Lecture 6. Derivatives: Futures and Options

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Topics Covered

- Introduction to derivatives
- Valuing derivatives: The no-arbitrage principle

We’ll do this in the context of **futures** and **options**.

Derivatives

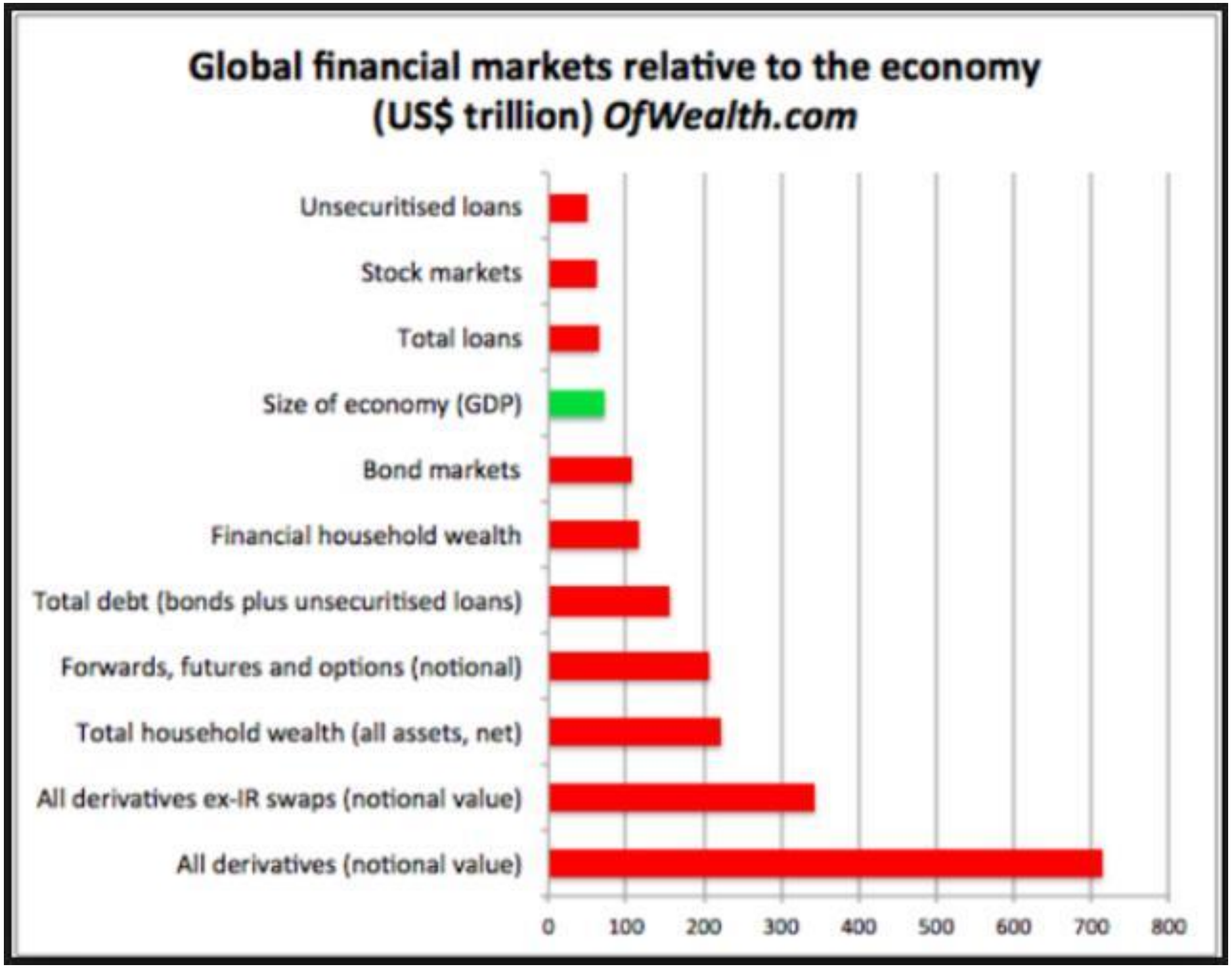
- A derivative security (“derivative”) is a security whose value derives from the price of some other security (“underlying security”).

- Derivatives are used for:
  - Hedging (reducing risk)
  - Speculating (taking risk to gain return)

