Problem 1.
Exercise Set 7.1

28. Student C tries to define a function $h : \mathbb{Q} \to \mathbb{Q}$ by the rule
$h\left(\frac{m}{n}\right) = \frac{m^2}{n}$, for all integers m and n with $n \neq 0$.
Student D claims that h is not well defined. Justify student D's claim.
$\frac{1}{3} = \frac{2}{6} \Rightarrow h\left(\frac{1}{3}\right) = h\left(\frac{2}{6}\right)$
$\left(h\left(\frac{1}{3}\right) = \frac{1}{3}\right) \neq \left(h\left(\frac{2}{6}\right) = \frac{4}{6} = \frac{2}{3}\right)$
Contradiction, therefore h is not well defined.

Problem 2.
Exercise Set 7.1

35. $F(A \cap B) \subseteq F(A) \cap F(B)$
Proof:
Suppose $y \in F(A \cap B)$.
Prove $y \in F(A) \cap F(B)$**:**
$y = F(x)$ for some $x \in A \cap B$.
$x \in A$ by definition of intersection, therefore $y \in F(A)$.
$x \in B$ by definition of intersection, therefore $y \in F(B)$.
$y \in F(A)$ and $y \in F(B)$, therefore $y \in F(A) \cap F(B)$.
Proof done.

Problem 3.
Exercise Set 7.1

41. For all subsets C and D of Y,
$F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$.
Proof:

Part 1: $F^{-1}(C - D) \subseteq F^{-1}(C) - F^{-1}(D)$
Suppose $x \in F^{-1}(C - D)$.
Prove $x \in F^{-1}(C) - F^{-1}(D)$**:**
$F(x) \in C - D$, therefore $F(x) \in C$ and $F(x) \notin D$.
$x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$, therefore $x \in F^{-1}(C) - F^{-1}(D)$.
Proof done.

Part 2: $F^{-1}(C) - F^{-1}(D) \subseteq F^{-1}(C - D)$
Suppose $x \in F^{-1}(C) - F^{-1}(D)$.
Prove $x \in F^{-1}(C - D)$**:**
$x \in F^{-1}(C)$ and $x \notin F^{-1}(D)$, therefore $F(x) \in C$ and $F(x) \notin D$.
$F(x) \in C - D$, therefore $x \in F^{-1}(C - D)$.
Proof done.

Both parts proven, therefore $F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$.

Problem 4.
Exercise Set 7.1

43. Given a set S and a subset A, the characteristic function of A, denoted χ_A, is the function defined from S to $\mathbb{Z}$ with the property that for all $u \in S$,

$$\chi_A(u) = \begin{cases} 1 & \text{if } u \in A \\ 0 & \text{if } u \notin A \end{cases}$$

$A \subseteq S$, $B \subseteq S$, $u \in S$.

(a) $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$

Case 1: $u \notin A$ and $u \notin B$	**Case 2:** $u \notin A$ and $u \in B$
$\chi_{A\cap B}(u) = 0$ $(\chi_A(u) = 0) \cdot (\chi_B(u) = 0) = 0$ $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$ Proof done.	$\chi_{A\cap B}(u) = 0$ $(\chi_A(u) = 0) \cdot (\chi_B(u) = 1) = 0$ $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$ Proof done.
Case 3: $u \in A$ and $u \notin B$	**Case 4:** $u \in A$ and $u \in B$
$\chi_{A\cap B}(u) = 0$ $(\chi_A(u) = 1) \cdot (\chi_B(u) = 0) = 0$ $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$ Proof done.	$\chi_{A\cap B}(u) = 1$ $(\chi_A(u) = 1) \cdot (\chi_B(u) = 1) = 1$ $\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$ Proof done.

$\chi_{A\cap B}(u) = \chi_A(u) \cdot \chi_B(u)$ is true in all cases.

(b) $\chi_{A\cup B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$

Case 1: $u \notin A$ and $u \notin B$	**Case 2:** $u \notin A$ and $u \in B$
$\chi_{A\cup B}(u) = 0$ $\chi_A(u) = 0, \chi_B(u) = 0$ $\chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u) = 0$ $\chi_{A\cap B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$ Proof done.	$\chi_{A\cup B}(u) = 1$ $\chi_A(u) = 0, \chi_B(u) = 1$ $\chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u) = 1$ $\chi_{A\cap B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$ Proof done.
Case 3: $u \in A$ and $u \notin B$	**Case 4:** $u \in A$ and $u \in B$
$\chi_{A\cup B}(u) = 0$ $\chi_A(u) = 1, \chi_B(u) = 0$ $\chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u) = 1$ $\chi_{A\cap B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$ Proof done.	$\chi_{A\cup B}(u) = 1$ $\chi_A(u) = 1, \chi_B(u) = 1$ $\chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u) = 1$ $\chi_{A\cap B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$ Proof done.

