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On the Hausdorff Dimension of Brownian Motion

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Contents

Abstract

In this thesis the dimensional aspects of a well known stochastic process, the Brownian motion, is explored using the geometrical instrument known as Hausdorff dimension.

We start recalling the definition of Brownian motion and introducing some of its interesting properties (such as the scaling invariance and the strong Markov property). These results are essential to study the geometric characteristics of the Brownian motion.

Then the fundamental tool of this thesis, i.e. the Hausdorff measure,is explored. So we find an upper and a lower bound for the Hausdorff dimension of the paths of the Brownian motion. As for the upper bound we exploit the fact that the paths are almost surely α-Hölder continuous for every $\alpha \in (0, \frac{1}{2})$, then we proceed proving that, given a d-dimensional Brownian motion $\mathcal{B}$, the 2-Hausdorff measure of the image of this Brownian motion is null. As for the lower bound for the Hausdorff dimension of the image of a d-dimensional Brownian motion the purpose is to develop potential methods for lower bounds of Hausdorff dimension such as the mass method and the energy method. This study leads us to find a lower bound, and this result is obtained in Taylor's theorem, that states that, if $\mathcal{B}$ is a d-dimensional Brownian motion (with $d \geq 2$) then it holds that

$$dim\,(\mathcal{B}([0,1])) = 2$$

Then we try to extend Taylor's theorem, to do that we introduce the Frostman lemma. In particular, using this lemma, we are going to prove that, if $d \geq 2$ and $\mathcal{B}$ is a d-dimensionl Brownian motion, then for all $A \subseteq \mathbb{R}$ we have almost surely:

$$dim\,(\mathcal{B}(A)) = 2dim\,(A)$$

This result is known as McKean's theorem. It is a first, strong, expansion of Taylor's theorem.

The aim of the last part of the thesis is to present the surprising extension of McKean's theorem due to Kaufman. This result let us switch the quantifiers in the Mckean's theorem, so that in dimension at least 2, almost surely, for any $A \subseteq \mathbb{R}$,

$$dim\,(\mathcal{B}(A)) = 2dim\,(A)$$

The study of this result and some of its corollaries ends this thesis.

Chapter 1

First Properties of Brownian Motion

In this chapter our aim will be to introduce the Brownian motion and to study some of its basic properties. In particular in the first part of this chapter we are going to recall the definition of Brownian motion and point out some basic characteristics of it. In the last part of this chapter we will introduce the Markov property of the Brownian motion, that will be essential in the next chapters.

1.1 Introduction to the Brownian motion

Let $(\Omega, \mathcal{F})$ be a σ-field on which a probability $\mathbb{P}$ is defined. A stochastic process is a collection of random variables $\{X_t(\omega)\}_{t\geq 0}$ which take values on a certain measurable space called the state space (in our case the state space will be $\mathbb{R}^d$ with the σ-field of Borel sets). For every fixed point $\omega \in \Omega$ the function $\omega \mapsto X_t(\omega)$ is called the sample path of the process X associated with the element $\omega \in \Omega$.

In order to define the Brownian motion we need another definition. We say that the random variable $X : \Omega \to \mathbb{R}$ is normally distributed with mean μ and variance σ (and we write $X \sim N(\mu, \sigma^2)$) if it holds:

$$\mathbb{P}\left(\{\omega \in \Omega : X(\omega) > x\}\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

Definition 1 (Standard one - dimensional Brownian motion)**.** A standard one - dimensional Brownian motion is a stochastic process $\{B_t\}_{t\geq 0}$ such that:

- $B_0 = 0$, i.e. B_0 is the null random variable. If we suppose $B_0 = x$, with $x \in \mathbb{R}, x \neq 0$, the stochastic process we obtain is called Brownian motion with start in x,

- for $0 \leq s < t$ it holds $B_t - B_s$ is independent from B_s (as random variables),

- for every $t \geq 0$ and $h > 0$ it holds that $B_{t+h} - B_t$ is normally distributed with mean 0 and variance h,

- almost surely the function $t \mapsto B_t$ is continuous.

The one - dimensional Brownian motion is sometimes called linear Brownian motion.

We are now going to recall some of the main properties of the Brownian motion. First of all it is necessary to remember a well-known and fundamental result due to Wiener, that in 1923 proved the existence of the standard Brownian motion.

Theorem 1 (Wiener, 1923)**.** *Standard Brownian motion exists*

Invariance and continuity properties of Brownian motion

One of the most interesting properties of the Brownian motion, and a result that we are going to use often in this thesis, is its scaling invariance. I.e. the fact that there are certain aspects about the Brownian motion that hold both locally and globally. The scaling invariance is useful because it allows us to study the Brownian motion only in the interval $[0, 1]$ and get results that hold generally in $[0, L]$, for any L.

Lemma 1 (Scaling invariance)**.** *Let $\{B_t\}_{t\geq 0}$ be a standard one dimensional Brownian motion and let $\lambda > 0$. It holds that $\{X_t\}_{t\geq 0} = \left\{\frac{1}{\lambda}B_{\lambda^2 t}\right\}_{t\geq 0}$ is also a standard Brownian motion.*

Proof. We have to prove that $\{X_t\}_{t\geq 0}$ respects the points stated before defining the Brownian motion. Some of them follows directly from the fact that B_t is a standard Brownian motion i.e. the independence of the increments, the continuity of the paths. Indeed, for any random path of the Brownian motion, we are just scaling the path, and multiplying a function for a linear factor does not interfere with continuity. The fact that the independence continues to hold follows from the fact that the random variable that is X_t is one of the random variables that are in the definition of $\{B_t\}_{t\geq 0}$ scaled by a factor $\frac{1}{\lambda}$. Then we have only to prove that the scaled random variable has increments that are normally distributed. Since it holds:

$$X_t - X_s = \frac{1}{\lambda}(B_{\lambda^2 t} - B_{\lambda^2 s})$$

And this is a random variable normally distributed with mean zero (this is trivial) and variance $t - s$ (to prove that the variance is $t - s$ is sufficient to consider that the variance of X_t is given by $\frac{1}{\lambda}(\lambda^2 t - \lambda^2 s) = t - s$). $\square$

As we said before, the scaling invariance is one of the most useful tools we have to work with Brownian motion. And this result also points out an interesting property of the Brownian motion, i.e. it links the Brownian motion to the fractals. Indeed with this result we are able to say that Brownian motion has a sort of self similarity, that is one of the fundamental characteristic aspects of fractals. It would be interesting, in a thesis about the geometrical aspect of the

Brownian motion, to study the relationship that characterizes these two structures and to study the Brownian motion from this point of view. A fascinating presentation of these aspects can be found in [1].

We are now going to study the continuity properties of the Brownian motion. In particular we need to recall the Hölder continuity of the paths of the Brownian motion. Later on we will use this result to find an upper bound for the Hausdorff dimension of the Brownian motion.

Theorem 2. *If $0 < \alpha < \frac{1}{2}$, almost surely Brownian motion is everywhere locally α-Hölder continuous.*

We also know that this result is optimal, that is, if $\alpha > \frac{1}{2}$, then almost surely the Brownian motion is not α-Hölder continuous.

Until now we recalled some known aspects of its regularity (scaling invariance and Hölder continuity. The following lemma will point out the fact that the Brownian motion is irregular, in some sense. And this, in some way, will give some insights on the fact that the Hausdorff measure of the Brownian motion is greater than one. Indeed we could think that a continuous function has to have an image of dimension at most one. The following results will show that, though every paths of the Brownian motion is continuous, there are only few random paths that are regular. In fact almost surely a random path of Brownian motion is nowhere differentiable and nowhere monotone.

Lemma 2. *Let $0 < a < b$, and $\{B_t\}_{t \geq 0}$ be a Brownian motion. Then the function $t \mapsto B_t$ is almost surely not monotone on the interval $[a, b]$.*

Proof. Without loss of generality we can suppose to have $B_a \leq B_b$. What we want to show is that the probability of the set $A \subseteq \Omega$ for which $t \mapsto B_t$ is monotone on the interval $[a, b]$ is zero. To show this, given $n \in \mathbb{N}$, pick n numbers $a < a_1 < \ldots < a_n < b$ between a and b. If we want the monotonicity of the Brownian motion on $[a, b]$ it has to hold that $B_a \leq B_{a_1} \leq \ldots \leq B_{a_n} \leq B_b$. But this happens only if $B_{a_1} - B_a \geq 0, \ldots, B_b - B_{a_n} \geq 0$. But now we can use the definition of Brownian motion to say that these increments are independent and the random variables we just showed are normally distributed with mean zero. And then the probability that $B_{a_i+1} - B_{a_i} \geq 0$ is exactly $\frac{1}{2}$. Then the probability p that the Brownian motion is monotone on any interval of positive length is less than $2 \cdot 2^{-(n+1)}$ for every $n \in \mathbb{N}$. Then almost surely the Brownian motion is not monotone on $[a, b]$. $\square$

The result we just proved would be enough to show the irregularity of this stochastic process and to justify the study of the Hausdorff dimension of its image. However there is another result, due to Wiener, that is similar to the last one and it is interesting in many ways.

Theorem 3. *Almost surely, Brownian motion is nowhere differentiable, furthermore it holds, almost surely and for every $t \geq 0$:*

$$\limsup_{h \to 0} \frac{B_{t+h} - B_t}{h} = +\infty$$

$$\liminf_{h \to 0} \frac{B_{t+h} - B_t}{h} = -\infty$$

1.2 Strong Markov property of Brownian motion

One of the most interesting aspects of the Brownian motion, and probably one of the most useful, is the fact that it is a strong Markov process. This property is not only interesting on its own (though being a Markov process does say a lot about Brownian motion) but will also be useful in our study of the Hausdorff dimension of its image. This section is entirely dedicated to the introduction of Markov processes and to the proof that the Brownian motion is one of them.

Brownian motion in higher dimensions and weak Markov property

In order to present the Markov property of the Brownian motion we are now going to extend our definition of Brownian motion to higher dimensions. The idea behind this definition is to have a stochastic process that takes values in $\mathbb{R}^d$ and whose projections are linear Brownian motions. This definition is interesting for mathematical reasons, but is also useful because it is an accurate description of many events of the real world.

Definition 2. Let $B^1, \ldots, B^d$ be independent, one dimensional, Brownian motions, starting respectively in $x_1, \ldots, x_d \in \mathbb{R}$. We call d-dimensional Brownian motion with start in $(x_1, \ldots, x_d)$ the stochastic process $\{B_t\}_{t\geq 0}$ defined as:

$$B_t = (B_t^1, \ldots, B_t^d) \text{ for every } t \geq 0$$

A d-dimensional Brownian motion is said to be standard if it does start in the origin (or equivalently if its projections are standard).

The property that we are going to introduce, the Markov property of Brownian motion, is of fundamental importance in the study of the Hausdorff dimension of the Brownian motion and gives a strong characterization of the Brownian motion as a stochastic process. In order to study the theorem that states that the strong Markov property holds for the Brownian motion we need to recall some definitions and preliminary results.

Theorem 4 ((Weak) Markov property)**.** *Let $\{B_t\}_{t\geq 0}$ be a d-dimensional standard Brownian motion. Let $S > 0$, then the process $\{B_{t+s} - B_s\}_{t\geq 0}$ is again a standard d-dimensional Brownian motion. And, most importantly, it is independent of the process $\{B_t\}_{t\geq 0}$.*

Proof. Is trivial to prove that the process $\{B_{t+s} - B_s\}_{t\geq 0}$ is a standard d-dimensional Brownian motion, simply considering the definition of Brownian motion and studying the components of the process. The restriction to an interval of a continuous function is still continuous, and the independence of the increments does not depend of where the stochastic process starts. The independence of the process from $\{B_t\}_{t\geq 0}$ follows from the independence of increments that defines the Brownian motion. □

The weak Markov property gives some insights about the Brownian motion; these insights regard an another aspect of its self similarity. The Brownian motions starts anew at any point, and its restrictions to intervals of the form $[L, \infty)$ is again a Brownian motion. Then the self similarity of Brownian motion is clear, both in its invariance meaning and in its starting anew at each time. But the weak Markov property is only a first step in the direction of the strong Markov property, and is a property that follows quite easily from the definition of Brownian motion. What we are going to do is to extend the independence part of the theorem: we will show that, under some particular conditions about s (that will be a random variable), the process $\{B_{t+s} - B_s\}_{t\geq 0}$ will be independent with respect to a certain filtration. But before proceeding we need to recall some useful terminology.

