A Glimpse of Young's Tableaux

From a book by R.P. Stanley

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Abstract

In this report we study methods to count the size of certain sets of walks in the Hasse diagram of the Young's lattice Y. For any walk in these sets we suppose fixed the starting point $\emptyset$ and, for a generic $\lambda \in Y$, the endpoint λ. We add some other restrictions, indeed we firstly count the Hasse walks which follow a given "pattern" or, as we say, of a given type; then we count walks of a given length between the two aforementioned vertices.

1 Introduction

1.1 Young's Lattice and Hasse walks

Let $N \in \mathbb{N}$ be a non-negative integer, $\lambda \in \mathbb{N}^{\mathbb{N}}$ is called a partition of N if $\lambda_0 \geq \lambda_1 \geq \dots$ and if we have $\sum_{k\geq 0} \lambda_k = N$ (we also write $N = |\lambda|$). We denote by Y_N the set of all partitions of N, which is a finite subset of $\mathbb{N}^{\mathbb{N}}$ (it is a subset of $[N]^{[N]}$). We call the disjoint union of such sets the Young's lattice of partitions $Y = Y_1 \uplus Y_2 \uplus \dots$, which is a countable set, moreover we sometimes call Y_N the N-th level of Y. The "natural" partial order over $\mathbb{N}^{\mathbb{N}}$ (the one for which $(a_1, a_2, \dots) \geq (b_1, b_2, \dots)$ if and only if $a_1 \geq b_1,\ a_2 \geq b_2, \dots$), endows Y with an induced "natural" partial order.

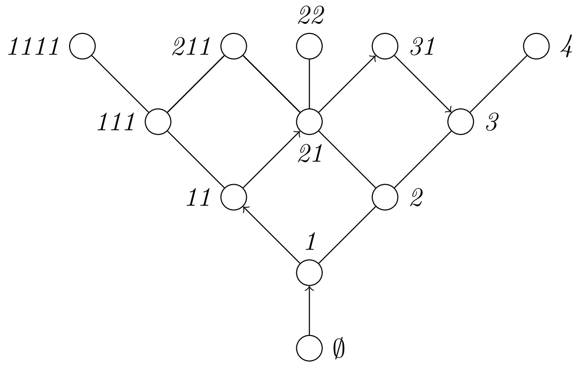
Let P be a partially ordered set with an infimum element $\bar{\mu}$, the Hasse diagram of P is the graph G_P defined as follows: the vertices of G_P are the elements of P ($V(G_P) = P$) and $(x, y) \in E(G_P)$ if $x \lessdot y$ (notation for: y covers x, i.e. $x < y$ and $\nexists z \in P$ s.t. $x < z < y$). We always follow the consolidated convention, in the drawings of Hasse diagrams, of drawing x below y if $x < y$; with this convention, the Hasse diagram uniquely determines the poset that represents (without it we would need to direct the edges of G_P, but it is of no concern for us). In particular we have that $x < y$ if and only if x is drawn below y and there is an ascending walk from x to y. Moreover, in the drawings of Y we usually draw at the same height elements of the same level Y_N.

It is time to notice that, if $x \lessdot y$ in Y, we have $y \in Y_{|x|+1}$ (indeed because $x < y$ we know $x_k \leq y_k$ and hence $\sum x_i \leq \sum y_i \,(< \infty)$; if $(\sum x_i) + 2 \leq \sum y_i$ we can find a $z \in Y$ with $x < z < y$ just considering the two possible cases). In particular, if $(x, y) \in E(G_Y)$, then $||x| - |y|| = 1$; hence there are no edges in $E(G_Y)$ between levels with index of distance 2 or more, or between two elements at the same level. Another remark to keep in mind is that Hasse diagrams are always simple graphs (there are no loops because $x \not\lessdot x$ and there are no multiple edges by definition), this allows us to represent Hasse walks simply as sequences of vertices.

1.2 Type of a walk and λ-valid words

Let $w = \lambda^0, \dots, \lambda^k$ be a Hasse walk on Y, we say that the i-th step of this walk is of type D (for "down") if $|\lambda^{i-1}| = |\lambda^i| + 1$, we say that the i-th step of w is of type U (for "up") if $|\lambda^{i-1}| = |\lambda^i| - 1$. More generally, we say that the walk w is of type $A = A_n A_{n-1} \dots A_1$, where each $A_i \in \{D, U\}$ (the notation from right to left is used because we shall consider D, U as linear maps, and A is what we call a finite word of length k in the alphabet $\{D, U\}$).

