Option Pricing Research Report

Author: Cong Cao

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1 Introduction:

1.1 Introduction of Option Pricing

Options are derivative contracts that give the holder (the "buyer") the right but no the obligation to buy or sell the underlying instrument at an agreed-upon price on or before a specified future date. Option pricing refers to the amount per share at which an option is traded. Although the holder of the options is not obligated to exercise the option, the option writer (the "seller") has an obligation to buy or sell the underlying instrument if the option is exercised. On the other hand, in derivatives analytics field, every major investment bank and buy-side decision maker in the financial market is concerned about the price of those options on the daily basis. Given price of liquidly traded options, those investors try to parameterize the market models in a way that replicates the observed option prices as well as possible and this activity is generally referred to as model calibration.

1.2 Brief introduction of option pricing models and Incentives

There are various kinds of options in the market and can be traded either over-the-counter or exchange, including vanilla options such as European call option and other exotics such as fixstrike lookback option. There are various option pricing models with respect to different types of options, and for Vanilla European Call Option the simple models we could think of is Black-Scholes-Merton Model and Bachelier Model; however, Black Scholes Model, for example, relies on unrealized assumptions and constraints; it assumes the price of heavily traded assets follows a geometric Brownian motion with constant drift and volatility, and it also assumes that the risk free rate remain unchanged and the time is continuous with no jumps. As a result, the errors might be caused when utilize Black-Scholes model to reflect the market price. so we decide to research on different option pricing models, and want to know whether or not they could reflect the market price better.

2 Option Pricing Algorithm

2.1 Model Selection

This model contains two parts: the stochastic volatility and the jump-diffusion; by combining both models, we get the SDE form of the new model that we want to test; then we are able to define the PDE approach of the European call options price; and by solving the equation, we get the solution below.

Framework of the model

1. Stochastic Volatility

By Heston (1993), assuming stochastic volatility, the SDE of underlying asset has following form:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dz_{1t}$$
$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dz_{2t}$$

The key to the application is the Fourier transform method. The characteristic function $\varphi^V(u,T)$ of stochastic volatility part is shown as follows:

$$\varphi^V(u,T) = e^{H_1(u,T)+H_2(u,T)v_0}$$

where:

$$c_1 \equiv \kappa\theta$$
$$c_2 \equiv -\sqrt{(\rho\sigma ui - \kappa)^2 - \sigma^2(-ui - u^2)}$$

$$c_3 \equiv \frac{\kappa - \rho\sigma ui + c_2}{\kappa - \rho\sigma ui - c_2}$$

$$H_1(u,T) \equiv ruiT + \frac{c_1}{\sigma^2}\left[(\kappa - \rho\sigma ui + c_2)T - 2\,log\left(\frac{1 - c_3 e^{c_2 T}}{1 - c_3}\right)\right]$$

$$H_2(u,T) \equiv \frac{\kappa - \rho\sigma ui + c_2}{\sigma^2}\frac{1 - e^{c_2 T}}{1 - c_3 e^{c_2 T}}$$

$$\rho \equiv corr(z_{1t}, z_{2t})$$

2. Jump-Diffusion

BSM model (1976) added a jump-diffusion part to the initial BS model. It has SDE with following form:

$$dS_t = (r - r_J)S_t dt + \sigma S_t dW_t + J_t S_t dN_t$$

where r: constant interest rate; r_J: adjustment of interest rate to jump-diffusion; σ: constant volatility; J_t: jump-diffusion part with log-normal distribution; N_t: Poisson process with parameter λ.

The characteristic function of the jump-diffusion part is a well-known function:

$$\varphi^J(u,T) = exp\left(\left(iu\omega - \frac{u^2\sigma^2}{2} + \lambda\left(e^{iu\mu_J - \frac{u^2\delta^2}{2}} - 1\right)\right)T\right)$$

where ω denotes the risk-neutral drift term and takes the form:

$$\omega = r - \frac{\sigma^2}{2} - \lambda\left(e^{\mu_J + \frac{\delta^2}{2}} - 1\right)$$

3. Valuation of the new model

Combining the stochastic volatility and jump-diffusion parts, we can get the SDE for underlying asset with following form:

$$dS_t = \left(r - r_J\right)S_t dt + \sqrt{v_t} S_t dZ_t^1 + J_t S_t dN_t$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dZ_t^2$$

where:

