

# Notes on the Parton Luminosities

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## Abstract

In this document I describe the definition and the implementation of the differential parton-luminosity functions.

## 1 Definition

The definition of the parton-luminosity functions is given by the expression of the total hadronic cross section  $\sigma$  in terms of the parton ditribution functions  $f_{i(j)}(x_{1(2)}, \mu)$  and the partonic cross sections  $\hat{\sigma}_{ij}$ :

$$\sigma = \sum_{ij} \int_0^1 dx_1 \int_0^1 dx_2 f_i(x_1, \mu) f_j(x_2, \mu) \hat{\sigma}_{ij} . \quad (1)$$

Now, defining the kinematic variable  $y$  (rapidity) and  $M_X$  (invariant mass of the partonic final state, usually normalized to the total center of mass squared energy  $s$ ) as:

$$\begin{aligned} \tau &= \frac{M_X^2}{s} = x_1 x_2 \\ y &= \frac{1}{2} \ln \left( \frac{x_1}{x_2} \right) \end{aligned} , \quad (2)$$

one can express the integral in eq. (1) in terms of  $y$  and  $\tau$ . In fact, by means of a change of variables and taking into account the fact that the Jacobian  $\partial(y, \tau)/\partial(x_1, x_2) = 1$ , we find that:

$$\sigma = \sum_{ij} \left( \int_{-\infty}^0 dy \int_0^{e^{2y}} d\tau + \int_0^{+\infty} dy \int_0^{e^{-2y}} d\tau \right) \frac{d^2 \mathcal{L}_{ij}}{dy d\tau} \hat{\sigma}_{ij} = \sum_{ij} \int_0^1 d\tau \left( \int_0^{-\frac{1}{2} \ln \tau} dy + \int_{\frac{1}{2} \ln \tau}^0 dy \right) \frac{d^2 \mathcal{L}_{ij}}{dy d\tau} \hat{\sigma}_{ij} , \quad (3)$$

where we have defined fully differential parton-luminosity functions as:

$$\frac{d^2 \mathcal{L}_{ij}}{dy d\tau} = f_i(\sqrt{\tau} e^y, \sqrt{s\tau}) f_j(\sqrt{\tau} e^{-y}, \sqrt{s\tau}) , \quad (4)$$

and where we have also set  $\mu = M_X = \sqrt{s\tau}$ . Considering that the the differential parton-luminosity functions and the partonic cross sections must be symmetric under  $y \leftrightarrow -y$ , eq. (3) can be written as:

$$\sigma = \sum_{ij} \int_{-\infty}^{+\infty} dy \int_0^{e^{-2|y|}} d\tau \frac{d^2 \mathcal{L}_{ij}}{dy d\tau} \hat{\sigma}_{ij} = \sum_{ij} \int_0^1 d\tau \int_{\frac{1}{2} \ln \tau}^{-\frac{1}{2} \ln \tau} dy \frac{d^2 \mathcal{L}_{ij}}{dy d\tau} \hat{\sigma}_{ij} , \quad (5)$$

that can be written in a unique way as:

$$\sigma = \sum_{ij} \int_{-\infty}^{+\infty} dy \int_0^1 d\tau \theta \left( -y - \frac{1}{2} \ln \tau \right) \theta \left( y - \frac{1}{2} \ln \tau \right) \frac{d^2 \mathcal{L}_{ij}}{dy d\tau} \hat{\sigma}_{ij} . \quad (6)$$

More in general, we can write the following equivalence at the level of phase space as:

$$\int_0^1 dx_1 \int_0^1 dx_2 = \int_{-\infty}^{+\infty} dy \int_0^1 d\tau \theta \left( -y - \frac{1}{2} \ln \tau \right) \theta \left( y - \frac{1}{2} \ln \tau \right) = \int_0^1 d\tau \int_{-\infty}^{+\infty} dy \theta \left( e^{-2|y|} - \tau \right) . \quad (7)$$