Example 1. *We present now a planar drawning of the Hasse diagram of the Young's lattice, and an example of Hasse walk with the study of its type.*



As we said, because Hasse diagrams are simple graphs, walks are determined by the ordered list of vectors they pass through. Therefore the walk in the figure is the walk $w = \emptyset, 1, 11, 21, 31, 3$*, which is of type* $DUUUU$ *or simply* DU^4*.*

We are now ready to pursue our first goal. We want to determine, for a fixed $\lambda \in Y$ and a given word A in the alphabet $\{D, U\}$, the number of Hasse walks of type A from $\emptyset$ to λ. Let's denote with $\alpha(A, \lambda)$ such number. Moreover, for the particular case in which $A = U^N$, we denote with f^λ the number $\alpha(U^{|\lambda|}, \lambda)$.

Remark. R1) For some general $A \in Seq(\{U, D\})$, $\lambda \in Y$, we may have $\alpha(A, \lambda) = 0$ (it is for example easy to see that any word having D to its foremost right position has that fate, independently from λ, because there are no walks from $\emptyset$ which start with a D-type step). If $\alpha(A, \lambda) \neq 0$ we say that A is a valid λ-word. In particular, for any $\lambda \in Y$, the only valid λ-word of the form U^N is the one in which $N = |\lambda|$.

Fix $\lambda \in Y$, a first natural step towards our goal is to find a way to determine whether a given word A is λ-valid or not. Such a method would allow us to focus our counting only on those words for which $\alpha(A, \lambda)$ is non-trivial. As the next result shows, the λ-validity of the word A, only depends on A and on $|\lambda|$.

Theorem 1. *Let* $A = D^{s_k} U^{r_k} \dots D^{s_2} U^{r_1} D^{s_1} U^{r_1}$ *be the standard expression of the word* A *(i.e.* $r_i, s_i \neq 0$ *with the possible exception of* r_1, s_k*), and let* $\lambda \in Y_N$*. There exists at least one Hasse walk of type* A *from* $\emptyset$ *to* λ *(i.e.* $\alpha(A, \lambda) \neq 0$*) if and only if:*

$$\sum_{i=1}^{k} (r_i - s_i) = N \qquad\qquad \forall 1 \le j \le k,\ \sum_{i=1}^{j} (r_i - s_i) \ge 0$$

This theorem is an important step towards our goal, because it allows us to say exactly for which A, λ we have $\alpha(A, \lambda) = 0$ (answering our question in countable cases). But it is still far from the result we would like to obtain, and it does not seem to help us to count walks (it is an existence result and a priori it gives no way of enumerating walks of a given type). In synthesis we lack a method for counting.

Remark. R2) $\alpha(A,\lambda)$ depends on λ, and not only on $|\lambda|$. As a nice example of this we can think of $\alpha(U^3, 111) = 1 \neq 2 = \alpha(U^3, 21)$. Moreover, there is no bound of the difference $\alpha(A,\lambda_1) - \alpha(A,\lambda_2)$ with both λ_1, λ_2 in the same level. For example it is not difficult to see that $\forall N \in \mathbb{N}$, $\exists x \in Y_N$ such that $\alpha(U^N, x) = f^x = 1$ (just consider the element $1\dots1 \in Y_N$), but also $\forall N \in \mathbb{N}$, $\exists y \in Y_N$ such that $\alpha(U^N, y) = f^y \geq N$ (just consider the element $21\dots1 \in Y_N$).

This last remark should point out the difficulty of what we are trying to do. Indeed any counting of $\alpha(A,\lambda)$ has to give full account of these behaviours.

1.3 How to use linear algebra to count walks

We want to give here the idea behind such a method, before stating our main theorem. Fix $y \in Y_i$ and a word W in the alphabet $\{U, D\}$, let $x_1, \dots, x_n \in Y_{i-1}$ be such that $(x_k, y) \in E(G_Y)$ and for no other $z \in Y_{i-1}$ we have $(z, y) \in E(G_Y)$; moreover suppose we know, for each $k \in \{1, \dots, n\}$, how many walks of type W there are between $\emptyset$ and x_k. Then the number of walks between $\emptyset$ and y of type UW is:

$$\alpha(UW, y) = \sum_{k=1}^{n} \alpha(W, x_k)$$

Alternatively, what we can write is:

$$\alpha(UW, y) = \sum_{x \in Y_{i-1}} \alpha(W, x)\mathbb{1}_{x \lessdot y} = \sum_{x \lessdot y} \alpha(W, x)$$

Therefore it seems useful to define a linear map U_{i-1} (or simply U, as we usually write when the index can be understood from the context) between $\mathbb{R}Y_{i-1}$ (the real vector space with basis Y_{i-1}) and $\mathbb{R}Y_i$ such that $[U]_{x,y} = \mathbb{1}_{x \lessdot y}$ (for strictly negative indices U is formally defined to be the null map between the proper spaces). This allows us to write:

$$\alpha(UW, y) = \sum_{x \lessdot y} \alpha(W, x) = \left[U \left(\sum_{x \in Y_{i-1}} \alpha(W, x)x \right) \right]_y$$

