1 && P(i) \text{ is true} \\ &= (k+1)^2 \end{aligned}$$
Proof done.</td></tr>
</table>

Basis and inductive hypothesis proven, therefore original statement is true.

Problem 3.

(a) Exercise Set 5.5, Problems 28, 29
$F_0, F_1, F_2, \ldots$ is the Fibonacci sequence.
$F_0 = 1$, $F_1 = 1$, $F_2 = 2$, $F_3 = 3$

28. Prove that $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$, for all integers $k \geq 1$.
Let property $P(n)$ be $F_{n+1}^2 - F_n^2 - F_{n-1}^2 = 2F_nF_{n-1}$.
Basis:
$P(1) : (F_2^2 - F_1^2 - F_0^2 = 2^2 - 1^2 - 1^2 = 2) = (2F_1F_0 = 2 \cdot 1 \cdot 1 = 2)$ is true.
Inductive hypothesis:
Assume $P(k) : F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_kF_{k-1}$ for $k \geq 1$ to be true.
Prove $P(k+1) : F_{k+2}^2 - F_{k+1}^2 - F_k^2 = 2F_{k+1}F_k$:
$$\begin{aligned} F_{k+2}^2 - F_{k+1}^2 - F_k^2 &= (F_{k+1} + F_k)^2 - (F_k + F_{k-1})^2 - F_k^2 \\ &= (2F_k + F_{k-1})^2 - (F_k + F_{k-1})^2 - F_k^2 \\ &= 4F_k^2 + 4F_kF_{k-1} + F_{k-1}^2 - (F_k^2 + 2F_kF_{k-1} + F_{k-1}^2) - F_k^2 \\ &= 2F_k^2 + 2F_kF_{k-1} \\ &= 2F_k(F_k + F_{k-1}) \\ &= 2F_kF_{k+1} \end{aligned}$$
Basis and inductive hypothesis proven, therefore original statement is true.

29. Prove that $F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}$, for all integers $k \geq 1$.
Let property $P(n)$ be $F_{n+1}^2 - F_n^2 = F_{n-1}F_{n+2}$.
Basis:
$P(1) : (F_2^2 - F_1^1 = 3) = (F_0F_3 = 3)$ is true.
Inductive hypothesis:
Assume $P(k) : F_{k+1}^2 - F_k^2 = F_{k-1}F_{k+2}$ for $k \geq 1 \in \mathbb{Z}$ to be true.
Prove $P(k+1) : F_{k+2}^2 - F_{k+1}^2 = F_kF_{k+3}$:
$$\begin{aligned} F_{k+2}^2 - F_{k+1}^2 &= (F_{k+1} + F_k)^2 - (F_k + F_{k-1})^2 \\ &= (2F_k + F_{k-1})^2 - (F_k^2 + 2F_kF_{k-1} + F_{k-1}^2) \\ &= 4F_k^2 + 4F_kF_{k-1} + F_{k-1}^2 - (F_k^2 + 2F_kF_{k-1} + F_{k-1}^2) \\ &= 3F_k^2 + 2F_kF_{k-1} \\ &= F_k(3F_k + 2F_{k-1}) \\ &= F_k(F_{k+1} + 2F_k + F_{k-1}) \\ &= F_k(F_{k+2} + F_k + F_{k-1}) \\ &= F_k(F_{k+2} + F_{k+1}) \\ &= F_kF_{k+3} \end{aligned}$$
Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.6, Problem 2(b,d)
The formula
$1 + r + r^2 + \cdots + r^n = \dfrac{r^{n+1} - 1}{r - 1}$
is true for all real numbers r except for $r = 1$ and for all integers $n \geq 0$.