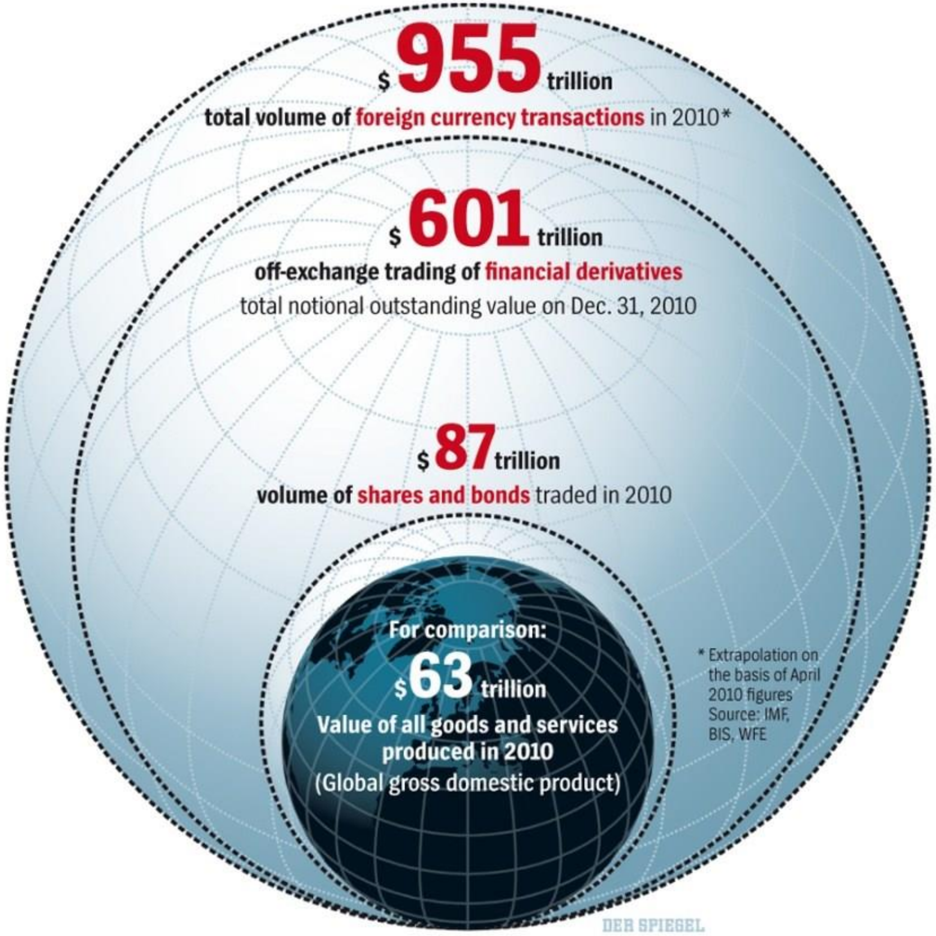
Hedging

- Risks to a business
  - Financial distress
  - Interest rate risk
  - Variable costs
  - Currency fluctuations
  - Political instability
  - Weather changes

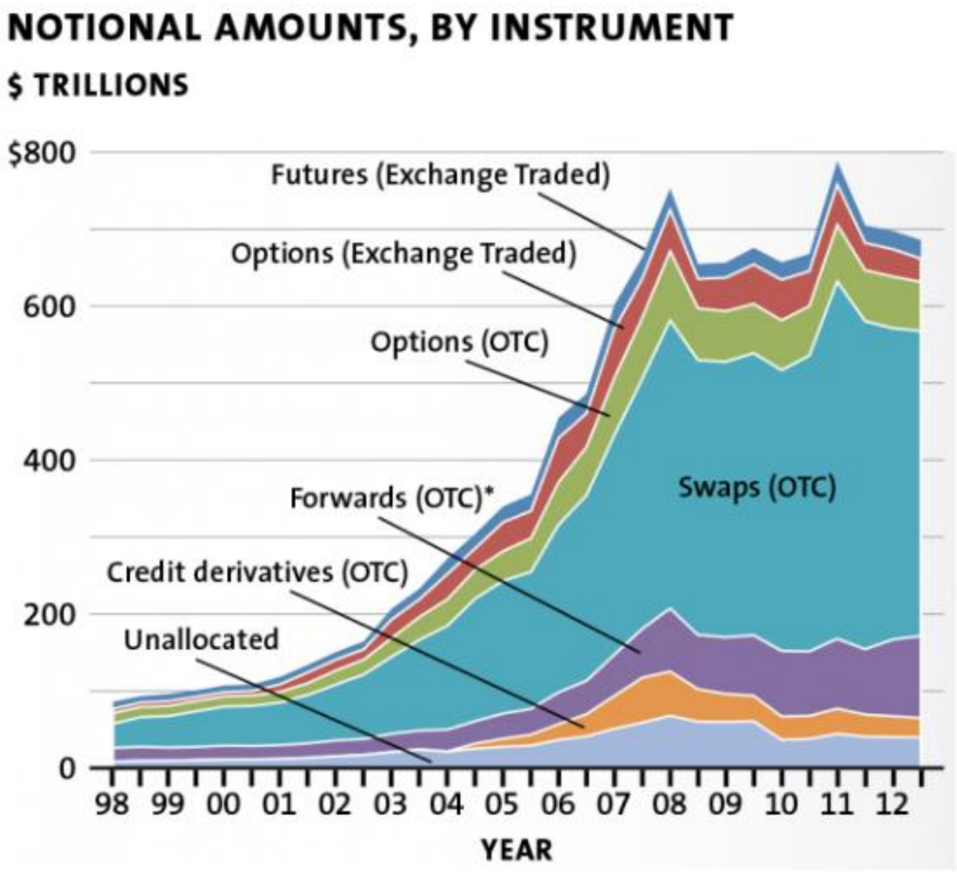
Derivatives in Context



Derivatives in Context



Types of derivatives



*Includes forex swaps, equity-linked swaps, and commodity swaps. The amounts outstanding for these categories are small and BIS reports their data with forwards.

SOURCE: Bank for International Settlements, June 2013

Forward Contracts

- **Spot Contract**: a contract for the immediate sale and delivery of an asset.

- **Forward Contract**: a contract for the delivery of an asset or commodity at a <u>predetermined</u> price on a <u>predetermined</u> date in the future.
  – At signing of contract, the delivery price (or the forward price) is chosen so that the value of the contract to both parties is zero, i.e., costs nothing to enter the contract;
  – No money/goods exchange until maturity.
  – Both parties are obliged to <u>honour</u> the contract.

Forward Contracts: Example

- A farmer is growing 1 ton of wheat for harvest, and plans to sell in September next year.

Forward Contracts: Example

- A cereal maker plans to purchase 1 ton of wheat in September to make cereal.

