MA210 Discrete Mathematics

Notes 2

30 January and 6 February 2022

Introduction to Recurrence Relations and Generating Functions

Recurrence Relations

An (infinite) *sequence* is a function f that maps natural numbers (or non-negative integers) to the set of real numbers. Instead of working with f itself, we write $a_n = f(n)$ and use the notation $(a_n)_{n\geq 1}$ or $(a_n)_{n\geq 0}$. We will mainly use the latter. We write $\mathbb{N}_0$ for the set of natural numbers together with 0 (i.e. the set $\{0, 1, 2, 3, \ldots\}$).

Definition 2.1 (Recurrence relation, initial conditions, solution, closed form)**.** *A* recurrence relation *is an equation of the form*

$$a_n = \varphi_n(n, a_{n-1}, a_{n-2}, \ldots, a_0),$$

which holds for all $n \geq N$ (for some $N \in \mathbb{N}$) for a sequence $(a_n)_{n\geq 0}$. Here φ_n can be any function. It doesn't have to be given by a simple formula. It can depend on any number of the previous terms, and there doesn't have to be any similarity between (say) φ_n and φ_{n+1}.

But in these notes we will (almost always) stick to recurrence relations where φ_n does not depend on n and only depends on the last few terms of the recurrence. So we will consider recurrence relations of the form

$$a_n = \varphi(n, a_{n-1}, a_{n-2}, \ldots, a_{n-N}).$$

We typically consider situations where the initial conditions *$a_0, a_1, \ldots, a_{N-1}$ are given with the recurrence relation. Together they define a unique sequence.*

A solution *of a recurrence relation with initial conditions, or a* closed form *of a sequence, is a function $F(n)$, depending only on n, such that for all $n \geq 0$ we have $a_n = F(n)$.*

Example 2.2. Show that the recurrence relation given by $a_n = 2a_{n-1}$, $n \geq 1$, and $a_0 = 1$, has a closed form $a_n = F(n) = 2^n$.

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Homogeneous Linear Recursions of Order 1 or 2

Definition 2.3. *A recurrence relation*

$$\begin{aligned} a_0 &= c\,, \\ a_n &= \alpha a_{n-1} \quad \textit{for} \quad n \geq 1\,, \end{aligned}$$

where α, c are real-valued constants, is called a homogeneous linear recursion of order 1 with constant coefficients.

Proposition 2.4. *A homogeneous linear recursion of order* 1 *with constant coefficients $a_n = \alpha a_{n-1}$, $n \geq 1$, $a_0 = c$, has the unique solution*

$$a_n = c \cdot \alpha^n \qquad \textit{for } n \geq 0.$$

Example 2.5. Find a solution of the recurrence relation $a_0 = 1$, $a_n = 7a_{n-1} - 3$ for $n \geq 1$.

Definition 2.6. *A recurrence relation*

$$\begin{aligned} &a_0 = c_0\,, \\ &a_1 = c_1\,, \\ &a_n = \alpha a_{n-1} + \beta a_{n-2} \quad \textit{for} \quad n \geq 2\,, \end{aligned}$$

where α, β, c_0, c_1 are real-valued constants, is called a homogeneous linear recursion of order 2 with constant coefficients.

Proposition 2.7. *Suppose we have a homogeneous linear recursion of order 2 with constant coefficients:*

$$a_0 = c_0\,, \quad a_1 = c_1\,, \quad \textit{and} \quad a_n = \alpha a_{n-1} + \beta a_{n-2} \textit{ for } n \geq 2\,, \quad \textit{where } \beta \neq 0\,.$$

Let r_1, r_2 be the roots of the equation $x^2 = \alpha x + \beta$.

(a) If these roots are distinct ($r_1 \neq r_2$), then the recurrence relation has a solution

$$a_n = k_1 \cdot r_1^n + k_2 \cdot r_2^n, \qquad \textit{for } n \geq 0,$$

where k_1 and k_2 are constants depending on the initial conditions:

$$\begin{aligned} c_0 &= a_0 = k_1 \cdot r_1^0 + k_2 \cdot r_2^0 = k_1 + k_2, \\ c_1 &= a_1 = k_1 \cdot r_1^1 + k_2 \cdot r_2^1 = k_1 r_1 + k_2 r_2. \end{aligned}$$

(b) If these roots are equal ($r_1 = r_2 = r$), then the recurrence relation has a solution

$$a_n = (k_1 + k_2 \cdot n) r^n, \qquad \textit{for } n \geq 0,$$

where k_1 and k_2 are constants depending on the initial conditions:

$$\begin{aligned} c_0 &= a_0 = (k_1 + k_2 \cdot 0) r^0 = k_1, \\ c_1 &= a_1 = (k_1 + k_2 \cdot 1) r^1 = (k_1 + k_2) r. \end{aligned}$$

Note that it is possible that the roots r_1, r_2 are **complex numbers** (since the quadratic equation $x^2 = \alpha x + \beta$ may have complex roots), even though all values a_n in the sequence are real numbers. It is (as a result) also possible that k_1, k_2 can be complex numbers.