This method allows us to use linear algebra to inductively count the number of ascending walks. So let's define a similar linear map for descending walks, indeed let D_{i+1} (or simply D) denote the linear map between $\mathbb{R}Y_{i+1}$ and $\mathbb{R}Y_i$ which is characterized by its sending an element $y \in Y_{i+1}$ in the sum of the elements of Y_i which are smaller than y (for indices not strictly positive we consider D to formally be the null map between the proper spaces). Equivalently D is the linear map between $\mathbb{R}Y_{i+1}$ and $\mathbb{R}Y_i$ which is determined by the matrix $[D]_{y,x} = \mathbb{1}_{x \lessdot y}$. Now fix $x \in Y_i$ and let $y_1, \dots, y_n$ be the set of all elements of Y_{i+1} greater than x. Moreover let W be a words and suppose to know the value of $\alpha(W, y_i)$ for any i. By the same reasoning we did above we can write:

$$\begin{aligned} \alpha(DW, x) &= \sum_{k=1}^{n} \alpha(W, y_n) = \sum_{y \gtrdot x} \alpha(W, y) \\ &= \left[D \left(\sum_{y \in Y_{i+1}} \alpha(W, y)y \right) \right]_x \end{aligned}$$

By linearity of the maps D, U and the operator of taking coefficients (the operator $[\cdot]_\lambda$) and by using induction we conclude that:

$$\alpha(A, \lambda) = [A(\emptyset)]_\lambda = [A]_{\emptyset,\lambda}$$

Remark. R3) It may seem that we abused the notation by using the same writing for words and combinations of linear maps. We could always, without fear of mistaking, distinguish the two cases, but there is no need for that. In what follows, as we did above, we are going to identify a word with the associated composition of operators that it represents, any ambiguity should be cleared by the context.

At this point we are ready to appreciate our main result.

Theorem 2. *Let $\lambda \in Y$ and A a valid λ-word of length n. For a given $i \leq n$, let a_i be the number of D's in A to the right of A_i, and b_i the number of U's in A to the right of A_i ($a_i - b_i$ is the level of the endpoint of any walk of type $A_{i-1} \dots A_1$ starting at $\emptyset$). Then we have:*

$$\alpha(A, \lambda) = f^\lambda \prod_{i : A_i = D} (b_i - a_i)$$

Other than being a complete answer to our problem, the aforementioned theorem also allow us to get one interesting collateral result.

Corollary 1. *Given $N \in \mathbb{N}$,*

$$N! = \alpha(D^N U^N, \emptyset) = \sum_{|\lambda| = N} (f^\lambda)^2$$

Proof. The first equality follows immediately from Theorem 2, the second one follows from considering what it means to choose a walks of type $D^N U^N$ with maximal point λ in terms of f^λ for any λ in the level Y_N. □

In the Appendix we are going to present, for this last Corollary, an alternative proof, which represents a nice example of double counting and which allows us to introduce the well known concept of Young Tableaux.

1.4 Counting walks of a given length

We successfully counted the walks starting from $\emptyset$ and with fixed endpoint and type. Another interesting question is: for a fixed $\ell \in \mathbb{N}$ and $\lambda \in Y$, how many walks of length ℓ there are between $\emptyset$ and λ? Let $\beta(\ell, \lambda)$ denote the number of such walks. There are some trivial cases, for example we can already note that it holds:

$$\beta(|\lambda|, \lambda) = \alpha(U^{|\lambda|}, \lambda) = f^\lambda$$

Even thought more general relations between $\alpha(A, \lambda), f^\lambda$ and $\beta(\ell, \lambda)$ are not easy to find, as before we can convince ourselves that there are certain cases in which $\beta(\ell, \lambda)$ takes trivial values. In this direction we point out the two following facts, which we are going to use in the second part of this report.

Remark. R4) Let λ, ℓ as above, if $\ell \not\equiv |\lambda| \mod 2$, then $\beta(\ell, \lambda) = 0$. This holds because at each step of our walk we change the parity of the index of the level in which we are in. Therefore if we start from $\emptyset$ and we make ℓ steps in the Hasse diagram of Y, we arrive in a level with an index of the same parity of ℓ.

R5) In the language of linear algebra we used above, let's consider the formal polynomial $(D+U)^\ell$. Then $\beta(\ell,\lambda)$ is the coefficient of λ in the standard expression of $(D+U)^\ell(\emptyset)$ as linear combination of partitions. Indeed, if $\mathcal{A}_\ell$ denotes the set of words in U, D of length ℓ, it holds:

$$\beta(\ell,\lambda) = \sum_{A\in\mathcal{A}_\ell} \alpha(A,\lambda)$$

Consider now that $\mathcal{A}_\ell$ is also the natural basis in which we can express $(D+U)^\ell$ as an explicit linear combination of words. Indeed we can convince ourselves that it holds: $(D+U)^\ell = \sum_{A\in\mathcal{A}_\ell} A$ (the product of linear maps is not commutative). And as we said before theorem 2, $\alpha(A,\lambda)$ equals the coefficient of λ in the standard expression of $A(\emptyset)$ as linear combination of partitions. Therefore:

$$\beta(\ell,\lambda) = \sum_{A\in\mathcal{A}_\ell} \alpha(A,\lambda) = \sum_{A\in\mathcal{A}_\ell} [A(\emptyset)]_\lambda = \left[(D+U)^\ell(\emptyset)\right]_\lambda$$

