MA210 Discrete Mathematics

Notes 1: Counting

16 January and 23 January, 2023

Theorem, Lemma, Proposition, Fact, Definition

In mathematics, a *definition* is a clear formal statement of what some concept means. It doesn't need a proof. It might (sometimes!) need an explanation of why the formal statement corresponds to what you intuitively think of for the concept.

A *theorem* is a formal statement which requires a proof. It is supposed to be interesting on its own (not just useful to prove other things).

A *lemma* is a formal statement which requires a proof. You might or might not find it interesting on its own, but its primary reason for existence is to help prove other theorem(s). Whether a statement is a theorem or a lemma is rather dependent on the lecturer's opinion (or yours, if you're writing it).

In this course, we have two other types of statement, which are more or less just other ways to write theorem/lemma but which are supposed to indicate difficulty. A *fact* is a statement which should be obviously true as soon as you look at the definitions. We won't always give proofs of 'facts', but you should always check you can see how to prove them. A *proposition* needs a proof, but this is expected to be easy (the first thing you think of should work, and it should be obvious that it will work). We will reserve 'theorem' for a statement whose proof needs an idea.

You should not try to devote energy to remembering whether statements in the course are theorems, lemmas, propositions or facts, or try to remember the numbers (which are there only to make my life easier referring to them). You might want to remember the names. What I will care about when marking work is whether you can produce a correct formal statement for the theorem/lemma/proposition/fact (which doesn't have to be the one I give in the notes, it just has to be equivalent) or whether it is obvious from your work that you know what that statement is.

Authors: P. Allen, J. van den Heuvel, J. Skokan, R. Kirsch

Introduction to counting

In this course the *natural numbers* $\mathbb{N}$ means the set $\{1, 2, \dots\}$. A common shorthand is to write $[k]$ for the set $\{1, 2, \dots, k\}$.

Another (standard) notation we will often use is: If $f : X \to Y$ is a function, and Z is a subset of Y, then we write $f^{-1}(Z)$ for the set

$$\big\{x \in X : f(x) \in Z\big\} .$$

Often Z will be a singleton set: $Z = \{z\}$. Then we will write $f^{-1}(z)$ as shorthand for $f^{-1}\big(\{z\}\big)$. It is important to remember $f^{-1}(z)$ is a *set* contained in X not an element of X.

We will also, if f is a bijection (and hence invertible) write f^{-1} for the inverse function, in which case $f^{-1}(z)$ could also mean the single element $x \in X$ such that $f(x) = z$. Yes, this is inconsistent: you are expected to recognise which is meant from the context. This is (unfortunately) standard practice.

Definition 1.1 (Cardinality of sets)**.** *Let X and Y be sets.*
If there is an injection $f : X \to Y$, we say that X is not larger than *Y, and write $|X| \le |Y|$.*
If there is a surjection $f : X \to Y$, we say that X is not smaller than *Y, and write $|X| \ge |Y|$.*
If both exist (and so there is also a bijection $f : X \to Y$) we write $|X| = |Y|$.

In this course we are only interested in *finite* sets. For $n \ge 1$, we define

$$\big|\{1, 2, \dots, n\}\big| = n$$

and write $\big|\emptyset\big| = 0$. So $|X| = n$ if and only if there is a bijection $f : X \to \{1, 2, \dots, n\}$.

What we are interested in, for the next couple of weeks, is: how can we figure out how big a finite set is?

Basic counting techniques

Proposition 1.2 (Addition Rule (AR))**.** *If X and Y are disjoint, we have $\big|X \cup Y\big| = |X| + |Y|$.*

The following generalisation is often used (which we therefore also call Addition Rule).

Proposition 1.3 (Addition Rule (AR))**.** *If $X_1, \dots, X_t$ are pairwise disjoint sets, then*

$$\big|X_1 \cup \cdots \cup X_t\big| = |X_1| + \cdots + |X_t| .$$

Example 1.4. There are 2 first-year students, 5 second-year students, and 6 third-year students in a class. How many students are in the class?

Example 1.5. There are 3 French speakers and 9 German speakers in a room. How many people are there in total?

