# About the Pentagon Conjecture

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MSc Thesis at ETH Zürich

# Outline

# Graph homomorphism

## A generalisation of graph colouring

Let $G$ be a graph, a colouring of $G$ is a way of assigning a label (colour) to each vertex of $G$ in such a way that the same label is never assigned to adjacent vertices.

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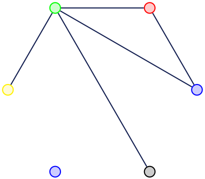

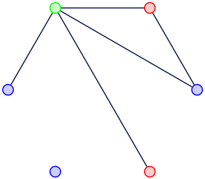

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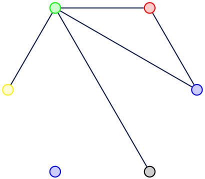
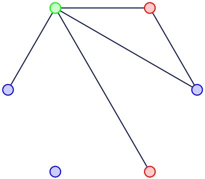




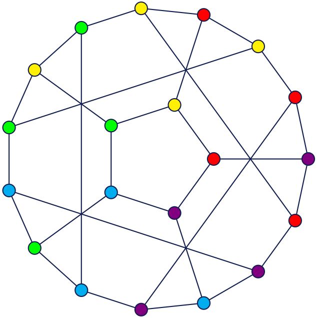
The chromatic number of $G$ (denoted with $\chi(G)$) is the number of colours required to colour $G$.

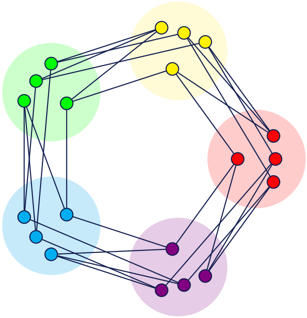
# Graph homomorphism

## A generalisation of graph colouring

### Definition of graph homomorphism

Let $H = (V_H, E_H)$ and $G = (V_G, E_G)$ be graphs. An homomorphism $\phi : G \to H$ (also called an $H$-colouring) is a map $\phi : V_G \to V_H$ such that if $uv \in E_G$ we have $\phi(u)\phi(v) \in E_H$.





# Graph homomorphism

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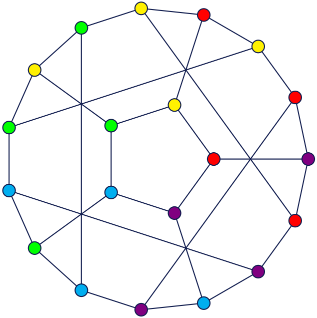
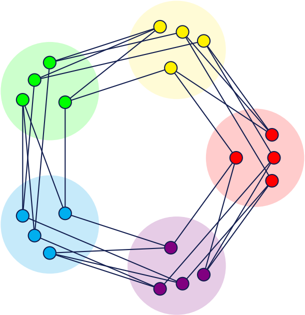
The concept of graph homomorphism is a generalisation of the concept of graph colouring. Indeed, it is still a colouring, but we additionally forbid edges between certain labels.





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There is a natural equivalence between a $k$-vertex-colouring and an homomorphism to $K_k$. So we can formulate results about colourings in the language of graph homomorphisms.

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### (Good) reasons to explore in this direction

- Constrain satisfaction problems,
- Important physical applications in the study of Ising Models,
- New point of view to study graph colourings.

# The Pentagon Conjecture

## Can we generalise Brook's Theorem?

**Theorem (Brook, 1941)**

If $G$ is a connected graph of maximum degree $\Delta$ other than a cycle or a complete graph, then $\chi(G) \leq \Delta$.

Let us consider a particular instance of this statement (recall that the girth of a graph is the length of its shortest cycle):

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**Corollary (Brook, 1941)**

Any 3-regular graph of girth at least 4 admits an homomorphism to $C_3$.

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**How a generalisation looks like?**

Any 3-regular graph of high enough girth admits an homomorphism to $H$.

How should $H$ look like and why the girth requirement?

- If $H$ contains $C_3$, Brook's theorem is enough,
- Larger cycles is the most natural generalisation:
  - Only bipartite graphs have homomorphisms to $C_{2k}$,
  - Only if $\texttt{girth}(G) \geq 2k+1$ we can have an homom. to $C_{2k+1}$.

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**Theorem**

The conjecture does not hold for $k \geq 3$.

- Kostochka and Nešetřil, 1998. Case $k = 5$,
- Wanless and Wormald, 1999. Case $k = 4$,
- Hatami, 1999. Case $k = 3$.

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  - Fix $g \in \mathbb{N}$. The probability that a random 3-regular graph over $n$ vertices has girth at least $g$ is strictly positive for $n$ big enough.
  - A.a.s. 3-regular graphs over $n$ vertices have independence ratio smaller than $\frac{6}{13}$.

# Neighbouring results
## Known approaches

It can be interesting to consider similar results.

