MA210 Discrete Mathematics

Notes and Exercises 4

27 February and 6 March, 2023

Trees, and Graph Colourings

Trees

Definition 4.1 (tree, forest, spanning tree). *A* tree *is a connected graph with no cycles.*

A forest *is a graph with no cycles.*

A spanning tree *of G is a set of edges of G which form a tree on $\big|V(G)\big|$ vertices.*

Proposition 4.2. *Let $T = (V, E)$ be a tree. Then we have:*

(a) the graph obtained from T by removing any edge has two components, and each component is a tree;

(b) $|E| = |V| - 1$;

(c) if $|V| \geq 2$, then T has at least two vertices of degree 1.

A vertex of degree 1 in a tree is often called a *leaf*.

Proposition 4.3. *A graph contains a spanning tree if and only if the graph is connected.*

Minimum cost spanning trees

Definition 4.4 (cost function). *Given a graph G, a* cost function *(or* weight function*) on $E(G)$ is a function $w : E(G) \to \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ denotes the non-negative real numbers.*

Definition 4.5 (tree cost, minimum cost spanning tree). *Given a graph G and a cost function $w : E(G) \to \mathbb{R}_{\geq 0}$, if T is a tree in G then we write $w(T)$ for the sum $\sum_{e \in E(T)} w(e)$, and we say $w(T)$ is the* cost *of T.*

A minimum cost spanning tree *T of G is a spanning tree of G such that for any other spanning tree T' of G we have $w(T) \leq w(T')$.*

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Theorem 4.6 (Kruskal's Algorithm)**.** *Let G be a connected graph on $n \geq 2$ vertices, with a cost $w(e)$ for each edge $e \in E(G)$. Then a minimum cost spanning tree of G can be found as follows.*

Let $W = \emptyset$ be a set containing no edges of G. We now run the following procedure, starting with $k = 0$.

(1) If $\{e_1, e_2, \ldots, e_k\} \cup W = E(G)$, then stop.
Otherwise, let f be an edge with minimum cost in $E(G) \setminus \big(\{e_1, e_2, \ldots, e_k\} \cup W\big)$.

(2) If we can form a cycle with edges from $e_1, e_2, \ldots, e_k, f$, then add f to W and go back to step (1).

If there is no cycle contained in $e_1, e_2, \ldots, e_k, f$, then set $e_{k+1} = f$, increase k by one, and go back to step (1).

When this algorithm stops (in step (1)) we have $k = n - 1$ and the edges $e_1, \ldots, e_{n-1}$ form a minimum cost spanning tree T of G.

Counting (labelled) trees

With one or two vertices, there is only one tree possible, isomorphic to the path P_1 or the path P_2 respectively.

Each tree on 3 vertices is isomorphic to the path P_3, but if the vertices are labelled, say $\{1, 2, 3\}$ is the vertex set, then the tree is uniquely determined by the label of the vertex in the middle of the path: sets $\{12, 23\}$, $\{13, 23\}$, and $\{21, 13\}$ are the only three possibilities for the edge set.

Example 4.7. Show that there are 16 different trees with vertex set $\{1, 2, 3, 4\}$ and 125 different trees with vertex set $\{1, 2, 3, 4, 5\}$.

We define $\Lambda = ()$ to be the empty sequence (the sequence with no entries).

Definition 4.8 (Prüfer code)**.** *Let T be be an n-vertex tree with vertex set $S \subseteq \mathbb{N}$, $n \geq 2$. The* Prüfer code *of T is the sequence $(a_1, \ldots, a_{n-2})$ generated by the following algorithm (which takes S and T as input).*

Algorithm 1: Prüfer(S, T)

if $|S| = 2$ **then**
  return Λ ;
else
  let L be the set of leaves of T ;
  let x be the smallest element of L ;
  let a_1 be the neighbour of x in T ;
  let $(b_1, \ldots, b_{n-3}) =$ Prüfer($S \setminus \{x\}, T - x$) ;
  return $(a_1, b_1, b_2, \ldots, b_{n-3})$;
end

We will usually simply write Prüfer(T) *rather than* Prüfer$\big(V(T), T\big)$ *for the sequence generated by Prüfer's algorithm on the tree T, when we don't want to make a point about the vertex set we are working with.*

Example 4.9. Let T be a tree with vertex set S. Explain why the leaves of T are exactly those vertices that do not appear in the Prüfer code of T.

Theorem 4.10 (Prüfer, 1918)**.** *Let S be a set of $n \geq 2$ distinct natural numbers. The Prüfer code is a bijection between the set of all trees with vertex set S and the set of all sequences of length $n-2$ with entries from S.*

Example 4.11. Given $S = \{1, 2, \ldots, 8\}$ and $\mathbf{a} = (7, 4, 4, 1, 7, 1)$. Find the tree T such that $f(T) = \mathbf{a}$.

