1. Graph the following functions using Theorem 1.3. Find the vertex, zeros and axis-intercepts (if any exist). Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

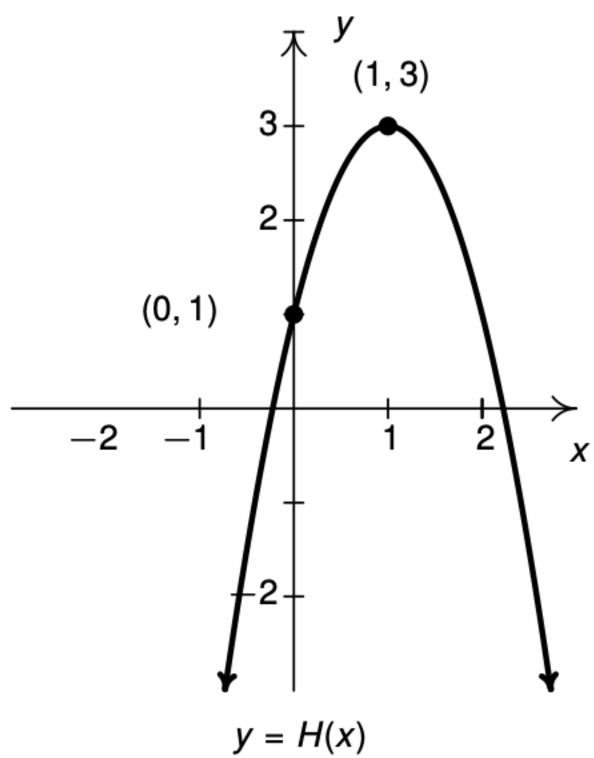
(a) $f(x) = \dfrac{(x-3)^2}{2}$

(b) $g(x) = (x+2)^2 - 3$

(c) $h(t) = -2(t-3)^2 + 1$

(d) $i(t) = \dfrac{(3-2t)^2 + 1}{2}$

2. Use Theorem 1.3 to write a possible formula for $H(x)$ whose graph is given below:



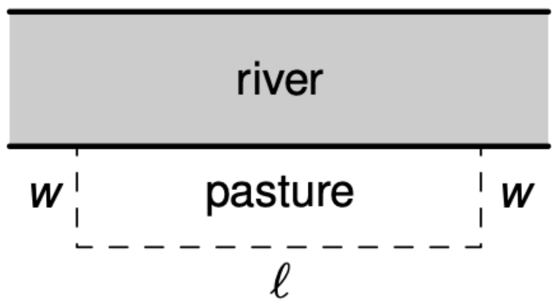
Example 1.4.2. Graph the following functions. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

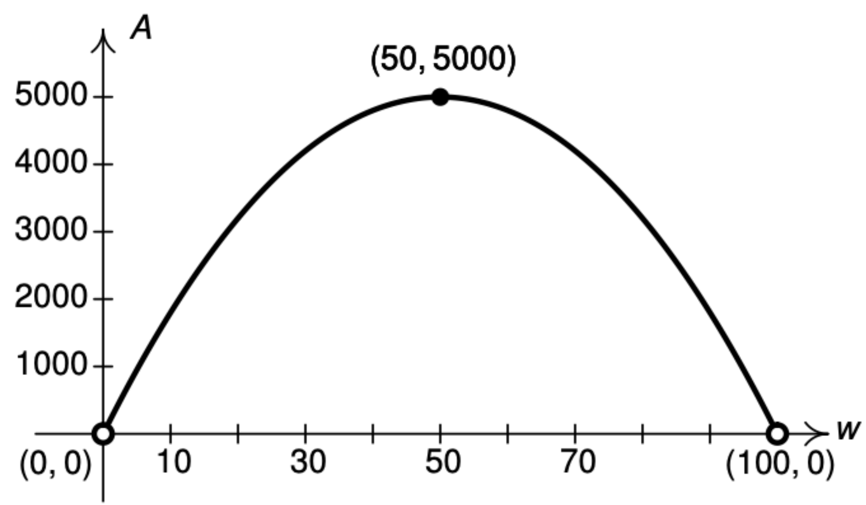
1. $f(x) = x^2 - 4x + 3.$
2. $g(t) = 6 - 4t - 2t^2$

