**Problem 1.**
Exercise Set 8.1

11. $A = \{3, 4, 5\}$, $B = \{4, 5, 6\}$
    $\forall (x, y) \in A \times B,\ x\ S\ y \Leftrightarrow x \mid y$
    $S = \{(3, 6), (4, 4), (5, 5)\}$
    $S^{-1} = \{(6, 3), (4, 4), (5, 5)\}$

**Problem 2.**
Exercise Set 8.1

20. $A = \{-1, 1, 2, 4\}$, $B = \{1, 2\}$
    $\forall (x, y) \in A \times B, x\ R\ y \Leftrightarrow |x| = |y|$
    $x\ S\ y \Leftrightarrow x - y$ is even
    $$A \times B = \begin{Bmatrix} (-1,1), (1,1), (2,1), (4,1), \\ (-1,2), (1,2), (2,2), (4,2) \end{Bmatrix}$$
    $R = \{(-1, 1), (1, 1), (2, 2)\}$
    $S = \{(-1, 1), (1, 1), (2, 2), (4, 2)\}$
    $R \cup S = \{(-1, 1), (1, 1), (2, 2), (4, 2)\} = S$
    $R \cap S = \{(-1, 1), (1, 1), (2, 2)\} = R$

**Problem 3.**
Exercise Set 8.2

10. $\forall x, y \in \mathbb{R},\ x\ C\ y \Leftrightarrow x^2 + y^2 = 1$
    $C$ is not reflexive because for $x = 1$, $x \not C\ x$.
    $C$ is symmetric because $\forall x, y \in \mathbb{R}$, $(x^2 + y^2 = 1) \equiv (y^2 + x^2 = 1)$.
    $C$ is not transitive because for $(x, y, z) = (1, 0, 1)$, $x\ C\ y$ and $y\ C\ z$, but $x \not C\ z$.

**Problem 4.**
Exercise Set 8.2

16. $\forall x, y \in \mathbb{R},\ x\ A\ y \Leftrightarrow |x| = |y|$
    $A$ is reflexive because $\forall x \in \mathbb{R},\ |x| = |x| \Leftrightarrow x\ R\ x$ is true.
    $A$ is symmetric because $\forall x, y \in \mathbb{R}$, $(|x| = |y|) \equiv (|y| = |x|)$, therefore $x\ A\ y \Leftrightarrow y\ A\ x$.
    $A$ is transitive because $\forall x, y, z \in \mathbb{R}$, if $x\ A\ y \Leftrightarrow |x| = |y|$ and $y\ A\ z \Leftrightarrow |y| = |z|$, then $|x| = |y| = |z|$, therefore $|x| = |z| \Leftrightarrow x\ A\ z$.

**Problem 5.**
Exercise Set 8.2

17. $\forall m, n \in \mathbb{Z},\ m\ P\ n \Leftrightarrow \exists$ a prime number $p$ such that $p \mid m$ and $p \mid n$
    $P$ is not reflexive because for $m = 1$, $m \not P\ m$.
    $P$ is symmetric because $\forall m, n \in \mathbb{Z}$, $(p \mid m$ and $p \mid n) \equiv (p \mid n$ and $p \mid m)$, therefore $m\ P\ n \Leftrightarrow n\ P\ m$.
    $P$ is not transitive because for $(x, y, z) = (3, 12, 4)$, $x\ P\ y$ for $p = 3$ and $y\ P\ z$ for $p = 2$, but $x \not P\ z$.

**Problem 6.**
Exercise Set 8.2

19. $\forall x, y \in \mathbb{R},\ x\ I\ y \Leftrightarrow x - y$ is irrational
    $I$ is not reflexive because for $x = 1$, $x \not I\ x$.
    $I$ is symmetric because $\forall x, y \in \mathbb{R}$, if $x - y$ is irrational, then $y - x = -(x - y)$ must also be irrational because the negation of an irrational number is also irrational, therefore $x\ I\ y \Leftrightarrow y\ I\ x$.
    $I$ is not transitive because for $(x, y, z) = (\pi, \sqrt{2}, \pi)$, $x\ I\ y$ and $y\ I\ z$, but $x \not I\ z$.

**Problem 7.**
Exercise Set 8.2

33. Let $A$ be the set of all lines in the plane.
$\forall l_1, l_2 \in A,\ l_1\, R\, l_2 \Leftrightarrow l_1 \perp l_2$
$R$ is not reflexive because no line is perpendicular to itself.
$R$ is symmetric because $\forall l_1, l_2 \in A$, if $l_1 \perp l_2$, then $l_2 \perp l_1 \Leftrightarrow l_2\, R\, l_1$.
$R$ is not transitive because if $a\, R\, b$ and $b\, R\, c$, then $a$ and $c$ are the same line, therefore $a \not R\, c$.

