

**Problem 1.**

(a) Exercise Set 5.2, Problem 11

$$1^3 + 2^3 + \cdots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2, \text{ for all integers } n \geq 1.$$

$$\text{Let property } P(n) \text{ be } \sum_{i=1}^n i^3 = \left[ \frac{n(n+1)}{2} \right]^2.$$

**Basis:**

$$P(1) : \left( \sum_{i=1}^1 i^3 = 1 \right) = \left( \left[ \frac{n(n+1)}{2} \right]^2 = 1 \right) \text{ is true.}$$

**Inductive hypothesis:**

$$\text{Assume } P(k) : \sum_{i=1}^k i^3 = \left[ \frac{k(k+1)}{2} \right]^2 \text{ for } k \geq 1 \in \mathbb{Z} \text{ is true.}$$

$$\text{Prove } P(k+1) : \sum_{i=1}^{k+1} i^3 = \left[ \frac{(k+1)(k+2)}{2} \right]^2 :$$

$$\sum_{i=1}^{k+1} i^3 \stackrel{?}{=} \left[ \frac{(k+1)(k+2)}{2} \right]^2$$

$$\left( \sum_{i=1}^k i^3 \right) + (k+1)^3 \stackrel{?}{=} \left[ \frac{k^2 + 3k + 2}{2} \right]^2$$

$$\left[ \frac{k(k+1)}{2} \right]^2 + k^3 + 3k^2 + 3k + 1 \stackrel{?}{=} \left[ \frac{k^2 + 3k + 2}{2} \right]^2$$

$$\left[ \frac{k^2 + k}{2} \right]^2 + k^3 + 3k^2 + 3k + 1 \stackrel{?}{=} \left[ \frac{k^2 + 3k + 2}{2} \right]^2$$

$$\frac{k^4 + 2k^3 + k^2}{4} + \frac{4k^3 + 12k^2 + 12k + 4}{4} \stackrel{?}{=} \frac{k^4 + 3k^3 + 2k^2 + 3k^3 + 9k^2 + 6k + 2k^2 + 6k + 4}{4}$$

$$\frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4} = \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.2, Problem 14

$$\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2, \text{ for all integers } n \geq 0.$$

Let property  $P(n)$  be  $\sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$ .

**Basis:**

$$P(0) : \left( \sum_{i=1}^1 i \cdot 2^i = 2 \right) = (0 \cdot 2^2 + 2 = 2) \text{ is true.}$$

**Inductive hypothesis:**

Assume  $P(k) : \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2$  for  $k \geq 0 \in \mathbb{Z}$  is true.

Prove  $P(k+1) : \sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$ :

$$\begin{aligned} & \sum_{i=1}^{k+2} i \cdot 2^i \stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\ & \left( \sum_{i=1}^{k+1} i \cdot 2^i \right) + (k+2) \cdot 2^{k+2} \stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\ & k \cdot 2^{k+2} + 2 + (k+2) \cdot 2^{k+2} \stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\ & k \cdot 2^{k+2} + 2 + k \cdot 2^{k+2} + 2^{k+3} \stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\ & 2k \cdot 2^{k+2} + 2 + 2^{k+3} \stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\ & k \cdot 2^{k+3} + 2 + 2^{k+3} \stackrel{?}{=} (k+1) \cdot 2^{k+3} + 2 \\ & (k+1) \cdot 2^{k+3} + 2 = (k+1) \cdot 2^{k+3} + 2 \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(c) Exercise Set 5.2, Problem 17

$$\prod_{i=0}^n \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2n+2)!}, \text{ for all integers } n \geq 0.$$

Let property  $P(n)$  be  $\prod_{i=0}^n \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right)$ .

**Basis:**

$$P(0) : \left( \prod_{i=0}^0 \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{2} \right) = \left( \frac{1}{(0+2)!} = \frac{1}{2} \right) \text{ is true.}$$

**Inductive Hypothesis:**

$$\text{Assume } P(k) : \prod_{i=0}^k \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2k+2)!} \text{ for } k \geq 0 \in \mathbb{Z} \text{ is true.}$$

$$\text{Prove } P(k+1) : \prod_{i=0}^{k+1} \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) = \frac{1}{(2(k+1)+2)!}:$$

$$\prod_{i=0}^{k+1} \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) \stackrel{?}{=} \frac{1}{(2(k+1)+2)!}$$

$$\prod_{i=0}^k \left( \frac{1}{2i+1} \cdot \frac{1}{2i+2} \right) \left( \frac{1}{2(k+1)+1} \cdot \frac{1}{2(2k+1)+2} \right) \stackrel{?}{=} \frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!} \left( \frac{1}{2k+3} \cdot \frac{1}{2k+4} \right) \stackrel{?}{=} \frac{1}{(2k+4)!}$$

$$\frac{1}{(2k+2)!(2k+3)(2k+4)} \stackrel{?}{=} \frac{1}{(2k+4)(2k+3)!}$$

$$\frac{1}{(2k+4)(2k+3)(2k+2)!} = \frac{1}{(2k+4)(2k+3)(2k+2)!}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(d) Exercise Set 5.3, Problem 10

$n^3 - 7n + 3$  is divisible by 3, for each integer  $n \geq 0$ .

Let property  $P(n)$  be  $n^3 - 7n + 3$  is divisible by 3.

