MA102 Mathematical Proof and Analysis
and
MA103 Introduction to Abstract Mathematics

Exercises 2

In the questions below you must take care to note whether the question deals with 'natural numbers' or 'integers'. Remember that a natural number is a positive integer; 'integers' includes zero and negative integers.

1 For each of the following pairs of sets A, B, determine whether the statement "$A \subseteq B$" is true or false, and also whether the statement "$A = B$" is true or false. In each case, make sure you explain why!

(a) $A = \{n \in \mathbb{N} \mid n \text{ is an odd integer}\}$, $B = \{n \in \mathbb{N} \mid n^2 \text{ is an odd integer}\}$;

(b) $A = \{n \in \mathbb{R} \mid n \text{ is an odd integer}\}$, $B = \{n \in \mathbb{R} \mid n^2 \text{ is an odd integer}\}$;

2 Consider the set $X = \{\, S \mid S \subseteq \{\, 0, 1 \,\} \,\}$. In words: X is the set consisting of those S which are subsets of $\{0, 1\}$.

(a) Exactly what does it mean to say that A is an *element* of X?
Exactly what does it mean to say that B is a *subset* of X?
(*If you are not sure, go back and look at the definitions of these terms.*)

(b) Is $\{0\}$ an *element* of X?

(c) Is $\{0\}$ a *subset* of X?

(d) Is there a set which is both an element of X and a subset of X?

Justify all your answers carefully.

3 Decide whether the following statement is true or false, and justify your assertion by means of a proof or a counterexample:
For all sets A, B and C, we have $A \cap (B \cup C) = (A \cap B) \cup C$.

4 The numbers 1 to 25 are arranged in a square array of five rows and five columns in an arbitrary way. The greatest number in each row is determined, and then the least number of these five is taken; call that number s. Next, the least number in each column is determined, and then the greatest number of these five is taken; this number is called t.

(a) Arrange the numbers 1 to 25 in a square array by writing $1, 2, 3, 4, 5$ in the first row, then $6, 7, 8, 9, 10$ in the second row, etc. Determine s and t for this arrangement. (You should have $s = t$.)

(b) Find a way to arrange the numbers 1 to 25 in a square array which leads to values s, t with $s \neq t$.

(c) Write out carefully a proof of the following statement:

For every possible arrangement in a square array of the numbers 1 to 25,
if we obtain s and t as described above, then we have $s \geq t$.