S_t： underlying price at date t;

r : constant risk-free rate；

r_J： $r_J \equiv \lambda \cdot \left(e^{\mu_J + \frac{\delta^2}{2}} - 1\right)$,drift correlation for jump；

v_t： variance at date t；

κ： speed of adjustment of v_t;

θ: the long-term average of the variance;

σ： volatility coefficient；

Z_t^n (n = 1, 2)： standard Brownian motions：

$dZ_t^1 dZ_t^2 \equiv \rho dt$；

J_t： jump diffusion at date t with：

$$log(1+J_t) \approx N\left(log\left(1+\mu_J\right) - \frac{\delta^2}{2}, \delta^2\right)$$

N_t : Poisson process with intensity λ

By PDE approach, take European call option price C_t for example, the option price process must satisfy that:

$$dC_t = \frac{\partial C_t}{\partial S_t}(m_t dt + v_t dZ_t^1 + j_t dN_t) + \frac{\partial C_t}{\partial v_t}(\overline{m}_t dt + \bar{v}_t dZ_t^2) + \frac{\partial^2 C_t}{\partial S_t \partial v_t} v_t \bar{v}_t \rho dt$$
$$+\frac{1}{2}\left(\frac{\partial^2 C_t}{\partial S_t^2} v_t^2 + \frac{\partial^2 C_t}{\partial v_t^2}\bar{v}_t^2 + \frac{\partial C_t}{\partial t}\right) dt$$
$$= \left(\frac{\partial C_t}{\partial S_t} m_t + \frac{\partial C_t}{\partial v_t}\overline{m}_t + \frac{1}{2}\frac{\partial^2 C_t}{\partial S_t^2} v_t^2 + \frac{1}{2}\frac{\partial^2 C_t}{\partial v_t^2}\bar{v}_t^2 + \frac{\partial^2 C_t}{\partial S_t \partial v_t} v_t \bar{v}_t \rho + \frac{\partial C_t}{\partial t}\right) dt$$
$$+\frac{\partial C_t}{\partial S_t} v_t dZ_t^1 + \frac{\partial C_t}{\partial v_t}\bar{v}_t dZ_t^2$$

where:

$$m_t = \left(r - r_J\right) S_t$$
$$\overline{m}_t = \kappa(\theta - v_t)$$
$$v_t = \sqrt{v_t} S_t$$
$$\bar{v}_t = \sigma\sqrt{v_t}$$
$$j_t = J_t S_t$$

and that:

$$\frac{1}{2}\, v_t S_t^2 \frac{\partial^2 C_t}{\partial S_t^2} + \left(r - r_J\right) S_t \frac{\partial C_t}{\partial S_t} + \rho\sigma v_t S_t \frac{\partial^2 C_t}{\partial S_t \partial v_t}$$
$$+\frac{1}{2}\,\sigma^2 \frac{\partial^2 C_t}{\partial v_t^2} + \kappa(\theta - v_t)\frac{\partial C_t}{\partial v_t} + \frac{\partial C_t}{\partial t} - rC_t$$
$$+\lambda E^Q\left[C\left(K, T; (1+J_t)S_t - C(K, T; S_t)\right)\right] = 0$$

finally, we can get a solution to this PDE:

$$C_t(K, T, S_t, v_t, r, t) = S_t \cdot \Pi_1(T; S_t, v_t, r, t) - e^{-r(T-t)} \cdot K \cdot \Pi_2(T; S_t, v_t, r, t)$$
$$\Pi_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-iulnK}\varphi_j(T; u)}{iu}\right] du, \qquad j = 1,2$$
$$\varphi_1(T; u) = \frac{\varphi(T; u - i)}{\varphi(T; -i)}, \quad \varphi_2(T; u) = \varphi(T; u)$$

where $\varphi(T; u)$ denotes the characteristic function for the model. Due to zero correlation between the H93 stochastic volatility part and the jump part, the characteristic function for model can be obtained by:

$$\varphi(u,T) = \varphi^V(u,T) \cdot \varphi^J(u,T)$$

2.2 Data Selection:

Since the option dataset is huge and contains various information within each option such as different strike prices, bid-ask prices and maturities, we decide to choose options prices that satisfy some specific requirements. First, since this model is the market-based valuation of index options, we decide to use this model to forecast the ETF XLB European call option in April 2017 with maturity in a month. To make our estimates as precise as possible, we select the European call option price by every day strike price from 90% to 110% of every day spot price in that month (1); Also the maturity of the European call option that we select is between May and June 2017 (2). Besides that, we notice that there are both bid and ask prices, and some of those prices are zero; so we delete the prices that either bid or ask is zero and take the average bid and ask prices for each contract in the remaining data (3).