$\chi_{A\cap B}(u) = \chi_A(u) + \chi_B(u) - \chi_A(u) \cdot \chi_B(u)$ is true in all cases.

Problem 5.
Exercise Set 7.2

23. Define $H : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:
$H(x,y) = (x+1, 2-y)$ for all $(x,y) \in \mathbb{R}^2$.

(a) Is H one-to-one?
Suppose (x_1, y_1) and $(x_2, y_2) \in \mathbb{R}^2$ and $H(x_1, y_1) = H(x_2, y_2)$.
H is one-to-one if $x_1 = x_2$ and $y_1 = y_2$.
$(x_1 + 1, 2 - y_1) = (x_2 + 1, 2 - y_2)$
$x_1 + 1 = x_2 + 1 \quad 2 - y_1 = 2 - y_2$
$x_1 = x_2 \qquad\qquad y_1 = y_2$
H is one-to-one.

(b) Is H onto?
Suppose $(a,b) \in \mathbb{R}^2$.
H is onto if there exists $(x,y) \in \mathbb{R}^2$ where $H(x,y) = (a,b)$.
$(x+1, 2-y) = (a,b)$
$(x,y) = (a-1, 2-b) \in \mathbb{R}^2$
$$\begin{aligned} H(x,y) &= H(a-1, 2-b) \\ &= ((a-1)+1, 2-(2-b)) \\ &= (a,b) \end{aligned}$$
H is onto.

Problem 6.
Exercise Set 7.2

29. Prove that all real numbers a, b, and x with b and x positive and $b \neq 1$, $\log_b(x^a) = a \log_b x$.
$\log_b(x^a) = a \log_b x$
$\exp_b(\log_b(x^a)) = \exp_b(a \log_b x)$
$x^a = (\exp_b(\log_b x))^a$
$x^a = x^a$

Problem 7.
Exercise Set 7.3

11. H and H^{-1} are both defined from $\mathbb{R} - \{1\}$ to $\mathbb{R} - \{1\}$ by the formula
$H(x) = H^{-1}(x) = \dfrac{x+1}{x-1}$, for all $x \in \mathbb{R} - \{1\}$.
$H(x) = H^{-1}(x)$ therefore $(H \circ H^{-1})(x) = (H^{-1} \circ H)(x) = (H \circ H)(x)$.
$$\begin{aligned} (H \circ H)(x) &= \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} \\ &= \frac{\frac{x+1}{x-1} + \frac{x-1}{x-1}}{\frac{x+1}{x-1} - \frac{x-1}{x-1}} \\ &= \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} \\ &= \frac{2x}{x-1} \cdot \frac{x-1}{2} \\ &= x \end{aligned}$$
Both compositions produce the identity function.

Problem 8.
Exercise Set 7.3

27. Let $f : X \to Y$ and $g : Y \to Z$. Is the following property true or false? For all subsets C in Z, $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$?
Let $h(x) = g(f(x))$.
$h(x) = g(f(x))$
$g^{-1}(h(x)) = f(x)$
$f^{-1}(g^{-1}(h(x))) = x$
$h^{-1}(x) = (f^{-1} \circ g^{-1})(x)$ because $h^{-1}(h(x)) = x$.
$$(g \circ f)^{-1}(C) \stackrel{?}{=} f^{-1}(g^{-1}(C))$$
$$(g(f(x)))^{-1}(C) \stackrel{?}{=} (f^{-1} \circ g^{-1})(C))$$
$$h^{-1}(C) = h^{-1}(C)$$
Property is true.

Problem 9.
Exercise Set 7.4

12. Let a and b be real numbers with $a < b$, and suppose that $W = \{x \in \mathbb{R} \mid a < x < b\}$. Prove that S and W have the same cardinality.
$S = (0, 1)$ and $W = (a, b)$.
Let $F(x) = \dfrac{x-a}{b-a}$.
When $x \in W$, $a < x < b$.
$F(a) < F(x) < F(b) \Rightarrow 0 < F(x) < 1$.
$F(W) = S$, therefore $F : W \to S$.