Forward Contracts: Example

- A farmer is growing 1 ton of wheat for harvest, and plans to sell in September next year.
- A cereal maker plans to purchase 1 ton of wheat in September to make cereal.
- They both want to lock in a price for the transaction of wheat in September next year rather than face uncertainty about the price at which they sell/buy.

→ They can sign a **forward contract**

Forward Contracts

- An example of a commodity forward contract:
  - Cereal maker promises to buy 1 ton of wheat from the farmer on September 1st for £100;
  - This is called "going long" or "buying" a forward contract, and would benefit from a price increase;
  - Farmer promises to sell the same amount for the same price;
  - This is called "going short" or "selling" the forward contract, and would benefit from a price decrease.

- No money exchange at contract signing.
  - The delivery price is set so that the contract has a present value of zero.

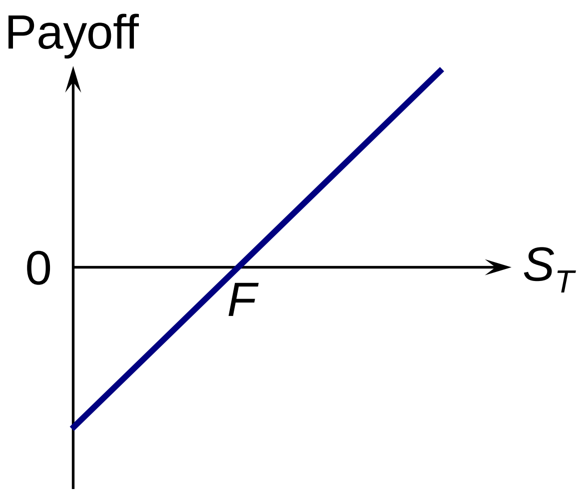
Forward Contracts

- On September 1st the spot price of wheat is £110 per ton.
- The cereal maker and farmer have a forward contract to trade at £100.
- Cash settlement rather than goods exchanges, to save transportation costs:
  - Farmer sells his wheat for £110 to his local store;
  - Cereal maker purchases wheat for £110 from his local depot;
  - The farmer sends the cereal maker a check for £10;
  - Both parties effectively “trade” at £100, as agreed.

Forwards on Non-Dividend-Paying Stocks

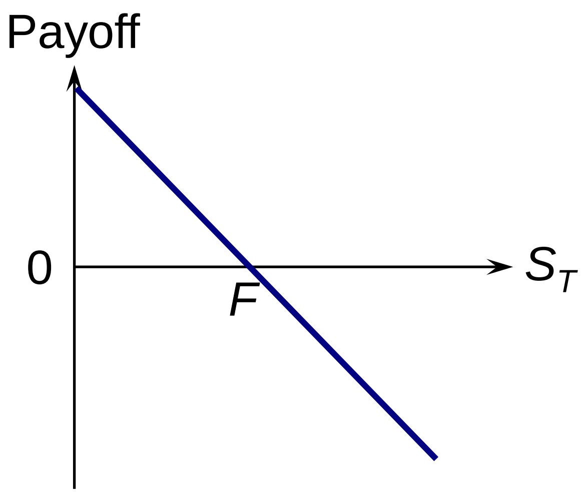
- The payoff diagrams are as follows:

Buy (Long) Forward



$$Payoff = S_T - F$$

Sell (Short) Forward



$$Payoff = -(S_T - F)$$

Forward Contracts

- A forward contract locks in the delivery price, but can lead to substantial losses to either party.
- Suppose, the farmer and the cereal maker agreed on forward price £100 per ton, but the spot price on the settlement date is £70.
  - The cereal maker "loses" £30 per ton and will incur higher production cost than other cereal makers
  - Knowing this, the cereal maker may choose to default rather than carry on with his business.

- Counterparty risk
  - In a forward contract, one of the two parties may default. The risk of contract default is called the counterparty risk.

Futures Contracts

- A futures contract is also an agreement to buy or sell an asset or commodity on a future day at a fixed price.
  - The delivery price (futures price) is chosen to make the futures contract have a zero value at initiation.
- Unlike forwards, futures contracts eliminate the counterparty risk:
  - Futures contracts are traded on exchanges (e.g., CME) and are standardized;
  - Futures contracts have margin requirement;
  - Futures contracts are marked to market.

Futures Contracts

- To prevent default, futures exchanges require buyers and sellers to deposit funds in <u>margin accounts</u>.
  - Funds in the margin account guarantee that investors have enough capital to honour their obligations;
  - Moreover, if one of the parties does not have enough money in the margin account, the investor receives a <u>margin call</u> and needs to urgently top up the margin account.
- There is negligible counter-party risk in futures contracts since the intermediary (“the clearing house”) is insured.
- Futures contracts can be easily sold in an exchange.

Forward vs Futures Contracts

	Forwards	Futures
(1)	Private contract between two parties	Traded on an exchange
(2)	Settled at end of contract (one cash flow at T)	Settled daily (cash flow every day)
(3)	Non-standardized	Standardized

Valuing Forwards

- Forward and futures contracts can be written on stocks, bonds, commodities, and other assets.

- Let's derive forward prices using the **no-arbitrage principle**.
  - No-arbitrage principle: Two assets/contracts with the same payoffs in all "states" of the future should have the same price.

Forwards on Non-Dividend-Paying Stocks

- These are two ways to invest money now to get a unit of a non-dividend-paying stock at time *T*
  - Buy a share of the stock now and hold it to time *T*. This costs *S* now (current price of the stock)
  - Buy a forward contract for the stock with maturity *T*. This forward price is *F*, which needs to be paid at *T*. Hence an investment of $F/(1+r_T)^T$ in a *T*-year zero-coupon bond with annualized rate r_T ensures that the funds are available at time *T*.