The purpose of the data selection process above is that we can hardly do the calibration and find a set of reasonable optimal parameters within a large dataset, since it may lead to over-fitting problem and the cost of computation is relatively high. After this data selection process, there won't be illiquidity or other problems that lead to unrealistic market prices. The second condition is due to the fact that our model, which combines stochastic volatility together with jump-diffusion part, has the best performance on contracts with 1- or 2- month maturity.

Eventually, we get approximately 216 European call option prices daily for 19 days in April 2017, and that is the data that we mainly used to calibrate parameters daily with daily quotes data of one month of daily data.

(All the ETF option data is obtained from Professor Eric Jacquier)

2.3 Calibration Process

2.3.1 Incentives and objectives

In simple terms, the problem of calibration is to find the parameters that minimize the deviation of model from the market, so that we can derive reasonable prices and other useful variables (option delta for example) via the calibrated model. To calibrate the model, one needs a performance metric for the quality of calibration, which is typically the minimum of an error function.

There are mainly two candidate objective functions:

(a). mean square error (MSE) of the price differences

(b). MSE of the implied volatility differences

In case of complexity of computation, we selected the MSE of relative price differences as our target function. Our objective function is:

$$\min_{p} \frac{1}{N} \sum_{n=1}^{N} \left(C_n^{*} - C_n^{model}(p) \right)^2$$

Where denotes the market price and denotes the model price of the options with parameter. In our model, the parameter has 8 input parameters: $\kappa_v, \theta_v, \sigma_v, \rho, v_0, \lambda, \mu, \delta$.

2.3.2 Calibration procedure

The BCC model is a combination of stochastic volatility model and Jump-Diffusion model, and the stochastic volatility and jump-diffusion parts are irrelevant according to the derivation of our model price. So, it's therefore reasonable to calibrate the model with 3 steps:

First, calibration of the stochastic volatility model without jump-diffusion part with the objective function as minimum MSE. From this step, we get the local optimal values for SV parameters $\kappa_v, \theta_v, \sigma_v, \rho, v_0$.

Second, Calibration of the jump-diffusion model without SV part. It is known that the calibration of the jump part suffers from problems of degeneracy and indeterminacy. Thus to get more stable parameters, we introduced the penalty function: $Penalty(p) \equiv \sqrt{(p_0 - p)^2}$ where p_0 is the initial value of input parameter. Also, we adjust the objective calibration function as the part modified by penalty function:

$$\min_p \frac{1}{N} \sum_{n=1}^{N} \left(C_n^* - C_n^{model}(p) \right)^2 + w * Penalty(p)$$

where w is the weighting parameter to decide more or less influence assigned to the penalty function, and we assume $w = 1$ as is commonly used. From this step, we get the local optimal of parameters λ, μ, δ.

Third, Calibration of the full BCC model with both SV and jump-diffusion parts. It takes the results from the previous two steps and gets complete calibration of all the parameters: $\kappa_v, \theta_v, \sigma_v, \rho, v_0, \lambda, \mu, \delta$.

2.3.3 Calibration Algorithm

Each step of calibration takes a 2-step optimization procedure: global minimization followed by local minimization. This approach is augmented by the 3-step calibration procedure. Note that we assume a constant risk-free rate and there is no concern for the correlation between interest rate process and the equity price process when we're calculating the minimum objective functions in separate steps. In addition, the cost of computation would be extremely high if we calibrate all the parameters in one single step; the local minimization after the global minimization is more robust compared with a single local minimization.