Proposition 1.6 (Multiplication Rule (MR)). *If a counting problem can be split into a number of stages, each of which involves choosing one of a number of options (not dependent on the previous choices), then the total number of possibilities can be found by multiplying together the number of options available at each stage.*

Formally: *Let X be any set, and let $t_1, \dots, t_k$ be natural numbers. Let Y be the set of vectors of the form $(a_1, \dots, a_k)$ where for each $1 \le i \le k$ the entry a_i is in the set $\{1, \dots, t_i\}$. (That is, $Y = [t_1] \times [t_2] \times \cdots \times [t_k]$.) Suppose that there is a bijection $f : Y \to X$. Then $|X| = |Y| = t_1 t_2 \cdots t_k$.*

Example 1.7. Given finite sets W and Z, how many functions $h : W \to Z$ are there? And how many of these functions are injections?

Proposition 1.8 (Multiple Counting (MC)). *If you have a method of counting elements of Y, which counts each element exactly k times, then you count to $k|Y|$.*

Formally: *Let k be a natural number, and let X and Y be sets. Suppose that we have a map $f : X \to Y$ with the following property. For each $y \in Y$, we have $f(x) = y$ for exactly k different $x \in X$ (equivalently, $\left|f^{-1}(y)\right| = k$). Then $|X| = k|Y|$.*

Example 1.9. Given a set of n people, how many un-ordered pairs of people are there?

Ordered selection

Basic problem

Given n distinct objects, how many ways are there to choose r (where $1 \le r \le n$) of these objects, where the order in which they are chosen is important, when

- repetition is allowed, or
- repetition is not allowed?

Definition 1.10 (Factorial, falling factorial). *For $n \ge 1$ we define $n!$ ('n factorial') to be $n! := n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$. We define $0! = 1$.*

Given $n \ge r \ge 1$, we define the 'falling factorial' $(n)_r := n \times (n-1) \times \cdots \times (n-r+1)$. We define $(n)_0 := 1$.

By definition, we have $(n)_r = \frac{n!}{(n-r)!}$.

Example 1.11. How many bijections $f : \{1, \dots, n\} \to \{1, \dots, n\}$ are there?

Example 1.12. How many 7-digit numbers are there (where digits are chosen from the set $\{0, \dots, 9\}$)? And how many of these have all digits distinct?

Binomial Coefficients: Unordered selection and the Binomial Theorem

Basic problem

Given n distinct objects, how many ways are there to choose r (where $1 \le r \le n$) of these objects, where the order in which they are chosen is irrelevant, when

- repetition is not allowed, or
- repetition is allowed?

Definition 1.13 (Binomial coefficient). *Given $n \geq 1$ and an integer r, we define $\binom{n}{r}$ ('n choose r') to be the number of subsets of $\{1, \ldots, n\}$ containing exactly r elements. We define $\binom{0}{0} = 1$.*

Fact 1.14. *For every integer $n \geq 0$ we have*

(1) $\binom{n}{0} = \binom{n}{n} = 1$,

(2) $\binom{n}{r} = 0$ *whenever* $r < 0$ *or* $r > n$, *and*

(3) $\binom{n}{r} = \binom{n}{n-r}$ *for every integer* r.

Theorem 1.15. *For every integer $n \geq 0$ and every integer r with $0 \leq r \leq n$, we have*

$$\binom{n}{r} = \frac{(n)_r}{r!} = \frac{n!}{(n-r)!r!}.$$

Example 1.16. A committee of 5 must be chosen from 15 faculty members and 60 students. In how many ways can this be done if:

(1) the committee must consist of 2 faculty members and 3 students?

(2) the committee should have at least 3 faculty members?

(3) the committee should have at least 2 faculty members and at least 1 student?

Theorem 1.17 (Binomial Theorem). *For any (complex) numbers a, b, and for all natural numbers n, we have*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}.$$

Fact 1.18. *We have*

$$2^n = \sum_{i=0}^{n} \binom{n}{i} \qquad \text{and} \qquad 0 = \sum_{i=0}^{n} (-1)^i \binom{n}{i}.$$

There are many such identities one can obtain using the Binomial Theorem with creative choices of a and b (and sometimes calculus).

Example 1.19. For any natural number n and $0 \leq i \leq n-1$, we have

$$(i+1)\binom{n}{i+1} = n\binom{n-1}{i}.$$

Multinomial coefficients and the Multinomial Theorem

Binomials are very often very useful. Here is a generalisation.

Basic Problem

We have n distinguishable objects, which we want to split into k (labelled) groups of specified sizes. How many ways are there to do it?