Corollary 4.12 (Cayley's Formula, 1889)**.** *For a set S of n distinct natural numbers, there are n^{n-2} trees with vertex set S.*

Graph colouring and the chromatic number

Definition 4.13 (colouring, proper colouring, colourable, chromatic number)**.** *Let $G = (V, E)$ be a graph and k a natural number. Then a k-*colouring *of G is a labelling $f : V \to \{1, 2, \ldots, k\}$. The labels are called* colours.

For a given colouring of G, a colour class *is a set of vertices that all have the same colour. In other words, if $f : V \to \{1, 2, \ldots, k\}$ is a k-colouring of G, then a* colour class *is some set $V_i = \{v \in V \mid f(v) = i\}$, for some colour i.*

A k-colouring is proper *if adjacent vertices have different labels (so for any $uv \in E$ we have $f(u) \neq f(v)$).*

*A graph is k-*colourable *if it has a proper k-colouring.*

The chromatic number *$\chi(G)$ of a graph G is the smallest k such that G is k-colourable.*

*If $\chi(G) = k$, then we also say that G is k-*chromatic.

Definition 4.14 (independent set, independence number, clique number)**.** *Let $G = (V, G)$ be a graph. An* independent set *in G (sometimes called a* stable set*) is a set of vertices $S \subseteq V$ so that there is* no *edge between any two vertices in S. The* independence number *$\alpha(G)$ of G is the maximum size of an independent set in G.*

An clique set *in G is a set of vertices $C \subseteq V$ so there is an edge between all pairs of vertices in C. The* clique number *$\omega(G)$ of G is the maximum size of a clique in G.*

It follows from the definition of a proper colouring that any colour class of a proper colouring is an independent set in the graph. This also allows for an alternative definition of k-colourable: a graph $G = (V, E)$ is k-colourable if and only if there exist k independent sets $S_1, S_2, \ldots, S_k$ in G such that $V = S_1 \cup S_2 \cup \cdots \cup S_k$.

In particular we see that a graph G is bipartite if and only if G is 2-colourable.

Definition 4.15 (greedy algorithm)**.** *The* greedy algorithm *to colour the vertices of an n-vertex graph G proceeds as follows. Initially, the set of colours is the set of natural numbers $\{1, 2, 3, \ldots\}$.*

(1) Choose some ordering $v_1, v_2, \ldots, v_n$ *of the vertices of* G.

(2) Colour v_1 *with colour* 1.

(3) Colour the remaining vertices in $n-1$ *steps: at step* j, *where* $j = 2, 3, \ldots, n$, *list the colours of all the neighbours that* v_j *has in the set* $\{v_1, v_2, \ldots, v_{j-1}\}$. *Give* v_j *the smallest colour not used in that list.*

(4) Once every vertex has been coloured, remove all colours from $\{1, 2, \ldots\}$ *that have not been used. (So at the end colours* $\{1, 2, \ldots, k\}$, *for some* $k \geq 1$, *are used.)*

The actual number of colours used by the greedy algorithm depends on the on the ordering of the vertices. In particular, you cannot assume that the greedy algorithm uses only $\chi(G)$ colours, it can use (many) more.

By analysing the greedy algorithm, we can prove the following result.

Theorem 4.16. *Let* $G = (V, E)$ *be a graph and let* $\Delta(G)$ *denote the* maximum degree *of* G, *so* $\Delta(G) = \max\limits_{v \in V} d(v)$. *Then we have* $\chi(G) \leq \Delta(G) + 1$.

Exercises

1. The complement of a graph $G = (V, E)$ is the graph $\overline{G}$ with the same vertex set V and, for every two vertices $u, v \in V$, uv is an edge in $\overline{G}$ if and only if uv is not an edge of G.

(a) Prove that if G is not connected, then $\overline{G}$ is connected.

(b) Is it the case that if G is connected, then $\overline{G}$ is not connected?

2. Let T be a tree on $n \geq 2$ vertices.

(a) Prove that for every pair of vertices u and v, there is a unique path between u and v.

(b) Prove that if the vertices u and v, $u \neq v$, are such that uv is *not* an edge of T, then adding uv to T will create exactly one cycle.

3. Show that every tree on at least two vertices has a vertex of degree 1.

4. Prove that if G is a connected graph with n vertices and $n - 1$ edges, then G is a tree.

5. Suppose that G is a forest with n vertices and $n - c$ edges, for some $c \geq 1$. Prove that G has c components.

6. A mouse intends to eat a $3 \times 3 \times 3$ cube of cheese. Being tidy-minded, it begins at a corner and eats the whole of a $1 \times 1 \times 1$ cube, before going on to an adjacent one.

Can the mouse end in the centre?

7. Prove by induction that every tree is a bipartite graph, without using Theorem 3.22.

8. There are five cities that have to be connected by some new roads. The cost of building a road directly between city i and city j is the entry $a_{i,j}$ in the matrix below.

$$\begin{pmatrix} 0 & 3 & 5 & 11 & 9 \\ 3 & 0 & 3 & 9 & 8 \\ 5 & 3 & 0 & 10^6 & 10 \\ 11 & 9 & 10^6 & 0 & 7 \\ 9 & 8 & 10 & 7 & 0 \end{pmatrix}.$$

Determine the minimum cost of making all the cities reachable from each other.