- These two strategies are equivalent so they must have the same price. Hence:

$S = F/(1+r_T)^T$ => $\boxed{F = S \times (1+r_T)^T,}$

Forwards on Non-Dividend-Paying Stocks

- Forward price is given by:

$$\boxed{F = S \times (1 + r_T)^T,}$$

where *S* is the current price of asset.

- Note that forward price depends on the <u>current asset price</u>.
  - The forward price comoves with the asset price.

- Forward price also depends on the discount rate.
  - The forward price comoves with the yield to maturity on the *T*-year bond.

Options

Option - The <u>right</u> to buy or sell a security at a specified price on or before a specified date.

Call Option - The right to <u>buy</u> a security at a specified price on or before a specified date.

Put Option - The right to <u>sell</u> a security at a specified price on or before a specified date.

American Option - Can be exercised at any time prior to and on the expiration date.

European Option - Can be exercised only on the expiration date.

Options

- The buyers and issuers of options have the following rights and obligations:

	Buyer	Seller (or "Writer")
Call option	Right to buy asset	Obligation to sell asset
Put option	Right to sell asset	Obligation to buy asset

Options

- **Example:** Prices of options on stock XYZ, July 12, 2018; current stock price = $19.56.

	Calls			Puts		
Strike price, $	21 Oct. 2018	20 Jan. 2019	21 Apr. 2019	21 Oct. 2018	20 Jan. 2019	21 Apr. 2019
15.00	4.650	4.950	5.150	0.025	0.150	0.275
17.50	2.300	2.775	3.150	0.125	0.475	0.725
20.00	0.575	1.175	1.650	0.875	1.375	1.700
22.50	0.075	0.375	0.725	2.950	3.100	3.300
25.00	0.025	0.125	0.275	5.450	5.450	5.450

Options

- We will use the following notations:
  - S = current price of the underlying security;
  - C = call option price;
  - P = put option price;
  - K = strike price;
  - T = maturity (expiration date);

- We will look at European options only.
- Moreover, we will assume that stocks do not pay dividends between $t=0$ and $t=T$.

Call Value at Maturity

- The payoff of an option at maturity is determined by its strike price and S_T.
- Call option:
  - Exercise when $S_T > K, C_T = S_T - K > 0,$ *in the money*;
  - Do nothing when $S_T \le K, C_T = 0,$ *out of the money*;
  - Call payoff at maturity:

$$\boxed{C_T = \max\left[0, S_T - K\right]}$$

Put Value at Maturity

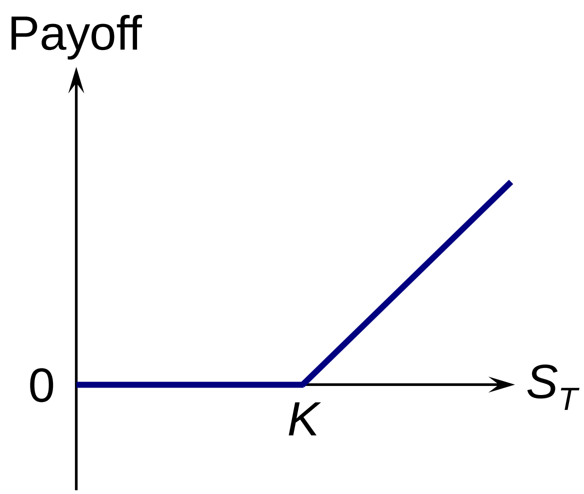
- Put option:
  - Exercise when $$K > S_T, P_T = K - S_T > 0,$$ *in the money*;
  - Do nothing when $$K \leq S_T, P_T = 0,$$ *out of the money*;
  - Put payoff at maturity:

$$\boxed{P_T = \max\left[0, K - S_T\right]}$$

Payoff Diagrams

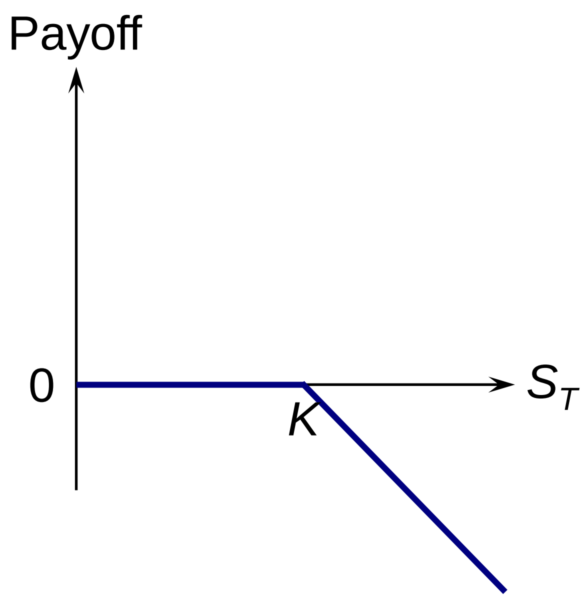
- The payoff diagrams (at maturity) for buying and selling calls:

Buy Call



$$Payoff = \max[0, S_T - K]$$

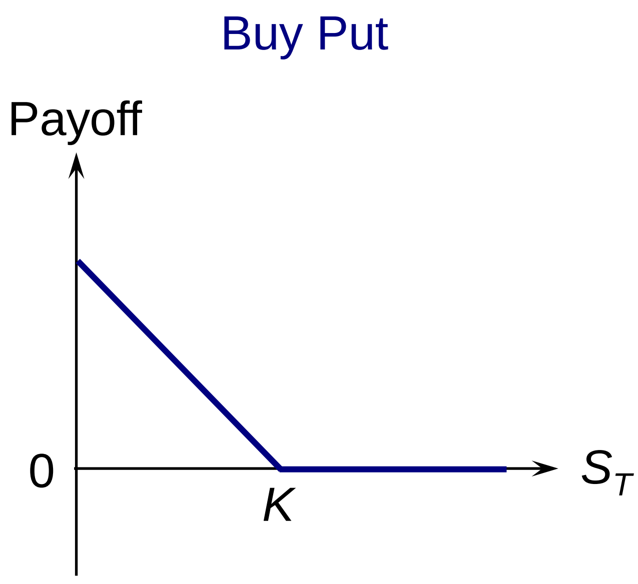
Sell Call



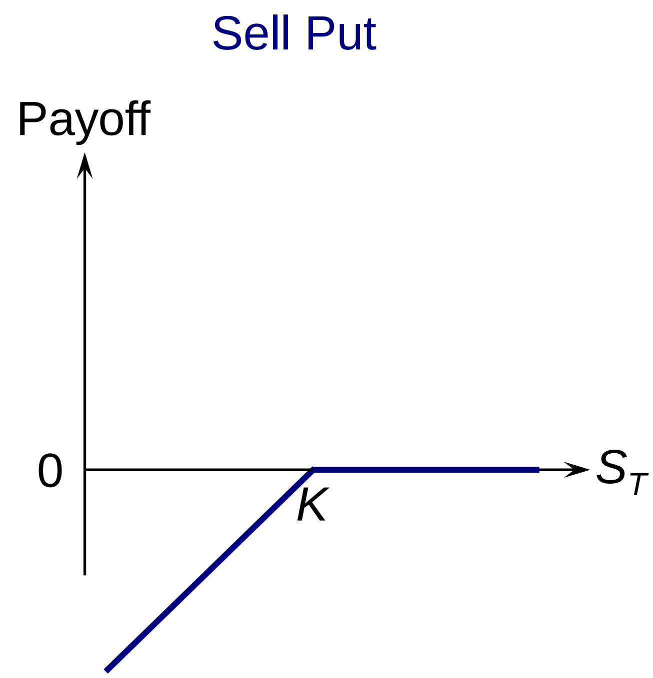
$$Payoff = -\max[0, S_T - K]$$

Payoff Diagrams

- The payoff diagrams (at maturity) for buying and selling puts:



$$Payoff = \max[0, K - S_T],$$



$$Payoff = -\max[0, K - S_T],$$

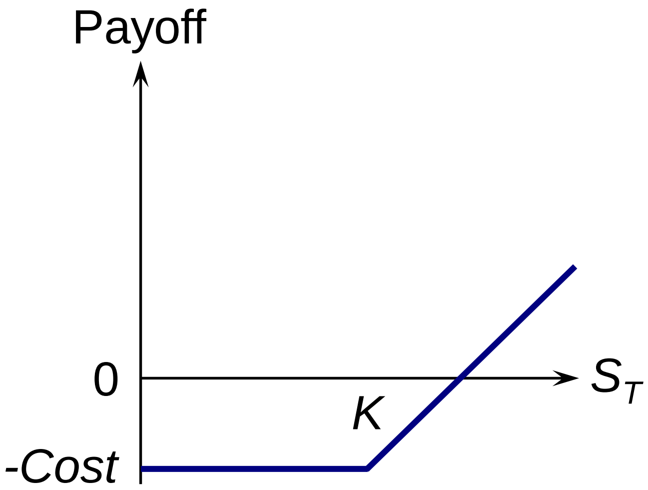
Options Strategies

- Covered call
- Married put
- Zero cost collar
- Straddle
- Butterfly

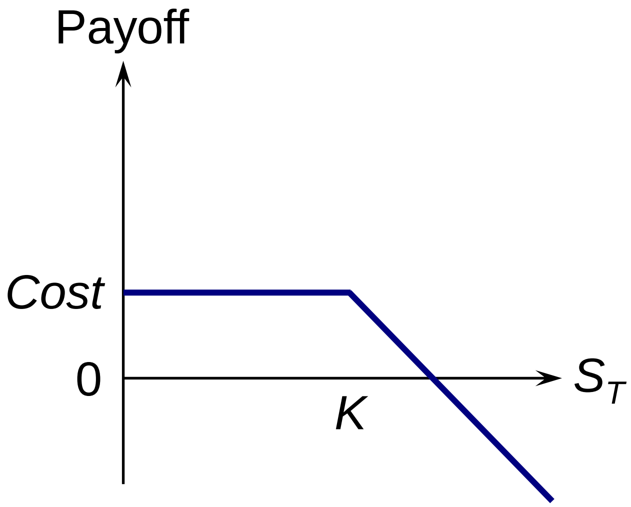
Net Payoffs

- Net Payoff = Payoff at maturity – Cost:

Buy Call



Sell Call



$$Payoff = \max[0, S_T - K] - Cost,$$

$$Payoff = -\max[0, S_T - K] + Cost,$$

Put-Call Parity

- Consider a call and a put with the same strike *K* and maturity *T*.
  - It can be shown that today's prices of the call, put, and stock satisfy the following equation:

$$\boxed{C = P + S - \underbrace{\frac{K}{(1+r_T)^T}}_{\text{PV of strike } K}}$$

  - Hence, if we know the put price, we can easily find the call price, and vice versa.
  - This is called put-call parity.