Definition 3 (Filtration)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ is a family $\left\{\mathcal{F}^t\right\}_{t\geq 0}$ of σ-fields such that, for every $s < t$, $\mathcal{F}^s \subset \mathcal{F}^t \subset \mathcal{F}$. A probability space over which is defined a filtration is called filtered probability space.

We are interested in probability spaces where stochastic processes are defined, and we are going to create some specific filtrations that permits us to describe adequately the properties of the process we are studying. In particular we are interested in the situation described by the following definition.

Definition 4 (Adapted stochastic process)**.** A stochastic process $\{X_t\}_{t\geq 0}$ defined on a filtered probability space $(\Omega, \left\{\mathcal{F}^t\right\}_{t\geq 0}, \mathbb{P})$ is called adapted if X_t is $\mathcal{F}^t$ measurable for any $t \geq 0$.

Filtration for which a stochastic process is adapted are useful instruments in the study of the stochastic process itself. They can be used to point out which one are the relevant events, up to a certain moment t, for the stochastic process that is under study. We are interested mainly in two specific filtration ($\mathcal{F}_0^t$ and $\mathcal{F}_+^s$), linked to the Brownian motion, for which the Brownian motion is adapted. So now is the occasion to define

$$\mathcal{F}_0^t = \sigma(\{B_s\}_{0\leq s\leq t})$$

the σ-field generated by the random variable B_s with $s \in [0, t]$. This is clearly a filtration, and is trivial to see that the Brownian motion is adapted to it. It is the smallest filtration for which the Brownian motion is adapted, but is not the only interesting filtration linked to Brownian motion. Another filtration, that we are going to use quite often, is the following:

$$\mathcal{F}_+^s = \bigcap_{t\geq s} \mathcal{F}_0^t$$

It's easy to prove that it is a filtration for which the Brownian motion is adapted (this last aspect follows trivially from the fact that it does contain $\mathcal{F}_0$, and the Brownian motion is adapted to $\mathcal{F}_0$), and it allows us to prove a first extension of the weak Markov property of Brownian motion.

Lemma 3. *Let $\{B_t\}_{t\geq 0}$ be a d-dimensional standard Brownian motion. Let $s > 0$, then the process $\{B_{t+s} - B_s\}_{t\geq 0}$ is independent of the σ-algebra $\mathcal{F}_+^s$.*

Proof. Let $s_1, \dots, s_n, \dots$ be a sequence such that $\lim_{n\to\infty} s_n = s$ and $\forall n,\ s_n > s$, i.e. converging to s from above. For the continuity of the paths of the Brownian motion we have that, almost surely and for every $t \geq 0$, the following holds:

$$B_{t+s} - B_s = \lim_{n\to\infty} B_{t+s_n} - B_{s_n}$$

So we can apply the weak Markov property we just proved to say that, for any $t_i \geq 0$ and for any n, it holds $(B_{t_1+s_n} - B_{s_n}, \dots, B_{t_d+s_n} - B_{s_n})$ are independent of $\mathcal{F}_+^s$. Then the limit for $j \to \infty$, that is $(B_{t_1+s} - B_s, \dots, B_{t_d+s} - B_s)$, is also independent of it. □

Stopping times and strong Markov property

The weak Markov property is a result on the independence of the Brownian motion of a deterministic time instance. The difference between the weak and the strong Markov property is the fact that the strong Markov property extends the result to a wider class of times, the stopping times.

Definition 5 (Stopping time)**.** Let T be a real random variable which takes only non negative values, defined on a filtered probability space $(\Omega, \{\mathcal{F}^t\}_{t\geq 0}, \mathbb{P})$. We say that T is a stopping time with respect to the filtration if for every $t \geq 0$, $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}^t$.

We are now going to show a couple of examples of stopping times, linked with the Brownian motion and the filtration we just defined. In the next chapters we will use the fact that the random variables we define here are indeed stopping times for the Brownian motion. Indeed this aspect will be essential when using the strong Markov property of the Brownian motion.

Example 1. *Let $C \subseteq \mathbb{R}^d$ be a closed set and let $\{B_t\}_{t\geq 0}$ be a d-dimensional Brownian motion with start in $x \in \mathbb{R}^d$. We can define the random variable T that represents the first hitting time of the set C. I.e. we can define*

$$T = \inf\{t \geq 0 : B_t \in C\}$$

Then we have that T is a stopping time with respect to $\{\mathcal{F}_0^t\}_{t\geq 0}$ (from this follows immediately that T is also a stopping time with respect to $\{\mathcal{F}_+^t\}_{t\geq 0}$, because, for every t, $\{\mathcal{F}_0^t\}_{t\geq 0} \subseteq \{\mathcal{F}_+^t\}_{t\geq 0}$). In fact we have that, if $\mathcal{S}(z, r)$ is the ball centered in z of ray length r:

$$\{\omega \in \Omega : T(\omega) \leq t\} = \bigcap_{n=1}^{\infty} \bigcup_{s\in\mathbb{Q}\cap(0,t)} \bigcup_{y\in\mathbb{Q}^d\cap C} \left\{\omega \in \Omega : B_s \in \mathcal{S}(y, \frac{1}{n})\right\}$$

We can do this because the Brownian motion is almost surely continuous, and the sets with null measure are measurable. The study of the first hitting time of *a set for the Brownian motion will be extremely important for our study of the Hausdorff dimension of the Brownian motion. We will be able to use the strong Markov property on the hitting times of closed sets because we just proved that the hitting times are stopping times with respect to both* $\left\{\mathcal{F}_+^t\right\}_{t\geq 0}$ *and* $\left\{\mathcal{F}_0^t\right\}_{t\geq 0}$.

There is another example that is even more useful than the last one. In the following example we will show that there is an important difference between $\left\{\mathcal{F}_+^t\right\}_{t\geq 0}$ and $\left\{\mathcal{F}_0^t\right\}_{t\geq 0}$. And this will lead us to choose the filtration we will conventionally study the Brownian motion with.

Example 2. *Let $A \subseteq \mathbb{R}^d$ be an open set and let $\{B_t\}_{t\geq 0}$ be a d-dimensional Brownian motion with start in $x \in \mathbb{R}^d$ then the first hitting time of the set A is a stopping time with respect to $\left\{\mathcal{F}_+^t\right\}_{t\geq 0}$ but not necessarily with respect to $\left\{\mathcal{F}_0^t\right\}_{t\geq 0}$.*
We can repeat the reasoning of before and write, for the continuity of Brownian motion (we now use the compact notation):

$$\{T \leq t\} = \bigcap_{s>t} \{T < s\} = \bigcap_{s>t} \bigcup_{v\in\mathbb{Q}\cap(0,s)} \{B_v \in A\} \in \mathcal{F}_+^t$$

From this follows that T is a stopping time with respect to $\left\{\mathcal{F}_+^t\right\}_{t\geq 0}$,
so there are cases in which it is not a stopping time with respect to $\left\{\mathcal{F}_0^t\right\}_{t\geq 0}$. Suppose that A is bounded and the starting point is not in the closure of A, i.e. $x \notin \bar{A}$. Then we may fix $\gamma : [0,t] \to \mathbb{R}^d$ with $\gamma((0,t)) \cap \bar{A} = \emptyset$ and $\gamma(t) \in \partial A$. With γ continuous. But then $\mathcal{F}_0^t$ contains no subsets of $\{\forall 0 \leq s \leq t,\ B_s = \gamma(s)\}$ different from the empty set and Ω itself. But then if we had that $\{T \geq t\} \in \mathcal{F}_0^t$ then the set $\{\forall 0 \leq s \leq t,\ B_s = \gamma(s)$ and $T = t\}$ would be in $\mathcal{F}_0^t$ and a nontrivial subset of Ω. But that is absurd.

We said before that the first hitting times of sets play an important role in our studies, and our last result shows that when dealing with Brownian motion it is preferable to work with stopping times with respect to the richer filtration. Therefore from now on we make the convention that we will always refer to the filtration $\left\{\mathcal{F}_+^t\right\}_{t\geq 0}$ (speaking about stopping times), as this filtration produces more stopping times. We are now ready for the strong Markov property.

Theorem 5 (Strong Markov property)**.** *Let T be a stopping time that is almost surely finite, i.e. a stopping time such that $T(\omega) < \infty$ for almost every ω. Then the process $\{B_{T+t} - B_T\}_{t\geq 0}$ is a standard Brownian motion independent of $\mathcal{F}_+^T$.*

Proof. We can define a sequence of stopping times that approximate T from above. Let then T_n be the decreasing sequence of stopping times defined as: $T_n = (m+1)2^{-n}$, where m is such that $m2^{-n} \leq T < (m+1)2^{-n}$. We are going to prove that this is true for any fixed T_n. We now consider two Brownian motions:

- $\left\{B_t^k\right\}_{t\geq 0} = \left\{B_{t+\frac{k}{2^n}} - B_{\frac{k}{2^n}}\right\}_{t\geq 0}$

- $\{B_t^*\}_{t\geq 0} = \{B_{t+T_n} - B_{T_n}\}_{t\geq 0}$

Suppose then that $E \in \mathcal{F}_+^{T_n}$, then it will hold that for every event $C = \{B^* \in A\}$ we can write:

$$\begin{aligned}\mathbb{P}\left(C \cap E\right) &= \sum_{k=0}^{\infty} \mathbb{P}\left(C \cap E \cap \left\{T_n = k2^{-n}\right\}\right) = \sum_{k=0}^{\infty} \mathbb{P}(C)\mathbb{P}(E \cap \left\{T_n = k2^{-n}\right\}) \\ &= \mathbb{P}(C) \sum_{k=0}^{\infty} \mathbb{P}(E \cap \left\{T_n = k2^{-n}\right\}) = \sum_{k=0}^{\infty} \mathbb{P}(C)\mathbb{P}(E \cap \left\{T_n = k2^{-n}\right\})\end{aligned}$$

The third equality follows from the fact that the event C is independent of $E \cap \{T_n = k2^{-n}\}$. The equality we just wrote shows that B^* is a Brownian motion and is independent of E, hence in generally independent of $\mathcal{F}_+^{T_n}$ and from $\mathcal{F}_+^{T}$ (because we have that $\mathcal{F}_+^{T} \subseteq \mathcal{F}_+^{T_n}$.
We then showed that the result holds for T_n, but we know also that T_n tends decreasing to T. Hence the increments

$$B_{s+t+T} - B_{t+T} = \lim_{n \to \infty} B_{s+t+T_n} - B_{t+T_n}$$

of the process $\{B_{t+T} - B_t\}_{t \geq 0}$ are independent and normally distributed with mean zero and variance s. But then the process is a Brownian motion, as the process is almost surely continuous. We have also that all the increments (and hence the process itself) are independent of $\mathcal{F}_+^{T}$. $\square$

Chapter 2

Hausdorff dimension and upper bound for Brownian motion

In this section we are going to use the results presented in the first chapter to get some first results about the Hausdorff dimension of Brownian motion.

First of all we need to introduce Hausdorff measure and Hausdorff dimension, two essential instruments to study Brownian motion from the point of view of the measure theory. In the first part of the section we are going to study some properties of the Hausdorff measure and some methods to find upper bounds.

In the second part we will apply these methods to the Brownian motion, and in particular we will obtain a first upper bound for the Hausdorff dimension of the Brownian motion.

In the end we will prove that the 2-Hausdorff measure of the Brownian motion is zero. This fundamental result leads naturally to the question: is the lower bound we found strict? Can we find a lower bound for the Hausdorff dimension of the Brownian motion?