Example 2.8. Find a solution of the recurrence relation $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$.

Generating Functions

Definition 2.9. *Given a sequence $(a_n)_{n\geq 0}$, we associate with it the* formal power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots .$$

The function $f(x)$ is called the generating function *of the sequence $(a_n)_{n\geq 0}$. We also say that "$f(x)$* generates the sequence $(a_n)_{n\geq 0}$*" or "*the sequence $(a_n)_{n\geq 0}$ is generated by the function $f(x)$*".*

Note that we do not make any assumptions on the convergence of $\sum_{n=0}^{\infty} a_n x^n$. It may not converge for some values of x; perhaps it even converges only if $x = 0$.

The main underlying principle of these formal power series we will use is the following.

Property 2.10. *Let $f(x)$ and $g(x)$ be two generating functions:*

$$f(x) = \sum_{n=0}^{\infty} a_n\, x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n\, x^n .$$

Then $f(x) = g(x)$ if and only if $a_n = b_n$ for all $n \geq 0$.

Example 2.11. The generating function of the sequence $\left(\binom{k}{n}\right)_{n\geq 0}$ is $(1+x)^k$.

Example 2.12. The generating function of the sequence $(1)_{n\geq 0}$ is $\frac{1}{1-x}$.

Example 2.13. What sequence is generated by $\dfrac{4x}{1-x^2}$?

Example 2.14. The function e^x is the generating function of $\left(\frac{1}{n!}\right)_{n\geq 0}$.

Example 2.15. Using $e^x = \sum_{n=0}^{\infty} \dfrac{1}{n!} x^n$ only, prove that $\dfrac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$.

Example 2.16. Use generating functions to find the closed form of the recurrence relation $a_n = 2a_{n-1}$, $n \geq 1$, $a_0 = 1$.

Example 2.17. Find a solution of the recurrence relation $a_n = a_{n-1} + 2n$, $n \geq 1$, $a_0 = 1$.

More on Generating Functions

The following theorem gives a list of possible ways that generating functions can be manipulated in general, so as to get new generating functions. Since we never discussed in detail what it means to be a formal power series, we can't prove the theorem. Nevertheless, the statements look acceptable if you believe that manipulations with infinite series are allowed.

Theorem 2.18. *Suppose that the sequence $(a_n)_{n\geq 0}$ has generating function $f(x)$, the sequence $(b_n)_{n\geq 0}$ has generating function $g(x)$. Then we have:*

(a) $f(x) + g(x)$ *generates* $(a_n + b_n)_{n\geq 0}$*;*

(b) for any integer $m \geq 0$*,* $x^m f(x)$ *generates* $\underbrace{0, \dots, 0}_{m}, a_0, a_1, \dots$*;*

(c) for any real number c*,* $cf(x)$ *generates* $(ca_n)_{n\geq 0}$*;*

(d) for any real number c*,* $f(cx)$ *generates* $(c^n a_n)_{n\geq 0} = a_0, ca_1, \dots, c^n a_n, \dots$*;*

(e) for any integer $m \geq 0$*,* $f(x^m)$ *generates* $a_0, \underbrace{0, \dots, 0}_{m-1}, a_1, \underbrace{0, \dots, 0}_{m-1}, a_2, \dots$*;*

(f) $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x)$ *generates* $a_1, 2a_2, 3a_3 \dots$*;*

(g) $\displaystyle\int_0^x f(t)\,\mathrm{d}t$ *generates* $0, a_0, \frac{1}{2}a_1, \frac{1}{3}a_2, \frac{1}{4}a_3, \dots$*.*

Theorem 2.19. *Suppose* $f(x)$ *is a formula for a formal power series* $\sum_{n=0}^{\infty} a_n x^n$*, for example* $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$*. If* $g(x)$ *is a formal power series (or a polynomial)* ***with constant term 0*** *then it is valid to write*

$$f\big(g(x)\big) = \sum_{n=0}^{\infty} a_n \big(g(x)\big)^n .$$

So for example we can write

$$\frac{1}{1 - (x^2 + 2x^3)} = \sum_{n=0}^{\infty} (x^2 + 2x^3)^n .$$

On the other hand, we should **NEVER EVER** do something like

$$\frac{1}{-1 - x} = \frac{1}{1 - (x+2)} = \sum_{n=0}^{\infty} (x+2)^n,$$

because the constant term of $x + 2$ is $2 \neq 0$.