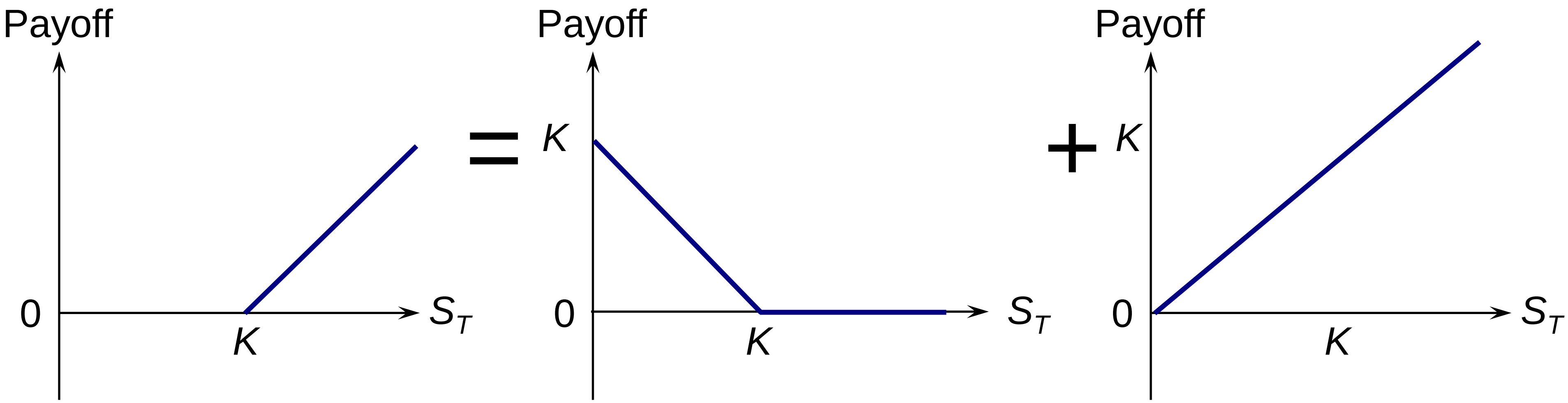
Put-Call Parity

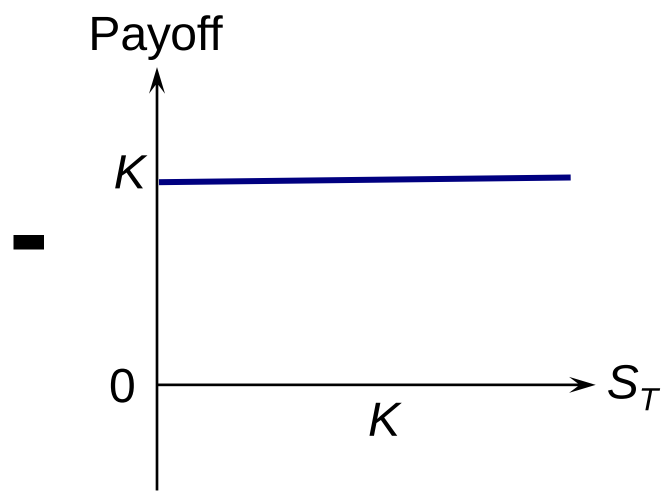
- We will prove put-call parity by replicating the cash flows of a call option, given **no-arbitrage**.
- The replicating portfolio is as follows:
  - Buy a put;
  - Buy a stock;
  - Sell *T*-year zero coupon bond with face value K (equivalent to borrowing $K/(1+r_T)^T$).

Put-Call Parity

$$C = P + S - \frac{K}{(1+r_T)^T}$$

Buy Call　　　Buy Put　　　Buy Stock (underlying)





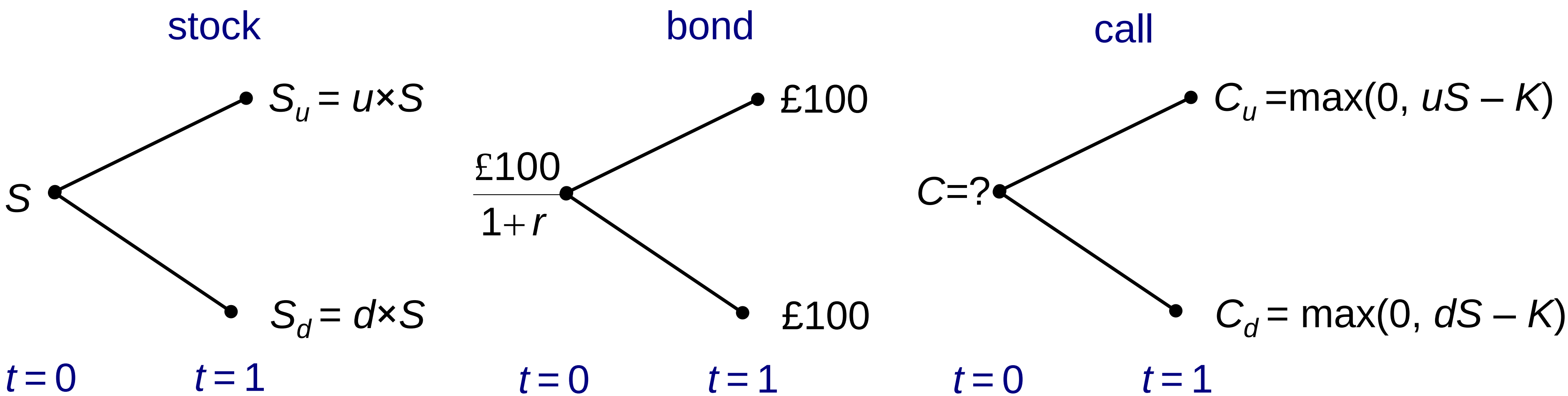
Sell T-year zero-coupon Bond whose value will be *K* at time *T* (hence value K $/(1+r_T)^T$ now)

Option Prices in a Binomial Model

- We will now solve the one-period option pricing problem for a stock that does not pay dividends.
  - Consider a call option with strike K.
- We will make the critical assumption that the stock price can take only <u>two</u> values at the call's maturity.
  - With probability p stock price S goes up: $S_u = u \times S,\ u > 1$;
  - With probability $1-p$ stock price S goes down: $S_d = d \times S,\ d < 1$.