(b) If n is an integer and $n \geq 1$, find a formula for the expression $3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1$.
$$\sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{2}$$

(d) If n is an integer and $n \geq 1$, find a formula for the expression
$2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n$.
$(-1)^0 2^n + (-1)^1 2^{n-1} + (-1)^2 2^{n-2} + (-1)^3 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2^1 + (-1)^n 2^0$.
$$\begin{aligned}
\sum_{i=0}^{n} (-1)^i 2^{n-i} &= \sum_{i=0}^{n} (-1)^i \frac{2^n}{2^i} \\
&= 2^n \sum_{i=0}^{n} \frac{(-1)^i}{2^i} \\
&= 2^n \sum_{i=0}^{n} \left(-\frac{1}{2}\right)^i \\
&= 2^n \frac{\left(-\frac{1}{2}\right)^{n+1} - 1}{-\frac{1}{2} - 1} \\
&= 2^n \frac{2((-\frac{1}{2})^{n+1} - 1)}{3} \\
&= -\frac{2^{n+1}((-\frac{1}{2})^{n+1} - 1)}{3} \\
&= -\frac{(-1)^{n+1} - 2^{n+1}}{3} \\
&= \frac{2^{n+1} - (-1)^{n+1}}{3}
\end{aligned}$$

(c) Exercise Set 5.6, Problems 9, 14, 15

9. $g_k = \dfrac{g_{k-1}}{g_{k-1} + 2}$ for all integers $k \geq 2$
$g_1 = 1$
$$g_2 = \frac{1}{1+2} = \frac{1}{1+2^1}$$
$$g_3 = \frac{\frac{1}{1+2}}{\frac{1}{1+2} + 2} = \frac{1}{1+2} \cdot \frac{1+2}{1+2(1+2)} = \frac{1}{1+2(1+2)} = \frac{1}{1+2^1+2^2}$$
$$g_4 = \frac{\frac{1}{1+2+4}}{\frac{1}{1+2+4} + 2} = \frac{1}{1+2+4} \cdot \frac{1+2+4}{1+2(1+2+4)} = \frac{1}{1+2(1+2+4)} = \frac{1}{1+2^1+2^2+2^3}$$
$$g_k = \frac{1}{\displaystyle\sum_{i=0}^{k-1} 2^i}$$
$$g_k = \frac{1}{\dfrac{2^{(k-1)+1} - 1}{2 - 1}}$$
$$g_k = \frac{1}{2^k - 1}$$

14. $x_k = 3x_{k-1} + k$, for all integers $k \geq 2$

$x_1 = 1$

$x_2 = 3 \cdot 1 + 2$

$x_3 = 3(3 \cdot 1 + 2) + 3 = 3^2 \cdot 1 + 3 \cdot 2 + 3$

$x_4 = 3(3^2 \cdot 1 + 3 \cdot 2 + 3) + 4 = 3^3 \cdot 1 + 3^2 \cdot 2 + 3 \cdot 3 + 4$

$$x_k = \sum_{i=0}^{k-1} 3^i(k-i) = \sum_{i=0}^{k-1} k3^i - \sum_{i=0}^{k-1} i3^i = k \cdot \frac{3^k - 1}{2} - \sum_{i=1}^{k-1} i3^i$$

Let $S = \sum_{i=1}^{k-1} i3^i = 1 \cdot 3 + 2 \cdot 3^2 + \cdots + (k-2)3^{k-2} + (k-1)3^{k-1}$

$$S = \left.\begin{array}{r} 3 + 3^2 + \cdots + 3^{k-2} + 3^{k-1} \\ +3^2 + \cdots + 3^{k-2} + 3^{k-1} \\ \vdots \\ +3^{k-2} + 3^{k-1} \\ +3^{k-1} \end{array}\right\} k-1 \text{ lines}$$

