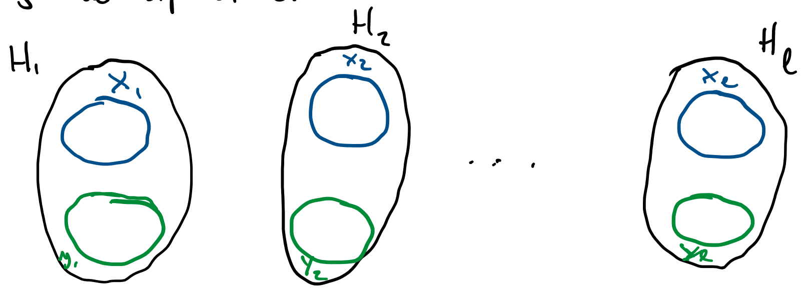
MA210 - Class 6

THM Let G be a graph. G is bipartite iff G has no odd cycle.

$\leadsto$ '$\Leftarrow$') Done in class for the cases in which G is connected.
So let us fix a graph G not connected and with no odd cycles.
We can partition G into connected components that we denote by $H_1, \dots, H_\ell$.
None of these H_i contains odd cycles, and all of them are connected, so we proved that all these H_i must be bipartite.



Let us denote by (X_i, Y_i) a bipartition of H_i.

CLAIM $X := \bigcup_{i=1}^{\ell} X_i \qquad Y := \bigcup_{i=1}^{\ell} Y_i$
forms a bipartition of G.

$\leadsto$ Let $a, b \in X$. If a, b are in the same X_i, there is no edge ab because X_i is an indep. set.
If a, b are in different components, there is no edge ab because there are no edges between distinct connected components (since they are maximal by definition).

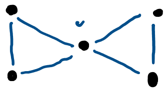
EXERCISE 11b) Let G be a graph with every vertex of even degree and let v be a vertex contained in a tour.
Then v is contained in a cycle.
$\leadsto$ Let S be a* tour of minimal length containing v. Let $S = vv_1, v_1v_2, \dots, v_kv$.

I claim that S is a cycle. Indeed, by definition, either this is the case or there exists i s.t. v_i is encountered twice in S. Which is, either S is a cycle or

$$S = vv_1, \dots, v_{i-1}v_i, v_iv_{i+1}, \dots, v_iv_{s+1}, \dots$$

In which case, $S' = vv_1, \dots, v_{i-1}v_i, v_iv_{s+1}, \dots$ is a strictly shorter tour containing v. $\lightning$

* there could be more than one:



EXERCISE 10 Let us fix G, C and E as in the text of the exercise. We can denote the vertices of C in such a way that $e = v_\ell v_1$ and $C = e, v_1v_2, \dots, v_{\ell-1}v_\ell$.

Now, let $x, y \in V$ be arbitrary vertices. Since G is connected, let us fix a walk $W = u_0u_1, \dots, u_{k-1}u_k$ between x and y ($x = u_0$, $y = u_k$). We want to show x and y are connected in $G' := G \setminus e$.

If W does not use e, we are done. Otherwise, we can create W' by SUBSTITUTING each occurrence of e with

$$e \longmapsto v_1v_2, \dots, v_{\ell-1}v_\ell \quad \text{(in the right direction)}$$

This will create a xy-walk in G'.

Question 2

(a) For $m, n \geq 3$, let $G_{m,n}$ be the graph formed in the following two steps. First take two cycles C_m and C_n, whose vertex sets are disjoint; then add edges from each vertex of C_m to each vertex of C_n.

We denote the vertex set of C_m by $X = \{x_1, x_2, \dots, x_m\}$, and the vertex set of C_n by $Y = \{y_1, y_2, \dots, y_n\}$.

(i) Make a sketch of $G_{4,3}$.

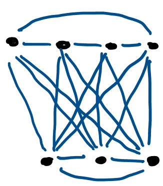
(ii) Formulate Euler's Theorem.

Use Euler's Theorem to decide the values of $m, n \geq 3$ for which $G_{m,n}$ has an Euler tour.

(iii) For what values of $m, n \geq 3$ does $G_{m,n}$ have a Hamilton cycle?

(iv) For what values of $m, n \geq 3$ is $G_{m,n}$ a bipartite graph?

i)



PAUSE TO THINK

ii) THM Let G be a non-trivial graph. Then G has an Euler tour iff G is connected AND all vertices in G have even degree.

CLAIM $\forall x \in X$, $d(x) = n+2$ and $\forall y \in Y$, $d(y) = m+2$.

$\rightsquigarrow$ ists the first equality.

Note that in the first construction step the degree of x is exactly 2. Then we add to its neighbour all the vertices in Y (and there are n of them).

CORO $G_{m,n}$ has an Euler tour iff $m, n \geq 3$ AND both m and n are even.

$\rightsquigarrow$ In this case $G_{n,m}$ is non trivial, connected, and all its vertices have even degree.

iii) Let $n, m \geq 3$ be arbitrary, then

$$W = \underbrace{x_1x_2, x_2x_3, \dots, x_{m-1}x_m}_{\text{edges in } C_m}, \underbrace{x_m y_1}_{\text{step 2}}, \underbrace{y_1 y_2, \dots, y_{n-1} y_n}_{\text{edges in } C_n}, \underbrace{y_n x_1}_{\text{step 2}}$$

is a Hamilton cycle in $G_{m,n}$

iv) Let $m, n \geq 3$. Then x_1x_2, x_2y_1, y_1x_1 is an odd cycle.