MA210 - Class 2

GENERAL REMARKS · Notes uploaded online

· Terrible solutions · No justification

· CHAOS.

→ WE ARE HERE TO LEARN

EXERCISE 19 A deck contains 52 distinct cards, each identified by a value in $\{1,2,\dots,10,J,Q,K\}$ and a suit in $\{H,D,S,C\}$. A k-hand is a set of k different cards from this deck.

b) # of 13-hands that contains at least one card from each suit. Hint: INCLUSION-EXCLUSION PRINCIPLE

Let V be the set of all 13-hands,
let A denote the set of 13-hands with at least one card for each suit.
Following the hint, and because by point a) we know $|V| = \binom{52}{13}$, we can compute $|A^C|$ to easily get to $|A|$.
↳ $|V \setminus A|$

exactly one is true

Rem Consider any 13-hand. It either is in A or there is a suit not represented in it.

Let us denote, for $x \in \{H,D,S,C\}$ with

$$A_x = \{\text{13-hands without cards from the suit } x\}.$$

Rem Our previous remark says:

$$V \setminus A = A_H \cup A_D \cup A_S \cup A_C$$

CLAIM ① $\forall x \in \{H,D,S,C\}$ $\quad |A_x| = \binom{39}{13}$

② $\forall x \neq y \in \{H,D,S,C\}$ $\quad |A_x \cap A_y| = \binom{26}{13}$

③ $\forall x,y,z \in \{H,D,S,C\}$ distinct, $|A_x \cap A_y \cap A_z| = 1$

④ $|A_H \cap A_D \cap A_S \cap A_C| = 0$

$\rightsquigarrow$ ④ Any of the 13 cards in a 13-hand has a suit so no 13-hand is in all the A_x

③ There are only 13 cards that are not associated to x, y or z. There is just one way of selecting 13 cards from a group of 13 cards

② There are $13 \times 2 = 26$ cards of suits different than x and y. There are $\binom{26}{13}$ ways of selecting 13 of them.

① There are $13 \times 3 = 39$ cards of suits different than x. There are $\binom{39}{13}$ ways of selecting 13 of them.

Finally, let us apply the I-E Principle

$$|A_S \cup A_D \cup A_H \cup A_C| \overset{\text{I.E. principle}}{=} \sum_{\substack{X \subseteq \{\ldots\} \\ X \neq \emptyset}} (-1)^{|X|+1} \left| \bigcap_{y \in X} A_y \right|$$

$$= \sum_{i=1}^{4} \sum_{\substack{X \subseteq \ldots \\ |X| = i}} (-1)^{|X|+1} \left| \bigcap_{y \in X} A_y \right|$$

$$\overset{\text{CLAIM}}{=} \sum_{i=1}^{4} \binom{13(4-i)}{13} (-1)^{i+1} \sum_{\substack{X \subseteq \\ |X| = i}} 1$$

$$= \binom{4}{1}\binom{39}{13} - \binom{4}{2}\binom{26}{13} + \binom{4}{3}\binom{13}{13} - 0$$

Finally, we get $|A| = |V| - |A^c| = \binom{52}{13} - \left[4\binom{39}{13} - 6\binom{26}{13} + 4\binom{13}{13} - 0 \right]$.

21.d) 4 married couple + 1 person. In how many ways can we assign the 5 people to 3 DISTINGUISHIBLE teams (R, B, G) in such a way that each team gets 3 players and no pair is together?

$\rightsquigarrow$ Let us number the couples $1,\dots,4$. We want to use IE princ.

So we let $V = \{$possible team assignments$\}$

$A = \{$team assign. without two partners in the same team$\}$

and for $i \in \{1,..,4\}$ $A_i = \{$te. as. with the couple i in the same team$\}$.

By our def. of those sets, $V \setminus A = A_1 \cup \dots \cup A_4$.

CLAIM ① $\forall x \in [4]$, $|A_x| = 7 \cdot 3 \cdot \binom{6}{3}$

② $\forall x,y \in [4]$ distinct $|A_x \cap A_y| = 3! \cdot 5 \cdot 4$

③ $\forall x,y,z \in [4]$ dist. $|A_x \cap A_y \cap A_z| = 3! \cdot 3!$

④ $|\bigcap_{[4]} A_i| = 0$

$\rightsquigarrow$ ④ By pigeonhole principle, there is no way to have every couple in the same team, since two couples cannot be in the same team.

