Product Schur Triples in the Integers

L. Mattos, **D. Mergoni Cecchelli**, O. Parczyk

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Thm. (Abbott and Moser, 1966)

For any k and l positive integers we have:

$$S(k+l) \geq 2S(k)S(l) + S(k) + S(l).$$

Paired with $S(5) = 161$ (Heule, 2018), this gives $S(k) \geq c \cdot 321^{k/5}$.

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The upper bound is $\lfloor k!(e - \frac{1}{24}) \rfloor$ and due to Irving (1974).

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How large is the largest subset of $[n]$ that can be partitioned into k sum-free sets?

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Abbott-Wang Conjecture (1977)

The Abbott-Wang construction is optimal. I.e. $n - \lfloor \frac{n}{H(k)} \rfloor$

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Graham, Rödl and Ruciński (1996)

Any 2-colouring of $[n]$ contains $n^2/19$ monochromatic sums.

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- For $p \gg n^{-2/3}$ we have $Var(X) \ll \mu^2$ as

$$Var(X) \sim n^2 p^3 (1 - p^3) + p^5 n^3 + p^4 n^2.$$

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Let $\alpha_n \in (0,1)$, what p_n guarantees that if $|C_n| \geq (1-\alpha_n)n$ then

$$\lim_{n\to\infty} \mathbb{P}[C_n \cup [n]_p \text{ is 2-Schur}] = 1?$$

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Thm. (Das, Knierim, and Morris, 2024)

If C is dense and $p \gg n^{-2/3}$, then w.h.p. every 2-colouring of $C \cup [n]_p$ contains a monochromatic sum.

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- Which $[n]_p$ cannot be partitioned into k sum-free sets?
- What's the interplay between deterministic and random?

What about PRODUCTS?

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Thm. (Mattos, MC, Parczyk, 2025)

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$$n - n^{1/S'(k)} \leq g_*(k,n) \leq n - (1-\varepsilon)n^{1/S(k)}.$$

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$\hat{p}_\alpha(n) = n^{-1/2+o(1)}$ is the threshold for the α-randomly perturbed product Schur property (for α in a wide range).

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- Any subset of $[n]$ that can be partitioned into k product-free sets must avoid an element of $P(a)$ for each a in A'.

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- Colour $a \in (n^{1/S'(k)}, n]$ with colour $\chi(\lceil S'(k) \cdot \log_n(a) \rceil - 1)$.
- If $ab = c$, then let $a' = \lceil S'(k) \cdot \log_n(a) \rceil - 1$, $b' = \lceil S'(k) \cdot \log_n(b) \rceil - 1$, and $c' = \lceil S'(k) \cdot \log_n(c) \rceil - 1$ and note that $\log_n(a) + \log_n(b) = \log_n(c)$ implies $a' + b' = c'$ or $a' + b' = c' - 1$.

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- **Lemma.** If $A \subseteq [2, n]$ has size $n - \frac{1}{2}\sqrt{n}$, it contains $n/8$ products.

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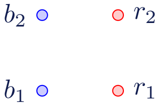

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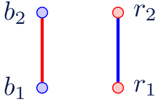

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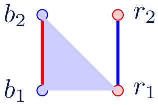
Let $\varepsilon > 0$. For n large enough, any 2-colouring of $[2, n]$ contains $n^{1/3-\varepsilon}$ monochromatic products.

Proof.

- **Lemma.** If $A \subseteq [2, n]$ has size $n - \frac{1}{2}\sqrt{n}$, it contains $n/8$ products.
- Let R be the red elements of $[n^{1/3}]$, B the blue ones. By Lemma, wlog we have $|R| \geq |B| \geq n^{1/6}/2$.
- Let $P_R := \{ab : a, b \in R\}$ and $P_B := \{ab : a, b \in B\}$.
- Note $|P_R|, |P_B| \geq n^{1/3-\varepsilon}$. Also, P_R contains $n^{1/3-\varepsilon}$ blue elements and P_B contains $n^{1/3-\varepsilon}$ red elements.



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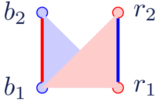

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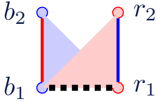

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- To each $\{r_1, r_2, b_1, b_2\}$ with $r_1, r_2 \in R$, $b_1, b_2 \in B$, $r_1 r_2$ blue and $b_1 b_2$ red we can associate a monochromatic product.

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- Take care of double-counting.

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Thm. (Aragão, Chapman, Ortega, Souza, 2024+)

For n large enough, any r-colouring of $[2, n]$ contains $n^{1/S(r-1)}$ monochromatic products.

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- For $p \ll (n \log(n))^{-1/3}$, there are no products.

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- This reduces to show $|A^2 \cap [2, n]| \gg 1/q$ w.h.p.
- A useful tool is that no c can be written as the product of elements of A in more than 2 ways w.h.p.

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- What is the k-colouring of n with the fewest monochromatic products?
- What is the threshold for any k-colouring of $[2,n]_p$ to contain a monochromatic product?


Thank
You!!