**Basis:**

$$P(0) : (0^3 - 7 \cdot 0 + 3 = 3) \text{ is divisible by 3 is true.}$$

**Inductive Hypothesis:**

$$\text{Assume } P(k) : k^3 - 7k + 3 \text{ is divisible by 3 where } k \geq 0 \in \mathbb{Z} \text{ is true.}$$

$$\text{Prove } P(k+1) : (k+1)^3 - 7(k+1) + 3 \text{ is divisible by 3:}$$

$$\begin{aligned} & (k+1)^3 - 7(k+1) + 3 \\ &= k^3 + 3k^2 + 3k + 1 - 7k - 7 + 3 \\ &= \underbrace{(k^3 - 7k + 3)}_m + 3k^2 + 3k + 1 - 7 \end{aligned}$$

$P(k)$  is true, therefore  $m$  is a multiple of 3 and  $m = 3a$  for some integer  $a$ .

$$= 3a + 3k^2 + 3k - 6$$

$$= 3(a + k^2 + k - 2)$$

Expression is a multiple of 3, therefore  $(k+1)^3 - 7(k+1) + 3$  is divisible by 3.

Basis and inductive hypothesis proven, therefore original statement is true.

(e) Exercise Set 5.3, Problem 17

$1 + 3n \leq 4^n$ , for every integer  $n \geq 0$ .

Let property  $P(n)$  be  $1 + 3n \leq 4^n$ .

**Basis:**

$P(0) : (1 + 3 \cdot 0 = 1) \leq (4^0 = 1)$  is true.

**Inductive hypothesis:**

Assume  $P(k) : 1 + 3k \leq 4^k$  for  $k \geq 0 \in \mathbb{Z}$  is true.

Prove  $P(k+1) : 1 + 3(k+1) \leq 4^{k+1}$ :

$$1 + 3(k+1) \stackrel{?}{\leq} 4^{k+1}$$

$$4 + 3k \stackrel{?}{\leq} 4^{k+1}$$

$$(1 + 3k) + 3 \stackrel{?}{\leq} 4^{k+1}$$

$$4^k + 3 \stackrel{?}{\leq} 4^{k+1} \quad P(k) \text{ is true}$$

$$3 \stackrel{?}{\leq} 4^{k+1} - 4^k$$

$$3 \stackrel{?}{\leq} 4^k(4 - 1)$$

$$3 \stackrel{?}{\leq} 3 \cdot 4^k$$

$$1 \leq 4^k$$

Last inequality holds true for all  $k \geq 0 \in \mathbb{Z}$ .

Basis and inductive hypothesis proven, therefore original statement is true.

(f) Exercise Set 5.3, Problem 21

$$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}, \text{ for all integers } n \geq 2.$$

$$\text{Let property } P(n) \text{ be } \sqrt{n} < \sum_{i=1}^n \frac{1}{\sqrt{i}}.$$

**Basis:**

$$P(2) : (\sqrt{2}) < \left( \sum_{i=1}^2 \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{\sqrt{2}}{2} \right) \text{ is true.}$$

**Inductive hypothesis:**

$$\text{Assume } P(k) : \sqrt{k} < \sum_{i=1}^k \frac{1}{\sqrt{i}} \text{ for } k \geq 2 \in \mathbb{Z} \text{ is true.}$$

$$\text{Prove } P(k+1) : \sqrt{k+1} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}:$$

$$\sqrt{k+1} < \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}}$$

$$\sqrt{k+1} < \sum_{i=1}^k \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}}$$

$$\sqrt{k+1} < \sqrt{k} + \frac{1}{\sqrt{k+1}} \quad P(k) \text{ is true}$$

$$\frac{k+1}{\sqrt{k+1}} < \frac{\sqrt{k}\sqrt{k+1} + 1}{\sqrt{k+1}}$$

$$k+1 < \sqrt{k}\sqrt{k+1} + 1$$

$$\sqrt{k}\sqrt{k} < \sqrt{k}\sqrt{k+1}$$

$$k < k+1$$

$$0 < 1$$

Basis and inductive hypothesis proven, therefore original statement is true.

(g) Exercise Set 5.3, Problem 22

$1 + nx \leq (1 + x)^n$ , for all real numbers  $x > -1$  and integers  $n \geq 2$ .

Let property  $P(n)$  be  $1 + nx \leq (1 + x)^n$  for all  $x > -1 \in \mathbb{R}$ .

**Basis:**

$P(2) : 1 + 2x \leq (1 + x)^2$  is true.

$$\Rightarrow 1 + 2x \leq 1 + 2x + x^2$$

$$\Rightarrow 0 \leq x^2$$

**Inductive hypothesis:**

Assume  $P(k) : 1 + kx \leq (1 + x)^k$  for  $k \geq 2 \in \mathbb{Z}$  and  $x > -1 \in \mathbb{R}$  is true.

Prove  $P(k+1) : 1 + (k+1)x \leq (1 + x)^{k+1}$ :

$$1 + (k+1)x \stackrel{?}{\leq} (1 + x)^{k+1}$$

$$1 + kx + x \stackrel{?}{\leq} (1 + x)^{k+1}$$

$$(1 + x)^k + x \stackrel{?}{\leq} (1 + x)^{k+1}$$

$$x \stackrel{?}{\leq} (1 + x)^k((1 + x) - 1)$$

$$x \stackrel{?}{\leq} (1 + x)^k x$$

**Case 1:  $x = 0$**

Prove  $x \leq (1 + x)^k x$ :

$$0 \leq (1 + 0)^k \cdot 0$$

$$0 \leq 0$$

Proof done.