2.1 Hausdorff measure

Let (X, ρ) be a metric space and $E \subseteq X$ [1], we can define the diameter of E as:

$$|E| = \sup \{\rho(x, y) : x, y \in E\}$$

We say that a countable sequence $C_1, C_2, \dots$ of subsets of X is a covering for X if $X \subseteq \bigcup_{i=1}^{\infty} C_i$. We will be interested only in countable covering as we have to sum the diameter of the sets that made the coverings. Given $\alpha > 0$ we define:

$$\mathcal{H}_\delta^\alpha (X) = \inf \left\{ \sum_{i=1}^{\infty} |C_i|^\alpha : C_1, C_2, \dots \text{ covers } X \text{ and } |C_i| \leq \delta \right\}$$

A covering is called a δ-covering if the diameter of its sets does not exceed δ. In the particular case $\delta = \infty$ [2] we speak of Hausdorff content of X and we write

[1] a metric space itself with the same distance

[2] when we take the infimum over every possible countable covering of X

$\mathcal{H}^{\alpha}_{\infty}(X)$, in all the other cases we speak of α-value of the δ-covering of X to refer to $\mathcal{H}^{\alpha}_{\delta}(X)$.

We may notice that $\mathcal{H}^{\alpha}_{\delta}(X)$ is a monotone decreasing function of δ, this holds because, if $0 < \epsilon < \delta$, every ϵ-covering of X is also a δ-covering for X, and we are taking the infimum over a smaller set.

Definition 6 (Hausdorff measure)**.** Let α be greater than zero and (X, ρ) be a metric space, we define the α-Hausdorff measure of X as:

$$\mathcal{H}^{\alpha}(X) = \sup_{\delta \geq 0} \mathcal{H}^{\alpha}_{\delta}(X) = \lim_{\delta \to 0} \mathcal{H}^{\alpha}_{\delta}(X)$$

the equivalence of the definitions follows from the fact we just pointed out that $\mathcal{H}^{\alpha}_{\delta}(X)$ is a monotone decreasing function of δ.

Even though $\mathcal{H}^{\alpha}_{\delta}(X)$ increases as δ decreases, there are some restrictions on this increasing. Take any $\alpha > 0$, if $\delta > 0$ is such that $\mathcal{H}^{\alpha}_{\delta}(X) = 0$, then $\mathcal{H}^{\alpha}(X) = 0$. There is a simple proof of this fact: let $\epsilon > 0$ and take $C_1, C_2, \ldots$ a δ -covering of X such that $\sum_{i=1}^{\infty} |C_i|^{\alpha} < \epsilon$, we know that a covering with this properties exists. Besides we can say that, for every $i \in \mathbb{N}$ we have $|C_i| \leq \epsilon^{\frac{1}{\alpha}}$. Then we can find a covering $C_1, C_2, \ldots$ that is a $\epsilon^{\frac{1}{\alpha}}$-covering for every $\epsilon > 0$ and for which $\sum_{i=1}^{\infty} |C_i|^{\alpha} < \epsilon$. This proves that $\mathcal{H}^{\alpha}(X) = 0$.

Another interesting property of Hausdorff's measure is the following: if we consider it as a function of α, almost all the values of the function are trivial. In particular, if $\alpha > 0$ and $\mathcal{H}^{\alpha}_{\delta}(X) < \infty$ we have that for all $\beta > \alpha$, $\mathcal{H}^{\beta}_{\delta}(X) = 0$. A simple proof of this fact is the following: let $\epsilon > 0$ and let $C_1, C_2, \ldots$ be a covering of X such that $\sum_{i=1}^{\infty} |C_i|^{\alpha} < \mathcal{H}^{\alpha}_{\delta}(X) + \epsilon$. Then

$$\mathcal{H}^{\beta}_{\delta}(X) \leq \sum_{i=1}^{\infty} |C_i|^{\beta} \leq \sum_{i=1}^{\infty} |C_i|^{\beta-\alpha} |C_i|^{\alpha} \leq \delta^{\beta-\alpha}(\mathcal{H}^{\alpha}_{\delta}(X) + \varepsilon)$$

We have our thesis for $\delta \to 0$.

Remark 1. It is appropriate to clarify why we called $\mathcal{H}^{\alpha}(X)$ the Hausdorff measure of XSo this remark is to show that Hausdorff's measure is an outer measure. Let $F^1, F^2, \ldots$ be a countable sequence of subsets of X and $U^j_1, U^j_2, \ldots$ be a δ-covering of F^j, then $\left\{U^j_i : i, j \in \mathbb{N}\right\}$ is a δ -covering of $\bigcup_{j=0}^{\infty} F^j$. Thus we can write:

$$\mathcal{H}^{\alpha}_{\delta}\left(\bigcup_{j=0}^{\infty} F^i\right) \leq \sum_{j \geq 0} \sum_{i \geq 0} \left|U^j_i\right|^{\alpha}$$

We can choose each $U^j_1, U^j_2, \ldots$ in a way that we have $\sum_{i \geq 0} \left|U^j_i\right|^{\alpha} \leq \mathcal{H}^{\alpha}_{\delta}\left(F^j\right) + \frac{\epsilon}{2^j}$, so we can now write:

$$\mathcal{H}^{\alpha}_{\delta}\left(\bigcup_{i=0}^{\infty} F_i\right) \leq \sum_{j \geq 0} \mathcal{H}^{\alpha}_{\delta}(F_j)$$

Letting $\delta \to 0$ we obtain

$$\mathcal{H}^{\alpha}_{\delta}\left(\bigcup_{i=0}^{\infty} F_i\right) \leq \sum_{j\geq 0} \mathcal{H}^{\alpha}\left(F_j\right)$$

That is our thesis.
We could now show that the Borel sets are Caratheodory measurable for this measure and that, for $\alpha \in \mathbb{N}$ the Hausdorff measure correspond to the Lebesgue measure. For further properties of Hausdorff measure and the proofs of these two one, please consult [1].

2.2 Hausdorff dimension

We did prove that, if $\alpha > 0$ and $\mathcal{H}^{\alpha}_{\delta}(X) < \infty$, we have $\forall \beta > \alpha$, $\mathcal{H}^{\beta}_{\delta}(X) = 0$. This is a first important result and by the same proof we can state that if $\alpha > 0$ and $\mathcal{H}^{\alpha}(X) > 0$, then we have, for every $0 < \beta < \alpha$ that it holds:

$$\mathcal{H}^{\beta}(X) = \infty$$

So, if we consider $\mathcal{H}^{\alpha}(X)$ as a function of α, we have that it takes at most three different values and we will have a certain $\xi \in \mathbb{R}$ for which:

$$\mathcal{H}^{\alpha}(X) = \begin{cases} \infty & \text{if } 0 < \alpha < \xi \\ \mathcal{H}^{\xi}(X) & \text{if } \alpha = \xi \\ 0 & \text{if } \alpha > \xi \end{cases}$$

This considerations lead us to define the Hausdorff dimension.

Definition 7 (Hausdorff dimension)**.** Let (X, ρ) be a metric space, we define Hausdorff dimension of X:

$$dim(X) = \inf\{\alpha > 0 : \mathcal{H}^{\alpha}(X) = 0\} = \sup\{\alpha > 0 : \mathcal{H}^{\alpha}(X) = \infty\}$$

It is important to notice that, given $X \subseteq \mathbb{R}^d$ a bounded set, $dim(X) \leq d$. This follows trivially from the fact that, for $\alpha \in \mathbb{N}$, the Hausdorff measure coincides with the Lebesgue measure. However this result can also be seen considering the fact that the Hausdorff measure is monotone and the cube $[-R, R]^d$ have Hausdorff dimension d for every $R > 0$. So if X is bounded there exists $R > 0$ such that $X \subseteq [-R, R]^d$; $\forall \alpha > d$ we have $\mathcal{H}^{\alpha}\left([-R, R]^d\right) = 0$ and then we have $\mathcal{H}^{\alpha}(X) = 0$. This gives $dim(X) \leq d$.

Remark 2. A powerful property of Hausdorff dimension need to be remarked: the countable stability property, i.e. if $F^1, F^2, \ldots$ is a countable sequence of sets, then

$$dim\left(\bigcup_{i=0}^{\infty} F^i\right) = \sup_{1\leq i\leq\infty}\left\{dim\left(F^i\right)\right\}$$

This follows immediately from the fact that the Hausdorff measure is an outer measure. So we can write:

$$\mathcal{H}^{\alpha}\left(\bigcup_{j=1}^{\infty} F^{j}\right) \leq \sum_{j=1}^{\infty} \mathcal{H}^{\alpha}\left(F^{j}\right)$$

let $\alpha > 0$ be the supremum of the set of the dimensions of the F^j, $\forall \epsilon > 0$, then

$$\mathcal{H}^{\alpha+\epsilon}\left(\bigcup_{j=1}^{\infty} F^{j}\right) \leq \sum_{j=1}^{\infty} \mathcal{H}^{\alpha+\epsilon}\left(F^{j}\right) = \sum_{j=1}^{\infty} = 0$$

From this we have the thesis.

2.2.1 A tool for the upper bound Hausdorff dimension

We are interested in finding an upper bound for the Hausdorff dimension of the Brownian motion. The simplest way of doing it would be to find, for all $\delta > 0$, a covering $C_1, C_2, \dots$ such that $|C_i| < \delta$ and $\mathcal{H}^{\alpha}_{\delta}\left(B_{[0,1]}\right) = 0$ for some α. Considering this and the countable stability of Hausdorff dimension we could extend the result to $B_{[0,\infty)}$. So the problem becomes: how can we find a suitable covering?

The underlying idea is the following: if we can, in some way, bound the region in which the Brownian motion can be, we will be able to cover the region of the possible locations of the Brownian motion. Of course we can only hope to find an upper bound for almost every path of the Brownian motion. So we are going to need some sort of regularity for the Brownian motion, that is a way to restraint the possible extension of the paths. For this purpose we proved that the Brownian motion is almost surely α-Hölder continuous, for every $\alpha < \frac{1}{2}$. So what we can do is to bound the image of a segment $[t_i, t_{i+1}]$ with a ball of diameter $L_\alpha \left|t_{i+1} - t_i\right|^{\alpha}$ for every $\alpha < \frac{1}{2}$ (where L_α is the α Hölder constant for the function). The purpose is to find an upper bound for the Hausdorff dimension; a natural way for doing this is to find a suitable covering. The existence of a covering "strict" enough is guaranteed by the α Hölder continuity that holds almost everywhere for the Brownian motion. The first step in this direction is the following lemma:

Lemma 4. *Let (X_1, ρ_1) and (X_2, ρ_2) be metric spaces. And let $f : X_1 \to X_2$ be an α-Hölder continuous surjective function with Hölder constant L. Then we have, for every $\beta > 0$,*

$$\mathcal{H}^{\beta}\left(X_2\right) \leq L^{\beta} \mathcal{H}^{\alpha\beta}\left(X_1\right)$$

Proof. Let $\delta, \epsilon > 0$ and take $C_1, C_2, \dots$ a δ -covering of X_1 such that

$$\sum_{i=1}^{\infty} \left|C_i\right|^{\alpha\beta} \leq \mathcal{H}^{\alpha\beta}_{\delta}\left(X_1\right) + \epsilon$$

We have that $f(C_1), f(C_2), \dots$ is a $L\delta^\alpha$ -covering for X_2. This is the suitable covering we spoke about. The interesting aspect of this covering is that each set

$$\mathcal{H}^{\beta}_{L\delta^\alpha}(X_2) \leq \sum_{i=1}^{\infty} |f(C_i)|^\beta \leq L^\beta \sum_{i=1}^{\infty} |C_i|^{\alpha\beta} \leq L^\beta \mathcal{H}^{\alpha\beta}_{\delta}(X_1) + L^\beta \epsilon$$

If we let decrease $\delta \to 0$ we obtain the limit β covering for the set and hence the β measure of X_2. Since L does not depend on δ and since we can choose an arbitrarily small ϵ, we can write:

$$\mathcal{H}^{\beta}(X_2) \leq L^\beta \mathcal{H}^{\alpha\beta}(X_1)$$

Simply because $L\delta^\alpha \to 0$ when $\delta \to 0$. □

From this lemma follows naturally an upper bound for the Hausdorff dimension of the image of Hölder continuous functions. We are interested in Brownian motion, so we can restrict our focus on functions $f : [0, 1] \to \mathbb{R}^d$. In detail we can only consider functions from $\mathbb{R} \to \mathbb{R}^d$ because these are the domain and the image space of the Brownian motion, but we can also restraint our attention to function with domain $[0, 1]$ because, as we pointed out before, if $F^1, F^2, \dots$ is a countable sequence of sets, then

$$dim\left(\bigcup_{i=0}^{\infty} F^i\right) = \sup_{1\leq i\leq\infty} \left\{dim\left(F^i\right)\right\}$$

so we can simply divide the image of Brownian motion in a countable sequence, made by the images of $[i, i + 1]$ with $i \in \mathbb{N}$.

