

MA210 - Class 1

INTRODUCTION

- ① ASSIGNMENTS:
- best preparation for the exam
 - performance in assignments correlated to performance in exams
- OBJECTIVES:
- clarity: solutions should be self-explanatory
 - content: correct use of the theory
- ② GRADESCOPE: • Please carefully select the correct pages in your gradescope submissions.

REVIEW of THEORY

- ① In definition 1.1 it says that if there exists an injection $f: X \rightarrow Y$ and a surjection $g: X \rightarrow Y$ then there exists a bijection $h: X \rightarrow Y$. THIS SHOULD NOT BE OBVIOUS. And it is known as CANTOR-SCHRÖDER-BERNSTEIN theorem.
- ② PROP: If X and Y are disjoint, $|X \cup Y| = |X| + |Y|$.

This is a special case of:

THM: Let A_1, A_2, \dots, A_m be sets, then

$$\left| \bigcup_{i=1}^m A_i \right| = \sum_{\substack{X \subseteq [m] \\ X \neq \emptyset}} (-1)^{|X|-1} \left| \bigcap_{j \in X} A_j \right|$$

③ PROP: $|A_1 \times A_2 \times \dots \times A_m| = \prod_{i=1}^m |A_i|$

NEW EXERCISES

- ① How many ways are there to assign 5 distinguishable fruits to 8 distinguishable children in such a way that every child gets at most 2 fruits?

→ Let $F = \{\text{fruits}\}$ $C = \{\text{children}\}$. We are counting the functions $f: F \rightarrow C$ such that $\forall z \in C, |f^{-1}(z)| \leq 2$.

Let $G = \{f: F \rightarrow C : \forall z \in C, |f^{-1}(z)| \leq 2\}$.

- ① Since the negation of $\forall z \in C, |f^{-1}(z)| \leq 2$ is $\exists z \in C, |f^{-1}(z)| > 2$ we have:

$$|G| = \underbrace{|\{f: F \rightarrow C\}|}_{|C|^{|F|} = 8^5} - \underbrace{|\{f: F \rightarrow C : \exists z \in C, |f^{-1}(z)| > 2\}|}_{\text{Call this } D}$$

- ② Compute $|D|$. Notice that $\forall f \in D$ there is

EXACTLY one element $z \in C$ s.t. $|f^{-1}(z)| > 2$. Indeed,

we have $|F| = 5$ and the preimage sets are all disjoint.

For any $f \in D$ we are in exactly one of the following:

CASE 1: $\exists z \in C, |f^{-1}(z)| = 5$. (which is, $f^{-1}(z) = F$)

There are 8 functions that satisfy this case: each of these functions is uniquely identified by the image $f(F)$.

CASE 2: $\exists z \in C, |f^{-1}(z)| = 4$.

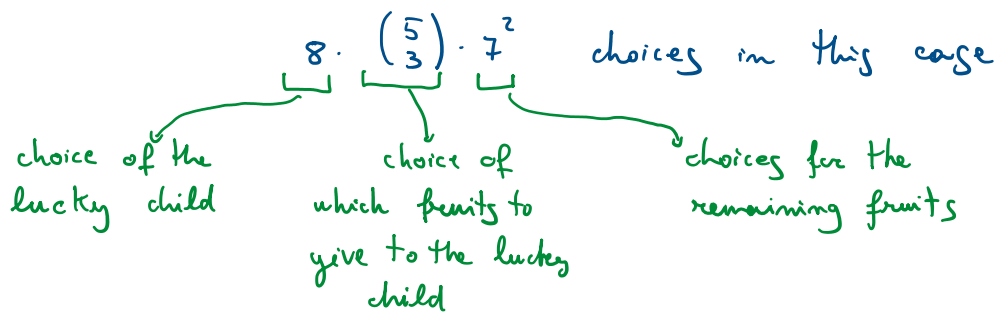
There are $8 \cdot \binom{5}{4} \cdot 7$ choices in this case

choice of the lucky child

choice of which fruits to give to the lucky child

choices for the remaining fruit

case 3: $\exists z \in C, |f^{-1}(z)| = 3$. There are



③ We have $|G| = |\{f: F \rightarrow C\}| - |D|$

$$= 8^5 - 8 \left(1 + 7 \binom{5}{4} + 7^2 \binom{5}{3} \right)$$

② Understand the reason why

$$\sum_{k=0}^m 2^k \binom{m}{k} = 3^m$$

and $3^m = \{1, 2, 3\}^m$.

Suppose we want to count $\{1, 2, 3\}^m$ in a less traditional way.

$$\{1, 2, 3\}^m = \bigcup_{j=0}^m \{x \in \{1, 2, 3\}^m \mid x \text{ contains } j \text{ copies of the digit } 3\}.$$

Because the RHS sets are disjoint,

$$|\{1, 2, 3\}^m| = \sum_{j=0}^m |\{x \in \{1, 2, 3\}^m \mid x \text{ contains } j \text{ copies of '3'}\}|$$

How many $x \in \{1, 2, 3\}^m$ have exactly j copies of 3?

$$\underbrace{\binom{m}{j}}_{\substack{\downarrow \\ \text{possible choices} \\ \text{for the positions} \\ \text{of the "3"s}}} \underbrace{2^{m-j}}_{\substack{\rightarrow \\ \text{possible choices} \\ \text{for the remaining} \\ m-j \text{ digits.}}}$$

this gives us

$$\begin{aligned} \mathfrak{Z}^m = |\{1,2,3\}^m| &= \sum_{j=0}^m |\{x \in \{1,2,3\}^m : \text{containing } j \text{ copies of '3'}\}| \\ &= \sum_{j=0}^m \binom{m}{j} 2^{m-j} = \sum_{j=0}^m \binom{m}{m-j} 2^{m-j} = \sum_{k=0}^m \binom{m}{k} 2^k \end{aligned}$$