**Case 2:  $x > 0$**

Prove  $x \leq (1 + x)^k x$ :

$$\begin{array}{l|l} x > 0 & x \stackrel{?}{\leq} (1 + x)^k x \\ 1 < 1 + x & 1 \stackrel{?}{\leq} (1 + x)^k \\ 1^k < (1 + x)^k & \\ 1 < (1 + x)^k & \end{array}$$

$$1 < (1 + x)^k$$

$$\therefore 1 \leq (1 + x)^k$$

$$\therefore x \leq (1 + x)^k x$$

Proof done.

**Case 3:  $-1 < x < 0$**

Prove  $x \leq (1 + x)^k x$ :

$$\begin{array}{l|l} 0 > x > -1 & x \stackrel{?}{\leq} (1 + x)^k x \\ 1 > 1 + x > 0 & 1 \stackrel{?}{\geq} (1 + x)^k \\ 1^k > (1 + x)^k > 0^k & \\ 1 > (1 + x)^k > 0 & \end{array}$$

$$1 > (1 + x)^k$$

$$\therefore 1 \geq (1 + x)^k$$

$$\therefore x \leq (1 + x)^k x$$

Proof done.

$P(k+1)$  is true in all cases.

Basis and inductive hypothesis proven, therefore original statement is true.

(h) Exercise Set 5.3, Problem 29

As each of a group of businesspeople arrives at a meeting, each shakes hands with all the other people present. Use mathematical induction to show that if  $n$  people come to the meeting then  $[n(n-1)]/2$  handshakes occur.

If the set of businesspeople has size  $n$ , then the number of handshakes is  $\binom{n}{2}$ .

For a set of 0 and 1 businesspeople, no handshakes occur.

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n!}{2(n-2)!}$$

$$\text{Let property } P(n) \text{ be } \begin{cases} 0 = \frac{n(n-1)}{2} & \text{if } n = 0, n = 1 \\ \frac{n!}{2(n-2)!} = \frac{n(n-1)}{2} & \text{if } n \geq 2 \end{cases}.$$

**Basis:**

$$P(0) : 0 = \left( \frac{0(0-1)}{2} = 0 \right) \text{ is true.}$$

$$P(1) : 0 = \left( \frac{1(1-1)}{2} = 0 \right) \text{ is true.}$$

$$P(2) : \left( \frac{2!}{2(2-2)!} = \frac{2}{2 \cdot 1} = 1 \right) = \left( \frac{2(2-1)}{2} = 1 \right) \text{ is true.}$$

**Inductive hypothesis:**

$$\text{Assume } P(k) : \frac{k!}{2(k-2)!} = \frac{k(k-1)}{2} \text{ for } k \geq 2 \in \mathbb{Z} \text{ to be true.}$$

$$\text{Prove } P(k+1) : \frac{(k+1)!}{2((k+1)-2)!} = \frac{(k+1)((k+1)-1)}{2}:$$

$$\frac{(k+1)!}{2((k+1)-2)!} \stackrel{?}{=} \frac{(k+1)((k+1)-1)}{2}$$

$$\frac{(k+1)k!}{2(k-1)!} \stackrel{?}{=} \frac{(k+1)k}{2}$$

$$\frac{(k+1)k(k-1)!}{2(k-1)!} \stackrel{?}{=} \frac{(k+1)k}{2}$$

$$\frac{(k+1)k}{2} = \frac{(k+1)k}{2}$$

Basis and inductive hypothesis proven, therefore original statement is true.

- (i) Prove that in an  $n$ -sided regular polygon, where  $n \geq 3$ , the number of diagonals is  $n(n-3)/2$ .

The number of possible vertex pairs in an  $n$ -sided regular polygon is  $\binom{n}{2}$ , and  $n$  of these vertex pairs are the edges of the polygon. The number of diagonals is  $\binom{n}{2} - n$ .

$$\binom{n}{2} - n = \frac{n!}{2!(n-2)!} - n = \frac{n!}{2(n-2)!} - n$$

Let property  $P(n)$  be  $\frac{n!}{2(n-2)!} - n = \frac{n(n-3)}{2}$ .

**Basis:**

$$P(3) : \left( \frac{3!}{2(3-2)!} - 3 = \frac{6}{2} - 3 = 0 \right) = \left( \frac{3(3-3)}{2} = 0 \right) \text{ is true.}$$

**Inductive hypothesis:**

Assume  $P(k) : \frac{k!}{2(k-2)!} - k = \frac{k(k-3)}{2}$  for  $k \geq 3 \in \mathbb{Z}$  to be true.

Prove  $P(k+1) : \frac{(k+1)!}{2((k+1)-2)!} - (k+1) = \frac{(k+1)((k+1)-3)}{2}$ :

$$\frac{(k+1)!}{2((k+1)-2)!} - (k+1) \stackrel{?}{=} \frac{(k+1)((k+1)-3)}{2}$$

$$\frac{(k+1)k!}{2(k-1)!} - k - 1 \stackrel{?}{=} \frac{(k+1)(k-2)}{2}$$

$$\frac{(k+1)k(k-1)!}{2(k-1)!} - k - 1 \stackrel{?}{=} \frac{(k+1)(k-2)}{2}$$

$$\frac{(k+1)k}{2} - \frac{2k+2}{2} \stackrel{?}{=} \frac{(k+1)(k-2)}{2}$$

$$\frac{k^2 + k - 2k - 2}{2} \stackrel{?}{=} \frac{k^2 - k - 2}{2}$$

$$\frac{k^2 - k - 2}{2} = \frac{k^2 - k - 2}{2}$$

Basis and inductive hypothesis proven, therefore original statement is true.

