Section for Statistical Theory

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Office Hour: Wednesday 09:30AM - 11:30AM

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Overview

1. Review
   - Preliminaries
   - Test Statistic: Neyman-Pearson Lemma
   - Uniformly Most Powerful Test & Monotone Likelihood Ratio Model
   - Likelihood Ratio Procedure
2. Problems

Preliminaries

Size of the Test (page 217 in [B&D, 2015])

If we have a test statistic T with critical value c, our test has size $\alpha(c)$ given by

$$\alpha(c) = \sup_{\theta \in \Theta_0} P_\theta [T(X) \geqslant c]. \tag{4.1.1}$$

Power Function (page 217 in [B&D, 2015])

Power function is defined for all $\theta \in \Theta$ by

$$\beta(\theta) = \beta(\theta, \delta) = P_\theta [\text{Rejection}] = P_\theta [\delta(X) = 1] = P_\theta [T(X) \geqslant c].$$

Test Statistic: Neyman-Pearson Lemma

- Simple Likelihood Ratio / Neyman-Pearson (NP) Statistic

$$L(x, \theta_0, \theta_1) = \frac{p(x, \theta_1)}{p(x, \theta_0)}.$$

- Likelihood Ratio / Neyman-Pearson (NP) Test (Function)

$$\varphi_k(x) = \begin{cases} 1 & L(x, \theta_0, \theta_1) > k \\ c & L(x, \theta_0, \theta_1) = k \\ 0 & L(x, \theta_0, \theta_1) < k \end{cases} \qquad 0 < c < 1 \ \& \ 0 \leqslant k \leqslant \infty.$$

Test Statistic: Neyman-Pearson Lemma

Theorem 4.2.1. (page 224 in [B&D, 2015])

(a) If $\alpha > 0$ and φ_k is a size α likelihood ratio test, then φ_k is MP in the class of level α tests.

(b) For each $0 \leqslant \alpha \leqslant 1$ there exists an MP size α likelihood ratio test provided that randomization is permitted, $0 < \varphi(x) < 1$, for some x.

(c) If φ is an MP level α test, then it must be a level α likelihood ratio test; that is, there exists k such that

$$P_\theta\left[\varphi(X) \neq \varphi_k(X), L(X, \theta_0, \theta_1) \neq k\right] = 0 \qquad (4.2.2)$$

for $\theta = \theta_0$ and $\theta = \theta_1$.

UMP & MLR

Definition 4.3.1 (page 227 in [B&D, 2015])

A level α test φ^* is uniformly most powerful (UMP) for $H : \theta \in \Theta_0$ versus $K : \theta \in \Theta_1$ if

$$\beta\left(\theta, \varphi^*\right) \geqslant \beta\left(\theta, \varphi\right) \tag{4.3.1}$$

for all $\theta \in \Theta_1$ and for any other level α test φ.

Definition 4.3.2 (page 228 in [B&D, 2015])

The family of models $\{P_\theta : \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}$ is said to be a monotone likelihood ratio (MLR) family in T if for $\theta_1 < \theta_2$ the distributions P_{θ_1} and P_{θ_2} are distinct and there exists a statistic $T\left(x\right)$ such that the ratio $p\left(x, \theta_2\right) / p\left(x, \theta_1\right)$ is an increasing function of $T\left(x\right)$.

Theorem 4.3.1 (page 228-229 in [B&D, 2015])

Suppose $\{P_\theta : \theta \in \Theta\}$ with $\Theta \subset \mathbb{R}$ is a MLR family in $T(x)$.

1. For each $t \in (0, \infty)$, the power funtion $\beta(\theta) = E_\theta \delta_t(X)$ is increasing in θ.
2. If $E_{\theta_0} \delta_t(X) = \alpha > 0$, then δ_t is UMP level α for testing $H : \theta \leqslant \theta_0$ versus $K : \theta > \theta_0$.

Here δ_t is the Neyman–Pearson (NP) test function given by

$$\delta_t(x) = \begin{cases} 1 & T(x) > t \\ c & T(x) = t \\ 0 & T(x) < t \end{cases} \qquad 0 < c < 1.$$

Note that δ_t equals the LRT $\varphi_{h(t)}$ for some increasing function h.

Likelihood Ratio Procedure

- Generlized Likelihood Ratio / Neyman-Pearson (NP) Statistic

$$L(\mathbf{x}) = \frac{\sup_{\theta \in \Theta_1} p(\mathbf{x}, \theta)}{\sup_{\theta \in \Theta_0} p(\mathbf{x}, \theta)}.$$

- Equivalently for $\Theta = \Theta_0 \cup \Theta_1$

$$\lambda(\mathbf{x}) = \frac{\sup_{\theta \in \Theta} p(\mathbf{x}, \theta)}{\sup_{\theta \in \Theta_0} p(\mathbf{x}, \theta)}.$$

Likelihood Ratio Procedure

1. Calculate the MLE $\widehat{\theta}$ of θ over Θ.
2. Calculate the MLE $\widehat{\theta}_0$ of θ over Θ_0.
3. Form
$$\lambda(\mathbf{x}) = \frac{p\left(\mathbf{x}, \widehat{\theta}\right)}{p\left(\mathbf{x}, \widehat{\theta}_0\right)}.$$
4. Find a function h that is strictly increasing on the range of λ such that $h(\lambda(\mathbf{X}))$ has a simple form and a tabled distribution under H. Because $h(\lambda(\mathbf{X}))$ is equivalent to $\lambda(\mathbf{X})$, we specify the size α likelihood ratio test through the test statistic $h(\lambda(\mathbf{X}))$ and its $(1-\alpha)$th quantile obtained from the table.

- Measure-Theoretic Probability

$$P[E] = \int_{\omega \in E} \mu_F(d\omega).$$

- Accumulation Point: if x is an accumulation point of a real sequence (x_n), then there exists a subsequence (x_{n_k}) converging to x.
- Denseness: a subset A of a topological space X is called dense in X if every point x in X either belongs to A or is a acuumulation point of A.
- Continuous Function: f is continuous at x_0 iff $f(x_n) \to f(x_0)$ for any sequence $x_n \to x_0$.

References

Bickel, Peter J., and Kjell A. Doksum. (2015) Mathematical statistics: basic ideas and selected topics, volume I. CRC Press.

Thanks for listening!