**Problem 8.**
Exercise Set 8.3

20. Let $A$ be the set of all statement forms in three variables $p$, $q$, and $r$.
$\forall P, Q \in A,\ P\, \mathbf{R}\, Q \Leftrightarrow P$ and $Q$ have the same truth table
$\mathbf{R}$ is reflexive because any statement has the same truth table as itself.
$\mathbf{R}$ is symmetric because $\forall P, Q \in A$, if $P\, \mathbf{R}\, Q$, then $P$'s truth table is the same as $Q$'s, which means $Q$'s truth table is the same as $P$'s, therefore $Q\, \mathbf{R}\, P$.
$\mathbf{R}$ is transitive because $\forall P, Q, R \in A$, if $P\, \mathbf{R}\, Q$ and $Q\, \mathbf{R}\, R$, then $P$ and $Q$ and $R$ all have the same truth tables, therefore $P\, \mathbf{R}\, R$.
$\mathbf{R}$ is reflexive, symmetric, and transitive therefore $\mathbf{R}$ is an equivalence relation.
There are $2^3 = 8$ distinct equivalence classes of $\mathbf{R}$. Each class contains an infinite amount of 3 variable boolean statements that share the same truth table.

**Problem 9.**
Exercise Set 8.3

26. $\forall (w, x), (y, z) \in \mathbb{R}^2,\ (w, x)\ Q\ (y, z) \Leftrightarrow x = z$
$Q$ is reflexive because $\forall (w, x) \in \mathbb{R}^2,\ x = x \Leftrightarrow (w, x)\ R\ (w, x)$ is true.
$Q$ is symmetric because $\forall (w, x), (y, z) \in \mathbb{R}^2,\ (x = z) \equiv (z = x)$, therefore $(w, x)\ Q\ (y, z) \Leftrightarrow (y, z)\ Q\ (w, x)$.
$Q$ is transitive because $\forall (a, b), (c, d), (e, f) \in \mathbb{R}^2$, if $(a, b)\ Q\ (c, d)$ and $(c, d)\ Q\ (e, f)$, then $b = d = f$, therefore $(a, b)\ Q\ (e, f)$.
$Q$ is reflexive, symmetric, and transitive therefore $Q$ is an equivalence relation.
There are an uncountably infinite number of distinct equivalence classes of $Q$. Each class contains an uncountably infinite number of points in $\mathbb{R}^2$ with the same y coordinate.

**Problem 10.**
Exercise Set 8.3

28. Let $A$ be the set of all straight lines in the Cartesian plane.
$\forall l_1, l_2 \in A,\ l_1 \parallel l_2 \Leftrightarrow l_1$ is parallel to $l_2$
$A$ is reflexive because all lines are parallel to themselves.
$A$ is symmetric because $\forall l_1, l_2 \in A$, if $l_1 \parallel l_2$, then $l_1$ is parallel to $l_2$, which means $l_2$ is parallel to $l_1$, therefore $l_2 \parallel l_1$.
$A$ is transitive because $\forall l_1, l_2, l_3 \in A$, if $l_1 \parallel l_2$ and $l_2 \parallel l_3$, then $l_1$, $l_2$, and $l_3$ are parallel to each other, therefore $l_1 \parallel l_3$.
$\parallel$ is reflexive, symmetric, and transitive therefore $\parallel$ is an equivalence relation.
There are an uncountably infinite number of distinct equivalence classes of $\parallel$. Each class contains an uncountably infinite number of mutually parallel lines.

**Problem 11.**
Exercise Set 8.4

11. $a, c, n \in \mathbb{Z}$ and $n > 1$ and $a \equiv c(\text{mod } n)$
    Prove $\forall m \geq 1 \in \mathbb{Z},\ a^m \equiv c^m(\text{mod } n)$:
    Let property $P(m)$ be $a^m \equiv c^m(\text{mod } n)$.
    **Basis:**
    $P(1) : (a^1 = a) \equiv (c^1 = c)(\text{mod } n)$ is true.
    **Inductive hypothesis:**
    Assume $P(k) : a^k \equiv c^k(\text{mod } n)$ for $k \geq 1 \in \mathbb{Z}$ is true.
    Prove $P(k+1) : a^{k+1} \equiv c^{k+1}(\text{mod } n)$:
    $a^{k+1} = a \cdot a^k$
    $a = c + sn$ because $a \equiv c(\text{mod } n)$.
    $a^k = c^k + tn$ because $P(k)$ is true.
    $$\begin{aligned} a^{k+1} &= (c + sn)(c^k + tn) \\ &= c^{k+1} + ctn + snc^k + stn^2 \\ &= c^{k+1} + n(\underbrace{ct + sc^k + stn}_{k}) \end{aligned}$$
    $k \in \mathbb{Z}$ and $a^{k+1} = c^{k+1} + kn$
    $a^{k+1} \equiv c^{k+1}(\text{mod } n)$
    Basis and inductive hypothesis proven, therefore original statement is true.