- (j) Prove that the number of permutations of the set  $\{1, 2, \dots, n\}$  with  $n$  elements is  $n!$ , for natural number  $n \geq 1$ .

A set of  $n = 1$  elements has  $1! = 1$  permutation.

Let property  $P(n)$  be  $\{1, 2, \dots, n\}$  has  $n!$  permutations.

**Basis:**

$P(1) : \{1\}$  has  $1! = 1$  permutation is true.

**Inductive hypothesis:**

Assume  $P(k) : \{1, 2, \dots, k\}$  has  $k!$  permutations for  $k \geq 1 \in \mathbb{Z}$  to be true.

In order to create a permuted set  $B_p$  of size  $k+1$ , one can insert  $k+1$  into  $A_p$ , an arbitrary permutation of set  $A$  of  $k$  elements.

This action is equivalent to permuting a set of size  $k+1$ .

There are  $k+1$  positions to insert such an element into  $A_p$ :  $k$  positions before each element and one position after the last element of  $A_p$ .

There are  $k!$  possible  $A_p$  made from  $A$ .

$(k+1)$  ways to insert into  $A_p \times (k!$  possible  $A_p) = (k+1)k! = (k+1)!$  ways to create  $B_p$ .

$P(k+1)$  is true.

Basis and inductive hypothesis proven, therefore original statement is true.



**Problem 2.**

(a) Exercise Set 5.4, Problem 8

Suppose that  $h_0, h_1, h_2, \dots$  is a sequence defined as follows:

$$h_0 = 1, h_1 = 2, h_2 = 3,$$

$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \text{ for all integers } k \geq 3.$$

(a) Prove that  $h_n \leq 3^n$  for all integers  $n \geq 0$ .

Let property  $P(n)$  be  $h_n \leq 3^n$ .

**Basis:**

$$P(0) : (h_0 = 1) \leq (3^0 = 1) \text{ is true.}$$

$$P(1) : (h_1 = 2) \leq (3^1 = 3) \text{ is true.}$$

$$P(2) : (h_2 = 3) \leq (3^2 = 9) \text{ is true.}$$

**Inductive hypothesis:**

Let  $k \geq 2$ .

Assume  $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq 3^i$  for  $0 \leq i \leq k$  and  $i \in \mathbb{Z}$  is true.

Prove  $P(k+1) : h_{k+1} = h_k + h_{k-1} + h_{k-2} \leq 3^{k+1}$ :

$$\begin{aligned} h_{k+1} &= h_k + h_{k-1} + h_{k-2} \\ &\leq 3^k + 3^{k-1} + 3^{k-2} && P(i) \text{ is true} \\ &\leq 3^{k-2} (3^2 + 3 + 1) \\ &\leq 13 \cdot 3^{k-2} && \leq (3^3 \cdot 3^{k-2} = 3^{k+1}) \\ &\leq 3^{k+1} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Suppose that  $s$  is any real number such that  $s^3 \geq s^2 + s + 1$ . (This implies that  $2 > s > 1.83$ .)

Prove that  $h_n \leq s^n$  for all  $n \geq 2$ .

Let property  $P(n)$  be  $h_n \leq s^n$ .

**Basis:**

$$P(2) : (h_2 = 2) \leq (3.34 < s^2 < 4) \text{ is true.}$$

$$P(3) : (h_3 = 6) \leq (6.12 < s^3 < 8) \text{ is true.}$$

$$P(4) : (h_4 = 11) \leq (11.21 < s^4 < 16) \text{ is true.}$$

**Inductive hypothesis:**

Let  $k \geq 4$ .

Assume  $P(i) : h_i = h_{i-1} + h_{i-2} + h_{i-3} \leq s^i$  for  $2 \leq i \leq k$  and  $i \in \mathbb{Z}$ .

Prove  $P(k+1) : h_{k+1} = h_k + h_{k-1} + h_{k-2} \leq s^{k+1}$ :

$$\begin{aligned} h_{k+1} &= h_k + h_{k-1} + h_{k-2} \\ &\leq s^k + s^{k-1} + s^{k-2} \\ &\leq s^{k-2} (s^2 + s + 1) \\ &\leq s^{k-2} s^3 && s^2 + s + 1 \leq s^3 \\ &\leq s^{k+1} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.4, Problem 9

Define a sequence  $a_1, a_2, a_3, \dots$  as follows:  $a_1 = 1$ ,  $a_2 = 3$ , and  $a_k = a_{k-1} + a_{k-2}$  for all integers  $k \geq 3$ .

Use strong mathematical induction to prove that  $a_n \leq \left(\frac{7}{4}\right)^n$  for all integers  $n \geq 1$ .

Let property  $P(n)$  be  $a_n \leq \left(\frac{7}{4}\right)^n$ .

**Basis:**

$P(1) : 1 \leq \frac{7}{4}$  is true.

$P(2) : 3 \leq \frac{49}{16}$  is true.

**Inductive hypothesis:**

Let  $k \geq 2$ .