Theorem 2.20 (Convolution Theorem)**.** *Suppose that the sequence* $(a_n)_{n\geq 0}$ *has generating function* $f(x)$ *and the sequence* $(b_n)_{n\geq 0}$ *has generating function* $g(x)$*. Then the product* $f(x)g(x)$ *generates the sequence* $(c_n)_{n\geq 0}$*, where* $c_n = \displaystyle\sum_{k=0}^{n} a_k b_{n-k}$*.*

Example 2.21. Suppose $k \geq 0$. Show that

$$(1 + x) \cdot \sum_{n=0}^{\infty} \binom{k}{n} x^n = \sum_{n=0}^{\infty} \binom{k+1}{n} x^n .$$

Partial Fractions

Suppose that we have a sequence (whose terms we do not know) $(a_n)_{n\geq 0}$ whose generating function is $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Suppose we find

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials whose coefficients we *do* know (as in Example 2.17). How can we find a formula for a_n for each $n \geq 0$?

Proposition 2.22. *Suppose $p(x)$ and $q(x)$ are polynomials. Then we can write*

$$\frac{p(x)}{q(x)} = a(x) + \frac{p^*(x)}{q(x)}$$

where $a(x)$ is a polynomial and $p^*(x)$ has degree less than $q(x)$.

Theorem 2.23. *Suppose $q(x)$ is any polynomial. We can write*

$$q(x) \;=\; \alpha(r_1 - x)^{m_1}(r_2 - x)^{m_2} \ldots (r_k - x)^{m_k}$$

where the (possibly complex) numbers $r_1, \ldots, r_k$ are the roots of the equation $q(x) = 0$ and for each $1 \leq i \leq k$ the positive integer m_i is the multiplicity of the root r_i.

Suppose $p^*(x)$ is any polynomial whose degree is smaller than that of $q(x)$. Then there are numbers $A_{1,1}, A_{1,2}, \ldots, A_{1,m_1}, A_{2,1}, A_{2,2}, \ldots, A_{2,m_2}, \ldots, \ldots, A_{k,1}, A_{k,2}, \ldots, A_{k,m_k}$ such that

$$\begin{aligned}
\frac{p^*(x)}{q(x)} \;=\; & \quad \frac{A_{1,1}}{r_1 - x} + \frac{A_{1,2}}{(r_1 - x)^2} + \cdots + \frac{A_{1,m_1}}{(r_1 - x)^{m_1}} \\
& + \frac{A_{2,1}}{r_2 - x} + \frac{A_{2,2}}{(r_2 - x)^2} + \cdots + \frac{A_{2,m_2}}{(r_2 - x)^{m_2}} \\
& \qquad\qquad\qquad \vdots \\
& + \frac{A_{k,1}}{r_k - x} + \frac{A_{k,2}}{(r_k - x)^2} + \cdots + \frac{A_{k,m_k}}{(r_k - x)^{m_k}} \,.
\end{aligned}$$

To calculate the $A_{i,j}$ you need to solve a system of linear equations. This gets rapidly more difficult as the system gets bigger: so it is a good idea to cancel any common factors of $p^*(x)$ and $q(x)$.

Proposition 2.24. *We have*

$$\frac{1}{(r - x)^m} = \sum_{n=0}^{\infty} r^{-m-n} \binom{n + m - 1}{m - 1} x^n \,.$$

Example 2.25. Let $(u_n)_{n\geq 0}$ be given by $u_0 = 0$, $u_1 = 1$ and $u_n = u_{n-1} + 6u_{n-2} + n - 2$, $n \geq 2$. Find a closed form for u_n.

The Binomial Theorem for Negative Exponents

The binomial coefficient $\binom{k}{r}$ is defined as the number of ways to choose r objects from k possible choices, without repetition and where order does not matter. In other words, $\binom{k}{r}$ is the number of r-element subsets of an n-element set. From this we deduced that

$$\binom{k}{r} = \frac{k!}{r!(k-r)!} = \frac{k(k-1)\cdots(k-(r-1))}{r!}.$$

Definition 2.26. *For a real number α and a natural number r, we define*

$$\binom{\alpha}{r} := \frac{\alpha(\alpha-1)\cdots(\alpha-(r-1))}{r!}.$$

For $r = 0$, we set $\binom{\alpha}{0} := 1$.