Then the last ingredient for upper bound is the fact that, if $f : [0, 1] \to \mathbb{R}^d$ is an α Hölder continuous function, then, for every $X \subseteq [0, 1]$,

$$dim\left(f(X)\right) \leq \frac{dim\left(X\right)}{\alpha}$$

This follows trivially from the lemma we just proved. It is sufficient, to find an upper bound, to take any $\beta > \frac{dim(X)}{\alpha}$, and from the fact that $\alpha \leq 1$ (if $\alpha > 1$ the function is constant) we get

$$\begin{aligned}\mathcal{H}^{\frac{dim(X)}{\alpha}+\epsilon}\left(f(X)\right) &\leq L^{\frac{dim(X)}{\alpha}+\epsilon}\mathcal{H}^{\alpha\left(\frac{dim(X)}{\alpha}+\epsilon\right)}(X)\\ &\leq L^{\frac{dim(X)}{\alpha}+\alpha\epsilon}\mathcal{H}^{dim(X)+\alpha\epsilon}(X) = 0\end{aligned}$$

The last equivalence follows from the definition of $dim\left(X\right)$, so our claim is proved. We are interested in the following consequence:

Corollary 1. *Almost surely, given $A \subseteq [0, \infty)$, the image of a d-dimensional Brownian motion satisfies,*

$$dim\left(B_A\right) \leq 2\ dim\left(A\right) \wedge d$$

Proof. This means that almost every random path of the Brownian motion, maps every set of Hausdorff dimension α in a set of $\mathbb{R}^d$ of dimension at most 2α. We showed in the first chapter that Brownian motion is almost surely γ-Hölder for every $\gamma < \frac{1}{2}$. So let's take a sequence $\gamma_1, \gamma_2, \ldots$ of values that converges to $\frac{1}{2}$ from below, such that $\forall n,\ \gamma_n \neq \frac{1}{2}$. Then for every n we have that $dim\left(B_A\right) \leq \frac{1}{\gamma_n} dim\left(A\right) \wedge d$. Letting n to infinity gives us the thesis. □

So we finally found an upper bound for the Hausdorff dimension of Brownian motion. Hence we can say that $\mathcal{H}^{\alpha}\left(B_A\right) = 0$ for every $\alpha > 2$ and for every $A \subseteq [0, \infty)$.

2.3 The value of $\mathcal{H}^2\left(B_{[0,\infty)}\right)$

We will now focus on finding the value of $\mathcal{H}^2\left(B_{[0,\infty)}\right)$. We are going to prove that $\mathcal{H}^2\left(B_{[0,\infty)}\right) = 0$. To do it we are going to find, for every $\delta \geq 0$, a suitable δ -covering for $B_{[0,\infty)}$. The first, essential, step towards our goal is the following lemma. Before it is important to remember a consequence of the scaling principle: let $a < 0 < b$, and consider $T(a,b) = \inf\left\{t \geq 0 : B_t = a \text{ or } B_t = b\right\}$ the first exit time of a one-dimensional Brownian motion starting in 0. We will prove that this method gives the results we are looking for but now we need to prove the following lemma: Then, if $X(t) = \frac{1}{a} B_{a^2 t}$ we have:

$$\mathbb{E}T(a,b) = a^2 \mathbb{E} \inf\left\{t \geq 0 : X(t) = 1 \text{ or } X(t) = \frac{b}{a}\right\} = a^2 \mathbb{E}T\left(1, \frac{b}{a}\right)$$

This property is called Brownian scaling, and will be used in the following lemma.

Lemma 5. *Suppose $d \geq 3$, $x \in \mathbb{R}^d \setminus [0,1]^d$ ($d(x, [0,1)^d) > 0$, where d is the Euclidean distance on $\mathbb{R}^d$) and $\{B_t\}_{t \geq 0}$ a Brownian motion with start in x. If we write $E = B_{[0,\infty)} \cap [0,1)^d$ we have:*

$$\mathcal{H}^2\left(E\right) = 0$$

Proof. The proof that follows is quite tedious and we need to introduce some notation before starting. We will try to define most of the notation in the first part of the proof:

- If $S = [a_1, a_1 + h) \times \ldots \times [a_d, a_d + h) \subseteq \mathbb{R}^d$ is a cube in $\mathbb{R}^d$, S^* will indicate the expanded cube made out of S, i.e. the cube that has the same centre as S (the point $(a_1 + \frac{h}{2}, \ldots, a_d + \frac{h}{2}) \in \mathbb{R}^d$) but side length $\frac{3h}{2}$.

- There is a random measure that is called occupation measure, defined on all the Borel sets by:
$$\mu(A) = \int_0^{\infty} 1_A(B_t) dt$$
This measure is locally finite and defined on every Borel set.

- We will work on the dyadic cubes, and in particular we will indicate the set of all dyadic cubes of side length 2^{-k} with:

$$\mathcal{D}_k = \left\{ \bigtimes_{j=1}^{d} \left[\frac{h_j}{2^k}, \frac{h_j+1}{2^k}\right) : \forall j,\ h_j \in \left\{0, \ldots, 2^k - 1\right\} \right\}$$

- We will also use: $I^d = [0,1)^d$ and $\mathrm{Ch}^d = [0, \frac{1}{2})^d$

- Given S, U cubes in $\mathbb{R}^d$ with $S \subseteq U$ we will write, for a Brownian motion started in $z \in \mathbb{R}^d$,

$$\begin{aligned}\tau(S,z) &= \inf\left\{t \geq 0 : B_t \in S\right\}\\ \chi(U,S,z) &= \inf\left\{t > \tau(S,z) : B_t \notin U\right\}\end{aligned}$$

  The first hitting time of S and the first exit time from U, provided that the Brownian motion hit S before.

We want to study the 2-Hausdorff measure of E, and in detail we want to show that $\mathcal{H}^2(E) = 0$, then, fixed $\delta > 0$, we want to find a suitable δ-covering for E. So let's fix $m \in \mathbb{N}$ such that $\sqrt{d}2^{-m} \leq \delta$. We will have that, $\forall k \geq m$, $\forall D \in \mathcal{D}_k$, $|D| \leq \sqrt{d}2^{-m} \leq \delta$. So we will take a covering of E made by elements of $\mathcal{D}_k$ with $k \geq m$.

Let's continue with our definitions, after having fixed $\epsilon > 0$:

- we say that $D \in \mathcal{D}_k$ is "full" if

$$\mu(D) \geq \frac{1}{\epsilon}2^{-2k}$$

  To indicate that D is full we will write $\mathfrak{B}(D)$

We are going to choose a covering made up by two parts: the first one is a collection of full sets, small enough to let the covering be a δ-covering, the second one made by smaller sets, used to complete the covering. We will then calculate the probability with which a certain dyadic cube is part of this covering to esteem $\mathcal{H}^2(E)$, to do this we will multiply the number of dyadic cubes for the probability that each one of this cube is part of the covering. For this reason we have to keep small the number of dyadic cubes we are considering, hence the need to choose $M \geq m$ (that will represent the maximum finesse we can reach with our covering, the diameter of the smaller sets of our covering) and then we can define:

$$\begin{aligned}\mathfrak{E}(M) = \{&D \in \mathcal{D}_k : m \leq k \leq M,\ \mathfrak{B}(D)\\ &\text{and } \nexists D' \in \mathcal{D}_h,\ h \geq m, \text{ such that } \mathfrak{B}(D')\\ &\text{and } D \subseteq D'\}\end{aligned}$$

The set of all maximal full dyadic cubes with side length not exceeding 2^{-m} and not smaller than 2^{-M}. This is the first part of our covering. The other part is:

$$\begin{aligned}\mathfrak{H}(M) = \big\{ D \in \mathcal{D}_M & \text{ such that } D \cap B_{[0,\infty)} \neq \emptyset \\ & \text{and } \nexists D' \in \mathcal{D}_{m \leq k \leq M} \text{ such that } \mathfrak{B}(D') \\ & \text{and } D \subseteq D' \big\}\end{aligned}$$

This part is made up by those dyadic cubes that do intersect $B_{[0,\infty)}$ but that are not full or contained in full cubes with side length smaller than 2^{-m}. Then we have, $\forall M > m$, that

$$\mathfrak{F}(M) = \mathfrak{H}(M) \cup \mathfrak{E}(M-1)$$

We will prove in the following that this method gives the results we are looking for but now we need to prove the following lemma: is a δ-covering for E.
What we want to do now is to estimate how much probable is that a generic dyadic cube is an element of $\mathfrak{F}(M)$. Take $\tilde{D} \in \mathcal{D}_M$, we can write $\tilde{D} = \tilde{D}_M \subset \tilde{D}_{M-1} \subset \ldots \subset \tilde{D}_m$ the ascending sequence with $\tilde{D}_k \in \mathcal{D}_k$. Before proceeding to the explicit esteem is useful to define,

$$q = \sup_{y \in \mathrm{Ch}^*} \mathbb{P}_y \left\{ \int_0^{\chi(I^d, I^d, y)} 1_{[0,1)^d}(B_t) dt \leq \frac{1}{\epsilon} \right\} < 1$$

Using the strong Markov property and applying the Brownian scaling to the times $\chi_M = \chi(\tilde{D}^*_M, \tilde{D}, x) < \ldots < \chi_{m+1} = \chi(\tilde{D}^*_{m+1}, \tilde{D}, x)$ we can write

$$\begin{aligned}
&\mathbb{P}_x \left\{ \mu(\tilde{D}_k) \leq \frac{2^{-2k}}{\epsilon} \text{ for all } m \leq k < M | \tau(\tilde{D}, x) < \infty \right\} \\
&\leq \mathbb{P}_x \left\{ \int_{\chi_{k+1}}^{\chi_k} 1_{\tilde{D}_k}(B_s) ds \leq \frac{2^{-2k}}{\epsilon} \text{ for all } M > k \geq m | \tau(\tilde{D}, x) < \infty \right\} \\
&\leq \prod_{k=m}^{M-1} \sup_{y \in \tilde{D}^*_{k+1}} \left\{ \mathbb{P}_y \left\{ 2^{2k} \int_0^{\chi_k} 1_{\tilde{D}_k}(B_s) ds \leq \frac{1}{\epsilon} \right\} \right\} \leq q^{M-m}
\end{aligned}$$

Where the last inequality follows from Brownian scaling. Now for the other upper bound, recall that there is a constant $c > 0$ depending only on d and the distance between x and $[0,1)^d$ for which it holds

$$\mathbb{P}_x \left\{ \tau(\tilde{D}, x) < \infty \right\} \leq c 2^{-M(d-2)}$$

Hence the probability that any given cube $\tilde{D} \in \mathcal{D}_M$ is in our covering (i.e. in $\mathfrak{H}(M)$) is:

$$\mathbb{P}_x \left\{ \neg \mathfrak{B}(\tilde{D}_k) \text{ for all } M > k \leq m, \ \tau(\tilde{D}, x) < \infty \right\} \leq c 2^{-M(d-2)} q^{M-m}$$

hence the expected 2-value from the cubes in $\mathfrak{G}(M)$ is:

$$d 2^{dM} 2^{-2M} \mathbb{P}_x \left\{ \neg \mathfrak{B}(\tilde{D}_k) \text{ for all } M > k \geq m, \ \tau(\tilde{D}, x) < \infty \right\} \leq c d q^{M-m}$$