For the specific algorithms we used in minimization, we introduced the *scipy* package in Python and the *brute* & *fmin* functions in *scipy.optimization*. We used the *brute* (i.e. brute force) function to find a global minimization with a given range of each parameter; afterwards, we used *fmin* function to find the local optimal values around the results yielded by brute force.

2.4 Model Validation

After Calibrating those parameters above, we hope to know whether or not our model is able to price the option better and more precisely than the Black-Sholes Model. Then dynamic delta hedging provides a prefect way to hedge against price changes of the options when the underlying of the potion is the only source of risk.

$$\Delta_t^P \equiv \frac{\partial P_t}{\partial S_t}$$

Basically we consider the delta hedging as a trading strategy of buying Δ shares of stocks and selling a call option. So we assume that in the first day of April we buy Δ1 shares of stocks and sell a European Call option at the end of that day; and Δ1 is calculated by the parameters that our model calibrated, and for the next several days, we adjust the shares of stocks at that day by the new Δ* by the parameters we calibrated at that day and the Δ calibrated yesterday; and they compare the change of shares of stocks with the change of call option prices in the market at the end of that day.

Day 1: Δ 1 * S 1 -C 1

Day 2: (Δ 2 – Δ 1) * S 2 - (C 2 – C 1)

Day n: (Δ n – Δ n-1) * S n - (C n - C n-1)

Since we price (April 2017) 19 days of the European call options price, at the end of the delta hedging process, we get 19 Δs and 19 profits with respect to the Δs, call option prices in the market and the stock prices. Then for comparison, we plot the 19 European Call Option price (Figure1), 19 days' 19 days' Δs of our model and BS-model (Figure2), profits graph (Figure3) and cumulative profits (Figure4).

By directly comparing the call price from the market, from our BCC model and from the BS model, we can easily see that the European call option price measured by our model and European call option prices in the market are more identical than comparing the prices measured by BS model and prices in the market; and it is reasonable since the option prices we get are supposed to be closer to the call price in the market than option price from BS-model. Besides that, the volatility of BS delta is 0.98329, which is higher than that of BCC delta (0.94828); also by looking at the Figure 2 comparison, we notice that the delta of our model seem to be more stable than the delta of BS model.

3. Conclusion

In summary, Although the daily profit of the delta (Figure 3) doesn't show any obvious difference between those two strategies, the cumulative profit of the 19 deltas (Figure 4) shows that the strategy using our model is more profitable than using the Black-Scholes model. Besides that, Because of the volatility of delta between two models above, we can conclude that our model provides better results and predictability in pricing European call option in the market with 1- or 2-month maturity than the Black-Scholes model.

Notwithstanding the better predictability of our option pricing model, there are some concerns and weakness of the model that hardly to ignore, firstly, our model is obviously more complex than the Black-Sholes model, which implies that there might be a tradeoff between how precise you want to price option and the complexity of the model. Considering the simplicity of Black-Scholes model, we can hardly say which model is better. Secondly, since our model need to calibrate 13 parameters to do the option pricing within both stochastic volatility and jump-diffusion parts, the parameters we calibrate may cause over-fitting and the results may not appear to be as good as we supposed. Last but not least, for call option with longer maturities or special situation such as big moves in the market, the model might lose the power of predictability and appear to be underperforming; the unrealistic assumptions including constant interest rate also need to be hold for our model, which require future improvements.