Definition 1.20 (Multinomial coefficients). *Given natural numbers n and k, and non-negative integers $r_1, \dots, r_k$ such that $\sum_{i=1}^k r_i = n$, we define the* multinomial coefficient $\binom{n}{r_1, r_2, \dots, r_k}$ *to be the number of collections of pairwise disjoint sets $(A_1, \dots, A_k)$ such that $|A_i| = r_i$ for each $1 \le i \le k$ and $A_1 \cup \dots \cup A_k = \{1, \dots, n\}$.*

Proposition 1.21. *For any natural number n and $0 \le r \le n$ we have*

$$\binom{n}{r, n-r} = \binom{n}{r}.$$

Example 1.22. How many different ways can 30 students be assigned to rows with 7 students in the first row, 7 in the second row, 7 in the third row, and 9 in the fourth row?

Proposition 1.23. *For any natural numbers n, k and nonnegative integers $r_1, \dots, r_k$ such that $\sum_{i=1}^k r_i = n$, we have*

$$\binom{n}{r_1, r_2, \dots, r_k} = \binom{n}{r_1}\binom{n-r_1}{r_2} \dots \binom{n - r_1 - r_2 - \dots - r_{k-1}}{r_k} = \frac{n!}{r_1! r_2! \dots r_k!}.$$

Theorem 1.24 (Multinomial Theorem). *For any (complex) numbers $x_1, \dots, x_k$ and for all natural numbers n, we have*

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{\substack{r_1, r_2, \dots, r_k \ge 0 \\ r_1 + r_2 + \dots + r_k = n}} \binom{n}{r_1, r_2, \dots, r_k} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}. \tag{1}$$

Example 1.25. How many different orderings of the letters MATHEMATICS are there?

Basic Problem II

Given a set of k different symbols, how many different ways are there to write a string with r_1 of the first symbol, r_2 of the second symbol, and so on up to r_k of the kth symbol, where $r_1 + r_2 + \dots + r_k = n$?

Unordered selection with repetition

Theorem 1.26. *Given a set X of n objects, the number of ways to choose r objects, where the order in which they are chosen is irrelevant and when repetition is allowed, is*

$$\binom{n+r-1}{n-1} = \binom{n+r-1}{r}.$$

Example 1.27. Show that $x_1 + x_2 + \dots + x_n = r$ has $\binom{n+r-1}{r}$ solutions where $x_1, \dots, x_n$ are all non-negative integers.

Example 1.28. In how many ways can we list p '0's and q '1's such that no consecutive '0's appear?

Set Theory and the Inclusion-Exclusion Principle

Proposition 1.29 (Generalised Pigeonhole principle)**.** *Let m, n, k be natural numbers such that $m > nk$. If m objects are distributed into n boxes, then at least one box must contain at least $k + 1$ objects.*

Formally: *Given sets X and Y of sizes $|X| = m$ and $|Y| = n$, and any map $f : X \to Y$, if $m > nk$ then there exists $y \in Y$ such that $\left|f^{-1}(y)\right| \geq k + 1$.*

How can we find exactly what $|A_1 \cup \cdots \cup A_n|$ is? In order to get somewhere with this, we need the following definition.

Given sets $A_1, \dots, A_n$, and a set $X \subseteq \{1, 2, \dots, n\}$, we define

$$A_X := \bigcap_{i \in X} A_i \,.$$

Theorem 1.30 (Inclusion-Exclusion Principle)**.** *For any natural number n, and sets $A_1, \dots, A_n$, we have*

$$\left|A_1 \cup A_2 \cup \cdots \cup A_n\right| = \sum_{\substack{X \subseteq \{1,2,\dots,n\} \\ X \neq \emptyset}} (-1)^{|X|-1} \left|A_X\right| .$$

Example 1.31. How many of the numbers between 1 and $1,000$ are not divisible by any of 3, 5 and 7?

Definition 1.32. *A* derangement *on n elements is a bijection $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that for no i do we have $f(i) = i$.*

Example 1.33. A professor has n students who write an exam. She wants the students to mark each other's tests. How many ways are there for her to hand back the papers such that no student receives their own exam for marking?

Additional reading and exercises

From *Biggs,* ***Discrete Mathematics***
– **Reading**: Sections 10.1, 10.4, 10.5; 11.1, 11.2, 11.3; 12.3.
– **Exercises**: Section 10.1: 1 – 4; Section 10.4: 1 – 4; Section 10.5: 1 – 4;
Section 10.7: 1, 2, 5 – 7, 10 – 12, 17 – 20; Section 11.1: 1 – 8;
Section 11.2: 1 – 4; Section 11.3: 1 – 5; Section 12.3: 1 – 5.

