

A TRANSFERENCE PRINCIPLE AND A COUNTING LEMMA FOR SPARSE HYPERGRAPHS

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ABSTRACT. The last two decades have witnessed a growing trend towards proving sparse random analogues of combinatorial theorems. One unified approach to proving such theorems, formalised by Conlon and Gowers, involves establishing a ‘transference principle’ which allows one to translate between robust properties in the dense setting to the sparse random setting, provided the density is not too small. Our results extend the previous literature by generalising a transference theorem of Conlon and Gowers, and applying this to obtain a counting result for hypergraphs, and graphs which are not necessarily strictly balanced. We obtain an asymptotically optimal probability of success.

1. INTRODUCTION

Extremal combinatorics is the branch of discrete mathematics dealing with finding conditions that force the existence of specified substructures. A rich area of study focuses on finding density conditions that guarantee that a subset of a complete set contains a specific structure.

Our goal in this work is to formalise and extend the Transference Principle, which is a method that can be traced back to the seminal paper of Green and Tao [11] and which was then further developed by Conlon and Gowers [5].

Intuitively, the transference principle is a method that allows one to translate “robust” counting results that are known in the dense regime to a random sparse regime. Let us see an example.

For graphs H and G , let us denote by $c(H, G)$ the number of copies of H in G . Moreover, let $m_2(H) = \max_{H' \subseteq H, |H'| \geq 3} \frac{e(H') - 1}{v(H') - 2}$ be the two-density of H . For any graph H , Erdős-Stone-Simonovits’ Theorem [9, 8] guarantees that for any $\varepsilon > 0$ there is N large enough such that any subgraph F of K_N with at least $(1 - \frac{1}{\chi(H)-1} + o(1)) \binom{N}{2}$ edges contains a copy of H . This result is “robust”, by which we mean that any subgraph of K_N with at least $(1 - \frac{1}{\chi(H)-1} + \varepsilon) \binom{N}{2}$ contains $\Omega(N^{v(H)})$ copies of H , as proved by Erdős and Simonovits [10] (see also [15] for a survey on the topic).

Now that we have an example of a robust counting result that is known in the dense regime, let us see how we can translate it to a sparse random regime. In this variation, we are interested in finding copies of H in subgraphs of $G_{N,p}$, the random graph over N vertices where each edge is selected independently with probability $p = p(N)$.

What the transference principle allows us to do is to reduce a counting in the sparse random regime to a counting in the dense regime, which we know how to do. It enables us to count the copies of H in a subgraph G of $\Gamma = G_{N,p}$ having sufficient relative density, by counting the copies of H in a dense model G' of G , which looks ‘similar’ to G under the appropriate point of view.

The formal translation from the sparse random regime to the dense regime in this setting (which is an application of our general transference principle) is as follows.

Theorem 1. *Let H be a fixed graph and $\varepsilon > 0$. Then there exists a constant $C > 0$ such that the following holds. Suppose that $p > CN^{-1/m_2(H)}$, and let η_N be the probability that the number of copies of H in $\Gamma = G_{N,p}$ exceeds $(1 + \frac{\varepsilon}{2})p^{e(H)}N^{v(H)}$. Then with probability at least $1 - \eta_N$, for every subgraph $G \subseteq \Gamma$ there exists a graph G' on $V(G)$ that satisfies:*

$$e(G)p^{-1} = e(G') \pm \varepsilon N^2 \quad \text{and} \quad c(H, G)p^{-e(H)} = c(H, G') \pm \varepsilon N^{v(H)}.$$

This result allows us to count copies of the fixed graph H in G in the following way. Given $p > CN^{-1/m_2(H)}$ and G a subset of $\Gamma = G_{N,p}$ with at least $(1 - \frac{1}{\chi(H)-1} + \varepsilon)p\binom{N}{2}$ edges, by applying theorem 1, with the right parameter $\varepsilon = \varepsilon'$, we can find a good model G' of G - that is, a subgraph G' of K_N with at least $(1 - \frac{1}{\chi(H)-1} + \varepsilon')\binom{N}{2}$ edges, and such that the number of copies of H in G is (up to rescaling) the number of copies of H in G' . In particular, Erdős and Simonovits' result [10] gives us that G' contains $\Omega(N^{v(H)})$ copies of H , which guarantees that G contains $\Omega(N^{v(H)}p_N^{e(H)})$ copies of H , which is a positive proportion of the expected number of copies of H in $G_{N,p}$.

The threshold for p is clearly optimal, since it is the threshold for $G(n, p)$ to contain a copy of H .

We also have that our “success probability” $1 - \eta_N$ is optimal, but we postpone to the proof of the theorem to see the details.

2. A GENERAL TRANSFERENCE PRINCIPLE AND ITS APPLICATIONS

We mentioned that our interest is to generalise and extend the transference principle. We start by seeing how we can translate Theorem 1 in a more abstract setting. We are then going to state and prove an extended version of this translation.

The main idea here is that we can formulate Theorem 1 as a statement about the set of edges of K_N . That is, we can consider $n = \binom{N}{2}$, and arbitrarily define a bijection between $[n]$ and the set of edges of K_N . Once we have done that, we can define the hypergraph S of all copies of H in $[n]$. By construction, S is a subset of $\binom{n}{k}$ (we have $k = 4$ as H has 4 edges), and has size of the order of $N^{v(H)}$. Counting copies of H in G is the same as counting elements of S contained in $[n]_{p_N}$.

Notice a similar procedure can also be done for counting copies of an r -uniform hypergraph (we would just need to consider $n = \binom{N}{r}$).

Given a set $[n]$, a uniform hypergraph S , and a subset Y of the random set $[n]_{p_N}$ (for appropriate values of p_N), our general transference principle allows us to find a dense model Z of Y such that the number of elements of S in Y is (up to scaling) close to the number of elements of S in Z .

Actually, our transference principle allows for a further layer of generality, for which we need additional notation. We now introduce the necessary notation and state the general version of our transference principle.

Given positive integers $n, k \geq 2$, a k -uniform ordered hypergraph S of size n is a k -uniform hypergraph on $[n]$ with an order associated to each of its edges. That is, each edge of a k -uniform ordered hypergraph is an (ordered) sequence of length k of elements of $[n]$. Given $x \in [n]$ and $i \in [k]$, we write $S_i(x)$ for the subset of S consisting of all edges whose i -th entry is x . Given such an hypergraph S , and a sequence \mathbf{x} of length k of elements of $[n] \cup \{*\}$, we write $\deg_S(\mathbf{x})$ for the number of edges of S which agree with \mathbf{x} at all positions which do not equal $*$. That is, those entries equalling $*$ are allowed to vary, while the others are fixed to the value they have in \mathbf{x} . For ℓ a positive integer, we write $\Delta_\ell(S)$ for the maximum value of $\deg_S(\mathbf{x})$

over all sequences \mathbf{x} with exactly ℓ entries not equal to $*$. This is the standard codegree in the ordered hypergraph setting, where ℓ vertices are fixed and the number of edges containing them is counted.

We call a function $\sigma : [n] \rightarrow [0, 1]$ over the set of vertices a *similarity function*. We call a function $\omega : S \rightarrow [0, 1]$ over the set of edges a *subcount*. We abuse notation by denoting with $\mathbf{1}$ any function that takes value 1 on its domain (whatever that might be). We write $\mathbf{1}$ to denote the indicator function of a proposition, which is, we write for example $\mathbf{1}(y \in Y)$ to be the function that has value 1 when ' $y \in Y$ ' is true, and value 0 when it is false (the domain is always clear from the context). For any real numbers x, y, z , we also write $x = y \pm z$ to indicate $y - z \leq x \leq y + z$.

Very importantly, we now introduce a general setting that accompanies us for the rest of this work. That is, we fix now the following quantities, and refer back to them frequently in the following.

Setting 2. Let $k, n \geq 2$ be fixed integers, let $c, p > 0$ be real numbers with $p \in (0, 1)$. Let S be a k -uniform ordered hypergraph on $[n]$, and let Σ and Ω be sets of respectively similarity functions on $[n]$ and subcounts of S . Let both Σ and Ω contain the $\mathbf{1}$ function that takes value 1 everywhere in their respective domains, and let Σ contain each of the n functions $f(x) \equiv \mathbf{1}(x = i)$.

We point out that this setting contains no conditions on any of these objects, which is why we need the following definition.

Definition (C -conditions). Let us be in Setting 2. For $C \geq 0$ a real number, we say that the C -conditions are satisfied if all the following inequalities are respected. We first ask $p \geq C(\log^{2k} n)n^{-1}$, and that for all $1 \leq \ell \leq k$, we have

$$\Delta_\ell(S) \leq cC^{1-\ell}p^{\ell-1}\frac{e(S)}{n}.$$

Here $e(S)$ is the number of edges of S . We also ask that Σ and Ω have at most $\exp(\frac{pn}{C})$ elements.

This setting and definition allow us to set up statements as follows. ‘‘Let us be in Setting 2. For every $\varepsilon > 0$ there is $C > 0$ such that, if the C -conditions are satisfied, then ...’’.

We need one more set of definitions.

Definition. Let us be in Setting 2. We say that $Z \subseteq [n]$ is an ε -good dense model for $Y \subseteq [n]$ if it satisfies the following:

(1) For each $\sigma \in \Sigma$, we have $\sum_{y \in [n]} p^{-1}\mathbf{1}(y \in Y)\sigma(y) = \sum_{z \in [n]} \mathbf{1}(z \in Z)\sigma(z) \pm \varepsilon n$, and

(2) For each $\omega \in \Omega$ and any Z_2, \dots, Z_k which are ε -good dense models for any $Y_2, \dots, Y_k \subseteq [n]$ respectively, we have

$$\sum_{s \in S} p^{-k} \mathbf{1}(s \in (Y, Y_2, \dots, Y_k))\omega(s) = \sum_{s \in S} \mathbf{1}(s \in (Z, Z_2, \dots, Z_k))\omega(s) \pm \varepsilon e(S).$$

Notice that whether Z is an ε -good dense model of Y depends on Ω and Σ even if this is not explicit from the notation. We say that Z is an ε -good dense lower model if the second equality of the definition is just a lower bound, i.e. if in 2 we have only

$$\sum_{s \in S} p^{-k} \mathbf{1}(s \in (Y, Y_2, \dots, Y_k))\omega(s) \geq \sum_{s \in S} \mathbf{1}(s \in (Z, Z_2, \dots, Z_k))\omega(s) - \varepsilon e(S).$$

We are now ready to introduce our general transference principle.

Theorem 3. *Let us be in Setting 2. For every $\varepsilon > 0$ there exists a constant $C > 0$ such that, if the C -conditions are satisfied, the following holds.*

- (1) **Lower bound:** With probability at least $1 - \exp(-\frac{pn}{C})$, every subset Y of the binomial random set $X = [n]_p$ has an ε -good dense lower model $Z \subseteq [n]$.
- (2) **Upper bound:** Let

$$\eta_n := \mathbb{P}(|\{s \in S : s \subseteq [n]_p\}| \geq (1 + \frac{\varepsilon}{2}) \cdot \mathbb{E}(|\{s \in S : s \subseteq [n]_p\}|)) + \exp(-\frac{pn}{C}).$$

With probability at least $1 - \eta_n$, every subset Y of the binomial random set $X = [n]_p$ has an ε -good dense model $Z \subseteq [n]$.

- (3) **Dense model with deletion:** With probability at least $1 - \exp(-\frac{pn}{C})$, there exists a subset \tilde{X} with at least $(1 - \varepsilon)pn$ elements of the binomial random set $X = [n]_p$ such that for every subset $Y \subseteq \tilde{X}$, there is an ε -good dense model $Z \subseteq [n]$ for Y .

Notice that the probabilities mentioned above are asymptotically optimal. Indeed, the failure probability has the same logarithmic order of magnitude as the probability that X contains no element of S at all for cases 1 and 3. Moreover, for case 2, η_n corresponds to the probability that X contains many more elements of S than expected, plus an error term of the order of magnitude of the probability that $[n]_p$ contains no element of S at all. In the case that $[n]_p$ contains many more than expected elements of S , taking $Y = X$ shows that we cannot ask for the existence of a good dense model for all subsets of X : even $[n]$ does not contain enough elements of S to satisfy the definition.

2.1. A further note about graphs. We now see that Theorem 1 follows from Theorem 3. Indeed, if we take Σ , and Ω to be minimal (as required by Setting 2), we obtain a counting result that is exactly Theorem 1. This is because for a subset Y of $[n]_p$, Theorem 3 gives us an ε -good dense model $Z \subseteq [n]$ such that $|Y|p^{-1} = |Z| \pm \varepsilon n$ and $\sum_{s \in S} \mathbb{1}(s \subseteq Y)p^{-k} = \sum_{s \in S} \mathbb{1}(s \subseteq Z) \pm \varepsilon n$. The first equality says that Z has the appropriate size, and the second allows us to know the number of copies of H in Y provided we can count the copies of H in Z .

2.2. Counting lemma for sparse hypergraphs. We provide a further application of our transference principle, which is a counting lemma for sparse hypergraphs. However, because much more notation is needed to state such a theorem, we postpone its statement to Section 12, where it is presented as Theorem 33. Because Section 12 is completely separated from the preceding sections, besides for the use of our transference principle, the interested reader can explore Section 12 independently from the rest of this work.

Theorem 33 is a strong counting result for hypergraphs in the sparse random regime. Indeed, it provides a more precise counting statement than the one obtained by Balogh, Morris, and Samotij [2], and by Saxton, and Thomason [17] with the container method. Similarly, with their version of the transference principle, Conlon, Gowers, Samotij, and Schacht [6] also obtained weaker lower bounds, and were able to obtain an upper bound only in the case of strictly-balanced graphs (while their work can probably be generalised to hypergraphs, no such generalisation has been completed).

2.3. The Deletion Version of our Transference Principle. Before Section 11, we focus on item 3 of Theorem 3, which is the deletion version of our Counting Lemma. In Section 11

we show how to obtain the rest of Theorem 3 from item 3. We restate now item 3 of Theorem 3 as an independent theorem and make explicit the notation.

Definition (ε -deletion). Let X be a sample of the binomial random set $[n]_p$. Given $\varepsilon > 0$, we say that \tilde{X} is an ε -deletion of X if \tilde{X} is a subset of X with at least $(1 - \varepsilon)pn$ elements.

¹

Theorem 4 (Case 3 of Theorem 3). *Let us be in Setting 2. For every $r \geq 2$ and $\varepsilon > 0$ there exists $C > 0$ such that, if the C -conditions are satisfied, then with probability at least $1 - \exp(-\frac{pn}{C})$, the binomial random set $X = [n]_p$ admits an ε -deletion \tilde{X} such that for any partition $Y_1 \cup \dots \cup Y_r = \tilde{X}$, there are ε -good dense models Z_1, \dots, Z_r for Y_1, \dots, Y_r respectively, such that Z_1, \dots, Z_r partitions $[n]$.*

Note that in the case where we have only one sparse set $Y \subseteq \tilde{X}$, we still obtain a dense model by adding an extra set $\tilde{X} \setminus Y$ and applying Theorem 4.

3. TOOLS

3.1. Concentration Inequalities. We start with some standard concentration inequalities. Theorem 5, Lemma 18 and Theorem 7 can be found in [21], respectively in Section 2.3, Section 2.8, and Section 2.9.

Theorem 5 (Chernoff's inequality). *Let X_1, \dots, X_n be independent Bernoulli random variables, let $Y = \sum_{i=1}^n X_i$, and let $\delta \in (0, 1)$. Then we have*

$$\mathbb{P}[Y \geq (1 + \delta)\mathbb{E}[Y]], \mathbb{P}[Y \leq (1 - \delta)\mathbb{E}[Y]] \leq \exp\left(-\frac{\delta^2}{3}\mathbb{E}[Y]\right).$$

The following result is known as Bernstein's inequality.

Lemma 6 (Bernstein's inequality). *Let Y_1, \dots, Y_n be independent random variables taking values in $[-M, M]$; let $S = Y_1 + \dots + Y_n$. For $\lambda \geq 0$ we have*

$$\mathbb{P}[|S - \mathbb{E}[S]| \geq \lambda] \leq 2 \exp\left(\frac{-\lambda^2/2}{\frac{M\lambda}{3} + \sum_i \text{Var}(Y_i)}\right).$$

The following result is due to McDiarmid.

Theorem 7 (McDiarmid's inequality). *Let X_1, \dots, X_n be independent real-valued random variables, and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. Assume that the value of $f(x)$ can change by at most $c_i > 0$ under an arbitrary change² of the i -th coordinate of $x \in \mathbb{R}^n$. Then, for every $\varepsilon > 0$ we have*

$$\mathbb{P}[|f(X_1, \dots, X_n) - \mathbb{E}[f(X_1, \dots, X_n)]| \geq \varepsilon] \leq 2 \exp\left(-\frac{2\varepsilon^2}{\sum_{j=1}^n c_j^2}\right).$$

We also require a further concentration inequality, due to Kim and Vu [13] which provides a concentration result for a multi-variable polynomial over independent Bernoulli random variables as follows. Let F be an hypergraph with $V(F) = \{1, \dots, n\}$ and edge set $E(F)$. Let us assume each edge e is associated to a weight $w(e) > 0$ and that each edge of F contains at most d vertices. Moreover, for any $A \subseteq V(F)$, let F_A denote the A -truncated sub-hypergraph of F , which is the hypergraph with vertex set $V(F) \setminus A$ and edge set $E(F_A) = \{e' \subseteq V(F_A) : e' \in E(F)\}$:

¹Colourful from here.

²This means that for any index i and any x_1, \dots, x_n, x'_i we have $|f(\dots, x_i, \dots) - f(\dots, x'_i, \dots)| \leq c_i$.

$e' \cup A \in E(F)\}$. Note that w extends in a unique way from $E(F)$ to $E(F_A)$, therefore we abuse notation and use w to denote either function.

Suppose now t_1, \dots, t_n are independent random variables, such that for each $i \in [n]$ there is $p_i \in [0, 1]$ such that t_i is either a Bernoulli $\{0, 1\}$ random variable with $\mathbb{E}(t_i) = p_i$, or the constant random variable $t_i \equiv p_i$. The following polynomial is a well-defined random variable

$$Y_F = \sum_{e \in E(F)} w(e) \prod_{t_i \in e} t_i.$$

Analogously we can define Y_{F_A} , where by convention $\prod_{t_i \in \emptyset} t_i = 1$.

In order to provide a concentration statement for Y_F , we need to introduce a language to describe its deviations. For $i \in \{0, \dots, d\}$ let $\mathbb{E}_i(Y_F) = \max_{A \subseteq V(F): |A|=i} \mathbb{E}(Y_{F_A})$. Note $\mathbb{E}_0(Y_F) = \mathbb{E}(Y_F)$ is just the expectation of Y_F . Let $\mathbb{E}'(Y_F) = \max_i \mathbb{E}_i(Y_F)$ and $\mathbb{E}''(Y_F) = \max_{i \geq 1} \mathbb{E}_i(Y_F)$.

Theorem 8 (Kim-Vu's inequality). *Let F , w , d , and $\{t_1, \dots, t_n\}$ be as above. For $\lambda > 1$ and $a_d := 8^d d!^{1/2}$, we have*

$$P[|Y_F - \mathbb{E}(Y_F)| > a_d (\mathbb{E}'(Y_F) \mathbb{E}''(Y_F))^{1/2} \lambda^d] = O(\exp(-\lambda + (d-1) \log n)).$$

The moral of this theorem is that if the average effect of any group of at most d random variables is considerable smaller than the expectation of Y_F , then Y_F is strongly concentrated.

3.2. Optimisation tools. Here and in the following, by *polytope* we mean a convex polytope, i.e. the convex hull of a finite set of points in a finite-dimensional Euclidean space. Given a polytope Φ , the vertex set of Φ is the³ minimal set V of points whose convex hull equals Φ . The reader should not confuse the vertex set of a polytope with the vertex set of a graph or hypergraph.

Lemma 9. *Consider $f : \mathbb{R}^n \rightarrow \mathbb{R}$ a polynomial in n variables that can be written in the form $f(x_1, \dots, x_n) = \sum_{i=1}^n a_i x_i^d$, where d is either 1 or any positive even integer, and where $a_1, \dots, a_n \geq 0$. Let Φ be a polytope in \mathbb{R}^n with vertex set V . Then f attains its maximum over Φ at a vertex of Φ , which is:*

$$\max_{x \in \Phi} f(x) = \max_{v \in V} f(v).$$

Proof. We show that if a maximiser is in the interior of a line segment in Φ , then all points on the line segment are also maximisers.

For distinct (y_1, \dots, y_n) and (z_1, \dots, z_n) in Φ , let us denote by (x_1, \dots, x_n) their middle point $\frac{1}{2}(y_1 + z_1, \dots, y_n + z_n)$. If $\sum_{i=1}^n a_i x_i^d$ is at least $\sum_{i=1}^n a_i y_i^d$ and strictly larger than $\sum_{i=1}^n a_i z_i^d$, then it is also larger than $\sum_{i=1}^n \frac{1}{2} a_i (y_i^d + z_i^d)$. By averaging, there exists i such that $a_i x_i^d > \frac{1}{2} a_i (y_i^d + z_i^d)$, so $x_i^d > \frac{1}{2} (y_i^d + z_i^d)$. But the function $x \rightarrow x^d$ is convex, a contradiction.

If x is in the interior of a face of Φ of some dimension D , by picking a line through x in this face, we see that there is a maximiser in a boundary face of dimension $D-1$, and iterating we reach a vertex which is a maximiser. \square

A *functional* is a function that has \mathbb{R} as codomain. Given a functional $h : X \rightarrow \mathbb{R}_{\geq 0}$, we say a functional $f : X \rightarrow \mathbb{R}_{\geq 0}$ is *h-bounded* if $0 \leq f(x) \leq h(x)$ for all $x \in X$. More generally, given a collection H of functionals from X to $\mathbb{R}_{\geq 0}$, we say that a functional $f : X \rightarrow \mathbb{R}_{\geq 0}$ is *H-bounded* if there exists a functional $h \in H$ such that f is *h-bounded*. Suppose that f is

³A proof of uniqueness follows by greedy selection.

H -bounded. We say that f is H -extreme if there is $h \in H$ such that for every $x \in X$ we have either $f(x) = 0$ or $f(x) = h(x)$.

A celebrated theorem about the existence of functionals is the Hahn-Banach theorem.

Theorem 10 (Hahn-Banach). *Let K be a closed convex set in \mathbb{R}^n and let f be a vector that does not belong to K . Then there is a linear functional ψ on \mathbb{R}^n such that $\psi(f) > 1$ and such that $\psi(g) \leq 1$ for every $g \in K$.*

Another celebrated result is the Stone-Weierstrass Theorem, which we present in its original form, proved by Weierstrass. We refer to Theorem 7.26 of [16].

Theorem 11 (Weierstrass Approximation). *If f is a continuous real function on $[a, b]$. For every $\varepsilon > 0$ there exists a polynomial P with real coefficients such that for every $x \in [a, b]$ we have $|P(x) - f(x)| \leq \varepsilon$.*

4. MAIN TECHNICAL THEOREM

The focus of this section is to rewrite our setting in the language of *functionals*, and create a parallelism between sets, functionals, and vectors. This is done following the example of Green and Tao [11], and Conlon and Gowers [5] after them.

4.1. Sets, functionals, and vectors. It is fundamental for understanding the rest of this work the idea that we can represent subsets of a specific set as functionals, and functionals as vectors, and that tools used in one of these scenarios often have a useful translation in one of the others.

We start by introducing the equivalence between subsets of $[n]$ and functionals from $[n]$ to $\mathbb{R}_{\geq 0}$. The statement of Theorem 4 is about random subsets of a given set $[n]$. Given a sample $X = [n]_p$ we write $\mu = \mu(X)$ for the scaled indicator function $x \mapsto p^{-1}\mathbb{1}(x \in X)$. This functional μ is our representation of X in the space of functionals $[n] \rightarrow \mathbb{R}$. Strictly speaking, we should not say this, since $\mu(X)$ depends not just on X but on the value of p used when X was chosen, but this is always clear from context. In this language, we think of a weighted subset of X as being a functional $f : [n] \rightarrow \mathbb{R}$ satisfying $0 \leq f(x) \leq \mu(x)$ for all $x \in [n]$. Also, the unweighted subset $Y \subseteq X$ corresponds to the scaled indicator function $p^{-1}\mathbb{1}(x \in Y)$ which takes value p^{-1} on Y and 0 elsewhere.

Often we also want to think of a functional $f : [n] \rightarrow \mathbb{R}$ as a vector of \mathbb{R}^n , in order to define more easily operations and norms over the space of such functionals. While quite standard, we give an explicit example of how this allows us to define an inner product on the set of said functionals by

$$(1) \quad \langle f, g \rangle := \frac{1}{n} \sum_{x \in [n]} f(x)g(x).$$

It is quite important in the following that this operation is indeed an inner-product, and therefore is, in particular, linear in each component and symmetric. We often apply real operations and operators to vectors, by which we always intend to apply them pointwise. For example, the product of two vectors, written fg , is the vector for which the component x has value the pointwise product $f(x)g(x)$. We also define, maybe in a less standard way, the operator \cdot^+ , which we apply to functionals and vectors alike, as follows. For f a functional,

$$f^+(x) = \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Also, for a positive integer d , the notation f^d indicates the pointwise product, so this indicates the functional $f^d(x)$, or equivalently the vector with d -powers at every component.

4.2. Further definitions towards our goal. The objective of this section is to define a norm over the set of functionals $[n] \rightarrow \mathbb{R}$. We want to use this norm to rewrite Theorem 4 in the language of functionals. More precisely, we want to define a norm such that, if the functional f representing Y and the functional g representing Z are close with respect to this norm, then Z is a good dense model for Y . We now proceed with introducing the necessary definitions, before moving to rewriting Theorem 4.

A fundamental operation for our work is the *convolution*, which is an operation on functionals that is dependent of S .