Example 1.4.5. Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives so the time is right for him to pursue his dream of raising alpaca. He wishes to build a rectangular pasture and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a river (so that no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

Solution. We are asked to find the dimensions of the pasture which would give a maximum area, so we begin by sketching the diagram seen below on the left. We let w denote the width of the pasture and we let ℓ denote the length of the pasture. The units given to us in the statement of the problem are feet, so we assume that w and ℓ are measured in feet. The area of the pasture, which we'll call A, is related to w and ℓ by the equation $A = w\ell$. Since w and ℓ are both measured in feet, A has units of feet^2, or square feet.

We are also told that the total amount of fencing available is 200 feet, which means $w + \ell + w = 200$, or, $\ell + 2w = 200$. We now have two equations, $A = w\ell$ and $\ell + 2w = 200$. In order to use the tools given to us in this section to *maximize* A, we need to use the information given to write A as a function of just *one* variable, either w or ℓ. This is where we use the equation $\ell + 2w = 200$. Solving for ℓ, we find $\ell = 200 - 2w$, and we substitute this into our equation for A. We get $A = w\ell = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A = f(w) = 200w - 2w^2 = -2w^2 + 200w$.





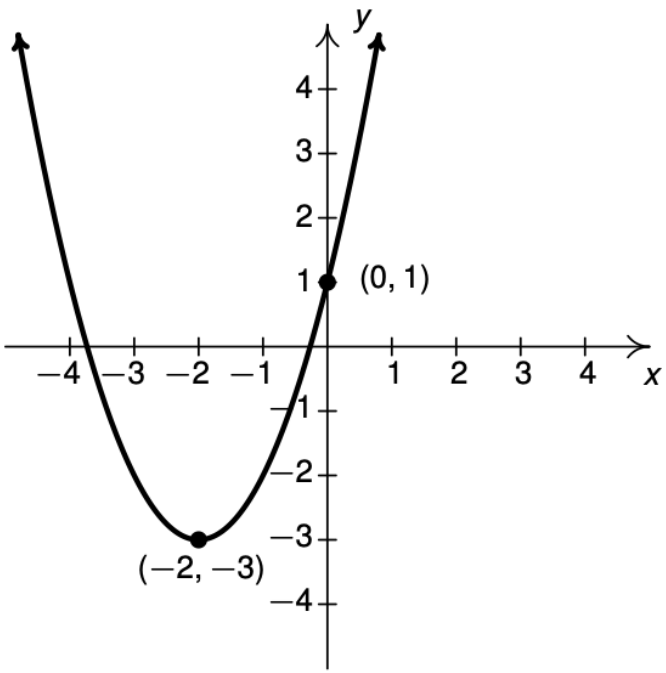
$A = f(w)$

In Exercises 1 - 9, graph the quadratic function. Find the vertex and axis intercepts of each graph, if they exist. State the domain and range, identify the maximum or minimum, and list the intervals over which the function is increasing or decreasing. If the function is given in general form, convert it into standard form; if it is given in standard form, convert it into general form.

1. $f(x) = x^2 + 2$
2. $f(x) = -(x+2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $g(t) = -2(t+1)^2 + 4$
5. $g(t) = 2t^2 - 4t - 1$
6. $g(t) = -3t^2 + 4t - 7$
7. $h(s) = s^2 + s + 1$
8. $h(s) = -3s^2 + 5s + 4$
9. $h(s) = s^2 - \dfrac{1}{100}s - 1$