The 2-value from the cubes in $\mathfrak{E}(M)$ is bounded by:

$$\sum_{k=m}^{M-1} d2^{-2k} \sum_{\tilde{D}\in\mathfrak{F}(M)\cap\mathcal{D}_k} 1(\mathfrak{B}(\tilde{D})) \leq d\epsilon \sum_{k=m}^{M-1} d2^{-2k} \sum_{\tilde{D}\in\mathfrak{F}(M)\cap\mathcal{D}_k} \mu(\tilde{D}) \leq d\epsilon\mu([0,1)^d)$$

As $\mathbb{E}\mu([0,1)^d) < \infty$ we have that the expected 2-value of our covering converges to zero for $\epsilon \to 0$ and a suitable choice of M. Hence a subsequence converges to 0 almost surely, and as m was arbitrary, this ensures that $\mathcal{H}^2\left(B_{[0,\infty)}\right) = 0$ almost surely. □

And now we have all the instruments to state the following

Theorem 6. *If $\{B_t\}_{t\geq 0}$ is a d-dimensional Brownian motion, with $d \geq 2$, with start in $x \in \mathbb{R}^d$, then almost surely it holds:*

$$\mathcal{H}^2\left(B_{[0,\infty)}\right) = 0$$

Proof. Without lost of generality we can say $x = 0$ and d greater than three[3]. Choose $\epsilon > 0$, we have:

$$\mathbb{R}^d = [\pm\epsilon, \pm 1)^d \cup S \bigcup_{n_i\in\mathbb{Z}\setminus\{0,-1\}} [n_i, n_i + 1)$$

with $\mathcal{H}^2(S) < C\epsilon$ for a fixed C depending only on d and decreasing to 0 with $\epsilon \to 0$. We have $\mathcal{H}^2\left(B_{[0,1)} \cap L\right) = 0$ for every L of the form we just presented. □

[3] If $d = 2$ we have just to project on $\mathbb{R}^2$ our result. And projections are Lipchitz maps, then they cannot increase the Hausdorff measure of a set that has Hausdorff measure zero

Chapter 3

Potential methods for a first lower bound

In the previous section we found an upper bound for the Hausdorff dimension of the Brownian motion. We also showed that $\mathcal{H}^2\left(B_{[0,\infty)}\right) = 0$. In this section we will introduce two techniques for calculating lower bounds for Housdorff dimension, widely used both in theory and practice. We will conclude this section by proving a result that implies that the Hausdorff dimension of $B_{[0,\infty)}$ is 2.

3.1 The mass method

To find an upper bound for the Hausdorff dimension it was necessary to find a proper δ-covering for our set and for an arbitrarily small δ. To study the lower bound we ought consider, for a fixed α, the α-value of every possible covering of the set. To do this we have to show that somehow the set we are considering is too big to be covered by sets with α-value zero. We are going to prove that, if we can define on the set a proper measure, the set will be big enough and we will not be able to find an arbitrarily small covering.

Let (E, d) be a metric space and μ a measure defined on the Borel sets of E. We call μ a mass distribution if $0 < \mu(E) < \infty$.

Theorem 7. *Let (X, ρ) be a metric space and $E \subseteq X$, $\alpha \geq 0$ and μ a mass distribution defined on E such that exist two constants L and δ such that*

$$\forall V \subseteq E,\ |V| \leq \delta \implies \mu(V) \leq L\,|V|^\alpha$$

Then we have $\mathcal{H}^\alpha(E) \geq \frac{\mu(E)}{L} > 0$. This clearly implies that, under the same conditions, $dim\,(E) \geq \alpha$.

Proof. Let $\eta > 0$ such that $\delta > \eta$ and let $C_1, C_2, \ldots$ be an η-covering for E. We can write:

$$0 < \mu(E) \leq \mu\left(\bigcup_{i=0}^{\infty} C_i\right) \leq \sum_{i=1}^{\infty} \mu(C_i) \leq L \sum_{i=1}^{\infty} |C_i|^\alpha$$

If μ is not defined on every C_i we can repeat the same proof using D_i, where D_i is the closure of C_i. Indeed we can notice that $|D_i| = |C_i|$. $\square$

Although it may seem difficult to find a measure with such properties (and often it is) there are many examples in which we have a set and a natural mass measure defined on it, and this measure has the desired property. Now we will make a quick deviation studying briefly a collateral property of Brownian motion, that is also a nice example of the use of the mass method.

Remark 3. Let $\{B_t\}_{t\in[0,1]}$ be a standard linear Brownian motion defined only in $[0,1]$ and let $M_t = \sup\{B_s : 0 \leq s \leq t\}$ be the associated maximal process. We define $Rec_{[0,1]} = \{t \in [0,1] : M_t = B_t\}$ the record set in $[0,1]$ for the linear Brownian motion. It holds:

$$dim\left(Rec_{[0,1]}\right) \geq \frac{1}{2}$$

Proof. We have that $t \mapsto M_t$ is a non decreasing function. We define on the Borel sets of $[0,1]$ the measure μ such that $\mu((a,b]) = M_a - M_b$, in particular this measure is supported by $Rec_{[0,1]}$ and almost surely is a mass measure. We know that almost surely we have that the Brownian motion is continuous and α Hölder for every $\alpha < \frac{1}{2}$. This means that, for every $0 < \alpha < \frac{1}{2}$ there is a random constant L_α for which we can write almost surely:

$$\forall a, b \in [0,1],\ M_b - M_a \leq \max_{0\leq h\leq b-a} B_{a+h} - B_a \leq L_\alpha(b-a)^\alpha$$

By the mass distribution principle we can now state that we have:

$$dim\left(Rec_{[0,1]}\right) \geq \alpha$$

for every $0 < \alpha < \frac{1}{2}$. This is enough to conclude. We could also show that $dim\left(Rec_{[0,1]}\right) = \frac{1}{2}$. To get the upper bound we can use a covering made by intervals of set length and make an estimation $\square$

3.2 The energy method

The mass method is a first, useful tool for the lower esteem, and the example we gave proves that it can be applied quite easily with strong results. But it does have a problem in its application, since we need to estimate the mass of a large number of small sets. The following method replaces this need and gives us a way for finding a lower bound simply checking the convergence of a certain integral.

Definition 8. Let (X,ρ) be a metric space, $E \subseteq X$, α a positive number and μ a mass distribution over E. We define the α-potential of a point $x \in E$ with respect to μ as:

$$\phi_\alpha(x) = \int \frac{d\mu(y)}{\rho(x,y)^\alpha}$$

We also define the α-energy of the mass distribution as:

$$I_\alpha(\mu) = \int \phi_\alpha(x)d\mu(x)$$

We are now ready to study the energy method.

Theorem 8 (Energy method). *Let (X, ρ) be a metric space, $E \subseteq X$ and $\alpha > 0$. If there is a mass distribution μ defined on E we have, for every $\delta > 0$:*

$$\mathcal{H}^{\alpha}_{\delta}(E) \geq \frac{\mu(E)^2}{\iint_{\rho(x,y)\leq\delta} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^{\alpha}}}$$

Proof. Choose $\epsilon > 0$, then, by the definition of Hausdorff measure, we can take $C_1, C_2, \ldots$ a δ-covering of E such that $\sum_{i=1}^{\infty} |C_i|^{\alpha} \leq \mathcal{H}^{\alpha}_{\delta}(E) + \epsilon$. The following inequalities provide us the result we were searching:

$$\begin{aligned} \mu(E)^2 &\leq \left(\sum_{i=1}^{\infty} \mu(C_i) \right)^2 = \left(\sum_{i=1}^{\infty} |C_i|^{\frac{\alpha}{2}} \frac{\mu(C_i)}{|C_i|^{\frac{\alpha}{2}}} \right)^2 \\ &\leq \left(\sum_{i=1}^{\infty} |C_i|^{\alpha} \right) \left(\sum_{i=1}^{\infty} \frac{\mu(C_i)^2}{|C_i|^{\alpha}} \right) \\ &\leq (\mathcal{H}^{\alpha}_{\delta}(E) + \epsilon) \left(\sum_{i=1}^{\infty} \iint_{C_i \times C_i} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^{\alpha}} \right) \\ &\leq (\mathcal{H}^{\alpha}_{\delta}(E) + \epsilon) \left(\iint_{\rho(x,y)\leq\delta} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^{\alpha}} \right) \end{aligned}$$

The inequality is obtained letting δ go to zero and dividing both sides by the integral. Furthermore, letting δ go to zero, if $\mathbb{E}I_{\alpha}(\mu) < \infty$, the integral converges to zero so that $\mathcal{H}^{\alpha}(E)$ diverges to infinity. □

More useful than this theorem is its corollary, that follows directly from it but it has a simpler application.

Corollary 2. *Let (X, ρ) be a metric space, $E \subseteq X$ and $\alpha > 0$. If there is a mass distribution μ defined on E such that*

$$I_{\alpha}(\mu) < \infty$$

then it holds $dim(E) \geq \alpha$.

Proof. We can use the energy method for α and a decreasing sequence of δ_i that has limit zero. By the energy method we get then $\mathcal{H}^{\alpha}(E) = \infty$. Then, by definition of Hausdorff dimension, we have our thesis. □

3.3 Taylor's Theorem

We are now ready to give the first lower bound for the Hausdorff dimension of the image of the Brownian motion. This result is due to Taylor (not the same one of the Taylor's series) and states the following:

Theorem 9 (Taylor 1953). *Let $\{B_t\}_{t\in[0,1]}$ be a d-dimensional Brownian motion, where $d \geq 2$, then almost surely we have:*

$$dim\left(B_{[0,1]}\right) = 2$$

Proof. We have to find a measure whose support is included in $B_{[0,1]}$. The natural measure over this set is the occupational measure, we can define it as the measure μ such that, for every $A \subseteq \mathbb{R}^d$ Borel set, we have:

$$\mu(A) = \int_0^1 1_A(B_t)dt$$

It is important to keep in mind that μ is a random measure, then we write $\mu(A)$ for $\mu_\omega(A)$. It is useful in this context to prove that it holds:

$$\int_{\mathbb{R}^d} f(x)d\mu(x) = \int_0^1 f(B_t)dt$$

To do this we start with the indicator functions of the Borel sets. So let $C \subseteq \mathbb{R}^d$ be a Borel set, then we can write:

$$\int_{\mathbb{R}^d} 1_C(x)d\mu(x) = \mu(C) = \int_0^1 1_c(B_t)dt$$

Then we have the other representation for Lebesgue's theorem. It is essential now to point out that, for every $\alpha \in (0,2)$ and for almost every $\omega \in \Omega$, it holds:

$$\begin{aligned} I_\alpha(\mu_\omega) &= \iint \frac{d\mu_\omega(x)d\mu_\omega(y)}{|x-y|^\alpha} = \int \left(\int \frac{d\mu_\omega(x)}{|x-y|^\alpha} \right) d\mu_\omega(y) \\ &= \int \left(\int_0^1 \frac{ds}{|B_s - y|^\alpha} \right) d\mu_\omega(y) = \int_0^1 \int_0^1 \frac{dsdt}{|B_s - B_t|^\alpha} \end{aligned}$$

To prove out thesis we are going to show that it holds:

$$\mathbb{E} I_\alpha(\mu) < \infty$$

But for what we just said this is equivalent to prove that:

$$\mathbb{E} \int_0^1 \int_0^1 \frac{dsdt}{|B_s - B_t|^\alpha} < \infty$$

If we can prove this we can use the energy method to prove that $\forall \alpha \in (0,2)$ the α-Hausdorff measure of our set is infinity. Then we can remember the lower bound of the last section to conclude and to reach our result.