③ we have:
- $3!$ ways of assigning 3 colours to the couples x,y,z
- $3!$ ways of assigning the last 3 people to the G, R, B teams

And these choices are done one after the other

② we have
- $3!$ ways of assigning 3 colours to the couples x,y
- 5 options for the third member of the team of x
- 4 options for the ~~———~~ y

And these choices, ...

① By point c)

Now we can apply the IE principle.

Question 1 SUMMER 2017

(a) A drawer contains six distinguishable pairs of gloves. each pair of a different colour. Six persons, named A, B, C, D, E, and F, choose a left-hand glove ~~at random~~ and a right-hand glove ~~at random~~ (without replacement, so all gloves are gone at the end).

(i) In how many ways can this selection be done ?

(ii) In how many ways can this be done such that A has a non-matching pair and D has a matching pair ?

(iii) In how many ways can this be done if exactly four people have a matching pair ?

(iv) In how many ways can this be done if no person has a matching pair ?

(b) Suppose you have n objects, $n-r$ of which are identical. What is the number of orderings of the n objects ? Prove your result.

i) Let's say that A chooses first, then B, then C, ...

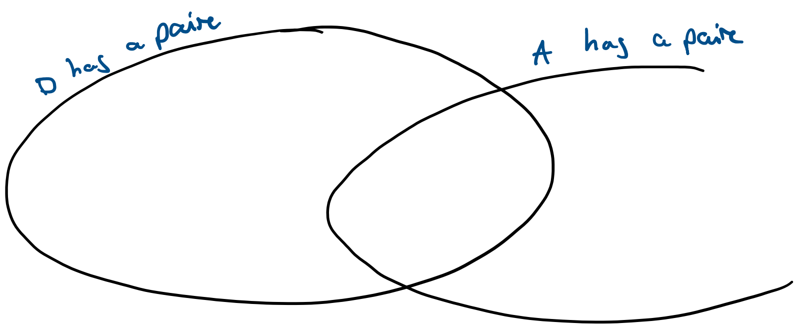
A has $6 \cdot 6$ choices, B has $5 \cdot 5$ choices, ...

$$(6!)^2$$

ii) There are $6 \cdot (5!)^2$ assignments so that D has a matching pair

$6 \cdot 5 \cdot (4!)^2$ assi. s.t. D&A has match. pairs

So we get $6(5!)^2 - 6 \cdot 5 \cdot (4!)^2$ s.t. D has a match but A does not.



iii) $\binom{6}{4}$ ways to select four people with matching pairs. We can choose the pairs to give them in $6 \cdot 5 \cdot 4 \cdot 3$ ways. The remaining two pairs must be given so that the people do not get a matching pair.

This can be done in 2 ways. So we get

$$\binom{6}{4} \cdot (6 \cdot 5 \cdot 4 \cdot 3) \cdot 2$$

iv) For $x \in \{A, \dots, F\}$, let $A_x = \{$as. s.t. x gets a matching pair$\}$.

By i), our answer is $(6!)^2 - |\bigcup_{x \in \{A,\dots,F\}} A_x|$.

To calculate $|\bigcup_{x\in\{A,\dots,F\}} A_x|$ we use IE-principle.

$$|\bigcup A_x| = \sum_{\substack{S \subseteq \{A,\dots,F\} \\ S \neq \emptyset}} (-1)^{|S|+1} \left|\bigcap_{y \in S} A_y\right|$$

Rem Let $S \subseteq \{A,\dots,F\}$, $S \neq \emptyset$ and let $i = |S|$. We have

$$\left|\bigcap_{y\in S} A_y\right| = \left(\prod_{\ell = 6-i+1}^{6} \ell\right)\left((6-i)!\right)^2$$

And therefore

$$|\bigcup A_x| = \sum_{\emptyset \notin S \subseteq \{A,\dots,F\}} (-1)^{|S|+1} \left(\prod_{\ell=6-|S|+1}^{6} \ell\right)\left((6-|S|)!\right)^2$$

$$= \sum_{i=1}^{6} (-1)^{i+1} \left(\prod_{\ell=7-i}^{6} \ell\right)\left((6-i)!\right)^2$$