Fact 2.27. *If $k \geq 0$ is an integer, we have*

$$\binom{-k}{r} = (-1)^r \binom{k+r-1}{r}.$$

Theorem 2.28 (Binomial Theorem for negative exponents)**.** *For any natural number k we have as a formal power series*

$$(1+x)^{-k} = \sum_{n=0}^{\infty} \binom{-k}{n} x^n .$$

Example 2.29. Use the Convolution Theorem for the product of two power series, to prove that

$$\left(\sum_{n=0}^{\infty} (-1)^n \binom{n+k-1}{k-1} x^n \right) \cdot \left(\sum_{n=0}^{\infty} \binom{k}{n} x^n \right) = 1.$$

Additional reading and exercises

From *Biggs*, ***Discrete Mathematics***
– **Reading**: Sections 4.5, 19.2, 25.1 – 25.6.
– **Exercises**: Section 4.5: 1 – 4; Section 19.2: 1 – 5; Section 19.7: 1 – 2;
Section 25.2: 1 – 2; Section 25.3: 1 – 4; Section 25.4: 1 – 4;
Section 25.5: 1 – 3; Section 25.6: 1 – 3; Section 25.7: 2 – 4, 8 – 18.

From *Cameron*, ***Combinatorics***
– **Reading**: Sections 4.1 – 4.3.

Exercises

1. Solve the following recurrence relation:

$$\begin{aligned} a_n &= 4a_{n-1} - 4a_{n-2}, \quad \text{for } n \geq 2; \\ a_0 &= 1; \\ a_1 &= 3. \end{aligned}$$

2. Solve the following recurrence relation:

$$\begin{aligned} b_n &= b_{n-1} + 6b_{n-2} + 6n, \quad \text{for } n \geq 2; \\ b_0 &= 1; \\ b_1 &= 1. \end{aligned}$$

(Hint: you may want to write $b_n = a_n + \alpha n + \beta$ and choose suitable α and β first.)

3. Let a_n denote the number of n-digit sequences in which each digit is either 0 or 1, and no two consecutive 0's are allowed.

(a) Show that $a_1 = 2$ and $a_2 = 3$. What would you say a_0 is?

(b) Show that for $n \geq 3$ we have $a_n = a_{n-1} + a_{n-2}$.

(c) Give a closed form expression for a_n.

4. Define the sequence $(b_n)_{n\geq 1}$ by $b_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \ldots$. (Remember that $\binom{k}{\ell} = 0$ if $\ell > k$.)

(a) Verify that $b_1 = 1$, $b_2 = 2$, and that, for every $n \geq 3$, we have $b_n = b_{n-1} + b_{n-2}$.

(b) Find a closed form expression for b_n.

5. Let f_n denote the n-th *Fibonacci number*, i.e., $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$.

Prove that for every $n \geq 2$ we have $f_n^2 - f_{n-1} \cdot f_{n+1} = (-1)^n$.

6. For $n \geq 1$, let a_n denote the number of n-digit sequences in which each digit is 0, 1, or -1, and no two consecutive 0's or two consecutive 1's are allowed.

(a) Show that $a_n = 2a_{n-1} + a_{n-2}$ for $n \geq 3$.

(b) Determine a_1 and a_2.

(c) Find a closed form expression for a_n.

7. On working through a problem, a student is said to be at the n-th stage if she or he is n steps from the solution. At any stage the student has five choices how to proceed. Two of these choices result in the student going to the $(n-1)$-th stage, and the remaining three of them are better and they take her or him directly to the $(n-2)$-th stage.

Let s_n be the number of ways the student can reach the solution if she or he starts from the n-th stage.

(a) If $s_1 = 2$, verify that $s_2 = 7$.

(b) Give a recurrence relation for s_n.

(c) Deduce that $s_n = \frac{1}{4}(3^{n+1} + (-1)^n)$.

8. Use generating functions to solve the following recurrence relation:

$$\begin{aligned} a_n &= 5a_{n-1} - 6a_{n-2}, \quad \text{for } n \geq 2; \\ a_0 &= 0; \\ a_1 &= 3. \end{aligned}$$

9. Suppose that $f(x)$ generates the sequence $a_0, a_1, a_2, \ldots$. Give expressions, in terms of f, for the generating functions of the following sequences:

(a) $0, a_0, 0, a_1, 0, a_2, 0, \ldots;$

(b) $1, a_0, a_1, a_2, \ldots;$

(c) $a_0, -a_1, a_2, -a_3, a_4, \ldots$.

