MA102 Mathematical Proof and Analysis
and
MA103 Introduction to Abstract Mathematics

Exercises 5

- Deadline Monday 1 November, 17:00. Always justify your answers. Questions 1, 2 and 4 will count for the class grade. You should check you can do question 3, but *it will not be marked*.

1 Let R be an equivalence relation on a set S. Prove that the following properties hold.

(a) For all $x, y \in S$ we have $xRy \iff [x] = [y]$.

(b) For all $x, y \in S$ we have $\neg xRy \iff [x] \cap [y] = \varnothing$.

2 In lectures, we gave a construction for the rational numbers. This started by looking at the set S of all pairs of the form (a, b), with $a, b \in \mathbb{Z}$ and $b \neq 0$, and then considering the relation Q on $S = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ defined by:

$$(a,b)Q(c,d) \qquad \text{if and only if} \qquad a\,d = b\,c.$$

(a) Explain why things would go badly wrong if we allow S to include pairs (a, b) with $b = 0$.

(Hint: the answer does not involve 'division by zero'.)

We defined the set $\mathbb{Q}$ to be the set of equivalence classes of the relation Q, and defined an "addition" operation on $\mathbb{Q}$ by setting $[(a,b)] \oplus [(e,f)] = [(af + be, bf)]$, for each $(a,b), (e,f) \in S$.

(b) Suppose that (a, b), (c, d) and (e, f) are in S, and that $(a,b)Q(c,d)$. Show that $(af + be, bf)Q(cf + de, df)$.

Your answer should *only* talk about operations with integers. If your answer involves writing any fractions at any stage, it is wrong.

Use this to show that if $(r,s)Q(t,u)$ and $(v,w)Q(x,y)$, then

$$[(r,s)] \oplus [(v,w)] = [(t,u)] \oplus [(x,y)]\,.$$

(This means that the addition operation defined on $\mathbb{Q}$ is *well-defined*.)

3 (a) Consider the equation $z^5 = a$, where a is a positive real number.

What can you say about the modulus of any solution z? What can you say about the principal argument?

Draw all the five solutions of $z^5 - 32 = 0$ on the Argand diagram.

(b) Write down a polynomial $P(z)$ such that

$$(z-2)P(z) = z^5 - 32.$$

What can you say about the solutions of $P(z) = 0$?

4 Consider the polynomial $P(z) = z^4 + z^2 - 2z + 6$.

Show that $z = 1 + i$ is a root of $P(z)$, i.e., a solution of $P(z) = 0$.

Hence find all the roots of $P(z)$.

The following question does **not** count for your class grade. But it is easier than starred questions in previous weeks - try it!

5$*$ If we take the formula $e^{x+y\,i} = e^x\,(\cos y + i\,\sin y)$ with $x = 0$ and $y = \frac{1}{2}\,\pi$ we get $e^{i\,\pi/2} = i$. This means we can deduce

$$i^i \;=\; (e^{i\,\pi/2})^i \;=\; e^{i^2\,\pi/2} \;=\; e^{-\pi/2} \;=\; \frac{1}{e^{\pi/2}} \approx 0.20787.$$

So it seems that i^i is a real number! But something even stranger is the case:

(a) Show that there are infinitely many real numbers r so that $i^i = r$.

(In other words, like $\sqrt{z}$ but more so, it's not uniquely defined.)

(b) Let $S = \{a^2 + b^2 \,:\, a, b \in \mathbb{Z}\}$. Show that if $n, m \in S$ then $nm \in S$.

(Hint: given a complex number z, what is $z\bar{z}$?)