Now, let us compute the single differential parton-luminosity functions integrating out one of the kinematic variables  $y$  and  $\tau$ . Let us start with  $y$ . In this case we have:

$$\begin{aligned}\frac{d\mathcal{L}_{ij}}{d\tau} &= \sum_{ij} \int_{-\infty}^{+\infty} dy \int_0^1 d\tau \theta\left(-y - \frac{1}{2} \ln \tau\right) \theta\left(y - \frac{1}{2} \ln \tau\right) \frac{d^2\mathcal{L}_{ij}}{dy d\tau} \delta(\tau - \tau') \\ &= \int_{\frac{1}{2} \ln \tau}^{-\frac{1}{2} \ln \tau} dy f_i\left(\sqrt{\tau'} e^y, \sqrt{s\tau'}\right) f_j\left(\sqrt{\tau'} e^{-y}, \sqrt{s\tau'}\right),\end{aligned}\tag{8}$$

that, performing the change of variable  $x = \sqrt{\tau'} e^y$  and taking into account that  $s\tau' = M_X^2$ , becomes:

$$\Phi_{ij} = \frac{d\mathcal{L}_{ij}}{dM_X^2} = \frac{1}{s} \int_{\tau'}^1 \frac{dx}{x} f_i(x, M_X) f_j\left(\frac{\tau'}{x}, M_X\right).\tag{9}$$

Now we integrate the same for  $\tau$ :

$$\begin{aligned}\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} &= \sum_{ij} \int_0^1 d\tau \int_{-\infty}^{+\infty} dy \theta\left(e^{-2|y|} - \tau\right) \frac{d^2\mathcal{L}_{ij}}{dy d\tau} \delta(y - y') \\ &= \int_0^{e^{-2|y'|}} d\tau f_i\left(\sqrt{\tau} e^{y'}, \sqrt{s\tau}\right) f_j\left(\sqrt{\tau} e^{-y'}, \sqrt{s\tau}\right),\end{aligned}\tag{10}$$

that, defining  $x = \sqrt{\tau} e^{y'}$  and assuming  $y' \geq 0$ , becomes:

$$\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} = 2e^{-2y'} \int_0^{e^{-y'}} dx x f_i(x, \sqrt{s} x e^{-y'}) f_j(x e^{-2y'}, \sqrt{s} x e^{-y'}).\tag{11}$$

Eq. (11) has the drawback that the integration range in  $x$  goes down to zero. In addition, also the scale in which PDFs are evaluate depends on  $x$  linearly. This implies that performing the integral requires the evaluation of PDFs in  $x = \mu = 0$  and this is clearly impossible. However, for converge reasons, the invariant mass of the final state is always taken to be bigger than a give threshold. At the LHC 14 TeV this threshold is typically of the order of a few tens of GeV. Here, for explicative purposes, we take  $M_{X,\text{cut}} = 10$  GeV and this results in the cutoff  $\tau_{\text{cut}} \simeq 6 \cdot 10^{-7}$ , so that eq. (10) becomes:

$$\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} = \int_{\tau_{\text{cut}}}^{e^{-2y'}} d\tau f_i\left(\sqrt{\tau} e^{y'}, \sqrt{s\tau}\right) f_j\left(\sqrt{\tau} e^{-y'}, \sqrt{s\tau}\right),\tag{12}$$

and thus:

$$\Psi_{ij} = \frac{d\mathcal{L}_{ij}}{dy} = 2e^{-2y'} \int_{\sqrt{\tau_{\text{cut}}} e^{y'}}^{e^{-y'}} dx x f_i(x, \sqrt{s} x e^{-y'}) f_j(x e^{-2y'}, \sqrt{s} x e^{-y'}).\tag{13}$$

Eq. (13) should allow for a practical implementation as the minimum of the integration range is now of the order of  $7 \cdot 10^{-4}$  and the minimum energy in which PDFs are evaluated is equal to  $M_{X,\text{cut}}$ . In addition, in order for the upper limit of the integral in eq. (13) to be smaller than the lower one, one should also require that:

$$y' \leq -\frac{1}{2} \ln \frac{M_{X,\text{cut}}}{\sqrt{s}} \simeq 3.6.\tag{14}$$