**Theorem (Galluccio, Goddyn and Hell, 2001)**

Let $F$ be any graph. The family $\mathcal{F}$ of the graphs avoiding $F$ as a minor is girth-bipartite.

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**Corollary**

For every $F$ graph and $k \in \mathbb{N}$, there exists $g \in \mathbb{N}$ such that every graph $G$ which is $F$-minor-free of girth at least $g$ has an homomorphism to $C_{2k+1}$.

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**Corollary**

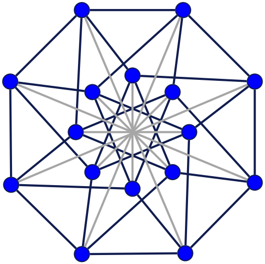
Every planar graph of high enough girth has an homomorphism to $C_{2k+1}$.

# Neighbouring results

## Known approaches

**Theorem (DeVos and Šámal, 2006)**

Every 3-regular graph with girth at least 17 admits an homomorphism to the Clebsh Graph.



**Theorem (DeVos and Šámal, 2006)**

Every 3-regular graph with girth at least 17 admits an homomorphism to the Clebsh Graph $PQ_4$.

Idea of the proof:

**Step one: an equivalent condition**

Let $G$ be a graph. The following are equivalent.

a) There are 4 pairwise disjoint cut complements,
b) There exists a homomorphism between $G$ and $PQ_4$,
c) There exists a cut-continuous mapping between $E(G)$ and $E(C_5)$.

**Theorem (DeVos and Šámal, 2006)**

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Idea of the proof:

**Step two: find 4 pairwise disjoint cut complements**

The main idea here is to work locally. Locally, every 3-regular graph of girth at least 17 looks similar. Given 4 cut complements $X_1, \ldots, X_4$, we define the weight of an edge as $|\{i : e \in X_i\}|$. From this, we define the cost of the 4 cut complements.

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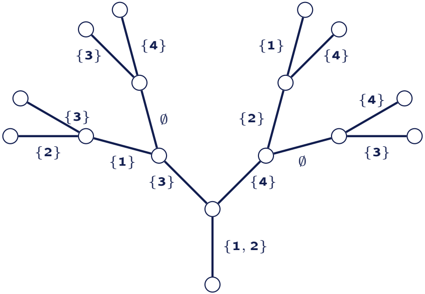

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### Step three: study the cut complements with minimal cost

If some edge is in two cut complements, there is a local operation that we can do to reduce the cost of $X_1, \ldots, X_4$ (not necessarily the sum of the weights thou).

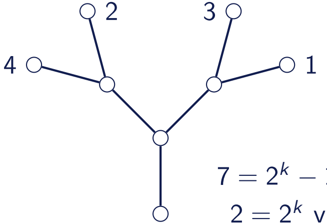
# Neighbouring results

## Known approaches

Can we use a similar approach to solve the Pentagon Conjecture?

**Limitation of this approach**

For every $k \geq 3$, there exists a map $h : V(2T_k) \to V(C_5)$ such that every modification $h'$ of $h$ which preserves the value of $h$ on the leaves is not a homomorphism.




$7 = 2^k - 1$ edges

$2 = 2^k$ violated edges

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**An approximate definition**

For a map $\phi : V(G) \to V(C_5)$, let $S_\phi$ be the set edges in $G$ that $\phi$ violates. Let $\omega_\phi := \frac{|S_\phi|}{|E(G)|}$ be the ratio of violated edges. Let $\omega_*(G)$ be the minimum that we can get for $G$.

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### Conjecture

For every $\epsilon > 0$ there exists $g \in \mathbb{N}$ such that any 3-regular graph with girth at least $g$, has $\omega_*(G) < \epsilon$.

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### Result

For evey $\epsilon > 0$ there exists $k \in \mathbb{N}$ such that if $G$ is a 3-regular graph of girth at least $2k+1$, then $\omega_*(G) < \frac{1}{4}(1+\epsilon)$.

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### Result

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Idea of the proof.

- Let $\phi$ be an homomorphism from $G$ to $C_5$ that violates the minimal number of edges.
- Suppose $\omega_\phi > \frac{1}{4}(1+\epsilon)$. Then we can find a local $2T_k$ in which $\phi$ violates more than $\frac{2^{k-2}}{2^k-1}$ of the edges.

# New possible approaches
## Physicists know better than us

**Schmidt, Guenther, Zdeborová; 2016**

"The results of the cavity method confirm that random graphs of degree three are indeed 5-circular colourable."

In the past, results that were "predicted" using the cavity method have been then proved with similar approaches. Therefore it is reasonable to suppose that the Pentagon Conjecture holds.

Introduction

- Definition and uses of graph homomorphism
- The Pentagon Conjecture

Two standard approaches

- A probabilistic upper bound
- Some similar result
  - Minor-avoiding graphs are girth-bipartite
  - Cubic graphs of high girth are homom. to the Clebsh graph


Possible approaches to the Conjecture

- An approximation
- The cavity method