MA103 - Class 3

INDUCTION Which of these are ways of stating induction?

1 $[\forall m \in \mathbb{N}, (P(m) \Rightarrow P(m+1))] \Rightarrow \forall m \in \mathbb{N}, P(m)$

2 $[(\forall m \in \mathbb{N}, P(m)) \Rightarrow P(m+1)] \Rightarrow \forall m \in \mathbb{N}, P(m)$

3 $\forall m \in \mathbb{N} \ [(P(m) \Rightarrow P(m+1)) \Rightarrow \forall m \in \mathbb{N}, P(m)]$

4 $[P(1) \wedge \forall m \in \mathbb{N} \ P(m) \Rightarrow P(m+1)] \Rightarrow \forall m \in \mathbb{N} \ P(m)$

5 $\forall m \in \mathbb{N} \ [P(1) \wedge (P(m) \Rightarrow P(m+1)) \Rightarrow \forall m \in \mathbb{N}, P(m)]$

6 $\forall S \subseteq \mathbb{N}, \exists s \in S, \forall v \in S, \quad s \leq v$

7 $\forall S \subseteq \mathbb{N} \ (S \neq \emptyset), \forall v \in S, \exists s \in S, \quad s \leq v$

8 $\forall S \subseteq \mathbb{N} \ (S \neq \emptyset), \exists s \in S, \forall v \in S, \quad s \leq v$

PROBLEM 1 a) Using induction, prove that $\forall m \in \mathbb{N}, \ 3 \mid m^3 + 5m$

b) —— " —— $\forall m \in \mathbb{N}, \ 6 \mid m^3 + 5m$

For every $m \in \mathbb{N}$, let
$P(m) := m^3 + 5m$.
Let us use induction.
i) We have $P(1) = 1 + 5 \cdot 1 = 6$.
So we have that 6 is a multiple of 6.
ii) Let the case $m = k$ be true by induction and let us show $m+1$.
We have
$(m+1)^3 + 5(m+1) = \ldots\ldots = 6 \cdot (\ldots)$
and therefore $P(k) \Rightarrow P(k+1)$.

We can conclude by induction

For $m \in \mathbb{N}$, let $P(m)$ be the statement $\forall m \in \mathbb{N}, \ 6 \mid m^3 + 5m$.
i) Since $6 \mid 1 + 5 \cdot 1$, we have that $P(1)$ holds.
ii) Assume that, for any m, $P(m)$ holds true. Let's prove $P(m+1)$.
We have
$(m+1)^3 + 5(m+1) = \ldots\ldots = 6 \cdot (\ldots)$
And therefore $P(m) \Rightarrow P(m+1)$ for all $m \in \mathbb{N}$.

We can conclude by induction

For $n \in \mathbb{N}$, let $P(n)$ be the statement $P(n) := $ "$6 \mid n^3 + 5n$".

i) Since $6 \mid 1 + 5 \cdot 1$, we have that $P(1)$ holds.

ii) Let us now show that $\forall n \in \mathbb{N}$, $P(n) \Rightarrow P(n+1)$.

Take a generic $n \in \mathbb{N}$, and assume $P(n)$. Then we have

$$(n+1)^3 + 5(n+1) = \ldots = 6 \cdot (\ldots)$$

which proves $P(n+1)$. We can conclude by induction

ALTERNATIVE Let us assume by contradiction that $\forall n \in \mathbb{N}$, $6 \mid n^3 + 5n$ is false. Then, $S = \{ n \in \mathbb{N} \mid 6 \nmid n^3 + 5n \}$ is a NON-EMPTY subset of $\mathbb{N}$.

Therefore, S has a least element, call it s.

We know $1 \notin S$, since $6 \mid 6$. Then call $\hat{s} := s - 1$. We know that $\hat{s} \notin S$ and $\hat{s} \in \mathbb{N}$! Note that we have

$$s^3 + 5s = \ldots\ldots = 6 \cdot (\ldots)$$

which is absurd, since $s \in S$.

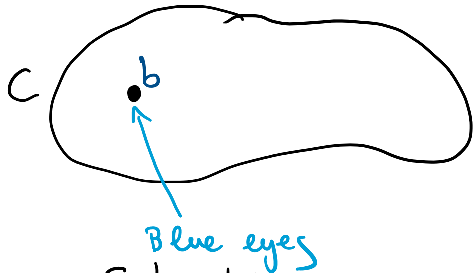
EXTRA PROBLEM If in this classroom there is one person with blue eyes, then everyone has blue eyes.

$\leadsto$ Let us prove this by induction on the number n of people in the room.

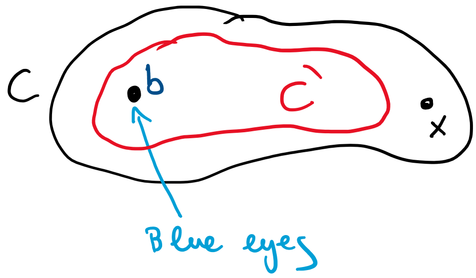
$n = 1$. This is trivial.

$n-1 \Rightarrow n$. Let us assume we have a classroom C with n people, at least one of which has blue eyes (call this person b)

We want to use the inductive hypothesis.



If everyone in C has blue eyes, we are done! Otherwise, let $x \in C$ be someone that does NOT have blue eyes.



And consider $C' := C \setminus \{x\}$.

By induction, everyone in C' has blue eyes. We can now use induction on $C'' := C' \setminus \{b\}$ and conclude that everyone in C'' has blue eyes. This is enough to conclude.

EXTRA PROBLEM 2 We play a tournament with n people. Everyone plays against everyone else. We say that a ranking $(i_1, i_2, \dots, i_n)$ is GOOD if, for every index j, i_j beated i_{j+1}.

Prove that, for every n, there exists a GOOD ranking.

$\rightsquigarrow$ $n = 1$

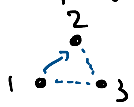
$n = 2$

$n = 3$ We can assume 1 beats 2.

3 beats : $- \phi$ $(1,2,3)$ $- \{2\}$ $(1,3,2)$

$- \{1\}$ $(3,1,2)$ $- \{1,2\}$ $(3,1,2)$

New way of representing the problem: $i \to j$ means that i beats j.



BY INDUCTION i) Base cases ✓

ii) Assume we had a tournament with n people, and let $(i_1, \dots, i_n)$ be a GOOD ranking for them.
Now let us have everyone playing with a new $(n+1)$ player P
We want a new good ranking.
$\to$ either P beats everyone. Then $(n+1, i_1, \dots, i_n)$ is good
$\to$ or there exists a worst player i_k that beats P.
then $(i_1, i_2, i_3, \dots, i_k, P, i_{k+1}, \dots)$ is a good ranking