Definition (Convolution). *Given our k -uniform ordered hypergraph S on $[n]$, let $i \in [k]$ be an index, let $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_k$ be non-negative functionals from $[n]$ to $\mathbb{R}_{\geq 0}$, and let $\omega : S \rightarrow [0, 1]$ be a subcount. The convolution $*_{i,S,\omega}(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_k)$ is the functional $[n] \rightarrow \mathbb{R}$ defined as follows. For $x \in [n]$,*

$$*_{i,S,\omega}(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_k)(x) := \frac{n}{e(S)} \sum_{s \in S_i(x)} \omega(s) \prod_{j \neq i} f_j(s_j).$$

In the following, we only write $*_{i,\omega}(f_1, \dots, f_k)$, as S is fixed. Moreover, we would write “let f_1, \dots, f_k be non-negative functionals” instead of “let $f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_k$ ” when we only need $k - 1$ functionals, for ease of indexing.

Note that the convolution operator is multilinear, i.e. it is linear in each of the f_1, \dots, f_k .

The reason we need such a definition is the following. Consider the expression

$$(2) \quad \langle f_i, *_{i,\omega}(f_1, \dots, f_k) \rangle = \frac{1}{|S|} \sum_{s \in S} \omega(s) \prod_{j=1}^k f_j(s_j).$$

If for each $j \in [k]$ we select f_j to be the scaled indicator function of the sparse subset Y , i.e. $f_j(x) = p^{-1} \mathbb{1}(x \in Y)$, then we obtain that equation (2) becomes the left-hand side of point 2 of Theorem 4 (without the error term). On the other hand, if we select as g_j the indicator function of the dense model Z , i.e. $g_j(x) = \mathbb{1}(x \in Z)$, then equation (2) becomes the right-hand side of point 2 of Theorem 4 (without the error term).

In order to say that the right and left side of point 2 of Theorem 4 are close to each others we bound a telescoping sum. Which is, we prove that the quantity

$$(3) \quad \langle f_i - g_i, *_{i,\omega}(f_1, \dots, f_{i-1}, g_{i+1}, \dots, g_k) \rangle$$

is small whenever g_i is a dense model of f_i . The reader should see this as a further example of taking advantage of the parallelism between the sets, functionals, and vectors formalisms.

We now go one step forward, and define a polytope Φ . The goal of this polytope is contain some *witness functionals* so that if $\langle f - g, \phi \rangle$ is small for all ϕ in the polytope, then the various (3) are also small.

Following Conlon and Gowers [5], we give a simplified version of their definition as follows.

Definition. *Let us be in Setting 2, and let H be a set of functionals $[n] \rightarrow \mathbb{R}$.*

*A functional ϕ is said to be H -anti-uniform if it is in Σ , or if it can be written in the form $*_{i,\omega}(f_1, \dots, f_k)$ for some H -bounded functionals f_1, \dots, f_k , some $i \in [k]$, and some $\omega \in \Omega$. The*

polytope $\Phi(H, \Sigma, \Omega)$ of H -anti-uniform functionals is the polytope in the space of functionals $\mathbb{R}^{[n]}$ defined by convex hull of the set containing all the H -anti-uniform functions ϕ and their inverses $-\phi$.

Because we use this definition only under Setting 2, we usually write $\Phi(H)$ instead of $\Phi(H, \Sigma, \Omega)$. Also, when we enumerate H we write $\Phi(h_1, \dots, h_s)$ instead of $\Phi(\{h_1, \dots, h_s\})$. E.g. we often write $\Phi(\tilde{\mu})$ or $\Phi(\mathbf{1})$ for $\Phi(\{\tilde{\mu}\})$ and $\Phi(\{\mathbf{1}\})$.

Because the convolution operator is multilinear, every vertex of $\Phi(H)$ is either in Σ , or it is a convolution $*_{i,\omega}(f_1, \dots, f_k)$ where each f_i is H -extreme. Moreover, if we have f_1, \dots, f_k non-negative functionals, respectively bounded by H -elements h_1, \dots, h_k , then we also have that for all $x \in [n]$ it holds

$$0 \leq *_{i,\omega}(f_1, \dots, f_k)(x) \leq *_{i,\mathbf{1}}(h_1, \dots, h_k)(x).$$

This justifies the following definition.

Definition. *For h_1, \dots, h_k in H , we say that an element of $\Phi(H)$ that can be written as $*_{i,\mathbf{1}}(h_1, \dots, h_k)$ is an H -largest anti-uniform functional of $\Phi(H)$ (or a largest anti-uniform functional of $\Phi(H)$).*

*Moreover, we call H -extreme anti-uniform functional a functional that is in Σ or of the form $*_{i,\omega}(h_1, \dots, h_k)$, where h_1, \dots, h_k are H -extreme.*

Remark 12. We consider a few properties of H -anti-uniform functionals and of $\Phi(H)$.

- $\Phi(H)$ is by definition centrally symmetric.
- Because Σ contains by definition all the standard basis vectors, which is all the functions of the form $f(x) \equiv \mathbb{1}(x = i)$, we have that $\Phi(H)$ is a full-dimensional polytope in $\mathbb{R}^{[n]}$.
- All H -largest anti-uniform functionals are H -extreme anti-uniform functionals, but not vice-versa.
- For any set H , and for any $v \in \Phi(H)$ a vertex of the polytope, there exist a H -largest anti-uniform functional $*_{i,\mathbf{1}}(h_1, \dots, h_k)$ such that we have $0 \leq v \leq *_{i,\mathbf{1}}(h_1, \dots, h_k)$ pointwise. This follows, as mentioned above, from multilinearity of the convolution operator.

As mentioned, the objective of this section is to rewrite in the language of functionals and vectors the statement of Theorem 4. The last technical step needed is the definition of a norm over $\mathbb{R}^{[n]}$. Our candidate is the following:

$$\|f\|_{\Phi(H)} := \max_{\phi \in \Phi(H)} \langle f, \phi \rangle.$$

To see that this is indeed a norm, we can consider that $\Phi(H)$ is a centrally symmetric polytope of dimension n , thus $\max_{\phi \in \Phi(H)} \langle f, \phi \rangle$ is zero if and only if $f = 0$ by the hyperplane separation Theorem (section 2.3 of [4]). Moreover, absolute homogeneity comes from equation (1). Finally we leave triangle inequality as an exercise for the reader. In the following, we write $\|\cdot\|$ when $\Phi(H)$ is clear from the context.

A useful bit of notation is as follows.

Notation 13. *In Setting 2, given \tilde{X} a subset of $[n]$, we denote by $\tilde{\mu}$ the functional $\tilde{\mu}(x) = p^{-1}(x \in \tilde{X})$ with domain $[n]$ and codomain $\{0, p^{-1}\} \subseteq \mathbb{R}$. We often denote with Φ the polytope $\Phi(\tilde{\mu}, \mathbf{1})$.*

We now have the language to state our main technical theorem, which is a functional version of Theorem 4.

Theorem 14. *Let us be in Setting 2. For every $r \geq 2$ and $\varepsilon > 0$ there exists $C > 0$ such that, if the C -conditions are satisfied, then with probability at least $1 - \exp(-\frac{pn}{C})$ the random set $X = [n]_p$ admits an ε -deletion \tilde{X} such that—using Notation 13—the following holds. For any choice of non-negative f_1, \dots, f_r with $f_1 + \dots + f_r = \mu$, there exist non-negative g_1, \dots, g_r with $g_1 + \dots + g_r = \mathbf{1}$, such that $\|f_i - g_i\|_{\Phi(\tilde{\mu}, \mathbf{1})} \leq \varepsilon$ for each $1 \leq i \leq r$.*

Something to note is that we have allowed the f_i to be $\tilde{\mu}$ -bounded functions (not just a scaled indicator function of a subset of \tilde{X} , which would be the exact translation of Theorem 4) but we also relaxed our conclusion to let the dense models g_i be $\mathbf{1}$ -bounded function, not necessarily $\{0, 1\}$ -valued. To prove Theorem 4, we need to return to integer-valued dense models, which is the subject of the next section.

5. INTEGER DENSE MODELS

5.1. Integer dense models suffice. The following result says that we can approximate the dense model g given by Theorem 14 by an integer-valued model.

Theorem 15. *Let us be in Setting 2. For every $r \geq 2$ and $\varepsilon > 0$ there is $C > 0$ such that, if the C -conditions are satisfied, then for any functionals $g_1, \dots, g_r : [n] \rightarrow [0, 1]$ such that $g_1 + \dots + g_r = \mathbf{1}$, there are functionals $g_1^*, \dots, g_r^* : [n] \rightarrow \{0, 1\}$ such that $\|g_i - g_i^*\|_{\Phi(\mathbf{1})} \leq \varepsilon$ for each $1 \leq i \leq r$ and such that $g_1^* + \dots + g_r^* = \mathbf{1}$.*

To prove Theorem 4 from this, we need one more lemma which establishes the connection between closeness in norm and counting.

Lemma 16. *Let us be in Setting 2. Given $\tilde{X} \subseteq [n]$ and $\varepsilon > 0$, we define $\tilde{\mu} = p^{-1}\mathbf{1}(x \in \tilde{X})$. Suppose that f_1, \dots, f_k are non-negative $\tilde{\mu}$ -bounded functionals, that g_1, \dots, g_k are non-negative $\mathbf{1}$ -bounded functionals, that g_1^*, \dots, g_k^* are $\{0, 1\}$ -valued functionals, and that for each $1 \leq i \leq k$ we have*

$$\|f_i - g_i\|_{\Phi(\tilde{\mu}, \mathbf{1})}, \|g_i - g_i^*\|_{\Phi(\mathbf{1})} < \frac{\varepsilon}{2k}.$$

Then we have for each $\omega \in \Omega$

$$\langle f_1, *_1, \omega(f_2, \dots, f_k) \rangle = \langle g_1^*, *_1, \omega(g_2^*, \dots, g_k^*) \rangle \pm \varepsilon.$$

Proof. Given $\omega \in \Omega$, we have the telescoping expression

$$\begin{aligned} \langle f_1, *_1, \omega(f_2, \dots, f_k) \rangle &= \langle g_1, *_1, \omega(f_2, \dots, f_k) \rangle \pm \frac{\varepsilon}{2k} = \langle f_2, *_2, \omega(g_1, f_3, \dots, f_k) \rangle \pm \frac{\varepsilon}{2k} \\ &= \dots = \langle g_k, *_k, \omega(g_1, \dots, g_{k-1}) \rangle \pm \frac{1}{2}\varepsilon, \end{aligned}$$

where we have in total k replacements of an f_i with a g_i (such as the first equality above), in each case using that the corresponding convolution and its negative are in $\Phi(\tilde{\mu}, \mathbf{1})$; and $k - 1$ rearrangements of terms (such as the second equality above), where the value does not change but the inner product is rewritten.

Repeating the same telescoping argument, but this time replacing each occurrence of g_i with g_i^* , and using that the corresponding convolutions are in $\Phi(\mathbf{1})$, we get

$$\begin{aligned} \langle g_k, *_k, \omega(g_1, \dots, g_{k-1}) \rangle &= \langle g_1, *_1, \omega(g_2, \dots, g_k) \rangle = \langle g_1^*, *_1, \omega(g_2, \dots, g_{k-1}) \rangle \pm \frac{\varepsilon}{2k} \\ &= \dots = \langle g_k^*, *_k, \omega(g_1^*, \dots, g_k^*) \rangle \pm \frac{1}{2}\varepsilon, \end{aligned}$$

Putting these two expressions together we have

$$\langle f_1, *_1, \omega(f_2, \dots, f_k) \rangle = \langle g_k^*, *_k, \omega(g_1^*, \dots, g_{k-1}^*) \rangle \pm \varepsilon,$$

an a final rearrangement of the second inner product completes the proof. \square

The proof of Theorem 4 from Theorem 14 and Theorem 15 is an exercise in functional analysis. We write the statement as functionals, then replace the sparse functional f representing \tilde{X} with its fractional dense model g by a telescoping sum, then the fractional dense model with its integer dense model g^* by another telescoping sum. This proof contains the type of argument needed when converting a statement to the functional setting.

Proof of Theorem 4. We are in Setting 2. Given $\varepsilon > 0$ we can take C such that both Theorem 14 and Theorem 15 hold in Setting 2 with $\frac{1}{2k}\varepsilon$ (instead of ε) if the the C -conditions are satisfied.

Suppose now that the likely event of Theorem 14 occurs for $X = [n]_p$, and let \tilde{X} be the set that this event provides. Given Y_1, \dots, Y_r partitioning \tilde{X} , let $f_i(y) = p^{-1}\mathbb{1}(y \in Y_i)$ for each $1 \leq i \leq r$. These are non-negative functions satisfying $f_1 + \dots + f_r = \tilde{\mu}$, so by Theorem 14 there are non-negative g_1, \dots, g_r such that $g_1 + \dots + g_r = \mathbf{1}$ and for each $1 \leq i \leq r$ we have

$$(4) \quad \|f_i - g_i\|_{\Phi(\tilde{\mu}, \mathbf{1})} \leq \frac{1}{2k}\varepsilon.$$

By Theorem 15, there are an integer $\mathbf{1}$ -bounded functions g_1^*, \dots, g_r^* such that $g_1^* + \dots + g_r^* = \mathbf{1}$ and for each $1 \leq i \leq r$ we have

$$(5) \quad \|g_i - g_i^*\|_{\Phi(\mathbf{1})} \leq \frac{1}{2k}\varepsilon.$$

Let $Z_i = \{z \in [n] : g_i^*(z) = 1\}$ for each $1 \leq i \leq r$; by definition, Z_1, \dots, Z_r form a partition of $[n]$ as required.

Fix now $1 \leq i \leq r$. Given $\sigma \in \Sigma$, since σ and $-\sigma$ are in $\Phi(\tilde{\mu}, \mathbf{1})$ and in its subset $\Phi(\mathbf{1})$, the inequalities (4) and (5) give us

$$\langle f_i, \sigma \rangle = \langle g_i, \sigma \rangle \pm \frac{\varepsilon}{2k} = \langle g_i^*, \sigma \rangle \pm \frac{\varepsilon}{k}$$

which, multiplying by n and filling in the definitions of inner product, f_i and g_i^* , gives part 1 of the definition of ε -good dense model.

The required counting statement (for any k -tuple of sets from Y_1, \dots, Y_r) now follows from (4), (5), Lemma 16, and the definitions of the inner product, convolution and Y_i and Z_i . \square

5.2. Random splitting: a useful technique. In this section we prove Theorem 15. We start by giving a sketch of the approach, as some of the ideas reappear later. In particular, we use a refinement of similar techniques to prove Theorem 20.

We start by defining g_i^* via *randomised rounding*. That is, independently for each x , we generate $g_i^*(x)$ by choosing 1 with probability $g_i(x)$ and 0 otherwise. We then argue that the required closeness in norm is likely, giving the following lemma.

Lemma 17. *Let us be in Setting 2. For every $\varepsilon > 0$ there is $C > 0$ such that, if the C -conditions are satisfied, then for any functional $g : [n] \rightarrow [0, 1]$, with high probability the randomised rounding $g^* : [n] \rightarrow \{0, 1\}$ satisfies $\|g - g^*\|_{\Phi(\mathbf{1})} \leq \varepsilon$.*

Before we prove this lemma, we use this to prove Theorem 15.

Proof of Theorem 15. Given $\varepsilon > 0$ and r , let C be returned by Lemma 17. Let us be in Setting 2 with the C -conditions satisfied.

Given g_1, \dots, g_r non-negative such that $g_1 + \dots + g_r = \mathbf{1}$, we define g_1^*, \dots, g_r^* as follows. For each $x \in [n]$ independently, choose $i \in [r]$ with probability $g_i(x)$. We set $g_i^*(x) = 1$ for the chosen i , and the other functions at x to 0.

Observe that for each i , the function g_i^* is distributed as the randomised rounding of g_i , so by Lemma 17, for each i with high probability we have $\|g_i - g_i^*\|_{\Phi(\mathbf{1})} \leq \varepsilon$. Taking the union bound over r , with high probability the norm statement holds for each $1 \leq i \leq r$; in particular the required functions exist, proving Theorem 15. \square

We now explain how to prove Lemma 17.

A first intuitive approach would be to try leverage our optimization Lemma 9 and say that the extremal value is attained at a vertex. This would allow us to argue that for any given vertex ϕ of $\Phi(\mathbf{1})$, with high probability we have $\langle g - g^*, \phi \rangle < \varepsilon$ and then take a union bound over the choices of ϕ . The reason to believe this might work is that $g(x) - g^*(x)$ is, for each $x \in [n]$, a random variable in $[-1, 1]$ with mean zero, while ϕ is a fixed vector, so the inner product is a sum of independent mean zero random variables. Unfortunately, this fails by a technical detail: there are too many choices of vertex for the required union bound. To get around this, we now define a polytope which contains $\Phi(\mathbf{1})$ but has fewer vertices.

Definition (Random split). *Let L be a positive integer, and let $\chi : [n] \rightarrow [L]$ be a sample of the uniform random function. For $i \in [L]$ we then denote by ν_i the function on $[n]$ such that $\nu_i(x) = L$ if $\chi(x) = i$, and $\nu_i(x) = 0$ otherwise. We have $\mathbf{1} = \frac{1}{L} \sum_{i=1}^L \nu_i$. We call this a random split of $\mathbf{1}$.*

By linearity, every vertex of $\Phi(\mathbf{1})$ is a convex combination of vertices of $\Phi(\nu_1, \dots, \nu_L)$, so it suffices to show $\langle g - g^*, \phi \rangle < \varepsilon$ holds for all vertices $\phi \in \Phi(\nu_1, \dots, \nu_L)$.

It follows from the $\Delta_1(S)$ bounds of the C -conditions and from the definition of $\Phi(\mathbf{1})$ that any $\phi \in \Phi(\mathbf{1})$ only attains values with absolute value at most c . Unfortunately, no such bound holds for functionals in $\Phi(\nu_1, \dots, \nu_L)$, which can attain values as large as $L^{k-1}c$. Such large values spoil the concentration we require of the random variable $\langle g - g^*, \phi \rangle$. We deal with this by splitting up ϕ in two components ϕ^{small} and ϕ^{big} : we define $\phi^{\text{small}}(x) = \phi(x)\mathbb{1}(|\phi(x)| \leq 2c)$, and $\phi^{\text{big}} = \phi - \phi^{\text{small}}$.

We can now write $\langle g - g^*, \phi \rangle = \langle g - g^*, \phi^{\text{small}} \rangle + \langle g - g^*, \phi^{\text{big}} \rangle$. The point of this is that the random variable $\langle g - g^*, \phi^{\text{small}} \rangle$ does concentrate well, while we can use a high moment argument to show that $\langle g - g^*, \phi^{\text{big}} \rangle$ is tiny. Importantly, while our concentration argument needs to take a union bound over all vertices of $\Phi(\nu_1, \dots, \nu_L)$ (i.e. all $\{\nu_1, \dots, \nu_L\}$ -extreme functions and their negatives), we only need to bound high moments of the $\{\nu_1, \dots, \nu_L\}$ -largest anti-uniform functions.

We start with a technical lemma that has apparently nothing to do with the proof we want to show. We present this lemma separately because we use it also in a later section.

Definition. *Let us be in Setting 2. Let d be a positive integer, $x \in [n]$ and $i_1, \dots, i_d \in [k]$. A configuration with spine x and index tuple (i_1, \dots, i_d) is a tuple (s^1, \dots, s^d) of edges of S such that $s_{i_j}^j = x$. If $i_1 = \dots = i_d = i$, we call this a d -book with spine x .*

For $\mathbf{i} = (i_1, \dots, i_d)$ and t a positive integer, we denote by $\alpha(\mathbf{i}, t, x)$ the number of configurations (s^1, \dots, s^d) with spine x and index tuple \mathbf{i} such that $|\cup_i s^i \setminus \{x\}| = t$.

Lemma 18. *Let us be in Setting 2. Let $C > 0$ be a positive real number and let d and t be positive integers, with $k - 1 \leq t \leq d(k - 1)$. If the C -conditions are satisfied, then for any $x \in [n]$ and any $\mathbf{i} = (i_1, \dots, i_d)$ we have:*

$$\alpha(\mathbf{i}, t, x) \leq t^d \cdot (2^{dk}k!)^d \cdot c^d C^{d+t-kd} p^{kd-d-t} e(S)^d n^{-d}.$$

Moreover, for $t = d(k - 1)$ we have:

$$\alpha(\mathbf{i}, t, x) \leq c^d e(S)^d n^{-d}$$

Proof. Let us fix x, t, d and \mathbf{i} . We now describe a process that can generate any configuration with spine x , index tuple (i_1, \dots, i_d) , and covering t vertices besides x . By counting the number of choices we make until a specific configuration is selected, we can upper bound $\alpha(\mathbf{i}, t, x)$. We start by picking non-negative integers $m_1, \dots, m_d \leq k - 1$ with $m_1 = k - 1$. We choose s^1 to be an edge of S whose i_1 -th element is x . We then pick $k - m_2$ elements, including x , among the k elements of s^1 that are also to be contained in S_2 . We then fix a position of these elements in S_2 , which is an injection from these $k - m_2$ elements to $[k]$, making sure that x is assigned position i_2 in s^2 . We then select an element s^2 of S that satisfy these constraints. We repeat a similar procedure, fixing $k - m_3$ elements of S_2 to generate S_3 (fixing x in i_3 for s^3), and repeat the procedure until we get s^d .

In this procedure, the main contribution to the number of configurations constructed comes from choosing the $m_1 = k - 1$ new elements of s^1 , the m_2 new elements of s^2 , and so on; the number of ways to do the i -th step is a constant—that counts the number of ways we have to fix elements of the previous edges into the new one, and can be upper-bounded by $2^k k!$, a constant—multiplied by the codegree of S of the right magnitude $\Delta_{k-m_i}(S)$ for which we have by hypothesis the upper-bound $cC^{1+m_i-k} p^{l-1-m_i} \frac{e(S)}{n}$. The total number of elements of $[n] \setminus \{x\}$ our constructed book covers is at most $\sum_{i=1}^d m_i$ (we do not enforce that the ‘new’ elements are really distinct from the previously chosen ones). We can therefore ignore books which cover too few elements of $[n]$ and assume $\sum_{i=1}^d m_i = t$. This means that the product of codegrees we get is $c^d C^{d+t-kd} p^{kd-d-t} e(S)^d n^{-d}$. This gives an upper bound on $\alpha(\mathbf{i}, t, x)$ of

$$\alpha(\mathbf{i}, t, x) \leq t^d \cdot (2^{dk}k!)^d \cdot c^d C^{d+t-kd} p^{kd-d-t} e(S)^d n^{-d}.$$

Indeed, the t^d counts the ways to choose m_1, \dots, m_d , the factor $2^{dk}k!$ corresponds to picking a subset of used elements and an injection to $[k]$, and the final product is the product of codegrees. Note that in one special case we can do better: when $t = d(k - 1)$, we have $m_1 = \dots = m_d = k - 1$, and we do not have to pick any used elements (we must pick x and no other element every time) nor injection (x must be the i_j -th vertex of each S_j , and no other elements are repeated) and we get the upper bound $\alpha(\mathbf{i}, t, x) \leq c^d e(S)^d n^{-d}$ on the number of these books. \square

We are now ready for the proof of Lemma 17.

Proof of Lemma 17. We are in Setting 2. Given $\varepsilon > 0$, let $d \geq 4$ be an integer such that $2^{4-d}c^2 \leq \frac{1}{2}\varepsilon$. Let $L = \lceil 1000c^2d\varepsilon^{-2} \rceil$ be another integer, and set

$$C = 100(dk)^{d+1} (2^{dk}k!)^d L.$$

We can assume now that the C -conditions are satisfied in our setting. Let ν_1, \dots, ν_L be a random split of $\mathbf{1}$. In the following claim, recall that when ϕ is a vector, ϕ^d denotes the pointwise power.

Claim 19. *With high probability⁴, the following properties are satisfied. For each $i \in [L]$, we have the inequality $\langle \mathbf{1}, \nu_i \rangle \leq 2$; and in addition, for each $j \in [k]$ and $i_1, \dots, i_k \in [L]$, we have*

$$\langle \mathbf{1}, (*_{j,1}(\nu_{i_1}, \dots, \nu_{i_k}))^d \rangle \leq 2c^d.\text{⁵}$$

This claim is our bound on high moments of the $\{\nu_1, \dots, \nu_L\}$ -extreme functions.

Proof. For the first statement, fix i . As $\langle \mathbf{1}, \nu_i \rangle = \frac{1}{n} \sum_{x=1}^n \nu_i(x)$, we are asking for the probability that ν_i has more than $2n/L$ entries equal to L . If we consider $\sum_x \mathbb{1}(\nu_i(x) = L)$, this is a binomial random variable with mean n/L , so by Chernoff's inequality (Theorem 5) the probability that it exceeds $2n/L$ is at most $\exp(-\frac{1}{3}n/L)$. Considering an union bound over i , the probability of failure of the first statement is $o(1)$.

For the second statement, fix j and i_1, \dots, i_k . Let $Z = \langle \mathbf{1}, (*_{j,1}(\nu_{i_1}, \dots, \nu_{i_k}))^d \rangle$. We first argue that $\mathbb{E}[Z] \leq \frac{3}{2}c^d$. We have

$$\begin{aligned} Z &= \frac{1}{n} \sum_{x \in [n]} (*_{j,1}(\nu_{i_1}, \dots, \nu_{i_k}))^d(x) = \sum_{x \in [n]} \frac{1}{n} \frac{n^d}{e(S)^d} \left(\sum_{s \in S_j(x)} \prod_{t \neq j} \nu_{i_t}(s_t) \right)^d \\ &= \sum_{x \in [n]} \frac{1}{n} \frac{n^d}{e(S)^d} L^{d(k-1)} \left(\sum_{s \in S_j(x)} \prod_{t \neq j} \mathbb{1}(\chi(s_t) = i_t) \right)^d \\ &= \sum_{x \in [n]} \sum_{s^1, \dots, s^d \in S_j(x)} \frac{1}{n} \frac{n^d}{e(S)^d} L^{d(k-1)} \cdot \prod_{t \neq j} \prod_{h=1}^d \mathbb{1}(\chi(s_t^h) = i_t). \end{aligned}$$

Notice that the internal sum of our last equation is a sum of d -books with spine x . Each term of said sum takes value either zero or $\frac{1}{n} \cdot \frac{n^d}{e(S)^d} \cdot L^{(k-1)d}$. Let us fix a d -book s^1, \dots, s^d , and let us ask what is the probability that the internal sum takes the larger value. If we let $Q = \cup_{t=1}^d s^t \setminus \{x\}$ and $q = |Q|$, the probability depends only on q . Indeed, notice that for each element of Q , the random variable χ needs to attain a specific value, otherwise the whole term is set to zero. Therefore, given s^1, \dots, s^d , the probability that the corresponding element of the sum takes value $\frac{1}{n} \cdot \frac{n^d}{e(S)^d} \cdot L^{(k-1)d}$ is at most L^{-q} (it can be that the probability is zero, for example if we have $\nu_{i_1}(y)\nu_{i_2}(y)$ as a term in our sum). Notice that lower values of q imply larger probability that the corresponding element of the sum samples the higher value. We can use Lemma 18 to count the number of books as follows.