Assume  $P(i) : a_i = a_{i-1} + a_{i-2} \leq \left(\frac{7}{4}\right)^i$  for  $1 \leq i \leq k$  and  $i \in \mathbb{Z}$  is true.

Prove  $P(k+1) : a_k + a_{k-1} \leq \left(\frac{7}{4}\right)^{k+1}$ :

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} \\ &\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \\ &\leq \left(\frac{7}{4}\right)^{k-1} \left(1 + \frac{7}{4}\right) \\ &\leq \left(\frac{11}{4}\right) \left(\frac{7}{4}\right)^{k-1} = \frac{44}{16} \left(\frac{7}{4}\right)^{k-1} \leq \left(\frac{49}{16}\right) \left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^2 \left(\frac{7}{4}\right)^{k-1} = \left(\frac{7}{4}\right)^{k+1} \\ &\leq \left(\frac{7}{4}\right)^{k+1} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

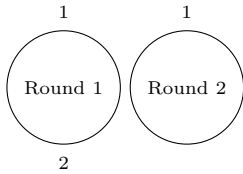
(c) Exercise Set 5.4, Problem 25(b)

Use mathematical induction to prove that for all integers  $n \geq 1$ , given any set of  $2^n$  people arranged in a circle and numbered consecutively 1 through  $2^n$ , if one starts from person #1 and goes repeatedly around the circle successively eliminating every second person, eventually only person #1 will remain. Let property  $P(n)$  be  $2^n$  people eliminated as given above will eventually leave only person #1.

Eliminating halves the number of people, so after one round there are  $2^n/2 = 2^{n-1}$  people.

**Basis:**

$P(1)$ :



is true.

**Inductive hypothesis:**

Assume  $P(k) : 2^k$  people eliminated as given above will eventually leave only person #1 for  $k \geq 1$  is true.

Prove  $P(k+1) : 2^{k+1}$  people eliminated as given above will eventually leave only person #1:

After round 1 there are  $2^k$  people.

Because  $P(k)$  is true, by round 2, when there are  $2^k$  people left, carrying out eliminations as given above must leave only person #1.

Basis and inductive hypothesis proven, therefore original statement is true.

(d) Exercise Set 5.4, Problem 30

It is a fact that every integer  $n \geq 1$  can be written in the form

$$c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0,$$

where  $c_r = 1$  or  $2$  and  $c_i = 0, 1$ , or  $2$  for all integers  $i = 0, 1, 2, \dots, r-1$ . Sketch a proof of this fact.

Let property  $P(n)$  be the property given above.

$P(n)$  says that any positive integer can be written as the sum of multiples of powers of 3.

**Basis:**

$$P(1) : (c_{r=0} = 1) \cdot 3^{r=0} = 1 \text{ is true.}$$

$$P(2) : (c_{r=0} = 1) \cdot 3^{r=0} + (c_0 = 1) = 2 \text{ is true.}$$

$$P(3) : (c_{r=1} = 1) \cdot 3^{r=1} = 3 \text{ is true.}$$

$$P(4) : (c_{r=1} = 1) \cdot 3^{r=1} + (c_{r=0} = 1) \cdot 3^{r=0} = 4 \text{ is true.}$$

**Inductive hypothesis:**

Let  $k \geq 4$ .

Assume  $P(i)$  for  $1 \leq i \leq k$  and  $i \in \mathbb{Z}$  to be true.

Prove  $P(k+1)$  can be written as the sum of multiples of powers of 3:

**Case 1:**  $k+1 \bmod 3 = 0$

$$\left. \begin{array}{l} 1 \leq (k+1)/3 \leq k \\ 2 \leq k \leq 3k-1 \\ k \geq 2 \end{array} \right\} \text{ is true.}$$

$$\frac{k+1}{3} = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \text{ because } P(i)$$

$$k+1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3$$

$$k+1 \text{ is the sum of multiples of powers of 3.}$$

Proof done.

**Case 2:**  $k+1 \bmod 3 = 1$

$$\left. \begin{array}{l} k \bmod 3 = 0 \\ 1 \leq k/3 \leq k \\ 3 \leq k \leq 3k \\ k \geq 3 \end{array} \right\} \text{ is true.}$$

$$\frac{k}{3} = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \text{ because } P(i)$$

$$k = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3$$

$$k+1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3 + (1 \cdot 3^0 = 1)$$

$$k+1 \text{ is the sum of multiples of powers of 3.}$$

Proof done.

**Case 3:**  $k+1 \bmod 3 = 2$

$$\left. \begin{array}{l} k-1 \bmod 3 = 0 \\ 1 \leq (k-1)/3 \leq k \\ 4 \leq k \leq 3k+1 \\ k \geq 4 \end{array} \right\} \text{ is true.}$$

$$\frac{k-1}{3} = c_r \cdot 3^r + c_{r-1} \cdot 3^{r-1} + \cdots + c_2 \cdot 3^2 + c_1 \cdot 3 + c_0 \text{ because } P(i)$$

$$k-1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3$$

$$k+1 = c_r \cdot 3^{r+1} + c_{r-1} \cdot 3^r + \cdots + c_2 \cdot 3^3 + c_1 \cdot 3^2 + c_0 \cdot 3 + (2 \cdot 3^0 = 2)$$

$$k+1 \text{ is the sum of multiples of powers of 3.}$$

Proof done.

$P(k+1)$  is true in all cases.