Starting from this second remark we just made, we can prove the following:

Theorem 3. *Fix $N\in\mathbb{N}$ and $\lambda\in Y_N$, let $\ell\geq N$ such that $\ell\equiv N \mod 2$. Then:*

$$\beta(\ell,\lambda) = \binom{\ell}{N}(1\cdot 3\cdot 5\cdot\ldots\cdot(\ell-N-1))f^\lambda$$

In particular using this theorem we are able to prove:

Corollary 2. *The total number of Hasse walks of Y of length $2N$ from $\emptyset$ to itself is:*

$$\beta(2N,\emptyset) = 1\cdot 3\cdot 5\cdot\ldots\cdot(2N-1)$$

1.5 Final remark

Our results seem partial ones, because we gave no insight on how to calculate f^λ. Therefore, for what we have said and for the nature of our results, it may seem that we are only changing the problem at hand. A complete formula for f^λ does exist and is known as the Hook Length Formula. But we are not really interested in such formula for two reasons: first of all it would represent a major deviation from our path (we would need to introduce some non-trivial objects), secondly, the actual knowledge of such formula does not add much to the study of the methods we present to count walks.

2 Proofs of the theorems

2.1 Proof of theorem 1

Theorem 1. *Let $A = D^{s_k}U^{r_k}\ldots D^{s_2}U^{r_1}D^{s_1}U^{r_1}$ be the standard expression of the word A (i.e. $r_i, s_i\neq 0$ with the possible exception of r_1, s_k), and let $\lambda\in Y_N$. There exists at least one Hasse walk of type A from $\emptyset$ to λ (i.e. $\alpha(A,\lambda)\neq 0$) if and only if:*

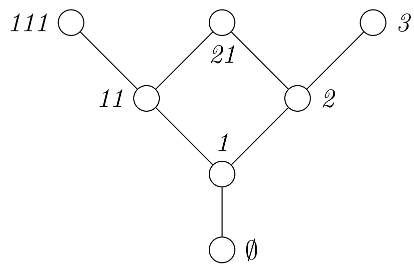
$$\sum_{i=1}^{k}(r_i - s_i) = N \qquad\qquad \forall 1\leq j\leq k,\ \sum_{i=1}^{j}(r_i-s_i)\geq 0 \tag{1}$$

Proof. We fist prove that (1) is a necessary condition. Let $w = (\emptyset = \lambda^0), \dots, \lambda^K$ be a walk of type A starting from $\emptyset$. We may notice, because w is of type A, that if λ^i is at level j and $A_{i+1} = U$, then λ^{i+1} is at level $j+1$; on the other hand, if λ^i is at level j and $A_{i+1} = D$, then λ^{i+1} is at level $j-1$. Therefore, by induction, the endpoint of such a walk w is on level $\sum_{i=1}^{k}(r_i - s_i)$ (recall that the starting point is on level 0). Moreover there are no elements in levels with negative index, and hence any word A which would imply for our walk to pass trough levels with negative index is immediately a non-valid λ-word, for any $\lambda \in Y$. This means that it is a necessary condition that $\forall 1 \leq j \leq k,\ \sum_{i=1}^{j}(r_i - s_i) \geq 0$.

Let $\lambda = (\lambda_1, \dots, \lambda_L), A$ be as in the statement of the lemma. We want to show that there exists a walk from $\emptyset$ to λ. Consider the chain (totally ordered subset of Y) C formed by the elements of the form $(\lambda_1, \dots, \lambda_i, \rho, 0, \dots)$ where $\rho \leq \lambda_{i+1}$ $(0 \leq i < L)$, and by elements of the form $(\lambda_1, \dots, \lambda_L, 1, \dots, 1, 0, \dots)$. It should be clear that there exists a (unique) walk between $\emptyset$ and λ of type A which uses only vertices from C (C is poset-isomorphic to $\mathbb{N}$). □

2.2 Proof of theorem 2

Example 2. *We study one example of how the linear method presented in the first section can be used to count walks in the Hasse diagram of Y.*



Consider the vector spaces $\mathbb{R}Y_0, \dots, \mathbb{R}Y_3$ and the aforementioned linear maps $U_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i+1}$ between them (usually we denote them just by U, forgetting the index). We want to study the image of the vector $\emptyset$ under $U^3 = U_2U_1U_0 : \mathbb{R}Y_0 \to \mathbb{R}Y_3$. As we said before, what we obtain is:

$$U^3 : \emptyset \mapsto 1 \cdot 111 + 2 \cdot 21 + 1 \cdot 3 = \alpha(U^3, 111) \cdot 111 + \alpha(U^3, 21) \cdot 21 + \alpha(U^3, 3) \cdot 3$$

As claimed, counting ascending walks between $\emptyset$ and λ is equivalent to count the coefficient of λ in the writing of $U^{|\lambda|}(\emptyset)$ as linear combination of partitions.

