MA210 - Class 8

EXERCISE 1 Calculate the # of labelled trees on $[n]$, where vertex 1 has degree 2 and vertex 2 has degree 3.

⇝ <u>IDEA</u> ① Prüfer codes ⟷ labelled trees. $(|\text{trees}| \leftrightarrow |\text{Prüfer codes}|)$

② $v \in T$ has degree $k \iff v$ appears EXACTLY $k-1$ times in Prüfer (T)

Hence, 1 has degree 2 in T iff 1 appears one in Prüfer (T).

$\left|\{T : d(1)=2 \wedge d(2)=3\}\right| = \left|\{\text{Prü. codes} : 1 \text{ appears once and } 2 \text{ appears twice}\}\right|$

$$= \underbrace{(n-2)}_{1 \text{ appears once}} \underbrace{\binom{n-2-1}{2}}_{2 \text{ appears twice}} \underbrace{(n-2)^{n-2-3}}_{\text{Freedom in the next } (n-2-3) \text{ vertices}}$$

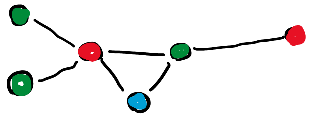
Rem In order to prove $\chi(G)=k$ we need to prove two things:

$\chi(G)=k$

→ $\chi(G) \le k$ We can colour G using k colours

→ $\chi(G) \ge k$ Any proper colouring uses at least k colours.

EXA



The colouring shows $\chi(G) \le 3$. $K_3 \subseteq G$ shows $\chi(G) \geq 3$.

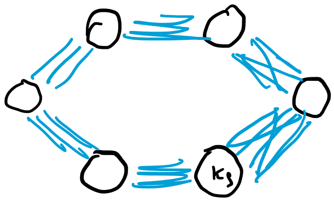
Q Can $\omega(G)$ and $\alpha(G)$ help us on this?

EXERCISE 2 (Blow-up of the even cycle).

→ start with the cycle on $2k$ vertices

→ replace each vertex of the cycle with K_s.

⇝ two vertices in different cliques are adj. iff the two original vertices were adjacent



$V = V_1 \cup V_2 \cup \dots \cup V_{2k}$ ⇝ $|V| = 2ks$

$|E(G_{2k,s})| = (2k)\binom{s}{2} + 2k(s^2)$

Prove that $\chi(G) = 2s$

- $\chi(G) \geq 2s$. Indeed, we need s colours to colour V_1 and s <u>new</u> colours to colour V_2
- $\chi(G) \leq 2s$.

EXTRA EXERCISE Find $\underbrace{\alpha(G)}_{\to k}$ and $\underbrace{\omega(G)}_{\to 2s}$

⇝ <u>IDEA</u> ① if we take v, w, z from three different cliques, we have at least one missing edge.

② If we take A an indep. set, no two vertices can come from adj. clusters.

Question 2

(a) For $m, n \geq 3$, let $G_{m,n}$ be the graph formed in the following two steps. First take two cycles C_m and C_n, whose vertex sets are disjoint; then add edges from each vertex of C_m to each vertex of C_n.

We denote the vertex set of C_m by $X = \{x_1, x_2, \dots, x_m\}$, and the vertex set of C_n by $Y = \{y_1, y_2, \dots, y_n\}$.

DONE

(i) Make a sketch of $G_{4,3}$.

(ii) Formulate Euler's Theorem.

Use Euler's Theorem to decide the values of $m, n \geq 3$ for which $G_{m,n}$ has an Euler tour.

(iii) For what values of $m, n \geq 3$ does $G_{m,n}$ have a Hamilton cycle?

(iv) For what values of $m, n \geq 3$ is $G_{m,n}$ a bipartite graph?

(v) Determine the chromatic number $\chi(G_{m,n})$ of $G_{m,n}$ for all values of $m, n \geq 3$.

⇝ We need to colour X and Y independently. which means

$$\chi(G_{m,n}) = \chi(C_m) + \chi(C_n)$$

(b) (i) Determine and display by means of a sketch the tree with vertex set $\{1, 2, \dots, 8\}$ whose Prüfer code is $(5, 7, 2, 6, 8, 5)$.

(ii) For every integer $n \geq 3$, find the number of trees with vertex set $\{1, 2, \dots, n\}$ and with exactly $n - 2$ leaves.

b)

$$\underbrace{\binom{n}{2}}_{\text{ways to choose the non-leaves}} \underbrace{(2^{n-2} - 2)}_{\text{\# of two digit seq. of length } n-2}$$

exactly two