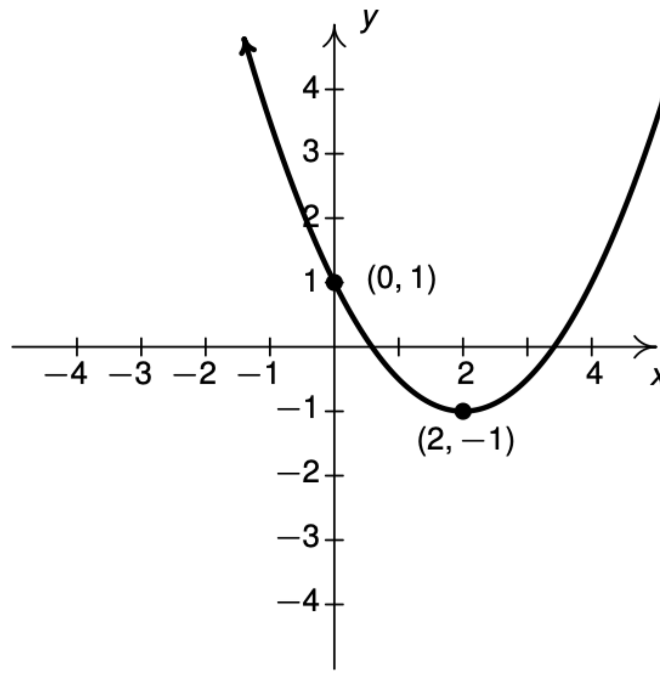
In Exercises 10 - 13, find a formula for each function below in the form $F(x) = a(x-h)^2 + k$.

10.



$y = F(x)$

11.



$y = F(x)$

In Exercises 30 - 34, cost and price-demand functions are given. For each scenario,

- Find the profit function $P(x)$.
- Find the number of items which need to be sold in order to maximize profit.
- Find the maximum profit.
- Find the price to charge per item in order to maximize profit.
- Find and interpret break-even points.

30. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, for $0 \leq x \leq 15$.

31. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, for $0 \leq x \leq 35$.

32. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x + 240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, for $0 \leq x \leq 30$.

33. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, for $0 \leq x \leq 24$.

34. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, for $0 \leq x \leq 70$.

Example 1.4.3. In Example 1.2.3 the cost to produce x PortaBoy game systems for a local retailer was given by $C(x) = 80x + 150$ for $x \geq 0$ and in Example 1.2.4 the price-demand function was found to be $p(x) = -1.5x + 250$, for $0 \leq x \leq 166$.

1. Find formulas for the associated revenue and profit functions; include the domain of each.

2. Find and interpret $P(0)$.

3. Find and interpret the zeros of P.

4. Graph $y = P(x)$. Find the vertex and axis intercepts.

5. Interpret the vertex of the graph of $y = P(x)$.

6. What should the price per system be in order to maximize profit?

7. Find and interpret the average rate of change of P over the interval $[0, 57]$.

Solution.

1. The formula for the revenue function is $R(x) = x\,p(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since the domain of p is restricted to $0 \leq x \leq 166$, so is the domain of R. To find the profit function $P(x)$, we subtract $P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150$. The cost function formula is valid for $x \geq 0$, but the revenue function is valid when $0 \leq x \leq 166$. Hence, the domain of P is likewise restricted to $[0, 166]$.