Before proceeding to prove theorem 2 we need some other results.

Lemma 1. *Fix $i \in \mathbb{N}^+$, let $\mathbb{R}Y_i$ denote the real vector space with basis Y_i; $U_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i+1}$ the linear map which is defined on the basis of Y_i as $U_i(\lambda) = \sum_{\mu > \lambda} \mu$ for every $\lambda \in Y_i$ and $D_i : \mathbb{R}Y_i \to \mathbb{R}Y_{i-1}$ the linear map which is defined on the basis of Y_i as $D_i(\lambda) = \sum_{\mu < \lambda} \mu$ for every $\lambda \in Y_i$. Then*

$$D_{i+1}U_i - U_{i-1}D_i = Id_i \tag{2}$$

Proof. Let us consider the left hand side of equation (2), and for sake of simplicity let us denote by L_i the linear map $D_{i+1}U_i - U_{i-1}D_i$. Hence by definition L_i is a linear map between $\mathbb{R}Y_i$ and itself, and therefore it is completely determined by the image of each $\lambda \in Y_i$. This means that if we can compute, for any given $\mu, \lambda \in Y_i$, the coefficient $[L_i(\lambda)]_\mu$ of μ in $L_i(\lambda)$, we are able to determine L_i by linearity. The possible cases are the following ones:

- μ, λ are distinct and exists unique $\{l, j\} \subseteq \mathbb{N}$ distinct indices such that $\mu_l = \lambda_l \pm 1,\ \mu_j = \lambda_j \pm 1$ and for $k \neq l, j$ we have $\mu_k = \lambda_k$. Consider now $\nu \in Y_{i+1}$, if $\nu > \lambda$ we have that ν, λ differ in at most one component and exactly by 1; i.e. if $\nu > \lambda$ we have $\nu = (\lambda_1, \dots, \lambda_{l-1}, \lambda_l + 1, \lambda_{l+1}, \dots)$. The same argument can be used for those $\xi \in Y_i$ such that $\xi < \nu$. We are now able to say $[D_{i+1}U_i(\lambda)]_\mu = 1$: we have exactly one way of obtaining μ from λ using $D_{i+1}U_i$. The same reasoning can be done for $[U_{i-1}D_i(\lambda)]_\mu$. The result is $[L_i(\lambda)]_\mu = 0$.

- μ, λ are distinct but we are not in the previous condition. By what we said above on the set of elements $\nu \in Y_{i+1}$ such that $\nu > \lambda$ we can conclude that $[D_{i+1}U_i(\lambda)]_\mu = 0$, and the same holds for $[U_{i-1}D_i(\lambda)]_\mu$.

- μ, λ are the same element. The coefficient of λ in $L_i(\lambda)$ is equal to: number of $\nu \in Y_{i+1}$ such that $\nu > \lambda$, minus the number of elements in Y_{i-1} smaller than λ. But take $\nu = (\lambda_1, \dots, \lambda_{l-1}, \lambda_l + 1, \lambda_{l+1}, \dots)$ in the first set; then, if $\lambda_l > 0$, there exists exactly one permutation of the elements of $(\lambda_1, \dots, \lambda_{l-1}, \lambda_l - 1, \lambda_{l+1}, \dots)$ such that the resulting vector is a partition; moreover the resulting partition is in Y_{i-1} and λ is strictly bigger than it. In this way we can actually reach every $\xi \in Y_{i-1}$ such that $\xi < \lambda$. And therefore we have just one element of the first set which we did not consider: the element in which we add 1 to a 0 entry.

By what we just said we conclude that $D_{i+1}U_i - U_{i-1}D_i(\lambda) = \lambda$ holds for every $\lambda \in Y_i$. $\square$

We need one last lemma to prove our main theorem.

Lemma 2. *For any $N \in \mathbb{N}^+$ the set $\left\{U^{i+N}D^i : i \in \mathbb{N}\right\}$ is a linearly independent set. By this we mean that if*

$$\sum_{i \in \mathbb{N}} d_i U^{i+N} D^i = \sum_{i \in \mathbb{N}} f_i U^{i+N} D^i$$

is an equality of linear maps between $\mathbb{R}Y_K$ and $\mathbb{R}Y_{K+N}$ for any generic K, then $d_i = f_i$ for every i.