Fix $\mathbf{i} = (j, \dots, j)$ a d -tuple with all entries equal to j . For the calculation of the expectation of Z , we need the following to bound the main term.

$$\sum_{x \in [n]} \sum_{s^1, \dots, s^d \in S_j(x)} \prod_{t \neq j} \prod_{h=1}^d \mathbb{1}(\chi(s_t^h) = i_t) \leq \sum_{x \in [n]} \sum_{q=k-1}^{d(k-1)} \alpha(\mathbf{i}, q, x) L^{-q}.$$

If we insert the bounds of Lemma 18 in the calculations we obtain:

$$\mathbb{E}[Z] \leq c^d + \frac{1}{2}c^d.$$

where the first term c^d is the $q = d(k-1)$, and by choice of C each other term in the sum contributes at most $\frac{1}{2}(dk)^{-1}c^d$.

⁴With probability tending to 0 as n tends to ∞ .

⁵The two $\mathbf{1}$ in this statement are functionals over different domains.

We next want to apply McDiarmid's inequality (Theorem 7) to Z . We therefore need to argue that Z does not vary a lot when just one component of the colouring χ is changed. For any fixed $y \in [n]$, consider that changing the colouring at y affects only the terms of the sum Z where y is in at least one edge of the book s^1, \dots, s^d . As before, we upper-bound the number of these terms by showing a procedure that can generate any such book containing y , and keeping track of the choices we made. We start by picking $i \in [d]$ and $i' \in [k]$ such that y is vertex number i' of s^i . Because the C -conditions are satisfied, $\Delta_1(S) \leq c \frac{e(S)}{n}$, and therefore there are at most $c \frac{e(S)}{n}$ choices of s^i containing y in position i' . For the same reason, the remaining $d - 1$ elements of the book (which all contain x at position j) can be chosen in at most $c^{d-1} \frac{e(S)^{d-1}}{n^{d-1}}$ ways. This means that the chance of value of χ at y can influence at most $dkc^d e(S)^d n^{-d}$ terms (we multiplied by d to take into account the choice of i and by k to take into account the choice of i'). Since each term takes value either 0 or $\frac{1}{n} \frac{n^d}{e(S)^d} L^{d(k-1)}$, the effect of changing the colouring at y is at most

$$\frac{1}{n} \frac{n^d}{e(S)^d} L^{d(k-1)} \cdot d k c^d e(S)^d n^{-d} = \frac{1}{n} L^{d(k-1)} d k c^d.$$

Applying McDiarmid's inequality, the probability that Z exceeds its expectation by $\frac{1}{2}c^d$ is at most

$$\exp\left(-\frac{2 \cdot \frac{1}{4} c^{2d}}{n \cdot (L^{d(k-1)} d k c^d n^{-1})^2}\right),$$

which tends to zero exponentially in n .

Taking the union bound over the at most kL^{d-1} choices of j and i_1, \dots, i_{k-1} , the failure probability for the second statement is $o(1)$. \square

Let g be a **1**-bounded functional from $[n]$ to $[0, 1]$. Our aim is to prove that there exists a functional $g^* : [n] \rightarrow \{0, 1\}$ such that $\|g - g^*\|_{\Phi(\mathbf{1})} \leq \varepsilon$. We take g^* to be a random rounding of g , which means that for each $x \in [n]$ we sample $g^*(x)$ independently at random to take the value 1 with probability $g(x)$.

By Claim 19, there exists ν_1, \dots, ν_L a random split such that the likely event of Claim 19 holds (otherwise it wouldn't hold with high probability). Fix such ν_1, \dots, ν_L . In order to prove that $\|g - g^*\|_{\Phi(\mathbf{1})} \leq \varepsilon$ we first show that for any ϕ an arbitrary vertex of $\Phi(\nu_1, \dots, \nu_L)$, we have $\langle g - g^*, \phi \rangle \leq \varepsilon$. We then show that this is enough because $\Phi(\mathbf{1}) \subseteq \Phi(\nu_1, \dots, \nu_L)$ and because linear functions attain their maximum over a polytope at a vertex (Lemma 9).

For a vertex ϕ of $\Phi(\nu_1, \dots, \nu_L)$, we write $\phi^{\text{small}}(x) := \phi(x) \mathbb{1}(|\phi(x)| \leq 2c)$ and $\phi^{\text{big}} = \phi - \phi^{\text{small}}$. We first prove that for each vertex ϕ of $\Phi(\nu_1, \dots, \nu_L)$ we have $\langle g - g^*, \phi^{\text{small}} \rangle \leq \frac{\varepsilon}{2}$, and then we prove a similar statement for ϕ^{big} .

For ϕ^{small} , we do this by union-bounding, for the choice of ϕ , the probability that we selected a g^* that is too far from g with respect to ϕ^{small} . To apply the union bound, we start by considering that the number of vertices of $\Phi(\nu_1, \dots, \nu_L)$ is at most

$$|\Sigma| + k \cdot |\Omega| \cdot L^{k-1} 2^{(k-1)2n/L}.$$

Indeed, every element of $\Phi(\nu_1, \dots, \nu_L)$ can be seen as the convex combination of elements of Σ (at most $|\Sigma|$ many) and ν_1, \dots, ν_L -extreme anti-uniform functionals (by Remark 12). The number of these latter functionals can be bounded by the fact that each of them is determined by being of the form $*_{i,\omega}(f_1, \dots, f_k)$, where there are k choices for the value i ; there are $|\Omega|$ choices for ω ; and each of the $k-1$ -many f_j comes from the selection of one of L -many elements

of $\{\nu_1, \dots, \nu_L\}$ and a subset of the at most $2n/L$ -many (by Claim 19) non-zero entries of the selected element of $\{\nu_1, \dots, \nu_L\}$.

We now observe that, considering g^* as a random variable with $\mathbb{E}[g^*(x)] = g(x)$, we have that $\langle g - g^*, \phi^{\text{small}} \rangle$ is a sum of n -many 0-mean random variables, each with range at most $2cn^{-1}$ and so variance at most c^2n^{-2} [3]. Applying Bernstein's inequality, we have

$$\mathbb{P}[\langle g - g^*, \phi^{\text{small}} \rangle > \frac{1}{2}\varepsilon] \leq \exp\left(-\frac{\varepsilon^2/4}{\frac{4}{3}\frac{1}{n}\frac{1}{2}\varepsilon + n \cdot c^2 n^{-2}}\right) \leq \exp\left(-\frac{\varepsilon^2 n}{32c^2}\right).$$

By choice of L , taking the union we obtain that with high probability we have $\langle g - g^*, \phi^{\text{small}} \rangle \leq \frac{1}{2}\varepsilon$ for every vertex ϕ of $\Phi(\nu_1, \dots, \nu_L)$. We claim that any such g^* satisfies the required norm statement, which implies the lemma statement. To prove this claim, fix such a g^* .

Let us now prove that $\langle g - g^*, \phi^{\text{big}} \rangle \leq \frac{\varepsilon}{2}$ for all ϕ in $\Phi(\nu_1, \dots, \nu_L)$. Take such a ϕ and let ψ be a $\{\nu_1, \dots, \nu_L\}$ -largest anti-uniform functional such that $\phi \leq \psi$ pointwise (which exists, as discussed in Remark 12). We have

$$\begin{aligned} |\langle g - g^*, \phi^{\text{big}} \rangle| &\leq |\langle g, \phi^{\text{big}} \rangle| + |\langle g^*, \phi^{\text{big}} \rangle| \\ &\leq \langle g, |\phi^{\text{big}}| \rangle + \langle g^*, |\phi^{\text{big}}| \rangle \leq 2\langle \mathbf{1}, |\phi^{\text{big}}| \rangle \\ &\leq \langle \mathbf{1}, (\phi^{\text{big}})^2 \rangle \leq 2 \cdot (2c)^{2-d} \langle \mathbf{1}, (\phi^{\text{big}})^d \rangle \\ &\leq 2 \cdot (2c)^{2-d} \langle \mathbf{1}, \psi^d \rangle \leq 4c^d (2c)^{2-d} \leq \frac{\varepsilon}{2}, \end{aligned}$$

where the first line holds by the triangle inequality, the second line holds by non-negativity of g , g^* and $|\phi^{\text{big}}|$, the third line holds because $|\phi^{\text{big}}|$ is bounded pointwise by $(\phi^{\text{big}})^2$ and because all these entries are either zero or at least $2c$. The final line follows since $\phi^{\text{big}} \leq \phi \leq \psi$ pointwise, and then uses Claim 19. By choice of d , this final number is at most $\frac{1}{2}\varepsilon$.

Putting these two estimates together, we have for every vertex ϕ of $\Phi(\nu_1, \dots, \nu_L)$ the bound

$$\langle g - g^*, \phi \rangle \leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.$$

Since linear functionals over a polytope are maximised at vertices, we conclude the same bound holds for every $\phi \in \Phi(\nu_1, \dots, \nu_L)$.

To complete the proof, we now show $\Phi(\mathbf{1}) \subseteq \Phi(\nu_1, \dots, \nu_L)$. Because both sets are polytopes, it is enough to show that all vertices of $\Phi(\mathbf{1})$ are in $\Phi(\nu_1, \dots, \nu_L)$. Given a vertex ϕ of $\Phi(\mathbf{1})$, either $\phi \in \Sigma$ —in which case $\phi \in \Phi(\nu_1, \dots, \nu_L)$ and we are done—, or $\phi = *_{i,\omega}(f_1, \dots, f_k)$ for some $\mathbf{1}$ -bounded functions f_1, \dots, f_k . For each $j \in [k]$ and $t \in [L]$, let $f_{j,t}(x) := f_j(x)\nu_t(x)$, which is ν_t -bounded. By definition of random split, $f_j = \frac{1}{L} \sum_{t \in [L]} f_{j,t}$. Therefore, we have by linearity

$$*_{i,\omega}(f_1, \dots, f_k) = L^{1-k} \sum_{t_1, \dots, t_k \in [L]} *_{i,\omega}(f_{1,t_1}, \dots, f_{k,t_k}),$$

which is a convex combination of elements of $\Phi(\nu_1, \dots, \nu_L)$. \square

6. REDUCTION TO ANTI-CORRELATION

We now show that an anti-correlation statement implies Theorem 14. Which is, we reduce Theorem 14 to the following.

Theorem 20. *Let us be in Setting 2. For every $r \geq 2$ and $\varepsilon > 0$ there exists $C > 0$ such that, if the C -conditions are satisfied, then with probability at least $1 - \exp(-\frac{pn}{C})$ the random set $X = [n]_p$ admits an ε -deletion \tilde{X} such that —using Notation 13— for every*

$\phi, \phi_1, \dots, \phi_{r-1} \in \Phi(\tilde{\mu}, \mathbf{1})$ we have $\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle < \varepsilon$ and $|\langle \tilde{\mu}, \phi \rangle|, |\langle \mathbf{1}, \phi \rangle| \leq 2c$. In addition we have $\|\tilde{\mu} - \mathbf{1}\|_{\Phi(\tilde{\mu}, \mathbf{1})} < \varepsilon$.

We now show, following closely the proof of Lemma 2.6 of Conlon and Gowers [5], that Theorem 20 implies Theorem 14.

Proof of Theorem 14. We are in Setting 2. Given $\varepsilon > 0$, let $\delta = \frac{1}{8c(r-1)^2\varepsilon^{-1}+2}\varepsilon$ ⁶. Take C such that Theorem 20 holds in Setting 2 with δ (in place of ε) if the C -conditions are satisfied. Suppose that the likely event of Theorem 20 occurs; that is, we are given $\tilde{\mu}$ such that $|\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle| < \delta$ and $|\langle \mathbf{1}, \phi \rangle| \leq 2$ for every $\phi, \phi_1, \dots, \phi_{r-1} \in \Phi(\tilde{\mu}, \mathbf{1})$. For the rest of the proof, we only consider the polytope $\Phi(\tilde{\mu}, \mathbf{1})$ and simply denote it by Φ .

Suppose now for a contradiction that f_1, \dots, f_{r-1} are non-negative functions with $f_1 + \dots + f_{r-1} \leq \tilde{\mu}$ such that we cannot write $\frac{1}{1+2(r-1)\varepsilon^{-1}\delta}f_i = g_i + h_i$ where $g_1 + \dots + g_{r-1} \leq \mathbf{1}$ and $\|h_i\|_{\Phi} < \frac{1}{2(r-1)}\varepsilon$ for every $1 \leq i \leq r$.

The set K of $(r-1)$ -tuples of functions of the form $(g_1 + h_1, \dots, g_{r-1} + h_{r-1})$, where each g_i is non-negative and satisfies $g_1 + \dots + g_{r-1} \leq \mathbf{1}$, and $\|h_i\|_{\Phi} \leq \frac{1}{2(r-1)}\varepsilon$ for each i , is a convex set containing the tuple of all zero functions (since we can choose all $g_i = h_i$ identically zero within the constraints). By the Hahn-Banach Theorem (Theorem 10), if $\frac{1}{1+2(r-1)\varepsilon^{-1}\delta}(f_1 + \dots + f_{r-1})$ is not in K there is a hyperplane separation. Because linear functionals can be represented as scalar products, this means that there are $\psi_1, \dots, \psi_{r-1} \in \mathbb{R}^{[n]}$ such that

$$\begin{aligned} \langle \frac{1}{1+2(r-1)\varepsilon^{-1}\delta}(f_1, \dots, f_{r-1}), (\psi_1, \dots, \psi_{r-1}) \rangle &> 1 \quad \text{but} \\ \langle (g_1 + h_1, \dots, g_{r-1} + h_{r-1}), (\psi_1, \dots, \psi_{r-1}) \rangle &\leq 1 \end{aligned}$$

for all $(g_1 + h_1, \dots, g_{r-1} + h_{r-1}) \in K$.

From the first line we get $\frac{1}{r-1} \sum_{i=1}^{r-1} \langle f_i, \psi_i \rangle > 1 + 2(r-1)^2\varepsilon^{-1}\delta$.

From the second, for each i , taking all other terms to be zero we see $\langle h, \psi_i \rangle \leq r-1$ for all h such that $\|h\|_{\Phi} \leq \frac{1}{2(r-1)}\varepsilon$. Thus the dual norm (which exists since Φ is full-dimensional) $\|\psi_i\|_{\Phi}^*$ is at most $2\varepsilon^{-1}(r-1)^2$, so $\phi_i = \frac{1}{2(r-1)^2}\varepsilon\psi_i$ is in Φ . Since we have $\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle < \delta$ from Theorem 20, we conclude that $\langle \tilde{\mu} - \mathbf{1}, \max(0, \psi_1, \dots, \psi_{r-1}) \rangle < 2(r-1)^2\varepsilon^{-1}\delta$.

Taking now in the second line each h_i to be identically zero, we choose each $g_i(x) \in \{0, 1\}$, with at most one of these being 1 for any given x , such that $\sum_{i=1}^{r-1} g_i \psi_i = \max(0, \psi_1, \dots, \psi_{r-1})$. Then we get $\langle (g_1, \dots, g_{r-1}), (\psi_1, \dots, \psi_{r-1}) \rangle = \frac{1}{r-1} \langle \mathbf{1}, \max(0, \psi_1, \dots, \psi_{r-1}) \rangle \leq 1$.

Letting $\psi_+ = \max(0, \psi_1, \dots, \psi_{r-1})$, putting these pieces together we can now write

$$\begin{aligned} (6) \quad 1 + 2(r-1)^2\varepsilon^{-1}\delta &< \frac{1}{r-1} \sum_{i=1}^{r-1} \langle f_i, \psi_i \rangle \leq \frac{1}{r-1} \sum_{i=1}^{r-1} \langle f_i, \psi_+ \rangle = \frac{1}{r-1} \left\langle \sum_{i=1}^{r-1} f_i, \psi_+ \right\rangle \leq \frac{1}{r-1} \langle \tilde{\mu}, \psi^+ \rangle \\ &\leq \frac{1}{r-1} \langle \mathbf{1}, \psi^+ \rangle + 2(r-1)^2\varepsilon^{-1}\delta \leq 1 + 2(r-1)^2\varepsilon^{-1}\delta \end{aligned}$$

Here the first and final inequalities are the deductions we made above, the second is since $\psi_i \leq \psi_+$ and $f_i \geq 0$, the second from considering that f is non-negative, the third from the fact that f is $\tilde{\mu}$ -bounded and $\psi^+ \geq 0$, and the fourth uses the statement from Theorem 20 that $\tilde{\mu} - \mathbf{1}$ does not correlate with ψ_+ . But (6) now reads $1 + 2(r-1)^2\varepsilon^{-1}\delta < 1 + 2(r-1)^2\varepsilon^{-1}\delta$, a contradiction.

⁶fix

We can therefore write $f_i = g_i + 2(r-1)^2\epsilon^{-1}\delta g_i + h_i$, where g_i is **1**-bounded and $\|h_i\|_\Phi < \frac{1}{2(r-1)}\epsilon$, for each $1 \leq i \leq r$, with $g_1 + \dots + g_{r-1} \leq \mathbf{1}$.

We let $g_r = \mathbf{1} - g_1 - \dots - g_{r-1}$. This gives us our non-negative functions g_1, \dots, g_r whose sum is **1** as required: it remains to establish closeness in norm to the f_i . We begin with f_r and g_r . We have

$$\begin{aligned} f_r - g_r &= \tilde{\mu} - f_1 - \dots - f_{r-1} - \mathbf{1} + (1 + 2(r-1)^2\epsilon^{-1}\delta)(g_1 + \dots + g_{r-1}) \\ &= \tilde{\mu} - \mathbf{1} - h_1 - \dots - h_{r-1} - 2(r-1)^2\epsilon^{-1}\delta(g_1 + \dots + g_{r-1}) =: h_r. \end{aligned}$$

By the triangle inequality, and using $\|g_1 + \dots + g_{r-1}\|_\Phi \leq \|\mathbf{1}\|_\Phi \leq 2c$, we have

$$\|h_r\|_\Phi \leq \delta + (r-1) \cdot \frac{1}{2(r-1)}\epsilon + 2(r-1)^2\epsilon^{-1}\delta \cdot 2c = \delta + \frac{1}{2}\epsilon + 4c(r-1)^2\epsilon^{-1}\delta = \epsilon,$$

where the final equality is by choice of δ . This establishes $\|f_r - g_r\|_\Phi \leq \epsilon$ as required.

Similarly, for each i we have $f_i - g_i = h_i + 2(r-1)^2\epsilon^{-1}\delta g_i$, and applying the triangle inequality and the fact that $0 \leq g_i \leq \mathbf{1}$ we get $\|f_i - g_i\|_\Phi \leq \frac{1}{2(r-1)}\epsilon + 2(r-1)^2\epsilon^{-1}\delta \cdot 2c \leq \epsilon$, again by choice of δ . \square

In this proof we did not use the conclusion $\|\tilde{\mu} - \mathbf{1}\|_{\Phi(\tilde{\mu}, \mathbf{1})} < \epsilon$ of Theorem 20, however this is a convenient fact to record.

The rest of this work is concerned with proving Theorem 20. The proof of this result follows the same broad lines that we followed in proving Theorem 15. There are however some important differences. Before entering in details in the next sections, we give a broad informal outlook of these differences.

First, in Theorem 20 we need to optimize for ϕ^+ , which is not linear in ϕ . This means that that we cannot assume $\langle \tilde{\mu} - \mathbf{1}, \phi^+ \rangle$ is maximised at a vertex of Φ as we did in Theorem 15. In the following Section 7 we show how to reduce this problem to a linear (and thus maximised at a vertex) optimisation problem over a different polytope.

Second, in Theorem 15 we were approximating a $[0, 1]$ -valued functional via a random rounding. In Theorem 20 we have to obtain concentration inequalities for $\tilde{\mu}$, which is a sparse random function. Therefore the kind of concentration we can hope for is much weaker. However, we still need an optimisation over $\Phi(\tilde{\mu}, \mathbf{1})$, which has $2^{\Omega(n)}$ vertices. Thus, the same union bounds that we used in Theorem 15 would simply not work here.

Third, $\Phi(\tilde{\mu}, \mathbf{1})$ itself depends on the randomness in $\tilde{\mu}$. Therefore, one cannot fix a vertex ϕ of Φ before revealing $\tilde{\mu}$.

It turns out that a concept similar to the previously-defined ‘random splitting’ deals with both these second and third problems; we describe the random splitting in Section 8 and prove it does the job in Section 9.

Finally, entries of ϕ^+ can be as large as $\log n$, which makes bounding inner products more difficult. However, the same idea that worked for Theorem 15—applying moment bounds to control exceptionally large entries—works just as well here. We prove the required moment bounds hold with high probability in Section 10, and in Section 11 we show that this high probability can (at the cost of some deletion) be upgraded to exponentially high probability.

7. A LINEAR APPROXIMATION

Part of proving Theorem 20 is to show that for every $\phi_1, \dots, \phi_{r-1} \in \Phi(\tilde{\mu}, \mathbf{1})$ we have $\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle < \varepsilon$. The difficulty in proving this statement is that the function $(\phi_1, \dots, \phi_{r-1}) \rightarrow \max(0, \phi_1, \dots, \phi_{r-1})$ makes this a non-linear optimisation problem over $\Phi(\tilde{\mu}, \mathbf{1})^{r-1}$. Thus, we cannot use, out-of-the-box, that $\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle$ is maximised—as a function of $(\phi_1, \dots, \phi_{r-1})$ in $\Phi(\tilde{\mu}, \mathbf{1})$ —with all terms at vertices of $\Phi(\tilde{\mu}, \mathbf{1})$. We show in this section that we can get around this by using the Weierstrass Approximation Theorem (Theorem 11) to approximate $(\phi_1, \dots, \phi_{r-1}) \rightarrow \max(0, \phi_1, \dots, \phi_{r-1})$ with a polynomial. As we now see, this translates our optimisation problem to a linear one over the *product polytope* $\Phi^d := \{\prod_{i=1}^d \phi'_i : \phi'_i \in \Phi(\tilde{\mu}, \mathbf{1})\}$, with $d \in \mathbb{N}$ determined by Weierstrass' Approximation Theorem. Since the constant $\mathbf{1}$ function is in Σ , and therefore in $\Phi(\tilde{\mu}, \mathbf{1})$, any product of at most d elements of $\Phi(\tilde{\mu}, \mathbf{1})$ is in Φ^d . Any vertex of Φ^d is a product of d vertices of $\Phi(\tilde{\mu}, \mathbf{1})$.

We need to be careful because the Weierstrass Approximation Theorem allows us to approximate well the function $(x_1, \dots, x_{r-1}) \rightarrow \max(0, x_1, \dots, x_{r-1})$ only within a closed and bounded set: we use the set $[-2c, 2c]^{r-1}$. We show using high moment bounds that the contribution to the inner product where $(\phi_1, \dots, \phi_{r-1})$ lies outside of this set is negligible. This argument is broadly similar to the one used in the proof of Theorem 15.

To this end, for any function ϕ on $[n]$, write ϕ^{big} for the function which takes the value $\phi(x)$ on $x \in [n]$ whenever $|\phi(x)| > 2c$, and 0 otherwise, and $\phi^{\text{small}} = \phi - \phi^{\text{big}}$. That is,

$$\phi^{\text{small}}(x) = \begin{cases} \phi(x) & \text{if } \phi(x) \in [-2c, 2c] \\ 0 & \text{otherwise} \end{cases} \quad \phi^{\text{big}}(x) = \begin{cases} 0 & \text{if } \phi(x) \in [-2c, 2c] \\ \phi(x) & \text{otherwise} \end{cases}.$$

Note that ϕ^{big} and ϕ^{small} have disjoint support. The aim of this section is to prove the following deterministic reduction of Theorem 20, which tells us that the above sketched approach works.

Lemma 21. ⁷ *Let us be in Setting 2 and let \tilde{X} be a subset of $[n]$. Let us use Notation 13. For any $\varepsilon' > 0$ and $r \geq 2$, there exist $\varepsilon > 0$ and d, d' , such that if the following holds:*

- (1) *For all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})^d$, we have $|\langle \tilde{\mu} - \mathbf{1}, \phi \rangle| < \varepsilon$,*
- (2) *For all $1 \leq j \leq 2d$, all $1 \leq j' \leq d'$ and all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})^j$ we have $|\langle \tilde{\mu}, \phi^{j'} \rangle|, |\langle \mathbf{1}, \phi^{j'} \rangle| \leq 2c^{jj'}$,*

then for all $(\phi_1, \dots, \phi_{r-1}) \in \Phi(\tilde{\mu}, \mathbf{1})$ we have $|\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle| < \varepsilon'$.