Figure 1

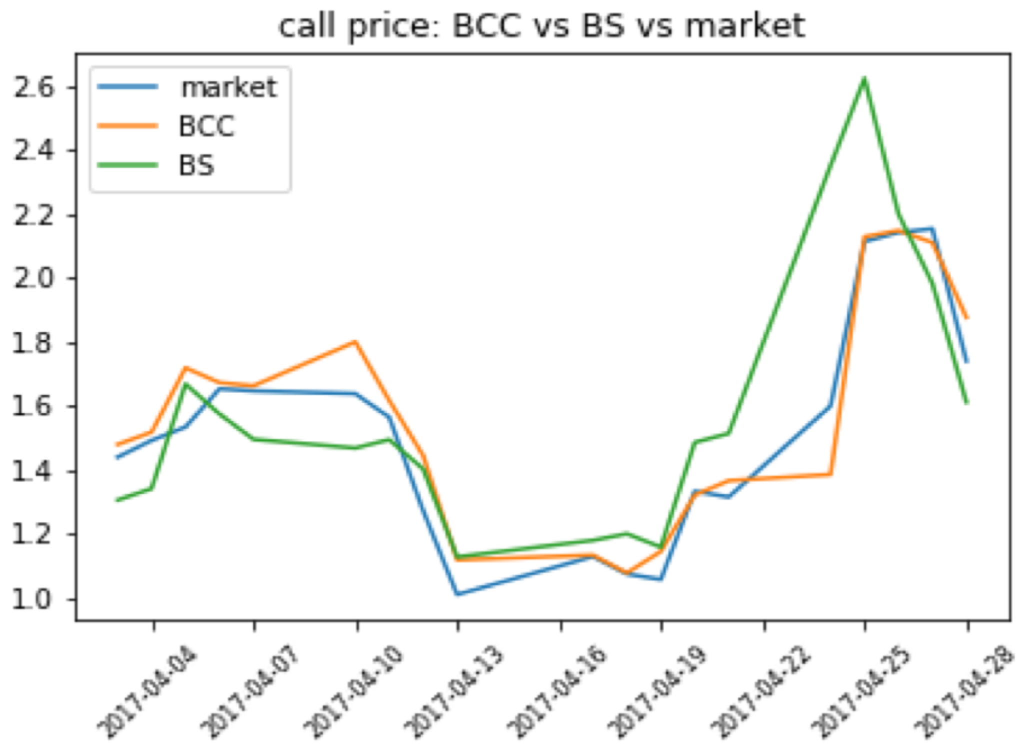


Figure 2

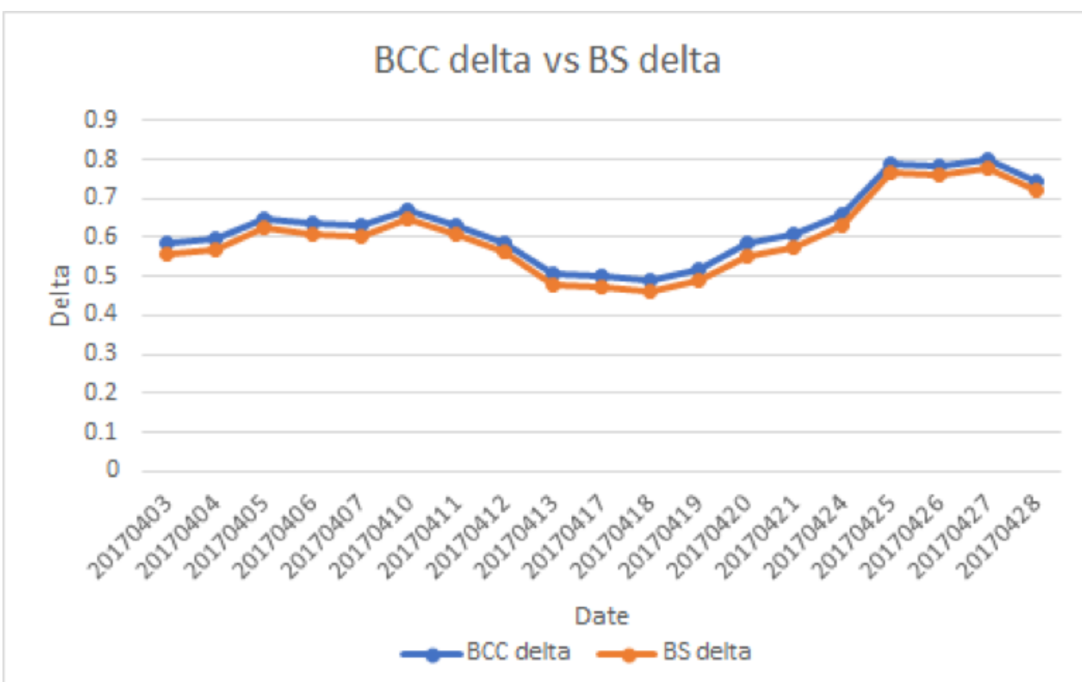


Figure 3