Proof. Suppose that for any $K \in \mathbb{N}$, $\sum_{i \in \mathbb{N}} d_i U^{i+N} D^i = 0$ as a linear map. Let L be the least index for which $d_L \neq 0$ and let $\mu \in Y_L$. Because $D^{L+h}(\mu) = 0$ for any strictly positive $h \in \mathbb{N}$, we have that, for any such μ,

$$\sum_{i \in \mathbb{N}} d_i U^{i+N} D^i(\mu) = \sum_{i \leq L} d_i U^{N+i} D^i(\mu) = d_L U^{N+L} D^L(\mu) = 0$$

Because $U^{N+L}D^L$ is not the null map from $\mathbb{R}Y_L$ to $\mathbb{R}Y_{L+N}$ we have that d_L has to be 0. $\square$

We are now ready to prove our main result.

Theorem 2. *Let $\lambda \in Y$ and A a valid λ-word of length n. For a given $i \leq n$, let a_i be the number of D's in A to the right of A_i, and b_i the number of U's in A to the right of A_i ($a_i - b_i$ is the level of the endpoint of any walk of type $A_{i-1} \dots A_1$). Then we have:*

$$\alpha(A, \lambda) = f^\lambda \prod_{i : A_i = D} (b_i - a_i)$$

Proof. We have to keep in mind that our goal is to study the coefficient of λ in the expansion of $A(\emptyset)$ as sum of partitions. The first thing we do is to express A as a linear combination $\left\{U^{i+N}D^i : i \in \mathbb{N}\right\}$. Recall that we proved that the equality $DU = UD + I$ holds at every level. And therefore we can replace any DU in A with $UD + I$ considering A as a linear map. Here it is an example:

$$DU^3D = UDU^2D + U^2D = U^2DUD + 2U^2D = U^3D^2 + 3U^2D$$

We can iterate our procedure, until we can express the linear map A as a sum of linear maps for which there are no U's to the right of any D's, or equivalently until we have something of the form: $A = \sum_{i,j} d_{i,j}(A)U^iD^j$. At most finitely many $d_{i,j}(A)$ are different from 0. We can say more: the difference between the number of U's and the number of D's in the components of A as a sum has to be N (in the above example DU^3D we had that this difference is 1 as it is in U^3D^2 and U^2D). This is because at each step we preserve this difference (equivalently considering the map A as a map between $\mathbb{R}Y_K$ and $\mathbb{R}Y_{K+N}$, we are writing it as a non null sum, therefore each component has to be a map between the same spaces). Therefore we can write:

$$A = \sum_{i \geq 0} r_i^N(A)U^{i+N}D^i$$

Let's study a general case: take W to be a generic word in U, D such that it represents a map between $\mathbb{R}Y_\ell$ and $\mathbb{R}Y_{\ell+M}$. For what we said above we can write:

$$W = \sum_{i \geq 0} r_i^M(W)U^{i+M}D^i$$

The following holds:

- UW is a map between $\mathbb{R}Y_\ell$ and $\mathbb{R}Y_{\ell+M+1}$, and we have:

  $$UW = \sum_{i \geq 0} r_i^M(W)U^{i+1+M}D^i$$

  which clearly implies $r_i^{M+1}(UW) = r_i^M(W)$. In particular we get: $r_0^{M+1}(UA) = r_0^M(W)$.

- DW is a map between $\mathbb{R}Y_\ell$ and $\mathbb{R}Y_{\ell+M-1}$, and we have (by induction on i):

  $$\begin{aligned} DW &= \sum_{i \geq 0} r_i^M(W)DU^{i+M}D^i \\ &= \sum_{i \geq 0} r_i^M(W)(U^{i+M}D + (i+M)U^{i+M-1})D^i \end{aligned}$$

  From which we obtain:

  $$r_i^{M-1}(DW) = r_{i-1}^M(W) + (i + M + 1)r_i^M(W)$$

  In particular we get: $r_0^{M-1}(DA) = (M + 1)r_0^M(A)$.

Now we can apply this inductive construction in the case of our theorem. Because we are interested in the coefficient of λ in $A(\emptyset)$, and because $D(\emptyset) = 0$, we have that:

$$A(\emptyset) = \sum_{i \geq 0} r_i^N(A)U^{i+N}D^i(\emptyset) = r_0^N(A)U^N(\emptyset)$$

Therefore the result we wanted is given by:

$$\alpha(A, \lambda) = [A(\emptyset)]_\lambda = r_0^N(A)[U^N(\emptyset)]_\lambda = r_0^N(A)f^\lambda$$

By a simple inductive reasoning and using the base case formulas for $r_0^{M-1}(DA)$ and $r_0^{M+1}(UA)$ we get that $r_0^N(A) = \prod_{i:A_i=D}(b_i - a_i)$. $\square$

2.3 Proof of theorem 3

Considering what we did to prove Theorem 2, and our previous remark R9, we know that for any fixed ℓ, it is well defined:

$$(D+U)^\ell = \sum_{i,j} b_{i,j}(\ell) U^i D^j$$

Moreover we know we are interested in the coefficients of $(D+U)^\ell(\emptyset)$, therefore we can use that $D(\emptyset) = 0$ to impose $j = 0$ and write (denoting $b_{i,0}(\ell)$ by $b_i(\ell)$):

$$(D+U)^\ell(\emptyset) = \sum_{i,j} b_{i,j}(\ell) U^i D^j(\emptyset) = \sum_{i\geq 0} b_i(\ell) U^i(\emptyset)$$