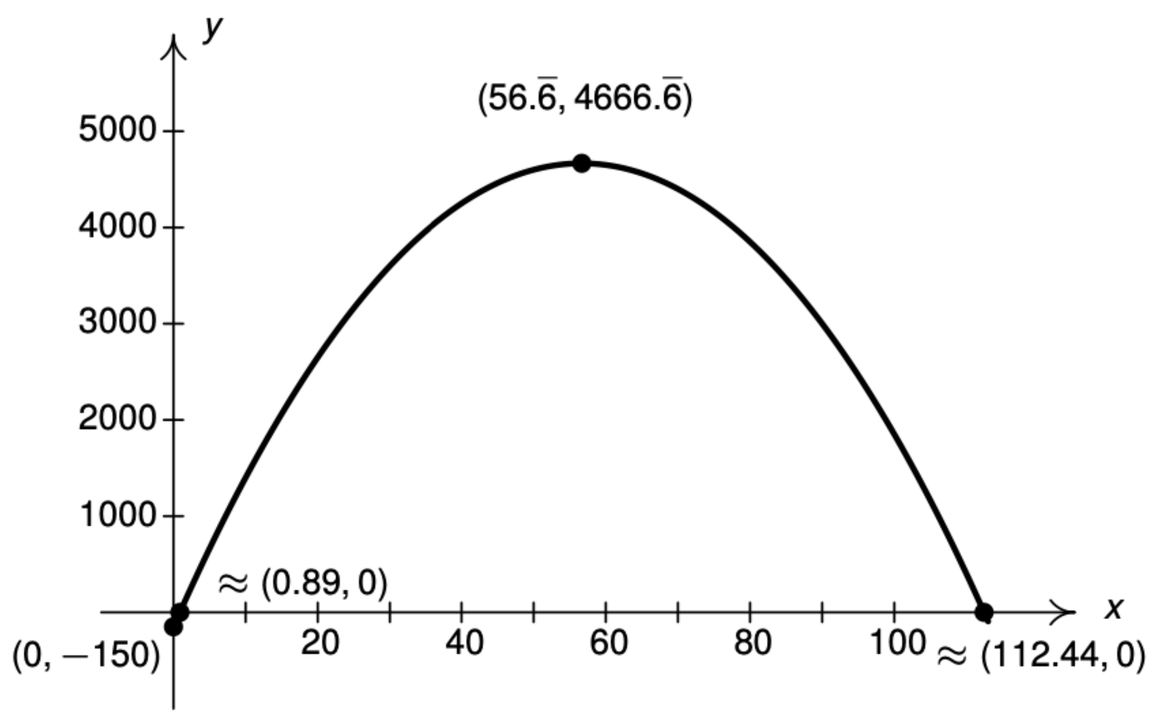
2. We find $P(0) = -1.5(0)^2 + 170(0) - 150 = -150$. This means that if we produce and sell 0 PortaBoy game systems, we have a profit of $-\$150$. Since profit = (revenue) − (cost), this means our costs exceed our revenue by \$150. This makes perfect sense, since if we don't sell any systems, our revenue is \$0 but our fixed costs (see Example 1.2.3) are \$150.

3. To find the zeros of P, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. Factoring here would be challenging to say the least, so we use the Quadratic Formula, Equation 1.3. Identifying $a = -1.5$, $b = 170$ and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \\ &\approx 0.89,\ 112.44. \end{aligned}$$

Given that profit = (revenue) − (cost), if profit = 0, then revenue = cost. Hence, the zeros of P are called the 'break-even' points - where just enough product is sold to recover the cost spent to make the product. Also, x represents a number of game systems, which is a whole number, so instead of using the exact values of the zeros, or even their approximations, we consider $x = 0$ and $x = 1$ along with $x = 112$ and $x = 113$. We find $P(0) = -150$, $P(1) = 18.5$, $P(112) = 74$ and $P(113) = -93.5$. These data suggest that, in order to be profitable, at least 1 but not more than 112 systems should be produced and sold, as borne out in the graph below.

4. Knowing the zeros of P, we have two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right) \approx (0.89, 0)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right) \approx (112.44, 0)$. Since $P(0) = -150$, we get the y-intercept is $(0, -150)$. To find the vertex, we appeal to Equation 1.2. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3} = 56.\overline{6}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3} = 4666.\overline{6}$. Hence, the vertex is $(56.\overline{6}, 4666.\overline{6})$. The domain is restricted $0 \leq x \leq 166$ and we find $P(166) = -13264$. Attempting to plot all of these points on the same graph to any sort of scale is challenging. Instead, we present a portion of the graph for $0 \leq x \leq 113$. Even with this, the intercepts near the origin are crowded.



$y = P(x)$

5. From the vertex, we see that the maximum of P is $4666.\overline{6}$ when $x = 56.\overline{6}$. As before, x represents the number of PortaBoy systems produced and sold, so we cannot produce and sell $56.\overline{6}$ systems. Hence, by comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

6. We've determined that we need to sell 57 PortaBoys to maximize profit, so we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. In other words, to sell 57 systems, and thereby maximize the profit, we should set the price at \$164.50 per system.

7. To find the average rate of change of P over $[0, 57]$, we compute

$$\frac{\Delta[P(x)]}{\Delta x} = \frac{P(57) - P(0)}{57 - 0} = \frac{4666.5 - (-150)}{57} = 84.5.$$

This means that as the number of systems produced and sold ranges from 0 to 57, the average profit per system is increasing at a rate of \$84.50. In other words, for each additional system produced and sold, the profit increased by \$84.50 on average. $\square$