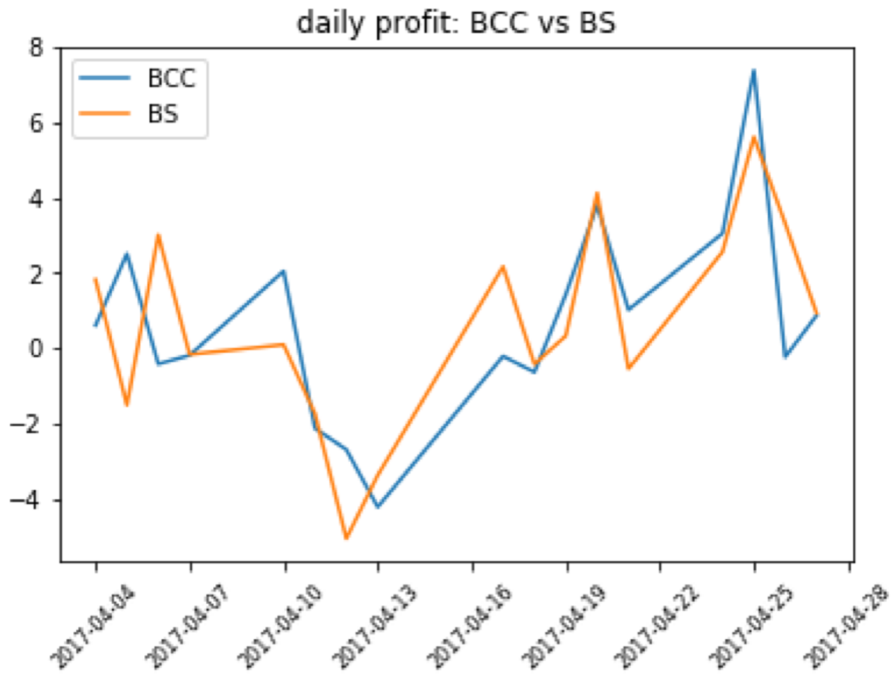
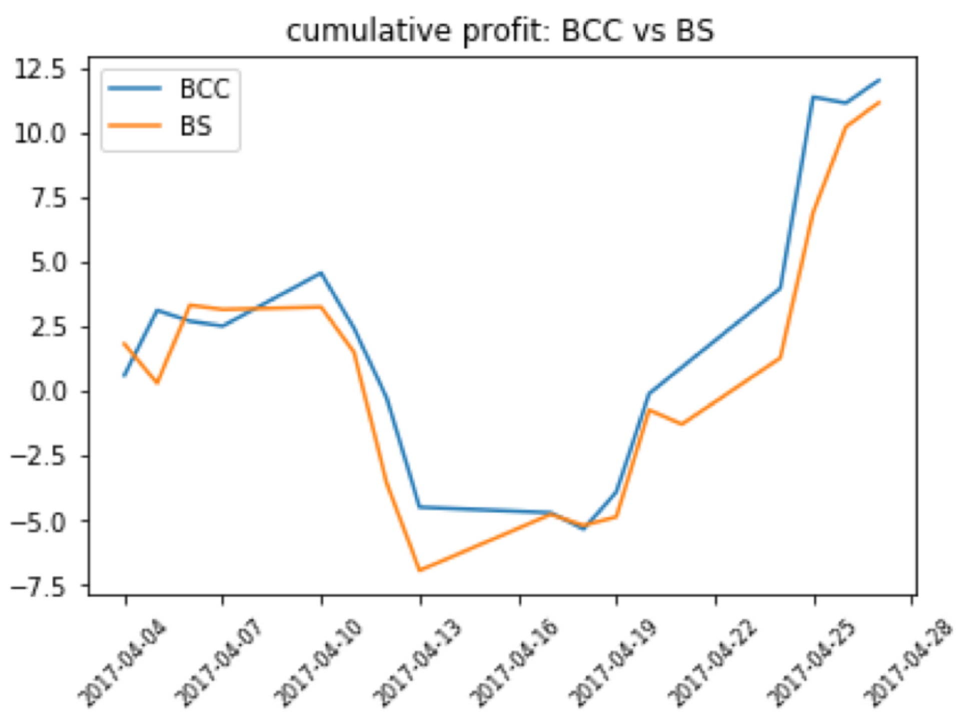


Figure 4



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