To prove theorem 3 we need another result:

Lemma 3. *If $\ell - i$ is odd, $b_i(\ell) = 0$. If $\ell - i = 2m$ we have:*

$$b_i(\ell) = \frac{\ell!}{2^m i! m!}$$

Proof. The first case is equivalent to remark R8. Assume $\ell - i = 2m$. Because the case $\ell = 1$ it is easily verified, let's prove the result for $\ell + 1$ assuming its validity for ℓ. Consider (using an equality that we already saw does hold by induction):

$$\begin{aligned}\sum_{i\geq 0} b_i(\ell+1) U^i(\emptyset) = (D+U)^{\ell+1}(\emptyset) &= (D+U) \sum_{i\geq 0} b_i(\ell) U^i(\emptyset) \\ &= \sum_{i\geq 0} b_i(\ell)(DU^i + U^{i+1})(\emptyset) \\ &= \sum_{i\geq 0} b_i(\ell)(U^i D + iU^{i-1} + U^{i+1})(\emptyset) \\ &= \sum_{i\geq 0} b_i(\ell)(iU^{i-1} + U^{i+1})(\emptyset)\end{aligned}$$

And by uniqueness of the coefficients in the expression of one linear map we obtain:

$$b_i(\ell+1) = (i+1) b_{i+1}(\ell) + b_{i-1}(\ell)$$

Because the function $\frac{\ell!}{2^m i! m!}$ also satisfies this recurrence condition and it has the same initial condition, we can conclude. □

We are now ready to prove our theorem.

Theorem 3. *Let $\ell \geq N$ and $\lambda \in Y$ such that $|\lambda| = N$ and $\ell \equiv N \mod 2$. Then:*

$$\beta(\ell, \lambda) = \binom{\ell}{N} (1 \cdot 3 \cdot 5 \cdot \ldots \cdot (\ell - N - 1)) f^\lambda$$

Proof. By the above notation we have:

$$\begin{aligned}(D+U)^\ell(\emptyset) &= \sum_{i\geq 0} b_i(\ell) U^i(\emptyset) \\ &= \sum_{i\geq 0} b_i(\ell) \sum_{\lambda \in Y_i} f^\lambda \lambda\end{aligned}$$

Since our last lemma gives us a close formula for

$$b_i(\ell) = \binom{\ell}{i} (1 \cdot 3 \cdot \ldots \cdot (\ell - i - 1))$$

when $\ell - i$ is even, the proof follows from remark R9. □

3 Appendix: The RSK Algorithm

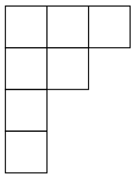
In the report we presented one interesting application of theorem 2:

Corollary 1. *Given $N \in \mathbb{N}$,*

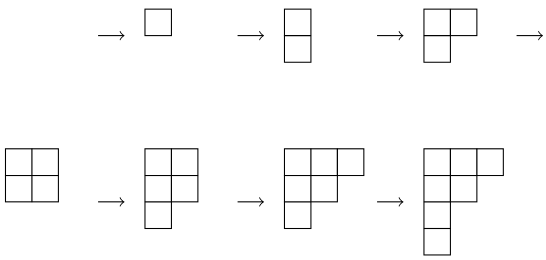
$$N! = \alpha(D^N U^N, \emptyset) = \sum_{|\lambda|=N} (f^\lambda)^2$$

There is another interesting proof of this result which uses a method completely different from the ones we studied. We give here such a proof because it is a nice example of double counting, and because it gives us the opportunity to present ascending walks starting from $\emptyset$ in the Hasse diagram of the Young's Lattice as Young's Tableaux.

Let's take as first example the element $(3, 2, 1, 1) \in Y_7$. We can represent it as a gathering of squares in the following way:



What we did is equivalent to select a subset T of $\mathbb{N} \times \mathbb{N}$ in the following way: if $(x_1, y_1) \in T$ then also $(x_2, y_2) \in T$ if $x_2 \leq x_1$ and $y_2 \leq y_1$. In the drawing of such a subset we prefer to draw squares instead of points because we are going to write numbers in them. Such a construction, often called Young's diagram of $(3, 2, 1, 1)$, can be carried out for any generic $\lambda \in Y$. Let's study how an ascending walk starting from $\emptyset$ looks like in terms of Young's diagrams. For example let's study the walk: $\emptyset, 1, 11, 21, 22, 221, 321, 3211$. We obtain something like:



It should be clear that any walk is completely determined by the ordered sequence of the Young's diagrams of the vertices it passes through. But in particular there is a more efficient way of presenting ascending walks. The previous walk, for example, is completely determined by the following writing:

1	3	6
2	4	
5		
7		

Such an object (a Young diagram of λ with numbers in it representing an ascending path from $\emptyset$ to λ) is called the Standard Young's Tableau (SYT) of the walk $\emptyset, 1, 11, 21, 22, 221, 321, 3211$, and 3211 is called shape of the SYT. The SYT can be constructed for any ascending walk starting from $\emptyset$, simply drawing the Young's diagram of the endpoint and inserting in each square the number corresponding to the order in which the square has been added (1 for the first square, 2 for the second and so on). Let's point out some relevant facts:

Remark. R6) Let consider a Young diagram of N squares with some numbers written inside its squares. If these numbers are $\{1, \ldots, N\}$ (and hence each one of them appearing exactly once) and if every row (from right to left) and every column (from up to down) contains numbers in an increasing order, then the drawing is a SYT for some ascending walk (which is completely determined by it).

R7) Given some squares ordered in lines in such a way that each row has fewer elements than the one above, and given some association of positive integers $\{1, \ldots, M\}$ to such squares, what we have is a SYT if and only if: each number appears in exactly one square and each row and column has numbers in ascending order. Independently from the shape of the SYT.

R8) From now on we can think of f^λ also as the number of valid SYT which have shape λ (we already said that it is not in our interest to explicitly study such a number, but it can be useful to have a different presentation of f^λ).

R9) The number of ascending walks from $\emptyset$ to λ is equal to the number of descending walks from λ to $\emptyset$, because we are actually counting the same walks. This equivalence is the central point of the equality $\alpha(D^N U^N, \emptyset) = \sum_{|\lambda|=N} (f^\lambda)^2$. But it also tells us that counting $\alpha(D^N U^N, \emptyset)$ is equivalent to count ordered pairs of Young's tableaux with N squares and with the same shape.

The idea behind our second proof of the Corollary is the same as R9. We are going to find a bijection between $\mathfrak{S}_N$ and ordered couples of SYT with N squares. The algorithm which gives us such a bijection is called RSK algorithm.

Let $\pi \in \mathfrak{S}_N$ be a fixed permutation, let's indicate with (C^π, D^π) the couple of SYD which is given by the RSK algorithm on π and let's denote with λ^π the shape of both C^π and D^π. There is not an easy way of telling λ^π from π. Therefore what we are going to do is to imagine two copies C and D of a countably infinite chessboard with just one angle (something isomorphic to $\mathbb{N}^+ \times \mathbb{N}^+$). As basis step we impose $C_{i,j}, D_{i,j} = 0$, at each step of our algorithm we add one of the numbers $\{1, \ldots, N\}$ in the squares $C_{i,j}$ and $D_{i,j}$ of such chessboards. When the algorithm stops we have two chessboards with non-null elements in squares which form the same shape ($C_{i,j} \neq 0$ if and only if $D_{i,j} \neq 0$). Restricting our attention to the non-null elements of C, D we get C^π and D^π (not only they have the same shape, but they both are SYT).

At its first step, the algorithm sets: $C_{1,1} = \pi(1)$, $D_{1,1} = 1$. Suppose the algorithm already processed $k-1$ steps. Consider $\pi(k)$, we are going to substitute an element of the first row of C with $\pi(k)$ in the only way in which the non-null entries of the first row are ordered in ascending order. Let s_1 be the substituted element of the first row. We repeat the same procedure with s_1 taking the place of $\pi(k)$ and starting from the second row of C. Recursively repeat this procedure. At the end of such a change we have just one more non-null entry of C (i.e. exists some unique

i, j such that $C_{i,j}$ passed from being 0 to be different from 0). Define $D_{i,j} = k$ for those i, j.

By remark R6, both C, D are SYT of some ascending walk in Y from $\emptyset$. By construction (because D has the same shape of C), the walks represented by C, D have the same endpoint.

Theorem 4. *The RSK algorithm defines a bijection between $\mathfrak{S}_N$ and the set of walks of type $D^N U^N$ from $\emptyset$ to itself.*

Proof. We mentioned how the set of walks of type $D^N U^N$ from $\emptyset$ to itself is in bijection with the set of ordered couples of SYT such that both elements of the couple have the same shape. The RSK algorithm we presented is a map between $\mathfrak{S}_N$ and the set of such couples. We can conclude simply showing an inverse of such an algorithm. The idea for such an inverse algorithm is the following: from Q we can understand which entry of C passed from null to not-null in the N-th step of the algorithm. Suppose the element K is in such entry of C, and in the row i_N. If $i_N = 1$ then $\pi(N) = K$, if not we can find which element of the $i_N - 1$-th row took the place of K in the N-th step of the RSK algorithm. Repeating such a procedure tells us the value of $\pi(N)$ (the element added in the first line at the N-th step). Repeat for $N - 1$ and so on. $\square$

References

[1] Richard P. Stanley. *Algebraic combinatorics.* Springer-Verlag, Second edition, 2018.