Proof. Consider the function $(x_1, \dots, x_{r-1}) \rightarrow \max(0, x_1, \dots, x_{r-1})$ on $[-2c, 2c]^{r-1}$. This is a continuous function on a closed and bounded subset of \mathbb{R}^{d-1} to \mathbb{R}^+ . By the Weierstrass Approximation Theorem (Theorem 11 ⁸), for any given $\varepsilon' > 0$, we can find a polynomial $P(x_1, \dots, x_{r-1})$ of maximum degree d (for some $d = d(r, \varepsilon')$) such that for any $(x_1, \dots, x_{r-1}) \in [-2c, 2c]^{r-1}$ we have $|P(x_1, \dots, x_{r-1}) - \max(0, x_1, \dots, x_{r-1})| < \frac{\varepsilon'}{16c}$. Define now M to be the maximum of the absolute value of the coefficients of P and the number of monomials in P . We now set $\varepsilon = \frac{1}{4}M^{-2}\varepsilon'$ and choose d' sufficiently large that

$$2(2^{r-1}M + 2(r-1)) \cdot (2c)^d M \cdot 2^{2d+1}c^{2d} \cdot 2^{-2d'} \leq \frac{1}{2}\varepsilon'.$$

⁷changed (2)

⁸need the multidimensional version

What we want to do is to upper bound $|\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle|$ given arbitrary $\phi_1, \dots, \phi_{r-1} \in \Phi$.

We split this inner product into several parts using linearity and the triangle inequality. The main term is

$$|\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1^{\text{small}}, \dots, \phi_{r-1}^{\text{small}}) \rangle|,$$

and what remains is to deal with the $x \in [n]$ where $\max(0, \phi_1, \dots, \phi_{r-1})(x) > 2c$. Necessarily, these are in the support of at least one ϕ_i^{big} . Using the fact $2c > 1$, the total contribution is upper bounded by

$$\sum_{i=1}^{r-1} |\langle \tilde{\mu}, (\phi_i^{\text{big}})^2 \rangle| + |\langle \mathbf{1}, (\phi_i^{\text{big}})^2 \rangle|,$$

where for the inequality we use the fact $\phi_i^2 \geq 0$ and non-negativity of $\tilde{\mu}, \mathbf{1}$. We will shortly argue that this final sum is guaranteed to be small: the important fact is that the terms on the rights of the inner product all have terms that are either 0 or at least $(2c)^2$.

We now explain how to deal with the main term. Consider the polynomial $P(\phi_1^{\text{small}} + \phi_1^{\text{big}}, \dots, \phi_{r-1}^{\text{small}} + \phi_{r-1}^{\text{big}})$. Expanding this out, we obtain $P(\phi_1^{\text{small}}, \dots, \phi_{r-1}^{\text{small}}) + S$, where S is a collection of monomials, each of which contains at least one term ϕ_i^{big} for some i . The absolute value of the coefficient of any given monomial is at most M , as in P , and the total number of monomials is at most $2^{r-1}M$: from any given monomial in $P(x_1, \dots, x_{r-1})$ (of which there are at most M) we can choose to replace x_i with either ϕ_i^{big} or ϕ_i^{small} , but not both because the pointwise product $\phi_i^{\text{big}}\phi_i^{\text{small}}$ is the zero function and these terms therefore disappear. By the approximation property of P , and since the entries of each ϕ_i^{small} are in $[-2c, 2c]$, we have

$$\max(0, \phi_1^{\text{small}}, \dots, \phi_{r-1}^{\text{small}}) = P(\phi_1^{\text{small}}, \dots, \phi_{r-1}^{\text{small}}) + h = P(\phi_1, \dots, \phi_{r-1}) + h - S$$

where the entries of h have absolute value at most $\frac{1}{16c}\varepsilon'$.

We therefore get

$$|\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1^{\text{small}}, \dots, \phi_{r-1}^{\text{small}}) \rangle| \leq |\langle \tilde{\mu} - \mathbf{1}, P(\phi_1, \dots, \phi_{r-1}) \rangle| + |\langle \tilde{\mu}, h \rangle| + |\langle \mathbf{1}, h \rangle| + |\langle \tilde{\mu}, S \rangle| + |\langle \mathbf{1}, S \rangle|.$$

We claim all these terms are small. For the first term, splitting P into its at most M monomials, all of which are members of Φ^d , and using assumption 1 of the lemma, we obtain an upper bound $|\langle \tilde{\mu} - \mathbf{1}, P(\phi_1, \dots, \phi_{r-1}) \rangle| \leq M^2\varepsilon$. For the two inner products with h , observe that the worst case is $h = \frac{1}{16c}\varepsilon'\mathbf{1}$ (the upper bound on the absolute values) and since $\mathbf{1} \in \Phi$, both terms are bounded by $\frac{1}{16c}\varepsilon' \cdot 2c = \frac{1}{8}\varepsilon'$ by assumption 2 of the lemma. For the other two terms, we split S into its monomials. Consider one such monomial $\phi'_1 \dots \phi'_{\ell'}$, where each ϕ'_i is either ϕ_i^{small} or ϕ_i^{big} for some i , and $1 \leq \ell' \leq d$. Without loss of generality, we can suppose the first ℓ' of these terms are the big terms and the rest are the small terms; as observed before, we have $\ell' \geq 1$. Then we have

$$|\phi'_1 \dots \phi'_{\ell'}| \leq (2c)^{\ell' - \ell'} (\phi'_1 \dots \phi'_{\ell'})^2$$

pointwise, since each entry of any ϕ'_i with $i > \ell'$ has absolute value at most $2c$, while those with $i \leq \ell'$ are either 0 or have absolute value at least $2c \geq 1$. Importantly, the entries of $\phi'_1 \dots \phi'_{\ell'}^2$ are all either zero or at least $(2c)^{2\ell'}$. Write S' for the sum of monomials of this latter form: then S' is a sum of at most $2^{r-1}M$ monomials whose entries are all either 0 or at least $(2c)^{2\ell'}$ (where ℓ' depends as above on the monomial), multiplied by coefficients bounded in absolute value by $(2c)^d M$.

Putting the pieces together, we have

$$\begin{aligned} |\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle| &\leq M^2 \varepsilon + \frac{1}{4} \varepsilon' + |\langle \tilde{\mu}, S' \rangle| + |\langle \mathbf{1}, S' \rangle| \\ &\quad + \sum_{i=1}^r |\langle \tilde{\mu}, (\phi_i^2)^{\text{big}} \rangle| + |\langle \mathbf{1}, (\phi_i^2)^{\text{big}} \rangle|. \end{aligned}$$

The remaining four inner products are, as we just argued, a total of $2(2^{r-1}M + 2(r-1))$ terms. Each of these terms is, by construction, obtained as follows: take a member ψ of Φ^{2j} for some $1 \leq j \leq d$, all of whose entries are non-negative, and set all entries smaller than $(2c)^{2j}$ to zero to obtain ψ' ; then the term is either $\langle \mathbf{1}, \psi' \rangle$ or $\langle \tilde{\mu}, \psi' \rangle$, multiplied by a coefficient whose absolute value is bounded by $(2c)^d M$.

Observe that, for each such term, since the entries of ψ' are either 0 or at least $(2c)^{2j}$, we can write

$$\psi' \leq (2c)^{2j(1-d')} (\psi')^{d'} \leq (2c)^{2j(1-d')} \psi^{d'},$$

where the second inequality uses that ψ has only non-negative entries. Now applying assumption 2 of the lemma, we have

$$\langle \tilde{\mu}, \psi' \rangle \leq (2c)^{2j(1-d')} \langle \tilde{\mu}, \psi^{d'} \rangle \leq (2c)^{2j(1-d')} 2c^{2jd'} \leq 2^{2d+1} c^{2d} \cdot 2^{-2d'}.$$

Thus we obtain a final bound

$$|\langle \tilde{\mu} - \mathbf{1}, \max(0, \phi_1, \dots, \phi_{r-1}) \rangle| \leq M^2 \varepsilon + \frac{1}{4} \varepsilon' + 2(2^{r-1}M + 2(r-1)) \cdot (2c)^d M \cdot 2^{2d+1} c^{2d} \cdot 2^{-2d'} \leq \varepsilon',$$

where the final inequality is by choice of ε and d' . \square

8. MORE INDEPENDENCE, LESS VERTICES

We introduced Lemma 21 to be of use in the proof of Theorem 20. When proving Theorem 20 we start by showing that $\tilde{\mu} - \mathbf{1}$ is unlikely to correlate with any $\phi \in \Phi(\tilde{\mu}, \mathbf{1})^d$, for some large fixed d given to us by Weierstrass Approximation Theorem. Much as in Section 5, a problem we encounter when doing so is that Φ^d has too many vertices, and therefore we cannot directly apply a union bound. As in Section 5, the solution to this problem is to randomly split $\tilde{\mu}$ and $\mathbf{1}$. An additional problem that exists in this section, which was not present in Section 5, is that in order to write down a vertex ϕ of $\Phi(\tilde{\mu}, \mathbf{1})^d$ we need to know $\tilde{\mu}$. Therefore, we cannot then ask for $\tilde{\mu} - \mathbf{1}$ to be independent of ϕ , if ϕ is a vertex of $\Phi(\tilde{\mu}, \mathbf{1})^d$. It turns out that random splitting deals with this problem as well.

We now introduce a finer notation for dealing with random splitting, and then prove that anti-correlation over $\Phi(\tilde{\mu}, \mathbf{1})^d$ is implied by anti-correlation over a new polytope with fewer vertices.

Notation 22. Let us be in Setting 2. We assume we are using Notation 13 throughout whenever needed.

If it is given a function $\chi_{\mathbf{1}} : [n] \rightarrow \{1, \dots, \lceil Lp^{-1} \rceil\}$ —called **1-colouring**—, we denote by ν_i (for $i \in [\lceil Lp^{-1} \rceil]$) the functional:

$$\nu_i(x) = \begin{cases} \lceil Lp^{-1} \rceil & \text{if } \chi_{\mathbf{1}}(x) = i \\ 0 & \text{else} \end{cases}.$$

If it is given a function $\chi_\mu : [n] \rightarrow \{1, \dots, L\}$ —called μ -colouring— and a subset X of $[n]$, we denote by μ_i (for $i \in [L]$) the functional:

$$\mu_i(x) = \begin{cases} Lp^{-1} & \text{if } x \in X \text{ and } \chi_\mu(x) = i \\ 0 & \text{else} \end{cases}.$$

We call each pre-image $\chi_\mu^{-1}(x)$ a part of the μ -colouring (similarly for χ_1), and we call colours the codomains of χ_1 and χ_μ .

If, in addition to χ_μ , it is given \tilde{X} a subset of $[n]$, we denote by $\tilde{\mu}_i$ (for $i \in [L]$) the functional:

$$\tilde{\mu}_i(x) = \begin{cases} Lp^{-1} & \text{if } x \in \tilde{X} \text{ and } \chi_\mu(x) = i \\ 0 & \text{else} \end{cases}.$$

If, in addition to χ_μ , χ_1 , and \tilde{X} , it is given a positive integer d , we denote by Φ' the polytope $\Phi' := \Phi(\tilde{\mu}_1, \dots, \tilde{\mu}_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^d$.

Also, for any ϕ a vertex of Φ' , let $Q_\phi \subseteq [L]$ be the minimum set of μ -colours such that we can write ϕ as a product of at most d functions which are $\{\tilde{\mu}_j : j \in Q_\phi\} \cup \{\nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil}\}$ -anti-uniform. We denote by $Y(\phi)$ the revealed part of ϕ , i.e. the set $\{x \in [n] : \chi_\mu(x) \in Q_\phi\}$.

Finally, for $\phi \in \Phi(\tilde{\mu}, \mathbf{1})^d$, we write $\phi^{\text{small}}(x) := \phi(x)\mathbb{1}(|\phi(x)| \leq 2c^d)$ and $\phi^{\text{big}}(x) = \phi(x) - \phi^{\text{small}}(x)$.

We now prove that, with the above notation, Φ' contains Φ^d in the following deterministic lemma.

Lemma 23. Let us be in Setting 2. Let $\chi_1 : [n] \rightarrow \{1, \dots, \lceil Lp^{-1} \rceil\}$ and $\chi_\mu : [n] \rightarrow \{1, \dots, L\}$ be an arbitrary $\mathbf{1}$ - and μ -colouring respectively, and let \tilde{X} be an arbitrary subset of $[n]$. Let us use Notation 22. For all $d \geq 1$, we have the set inclusion $\Phi(\tilde{\mu}, \mathbf{1})^d \subseteq \Phi'$.

Proof. It is enough to show that the vertices of Φ^d are in Φ' . By definition of Φ^d and Φ' (as the power of polytopes generated by anti-uniform functionals and their negatives), we can consider a vertex $\phi \in \Phi^d$ which is a product of at most d of the $\{\tilde{\mu}, \mathbf{1}\}$ -anti-uniform functions. We say at most d and not exactly d because $\mathbf{1} \in \Phi$.

By definition, every vertex of Φ is either in Σ or is of the form $*_{i,\omega}(f_1, \dots, f_k)$, where $i \in [k]$, $\omega \in \Omega$, and f_1, \dots, f_k are $\{\tilde{\mu}, \mathbf{1}\}$ -bounded, or is the negative of such a vertex. Therefore, we can write our vertex ϕ as

$$\phi = \prod_{j=1}^{\ell} *_{i_j, \omega_j}(f_1^{(j)}, \dots, f_k^{(j)}) \prod_{j=1}^{\ell'} \sigma_j$$

where $\ell + \ell' \leq d$ and where each σ_j is in Σ . Consider a specific $j \in [\ell]$ and $j' \in [k] \setminus \{i_j\}$. Then $f_{j'}^{(j)}$ is bounded either by $\tilde{\mu}$ or by $\mathbf{1}$.

If $f_{j'}^{(j)}$ is bounded by $\tilde{\mu}$, then we can write

$$f_{j'}^{(j)} = \frac{1}{L} \sum_{j'' \in [L]} f_{j'}^{(j)} \frac{\tilde{\mu}_{j''}}{\tilde{\mu}}.$$

Where the fraction $\frac{\tilde{\mu}_{j''}}{\tilde{\mu}}$ is to be interpreted pointwise and if $\tilde{\mu}(x) = 0$ (so $\tilde{\mu}_{j''}(x) = 0$ too) then we define the result to be 0.

On the other hand, if $f_{j'}^{(j)}$ is bounded by $\mathbf{1}$, then we can write

$$f_{j'}^{(j)} = \frac{1}{\lceil Lp^{-1} \rceil} \sum_{j'' \in [\lceil Lp^{-1} \rceil]} f_{j'}^{(j)} \nu_{j''}.$$

Recall that $*_{i_j, \omega_j}(f_1^{(j)}, \dots, f_k^{(j)})$ is linear in each argument. Therefore, substituting the two equations above into the definition of ϕ , and pulling the sums and coefficients $\frac{1}{L}$ and $\frac{1}{\lceil Lp^{-1} \rceil}$ out by linearity, we have written ϕ as a weighted sum of vertices of Φ' . The sum has $L^q \lceil Lp^{-1} \rceil^{q'}$ terms, where q is the number of functions bounded by $\tilde{\mu}$ and q' the number bounded by $\mathbf{1}$. Each term in the sum has the same coefficient $L^{-q} \lceil Lp^{-1} \rceil^{-q'}$, so that this weighted sum is a convex combination and we proved $\phi \in \Phi'$. \square

9. THE FINAL PROBABILISTIC ESTIMATE

In this section, we finally show that, assuming some moment bounds, it is likely that $|\langle \tilde{\mu} - \mathbf{1}, \phi \rangle| < \varepsilon$ for all $\phi \in \Phi'$.

This proof looks quite similar to the corresponding statement from the proof of Theorem 15 in Section 5. As before, it is enough to prove anti-correlation for vertices of Φ' . And as before, we split the anti-correlation into anti-correlation with $\phi^{\text{small}}(x)$ and the remaining ϕ^{big} .

Much as in Section 5, we can show that $\langle \tilde{\mu} - \mathbf{1}, \phi^{\text{big}} \rangle$ is small by applying some moment bounds. However, proving $\langle \tilde{\mu} - \mathbf{1}, \phi^{\text{small}} \rangle$ is small requires some new ideas. There are two reasons for this: first, the entries of $\tilde{\mu}$ are not independent random variables, and second, in order to describe a vertex of ϕ we first need to reveal some entries of $\tilde{\mu}$.

In the case when \tilde{X} is an ε -deletion of X , we have that μ and $\tilde{\mu}$ are equal in most components. Therefore, we show that it suffices to prove $\langle \mu - \mathbf{1}, \phi^{\text{small}} \rangle$ is small. We then show that this holds as this inner product is a sum of independent mean zero random variables. It turns out we do not need to reveal many entries of μ in order to describe ϕ . We give more details of why this is later, but the idea is as follows.

Given a vertex ϕ of Φ' , recall that ϕ is a product of some at most d functions which are $\{\tilde{\mu}_1, \dots, \tilde{\mu}_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil}\}$ -anti-uniform. Therefore, with Notation 22, we have that $|Q_\phi| \leq d(k-1)$, and hence that $Y(\phi)$ is a small subset of $[n]$. The idea is to split the inner product $\langle \mu - \mathbf{1}, \phi^{\text{small}} \rangle$ into the contribution from $Y(\phi)$, which we can bound using moment bounds, and the contribution from the remainder, which we can bound using Bernstein's inequality. We do this latter bound in the next lemma. That is, we now show how to apply Bernstein's inequality to the contributions not from $Y(\phi)$.

Lemma 24. *Let us be in Setting 2. Given d be a positive integer, and $\delta > 0$, let $C = 100c^{2d}dk\delta^{-2}$. If $L \geq 16C$ and the C -conditions are satisfied in Setting 2, then with probability at least $1 - \exp(-\frac{1}{10}\delta^2pn)$ over the uniform and independent choices of $X = [n]_p$, and $\chi_\mu : [n] \rightarrow \{1, \dots, L\}$, and $\chi_1 : [n] \rightarrow \{1, \dots, \lceil Lp^{-1} \rceil\}$, the following holds. For any given $\tilde{X} \subseteq X$, let us use Notation 22. For each vertex ϕ of Φ' , we have*

$$(7) \quad |\langle \mu - \mathbf{1}, \phi^{\text{small}} \cdot \mathbb{1}([n] \setminus Y(\phi)) \rangle| < \delta.$$

Something we need to be a little careful about in the above statement is that in order to know any vertices of Φ' , we need to reveal all of μ (because Φ' depends on $\tilde{\mu}$). We actually show the above bound for vertices of $\Phi(\mu_1, \dots, \mu_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^d$, and deduce the required statement for Φ' .

Proof. By Chernoff's inequality and by doing a union bound, we get that with probability at least $1 - 2Lp^{-1} \exp(-\frac{1}{300}pn)$ each part of the μ -colouring of $[n]$ has size at most $\frac{1.1n}{L}$, and each part of the $\mathbf{1}$ -colouring has size at most $\frac{2pn}{L}$. Suppose this likely event occurs, and reveal the μ - and $\mathbf{1}$ -colourings χ_μ and χ_1 .

Without revealing X , we know that every vertex ϕ of Φ' has a revealed part $Y(\phi)$ which is the union of some at most $d(k-1)$ parts of the μ -colouring. We can therefore prove the lemma by a union bound over the possible choices of Y ; which is, over the choices of at most $d(k-1)$ -many μ -colours of $[L]$.

Let Q be a set of at most $d(k-1)$ colours in $[L]$, and let Y be the union of the parts of the μ -colouring that are mapped to Q . We can now consider the random variable $X \cap Y$ (where Y is fixed and X needs to be sampled). By Chernoff's inequality and by doing a union bound, we obtain that with probability at least $1 - |Q| \exp(-\frac{1}{8}pn)$, for each $q \in Q$ the number of elements of X with μ -colour q is at most $\frac{2pn}{L}$. Suppose that this likely event occurs, and reveal $X \cap Y$. We now can define the set $\Psi(Q)$ of $\{\mu_j : j \in Q\} \cup \{\nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil}\}$ -extreme anti-uniform functions. For this proof only, let us denote with $H_Q = \{\mu_j : j \in Q\} \cup \{\nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil}\}$.

We can upper bound $|\Psi(Q)|$ as follows. First consider that by definition every vertex of $\Psi(Q)$ is either in Σ , or of the form $*_{i,\omega}(f_1, \dots, f_k)$ where f_1, \dots, f_k are H_Q -extreme. By definition, f_j is H_Q -extreme if there is $h_j \in H_Q$ such that for every $x \in [n]$ we have either $f(x) = 0$ or $f(x) = h(x)$. Therefore, to upper bound $|\Psi(Q)|$ we can consider that every H -extreme anti-uniform function can be in Σ , or of the form $*_{i,\omega}(f_1, \dots, f_k)$ obtained as follows. We first select $\omega \in \Omega$ and a sequence of $k-1$ bounding functions h_1, \dots, h_k from H ; we then choose for each bounding function h_j , from the at most $\frac{2pn}{L}$ non-zero entries, the non-zero entries of f_j (which by definition of 'extreme' are equal to the corresponding entries of h_j). The total number of elements of $\Psi(Q)$ is therefore is at most

$$|\Sigma| + |\Omega|(2Lp^{-1})^{k-1} \cdot 2^{\frac{2pn}{L}(k-1)}.$$

We now select a function ϕ which is a product of at most d elements of $\Psi(Q)$. The number of possible choices for ϕ is at most

$$d \left(|\Omega|(2Lp^{-1})^{k-1} \cdot 2^{\frac{2pn}{L}(k-1)} + |\Sigma| \right)^d \leq d(2Lp^{-1})^{d(k-1)} 2^{\delta^2 pn / 16} 2^{\frac{4pn}{L}d(k-1)}.$$

By definition, the entries of $\phi^{\text{small}} \cdot \mathbb{1}([n] \setminus Y(\phi))$ are in $[-2c^d, 2c^d]$, and only the entries outside $Y(\phi)$ can be non-zero. Thus, the quantity

$$\langle \mu - 1, \phi^{\text{small}} \cdot \mathbb{1}([n] \setminus Y(\phi)) \rangle$$

is a sum of $n - |Y(\phi)| \leq n$ independent random variables

$$\frac{1}{n} (\mathbb{1}(x \in X)p^{-1} - 1)c_x$$

where the number $c_x = \phi^{\text{small}}(x) \cdot \mathbb{1}(x \in [n] \setminus Y(\phi))$ is in $[-2c^d, 2c^d]$. Since the probability of $x \in X$ is p , these random variables all have mean zero, and are bounded between $\frac{-2c^d}{n}$ and $\frac{2c^d}{n}p^{-1}$. It remains to calculate the variance. We have

$$\text{Var}(\mathbb{1}(x \in X)p^{-1} - 1) = p(p^{-1} - 1)^2 + (1-p)(-1)^2 = p^{-1} - 1 \leq p^{-1},$$

so that the variance of each of our random variables is at most $\frac{4c^{2d}}{n^2}p^{-1}$.

By Bernstein's inequality (Lemma 6), the probability that when we reveal $X \setminus Y(\phi)$ we get

$$|\langle \mu - 1, \phi^{\text{small}} \cdot \mathbb{1}([n] \setminus Y(\phi)) \rangle| \geq \delta$$

is at most

$$2 \exp \left(- \frac{\delta^2/2}{2p^{-1}\delta/(3n) + n \frac{4c^{2d}}{n^2} p^{-1}} \right) \leq 2 \exp \left(- \frac{1}{16c^{2d}} \delta^2 pn \right).$$

Taking a union bound over the choices of ϕ , the probability that there exists any product ϕ of at most d elements of $\Psi(Q)$ with

$$|\langle \mu - 1, \phi^{\text{small}} \cdot \mathbb{1}([n] \setminus Y(\phi)) \rangle| \geq \delta$$

is at most

$$d(2Lp^{-1})^{d(k-1)} \cdot 2^{\frac{4pn}{L}d(k-1)+\delta^2pn/16} \cdot 2 \exp \left(- \frac{1}{16c^{2d}} \delta^2 pn \right) + d(k-1) \exp(-\frac{1}{8}pn).$$

Finally, taking a union bound over the choices of Q , the probability that there exists Q and a product ϕ of at most d elements of $\Psi(Q)$ such that

$$|\langle \mu - 1, \phi^{\text{small}} \cdot \mathbb{1}([n] \setminus Y(\phi)) \rangle| \geq \delta$$

is at most

$$2^L d(2Lp^{-1})^{d(k-1)} \cdot 2^{\frac{4pn}{L}d(k-1)} 2^{\delta^2pn/16} \cdot 4 \exp \left(- \frac{1}{16c^{2d}} \delta^2 pn \right) \leq \exp \left(- \frac{1}{100c^2} \delta^2 pn \right),$$

where the final inequality is by choice of L and since $pn \geq 100c^{2d}\delta^{-2}dk \log n$.

Suppose now that X is such that this unlikely event does not occur. Given \tilde{X} , we can now calculate the polytope Φ' . Let ϕ be a vertex of this polytope: then ϕ is a product of at most d extreme restricted anti-uniform functions. Letting Q be the set of μ -colours bounding ϕ , we see ϕ is a product of at most d members of $\Psi(Q)$, because for each j the function $\tilde{\mu}_j$ is pointwise either equal to μ_j or equal to zero. The lemma statement follows. \square

There is a last anti-correlation lemma we need. But before introducing that, we state a moment bound lemma (Lemma 25) which is needed in its proof. The proof of Lemma 25 is left for a later section.

Lemma 25. *Given $\delta > 0$, and d' an even positive integer, there exists L_0 such that, if $L \geq L_0$, then there exist C such that, if the C -conditions are satisfied in Setting 2, then with probability at least $1 - 3 \exp(-\frac{1}{8}\delta^2pn)$ over the choice of $X = [n]_p$, the following happens. With probability at least 0.9 over the choice of $\chi_\mu : [n] \rightarrow [L]$ and $\chi_1 : [n] \rightarrow [\lceil Lp^{-1} \rceil]$ independent and uniform at random, there is a δ -deletion \tilde{X} of X such that the following happens. Let us use Notation 22. For any $1 \leq \ell \leq d'$ and ψ an largest anti-uniform functional in either $\Phi(\tilde{\mu}, \mathbf{1})^\ell$ or $\Phi(\tilde{\mu}_1, \dots, \tilde{\mu}_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^\ell$,*

$$\langle \tilde{\mu}, \psi \rangle \leq 2c^\ell \quad \text{and} \quad \langle \mathbf{1}, \psi \rangle \leq 2c^\ell.$$

In addition, if ψ is any largest anti-uniform functional in $\Phi(\tilde{\mu}_1, \dots, \tilde{\mu}_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^\ell$, and $1 \leq j \leq L$ and $1 \leq j' \leq \lceil Lp^{-1} \rceil$ then we have

$$\langle \tilde{\mu}_j, \psi \rangle \leq 2c^\ell \quad \text{and} \quad \langle \nu_{j'}, \psi \rangle \leq 2c^\ell.$$

We are now in a position to state and prove the final anti-correlation lemma we need: Lemma 26. The main probabilistic inputs to this lemma are the above Lemma 24 and the moment bounds Lemma 25, which we prove in a following section.