Basis and inductive hypothesis proven, therefore original statement is true.

(e) Exercise Set 5.5, Problem 30

$F_0, F_1, F_2, \dots$  is the Fibonacci sequence.

Use mathematical induction to prove that for all integers  $n \geq 0$ ,  $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$

Let property  $P(n)$  be  $F_{n+2}F_n - F_{n+1}^2 = (-1)^n$ .

**Basis:**

$$F_0 = 1, F_1 = 1, F_2 = 2$$

$$P(0) : (F_2F_0 - F_1^2 = 2 \cdot 1 - 1^2 = 1) = ((-1)^0 = 1) \text{ is true.}$$

**Inductive hypothesis:**

Assume  $P(k) : F_{k+2}F_k - F_{k+1}^2 = (-1)^k$  for  $k \geq 0 \in \mathbb{Z}$  to be true.

Prove  $P(k+1) : F_{k+3}F_{k+1} - F_{k+2}^2 = (-1)^{k+1}$ :

$$\begin{aligned} F_{k+3}F_{k+1} - F_{k+2}^2 &= (F_{k+2} + F_{k+1})F_{k+1} - (F_{k+1} + F_k)^2 \\ &= (2F_{k+1} + F_k)F_{k+1} - (F_{k+1}^2 + 2F_{k+1}F_k + F_k^2) \\ &= 2F_{k+1}^2 + F_kF_{k+1} - (F_{k+1}^2 + 2F_{k+1}F_k + F_k^2) \\ &= F_{k+1}^2 - F_kF_{k+1} - F_k^2 \\ &= F_{k+1}^2 - F_k(F_{k+1} + F_k) \\ &= F_{k+1}^2 - F_kF_{k+2} = -(F_{k+2}F_k - F_{k+1}^2) = -(-1)^k = (-1)^{k+1} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(f) Let  $f$  be a function on whole numbers satisfying

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 4f(n/2) & \text{if } n > 0 \text{ and even} \\ f(n-1) + 2n - 1 & \text{if } n > 0 \text{ and odd} \end{cases}$$

Prove that  $f(n) = n^2$  for all  $n \geq 0$ .

Let property  $P(n)$  be  $f(n) = n^2$ .

**Basis:**

$$P(0) : (f(0) = 0) = (0^2 = 0) \text{ is true.}$$

$$P(1) : (f(0) + 2 \cdot 1 - 1 = 1) = (1^2 = 1) \text{ is true.}$$

**Inductive hypothesis:**

Let  $k \geq 1$ .

Assume  $P(i) : f(i) = i^2$  for  $0 \leq i \leq k$  and  $i \in \mathbb{Z}$  to be true.

Prove  $P(k+1) : f(k+1) = (k+1)^2$ :

Case 1: $k+1$ is even	Case 2: $k+1$ is odd
$\left. \begin{aligned} 0 \leq (k+1)/2 \leq k \\ 0 \leq k+1 \leq 2k \\ -k \leq 1 \leq k \\ k \geq 1 \end{aligned} \right\} \text{ is true.}$ $\begin{aligned} f(k+1) &= 4f((k+1)/2) \\ &= 4 \left( \frac{k+1}{2} \right)^2 & P(i) \text{ is true} \\ &= (k+1)^2 \end{aligned}$ <p>Proof done.</p>	$\left. \begin{aligned} 0 \leq k/2 \leq k \\ 0 \leq k \leq 2k \\ -k \leq 0 \leq k \\ k \geq 0 \end{aligned} \right\} \text{ is true.}$ $\begin{aligned} f(k+1) &= f((k+1)-1) + 2(k+1) - 1 \\ &= f(k) + 2k + 1 \\ &= k^2 + 2k + 1 & P(i) \text{ is true} \\ &= (k+1)^2 \end{aligned}$ <p>Proof done.</p>

Basis and inductive hypothesis proven, therefore original statement is true.

**Problem 3.**

- (a) Exercise Set 5.5, Problems 28, 29

$F_0, F_1, F_2, \dots$  is the Fibonacci sequence.

$F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3$

28. Prove that  $F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_k F_{k-1}$ , for all integers  $k \geq 1$ .

Let property  $P(n)$  be  $F_{n+1}^2 - F_n^2 - F_{n-1}^2 = 2F_n F_{n-1}$ .

**Basis:**

$P(1) : (F_2^2 - F_1^2 - F_0^2 = 2^2 - 1^2 - 1^2 = 2) = (2F_1 F_0 = 2 \cdot 1 \cdot 1 = 2)$  is true.

**Inductive hypothesis:**

Assume  $P(k) : F_{k+1}^2 - F_k^2 - F_{k-1}^2 = 2F_k F_{k-1}$  for  $k \geq 1$  to be true.

Prove  $P(k+1) : F_{k+2}^2 - F_{k+1}^2 - F_k^2 = 2F_{k+1} F_k$ :

$$\begin{aligned} F_{k+2}^2 - F_{k+1}^2 - F_k^2 &= (F_{k+1} + F_k)^2 - (F_k + F_{k-1})^2 - F_k^2 \\ &= (2F_k + F_{k-1})^2 - (F_k + F_{k-1})^2 - F_k^2 \\ &= 4F_k^2 + 4F_k F_{k-1} + F_{k-1}^2 - (F_k^2 + 2F_k F_{k-1} + F_{k-1}^2) - F_k^2 \\ &= 2F_k^2 + 2F_k F_{k-1} \\ &= 2F_k(F_k + F_{k-1}) \\ &= 2F_k F_{k+1} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

29. Prove that  $F_{k+1}^2 - F_k^2 = F_{k-1} F_{k+2}$ , for all integers  $k \geq 1$ .

Let property  $P(n)$  be  $F_{n+1}^2 - F_n^2 = F_{n-1} F_{n+2}$ .