10. Show that the generating function of the sequence $a_n = n$, $n \geq 0$, is $f(x) = \dfrac{x}{(1-x)^2}$.

11. Find the generating function for the sequence $b_n = n^2$, $n \geq 0$.
(Hint: differentiation.)

12. Find the sequences generated by the following functions:

(a) $f(x) = \dfrac{x^3}{1+x}$;

(b) $g(x) = \dfrac{x}{1 - 7x + 12x^2}$;

(c) $h(x) = \dfrac{x^7}{2 - x^7}$;

(d) $k(x) = e^{-2x}$.

13. Let a_n be the number of n-letter words formed from the 26 letters of the alphabet, in which the five vowels A, E, I, O, U together occur an even number of times. (By a word we mean simply any string of letters.) For example, when $n = 8$, such a word is APQIITOW since four (an even number) of the positions contain vowels.

(a) Show that $a_1 = 21$ and that, for $n \geq 2$, we have

$$a_n = 16a_{n-1} + 5 \cdot 26^{n-1}.$$

What would you say that a_0 is?

(b) Find the generating function for the sequence $a_0, a_1, \ldots$.

(c) Use this generating function to find a closed form expression for a_n.

14. The language of Verwegistan has words consisting of the letters A,E,O,U,B,P, and X. Words are formed according to the following rules: the vowels (A,E,O,U) always appear in pairs of the form AA, EE, OO, or UU, and they appear in a word before all non-vowels (if any). For instance, AAEEPXP and AAAA are words, but UUUB, AAXBAAX, and AEXX are not.

Let a_n denote the number of words of length n in the language.

(a) Show that $a_0 = 1$, $a_1 = 3$, and

$$a_n = 4a_{n-2} + 3^n, \quad \text{for } n \geq 2.$$

(b) Let $f(x)$ be the generating function of the sequence $a_{0,1}, \ldots$. Show that $f(x) = \dfrac{1}{(1-3x)(1-4x^2)}$.

(c) Use this generating function to find a general expression for a_n.

15. Let $f(x)$ be the generating function for the sequence $a_0, a_1, \ldots$.

Find the sequences whose generating functions are $(1+x)f(x)$ and $\dfrac{f(x)}{1-4x^2}$.

16. Let d_n denote the number of selections of n letters from $\{a, b, c\}$, with repetitions allowed, in which the letter a is selected an even number of times. (Note that these selections are unordered.)

(a) Show that the total number of unordered selections of n letters from $\{a, b, c\}$ with repetitions allowed is $\binom{n+2}{2}$.

(b) Use the result in (a) to prove that for $n \geq 2$,

$$d_n = \binom{n+2}{2} - d_{n-1} = \tfrac{1}{2}(n+2)(n+1) - d_{n-1}.$$

(c) Show that the sequence $d_0, d_1, \ldots$ has the generating function

$$f(x) = \frac{1}{(1-x^2)(1-x)^2} = \frac{1}{(1+x)(1-x)^3}.$$

(d) Use this generating function to prove that

$$d_n = \begin{cases} \frac{1}{4}(n+2)^2, & \text{if } n \text{ is even;} \\ \frac{1}{4}(n+1)(n+3), & \text{if } n \text{ is odd.} \end{cases}$$

17. (a) Suppose we roll a six-sided die (English: one die, many dice... even if not all native speakers know this). Let d_n be the number of possible ways to roll a die so that the outcome is n.

Explain why the generating function of the sequence $d_0, d_1, \ldots$ is given by

$$f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6.$$

(b) Suppose that we roll four (distinguishable) dice. Let a_n be the number of throws such that the sum of outcomes is equal to n.

Explain why the generating function of the sequence $a_0, a_1, \ldots$ is given by

$$g(x) \;=\; (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

(c) Now let b_n be the number of throws with any number of (distinguishable) dice such that the sum of outcomes is equal to n.

Explain why the generating function of the sequence $b_0, b_1, \ldots$ is given by

$$h(x) \;=\; \sum_{n=0}^{\infty}(x + x^2 + x^3 + x^4 + x^5 + x^6)^n.$$

Prove that $h(x) = (1 - x - x^2 - x^3 - x^4 - x^5 - x^6)^{-1}$.

18. The British coin system has 1p, 2p, 5p, 10p, 20p, 50p, £1 = 100p, and £2 = 200p coins.

(a) Let a_n count the number of different ways that you can pay a sum of n pennies with 10p coins.

Find the generating function of the sequence $a_0, a_1, \ldots$.

(b) Let b_n count the number of different ways that you can pay a sum of n pennies with 1p and 2p coins.

Find the generating function of the sequence $b_0, b_1, \ldots$ and determine the closed formula for b_n.

(c) Let c_n count the number of different ways that you can pay a sum of n pennies.

Find the generating function of the sequence $c_0, c_1, \ldots$.