Lemma 26. *Let us be in Setting 2. Given d, d' positive integers with d' even, given $\varepsilon > 0$, there exist L_0 such that, if $L \geq L_0$, there exists C such that, if the C -conditions are satisfied in Setting 2, then with probability at least $1 - \exp(-\frac{pn}{C})$ over the choice of $X = [n]_p$, there*

is an ε -deletion \tilde{X} of X such that the following happens. There exist functions $\chi_\mu : [n] \rightarrow [L]$ and $\chi_1 : [n] \rightarrow \lceil Lp^{-1} \rceil$ such that, using Notation 22, we have $\langle \tilde{\mu} - \mathbf{1}, \phi \rangle < \varepsilon$ for all $\phi \in \Phi(\tilde{\mu}_1, \dots, \tilde{\mu}_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^d$. In addition, for all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})$, we have $|\langle \tilde{\mu}, \phi \rangle|, |\langle \mathbf{1}, \phi \rangle| \leq 2c$ and $\langle \tilde{\mu}, \phi^{d'} \rangle, \langle \mathbf{1}, \phi^{d'} \rangle \leq 2c^{d'}$.

Proof. In this proof, let us use the notation $\tilde{H} = \{\tilde{\mu}_1, \dots, \tilde{\mu}_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil}\}$ and $H = \{\mu_1, \dots, \mu_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil}\}$.

Given $d, d', \varepsilon > 0$ with d' even, we set $\delta = \frac{\varepsilon}{12c^d}$ and $d'' = \max(d', d(1 + \lceil \log_2 \frac{2c^d}{\delta} \rceil))$. Let $L_0 = 1600c^{2d}dk\delta^{-2}$, which guarantees that if $L \geq L_0$, then it satisfies the conditions for Lemma 24 with input δ . Let C be large enough for Lemma 25 and Lemma 24. Without loss of generality we assume $C \geq 16d\delta^{-2}$ and that the C -conditions are satisfied.

Chernoff's inequality tells us that with probability at least $1 - \exp(-\frac{1}{3}pn)$, the set $X = [n]_p$ has at most $2pn$ elements. Moreover, Lemma 24, with input δ , tells us that with probability (over the product probability space of $[n]_p$ and the μ - and $\mathbf{1}$ -colourings) at least $1 - \exp(-\frac{1}{10}\delta^2pn)$ we have, for each vertex $\phi \in \Phi'$, the following inequality holds:

$$(8) \quad |\langle \mu - \mathbf{1}, \phi^{\text{small}} \cdot \mathbb{1}([n] \setminus Y(\phi)) \rangle| < \delta.$$

In particular, with probability at least $1 - \exp(-\frac{1}{20}\delta^2pn)$ over $[n]_p$, the probability of the μ - and $\mathbf{1}$ -colourings having this property is at least 0.9.

In addition, because of the conditions on L and d'' and C , Lemma 25 tells us that with probability at least $1 - \exp(-\frac{1}{8}\delta^2pn)$ (over the choice of $[n]_p$), the set $X = [n]_p$ has the following property. There exists a δ -deletion \tilde{X} of X such that we have, with probability at least 0.9 (over the random choice of χ_μ and χ_1) that for any $1 \leq \ell \leq d''$ and ψ' a \tilde{H} -largest anti-uniform functional in $\Phi(\tilde{H})^\ell$, and $1 \leq j \leq L$, it holds

$$(9) \quad \langle \tilde{\mu}_j, \psi' \rangle, \langle \tilde{\mu}, \psi' \rangle, \langle \mathbf{1}, \psi' \rangle \leq 2c^\ell.$$

Suppose now that X is such that all three likely events occur, which by the union bound has probability at least

$$1 - \exp(-\frac{1}{3}pn) - \exp(-\frac{1}{20}\delta^2pn) - \exp(-\frac{1}{8}\delta^2pn) \geq 1 - \exp(-\frac{1}{30}\delta^2pn).$$

Fix \tilde{X} a δ -deletion witnessing the likely event occurring. The probability that the μ - and $\mathbf{1}$ -colourings are such that their likely events occur is by the union bound at least 0.8. Suppose this likely event occurs: this gives us that there exist χ_μ and χ_1 as in the lemma statement.

We next establish the anti-correlation claimed in the lemma.

Claim 27. *For each \tilde{H} -largest anti-uniform functional $\psi \in \Phi'$ and each $j \in [L]$, we have*

$$(10) \quad \langle \tilde{\mu}_j, \psi \rangle \leq 2c^d,$$

$$(11) \quad \langle \tilde{\mu}, \psi^{\text{big}} \rangle \leq \delta,$$

$$(12) \quad \langle \mathbf{1}, \psi^{\text{big}} \rangle \leq \delta.$$

Proof. Equation (10) is immediate from (9) taking $\psi' = \psi$ with $\ell \leq d$ and using $c \geq 1$.

For the remaining two equations, choose ℓ minimal such that ψ is a \tilde{H} -largest anti-uniform functional in $\Phi(\tilde{H})^\ell$, and note $\ell \leq d$. Let $a = \lceil \log_2 \frac{2c^d}{\delta} \rceil$, and note $(1+a)\ell \leq (1+a)d \leq d'$.

For (11), observe that by definition of \tilde{H} -largest anti-uniform functional (with this specific \tilde{H}), if $\psi \neq 0$ for some x then $\psi(x) > 2c^d$. It follows that

$$\langle \tilde{\mu}, \psi^{\text{big}} \rangle \cdot (2c^d)^a \leq \langle \tilde{\mu}, (\psi^{\text{big}})^{1+a} \rangle \leq \langle \tilde{\mu}, \psi^{1+a} \rangle \leq 2c^{(1+a)\ell},$$

where the final inequality is by (9) with $\psi' = \psi^{1+a}$. By choice of a , we have the upper bound $2c^{(1+a)\ell}(2c^d)^{-a} \leq \delta$, giving (11). Swapping $\mathbf{1}$ for $\tilde{\mu}$ in the above calculation establishes (12). \square

By Lemma 9, the maximum $\max_{\phi \in \Phi'} |\langle \tilde{\mu} - \mathbf{1}, \phi \rangle|$ is attained in one of the vertices of Φ' . By central symmetry in the definition of Φ' and linearity of the inner product, the maximum value of $|\langle \tilde{\mu} - \mathbf{1}, \phi \rangle|$ over Φ' is also an extremal value of $|\langle \tilde{\mu} - \mathbf{1}, \phi \rangle|$ over the vertices of Φ' which are products of d restricted anti-uniform functions (and not their opposites).

Let us therefore fix such a vertex ϕ in Φ' , and let $Y = Y(\phi)$. Our goal is to show that $|\langle \tilde{\mu} - \mathbf{1}, \phi \rangle| < \varepsilon$. We use linearity of the inner product and the triangle inequality to split this up. Write $\tilde{\mu}' = \tilde{\mu}\mathbb{1}([n] \setminus Y)$ and $\tilde{\mu}'' = \tilde{\mu}\mathbb{1}(Y)$; define similarly μ' , μ'' and $\mathbf{1}'$ and $\mathbf{1}''$. We obtain

$$|\langle \tilde{\mu} - \mathbf{1}, \phi \rangle| \leq |\langle \tilde{\mu}' - \mathbf{1}', \phi \rangle| + |\langle \tilde{\mu}'' - \mathbf{1}'', \phi \rangle|.$$

We can further split the first term

$$|\langle \tilde{\mu}' - \mathbf{1}', \phi \rangle| \leq |\langle \mu' - \mathbf{1}', \phi^{\text{small}} \rangle| + |\langle \mu' - \tilde{\mu}', \phi^{\text{small}} \rangle| + |\langle \tilde{\mu}' - \mathbf{1}', \phi^{\text{big}} \rangle|.$$

Of these terms, (8) tells us that the first term is bounded by δ . Since $\tilde{\mu}$ and μ differ in at most δpn places, $\mu' - \tilde{\mu}'$ is equal to p^{-1} in at most δpn places and otherwise equal to zero, while $|\phi^{\text{small}}|$ is bounded by $2c^d$, so the second term is at most $\frac{1}{n} \cdot p^{-1} \cdot \delta pn \cdot 2c^d = 2c^d \delta$. Splitting the third term

$$|\langle \tilde{\mu}' - \mathbf{1}', \phi^{\text{big}} \rangle| \leq \langle \tilde{\mu}', \phi^{\text{big}} \rangle + \langle \mathbf{1}', \phi^{\text{big}} \rangle \leq \langle \tilde{\mu}, \phi^{\text{big}} \rangle + \langle \mathbf{1}, \phi^{\text{big}} \rangle,$$

where in the final two inner products, all terms are non-negative.

Returning to split

$$|\langle \tilde{\mu}'' - \mathbf{1}'', \phi \rangle| \leq \langle \tilde{\mu}'', \phi \rangle + \langle \mathbf{1}'', \phi \rangle,$$

again all the terms in the inner products are non-negative. In particular, if ψ is any function which is pointwise greater than or equal to ϕ , replacing ϕ with ψ gives an upper bound on all these non-negative inner products. Let ψ then be a largest restricted anti-uniform function which is pointwise at least ϕ . By (11), (12), we have $\langle \tilde{\mu}, \psi^{\text{big}} \rangle, \langle \mathbf{1}, \psi^{\text{big}} \rangle < \delta$.

We apply (10) to obtain $\langle \tilde{\mu}_j, \psi \rangle \leq 2c^d$ where $\tilde{\mu}_j$ is revealed by ϕ , that is, $\chi_\mu^{-1}(j) \subseteq Y$. Recall that the normalisation of $\tilde{\mu}_j$ is $p^{-1}L$, so that $\tilde{\mu}'' = \frac{1}{L} \sum_j \tilde{\mu}_j$, where the sum ranges over j with $\chi_\mu^{-1}(j) \subseteq Y$. This gives

$$\langle \tilde{\mu}', \phi \rangle \leq \langle \tilde{\mu}'', \psi \rangle = \frac{1}{L} \sum_j \langle \tilde{\mu}_j, \psi \rangle \leq \frac{2c^d d(k-1)}{L}.$$

Finally, we come to $\langle \mathbf{1}'', \psi \rangle$. Here we split $\psi = \psi^{\text{small}} + \psi^{\text{big}}$, and write

$$\langle \mathbf{1}'', \psi \rangle = \langle \mathbf{1}'', \psi^{\text{small}} \rangle + \langle \mathbf{1}'', \psi^{\text{big}} \rangle \leq \langle \mathbf{1}'', \psi^{\text{small}} \rangle + \langle \mathbf{1}, \psi^{\text{big}} \rangle.$$

To deal with the first term of this, observe that $\mathbf{1}''$ takes the value 1 in at most $\frac{2d(k-1)n}{L}$ places, and zero elsewhere, while ψ^{small} is bounded by $2c^d$, so that the first inner product is at most $\frac{1}{n} \cdot \frac{2d(k-1)n}{L} \cdot 2c^d = \frac{4c^d d(k-1)}{L}$. The second inner product is one we have already bounded, using (12), by δ .

Putting the pieces together, we have

$$|\langle \tilde{\mu} - \mathbf{1}, \phi \rangle| \leq \delta + 2c^d \delta + \delta + \delta + \frac{2c^d d(k-1)}{L} + \frac{4c^d d(k-1)}{L} + \delta \leq \varepsilon,$$

as required.

Finally, we need to prove the moment bounds required in the lemma. By Lemma 23, we have $\Phi(\tilde{\mu}, \mathbf{1}) \subseteq \Phi(\mu_1, \dots, \mu_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})$, so it suffices to prove the required moment bounds hold for all ϕ in the latter polytope.

Consider the optimisation problem $\max_{\phi} \langle \tilde{\mu}, \phi^{d'} \rangle$, over $\phi \in \Phi(H)$. By Fact 9, the maximum is attained at a vertex of $\Phi(H)$. Since $\tilde{\mu}$ is a non-negative vector, the vertex in question is a H -anti-uniform function ψ (and not a negation). If $\psi \in \Sigma$, then since $0 \leq \psi \leq 1$ we have $\langle \tilde{\mu}, \psi \rangle \leq \langle \tilde{\mu}, \mathbf{1} \rangle \leq 2$ since X has at most $2pn$ elements, which is sufficient. So we may assume ψ is not in Σ . Again since $\tilde{\mu}$ is non-negative, we may assume this anti-uniform function is pointwise maximised, in other words it is an H -largest anti-uniform functional in $\Phi(H)$, and therefore $\psi^{d'}$ is an H -largest anti-uniform functional in $\Phi(H)^{d'}$. Applying (9), we have $\langle \tilde{\mu}, \psi^{d'} \rangle \leq 2c^{d'}$ as required.

A similar argument applies to the optimisation problem $\max_{\Phi} |\langle \tilde{\mu}, \phi \rangle|$. Since Φ is centrally symmetric, the maximum is the same as for the linear problem $\max_{\Phi} \langle \tilde{\mu}, \phi \rangle$; as above, this is attained for ϕ an H -largest anti-uniform functional in $\Phi(H)$, and (9) gives $\langle \tilde{\mu}, \psi \rangle \leq 2c$ for such functionals.

The same argument, replacing $\tilde{\mu}$ with $\mathbf{1}$, gives the other required moment bounds. \square

Finally, we are in a position to prove Theorem 20: at this stage, this simply amounts to putting together the lemmas we showed in the last two sections.

Proof of Theorem 20. Given $\varepsilon > 0$, let $\varepsilon_1 > 0$ and d, d' be returned by Lemma 21 for input ε . Without loss of generality, we may assume $\varepsilon_1 \leq \varepsilon$. Note that d' is guaranteed to be even. We now input d, d' , and ε_1 to Lemma 26, which returns L_0 , and, provided $L \geq L_0$ also C .

Now, assume that our setting satisfies the C -conditions. In particular, the conditions of Lemma 26 are satisfied, so with probability at least $1 - \exp(-\frac{pn}{C})$, the set $X = [n]_p$ has an ε_1 -deletion \tilde{X} such that there exist χ_μ, χ_1 for which the following hold, with Notation 22. For all $\phi \in \Phi'$, we have $\langle \tilde{\mu} - \mathbf{1}, \phi \rangle < \varepsilon_1$, and in addition for all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})$ we have $|\langle \tilde{\mu}, \phi \rangle|, |\langle \mathbf{1}, \phi \rangle| \leq 2c$ and $\langle \tilde{\mu}, \phi^{d'} \rangle, \langle \mathbf{1}, \phi^{d'} \rangle \leq 2c^{d'}$. Suppose X satisfies the likely event, and fix \tilde{X} and χ_μ, χ_1 witnessing this.

The inequalities $|\langle \tilde{\mu}, \phi \rangle|, |\langle \mathbf{1}, \phi \rangle| \leq 2c$ for all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})$ are as required for Theorem 20, while the inequalities $\langle \tilde{\mu}, \phi^{d'} \rangle, \langle \mathbf{1}, \phi^{d'} \rangle \leq 2c^{d'}$ for $\phi \in \Phi(\tilde{\mu}, \mathbf{1})$ verify 2 of Lemma 21.

Applying Lemma 23, we have $\Phi(\tilde{\mu}, \mathbf{1})^d \subseteq \Phi'$, so in particular we obtain $\langle \tilde{\mu} - \mathbf{1}, \phi \rangle < \varepsilon_1$ for all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})^d$. This verifies 1 of Lemma 21, and hence we obtain the conclusion that $|\langle \tilde{\mu} - \mathbf{1}, \phi^+ \rangle| < \varepsilon$ for all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})$. In addition, since $\Phi(\tilde{\mu}, \mathbf{1}) \subseteq \Phi(\tilde{\mu}, \mathbf{1})^d$, we have $\langle \tilde{\mu} - \mathbf{1}, \phi \rangle < \varepsilon_1 \leq \varepsilon$ for all $\phi \in \Phi(\tilde{\mu}, \mathbf{1})$, which is the same as $\|\tilde{\mu} - \mathbf{1}\|_{\Phi(\tilde{\mu}, \mathbf{1})} < \varepsilon$, completing the proof of Theorem 20. \square

It remains to prove Lemma 25.

10. MOMENT ESTIMATES

Ultimately, as per the reduction in Theorem 20, we seek that our random subset X contains, outside of an event with exponentially small probability, a large subset \tilde{X} whose corresponding functional $\tilde{\mu}$ satisfies certain anti-correlation and moment bound properties with the functions in the polytope Φ' . In this section we show that these properties hold with a reasonably high probability for X itself; we use this to prove that a subset with these properties $\tilde{X} \subseteq X$ exists with the required exponential probability in the next section.

To state the precise lemma, we need the following definition.

Definition ((q, d)-special product). Let us be in Setting 2. A (q, d) -special product is a random functional $\psi : [n] \rightarrow \mathbb{R}$ obtained as the product of at most d convolution functions $*_{i,1}(f_1, \dots, f_k)$, in which each of the f_j is either equal to the $\mathbf{1}$ function, or it is a scaled copy of the random set $[n]_q$ (having entries valued 0 or q^{-1}). Moreover the copies of $[n]_q$ in the product comprising ψ have the property that each is either identical to, or completely independent from, any of the other copies of $[n]_q$ used in ψ .

The technical lemma we require is as follows.

Lemma 28. Let us be in Setting 2. Given $d' \in \mathbb{N}$ and $\alpha \geq 0$, there exists a C such that the following holds if the C -conditions are satisfied. Let q be at least $C \log^{2k} n^{-1}$. Then with probability at least $1 - \frac{1}{n^{\alpha k}}$ over the sample of a (q, d') -special product ψ , and over the choice of X as a copy of $[n]_q$ which is either identical to a copy of $[n]_q$ in ψ , or completely independent from all copies, we have the following:

$$\langle \mu, \psi \rangle \leq 2c^{d'} \quad \text{and} \quad \langle \mathbf{1}, \psi \rangle \leq 2c^{d'}.$$

Proof. Let $C = (1 + \alpha)^{2k} d'^{d'} + 1 k^{3k+d'} 2^{kd'^2+8k}$. As ψ is a (q, d') -special product we have for some $d \leq d'$ that $\psi(x) = \prod_{j \in [d]} *_{i_j,1}(f_1^{(j)}, \dots, f_k^{(j)})$ for some $i_1, \dots, i_d \in [k]$ and some $f_\ell^{(j)}$ that are either equal to the $\mathbf{1}$ function, or to a scaled copy of independent random sets $[n]_q$ (with possibility of two functionals being the same, but all different samples taken independently). For such i_j and $f_\ell^{(j)}$ we can therefore write explicitly

$$\psi(x) = \left(\frac{n}{e(s)}\right)^d \prod_{j=1}^d \sum_{s \in S_{i_j}(x)} \prod_{\ell \neq i_j} f_\ell^{(j)}(s_\ell).$$

It is helpful to refer to each of the terms in this summation individually. To this end we use the following notation

$$\hat{\psi}(x; s^{(1)}, \dots, s^{(d)}) = \prod_{j=1}^d \prod_{\ell \neq i_j} f_\ell^{(j)}(s_\ell^{(j)}).$$

We require that with high probability $\langle \mathbf{1}, \psi \rangle \leq 2c^{d'}$ and $\langle \mu, \psi \rangle \leq 2c^{d'}$. Since we may assume $c \geq 1$, it is enough to show that $\langle \mathbf{1}, \psi \rangle \leq 2c^d$ and $\langle \mu, \psi \rangle \leq 2c^d$. To do so, we prove the concentration of the following polynomials around their expectations. We have:

$$\begin{aligned} Y_{\mathbf{1}} &= \langle \mathbf{1}, \psi \rangle = \frac{1}{n} \left(\frac{n}{e(S)}\right)^d \sum_{x \in [n]} \sum_{\substack{s^{(1)} \in S_{i_1}(x) \\ \vdots \\ s^{(d)} \in S_{i_d}(x)}} \hat{\psi}(x; s^{(1)}, \dots, s^{(d)}) \\ Y_\mu &= \langle \mu, \psi \rangle = \frac{1}{n} \left(\frac{n}{e(S)}\right)^d \sum_{x \in [n]} \sum_{\substack{s^{(1)} \in S_{i_1}(x) \\ \vdots \\ s^{(d)} \in S_{i_d}(x)}} \mu(x) \hat{\psi}(x; s^{(1)}, \dots, s^{(d)}). \end{aligned}$$

It is extremely important to notice that all of these terms make use of the same set of functionals $f_\ell^{(j)}$ (evaluated in different points of different edges). Thus, the difference between terms is not given by a difference in functionals, which are always the same, but a different in indices and hyperedges. This justifies the following notation: we denote by l the number

of the $d(k - 1)$ functions $f_\ell^{(j)}$ comprising ψ (and thus each of the $\hat{\psi}$) that are copies of $[n]_q$, and $l' = d(k - 1) - l$ are copies of $\mathbf{1}$. Moreover, recalling that any copies of $[n]_q$ must be either identical or completely independent from each other, we denote by $w \leq l$ the number of independent copies of $[n]_q$ in ψ . As mentioned above, it is important to keep in mind that l, l' and w hold term-by term, as the functionals do not change in between terms.

Our plan now is to first calculate the expectation of Y_1 and Y_μ . We then use Kim-Vu's inequality (Theorem 8) to prove the concentration. This result applies since we may form new polynomials $\tilde{Y}_1, \tilde{Y}_\mu$, having the same value as Y_1, Y_μ , but consisting of independent Bernoulli random variables (as required by Theorem 8) by factoring out q^{-1} from each $\{0, q^{-1}\}$ valued Bernoulli variable into a collective weight, and dropping any repeat copies of the now $\{0, 1\}$ valued Bernoulli variables within a configuration, we obtain the polynomials $\tilde{Y}_1, \tilde{Y}_\mu$ as required. The details are as follows.

Observe that each term in Y_1 and in Y_μ corresponds to a tuple $(s^{(1)}, \dots, s^{(d)})$ of d hyperedges of S for which the i_1, \dots, i_d -th vertices within the corresponding hyperedge are the same element $x \in [n]$. We thus refer to the terms within these polynomials as (linked hyper-edge) configurations. Each configuration is completely determined by $(s^{(1)}, \dots, s^{(d)})$.

Notice that each of the $f_j^{(r)}(s_j^{(r)})$ with $r \in [d], j \in [k] \setminus \{i_r\}$, is a random variable taking a value of 1 if $f^{(r)} = \mathbf{1}$ is the constant one function, or else q^{-1} with a probability q , and 0 with probability $1 - q$, if $f^{(r)}$ is a copy of $[n]_q$. Since ψ is a (q, d') -special product, any two of the random variables, $f_{j_1}^{(i_1)}(s_{j_1}^{(i_1)})$ and $f_{j_2}^{(i_2)}(s_{j_2}^{(i_2)})$, are dependent if and only if $f_{j_1}^{(i_1)}, f_{j_2}^{(i_2)}$ are the same copy of $[n]_q$ (if one of them is $\mathbf{1}$, or if they're two distinct copies of $[n]_q$, they'd be independent) and $s_{j_1}^{(i_1)} = s_{j_2}^{(i_2)}$ (every entry of $[n]_q$ is selected independently). In this case, they are identical. The second condition $s_{j_1}^{(i_1)} = s_{j_2}^{(i_2)}$, corresponds to the hyperedges $s^{(i_1)}, s^{(i_2)}$ overlapping on their j_1 -th and j_2 -th elements respectively. We again point out that the number of independent variables in each configuration is not given by any choice of functionals (which are always the same), but rather by how much the corresponding hyperedges of the configuration intersect one another (thus possibly allowing for two identical functionals to be evaluated at the same value).

We first calculate the quantities $\mathbb{E}(Y_1), \mathbb{E}(Y_\mu)$ whose concentration we wish to establish. Let us consider the polynomial Y_1 . We want to calculate $\mathbb{E}(Y_1)$ applying the linearity of expectation and summing the contribution from each edge configuration. As mentioned above, every term of Y_1 makes use of always the same functions $f_\ell^{(j)}$ which never change. Thus, in Y_1 , the number of independent variables $f_j^{(r)}(s_j^{(r)})$ within a configuration is at least $\max(k - 1, l' + w)$ and at most $d(k - 1)$. The lower bound $k - 1$ holds since each hyperedge $s^{(r)}$ contains k distinct elements of $[n]$, so for any fixed $r \in [d]$, the set $\{f_j^{(r)}(s_j^{(r)})\}, j \in [k] \setminus \{i_r\}$ is a set of mutually independent (possibly constant) random variables. The lower bound $l' + w$, holds because, as mentioned above, each term comprises of $l' + w$ independent random functionals (evaluated at some of their points, possibly the same). That is, if $\{f^{(u_1)}, \dots, f^{(u_{w+l'})}\}$ is a maximal independent set of functionals —containing w independent copies of $[n]_q$ and l' copies of constant $\mathbf{1}$ — for ψ (and thus for the specific term we are considering) then any set of random variables formed taking one argument from each of these, is a set of mutually independent variables. Note for $\mathbb{E}(Y_\mu)$ the corresponding bounds are $\max(k - 1, l' + w) + 1$ and $d(k - 1) + 1$ since $\mu(x)$ contributes one variable independent from all the others.

For $\mathbb{E}(Y_1)$, suppose that, in a given configuration the edges are such that t of the variables are mutually independent with $\max(k-1, l'+w) \leq t \leq d(k-1)$. The number of repeat $\{0, q^{-1}\}$ -valued Bernoulli variables is then $d(k-1) - t$, each of which contributes an extra factor of q^{-1} to the expectation of this configuration. To see this, consider the product of s such identical variables $Z = x_1 \dots x_s$. We have that Z takes value q^{-s} with probability q and 0 otherwise, so $\mathbb{E}(Z) = q^{-(s-1)}$, whereas the expectation of each of the x_i is 1.