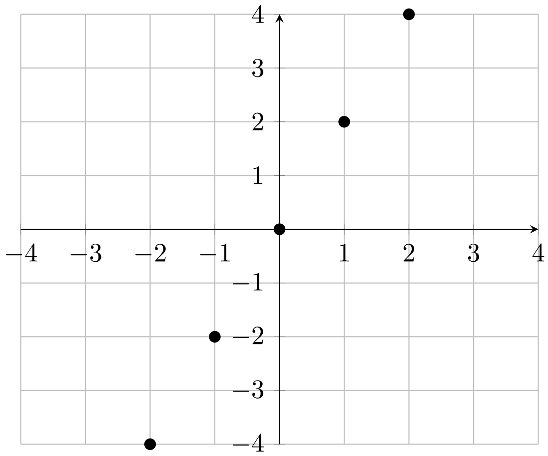
Injective	**Surjective**
Suppose x_1 and $x_2 \in \mathbb{R}$ and $F(x_1) = F(x_2)$. F is one-to-one if $x_1 = x_2$. $F(x_1) = F(x_2)$ $\dfrac{x_1 - a}{b-a} = \dfrac{x_2 - a}{b-a}$ $x_1 - a = x_2 - a$ $x_1 = x_2$ F is one-to-one.	Suppose $y \in S$. F is onto if there exists $x \in W$ where $F(x) = y$. $\dfrac{x-a}{b-a} = y$ $x = y(b-a) + a$ $0 < y < 1$ $0 < y(b-a) < b-a$ $a < y(b-a) + a < b$ $a < x < b$ $x \in W$ $F(y(b-a)+a) = \dfrac{(y(b-a)+a)-a}{b-a}$ $= \dfrac{y(b-a)}{b-a}$ $= y$ F is onto.

$F : W \to S$ and is one-to-one and onto, therefore W and S have the same cardinality.

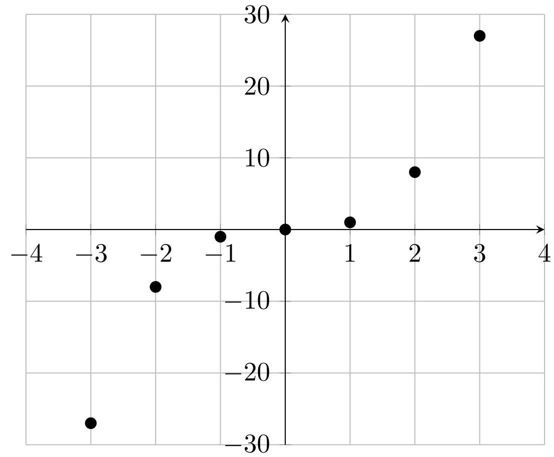
Problem 10.
Exercise Set 7.4

20. Give two examples of functions from $\mathbb{Z}$ to $\mathbb{Z}$ that are one-to-one but not onto.

$f(x) = 2x$

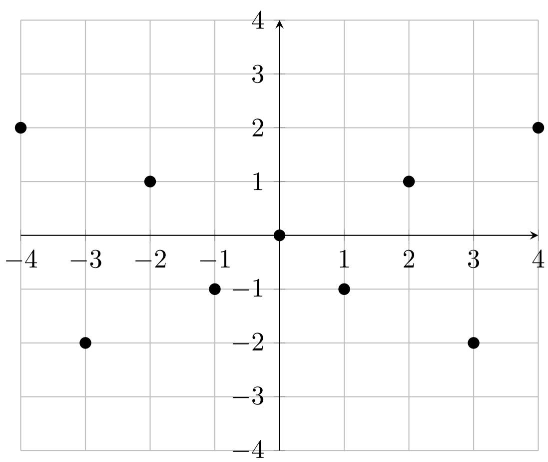


$f(x) = x^3$



21. Give two examples of functions from $\mathbb{Z}$ to $\mathbb{Z}$ that are onto but not one-to-one.

$$f(x) = \begin{cases} |x|/2 & \text{if } x \text{ is even} \\ -(|x|+1)/2 & \text{if } x \text{ is odd} \end{cases}$$



$$f(x) = \begin{cases} x+1 & x < -1 \\ 0 & -1 \le x \le 1 \\ x-1 & 1 < x \end{cases}$$