**Basis:**

$P(1) : (F_2^2 - F_1^2 = 3) = (F_0 F_3 = 3)$  is true.

**Inductive hypothesis:**

Assume  $P(k) : F_{k+1}^2 - F_k^2 = F_{k-1} F_{k+2}$  for  $k \geq 1 \in \mathbb{Z}$  to be true.

Prove  $P(k+1) : F_{k+2}^2 - F_{k+1}^2 = F_k F_{k+3}$ :

$$\begin{aligned} F_{k+2}^2 - F_{k+1}^2 &= (F_{k+1} + F_k)^2 - (F_k + F_{k-1})^2 \\ &= (2F_k + F_{k-1})^2 - (F_k^2 + 2F_k F_{k-1} + F_{k-1}^2) \\ &= 4F_k^2 + 4F_k F_{k-1} + F_{k-1}^2 - (F_k^2 + 2F_k F_{k-1} + F_{k-1}^2) \\ &= 3F_k^2 + 2F_k F_{k-1} \\ &= F_k(3F_k + 2F_{k-1}) \\ &= F_k(F_{k+1} + 2F_k + F_{k-1}) \\ &= F_k(F_{k+2} + F_k + F_{k-1}) \\ &= F_k(F_{k+2} + F_{k+1}) \\ &= F_k F_{k+3} \end{aligned}$$

Basis and inductive hypothesis proven, therefore original statement is true.

(b) Exercise Set 5.6, Problem 2(b,d)

The formula

$$1 + r + r^2 + \cdots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

is true for all real numbers  $r$  except for  $r = 1$  and for all integers  $n \geq 0$ .

(b) If  $n$  is an integer and  $n \geq 1$ , find a formula for the expression  $3^{n-1} + 3^{n-2} + \cdots + 3^2 + 3 + 1$ .

$$\sum_{i=0}^{n-1} 3^i = \frac{3^n - 1}{2}$$

(d) If  $n$  is an integer and  $n \geq 1$ , find a formula for the expression

$$2^n - 2^{n-1} + 2^{n-2} - 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2 + (-1)^n.$$

$$(-1)^0 2^n + (-1)^1 2^{n-1} + (-1)^2 2^{n-2} + (-1)^3 2^{n-3} + \cdots + (-1)^{n-1} \cdot 2^1 + (-1)^n 2^0.$$

$$\begin{aligned} \sum_{i=0}^n (-1)^i 2^{n-i} &= \sum_{i=0}^n (-1)^i \frac{2^n}{2^i} \\ &= 2^n \sum_{i=0}^n \frac{(-1)^i}{2^i} \\ &= 2^n \sum_{i=0}^n \left(-\frac{1}{2}\right)^i \\ &= 2^n \frac{\left(-\frac{1}{2}\right)^{n+1} - 1}{-\frac{1}{2} - 1} \\ &= 2^n \frac{2\left(\left(-\frac{1}{2}\right)^{n+1} - 1\right)}{3} \\ &= -\frac{2^{n+1}\left(\left(-\frac{1}{2}\right)^{n+1} - 1\right)}{3} \\ &= -\frac{(-1)^{n+1} - 2^{n+1}}{3} \\ &= \frac{2^{n+1} - (-1)^{n+1}}{3} \end{aligned}$$

(c) Exercise Set 5.6, Problems 9, 14, 15

$$9. g_k = \frac{g_{k-1}}{g_{k-1} + 2} \text{ for all integers } k \geq 2$$

$$g_1 = 1$$

$$g_2 = \frac{1}{1+2} = \frac{1}{1+2^1}$$

$$g_3 = \frac{\frac{1}{1+2}}{\frac{1}{1+2} + 2} = \frac{1}{1+2} \cdot \frac{1+2}{1+2(1+2)} = \frac{1}{1+2(1+2)} = \frac{1}{1+2^1+2^2}$$

$$g_4 = \frac{\frac{1}{1+2+4}}{\frac{1}{1+2+4} + 2} = \frac{1}{1+2+4} \cdot \frac{1+2+4}{1+2(1+2+4)} = \frac{1}{1+2(1+2+4)} = \frac{1}{1+2^1+2^2+2^3}$$

$$g_k = \frac{1}{\sum_{i=0}^{k-1} 2^i}$$

$$g_k = \frac{1}{\frac{2^{(k-1)+1} - 1}{2 - 1}}$$

$$g_k = \frac{1}{2^k - 1}$$

14.  $x_k = 3x_{k-1} + k$ , for all integers  $k \geq 2$

$$x_1 = 1$$

$$x_2 = 3 \cdot 1 + 2$$

$$x_3 = 3(3 \cdot 1 + 2) + 3 = 3^2 \cdot 1 + 3 \cdot 2 + 3$$

$$x_4 = 3(3^2 \cdot 1 + 3 \cdot 2 + 3) + 4 = 3^3 \cdot 1 + 3^2 \cdot 2 + 3 \cdot 3 + 4$$