To calculate the expectation of Y_1 , it is therefore enough to enumerate the configurations having the same number t of independent variables. To this end, let $S(\psi, t, x) \subseteq S_{i_1}(x) \times S_{i_2}(x) \times \dots \times S_{i_d}(x)$ be those d -tuples $(s^{(1)}, s^{(2)}, \dots, s^{(d)})$ in which exactly t of the $\{f_j^{(i)}(s_j^{(i)})\}$ are mutually independent (counting also those for which $f_j^{(i)}$ is the constant functional $\mathbf{1}$), and let $\alpha(\psi, t, x) = |S(\psi, t, x)|$. For the sake of notational brevity, let $t' = d(k-1)$ and $a = \max(k-1, l'+w)$. We have

$$\mathbb{E}(Y_1) = \frac{1}{n} \left(\frac{n}{e(S)} \right)^d \sum_{x \in [n]} \sum_{a \leq t \leq t'} (q^{-1})^{(t'-t)} \alpha(\psi, t, x).$$

Note that $\mathbb{E}(Y_1) = \mathbb{E}(Y_\mu)$ as $\mu(x)$ is independent from all other variables within a given configuration, since the k elements in each edge s of S are distinct; therefore the argument of μ does not occur as the argument of any other $f_j^{(i)}$ within this configuration, and for any $x \in [n]$ we have that $\mu(x)$ contributes an expectation factor of $\mathbb{E}(\mu(x)) = 1$.

To obtain an upper bound for $\alpha(\psi, t, x)$ with $a \leq t \leq t'$, first note that when $t = t' = d(k-1)$, we may take the crude upper bound fixing only x in each $S_i(x)$, thus $\alpha(\psi, d(k-1), x) \leq \prod_{j \in [d]} |S_{i_j}(x)| \leq (\Delta_1)^d \leq \left(c \frac{e(S)}{n} \right)^d$. If the number of independent variables t is less than $d(k-1)$ in a configuration, we have at least two random variables that are identical, say $f_{j_1}^{(i_1)}(s_{j_1}^{(i_1)}) = f_{j_2}^{(i_2)}(s_{j_2}^{(i_2)})$. Note that this can only occur if $s_{j_1}^{(i_1)} = s_{j_2}^{(i_2)}$. Thus, in order to have precisely t independent variables in the configuration $(s^{(1)}, s^{(2)}, \dots, s^{(d)}) \in S(\psi, t, x)$, it must be that the union of underlying hyperedges, $\cup_{i \in [d]} s^{(i)}$, together covers at most t vertices other than x . But this calculation is exactly what is given by Lemma 18 setting $\mathbf{i} = (i_1, \dots, i_d)$ and forgiving the horrible notation $\alpha(\psi, t, x) \leq \alpha(\mathbf{i}, t, x)$ (which is only needed here). We therefore obtain:

$$\begin{aligned} \mathbb{E}(Y_1) &= \left(\frac{n}{e(S)} \right)^d \frac{1}{n} \sum_{x \in [n]} \sum_{a \leq t \leq t'} (q^{-1})^{(t'-t)} \alpha(\psi, t, x) \\ &\leq \left(\frac{n}{e(S)} \right)^d \frac{1}{n} \sum_{x \in [n]} \alpha(\mathbf{i}, t', x) + \left(\frac{n}{e(S)} \right)^d \frac{1}{n} \sum_{x \in [n]} \sum_{a \leq t \leq t'-1} (q^{-1})^{(t'-t)} \alpha(\mathbf{i}, t, x) \\ &\leq c^d + \sum_{k-1 \leq t \leq t'-1} (q^{-1})^{(t'-t)} 2^{kd^2} c^d t^d C^{-(t'-t)} q^{t'-t} \\ &= c^d + 2^{kd^2} c^d \sum_{a \leq t \leq t'-1} t^d C^{-(t'-t)} \\ &= c^d + 2^{kd^2} c^d \sum_{1 \leq s \leq t'-a} (t' - t)^d C^{-s} \\ &\leq c^d + 2^{kd^2} c^d (kd)^{(d+1)} C^{-1} \leq c^d (1 + k^{1-3k}) \leq 3c^d / 2. \end{aligned}$$

Where the last line follows because of our lower bound on C .

We now advise the reader to familiarise themselves with the notation of Kim-Vu's inequality (Theorem 8), which we now want to apply. It is not difficult to see, as claimed above, that Y_1 is a polynomial in random variables exactly as the one studied by Kim-Vu's inequality (up to a scaling factor). We use here the notation introduced for Kim-Vu's inequality.

Consider now the calculation for $\mathbb{E}_i(Y_1)$ with $i \geq 1$. Suppose we fix the variables $A \subseteq \{f_j^{(u)}(x) : u \in [d], j \in [k] \setminus \{i_u\}, x \in [n]\}$. Note that if we have $f_j^{(u)}(a), f_j^{(u)}(b) \in A$ with $a \neq b$ then the expectation is 0 since no term in the sum contains both. Thus, also $\mathbb{E}_i(Y_1) = 0$ whenever $i > d(k - 1)$, and we may equivalently describe any subset of variables A for which $\mathbb{E}(Y_{1_A})$ is non-vanishing, by specifying the elements $s_j^{(i)}$ held fixed in the corresponding functions $f_j^{(i)}$ in ψ . To this end, we introduce the following notation. Write \mathbf{m}_i for the vector of length k , and whose entries may be empty, for the elements $s_j^{(i)}$ held fixed in $s^{(i)}$. Let m_i be the number of non-empty elements in \mathbf{m}_i and $M = \sum m_i \leq l$ (the total number of elements held fixed). For instance, suppose $\psi = *_{i_1,1}(f_1^{(1)}, \dots, f_k^{(1)}) \dots *_{i_d,1}(f_1^{(d)}, \dots, f_k^{(d)})$ and we fix $f_1^{(1)}(s_1^{(1)}), f_k^{(1)}(s_k^{(1)}), f_1^{(2)}(s_1^{(2)})$, with all other variables allowed to vary. Then letting \star denote the empty element, we have $\mathbf{m}_1 = (s_1^{(1)}, \star, \dots, \star, s_k^{(1)}), \mathbf{m}_2 = (s_1^{(2)}, \star, \dots, \star)$ and for all other $3 \leq j \leq l$, \mathbf{m}_j is the empty vector of length k . Write $\mathbf{M} = (\mathbf{m}_1, \dots, \mathbf{m}_d)$ for the collection of \mathbf{m}_i and $H_{\mathbf{M}}$ for the truncated polynomial retaining those configurations fixing \mathbf{M} . We use the notation $S(\mathbf{m})$ for the collection of hyperedges fixing \mathbf{m} , and $S_i(x; \mathbf{m})$ for the set of hyperedges with x in the i -th position, and the elements of the \mathbf{m} in the positions in which they occur. Note that wherever we use this notation, we have \mathbf{m} empty in the i -th position, so there is no potential conflict here. Recalling that l is the number of copies of $[n]_q$ in ψ , let $\tilde{l} = \tilde{l}(\mathbf{M})$ be the number of these whose entries are fixed in \mathbf{M} . In this notation, the truncated polynomial $H_{1,\mathbf{M}}$ fixing $\mathbf{m}_1, \dots, \mathbf{m}_d$ in the edges $s^{(1)}, \dots, s^{(d)}$ takes the form

$$(13) \quad H_{1,\mathbf{M}} = \frac{q^{-\tilde{l}}}{n} \left(\frac{n}{e(S)} \right)^d \sum_{x \in [n]} \sum_{\substack{s^{(1)} \in S_1(x; \mathbf{m}_1) \\ \dots \\ s^{(d)} \in S_d(x; \mathbf{m}_d)}} g_1(\cdot) \dots g_{l-M}(\cdot).$$

where we have omitted explicitly writing any variables corresponding to constant one functions since it does not change the value of the polynomial. $g_1(\cdot) \dots g_{l-M}(\cdot)$ denote only the $l - M$ unfixed variables $f_j^{(i)}(\cdot)$ where $f_j^{(i)}$ is a copy of $[n]_q$ and \cdot the appropriate element $x \in [n]$ at which it is evaluated within the configuration.

In this form, it is clear that fixing additional constant-one valued variables decreases the polynomial expectation since it only amounts to dropping terms, each having a strictly positive expectation. Specifically given any \mathbf{M}' , if \mathbf{M}' fixes all the elements of \mathbf{M} , along with elements $s_{j_1}^{(i_1)}, \dots, s_{j_a}^{(i_a)}$ for which the corresponding $f_{j_1}^{(i_1)}, \dots, f_{j_a}^{(i_a)}$ are all constant-one valued, then the polynomial $H_{\mathbf{M}'}$ retains only those configurations (if any) of $H_{\mathbf{M}}$ which fix also $s_{j_1}^{(i_1)}, \dots, s_{j_a}^{(i_a)}$ in $s^{(i_1)}, \dots, s^{(i_a)}$ respectively and so $\mathbb{E}(H_{\mathbf{M}}) \geq \mathbb{E}(H_{\mathbf{M}'})$. We may therefore assume that no constant-one variables are fixed for the purpose of maximising $\mathbb{E}_i(Y_1)$ to apply Theorem 8. Recalling that ψ has l of its $d(k - 1)$ comprising functions being copies of $[n]_q$ and the rest being constant one, we are interested only in $\mathbb{E}_i(Y_1)$ with $1 \leq i \leq l$. In general, the greater the number i of 0, q^{-1} valued variables being fixed, the larger the premultiplying coefficient q^{-i} . However, fixing these variables also reduces the number of contributing configurations by a factor of order $(C^{-1}q)^i < q^i$ (arising from the maximum co-degree condition), meaning

an overall reduction in $\mathbb{E}_i(Y_1)$ with greater i . Moreover, where a fixed variable corresponds to an $f_j^{(i)}$ which is identical to an unfixed copy of $[n]_q$, there is a further reduction from the interdependence. Although in almost all configurations, the arguments for these indicator functions differs and thus the corresponding variables are independent. In this way one expects that $\mathbb{E}_i(Y_1)$ is greatest for $i = 1$, decreasing by a factor of about C^{-1} per variable fixed.

Recalling $1 \leq M \leq l$. When $M = l$,

$$\begin{aligned} E(H_{1,\mathbf{M}}) &= \frac{q^{-l}}{n} \sum_{x \in [n]} \left(\frac{n}{e(S)} \right)^d \sum_{\substack{s^{(1)} \in S_{i_1}(x; \mathbf{m}_1) \\ \dots \\ s^{(d)} \in S_{i_d}(x; \mathbf{m}_d)}} \leq \frac{q^{-l}}{n} \cdot \left(\frac{n}{e(S)} \right)^d \sum_{\substack{s^{(1)} \in S(\mathbf{m}_1) \\ \dots \\ s^{(2)} \in S_{i_2}(s^{(1)}_{i_1}; \mathbf{m}_2) \\ \dots \\ s^{(d)} \in S_{i_d}(s^{(1)}_{i_1}; \mathbf{m}_d)}} 1 \\ &\leq \frac{q^{-l}}{n} \cdot \left(\frac{n}{e(S)} \right)^d \Delta_{m_1} \Delta_{m_2+1} \dots \Delta_{m_d+1} \\ &\leq \frac{q^{-l}}{n} \cdot \left(\frac{n}{e(S)} \right)^d c^d C^{d-(M+d-1)} q^{(M+d-1)-d} \left(\frac{e(S)}{n} \right)^d \leq \frac{q^{-1} c^d C^{1-l}}{n} \\ &\leq \frac{c^d C^{-l}}{\log^{2k}(n)}. \end{aligned}$$

Thus, $\mathbb{E}_l(Y_1) \leq \frac{c^d C^{-l}}{\log^{2k}(n)}$. Otherwise, $M < l$ and within each configuration in $H_{\mathbf{M}}$ there are $l - M$ Bernoulli- $\{0, q^{-1}\}$ variables, which may or may not be independent. As with the calculation for $\mathbb{E}_0(Y_1)$, we allow that any configuration fixing $\mathbf{m}_1, \dots, \mathbf{m}_2$ in $s^{(1)}, \dots, s^{(d)}$ may result in precisely $t \in [l - M]$ of the $g_i(\cdot)$ being mutually independent. We follow a similar approach to that for $\mathbb{E}_0(Y_1)$, counting configurations which have the same number of mutually independent $g_i(\cdot)$. Note first that

$$\begin{aligned} \mathbb{E}(H_{1,\mathbf{M}}) &= \frac{q^{-M}}{n} \left(\frac{n}{e(S)} \right)^d \sum_{x \in [n]} \sum_{\substack{s^{(1)} \in S_{i_1}(x; \mathbf{m}_1) \\ \dots \\ s^{(d)} \in S_{i_d}(x; \mathbf{m}_d)}} \mathbb{E} g_1(\cdot) \dots g_{M-l}(\cdot) \\ &= \frac{q^{-M}}{n} \left(\frac{n}{e(S)} \right)^d \sum_{s^{(1)} \in S(\mathbf{m}_1)} \sum_{\substack{s^{(2)} \in S_{i_2}(s^{(1)}_{i_1}; \mathbf{m}_2) \\ \dots \\ s^{(d)} \in S_{i_d}(s^{(1)}_{i_1}; \mathbf{m}_d)}} \mathbb{E} g_1(\cdot) \dots g_{M-l}(\cdot). \end{aligned}$$

For a given $s^{(1)} \in S(\mathbf{m}_1)$, let $\alpha(\psi, \mathbf{M}, t, s^{(1)})$ be the size of the set $S(\psi, \mathbf{M}, t, s^{(1)}) \subseteq \{s^{(1)}\} \times S_{i_2}(s^{(1)}; \mathbf{m}_2) \times \dots \times S_{i_d}(s^{(1)}; \mathbf{m}_d)$ for which t of the $g_i(\cdot)$ are mutually independent. The expectation of the product $g_1(\cdot) \dots g_{l-M}(\cdot)$ for such a configuration is $q^{-(l-M-t)}$. We have

$$\begin{aligned}
\alpha(\psi, \mathbf{M}, t, s^{(1)}) &\leq \sum_{\substack{r_2+...+r_d \\ =l-M-t}} \binom{(k-1)}{r_2} \Delta_{m_2+1+r_2} \binom{2(k-1)-r_2}{r_3} \Delta_{m_3+1+r_3} \cdots \\
&\quad \cdots \cdot \binom{(d-2)(k-1) - \sum_{i \in [d-1] \setminus \{1\}} r_i}{r_d} \Delta_{m_d+1-r_d} \\
&\leq 2^{kd^2} (l-M-t)^{d-1} c^{d-1} C^{-l+t+m_1} q^{l-t-m_1} \left(\frac{e(S)}{n} \right)^{d-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbb{E}(H_{1,\mathbf{M}}) &\leq \frac{q^{-M}}{n} \left(\frac{n}{e(S)} \right)^d \sum_{s^{(1)} \in S(\mathbf{m}_1)} \sum_{t \in [l-M]} q^{-(l-M-t)} 2^{kd^2} (l-M-t)^{d-1} c^{d-1} \\
&\quad \cdot C^{-l+t+m_1} q^{l-t-m_1} \left(\frac{e(S)}{n} \right)^{d-1} \\
&\leq \frac{q^{-1}}{n} 2^{kd^2} (l-M)^d c^d C^{1-M} \\
&\leq \frac{c^d 2^{kd^2} (l-M)^d C^{-M}}{\log^{2k}(n)}.
\end{aligned}$$

Hence, for $M \geq 1$, $\mathbb{E}(H_{\mathbf{M}})$ is maximised when $M = 1$. Using the notation as in Theorem 8 we have $\mathbb{E}'(Y_1) \leq \frac{c^d 2^{kd^2} (l-1)^d C^{-1}}{\log^{2k}(n)} \leq \frac{c^d}{k^{3k} \log^{2k}(n)}$ and using the lower bound for q and C . Clearly $\mathbb{E}'(Y_1) \ll 1 < \mathbb{E}_0(Y_1) = \mathbb{E}(Y_1)$.

Using (Theorem 8) we have

$$P[|Y_1 - \mathbb{E}_0(Y_1)| > (8^k \cdot k!^{1/2}) (\mathbb{E}(Y_1) \mathbb{E}'(Y_1))^{1/2} \lambda^k] = O(e^{(-\lambda + (k-1) \log(n))}).$$

To achieve the concentration we require, take $\lambda = k(1 + \alpha) \log(n)$. It then follows that we have $e^{(-\lambda + (k-1) \log(n))} \leq e^{-(\alpha k + 1) \log(n)} \leq \frac{1}{n^{\alpha k}}$. We then have

$$\begin{aligned}
(8^k \cdot k!^{1/2}) (\mathbb{E}(Y_1) \mathbb{E}'(Y_1))^{1/2} \lambda^k &\leq (8^k \cdot k^{k/2}) \left(\frac{3c^{2d}}{2k^{3k} \log^{2k}(n)} \right)^{1/2} (k(1 + \alpha) \log(n))^k \\
&\leq 2c^d 8^k (1 + \alpha)^k.
\end{aligned}$$

We now turn to the concentration of $Y_\mu = \langle \mu, \psi \rangle$. Observe first that since the argument of μ is distinct from the argument of any other $f_j^{(i)}$ within a configuration, then $\mu(x)$ is independent of all other variables in the configuration. Since $\mathbb{E}(\mu(x)) = 1$ for any x , we have $\mathbb{E}_0(Y_\mu) = \mathbb{E}_0(Y_1) \leq 3/2$.

For any subset A of variables that we fix in Y_1 to calculate $\mathbb{E}_A(Y_1)$, we have $\mathbb{E}(Y_{1_A}) = \mathbb{E}_A(Y_{\mu_A})$ and for a fixed $x \in [n]$ and $B = A \cup \{\mu(x)\}$, we have $\mathbb{E}(Y_{\mu_B}) \leq \mathbb{E}(Y_{\mu_A})$ since we merely sum the same expectations over fewer configurations. Thus, $\mathbb{E}_i(Y_\mu)$ is maximised for $i = 1$ as for $\mathbb{E}_i(Y_1)$. Note that if $A = \{f_j^{(i)}(s_j^{(i)})\}$ then $\mathbb{E}(Y_{1_A}) = \mathbb{E}(Y_{\mu_A})$. If $A = \{\mu(x)\}$ for some fixed $x \in [n]$ then the calculation $\mathbb{E}(Y_{1_A})$ proceeds just as for $\mathbb{E}_0(Y_1)$ except with the summation over $x \in [n]$ dropped. That is, for $A = \{\mu(x)\}$, we have $\mathbb{E}_A(Y_{\mu_A}) \leq 3/2n \leq \frac{q^{-l}}{n} 2^{kd^2} l^d c^d$, thus the upper bound for $\mathbb{E}'(Y_1)$ holds for $\mathbb{E}'(Y_\mu)$ also, and the concentration obtained from the Kim-Vu inequality applies.

□

We deduce the following corollary.

Corollary 29. *Given d positive integer, there exist C and L_0 such that, if the C -conditions are satisfied in Setting 2 and $L \geq L_0$, then with high probability over the choice of $X = [n]_p$, and independently $\chi_\mu : [n] \rightarrow \{1, \dots, L\}$ uniformly at random, and independently $\chi_{\mathbf{1}} : [n] \rightarrow \{1, \dots, \lceil Lp^{-1} \rceil\}$ uniformly at random, the following holds. Let us use Notation 22. For any $1 \leq \ell \leq d$ and ψ a largest anti-uniform functional in $\Phi(\mu, \mathbf{1})^\ell$ or $\Phi(\mu_1, \dots, \mu_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^\ell$.*

$$\langle \mu, \psi \rangle \leq 2c^\ell \quad \text{and} \quad \langle \mathbf{1}, \psi \rangle \leq 2c^\ell.$$

In addition, if ψ is any largest anti-uniform functional in $\Phi(\mu_1, \dots, \mu_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^\ell$, and $1 \leq j \leq L$ and $1 \leq j' \leq \lceil Lp^{-1} \rceil$ then we have

$$\langle \mu_j, \psi \rangle \leq 2c^\ell \quad \text{and} \quad \langle \nu_{j'}, \psi \rangle \leq 2c^\ell.$$

The idea of the proof is to take a union bound over choices of ψ and j , of which there are only polynomially many, and use Lemma 28 to obtain the correlation bounds.

The only place where we need to be a bit careful is that the μ_j are not independent; and similarly the $\nu_{j'}$. We find some related functions $\hat{\mu}_j$ and $\hat{\nu}_{j'}$ which are independent and to which we apply Lemma 28, and deduce the required correlation bounds from these.

Proof. Let $H = \{\mu_1, \dots, \mu_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil}\}$, let α be such that both $dk^d \cdot 2^{(k-1)d} \frac{1}{n^{\alpha k}}$ and $dk^d (2Lp^{-1})^{d(k-1)+1} \frac{1}{n^{\alpha k}}$ are $o(1)$. Let C be large enough so that Lemma 28 works for the choice $q = p$, α , and $d' = d$. Assume the C -conditions are satisfied.

We first establish bounds on $\langle \mu, \psi \rangle$ and $\langle \mathbf{1}, \psi \rangle$ for $\psi \in \Phi(\mu, \mathbf{1})^\ell$. Given $1 \leq \ell \leq d$, if ψ is a largest anti-uniform functional in $\Phi(\mu, \mathbf{1})^\ell$, then $\psi = \prod_{j=1}^\ell *_{i_j, 1}(f_{j,1}, \dots, f_{j,k-1})$, where $1 \leq i_j \leq k$ for each j and each $f_{j,j'}$ is either μ or $\mathbf{1}$. For any such function, the probability of

$$\langle \mu, \psi \rangle > 2c^\ell \quad \text{or} \quad \langle \mathbf{1}, \psi \rangle > 2c^\ell$$

is, by Lemma 28, at most $\frac{1}{n^{\alpha k}}$. Taking the union bound over the at most $dk^d \cdot 2^{(k-1)d}$ choices of ℓ , i_j and $f_{j,j'}$, we see that the probability of any of these events occurring is at most $dk^d \cdot 2^{(k-1)d} \frac{1}{n^{\alpha k}}$, which is $o(1)$ because of our choice of α .

We now establish corresponding bounds on $\langle \mu_j, \psi \rangle$ and $\langle \nu_j, \psi \rangle$. Observe that, as before, we can describe any largest anti-uniform functional ψ in $\Phi(H)^\ell$ as follows. We choose i_1, \dots, i_ℓ , and for each of the $\ell(k-1)$ functions in the product, we must choose one of H . Finally, to describe the entire inner product, we must choose the left term in the inner product (either μ_j or ν_j) from H . In total, the number of choices is at most $dk^d (2Lp^{-1})^{d(k-1)+1}$.

Fix now one such set of choices. Let T denote a collection of $d(k-1) + 1$ indices in $[L]$ such that μ_t is one of the chosen functions for each $t \in T$, and T' a subset of $[\lceil Lp^{-1} \rceil]$ of size $d(k-1)$ such that ν_t is chosen for each $t \in T'$.

Consider the following random experiment. For each $t \in T$, we first generate independent binomial random subsets $Z_t = [n]_q$, with $0 < q < 1$ chosen such that $(1-q)^{|T|} = 1 - t \frac{p}{L}$. We now obtain sets Z'_t for $t \in T$ as follows. For each $x \in \bigcup_{t \in T} Z_t$ independently, pick t uniformly at random from the set $\{t : x \in Z_t\}$, and let $x \in Z'_t$.

By definition of q , for a given $x \in [n]$ the probability that $x \in \bigcup_{t \in T} Z_t$ is $\frac{tp}{L}$, and conditioning on this event occurring, the events $x \in Z'_t$ are disjoint over $t \in T$, and x is equally likely to appear in any given Z'_t for $t \in T$, so that probability of $x \in Z'_t$ is $\frac{p}{L}$. Observe that this is the same probability as the event that $x \in X$ and $\chi_\mu(x) = t$, which are also disjoint

events over $t \in T$. It follows that the distribution of $(Z_t)_{t \in T}$ is the same as the distribution of $(X \cap \{x : \chi_\mu(x) = t\})_{t \in T}$, so we can consider the coupling in which the latter sets are generated according to the above random experiment.

Let $\hat{\mu}_t(x) = q^{-1} \mathbb{1}(x \in Z_t)$. By construction, we have $0 \leq \mu_t(x) \leq Lp^{-1}q\hat{\mu}_t(x)$.

We now perform a similar, independent, random experiment. For each $t \in T'$, we generate independently $W_t = [n]_q$ where q is as defined above. Letting now $0 < q' < 1$ solve $(1-q')^{|T'|} = 1 - t \frac{1}{\lceil Lp^{-1} \rceil}$, we observe $q' \leq q$. We generate W'_t by sampling the elements of W_t independently with probability $\frac{q'}{q}$, so that the W'_t are independent copies of $[n]_{q'}$. Finally, we generate W'_t by, as above, picking t from $\{t : x \in W'_t\}$ independently and uniformly and letting $x \in W'_t$.

As before, the distribution of $(W'_t)_{t \in T'}$ is identical to the distribution of $(\{x : \chi_1(x) = t\})_{t \in T'}$ and we consider the coupling in which the latter sets are generated by the above random experiment. Letting $\hat{\nu}_t(x) = q^{-1} \mathbb{1}(x \in W_t)$, we have $0 \leq \nu_t(x) \leq \lceil Lp^{-1} \rceil q \hat{\nu}_t(x)$.