From *Cameron,* ***Combinatorics***
– **Reading**: Sections 3.1, 3.2, 3.3, 3.7; 5.1.
– **Exercises**: Section 10.1: 1 – 4; Section 3.13: 1 – 3, 10; Section 5.6: 1.

Exercises

1. (a) There are 5 seats in a row and 11 people to choose from. How many different seatings are possible if every seat must be occupied by a person?

(b) Now there are 11 seats in a row and 5 people to be seated. If every person must sit in a seat, how many different seatings are possible?

2. How many numbers between 4,000 and 7,000 can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8 if repetition of digits is

(i) allowed;

(ii) not allowed?

3. Let A be a non-empty set of size n and B be a non-empty set of size m. Let $A \times B$ be the set of all pairs (a, b) such that $a \in A$ and $b \in B$.
Prove (in any way) that $|A \times B| = nm$.

4. Let A be a non-empty set of size n, B be a non-empty set of size m, and C be a non-empty set of size k. Let $A \times B \times C$ be the set of all triples (a, b, c) such that $a \in A$, $b \in B$, $c \in C$.
Prove (in any way) that $|A \times B \times C| = nmk$.

5. You have an apple, a banana, a peach, a pear, and an apricot.

(a) In how many ways can you distribute these pieces of fruit among 8 children if no child can receive more than one piece?

(b) How many ways are there if any child can receive any number of pieces?

6. Suppose there are m girls and n boys in a class. What is the number of ways of arranging them in a line so that all the boys are together?

7. The names of the 12 months of the year are listed in a certain order. Given that July and August are *not* next to each other, how many such lists are possible?

8. Seven people are to be seated around a circular table.

(a) How many ways are there to achieve this?

(b) And in how many ways can it be done if there are two people who refuse to sit next to each other?

(c) And in how many ways can it be done if there are three people who want to sit together (i.e. in three consecutive seats)?

9. An eight-person committee is to be formed from a group of 10 people from London and 15 people from outside the city.

(a) In how many ways can the committee be chosen if the committee must contain 4 people from each of the 2 groups?

(b) In how many ways can the committee be chosen if the committee must contain more people from outside London than from the city?

(c) In how many ways can the committee be chosen if the committee must contain at least 2 Londoners?

10. A certain game is played by 13 people using a (standard) deck of 52 cards: there are four suits (Spades, Diamonds, Hearts, and Clubs) and 13 distinguishable cards (Ace, Two, ..., Ten, Jack, Queen, King) in each of the suits. A *k-hand* is a set of k different cards from this deck.

(a) How many 4-hands are there?

(b) A player looks at her 4 cards and sees that she has only cards from one suit.
How many 4-hands that contain only cards from one suit are there?

(c) Another player looks at his 4 cards and sees that he has one card from each suit.
How many 4-hands that contain one card from each suit exist?

(d) The player from (b) peeps at her left neighbour's cards and sees that this neighbour has only cards of one suit, but different from the one she has. Then she looks at her right neighbour's cards and, to her surprise, again sees only cards of one suit, and different from the suit she or the left neighbour has.
How many ways are there to divide 52 cards into thirteen 4-hands such that player 1 has cards of exactly one suit; player 2 has cards of exactly one suit, but not the same suit as player 1; player 3 has cards of exactly one suit, but not the same suit as player 1 or 2; and there are no restrictions for the hands of the other players?

11. Prove that if any ten points are chosen within an equilateral triangle of side-length 1, then there are two of them whose distance apart is at most $1/3$.

12. (a) How many solutions are there of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 23$, with the variables x_1, x_2, x_3, x_4, x_5 non-negative integers?

(b) How many integer solutions are there of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 23$, with $x_1 \geq 1$, $x_2 \geq 2$, $x_3 \geq 3$, $x_4 \geq 4$, $x_5 \geq 5$?

13. A baseball team is made up of a pitcher and eight other players. The manager must choose the team from a group of 20 players, six of whom are pitchers. Note that non-pitchers cannot play at the pitcher's position.

(a) In how many ways can the manager pick the team if the pitchers do not want to play in any other position than as a pitcher?

(b) In how many ways can the team be formed if the pitchers can also play as one of the other players?