9. Let the graph K_n have vertices $\{1, 2, \ldots, n\}$ and suppose that for each $u, v \in \{1, 2, \ldots, n\}$, $u \neq v$, the edge uv has weight $w(uv) = u + v$.

Determine the minimal spanning tree of this graph. What is the total cost for this minimal spanning tree?

10. For natural numbers n and p, let G be the complete graph with vertex set $\{1, 2, \ldots, n\}$, and let the weight of the edge ij be given by $w(ij) = |i-j| \mod p$. (So $w(ij) \in \{0, 1, \ldots, p-1\}$.)

For every n and p, determine the minimum weight of a spanning tree in G. (Do not expect to be able to write down the answer just like that; **try** it for **a few small values** of n and p to see what is going on. The **answer** also **depends on** which of n and p is larger.)

11. (a) How many spanning trees does the (labelled) graph P_n have?

(b) How many spanning trees does the (labelled) graph C_n have?

(c) How many spanning trees does the (labelled) graph K_n have?

12. (a) Let T be a tree with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and edge set $\{12, 17, 18, 19, 23, 24, 59, 67\}$. Draw T and find its Prüfer code.

(b) Find the tree with vertex set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ whose Prüfer code is $(2, 3, 5, 7, 5, 3, 2)$.

(c) Determine which trees with vertex set $\{1, 2, 3 \ldots, n\}$ have a Prüfer code that contains only two values.

13. (a) Let T be a tree and v be any of its vertices.

Prove that v appears $d_T(v) - 1$ times in the Prüfer code for T. (Here $d_T(v)$ is the degree of v in T.)

(b) Let T be a tree with vertex set S, $|S| = n$, and suppose that $(a_1, \ldots, a_{n-2})$ is the Prüfer code of T. Let v be the smallest leaf adjacent to a_1 in T.

Prove that $(a_2, \ldots, a_{n-2})$ is the Prüfer code of the tree $T - v$ with vertex set $S \setminus \{v\}$.

(c) Determine the number of labelled trees with vertex set set $\{1, 2, 3, \ldots, 8\}$ whose degree sequence is $3, 3, 2, 2, 1, 1, 1, 1$.

14. (a) What is the chromatic number of the complete graph K_n on $n \geq 2$ vertices?

(b) What is the chromatic number of the path P_n on $n \geq 1$ vertices?

(c) What is the chromatic number of the cycle C_n on $n \geq 3$ vertices?

15. For two natural numbers k, s, let $H_{k,s}$ be the graph obtained by taking a cycle C_{2k+1} on $2k + 1$ vertices, a complete graph K_s on s vertices, and putting and edge between every vertex of C_{2k+1} and every vertex of K_s (so C_{2k+1} and K_s do not have any common vertices).

(a) Draw the graphs $H_{1,3}$ and $H_{2,4}$.

(b) For every k and s, what is the chromatic number of the graph $H_{k,s}$?

(c) For every k and s, what is the clique number of the graph $H_{k,s}$?

(d) For every k and s, what is the independence number of the graph $H_{k,s}$?

16. For two positive integers k and n, let $G_{n,k}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{ij \mid |i - j| \leq k\}$.

(a) Draw the graphs $G_{4,3}$ and $G_{5,2}$.

(b) Determine the chromatic numbers $\chi(G_{4,3})$ and $\chi(G_{5,2})$. Justify your answers.

(c) For all values of $k \geq 1$ and $n \geq 1$, determine the chromatic number $\chi(G_{n,k})$.

(d) For what values of $k \geq 1$ and $n \geq 1$ is the graph $G_{n,k}$ Eulerian?

(Hint: what are the degrees of vertices 1 and 2 in $G_{n,k}$?)

17. Let $G = (V, E)$ and $H = (V', E')$ be two graphs with disjoint vertex sets, i.e. $V \cap V' = \varnothing$. Denote by $G + H$ the graph with vertex set $V \cup V'$ and edge set $E \cup E'$.

Prove that for every two vertex disjoint graphs G and H we have $\chi(G+H) = \max\{\chi(G), \chi(H)\}$.

18. Prove or disprove the following statements.

(a) Every graph G with chromatic number k has a proper k-colouring in which some colour class has $\alpha(G)$ vertices.

(b) For every graph G with chromatic number k there exists some ordering of the vertices of G such that the greedy algorithm uses exactly k colours.

Additional reading and exercises

From *Biggs*, ***Discrete Mathematics***
– **Reading**: Sections 15.5 – 15.7; 16.3.
– **Exercises**: Section 15.5: 1 – 4; Section 15.6: 1 – 3; Section 15.7: 1 – 4;
Section 15.8: 11, 15, 22; Section 16.3: 1 – 3; Section 16.7: 6.

From *Cameron*, ***Combinatorics***
– **Reading**: Sections 11.2, 11.3.