Option Prices in a Binomial Model

- The payoff of a call option can be replicated as follows:
  - Buy Δ units of stock and B units of bond



- Note that to avoid arbitrage we must have:

$$d < 1 + r < u$$

Option Prices in a Binomial Model

- We construct the replicating portfolio by matching the payoffs at maturity:

	Matching payoffs at maturity		
	Replicating Portfolio		Call Option
up:	$\Delta \times uS + B \times £100$	=	C_u
down:	$\Delta \times dS + B \times £100$	=	C_d

- Solving these equations we find the replication strategy:

$$\Delta = \frac{C_u - C_d}{(u-d) \times S}, \quad B = \frac{uC_d - dC_u}{(u-d) \times 100}$$

Option Prices in a Binomial Model

- Call price is then given by:

$$C = \Delta \times S + B \times \frac{100}{1+r}$$

$$= \frac{C_u - C_d}{(u-d)\times S} \times S + \frac{uC_d - dC_u}{(u-d)\times 100} \times \frac{100}{1+r}$$

$$= \frac{qC_u + (1-q)C_d}{1+r}, \text{ where } q = \frac{1+r-d}{u-d}.$$

$$\boxed{C = \frac{qC_u + (1-q)C_d}{1+r}, \text{ where } q = \frac{1+r-d}{u-d}.}$$

- Put price can be derived analogously.

Risk-Neutral Probabilities

- Since $d < 1 + r < u$, it follows that:

$$0 < q = \frac{1 + r - d}{u - d} < 1.$$

- Therefore, q can be given an interpretation of probability.
  - q is called the risk neutral probability (RNP) of the up state, and is NOT related to the true probability;
  - 1– q is the risk-neutral probability of the down state.

- The expression for the option price can be interpreted as discounted expected option payoff under RN framework:

$$\boxed{C = \frac{qC_u + (1-q)C_d}{1+r} = \frac{E^*[C_1]}{1+r}}$$

Risk-Neutral Probabilities

- Similarly, today's stock price is given by:

$$S = \frac{qS_u + (1-q)S_d}{1+r} = \frac{E^*[S_1]}{1+r}.$$

- Consider a hypothetical world in which real probabilities are replaced by risk-neutral ones.
  - In this world we can price risky assets (not only options) assuming that investors are risk neutral.
  - This is because in this world investors do not require risk-premium:

$$\frac{E^*[S_1] - S}{S} = r.$$

Risk-Neutral Probabilities

- **Example:** Consider again a call option with $K = £100$ and $T = 1$, $S = £100$, $u = 1.1$, $d = 0.9$, $r = 0.05$.

  – The RN Probability of up move is given by:

$$q = \frac{1 + r - d}{u - d} = \frac{1 + 0.05 - 0.9}{1.1 - 0.9} = 0.75.$$

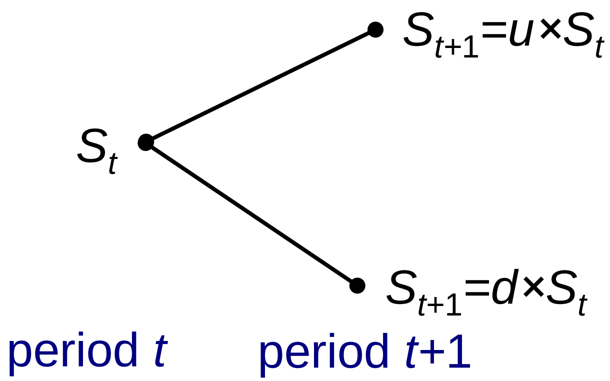
  – $C_u = \max(0, uS - K) = £10$, $C_d = \max(0, dS - K) = £0$;

  – Call price is given by:

$$C = \frac{qC_u + (1-q)C_d}{1+r} = \frac{0.75 \times £10 + 0.25 \times £0}{1 + 0.05} = £7.14$$