Let $\hat{\psi}$ denote the function obtained by replacing each μ_t with $\hat{\mu}_t$ for $t \in T$, and each ν_t with $\hat{\nu}_t$ for $t \in T'$, in the product defining ψ . Then we have

$$\langle \mu_j, \psi \rangle \leq (\lceil Lp^{-1} \rceil q)^{d(k-1)+1} \langle \hat{\mu}_j, \hat{\psi} \rangle \quad \text{and} \quad \langle \nu_j, \psi \rangle \leq (\lceil Lp^{-1} \rceil q)^{d(k-1)+1} \langle \hat{\nu}_j, \hat{\psi} \rangle.$$

Now, $\hat{\psi}$ is a (q, d) -special product. Assume C is also large enough that so that Lemma 28 holds for $d' = d$, our α , and $q = q$.

$$\langle \hat{\mu}_j, \hat{\psi} \rangle > \frac{7}{4}c^\ell \quad \text{and} \quad \langle \hat{\nu}_j, \hat{\psi} \rangle > \frac{7}{4}c^\ell$$

each have probability at most $\frac{1}{n^{\alpha k}}$ by Lemma 28. Since $\frac{7}{4}(\lceil Lp^{-1} \rceil q)^{d(k-1)+1} < 2$, the same probability bounds hold on the events

$$\langle \mu_j, \psi \rangle > 2c^\ell \quad \text{and} \quad \langle \nu_j, \psi \rangle > 2c^\ell.$$

Finally taking the union bound, the probability that any one of these events fails is $o(1)$ by our choice of α .

Suppose that none of the above bad events occur. We deduce, deterministically, the remaining bounds of Corollary 29. We begin with $1 \leq \ell \leq d$ and $\psi \in \Phi(H)^\ell$, for which we have

$$\langle \mu, \psi \rangle = \frac{1}{L} \sum_{i=1}^L \langle \mu_i, \psi \rangle \leq \frac{1}{L} \sum_{i=1}^L 2c^\ell = 2c^\ell.$$

Similarly, we have

$$\langle \mathbf{1}, \psi \rangle = \frac{1}{\lceil Lp^{-1} \rceil} \sum_{i=1}^{\lceil Lp^{-1} \rceil} \langle \nu_i, \psi \rangle \leq \frac{1}{\lceil Lp^{-1} \rceil} \sum_{i=1}^{\lceil Lp^{-1} \rceil} 2c^\ell = 2c^\ell. \quad \square$$

11. DELETION METHOD

11.1. A general deletion method. In this section we prove that the required \tilde{X} satisfying moment estimates exists with exponentially small failure probability. This follows from the Harris inequality. Recall that a subset \mathcal{D} of $\mathcal{P}([n])$ is called *decreasing* if whenever $S'' \subseteq S \in \mathcal{D}$ we have $S' \in \mathcal{D}$, and *increasing* if the same statement holds with \subseteq replaced by \supseteq .

Theorem 30 (Harris [12]). *For any $p \in [0, 1]$ and n , let \mathcal{A} and \mathcal{B} be two subsets of $\mathcal{P}([n])$, which are both decreasing. Then*

$$\mathbb{P}([n]_p \in \mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}([n]_p \in \mathcal{A}) \mathbb{P}([n]_p \in \mathcal{B}).$$

Spöhel, Steger and Warnke [18] deduced the following theorem. They state their result for the specific case $[n] = \binom{[m]}{2}$ (i.e. for the random graph), but their proof works verbatim in the more general situation. For completeness, we give the details.

Theorem 31 ([18, Theorem 4]). *Let \mathcal{D} be a decreasing subset of $\mathcal{P}([n])$. Given $\alpha, \delta \in (0, 1]$, let $p \in (0, 1]$ be such that $\mathbb{P}([n]_p \in \mathcal{D}) \geq \delta$. Then with probability at least $1 - \delta^{-1} \exp(-\frac{1}{2}\alpha^2 pn)$, there is a subset of $[n]_p$ with at least $(1 - \alpha)pn$ elements which is in \mathcal{D} .*

Proof. Let \mathcal{I} be the subset of $\mathcal{P}([n])$ consisting of sets with at least $(1 - \alpha)pn$ elements. Let \mathcal{S} be the subset of sets $S \in \mathcal{P}([n])$ such that S has a subset in $\mathcal{I} \cap \mathcal{D}$, which is clearly increasing, so $\overline{\mathcal{S}}$ is decreasing. By Theorem 30, we have $\mathbb{P}([n]_p \in \overline{\mathcal{S}})\mathbb{P}([n]_p \in \mathcal{D}) \leq \mathbb{P}([n]_p \in \overline{\mathcal{S}} \cap \mathcal{D})$. Rearranging, and observing $\overline{\mathcal{S}} \cap \mathcal{D} \subseteq \overline{\mathcal{I}}$, we get

$$\mathbb{P}([n]_p \in \overline{\mathcal{S}}) \leq \frac{\mathbb{P}([n]_p \in \overline{\mathcal{S}} \cap \mathcal{D})}{\mathbb{P}([n]_p \in \mathcal{D})} \leq \delta^{-1} \mathbb{P}([n]_p \in \overline{\mathcal{I}}).$$

Chernoff's inequality now gives $\mathbb{P}([n]_p \in \overline{\mathcal{I}}) \leq \exp(-\frac{1}{2}\alpha^2 pn)$, which gives the required probability bound. \square

We now have the tools to prove the last remaining Lemma, i.e. Lemma 25.

Proof of Lemma 25. We are in Setting 2. Let C and L be large enough so that Corollary 29 works for our choice of $d = d'$. Assume the C -conditions are satisfied.

Let $\mathcal{D} \subseteq \mathcal{P}([n])$ be the set of subsets $Y \subseteq [n]$ satisfying the following. Letting $\mu(x) = p^{-1}\mathbf{1}(x \in Y)$, for uniform random choices of χ_μ and $\chi_{\mathbf{1}}$, with probability at least 0.9, for all $1 \leq \ell \leq d$ and all largest anti-uniform functionals ψ either in the set $\Phi(\mu, \mathbf{1})^\ell$ or in the set $\Phi(\mu_1, \dots, \mu_L, \nu_1, \dots, \nu_{\lceil Lp^{-1} \rceil})^\ell$, we have

$$\langle \mu, \psi \rangle \leq 2c^\ell \quad \text{and} \quad \langle \mathbf{1}, \psi \rangle \leq 2c^\ell.$$

In addition, if ψ is any largest anti-uniform functional in $\Phi(H)^\ell$, and $1 \leq j \leq L$ and $1 \leq j' \leq \lceil Lp^{-1} \rceil$ then we have

$$\langle \mu_j, \psi \rangle \leq 2c^\ell \quad \text{and} \quad \langle \nu_{j'}, \psi \rangle \leq 2c^\ell.$$

Observe that since all the left hand sides of these conditions are increasing in X , the event \mathcal{D} is a decreasing event. Furthermore, Corollary 29 states that $\mathbb{P}(\mathcal{D}) = 1 - o(1) \geq \frac{1}{2}$.

We now apply Theorem 31 with this \mathcal{D} , with $\alpha = \frac{1}{2}\delta$, and with $\mathbb{P}(\mathcal{D}) \geq \frac{1}{2}$, to deduce that with probability at least $1 - 2 \exp(-\frac{1}{8}\delta^2 pn)$ there is a subset \tilde{X} of X which is in \mathcal{D} and which has at least $(1 - \frac{1}{2}\delta)pn$ elements. Additionally, the probability that $[n]_p$ has more than $(1 + \frac{1}{2}\delta)pn$ elements is by Theorem 5 at most $\exp(-\frac{1}{8}\delta^2 pn)$. Suppose that X satisfies both conditions, which occurs with probability at least $1 - 3 \exp(-\frac{1}{8}\delta^2 pn)$ by the union bound. Then $|X \setminus \tilde{X}| \leq \delta pn$ as required. \square

11.2. Transference Principle without Deletion. We are finally ready to prove items 1 and 2 of Theorem 3.

1 and 2 of Theorem 3. We are in Setting 2. We have that 1 follows immediately from 3 as an ε -good dense model for \tilde{X} provides an ε -good lower dense model for X .

Let us now show how to get 2 from 3 in Theorem 3. First, we may assume without loss of generality that $\bar{\Omega} = \{\bar{\omega} = \mathbf{1} - \omega : \omega \in \Omega\}$ is contained in Ω . This is because this assumption at most doubles the size of Ω , and therefore doesn't affect the order of magnitude of its size.

Let C be large enough to guarantee that 3 works for $\varepsilon = \frac{\varepsilon}{2^{k+2}}$. Assume the C conditions are satisfied. Let X be a sample of $[n]_p$ such that $|\{s \in S : s \subseteq X\}| \leq (1 + \frac{\varepsilon}{2})\mathbb{E}[|\{s \in S : s \subseteq [n]_p\}|]$ and such that X admits an $\frac{\varepsilon}{2^{k+2}}$ -deletion \tilde{X} such that all subsets of \tilde{X} have an $\frac{\varepsilon}{2^{k+2}}$ -good dense model. This happens with probability at most $1 - \eta_n$ on the choice of $X = [n]_p$. Notice that we have $|\{s \in S : s \subseteq X \setminus \tilde{X}\}| \leq \frac{\varepsilon}{2}\mathbb{E}[|\{s \in S : s \subseteq [n]_p\}|]$ because of our upper bound on $|\{s \in S : s \subseteq X\}|$ and because by Theorem 20 we have that $\mathbf{1}$ is a dense model of \tilde{X} . Let μ and $\tilde{\mu}$ be the p^{-1} scaled indicator functions of X and \tilde{X} respectively.

Let us now consider a subset Y of X . Let $\tilde{Y} = Y \cap \tilde{X}$, and let $\bar{Y} = \tilde{X} \setminus Y$. Let f be the scaled indicator function of \tilde{Y} . We use \bar{f} for the complements in \tilde{X} .

Fix an arbitrary $\omega \in \Omega$, and let $\bar{\omega} = \mathbf{1} - \omega$. We have the following.

$$\langle \tilde{\mu}, *_i \mathbf{1}(\tilde{\mu}, \dots, \tilde{\mu}) \rangle = \langle \tilde{\mu}, *_i \omega(\tilde{\mu}, \dots, \tilde{\mu}) \rangle + \langle \tilde{\mu}, *_i \bar{\omega}(\tilde{\mu}, \dots, \tilde{\mu}) \rangle$$

Consider now that we can split the set of edges of S contained in \tilde{X} by grouping together edges depending on what are the indices corresponding to elements of Y and which to element of \bar{Y} .

$$\langle \tilde{\mu}, *_i \omega(\tilde{\mu}, \dots, \tilde{\mu}) \rangle = \sum_{\mathbf{f} \in \{f, \bar{f}\}^k} \langle \mathbf{f}_i, *_i \omega(\mathbf{f}_1, \dots, \mathbf{f}_k) \rangle$$

Because \tilde{Y} is a subset of \tilde{X} , we can ask for an $\frac{\varepsilon}{2^{k+2}}$ -good dense model $Z_{\tilde{Y}}$ of \tilde{Y} . Let g be the scaled indicator function of its model $Z_{\tilde{Y}}$ and define \bar{g} as $\mathbf{1} - g$. Because $Z_{\tilde{Y}}$ is a good model of \tilde{Y} we have $\|f - g\|_{\Phi(\mathbf{1})} \leq \frac{\varepsilon}{2^{k+2}}$. We therefore have:

$$\sum_{\mathbf{f} \in \{f, \bar{f}\}^k} \langle \mathbf{f}_i, *_i \omega(\mathbf{f}_1, \dots, \mathbf{f}_k) \rangle = \sum_{\mathbf{g} \in \{g, \bar{g}\}^k} \langle \mathbf{g}_i, *_i \omega(\mathbf{g}_1, \dots, \mathbf{g}_k) \rangle \pm \frac{\varepsilon}{4}.$$

We can substitute this to obtain the following:

$$\begin{aligned} \langle \tilde{\mu}, *_i \mathbf{1}(\tilde{\mu}, \dots, \tilde{\mu}) \rangle &\geq \langle \tilde{\mu}, *_i \bar{\omega}(\tilde{\mu}, \dots, \tilde{\mu}) \rangle + \langle f, *_i \omega(f, \dots, f) \rangle - \langle g, *_i \omega(g, \dots, g) \rangle \\ &\quad + \sum_{\mathbf{g} \in \{g, \bar{g}\}^k} \langle \mathbf{g}_i, *_i \omega(\mathbf{g}_1, \dots, \mathbf{g}_k) \rangle - \frac{\varepsilon}{4}. \end{aligned}$$

Considering now that $g + \bar{g} = \mathbf{1}$, and that, by Theorem 20 we have $\|\tilde{\mu} - \mathbf{1}\|_{\Phi(\tilde{\mu}, \mathbf{1})} < \frac{\varepsilon}{2^{k+2}}$, we obtain:

$$\begin{aligned} \langle \tilde{\mu}, *_i \mathbf{1}(\tilde{\mu}, \dots, \tilde{\mu}) \rangle &\geq \langle \tilde{\mu}, *_i \bar{\omega}(\tilde{\mu}, \dots, \tilde{\mu}) \rangle + \langle f, *_i \omega(f, \dots, f) \rangle - \langle g, *_i \omega(g, \dots, g) \rangle \\ &\quad + \langle \tilde{\mu}, *_i \omega(\tilde{\mu}, \dots, \tilde{\mu}) \rangle - \frac{\varepsilon}{2}. \end{aligned}$$

By cancelling out the terms (which we can do as $\mathbf{1} = \omega + \bar{\omega}$), we obtain:

$$\langle f, *_i \omega(f, \dots, f) \rangle \leq \langle g, *_i \omega(g, \dots, g) \rangle + \frac{\varepsilon}{2}.$$

Returning to the definition of inner product (i.e. noticing that by definition we have that $\langle f, *_i \omega(f, \dots, f) \rangle = \frac{p^k}{e(S)} \sum_{s \in S} \omega(s) \mathbb{1}(s \subseteq Y)$ and similarly for $\tilde{\mu}$ and \tilde{X} , f and \tilde{Y} , and X and μ) we conclude. \square

12. A SPARSE COUNTING LEMMA

In this section we prove Theorem 33.⁹ This turns out to be a fairly easy application of Theorem 3; most of what follows is simply dealing with the somewhat complicated hypergraph regularity setup and handling constant choices.

Let k be a positive integer. A k -complex is a down-closed hypergraph in which all edges have size at most k . Given a k -complex H with at least $k+1$ vertices, we define its k -density as $d_k(H) := \frac{e_k(H)-1}{v(H)-k}$, where $e_k(H)$ is the number of edges of size k in H , and $v(H)$ denotes the number of vertices of H . We also define $m_k(H) := \max_{H' \subseteq H} d_k(H')$, where the maximum is taken over all sub- k -complexes H' of H with at least $k+1$ vertices.

Given a vertex set $[N]$, a k -partition with ℓ clusters \mathcal{V} consists of a family of disjoint subsets $V_{\{1\}}, \dots, V_{\{\ell\}} \subseteq [N]$ called *clusters*, together with, for each integer $2 \leq i \leq k$ and each subset $E \subseteq [\ell]$ of size i , a collection V_E of subsets of $[N]$ of size i , called *i -edges*. These V_E must satisfy the following compatibility condition: for every $e \in V_E$ and every $j \in E$, the set e intersects the cluster $V_{\{j\}}$ in exactly one element, and the remaining $i-1$ elements of e form an $(i-1)$ -edge in $V_{E \setminus \{j\}}$. The *supporting $(i-1)$ -graphs* of V_E is the $(i-1)$ -uniform hypergraph consisting of all $(i-1)$ -sets that arise in this way from some edge of V_E .

Let $E \subseteq [\ell]$ be given, with $|E| = i \geq 2$, and suppose V_E is given along with its supporting $(i-1)$ -graphs W_1, \dots, W_i . For any subsets $Q_1 \subseteq W_1, \dots, Q_i \subseteq W_i$, define $R(Q_1, \dots, Q_i)$ to be the collection of i -element subsets of $[N]$ that contain one element from each Q_j . In particular, $R(W_1, \dots, W_i)$ contains V_E . If $R(W_1, \dots, W_i)$ is nonempty, and given $p \in (0, 1]$, we define respectively the *relative density* of V_E and the *relative p -density* of V_E as follows:

$$d^*(V_E) := \frac{|V_E|}{|R(W_1, \dots, W_i)|} \quad \text{and} \quad d_p^*(V_E) := \frac{|V_E|}{p \cdot |R(W_1, \dots, W_i)|}.$$

Finally, for singleton sets, we define $d^*(V_{\{i\}}) := |V_{\{i\}}|N^{-1}$.

Let $E \subseteq [\ell]$ be a set of size i , with $2 \leq i \leq k$, and let $p \in (0, 1]$. Consider V_E and let W_1, \dots, W_i denote the supporting $(i-1)$ -graphs of V_E . We say that V_E is (ε, r, p) -regular with respect to its supporting $(i-1)$ -graphs if the following holds. For any set R^* of the form $R^* = \bigcup_{j=1}^r R(Q_1^{(j)}, \dots, Q_i^{(j)})$ where $Q_i^{(j)} \subseteq W_i$, we have that if $|R^*| \geq \varepsilon|R(W_1, \dots, W_i)|$, then

$$\frac{|V_E \cap R^*|}{p|R^*|} = d_p^*(V_E) \pm \varepsilon.$$

If either of the parameters r , p , or both are omitted, they are understood to be equal to 1.

So far we gave the standard setup for defining the so-called strong regularity. At this point we generalise slightly and consider s -regularity, for some $1 \leq s \leq k-1$. Here $s = k-1$ will be the usual strong regularity, while $s = 1$ is normally called weak regularity (and in this context it is normal to take $r = 1$). Given a k -partition \mathcal{V} with ℓ clusters, and a k -set $E \subseteq [\ell]$, we say the *supporting s -graphs* of V_E are the collection of s -graphs $V_{E'}$ for $E' \subseteq E$ with $|E'| = s$. For each $E' \subseteq E$ with $|E'| = s$, choose $Q_{E'} \subseteq V_{E'}$: then $R(\{Q_{E'} : E' \subseteq E, |E'| = s\})$ is the collection of k -element subsets e of $[N]$ with one member in each cluster of V_E such that the intersection of e with the clusters of $V_{E'}$ is in $Q_{E'}$ for each $E' \subseteq E$ with $|E'| = s$. We define

⁹PA: Generalised regularity here.

the relative $(p; s)$ -density of V_E to be

$$d_{p;s}^*(V_E) := \frac{|V_E|}{|R(\{V_{E'} : E' \subseteq E, |E'| = s\})|},$$

and we say V_E is $(\varepsilon, r, p; s)$ -regular with respect to $\{V_{E'} : E' \subseteq E, |E'| = s\}$ if the following holds. For any set R^* of the form $R^* = \bigcup_{j=1}^r R(Q_{E'}^j : E' \subseteq E, |E'| = s)$, we have that if $|R^*| \geq \varepsilon |R(\{V_{E'} : E' \subseteq E, |E'| = s\})|$ then

$$\frac{|V_E \cap R^*|}{p|R^*|} = d_{p;s}^*(V_E) \pm \varepsilon.$$

Observe that for $s = k - 1$, this is simply repeating the definition of (ε, r, p) -regularity, while for $s = r = 1$ we obtain weak regularity: we are counting edges supported by given vertex subsets of the clusters.

A k -partition is said to be $(\varepsilon_k, \varepsilon, d_1, \dots, d_s, d_k, r, p; s)$ -regular if the following conditions hold:

- For each $i \in [\ell]$, we have $|V_{\{i\}}| = d_1 N$;
- For every $E \subseteq [\ell]$ with $2 \leq |E| \leq s$, the set V_E is ε -regular with respect to its supporting $(|E| - 1)$ -graphs, and its relative density satisfies $d^*(V_E) = d_{|E|} \pm \varepsilon$;
- For every $E \subseteq [\ell]$ with $s + 1 \leq |E| \leq k - 1$, we have $V_E = R(W_1, \dots, W_{|E|})$, where $W_1, \dots, W_{|E|}$ are the supporting $|E| - 1$ -graphs.
- For every $E \subseteq [\ell]$ with $|E| = k$, the set V_E is $(\varepsilon_k, r, p; s)$ -regular with respect to its supporting s -graphs, and its relative $(p; s)$ -density satisfies $d_{p;s}^*(V_E) \geq d_k$.

Note that the third item enforces $d^*(V_E) = 1$ for any E with $s + 1 \leq |E| \leq k - 1$.

Let H be a k -complex. An injective map $\phi : V(H) \rightarrow [\ell]$ is called a *k -complex homomorphism* if for every edge $e \in E(H)$, the image $\phi(e)$ has size $|e|$. That is, ϕ maps the vertices of each edge to distinct cluster indices. Given a k -partition \mathcal{V} with ℓ clusters over the vertex set $[N]$, a map $\psi : V(H) \rightarrow [N]$ is said to be a *ϕ -partite copy of H in \mathcal{V}* if ψ is injective and for every edge $e \in E(H)$, the image $\psi(e)$ is an element of $V_{\phi(e)}$.

We are finally ready to introduce the hypergraph counting statements. We begin with a dense counting assumption, that H -counting in the dense case follows from s -regularity. We will transfer this statement to the sparse setting: if (H, s) satisfies the dense assumption, there is a corresponding sparse version.

Assumption 32 (s -regularity implies H -counting). *Given $k \geq 2$ and $1 \leq s \leq k - 1$, a fixed k -complex H , and $\delta, d_k > 0$, there exists $\varepsilon_k > 0$ such that for any $d_2, \dots, d_s > 0$ with $d_2^{-1}, \dots, d_s^{-1} \in \mathbb{N}$ there exist $\varepsilon > 0$ and $r \in \mathbb{N}$ such that for any $d_1 > 0$ the following holds for all sufficiently large N .*

Given any $k \leq \ell \leq v(H)$ and $(\varepsilon_k, \varepsilon, d_1, \dots, d_s, d_k, r; s)$ -regular k -partition \mathcal{V} with ℓ clusters on $[N]$, and given any k -complex homomorphism $\phi : v(H) \rightarrow [\ell]$, the number of ϕ -partite copies of H in \mathcal{V} is

$$(1 \pm \delta)N^{v(H)} \prod_{e \in E(H)} d^*(V_{\phi(e)}).$$

Assumption 32, for a given pair (H, S) where H is a k -complex which is not the down-closure of a k -uniform hypergraph, follows from Assumption 32 for (H', S) where H' is obtained by, for each maximal edge of H of size smaller than k , adding new vertices to increase the edge to size k and then taking the down-closure of the resulting hypergraph. To see that this is

true, given a required instance of (H, s) -counting on $[N]$, we construct an instance of (H', s) -counting on $[v(H')N]$ by adding one new cluster of size N (disjoint from all others) for each vertex $V(H') \setminus V(H)$, and for each V_E with $E \in E(H') \setminus E(H)$ of size between 2 and k (in order of increasing $|E|$), letting V_E consist of all $|E|$ -sets supported by the collection $V_{E'}$ for $E' \subseteq E$ of size $|E| - 1$. It is straightforward to check that the constructed k -partition is $(\varepsilon, k, \varepsilon, d_1, \dots, d_s, d_k, r; s)$ -regular if and only if the original k -partition is. Letting ϕ' be the obvious extension of ϕ mapping the new vertices to their clusters, there is an obvious $N^{v(H')-v(H)}$ -to-1 map between ϕ' -partite copies of H' in the constructed instance and ϕ -partite copies of H in the original instance, so that if Assumption 32 holds for (H', s) then it holds for (H, s) . Note that H' is the down-closure of a k -uniform hypergraph.

By definition of a k -partition, if H is the down-closure of a k -uniform hypergraph, a given map $\psi : V(H') \rightarrow [N]$ is a ϕ -partite copy of H in \mathcal{V} if and only if it is ϕ -partite and maps the k -edges of H' to edges of $\bigcup_E V_E$, where E runs over the edges of size k . Putting these two observations together, to prove Assumption 32 for a given pair (H, s) , it is enough to know the corresponding counting lemma for (H'', s) where H'' is the k -uniform hypergraph obtained as above from H . This is the setting which has been considered in the literature.

A theorem of Kohayakawa, Nagle, Rödl and Schacht [14, Lemma 10] tells us that 1-regularity implies H -counting for any *linear* H : that is, for any two maximal edges of H , the intersection of the two edges contains at most one vertex. Observe that the transformation described above preserves linearity: formally, they stated it specifically for counting k -uniform hypergraphs, but this implies that Assumption 32 holds for $(H, 1)$ whenever H is a linear k -complex. In fact, they show that in this setting we can even take $r = 1$.

For $s = k - 1$, Assumption 32 was proved by Cooley, Kühn, Fountoulakis and Osthus [7, Lemma 4]. Formally, they proved the special case where d_k^{-1} is an integer and $d^*(V_E) = d_k \pm \varepsilon_k$ holds for all $|E| = k$. However, a standard reduction gives the more general statement; we sketch the reduction in Appendix A.

Theorem 33 (Counting lemma for sparse hypergraphs). *Given $k \geq 2$ and $1 \leq s \leq k - 1$, a fixed k -complex H , and $\delta, d_k > 0$, there exists $\varepsilon_k > 0$ such that for any $d_2, \dots, d_s > 0$ with $d_2^{-1}, \dots, d_s^{-1} \in \mathbb{N}$, there exist $\varepsilon > 0$ and $r \in \mathbb{N}$ such that for any $d_1 > 0$ there exists C^* with the following property. Suppose Assumption 32 holds for H and s . Suppose that N is sufficiently large, and $p \geq \max(C^* N^{s-k}, C^* N^{-1/m_k(H)})$. With high probability, the random k -uniform hypergraph $\Gamma = G^{(k)}(N, p)$ has the following property.*

Given any $k \leq \ell \leq v(H)$ and $(\varepsilon_k, \varepsilon, d_1, \dots, d_s, d_k, r, p; s)$ -regular k -partition \mathcal{V} with ℓ clusters on $[N]$, such that for each $E \subseteq [\ell]$ with $|E| = k$ we have $V_E \subseteq \Gamma$, and given any k -complex homomorphism $\phi : v(H) \rightarrow [\ell]$, the number of ϕ -partite copies of H in \mathcal{V} is

$$(1 \pm \delta)N^{v(H)} \prod_{e \in E(H)} d^*(V_{\phi(e)}).$$

Assumption 32 is not true for an arbitrary pair (H, s) . A standard example shows that we need the condition that any two maximal edges of H intersect in at most s vertices, and this could well be sufficient too (as it is for $s = 1$). If this is the case, then Theorem 33 gives an optimal sparse counting lemma for any H , with s -regularity sufficing whenever two maximal edges of H intersect in at most s vertices.