$$x_k = \sum_{i=0}^{k-1} 3^i(k-i) = \sum_{i=0}^{k-1} k3^i - \sum_{i=0}^{k-1} i3^i = k \cdot \frac{3^k - 1}{2} - \sum_{i=1}^{k-1} i3^i$$

$$\text{Let } S = \sum_{i=1}^{k-1} i3^i = 1 \cdot 3 + 2 \cdot 3^2 + \cdots + (k-2)3^{k-2} + (k-1)3^{k-1}$$

$$S = \left. \begin{array}{l} 3 + 3^2 + \cdots + 3^{k-2} + 3^{k-1} \\ + 3^2 + \cdots + 3^{k-2} + 3^{k-1} \\ \vdots \\ + 3^{k-2} + 3^{k-1} \\ + 3^{k-1} \end{array} \right\} k-1 \text{ lines}$$

$$\begin{aligned} S &= \sum_{i=1}^{k-1} \sum_{j=i}^{k-1} 3^j \\ &= \sum_{i=1}^{k-1} \left( \sum_{j=1}^{k-1} 3^j - \sum_{j=1}^{i-1} 3^j \right) \\ &= \sum_{i=1}^{k-1} \left( \frac{3^k - 3}{2} - \frac{3^i - 3}{2} \right) \\ &= (k-1) \frac{3^k - 3}{2} - \sum_{i=1}^{k-1} \frac{3^i - 3}{2} \\ &= (k-1) \frac{3^k - 3}{2} - \frac{1}{2} \left( \sum_{i=1}^{k-1} 3^i - 3(k-1) \right) \\ &= (k-1) \frac{3^k - 3}{2} - \frac{1}{2} \sum_{i=1}^{k-1} 3^i + \frac{1}{2} (3k - 3) \\ &= \frac{k \cdot 3^k - 3^k - 3k + 3}{2} - \frac{1}{2} \cdot \frac{3^k - 3}{2} + \frac{3k - 3}{2} \\ &= \frac{2k \cdot 3^k - 2 \cdot 3^k - 6k + 6}{4} - \frac{3^k - 3}{4} + \frac{6k - 6}{4} \\ &= \frac{2k \cdot 3^k - 3^{k+1} + 3}{4} \\ x_k &= k \cdot \frac{3^k - 1}{2} - S \\ x_k &= \frac{k \cdot 3^k - k}{2} - \frac{2k \cdot 3^k - 3^{k+1} + 3}{4} \\ x_k &= \frac{2k \cdot 3^k - 2k}{4} - \frac{2k \cdot 3^k - 3^{k+1} + 3}{4} \\ x_k &= \frac{3^{k+1} - 2k - 3}{4} \end{aligned}$$

$$\begin{aligned} 1 + r + r^2 + \cdots + r^n &= \frac{r^{n+1} - 1}{r - 1} \\ r + r^2 + \cdots + r^n &= \frac{r^{n+1} - 1}{r - 1} - 1 \\ r + r^2 + \cdots + r^n &= \frac{r^{n+1} - r}{r - 1} \\ \sum_{i=1}^n r^i &= \frac{r^{n+1} - r}{r - 1} \end{aligned}$$

15.  $y_k = y_{k-1} + k^2$ , for all integers  $k \geq 2$

$$y_1 = 1$$

$$y_2 = 1 + 2^2 = 5$$

$$y_3 = (1 + 2^2) + 3^2 = 14$$

$$y_4 = (1 + 2^2 + 3^2) + 4^2 = 30$$

$$y_k = \sum_{i=1}^k i^2$$

$$y_k = \left. \begin{array}{l} 1 + 2 + 3 + 4 + \cdots + (k-1) + k \\ \quad + 2 + 3 + 4 + \cdots + (k-1) + k \\ \quad \quad + 3 + 4 + \cdots + (k-1) + k \\ \quad \quad \quad + 4 + \cdots + (k-1) + k \\ \quad \quad \quad \quad \vdots \\ \quad \quad \quad \quad + (k-1) + k \\ \quad \quad \quad \quad \quad + k \end{array} \right\} k \text{ lines}$$

$$\begin{aligned} y_k &= \sum_{i=1}^k \sum_{j=i}^k j \\ &= \sum_{i=1}^k \left( \sum_{j=1}^k j - \sum_{j=1}^{i-1} j \right) \\ &= \sum_{i=1}^k \left( \frac{k(k+1)}{2} - \frac{i(i-1)}{2} \right) \\ &= k \cdot \frac{k(k+1)}{2} - \sum_{i=1}^k \frac{i(i-1)}{2} \\ &= \frac{k^2(k+1)}{2} - \sum_{i=1}^k \left( \frac{i^2}{2} - \frac{i}{2} \right) \\ &= \frac{k^2(k+1)}{2} + \sum_{i=1}^k \frac{i}{2} - \sum_{i=1}^k \frac{i^2}{2} \\ &= \frac{k^2(k+1)}{2} + \frac{1}{2} \sum_{i=1}^k i - \frac{1}{2} \sum_{i=1}^k i^2 \\ y_k &= \frac{k^2(k+1)}{2} + \frac{1}{2} \cdot \frac{k(k+1)}{2} - \frac{1}{2} y_k \\ 4y_k &= 2k^2(k+1) + k(k+1) - 2y_k \\ 6y_k &= (k+1)(2k^2 + k) \\ y_k &= \frac{k(k+1)(2k+1)}{6} \end{aligned}$$