Multi-Period Binomial Model

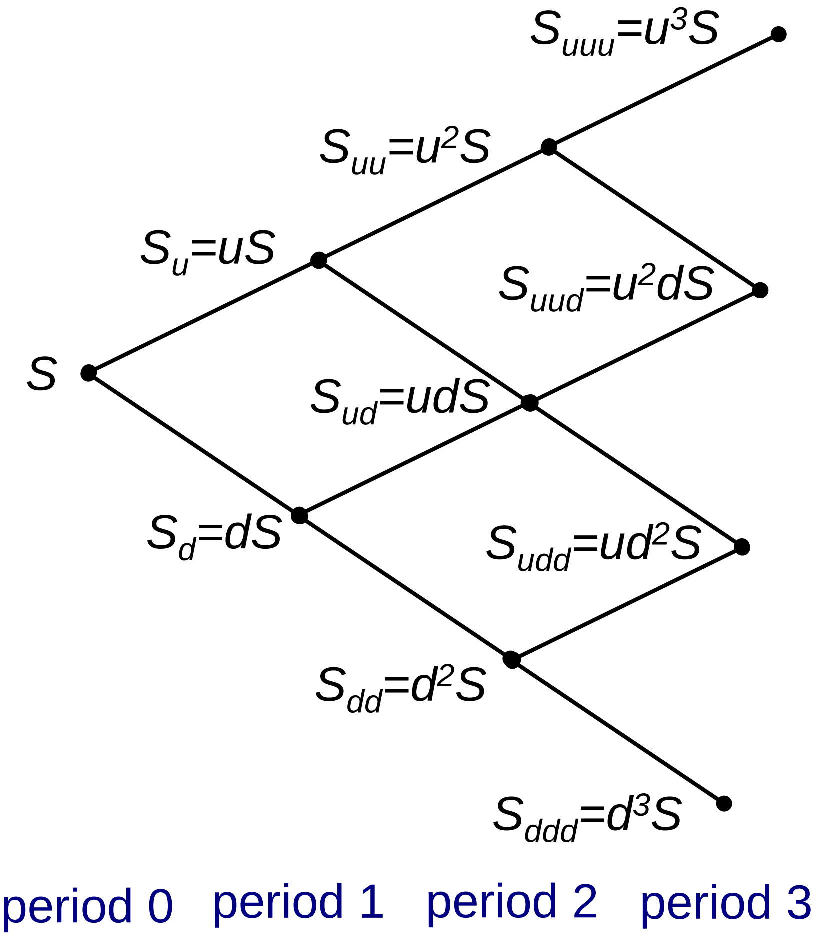
- Multi-Period Model:
  - There are N periods;
  - In each period, stock price follows a binomial process:



- Risk-neutral probability is the same at each time period, as long as u and d are the same in each period.

Multi-Period Binomial Model

- The tree for a three period stock price process:



Black-Scholes Option Valuation

- We can add more and more periods, eventually, we converge to the Black-Scholes formula:

$$C = S\,N(d_1) - Ke^{-rT}N(d_2)$$

$$d_1 = \frac{\ln(S/K)+rT}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Black-Scholes Option Valuation

C = Call option price

S = Stock price

K = Exercise price

T = time to maturity of the option in years

r = Risk-free interest rate (annualized continuously compounded with the same maturity as the option)

σ = Volatility: Standard deviation of annualized cont. compounded rate of return on the stock

N(d) = Cumulative normal distribution function

Call Option Example

$S = 100$ $\quad$ $K = 95$

$r = .10$ $\quad$ $T = .25$ (quarter)

$\sigma = .50$

$d_1 = [\ln(100/95) + (.10+(.5^2/2))*.25] / (.5\ .25^{1/2}) = .43$

$d_2 = .43 - (.5)(.25^{1/2}) = .18$

$N(.43) = .6664$

$N(.18) = .5714$

$$C = S\,N(d_1) - Ke^{-rT}N(d_2)$$
$$= 100\,(.6664) - 95\,e^{-.10\,(.25)}\,(.5714)$$
$$= 13.70$$

Appendix

- More details on futures/forwards contracts

- More on proving put-call parity

Futures Contracts

- Futures contracts are settled each day based on the market price (mark-to-market).
- Suppose, in our previous example the agreed delivery futures price is £100.
  - If next day, the futures value goes up by £2, cereal maker gains £2 and farmer loses £2;
  - £2 is immediately transferred from the farmer's margin account to the cereal maker's margin account;
  - If a day later the futures price goes down by £1 then £1 is transferred from cereal maker's to farmer's margin account;
  - Cash transfers are arranged by futures exchange to ensure both parties can honour their contracts.

Futures Contracts

- Total gains/losses over the life of the contract are determined by the difference between the futures price (£100) and the spot price at expiration (£95);
- On the delivery day, as in the case with forwards, farmer sells while cereal maker buys corn at £95 in the market;
  - Farmer receives £100 in total (sells corn for £95 and receives £5 from cereal maker);
  - Cereal maker pays £100 in total (buys corn for £95 and pays the farmer £5);

Forwards on Non-Dividend-Paying Stocks

- After date zero the value of a forward contract may deviate from zero.
  - It follows from no-arbitrage pricing that the value of a forward contract at time *t* (i.e., there are *T–t* years to maturity) is given by:

    $$V_t = S_t - \frac{F}{(1+r_{t,T})^{T-t}},$$

    where $r_{t,T}$ is the spot rate between years *t* and *T*, while *F* is the forward price determined at time 0.

  - Higher S_t => larger V_t

Forwards on Dividend-Paying Stocks

- Consider now a *T*-year forward contract on a dividend paying stock with current price *S*.

- Total payoff to the buyer after *T* years is $S_T - F$;

$$V = PV(S_T - F) = (S - PV(Div)) - \frac{F}{(1+r_T)^T};$$

- Given that V=0

$$F = (S - PV(Div)) \times (1 + r_T)^T.$$

Forwards on Commodities

- If we are dealing with commodity forwards or futures then there might be some benefits (e.g., a forward contract on cars) and costs (e.g., storage costs) associated with holding the commodity.
  - Benefits can be interpreted as positive "dividends" while storage costs can be interpreted as negative "dividends";
  - Therefore, the formula for one-year futures price with benefits and storage cost will be given by:

$$F = \left(S - PV(\text{benefits}) + PV(\text{storage cost})\right) \times (1 + r).$$

Forwards on Commodities

- Consider a one-year commodity forward, and denote by B and C time $t=1$ value of benefits and costs.
  - Then, the formula for forward price can be rewritten as follows: $$F = \left(S - \frac{B}{1+r} + \frac{C}{1+r} \right) \times (1+r).$$
  - *convenience yield* $y = B / S.$
  - *net convenience yield* $y = (B-C) / S.$

$$\boxed{\begin{aligned} F &= \left(S - \frac{yS}{1+r} \right) \times (1+r) \\ &= S \times (1+r-y). \end{aligned}}$$