Using the Brownian scaling we can evaluate the following expectation:

$$\mathbb{E} |B_t - B_s|^{-\alpha} = \mathbb{E} \left(|t-s|^{\frac{1}{2}} |B_1| \right)^{-\alpha} = |t-s|^{-\frac{\alpha}{2}} \int \frac{K_d}{|v|^\alpha} e^{-\frac{|v|^2}{2}} dv$$

where in the last equality we used the definition of multidimensional Brownian motion and the parameter K_d is finite and depends only on the dimension d in which the Brownian motion takes place. We could calculate the value of the last integral, but the only thing we need to know is that it is finite, and it will take a certain value K'_d, finite. Then we can write:

$$\begin{aligned} \mathbb{E} I_\alpha &= \mathbb{E} \int_0^1 \int_0^1 \frac{dsdt}{|B_s - B_t|^\alpha} \leq K'_d \int_0^1 \int_0^1 \frac{dsdt}{|t-s|^{\frac{\alpha}{2}}} \\ &\leq 2K'_d \int_0^1 \frac{dt}{t^{\frac{\alpha}{2}}} < \infty \end{aligned}$$

Then we have that it must hold $I_\alpha < \infty$ almost surely, and then for the energy method we can state that $dim\left(B_{[0,1]}\right) \geq 2$. For what we said in the last section we have that also the other inequality holds. $\square$

We can use the countable stability property of the Hausdorff dimension to state that the Hausdorff dimension of the image of $\mathbb{R}$ with respect to the Brownian motion is exactly 2. This result answers the question we asked ourselves starting this study. But the problem we just solved can be extended in interesting ways.

Chapter 4

The Frostman's Lemma and McKean's Theorem

In the previous section we finally gave an answer to our first question: which is the Hausdorff dimension of a multidimensional Browinan motion? The theorem due to Taylor gave us a complete answer. But now we are interested in another similar problem: what can we say about the dimension of the image (again with respect to the Brownian motion) of specific subsets of $\mathbb{R}$? I.e. we proved that, almost surely $dim\left(B_{[0,1]}\right) = 2$, now we are going to study the value of $dim\left(B_A\right)$, where A is a general subset of $\mathbb{R}$.

The main instrument in this section will be Frostman's lemma. This lemma is in some way the converse of the mass principle: given a lower bound we will find a proper mass distribution. It may seems strange to do such a thing if our goal is to find lower bound, but the aim is to find a lower bound for a set C that is a transformation of a set A that we do know. So we will find a measure over A using our knowledge of the Hausdorff dimension of A and Frostman's lemma, then we will use the same measure over C, using the transformation to transform also the measure.

4.1 The Max-Flow Min-Cut Theorem

Frostman's lemma has its foundation in the max-flow min-cut theorem, an important logic result about trees that uses compactness arguments. We will suppose that the reader is familiar with this logical procedure. We are going to need some notation before to state the theorem.

Definition 9. A tree T is a triple (V, E, ρ), where V is a non empty and (at most) countable set of elements called vertices, $\rho \in V$ a special element called root of the tree, and E is the set of edges, i.e. $E \subseteq V \times V$ is a set with the following properties:

- $\forall v \in V, v \neq \rho,\ \exists! w \in W$ such that $(w, v) \in E$, this element is called the parent of v. And there is no element w in V for which $(w, \rho) \in E$.

- $\forall v \in V,\ \{w \in V : (v, w) \in E\}$ is finite. This is the set of children of v.

- For every $v \in V$, there is only one finite self-avoiding path in E from ρ to v. The number of edges in this path is called the order of the vertex v and indicated with $|e|$. We extend this definition to edges: if $e \in E$, $|e| = |(v, w)| = |w|$ the order of its last vertex.

Given two vertex $v, w \in V$ we say that v is an ancestor of w, and write $v \leq w$, if v is in the path between ρ and w, i.e. v is a vertex of one of the edges of the only finite self-avoiding path between ρ and w. We indicate with $v \wedge w \in V$ the last common ancestor between v and w, i.e. the $t \in V$ with maximal order such that t is in the path between ρ and v and also in the path between ρ and w. This t is unique and does exist for every v, w [1] and we have obviously, for every $v, w \in V$, $v \wedge w = w \wedge v$.

Remark 4. Let $T = (V, E, \rho)$ be a countable tree. We call ray an infinite self avoiding path that starts in ρ, we can define the last common ancestor between two rays, i.e. if ξ, η are two rays we call $\xi \wedge \eta$ the vertex $v \in V$ of maximum order such that v is in η and also in ξ.
The set of all rays is indicated by ∂T and called the boundary of T. We can define on this set a distance with which the set becomes a metric space. Given $\xi, \eta \in \partial T$ we define:

$$d(\xi, \eta) = 2^{-|\xi \wedge \eta|}$$

It's trivial to prove that this is a distance: it is symmetric as it is the definition of last common ancestor, if two rays are different they have a last common ancestor of finite order, and the triangular inequality follows from a simple consideration. But what's interesting is that with this distance the metric space is compact[2].
$\Pi \subset E$ is called a cutset if every ray includes an edge of Π, i.e. if

$$\forall \xi \in \partial T, \ \exists e \in \xi \cap \Pi$$

We are interested in how we could define flows over trees. To understand the max-flow min-cut theorem we need one last definition:

Definition 10. Let $T = (V, E, \rho)$ be a countable tree, a capacity defined on T is a function $C : E \to [0, \infty)$. Given (T, C) a tree on which is defined a capacity, we say that $\theta : E \to [0, s]$ is a flow of strength s through the tree T of capacity C if the following holds:

- $\sum_{(\rho,v) \in E} \theta((\rho, v)) = s$.
- for every $v \in V$, if $(\bar{v}, v) \in E$,

$$\sum_{(v,w) \in E} \theta((v, w)) = \theta((\bar{v}, v))$$

  i.e. the flow is conserved through every vertex.
- for every $e \in E$, $\theta(e) \leq C(e)$.

[1] We can prove the existence and uniqueness by induction on the orders of v and w

[2] This follows from the fact that every vertex have at most a finite number of children. Then there is a finite number of vertex with order that not exceed a fixed $r \in \mathbb{N}$

Given θ a flow with capacity C we write $\sigma(\theta)$ to indicate its strength.

We are now ready for the max-flow min-cut theorem.

Theorem 10 (Max-flow min-cut)**.** *Let (V, E, ρ) be a countable tree on which is defined a capacity C. Then we have:*

$$\Sigma = \sup \{\sigma(\theta) : \theta \textit{ flow with capacity } C\} = \inf \left\{ \sum_{e \in \Pi} C(e) : \Pi \textit{ cutset} \right\} = \Xi$$

And the sup *and* inf *are reached.*

Proof. This proof can be divided in three basic steps: we prove that the sup and the inf are in fact a max and a min, then we show that for every θ flow and for every Π cutset it holds $\sigma(\theta) \leq \sum_{e \in \Pi} C(e)$, in the end we will show that the limit of a particular sequence θ_n of flows satisfies the equality.
Take $\theta_1, \theta_2, \dots$ a sequence of flows on T such that $\sigma(\theta_i) \uparrow \Sigma$, then we can use the Cantor diagonal argument to get a flow whose strength is the limit of the strengths of the sequence. To do this we can enumerate the set E (that is countable) and write $E = \{e_1, e_2, \dots\}$, then we can find a subsequence $\theta_{1,1}, \theta_{1,2}, \dots$ of $\theta_1, \theta_2, \dots$ for which $\theta_{1,1}(e_1), \theta_{1,2}(e_1), \dots$ does converge. We can repeat this procedure and get a sequence $\tilde{\theta}_1, \tilde{\theta}_2, \dots$ that does converge on every $e \in E$. The limit of this sequence does exist (is defined for every $e \in E$) and it is easy to see that does respect the requests for being a flow with respect to the capacity C. We will have also that the strength of the flow that we will get will be the limit of the sequence of strengths.

To reach the first inequality we first have to notice an interesting property of cutsets: if Π is a cutset, there is a $\Pi' \subseteq \Pi$ finite that is also a cutset. In fact suppose that there is no such sub cutset, then for any arbitrarily large n we would have that there exists a ray $\xi_n \in \partial T$ such that the first n edges of ξ_n are not in Π (if there is an M for which for every $\eta \in \partial T$, an edge of η of order less than M was in Π we can find a finite subset using the fact that every vertex has at most a finite number of children). But then, if we have that we can have an arbitrarily long self-avoiding path with no edges in common with Π, we could use the compactness argument to find a ray that has no edges in Π, but that's absurd.

Now let θ be a flow and Π an arbitrary cutset. And let $A_\Pi \subseteq V$ the set of vertices we can reach from the root not passing through any edge of Π, i.e. A_Π is the set of every vertex v for which the only finite self avoiding path that leads from the root to v does not intersect Π. We have that for every Π cutset, A_Π is finite (we can use the same arguments as before). Le's define the function $\psi : V \times E \to \{-1, 1, 0\}$ that indicates if a vertex is part of an edge, i.e.:

$$\psi(v, e) = \begin{cases} 1 & \text{if } e = (v, w) \text{ for some } w \in V \\ -1 & \text{if } e = (w, v) \text{ for some } w \in V \\ 0 & \text{otherwise} \end{cases}$$

We are now ready to prove the first inequality:

$$\begin{aligned}\sigma(\theta) &= \sum_{e\in E} \psi(\rho, e)\theta(e) = \sum_{v\in A_\Pi} \sum_{e\in E} \psi(v, e)\theta(e)\\ &= \sum_{e\in E} \theta(e) \sum_{v\in A_\Pi} \psi(v, e) \leq \sum_{v\in A_\Pi} \theta(e) \leq \sum_{e\in \Pi} C(e)\end{aligned}$$

The second passage of this last expression is justified because it holds, if $v \neq \rho$, $\sum_{e\in E} \psi(v, e) = 0$ by definition of flow.

We have only to prove the inverse inequality. So we have to introduce some more notation, in particular we want to construct a flow with the desired strength, and to define such a flow we are going to work with finite trees. Our aim will be to define proper flows in sub-trees of T and then extend the flows we have obtained to T. So let's define, for every $n \geq 2$, $T_n = (V_n, E_n, \rho)$ the tree where $V_n \subseteq V$ is the set of every vertex of order that does not exceed n and E_n is the restriction of E to V_n. We define in the same way the capacity C_n of T_n. We may take $\Pi_n \subseteq E_n$ the intersection between a cutset and E_n. We have also to redefine the concept of flow. We will say that $\theta_n : E_n \to [0, s]$ is a flow of strength s if is a restriction of a flow θ on E_n with strength s. The only thing that really changes is that the vertices of order n do not have the property for which the flow that enters them has the same value of the sum of the values of the flows that exit from them.
So we have now to prove, for every $n \geq 2$, that there exists θ_n a flow in T_n and a cutset Π_n such that the strength of θ_n is greater than the flow through the cutset. So let θ_n be a flow of maximal strength Σ_n (we proved before it does exists) with respect to the capacity C_n. We call a sequence of vertices $v_0, v_1, \ldots, v_n$ in V_n such that $v_0 = \rho$ and v_{i+1} is child of v_i for every i an augmenting sequence if $\theta(v_i, v_{i+1}) < C(v_i, v_{i+1})$. The whole point of this third step is that if we have a flow θ_n that does contain augmenting sequences, we can construct a flow $\bar{\theta}_n$ of strength greater than Σ_n. We can construct $\bar{\theta}_n$ by increasing the flow through every edge of an ϵ. Is now also clear why we restricted ourselves to finite trees: we can increase the flow of the smallest quantity that let $\bar{\theta}_n$ be a flow, this could not be possible with infinite trees.
Then we can construct a cutset $\Pi_n \subset E_n$ such that, $\forall e \in \Pi_n,\ \theta(e) \geq C(e)$. We can then write (defining $A^n_{\Pi_n}$ as the restriction to E_n of A_{Π_n}):

$$\sigma(\theta) = \sum_{e\in E} \theta(e) \sum_{v\in A^n_{\Pi_n}} \phi(v, e) = \sum_{e\in \Pi_n} \theta(e) \geq \sum e \in \Pi_n C(e)$$

Now we have θ_n a sequence of flows defined on T_n. We can use the diagonal argument, just like we did before, and build a subsequence of θ_n that does converge for every $e \in E$. We can take the flow that, in every edge, is defined as the limit of this sequence. What we obtain is a flow with the desired properties.
□

4.2 Frostman's lemma

We are now ready to present Frostman's lemma, a central tool for finding lower bound of Hausdorff dimension. This instrument allows us to find a proper mass distribution over a closed set with positive α-Hausdorff measure and does represent a converse for the mass method. If with the mass method we could find a lower bound for the Hausdorff dimension using the existence of a mass distribution with certain qualities, this result will let us find a mass distribution only knowing that the α-Hausdorff measure of this set is positive.