14. Let k and n be integers such that $1 \leq k \leq n$. Prove, by using a counting argument (i.e. you cannot use that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$), that

$$\binom{n+2}{k} = \binom{n}{k-2} + 2\binom{n}{k-1} + \binom{n}{k}.$$

15. A domino has two numbers from $\{0, 1, 2, 3, 4, 5, 6\}$, one on each of its (indistinguishable) ends. So, for example, there is a 3-3 domino, with a 3 at each end; and a 4-0 domino, which is the same as 0-4 domino. A set of dominoes contains one copy of each different domino.

(a) How many dominoes are there in a complete set?

(b) In how many ways can one choose 2 dominoes from the complete set so that at least one domino contains a 1 and at least one contains a 6?

16. Prove the identity

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

in the following two ways.

(a) Apply the Binomial Theorem to both sides of the identity

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n},$$

and look at the coefficient of x^n.

(b) Consider two disjoint sets A and B, each of size n, and count the number of subsets of $A \cup B$ with n elements. (Hint: remember that $\binom{n}{i} = \binom{n}{n-i}$.)

17. What is the coefficient of $x^5y^2z^3$ in $(x + y + z)^{10}$? Explain!

18. In an experiment on the effects of fertiliser on 27 plots of new breed of tomatoes, 8 plots are given nitrogen, phosphorus and potash fertiliser; 12 plots are given at least nitrogen and phosphorus, 12 plots are given at least phosphorus and potash; and 12 plots are given at least nitrogen and potash. Also, 18 plots receive nitrogen; 18 plots receive phosphorus; and 18 plots receive potash. How many plots were left unfertilised?

19. A (standard) deck of cards consists of 52 cards: there are four suits (Spades, Diamonds, Hearts and Clubs) and 13 distinguishable cards (Ace, Two, ..., Ten, Jack, Queen, King) in each of the suits. A k-hand is a set of k different cards from this deck.

(a) How many 13-hands are there?

(b) Use the Inclusion-Exclusion Principle to find the number of 13-hands that contain at least one card from each suit.

(c) What is the chance that you obtain a 13-hand in which there are at most three suits?

20. The game of korfball is played with teams of 8 players. Each team contains 4 female and 4 male players. A coach must select a korfball team from 8 male and 7 female candidates.

(a) How many choices are there to choose the team?

In fact, the rules for forming a korfball team of 8 players are a bit more complicated. A team actually consists of two groups of 4 players, both groups containing 2 female and 2 male players. One group will play in the attack; the other group is defending. Our coach still has 8 male and 7 female candidates.

(b) In how many ways can the coach choose two groups of 4 players (one attacking group, one defending group), with each group consisting of 2 female and 2 male players?

A coach of a different club has 7 female and 7 male players to form a korfball team. These players are actually 7 married couples. The second coach knows from experience that a married couple should not be together in one of the two groups in a team.

(c) Determine, using the Inclusion-Exclusion Principle or otherwise, in how many ways the second coach can choose two groups of 4 players (one attacking group, one defending group), with each group consisting of 2 female and 2 male players, and so that none of the groups contains any of the married couples.

21. Five married couples attend a dinner party, and are to be seated around a rectangular table, with the host at one end, the hostess at the other, and four other places on either side. Men and women are to alternate around the table, and no guest is to sit on the same side of the table as their spouse.

(a) How many different possible seating arrangements are there?

After dinner, while the host makes coffee, the other nine people are to be arranged in three teams (Red, Blue, Green) of three for a party game.

(b) In how many ways can the teams be selected?

(c) In how many ways can the teams be selected if Mr. and Mrs. Smith refuse to be on the same team?

(d) In how many ways can the teams be selected if nobody wants to be on the same team as their spouse?

22. How many orderings are there of the numbers $1, 2 \ldots, 8$ in which none of the patterns 12, 34, 56, or 78 appears?

23. How many integers from 1 to 1000 are divisible by none of 2, 3, 5, 7, 11?

24. *(Hard!)* A sociologist observes that in all the big friendship networks she studies, there is either a group of 50 people who are all friends with each other or a group of 50 people none of whom know each other. She observes that this is not true in smaller networks.

Prove that there exist friendship networks with 2^{24} people in which there is no group of 50 people who are all friends with each other, nor any group of 50 people none of whom know each other.

Prove that in any friendship network with at least 2^{100} people, there is either a group of 50 people who are all friends with each other or a group of 50 people none of whom know each other.