In the above theorem, we do allow for the possibility that some edges of H of uniformity smaller than k are not contained in any k -edges of H ; that is, H need not be just the down-closure of a k -uniform hypergraph. This turns out to be required in some applications for $k \geq 3$; for $k = 2$ this extra generality is not interesting.

Proof of Theorem 33. The case $e_k(H) = 0$ of Theorem 33 is simply Assumption 32, viewing H as a $(k - 1)$ -complex.

The case $e_k(H) = 1$ is standard and does not require Theorem 3. We give only a sketch. Let H' be the $(k - 1)$ -complex H with the one k -edge removed. An application of the Extension Lemma [7, Lemma 5] shows that all but a tiny fraction of k -sets supported by any given $(k - 1)$ -partition \mathcal{V}' are in roughly the same number of ϕ -partite copies of H' , and that the exceptional k -sets account for only a tiny fraction of all ϕ -partite copies of H' . A standard application of Chernoff's inequality shows that with very high probability, when $G_N^{(k)}$ is revealed, there are very few edges on these exceptional k -sets and the number of ϕ -partite H -copies they generate is tiny compared to those on typical k -sets. Critically, this 'very high probability' is sufficient for a union bound over choices of \mathcal{V}' and ϕ . Supposing now this likely event occurs, given any regular \mathcal{V} , letting \mathcal{V}' be the $(k - 1)$ -partition obtained by removing the k layer, we see that (using the fact that ε_k is much smaller than d_k) most of the k -edges of \mathcal{V} are on typical k -sets and a short calculation gives the desired count of ϕ -partite H -copies.

We now assume $e_k(H) \geq 2$, which is the interesting case. Let $2\varepsilon_k > 0$ be small enough for Assumption 32 with input $\frac{1}{2}\delta$. Given $d_2, \dots, d_s, d_k > 0$, let ε and r be given by Assumption 32 for input $d_2, \dots, d_s, \frac{1}{2}d_k$. Let finally $d_1 > 0$ be given.

We set $c = 2v(H)!$, and apply Theorem 3 with input uniformity $e_k(H)^{10}$, c and error parameter

$$\varepsilon^* = \frac{\delta d_k \varepsilon_k^2}{10v(H)!} \prod_{e \in E(H)} d_{|e|}.$$

Let C be the constant returned by Theorem 3. Order arbitrarily the k -edges of H . Let $n = \binom{N}{k}$ enumerate the edges of $K_N^{(k)}$, and let S consist of the ordered subsets of $[n]$ corresponding to $e_k(H)$ -sets in $[N]$ which form isomorphic copies of the k -uniform edges of H , in the chosen order.

Let $C^* = 10r2^kCk!$. We now verify the maximum degree condition on S holds for $p \geq C^*N^{-1/m_k(H)}$. To begin with, we estimate $e(S)$. Let $q(H)$ be the number of vertices of H which are not in any k -uniform edge of H . There are $(1 + o(1))N^{v(H)-q(H)}$ injective maps from the vertices of H which are in k -edges to $[N]$, each of which gives one element of S , so $e(S) = (1 + o(1))N^{v(H)-q(H)}$.

Given $1 \leq \ell \leq e_k(H)$, let \mathbf{x} be a sequence of length $e_k(H)$ from $[n] \cup \{*\}$ with exactly ℓ entries not equal to *. For $\ell = 1$, by symmetry we have $\deg_S(\mathbf{x}) = \frac{e(S)}{n}$, which is as required. We now assume $\ell \geq 2$. Let $W \subseteq [N]$ be the vertices of $K_N^{(k)}$ which are contained in some edge in \mathbf{x} . By definition, if \mathbf{x} has two identical non-* entries, then $\deg_S(\mathbf{x}) = 0$, so we can assume that \mathbf{x} has at least two distinct non-* entries, and hence $|W| \geq k + 1$. By definition of $m_k(H)$, we have

$$\frac{\ell-1}{|W|-k} \leq m_k(H), \quad \text{so} \quad |W| \geq \frac{\ell-1}{m_k(H)} + k.$$

To obtain a member of S which agrees with \mathbf{x} at the non-* coordinates, we can at most pick a further $v(H) - |W| - q(H)$ vertices in k -edges of H and one of the at most $v(H)!$ maps from

¹⁰This is bad notation, but k is only used in this proof as in the statement of Theorem 33, we have it here $k = e_k(H)$ because k also has a meaning in Theorem 3.

the vertices of H in k -edges to the picked vertices together with W . Thus, we have

$$\begin{aligned} \deg_S(\mathbf{x}) &\leq N^{v(H)-|W|-q(H)} v(H)! \\ &\leq 2v(H)!e(S)N^{-|W|} \\ &\leq 2v(H)!e(S)N^{-\frac{\ell-1}{m_k(H)}-k} \\ &= 2v(H)!\frac{e(S)}{N^k}(N^{-1/m_k(H)})^{\ell-1} \\ &\leq 2v(H)!\frac{e(S)}{n}(p/C^*)^{\ell-1}, \end{aligned}$$

which is the required bound.

We next give the set Σ of similarity functions. These are the ‘obvious’ functions which allow us to transfer s -regularity: that is, for any collection of $\binom{k}{s}$ hypergraphs $Q_1, \dots, Q_{\binom{k}{s}}$ on $[N]$ of uniformity s , we consider the set $R(Q_1, \dots, Q_{\binom{k}{s}})$ of k -sets in $[N]$ which contain exactly one edge of each Q_i . We let Σ consist of the indicator functions of the unions of any up to r sets of the form $R(Q_1, \dots, Q_{\binom{k}{s}})$. Then we have

$$|\Sigma| \leq 2 \cdot 2^{r \binom{k}{s} N^s} \leq \exp\left(\frac{pn}{C}\right),$$

where we use the simple bound that the number of s -uniform hypergraphs on $[N]$ is at most 2^{N^s} , and the inequality uses $p \geq C^* N^{s-k}$ and the choice of C^* .

Finally, we give the set Ω of subcounts. For each $k \leq \ell \leq v(H)$, consider each choice of a k -partition \mathcal{V} with ℓ clusters whose $s+1, \dots, k$ levels are complete (i.e. each V_E with $s+1 \leq |E| \leq k$ is equal to $R(W_1, \dots, W_{|E|})$ where $W_1, \dots, W_{|E|}$ are the supporting $(|E|-1)$ -graphs), and each $\phi : v(H) \rightarrow [\ell]$. For each such $(\ell, \mathcal{V}, \phi)$ we construct a subcount ω as follows. For each member \mathbf{s} of S (recall \mathbf{s} is an ordered tuple of $e_k(H) \geq 2$ members of $[n] = \binom{[N]}{k}$), we count the number $w(\mathbf{s})$ of ϕ -partite copies ψ of H in \mathcal{V} such that the i -th edge of H is mapped to the i -th member of \mathbf{s} , for each $1 \leq i \leq e_k(H)$. Observe that necessarily $0 \leq w(\mathbf{s}) \leq (v(H))!N^{q(H)}$, where $q(H)$ is the number of vertices of H not in any k -edge of H . We define $\omega(s) = \frac{1}{(v(H))!N^{q(H)}}w(s)$, which is therefore in $[0, 1]$. We say this is the subcount corresponding to $(\ell, \mathcal{V}'', \phi)$ for any choice \mathcal{V}'' of a k -partition which is identical to \mathcal{V} on any level except perhaps the k -th. We now upper bound the size $|\Omega|$ of the set of all such subcounts. There are $v(H)$ choices of ℓ , and at most $v(H)^\ell \leq v(H)^{v(H)}$ choices of ϕ . What remains is to estimate the number of choices of \mathcal{V} . Observe that \mathcal{V} is defined by the choices of V_E for $1 \leq |E| \leq s$. There are at most $N^{v(H)+1}$ ways to choose the clusters, since the clusters are disjoint. Again since the clusters are disjoint, to define V_E for each $2 \leq |E| \leq s$ it suffices to choose a subset of each of $\binom{[N]}{2}$ through $\binom{[N]}{s}$, which can be done in at most $2^{N^2} \cdots 2^{N^s}$ ways. We conclude

$$|\Omega| \leq v(H)^{v(H)+1} N^{v(H)+1} 2^{sN^s} \leq \exp\left(\frac{pn}{C}\right),$$

where as before the inequality uses $p \geq C^* N^{s-k}$ and the choice of C^* , and this time also that N is sufficiently large. This verifies the C -conditions hold for $(\varepsilon^*, S, \Sigma, \Omega)$.

Suppose now that $X = [n]_p$ satisfies the likely event of Theorem 3 for this ε^* , S , Σ and Ω . Let Γ be the corresponding instance of $G^{(k)}(N, p)$.

Given $k \leq \ell \leq v(H)$ and an $(\varepsilon_k, \varepsilon, d_1, \dots, d_s, d_k, r, p; s)$ -regular k -partition \mathcal{V} with ℓ clusters on N , such that for each V_E with $|E| = k$ we have $V_E \subseteq \Gamma$, let Y be the subset of X consisting of elements in any V_E with $|E| = k$. Let Z be the dense model guaranteed by the likely event of Theorem 3, and let \mathcal{V}' be the k -partition with ℓ clusters on N obtained by replacing each

V_E where $|E| = k$ with V'_E corresponding to the elements of Z that are supported on the $(k-1)$ -graphs supporting V_E .

We claim that \mathcal{V}' is $(2\varepsilon_k, \varepsilon, d_1, \dots, d_s, d_{k-1}, \frac{1}{2}d_k, r, 1; s)$ -regular and that the relative densities of the top level are close to the relative p -densities of \mathcal{V} . The regularity of the levels from 1 to $k-1$ follows from the regularity of \mathcal{V} , and what needs to be proved is that each V'_E with $|E| = k$ is $(2\varepsilon_k, r, 1; s)$ -regular with density $d^*(V'_E) = (1 \pm \frac{\delta}{10e_k(H)})d_p^*(V_E) \geq \frac{1}{2}d_k$.

To see this holds, fix E and let $W_1, \dots, W_{\binom{k}{s}}$ be the supporting (s) -graphs of V_E (so also of V'_E). Let R^* be a union of at most r sets of the form $R(Q_1, \dots, Q_{\binom{k}{s}})$ (as defined where we described the set Σ of similarity functions), with the extra condition $Q_i \subseteq W_i$ for each $1 \leq i \leq \binom{k}{s}$. Abusing notation slightly, we think of R^* as both a subset of $\binom{[N]}{k}$ and of $[n]$. Because Z is a dense model of Y , using the similarity function $\sigma \in \Sigma$ which takes the value 1 precisely on R^* , we have

$$p^{-1}|R^* \cap Y| = |R^* \cap Z| \pm \varepsilon^* n.$$

Taking the particular case that R^* is all k -sets supported by $W_1, \dots, W_{\binom{k}{s}}$, this immediately says that

$$d^*(V'_E) = d_p^*(V_E) \pm \varepsilon^* n N^{-k} \prod_{i=1}^{k-1} d_i^{-\binom{k}{i}} = (1 \pm \frac{\delta}{10e_k(H)})d_p^*(V_E) \geq \frac{d_k}{2},$$

where the final equality is by choice of ε^* . Suppose now $|R^*|$ contains at least an ε_k -fraction of all k -edges supported by W_1, \dots, W_k . Because \mathcal{V} is regular, we have

$$|R^* \cap Y| = |R^* \cap V_E| = (1 \pm \varepsilon_k)d^*(V_E)|R^*| = (1 \pm \varepsilon_k)p d_p^*(V_E)|R^*|.$$

Putting these bits together, we have

$$\begin{aligned} |R^* \cap Z| &= (1 \pm \varepsilon_k)d_p^*(V_E)|R^*| \pm \varepsilon^* n \\ &= (1 \pm \varepsilon_k)(1 \pm \frac{\delta}{10e_k(H)})d^*(V'_E)|R^*| \pm \varepsilon^* n \\ &= (1 \pm 2\varepsilon_k)d^*(V'_E) \end{aligned}$$

which verifies $(2\varepsilon_k, r, 1; s)$ -regularity of V'_E . Here again the final inequality is by choice of ε^* .

Applying Assumption 32 to \mathcal{V}' , we see that the number of ϕ -partite copies of H in \mathcal{V}' is

$$(1 \pm \frac{1}{2}\delta)N^{v(H)} \prod_{e \in E(H)} d^*(V'_{\phi(e)}).$$

Letting ω be the subcount corresponding to $(\ell, \mathcal{V}, \phi)$, we have by definition of ω

$$\begin{aligned} \sum_{\mathbf{s} \in S} \mathbb{1}(\mathbf{s} \subseteq Z) \omega(\mathbf{s})(v(H))! N^{q(H)} &= (1 \pm \frac{1}{2}\delta)N^{v(H)} \prod_{e \in E(H)} d^*(V'_{\phi(e)}) \\ &= (1 \pm \frac{3}{4}\delta)N^{v(H)} p^{-e(H)} \prod_{e \in E(H)} d^*(V_{\phi(e)}). \end{aligned}$$

Here the final equality uses that $d_p^*(V_E) = p^{-1}d(V_E) = (1 \pm \frac{\delta}{8e_k(H)})d^*(V'_E)$ whenever $|E| = k$.

Since Z is a dense model of Y , we have

$$p^{-e_k(H)} \sum_{\mathbf{s} \in S} \mathbb{1}(\mathbf{s} \subseteq Y) \omega(\mathbf{s}) = \sum_{\mathbf{s} \in S} \mathbb{1}(\mathbf{s} \subseteq Z) \omega(\mathbf{s}) \pm \varepsilon^* e(S).$$

We therefore get

$$\begin{aligned} & (1 \pm \frac{3}{4}\delta)N^{v(H)}p^{-e_k(H)} \prod_{e \in E(H)} d^*(V_{\phi(e)}) \\ &= p^{-e_k(H)} \sum_{\mathbf{s} \in S} \mathbb{1}(\mathbf{s} \subseteq Y) \omega(\mathbf{s})(v(H))! N^{q(H)} \pm \varepsilon^* e(S)(v(H))! N^{q(H)} \\ &= p^{-e_k(H)} \sum_{\mathbf{s} \in S} \mathbb{1}(\mathbf{s} \subseteq Y) \omega(\mathbf{s})(v(H))! N^{q(H)} \pm \varepsilon^* N^{v(H)}(v(H))!, \end{aligned}$$

and so

$$\begin{aligned} \sum_{\mathbf{s} \in S} \mathbb{1}(\mathbf{s} \subseteq Y) \omega(\mathbf{s})(v(H))! N^{q(H)} &= (1 \pm \frac{3}{4}\delta)N^{v(H)} \prod_{e \in E(H)} d^*(V_{\phi(e)}) \\ &\quad \pm \varepsilon^*(v(H))! N^{v(H)} p^{e_k(H)} \\ &= (1 \pm \delta)N^{v(H)} \prod_{e \in E(H)} d^*(V_{\phi(e)}) \end{aligned}$$

by choice of ε^* . Since the left-hand side of this is, by definition of ω , the number of ϕ -partite copies of H in \mathcal{V} , this completes the proof. \square

13. A CANONICAL VAN DER WAERDEN THEOREM

In this section we show how to give a quick proof of the following recent result of Alvarado, Kohayakawa, Morris, Mota and Ortega [1].

Theorem 34 (Theorem 1, [1]). *Let $k \geq 3$ be an integer. There exists $C^* > 0$ such that the following holds with high probability for any $p \geq C^*n^{-1/(k-1)}$ and $X = [n]_p$. For any colouring of X , there exists a k -term arithmetic progression in X whose terms are either all of the same colour (monochromatic), or contain no colour more than once (rainbow).*

As they observe, it is a standard fact that the probability bound is optimal up to the value of C^* .

Our proof (much as theirs) uses the following idea. If there is a single colour which appears $\Omega(pn)$ times in the colouring, then the dense model of this colour will contain many monochromatic k -term progressions and we are done. If not, we will find a rainbow k -term progression. In the latter case, we can safely group colours together and assume that the total number of colours is a large constant.

Proof. Given k , let $\varrho = \frac{1}{16}k^{-4}$. Let $r = \lceil 2\varrho^{-1} + 1 \rceil$, and let $\delta > 0$ be returned by the supersaturation Szemerédi theorem [19, 20] (see [2, Lemma 4.2] for a statement in this form) for input k and $\frac{1}{2}\varrho$; that is, any subset of $[n]$ of size at least $\frac{1}{2}\varrho n$ contains at least δn^2 k -term progressions. Let $\varepsilon = \min(\frac{1}{2}\delta, \frac{1}{8}k^{-2}r^{-k})$.

Let $c = k^2$. Let C be returned by Theorem 3, and let $C^* = C$. Given n , let S be the ordered k -uniform hypergraph on $[n]$ whose elements are the k -term progressions (in order) on $[n]$; it is easy to see we have $e(S) \geq nk^{-2}$. We take Σ and Ω with the minimal requirement of Setting 2, in particular both contain their respective 1. Finally, given $p \geq C^*n^{-1/(k-1)}$, we now argue that the C -conditions are satisfied. For Σ and Ω this is trivial. For S , consider $1 \leq \ell \leq k$ and a k -tuple \mathbf{x} with exactly ℓ entries not equal to *. If $\ell \geq 2$, there is at most one k -term progression in $[n]$ passing through the given integers in the given positions, so $\Delta_\ell(S) \leq 1 \leq cC^{1-\ell}p^{\ell-1}\frac{e(S)}{n}$ holds easily: it is enough to check $cC^{1-k}p^{k-1}e(S) \geq n$, which

follows by choice of C^* . For $\ell = 1$, there are at most n choices for a given $*$ entry of \mathbf{x} and this then determines the element of S uniquely, so $\Delta_1(S) \leq n \leq ce(S)/n$ by choice of c .

Suppose that the likely event of Theorem 3 holds for $X = [n]_p$. Since $\mathbf{1} \in \Sigma$ and $\varepsilon < 1$, this implies that $|X| \leq 2pn$. Let $\chi' : X \rightarrow \mathbb{N}$ be any colouring of X .

If there is a colour class Y of χ' of size at least ϱpn , then consider the dense model $Z \subseteq [n]$ of Y as guaranteed by Theorem 3. Since $\mathbf{1} \in \Sigma$, we have $|Z| \geq \frac{1}{2}\varrho n$. By the supersaturation Szemerédi theorem, there are at least δn^2 k -term progressions in Z , and by choice of ε there are therefore at least $\frac{1}{2}\delta p^k n^2 > 0$ k -term progressions in Y ; in particular there is a monochromatic k -term progression in χ' .

Suppose now no colour class of χ' has size at least ϱpn . We iteratively identify pairs of colour classes each of size less than ϱpn until no such pairs remain. In the resulting colouring, all but at most one colour class has size at least ϱpn and so the number of colour classes is at most $\frac{2pn}{\varrho pn} + 1 = 2\varrho^{-1} + 1 \leq r$. Relabelling the colours, this gives a colouring $\chi : X \rightarrow [r]$. By construction, the colour classes Y_1, \dots, Y_r of χ have size at most $2\varrho pn$. Importantly, observe that if a given k -term progression is rainbow according to χ , it is also rainbow according to χ' .

Let Z_1, \dots, Z_r be the dense model of Y_1, \dots, Y_r as guaranteed by Theorem 3. Since $\mathbf{1} \in \Sigma$, we have $|Z_i| \leq 4\varrho n$ for each $i \in [r]$; let $\chi^* : [n] \rightarrow [r]$ be the corresponding colouring of $[n]$.

Consider the following random experiment: we choose a and d independently and uniformly at random in $[\lfloor n/k \rfloor]$. We obtain the (random) k -term progression $(a, a+d, \dots, a+(k-1)d)$ contained in $[n]$. Consider two elements $a+id, a+jd$ of this progression, with $0 \leq i < j \leq k-1$. These two elements form a pair in $\binom{[n]}{2}$, chosen uniformly at random from a set of size $[\lfloor n/k \rfloor]$. The total number of χ^* -monochromatic pairs in $\binom{[n]}{2}$ is at most $n \cdot 4\varrho n$: having chosen a first element of the pair in $[n]$, this element will be in some Z_q and there are at most $|Z_q|$ choices of a second element which creates a monochromatic pair. Thus the probability that $(a+id, a+jd)$ is monochromatic is at most $\frac{4\varrho n^2}{n^2/(2k^2)} \leq 8k^2\varrho$.

Taking a union bound over choices of i and j , the probability that the random progression contains a monochromatic pair is at most $8k^4\varrho = \frac{1}{2}$. Thus χ^* contains at least $\frac{1}{2}\lfloor \frac{n}{k} \rfloor^2 \geq \frac{1}{4}n^2k^{-2}$ rainbow k -term progressions. Fixing a colour pattern from the less than r^k total colour patterns which occurs most often as a rainbow k -term progression, there is a specific rainbow colour pattern such that the number of k -term progressions with the given colour pattern occurs at least $\frac{1}{4}n^2k^{-2}r^{-k}$ times in χ^* . Because χ^* is a dense model of χ , by choice of ε the same colour pattern occurs as a k -term progression at least $\frac{1}{8}p^k n^2 k^{-2} r^{-k} > 0$ times in χ . In particular, there is a rainbow k -term progression in χ and so in χ' , as required. \square

Our proof above gives $o(1)$ bounds on the failure of the canonical van der Waerden property. However it is trivial to enhance this to the optimal $e^{-O(pn)}$ bounds by replacing X with the ε -deletion \tilde{X} , matching [1]; obviously if every colouring of the latter set contains either a monochromatic or rainbow k -term progression, the same is true for the former set. We also note that our argument above showing χ^* contains many rainbow k -term progressions is essentially the same as [1, Lemma 3.5].

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APPENDIX A. DENSE COUNTING

Recall that the following special case of Assumption 32 with $s = k - 1$ is [7, Lemma 4]. We have $d_k^{-1} \in \mathbb{N}$ and $d^*(V_E) = d_k \pm \varepsilon_k$ holds for all $|E| = k$.

We now explain why this proves Assumption 32 for $s = k - 1$ with no further restriction. Given any $\delta, d_k > 0$, we set $\delta' = \frac{1}{4}\delta$ and $d'_k = 1/(2\lceil d_k^{-1} \rceil)$, so that $(d'_k)^{-1} \in \mathbb{N}$. Let ε'_k be returned by [7, Lemma 4] for input k, δ', d'_k . Let $\varepsilon_k \leq \frac{1}{4}\varepsilon'_k$ be such that $d'_k(d_k - \varepsilon_k)^{-1} < 1$.

For given d_2, \dots, d_{k-1} we let ε, r be returned by [7, Lemma 4] for input $k, \delta', d'_k, d_2, \dots, d_{k-1}$.

Given \mathcal{V} as in Assumption 32, we construct \mathcal{V}' as follows: for each E of size k and V_E in \mathcal{V} , we obtain V'_E by selecting edges of V_E independently with probability $d'_k(d^*(V_E))^{-1}$; by our assumption on ε_k , this number is indeed in $[0, 1]$. For E of size smaller than k , we set $V'_E = V_E$.

We claim that with probability $1 - o(1)$, the k -partition \mathcal{V}' is $(\varepsilon'_k, \varepsilon, d_1, \dots, d_{k-1}, d'_k, r; k-1)$ -regular with $d^*(V_E) = d'_k \pm \varepsilon'_k$ for each E of size k .

Assuming this claim is true, we now calculate the number M of ϕ -partite copies of H in \mathcal{V} as follows. On the one hand, by linearity of expectation, the expected number of ϕ -partite copies of H in \mathcal{V}' is

$$M \cdot \prod_{e \in H, |e|=k} \frac{d'_k}{d^*(V_E)}.$$

On the other hand, whenever \mathcal{V}' is $(\varepsilon'_k, \varepsilon, d_1, \dots, d_{k-1}, d'_k, r; k-1)$ -regular with $d^*(V'_E) = d'_k \pm o(1)$ for each E of size k , by [7, Lemma 4] the number of ϕ -partite copies of H in \mathcal{V}' is

$$(1 \pm \delta')N^{v(H)} \prod_{e \in E(H)} d^*(V'_{\phi(e)}).$$

Since the number of ϕ -partite copies of H in \mathcal{V}' is always between 0 and $N^{v(H)}$, the expected number of ϕ -partite copies of H in \mathcal{V}' is

$$(1 - o(1)) \cdot (1 \pm \delta')N^{v(H)} \prod_{e \in E(H)} d^*(V'_{\phi(e)}) \pm o(1)N^{v(H)} = (1 \pm 2\delta')N^{v(H)} \prod_{e \in E(H)} d^*(V'_{\phi(e)}).$$

Equating the two estimates of the expectation and rearranging, and by choice of δ' , we obtain the required estimate for M .

To establish the claim, consider some V_E with $|E| = k$. Without loss of generality, we can assume $E = [k]$. Let $W = R(V_{E \setminus \{1\}}, \dots, V_{E \setminus \{k\}})$; by the $(k-1)$ -uniform case of Assumption 32, $|W| = \Omega(N^k)$. By definition $\frac{|V_E|}{|W|} = d^*(V_E)$, so by Chernoff's inequality, $\frac{|V'_E|}{|W|} \neq d'_k \pm o(1)$ with probability $o(1)$.

Suppose now W' , which is a union of at most r sets of the form $R(Q_1^{(j)}, \dots, Q_k^{(j)})$ where each $Q_i^{(j)}$ is a subset of $V_{E \setminus \{i\}}$, is a witness of $(\varepsilon'_k, d'_k, r)$ -regularity failing. Then we have $|W'| \geq \varepsilon'_k |W|$, so $|V_E \cap W'| = (d^*(V_E) \pm \varepsilon_k) |W'|$. Again by Chernoff's inequality, the probability that

$$|V'_E \cap W'| \neq (1 \pm \varepsilon_k)(d^*(V_E) \pm \varepsilon_k) \frac{d'_k}{d^*(V_E)} |W'| = (d'_k \pm \varepsilon'_k) |W'|$$

is at most $\exp(-\Omega(N^k))$. Taking a union bound over the at most $\binom{\ell}{k}r2^{rk \cdot N^{k-1}}$ events, we see that the probability of any of these events occurring is as claimed $o(1)$.

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