Theorem 11 (Frostman's lemma)**.** *Let $\alpha > 0$ and $X \subseteq \mathbb{R}^d$ be a closed set such that $\mathcal{H}^\alpha(X) > 0$, then there exists μ, a mass measure defined on the Borel sets of $\mathbb{R}^d$ and supported in X, and $L > 0$ such that, for every D Borel set,*

$$\mu(D) \leq L\,|D|^\alpha$$

Proof. We want to build a measure, that is, in some sort of way, upper-bounded by the diameter of the set. But we have studied just now a theorem that allows us to to build a function on the edges of a tree, and the value of this function is upper-bounded by the value of an arbitrary function called capacity. But the max-flow min-cut theorem does more: the function it does build is maximal in some sort of way, and this will give us a non trivial function.
We can suppose that $E \subseteq [0,1]^d$[3]. We are now ready to construct our tree $T = (V, E, \rho)$, and we are going to do it by recursion. We define $\rho = [0,1]^d$, the children of ρ will be those of the the 2^d non-overlapping compact cubes of $\mathbb{R}^d$ with side length $\frac{1}{2}$ that does intersect X; i.e. we consider every compact cube of side length $\frac{1}{2}$, that has a vertex in $\left\{0, \frac{1}{2}\right\}^d$, and we will say that this is a child of ρ if it does intersect X. We can recursively define the tree T, and we will have that every vertex of order n is a compact cube of side length 2^{-n} and does intersect X.
On this tree we can define the capacity $C : E \to [0,\infty)$, for which $C(e) = \left(d^{\frac{1}{2}} 2^{|e|}\right)^\alpha$, and we recall that $|e|$ is the order of the edge that is, n if $e = (v,w)$ and w is a vertex of order n.
At this point we can define a map $\Psi : \partial T \to X$ in the obvious way: if $\xi \in \partial T$ then we define $\Psi(\xi) = \bigcap_{(v,w)\in\xi} v$, remember that the vertices of T are cubes, so we can intersect them. It's easy to see that this map is surjective (but it's not injective, because points as $(\frac{1}{2}, \ldots, \frac{1}{2})$ have more than one representation).
We now associate at every cutset Π of the tree, a covering of the set X, i.e. given $\Pi = \{e_1 = (v_1, w_2), e_2 = (v_2, w_2), \ldots\}$ we associate to Π the covering $\Sigma_\Pi = \{v_1, v_2, \ldots\}$ of X. It's not difficult to see that for every Π cutset, Σ_Π is a covering, indeed we can notice that Ψ is surjective, then for every $x \in X$, it does exists $\xi \in \partial T$ such that $\Psi(\xi) = x$, but there must be an edge of $\xi \in \Pi$, and hence we have that $x \in \bigcup_{v\in\Sigma_\Pi} v$, and then Σ_Π is a covering. But then we

[3] If this is not the case we can work with X like a countable union of compact sets X_i, each of them subsets of a certain cube of side length 1

can write:

$$\inf\left\{\sum_{e\in\Pi} C(e) : \Pi \text{ cutset}\right\} \geq \inf\left\{\sum_i |Y_i|^\alpha : X \subset \bigcup_i Y_i\right\}$$

But we know that $\mathcal{H}^\alpha(X) > 0$, then $\inf\left\{\sum_{e\in\Pi} C(e) : \Pi \text{ cutset}\right\} > 0$. Then we can apply the max-flow min-cut theorem to find a flow θ defined on the tree T with capacity C. This flow will have positive strength. We are now going to use this flow to define a measure on the Borel sets of $[0,1]^d$, and to do that we need just another notation: if $e = (v,w) \in E$ is an edge of T, we define $A(e)$ the set $A(e) \subseteq \partial T$ of rays of T that does contain e. Now we are ready to define the function:

$$\bar{\nu}(A(e)) = \theta(e)$$

using Dynkin lemma we can extend this function to a measure ν defined on the σ-algebra generated by the sets $A(e)$. But we want a measure defined on the Borel sets, then we can use $\mu = \nu\circ\Psi^{-1}$. This function is a mass measure, defined on the Borel sets of $[0,1]^d$, it obviously is supported by X and if $e = (v,w)$ is a vertex we have that $\mu(v) = \theta(e)$.
We are then ready to prove that does hold the inequality $\mu(D) \leq L\,|D|^\alpha$, for a certain constant L and for every D Borel set. Suppose n is such that $2^{-n} < |D| \leq 2^{-(n-1)}$. Then we can cover D with not more than 3^d cubes having side length 2^{-n}. Using this bound we have, considering that these sets have diameter $d^{\frac{1}{2}}2^{-n}$, that we can write:

$$\mu(D) \leq d^{\frac{\alpha}{2}}3^d 2^{-n\alpha} \leq d^{\frac{\alpha}{2}}3^d\,|D|^\alpha$$

So the measure μ we have found does respect the requirements we asked. □

4.3 Riesz capacity and McKean's theorem

In this last part of the chapter we are going to see one powerful application of Frostman's lemma: we are going to use it to generalize the result obtained with Taylor's lemma. To do that we are going to need first another result.

Definition 11 (Riesz capacity)**.** Let (X,ρ) be a metric space and $E \subseteq X$, let also $\alpha > 0$. Then we define the α-Riesz capacity of E as:

$$Cap_\alpha(E) = \left\{\frac{1}{I_{\alpha(\mu)}} : \mu \text{ mass distribution with } \mu(W) = 1\right\}$$

Before to prove McKean's theorem it's useful to give another way to find the Hausdorff dimension of closed set. What this result does is to assure us of the fact that the energy method is sharp. I.e. the estimation for the Hausdorff dimension we obtained with the energy method is the best estimation we can have, and does indicate the exact Hausdorff dimension. This is possible because of the Frosman lemma, that allows us to find a proper mass distribution, if the Hausdorff measure is not zero. So the following result is the bridge between the formal definition of Hausdorff dimension, and the presentation of the Hausdorff dimension as the limit value α for which, if $\beta > \alpha$, is not possible to find a non-trivial and mass measure with the desired qualities.

Theorem 12. *For any closed $E \subseteq \mathbb{R}^d$ it holds:*

$$dim\,(E) = \sup\{\alpha : Cap_\alpha\,(E) > 0\}$$

Proof. We have just to prove the $\leq$, because the other inequality follows from the energy method. To prove $\leq$ we have to find, for every $\alpha < dim\,(E)$, a probability measure ν, defined on the Borel sets of $\mathbb{R}^d$ with support in E, such that it holds: $I_\alpha(\mu) < \infty$. To do that first of all we suppose that E is contained on a ball of unit diameter[4], then we take $\beta > \alpha$ such that $\mathcal{H}^\beta\,(E) > 0$ (it exists, is enough to take $\beta < dim\,(E)$). Then, by Frosman's lemma, we have that does exists a mass measure μ, supported in E, and a constant L, such that $\mu(D) \leq L\,|D|^\beta$, for every D closed. We can also suppose that μ is a probability measure[5]. Fixed $x \in E$ and for $k \geq 1$ we can define $S_k(x) = \left\{y : |x-y| \in (2^{-k}, 2^{1-k}]\right\}$. But then we can write:

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{d\mu(y)}{|x-y|^\alpha} &= \sum_{k=1}^{\infty} \int_{S_k(x)} \frac{d\mu(y)}{|x-y|^\alpha}\\ &\leq \sum_{k=1}^{\infty} \mu(S_k(x))2^{k\alpha} \leq L \sum_{k=1}^{\infty} \left|2^{2-k}\right|^\beta 2^{k\alpha}\\ &= L' \sum_{k=1}^{\infty} 2^{k(\alpha-\beta)} < \infty\end{aligned}$$

But then we just proved that it holds $I_\alpha(\mu) < \infty$. □

Now we are finally ready to study the generalization of Taylor's theorem.

Theorem 13 (McKean)**.** *Let $\{B_t\}_{t\geq 0}$ be a d-dimensional Brownian motion, with $d \geq 2$, then, fixed $A \subseteq [0, \infty)$ a closed set, it holds, almost surely:*

$$dim\,(B_A) = 2dim\,(A)$$

Remark 5. It's fundamental to make a remark before proving the theorem. It is important to point out that the claim of the theorem is not that almost surely, for every A closed set, it holds the equality. What is the difference? We are going to prove now that, fixed A closed set, almost surely $dim\,(B_A) = 2dim\,(A)$. But this does not mean that for almost every random path of the Brownian motion, this path maps A in a set B_A of Hausdorff dimension $2dim\,(A)$. Then, with this result, we have a strong property of $\{B_t\}_{t\geq 0}$ as a stochastic process, but we cannot give any guarantee about what happens in almost every random path.

Proof. Let $\alpha < dim\,(A)$, then by our last theorem there exists μ mass measure such that $\mu(A) = 1$ and $I_\alpha(\mu) < \infty$. We need also to introduce the measure, defined on the Borel sets of $\mathbb{R}^d$ as:

$$\nu(D) = \mu(\{t \geq 0 : B_t \in D\})$$

[4]We can do that, because if holds the countable stability of the Hausdorff dimension

[5]If it is not we can normalize it

a sort of occupational measure, built upon the measure given by Frostman's lemma and not upon Lebesgue measure. Then it holds:

$$\mathbb{E}I_{2\alpha}(\nu) = \mathbb{E}\iint \frac{d\nu(x)d\nu(y)}{|x-y|^{2\alpha}} = \mathbb{E}\iint_{\mathbb{R}^2} \frac{dsdt}{|B_s - B_t|^{2\alpha}}$$

But $2\alpha < d$ and we have, for Brownian scaling, that $|B_s - B_t|^{2\alpha}$ has the same distribution as $|t-s|^{\alpha}\,|Z|^{2\alpha}$, where Z is a d-dimensional standard normal random variable. But then we can write:

$$\mathbb{E}\,|Z|^{-2\alpha} = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} |y|^{-2\alpha}\, e^{\frac{|y|^2}{2}}\, dy < \infty$$

And then, at least, for Fubini's theorem, we can write:

$$\mathbb{E}I_{2\alpha}(\nu) = \int_0^{\infty}\int_0^{\infty} \frac{1}{|t-s|^{\alpha}}\mathbb{E}\,|Z|^{-2\alpha}\,\mu(s)d\mu(t) \leq I_{\alpha}(\mu)\mathbb{E}\,|Z|^{-2\alpha} < \infty$$

But then we can conclude simply observing that ν is supported in B_A, because μ is supported on A. Then, for the energy method, we can say that $dim\,(B_A) \geq 2dim\,(A)$ almost surely. $\square$

Chapter 5

Kaufman's doubling theorem

McKean's theorem states that, for any $A \subseteq \mathbb{R}$ closed set, it holds almost surely that $dim\left(B_A\right) = 2dim\left(A\right)$ if B is a d-dimensional Brownian motion and $d \geq 2$. But, this theorem says little about every single path of the Brownian motion. Indeed, it would not be in contradiction with McKean's theorem the existence, for any random path, a random set A_ω for which $dim\left(B_{A_\omega}\right) \neq dim\left(A_\omega\right)$. Kaufman's doubling theorem is a strong and interesting extension of McKean's theorem because it let us switch the order of the quantifiers and gives us a statement about each single path (almost surely).

But Kaufman's theorem is also quite important because it has some strong consequences, i.e. there are plenty of results that become trivial after the proof of Kaufman's theorem. For this reason in the first part of this chapter we are going to introduce and prove Kaufman's theorem and in the last part we are going to cite some of its most important and geometrically interesting consequences.

Although Kaufman's theorem is not difficult to prove, it is useful to divide its proof in two parts: the case in which $d = 2$ and the case $d > 2$. For this reason we will immediately enunciate Kaufman's theorem, and then we will study separately the two different situations.