$$\begin{aligned}
S &= \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} 3^j \\
&= \sum_{i=1}^{k-1} \left(\sum_{j=1}^{k-1} 3^j - \sum_{j=1}^{i-1} 3^j \right) \\
&= \sum_{i=1}^{k-1} \left(\frac{3^k - 3}{2} - \frac{3^i - 3}{2} \right) \\
&= (k-1)\frac{3^k - 3}{2} - \sum_{i=1}^{k-1} \frac{3^i - 3}{2} \\
&= (k-1)\frac{3^k - 3}{2} - \frac{1}{2} \left(\sum_{i=1}^{k-1} 3^i - 3(k-1) \right) \\
&= (k-1)\frac{3^k - 3}{2} - \frac{1}{2} \sum_{i=1}^{k-1} 3^i + \frac{1}{2}(3k-3) \\
&= \frac{k \cdot 3^k - 3^k - 3k + 3}{2} - \frac{1}{2} \cdot \frac{3^k - 3}{2} + \frac{3k-3}{2} \\
&= \frac{2k \cdot 3^k - 2 \cdot 3^k - 6k + 6}{4} - \frac{3^k - 3}{4} + \frac{6k - 6}{4} \\
&= \frac{2k \cdot 3^k - 3^{k+1} + 3}{4}
\end{aligned}$$

$$\begin{aligned}
1 + r + r^2 + \cdots + r^n &= \frac{r^{n+1} - 1}{r - 1} \\
r + r^2 + \cdots + r^n &= \frac{r^{n+1} - 1}{r - 1} - 1 \\
r + r^2 + \cdots + r^n &= \frac{r^{n+1} - r}{r - 1} \\
\sum_{i=1}^{n} r^i &= \frac{r^{n+1} - r}{r - 1}
\end{aligned}$$

$$x_k = k \cdot \frac{3^k - 1}{2} - S$$

$$x_k = \frac{k \cdot 3^k - k}{2} - \frac{2k \cdot 3^k - 3^{k+1} + 3}{4}$$

$$x_k = \frac{2k \cdot 3^k - 2k}{4} - \frac{2k \cdot 3^k - 3^{k+1} + 3}{4}$$

$$x_k = \frac{3^{k+1} - 2k - 3}{4}$$

15. $y_k = y_{k-1} + k^2$, for all integers $k \geq 2$

$$y_1 = 1$$
$$y_2 = 1 + 2^2 = 5$$
$$y_3 = (1 + 2^2) + 3^2 = 14$$
$$y_4 = (1 + 2^2 + 3^2) + 4^2 = 30$$
$$y_k = \sum_{i=1}^{k} i^2$$

$$y_k = \left.\begin{array}{r} 1+2+3+4+\cdots+(k-1)+k \\ +2+3+4+\cdots+(k-1)+k \\ +3+4+\cdots+(k-1)+k \\ +4+\cdots+(k-1)+k \\ \vdots \\ +(k-1)+k \\ +k \end{array}\right\} k \text{ lines}$$

$$\begin{aligned}
y_k &= \sum_{i=1}^{k} \sum_{j=i}^{k} j \\
&= \sum_{i=1}^{k} \left(\sum_{j=1}^{k} j - \sum_{j=1}^{i-1} j \right) \\
&= \sum_{i=1}^{k} \left(\frac{k(k+1)}{2} - \frac{i(i-1)}{2} \right) \\
&= k \cdot \frac{k(k+1)}{2} - \sum_{i=1}^{k} \frac{i(i-1)}{2} \\
&= \frac{k^2(k+1)}{2} - \sum_{i=1}^{k} \left(\frac{i^2}{2} - \frac{i}{2} \right) \\
&= \frac{k^2(k+1)}{2} + \sum_{i=1}^{k} \frac{i}{2} - \sum_{i=1}^{k} \frac{i^2}{2} \\
&= \frac{k^2(k+1)}{2} + \frac{1}{2} \sum_{i=1}^{k} i - \frac{1}{2} \sum_{i=1}^{k} i^2
\end{aligned}$$

$$y_k = \frac{k^2(k+1)}{2} + \frac{1}{2} \cdot \frac{k(k+1)}{2} - \frac{1}{2} y_k$$
$$4y_k = 2k^2(k+1) + k(k+1) - 2y_k$$
$$6y_k = (k+1)(2k^2 + k)$$
$$y_k = \frac{k(k+1)(2k+1)}{6}$$