Theorem 14 (Kaufman, 1969)**.** *Let $d \geq 2$ and $\{B_t\}_{t \geq 0}$ be a Brownian motion. Almost surely, for any set $A \subseteq [0, \infty)$*

$$dim\left(B_A\right) = 2dim\left(A\right)$$

5.1 Kaufman's theorem in dimension $d > 2$

The proof of Kaufman's lemma in dimension $d > 2$ is based on the transience of the Brownian motion in dimension greater than 2. For this reason we can not use the same proof to prove the result in dimension $d = 2$.

Lemma 6. *Let $d \geq 3$, $x \in \mathbb{R}^d$ and $Q_x^r \subseteq \mathbb{R}^d$ be the cube centered in x of side length $2r$. And let also $\{B_t\}_{t \geq 0}$ be a d-dimensional Brownian motion. We then can define:*

$$\tau_1^{Q_x^r} = \inf\{t \geq 0 : B_t \in Q_x^r\}$$
$$\tau_{k+1}^{Q_x^r} = \inf\left\{t \geq \tau_k^{Q_x^r} + r^2 : B_t \in Q_x^r\right\}$$

We use the convention that $\inf \emptyset = \infty$*. Then it exists* $\theta \in (0,1)$ *dependent only on* d *such that,* $\forall z \in \mathbb{R}^d$ *and for every* $n \in \mathbb{N}$*,*

$$\mathbb{P}_z \left\{ \tau_{n+1}^{Q_x^r} < \infty \right\} \leq \theta^n$$

Proof. In this proof we will write τ_k for $\tau_k^{Q_x^r}$. To prove the result it is sufficient to prove that, for some $\theta \in (0,1)$, it holds:

$$\mathbb{P}_z \left\{ \tau_{k+1} = \infty \mid \tau_k < \infty \right\} > 1 - \theta$$

But, being in dimension $d \geq 3$, we can use the transience of the Brownian motion and the strong Markov property to write:

$$\begin{aligned} \mathbb{P}_z \left\{ \tau_{k+1} = \infty \middle| \, \tau_k < \infty \right\} \geq \mathbb{P}_z \left\{ \tau_{k+1} = \infty \middle| \, |B_{\tau_k + r^2} - x| > 3r, \tau_k < \infty \right\} \cdot \\ \cdot \mathbb{P}_z \left\{ |B_{\tau_k + r^2} - x| > 3r \middle| \, \tau_k < \infty \right\} \end{aligned}$$

But now we can bound from below each component of the second factor.

- We can bound $\mathbb{P}_z \left\{ \tau_{k+1} = \infty \middle| \, |B_{\tau_k + r^2} - x| > 3r, \tau_k < \infty \right\}$ with $\inf_{y \notin Q_x^{3r}} \mathbb{P}_y \left\{ \tau_1^{Q_x^r} = \infty \right\}$. This bound is positive and does not depend on the scaling factor r.
- We can also bound $\mathbb{P}_z \left\{ |B_{\tau_k + r^2} - x| > 3r \middle| \, \tau_k < \infty \right\}$, using the strong Markov property, with $\inf_{y \in Q_x^r} \mathbb{P}_y \left\{ |B_{r^2} - x| > 3r \right\}$ that also is positive and not dependent on r.

$\square$

We need another lemma in order to proceed to the proof of Kaufman's theorem.

Lemma 7. *Consider the setup of the last lemma. There exists a random variable* C *such that, almost surely, for all* m *and for every generic dyadic cube* Q *of side length* 2^{-m} *inside the cube* $[-\frac{1}{2}, \frac{1}{2}]^d$*, we have* $\tau^Q_{\lceil mC+1 \rceil} = \infty$

Proof. We can use the last lemma to write:

$$\sum_{m=1}^{\infty} \sum_{Q} \mathbb{P} \left\{ \tau^Q_{\lceil mC+1 \rceil} < \infty \right\} \leq \sum_{m=1}^{\infty} 2^{dm} \theta^{cm}$$

Where the sum of Q is done for every generic dyadic cube Q of side length 2^{-m} inside the cube $[-\frac{1}{2}, \frac{1}{2}]^d$. We can then choose c so large that $2^d \theta^c < 1$. Then, by Borel-Cantelli lemma, we have $\tau^Q_{\lceil mc+1 \rceil} = \infty$ for all but finitely many m and for every Q as before. Finally we can choose C such that, for every ω, $C(\omega) > c$ to handle the finitely many cubes that do not enter in the previous case. $\square$

We are now ready to prove Kaufman's lemma in dimension $d > 2$. We recall here (and we won't repeat it in the next chapter) that we need only to prove the lower bound, as the upper bound follows directly from the fact that the Brownian motion is α Hölder continuous for every $a \in (0, \frac{1}{2})$.

Proof. (Kaufman's theorem with $d > 2$) We are going to prove that almost surely, for every $L > 0$ and for every set $S \subseteq [-L, L]^d$, it holds:

$$dim\left(B_S^{-1}\right) \leq \frac{1}{2} dim\left(S\right)$$

Proved that we can simply choose $S = B_A \cap [-L, L]^d$ and take a divergent sequence $L_1, L_2, \ldots$ to prove the generic result. We do not need to take a generic L because, for the scaling invariance, we can without loss of generality, consider only the case $L = \frac{1}{2}$.
We are also going to prove our result only for those paths $\gamma : t \mapsto B_t$ that satisfy the conditions of the last lemma. This will give us the thesis since we just proved that these paths have full dimension. Let then γ be a random path of our Brownian motion that satisfies the conditions of the last lemma for a constant $C > 0$. Take also β such that $dim\left(S\right) < \beta$. For every $\epsilon > 0$ we can take a covering $E = \{E_1, E_2, \ldots\}$ of S made only by dyadic subcubes of $[-\frac{1}{2}, \frac{1}{2}]^d$ such that $\sum_{i \in \mathbb{N}} |E_i|^\beta < \epsilon$. But then, if N_m is the number of dyadic cubes of E of side length 2^{-n}, we have also that $\sum_{m=1}^{\infty} 2^{-m\beta} N_m < \epsilon$. But we took a sample path γ for which there is a covering of B_S^{-1} such that for each $m \geq 1$ it uses at most CmN_m intervals of length $r^2 = d2^{-2m}$. But now, if we take $\eta > \beta$ we can bound the $\frac{\eta}{2}$-dimensional Hausdorff content of B_S^{-1} from above by

$$\sum_{m=1}^{\infty} CmN_m (d2^{-2m})^{\frac{\eta}{2}} = Cd^{\frac{\eta}{2}} \sum_{m=1}^{\infty} mN_m 2^{-2m}$$

And we can bound from above this quantity choosing a suitable $\epsilon > 0$. Thus B_S^{-1} has Hausdorff dimension at most $\frac{\eta}{2}$. And therefore, letting $\eta \to dim\left(S\right)$, we can write:

$$dim\left(B_S^{-1}\right) \leq \frac{dim\left(S\right)}{2}$$

□

5.2 Kaufman's theorem in dimension $d = 2$ and a study in dimension $d = 1$

In $d = 2$ we can not use the transience of Brownian motion. We are then going to look at the random paths only up to a certain stopping time. We will find that a convenient class of stopping times is the one of the stopping times of the form $\tau_R^* = \min\{t : |B_t| = R\}$. We can then simply show that it holds, almost surely,

$$\forall A \subseteq [0, \infty),\ dim\left(B_A\right) \geq 2 dim\left(A \cap [0, \tau_R^*]\right)$$

We can use the same argument we used in $d > 3$, if we modify in a proper way the first lemma of this chapter. In dimension $d = 2$ we have that it holds.

Lemma 8. *Consider a square $Q \subset \mathbb{R}^2$ centered at x and of diameter $2r$. Let also R be such that Q is a subset of the ball of radius R centered at the origin.*

Let $\{B_t\}_{t\geq 0}$ be a planar Brownian motion. We can define, as before,

$$\tau_1^Q = \inf\{t \geq 0 : B_t \in Q\}$$
$$\tau_{k+1}^Q = \inf\left\{t \geq \tau_k^Q + r^2 : B_t \in Q\right\}$$

then there exists $c(R)$, function of R such that, if m is such that $2^{-m-1} < r < 2^{-m}$, for any $z \in \mathbb{R}^2$,

$$\mathbb{P}_z\left\{\tau_k^Q < \tau_R^*\right\} \leq (1 - \frac{c}{m})^k \leq e^{-\frac{ck}{m}}$$

Proof. As before we are going to bound $\mathbb{P}_z\left\{\tau_{k+1}^Q \geq \tau_R^* | \tau_k^Q < \tau_R^*\right\}$ from below, using:

$$\mathbb{P}_z\left\{\tau_{k+1}^Q \geq \tau_R^* | \tau_k^Q < \tau_R^*\right\} \geq \mathbb{P}_z\left\{\tau_{k+1}^Q \geq \tau_R^* | \left|B_{\tau_k^Q + R^2} - x\right| > 2r, \tau_k^Q < \tau_R^*\right\} \cdot$$
$$\cdot\, \mathbb{P}_z\left\{\left|B_{\tau_k^Q + R^2} - x\right| > 2r | \tau_k^Q < \tau_R^*\right\}$$

We can bound from below these two factors.

- $\mathbb{P}_z\left\{\tau_{k+1}^Q \geq \tau_R^* | \left|B_{\tau_k^Q + R^2} - x\right| > 2r, \tau_k^Q < \tau_R^*\right\}$ is bounded from below by the probability that planar Brownian motion started at any point in the boundary of $B(0, 2r)$ hits the boundary of $B(0, 2R)$ before hitting the boundary of $B(0, r)$. Using a well known property of Brownian motion this is bounded from below by $\frac{1}{\log_2 R + 2 + m}$ that is at least $\frac{c}{m}$ for some $c > 0$ that depends only on R.

- $\mathbb{P}_z\left\{\left|B_{\tau_k^Q + R^2} - x\right| > 2r | \tau_k^Q < \tau_R^*\right\}$ is bounded from below by a positive constant, independent of r and R.

$\square$

We can now complete the proof of the Kaufman lemma as we did in dimension $d > 3$. We will not repeat the proof because it is exactly the same of what we did before.
There is another interesting situation we did not treat until now. From the beginning of this thesis we focused on dimensions $d \geq 2$. And also Kaufman's doubling theorem is a result that holds only in these dimensions. But there is an interesting version of Kaufman's theorem also for linear Brownian motion. The aim of this thesis is to study some geometrical properties of the Brownian motion, for this reason this result is interesting, although our focus is on dimension 2 or greater. We are then going only to state the theorem.

Theorem 15. *Let $\{B_t\}_{t\geq 0}$ be a linear Brownian motion. Then, almost surely, for every nonempty closed sets $S \subseteq \mathbb{R}$, we have*

$$dim\,(B_S)^{-1} = \frac{1}{2} + \frac{dim\,(S)}{2}$$

Proof. We are not going to prove this theorem, but what follows is an idea of how to prove it.
In this case we have to show both the upper bound and the lower bound. We begin with the upper bound. Let $\{W_t\}_{t\geq 0}$ be a Brownian motion independent of $\{B_t\}_{t\geq 0}$. Then we have that (B_t, W_t) is a planar Brownian motion. We can apply Kaufman's theorem to $\tilde{B}_t = (B_t, W_t)$ and we can say that it holds, almost surely, that for every set $S \subseteq \mathbb{R}$,

$$dim\left(B^{-1}(S)\right) = dim\left(\tilde{B}^{-1}(S \times \mathbb{R})\right) \leq \frac{1}{2} dim\,(S \times \mathbb{R}) = \frac{1}{2} + \frac{1}{2} dim\,(S)$$

And this proves the upper bound.
To get the lower bound we have to use Frostman's lemma. We suppose then that $S \subseteq (-M, M)$ is closed and $dim\,(S) > \alpha$. Then, for Frostman's lemma, it exists a measure μ supported by S such that

$$\mu(\mathcal{B}(x,r)) \leq r^{\alpha} \ for\ all\ x \in S,\ 0 < r < 1$$

It can be proved that this measure satisfies the conditions of the mass distribution principle. The lower bound follows when $\alpha \to dim\,(S)$. $\square$

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