MA102 Mathematical Proof and Analysis
and
MA103 Introduction to Abstract Mathematics

Exercises 1

Each week, we will set some exercises which **you are expected to do** and which contribute to your grade for this course. The deadline will always be **Monday 17:00 London time** in the following week.

If you are not sure how to get started on a question, write out the *definitions* of all the concepts in the question from your notes. This guarantees that your work will be Acceptable for the course grade.

If you still are not sure what to do, try applying the definitions you just wrote down to the question. That is, having written '$a|b$ means that there is an integer k such that $b = ka$' (the definition), if you are supposed to prove that for all integers n we have $2|n^2 + 3n$, you might write '$2|n^2 + 3n$ means that there is an integer k such that $n^2 + 3n = 2k$'. Now think again about how you can show that this equation is true for all integers n.

If you are still not sure what to do, try small examples. What happens if $n = 1$? if $n = 2$? Can you see a pattern? Then you are most of the way to a solution.

Do not give up too easily. Mathematics is hard work, and doing mathematics means spending a lot of time thinking and being stuck. You may not like being stuck now, but you will soon become confident in your ability to eventually find a solution; you will get used to the idea that even if you are stuck now, in a few hours, or perhaps a few days, you will have solved the problem. What you are learning and training yourself to do — the single most important thing you want to get our of your Mathematics studies — is to be a person who can take difficult problems, work on them, and solve them. That skill is something you will profit from for the rest of your life. The problems you need to solve in future may not be mathematical, but they will be difficult and require persistence — most people will give up, you will not, and that is why you will be able to demand a good salary.

Some weeks we will also set a harder starred question. We will often give two weeks not one to solve this, and we recommend you work on it in with friends. This question does **not** count towards your course grade.

Here are some general remarks about writing mathematics:

- **Always** justify your answers, whatever the wording of the question.
- Most answers to exercises in this course will require a careful explanation of why something is true or false. You should give your argument in enough detail for somebody else to follow what you write.
- Write your answers in **English**. That is, don't just use symbols, but use **words** to explain how you get from each line to the next.
- Avoid using the symbols "$\therefore$" or "$\because$" (if you don't know what they mean, that's just fine), or arrows like "$\longrightarrow$" or "$\Longrightarrow$" in mathematical arguments. Use words!

The following four questions **are required for your coursework grade**. The deadline is **Monday of Week 2, 17:00** London time (17:00 on 4 October).

1 Consider the following (false) statement about natural numbers n :

If n is a multiple of 14, then n is not a multiple of 6.

What properties must the natural number n have, for n to be a *counterexample* to this statement ?

Find a counterexample to the given statement.

2 Show that the following statement about natural numbers n is true, by giving a proof :

If n is a multiple of 4, then $9n - 30$ is a multiple of 6.

3 For which natural numbers n is $4^n - 1$ a prime number ? Justify your answer.

4 (a) Explain what is wrong with the following proof of the statement :

If n and m are natural numbers, then $4 = 5$.

Proof : Denote the sum $n + m$ by t. Then we certainly have $n + m = t$.
This last statement can be rewritten as $(4\,n - 5\,n) + (4\,m - 5\,m) = 4\,t - 5\,t$.
Rearrangement leads to $4\,n + 4\,m - 4\,t = 5\,n + 5\,m - 5\,t$.
Taking out common factors on each side gives $4\,(n + m - t) = 5\,(n + m - t)$.
Removing the common term now leads to $4 = 5$. $\square$

(b) Explain what is wrong with the following proof of the statement :

If n and m are natural numbers, then $n = m$.

Proof : Denote the sum $n + m$ by t. Then we certainly have $n + m = t$.
Multiplying both sides by $n - m$ yields $(n + m)\,(n - m) = t\,(n - m)$.
Multiplying out gives $n^2 - m^2 = t\,n - t\,m$.
This can be rewritten as $n^2 - t\,n = m^2 - t\,m$.
Adding $\frac{1}{4}\,t^2$ to both sides gives $n^2 - t\,n + \frac{1}{4}\,t^2 = m^2 - t\,m + \frac{1}{4}\,t^2$.
This is the same as $(n - \frac{1}{2}\,t)^2 = (m - \frac{1}{2}\,t)^2$.
So we get $n - \frac{1}{2}\,t = m - \frac{1}{2}\,t$.
Adding $\frac{1}{2}\,t$ to both sides gives $n = m$. $\square$

This question is **not compulsory but encouraged**. We strongly suggest that you work on it with friends. The deadline is **Monday of Week 3, 5pm**.

5$*$ In a tug-of-war *pull*, two teams pull a rope in opposite directions, until one team reaches their winning line (there are no draws). In the Southern League, there are n village teams; each team pulls against each other team once. Teams receive 3 points for a win, and 0 points for a loss. The highest-scoring team (or teams) are declared the winners of the league.

When the league is over, the Little Watting captain notices that they have a *moral victory*. They beat Great Catford, who beat Moreton, who beat Kedington... and so on through all n teams.

Definition: a team T has a moral victory if the following is true. There is a way of listing all the teams in some order, with T first, such that each team (except the last one on the list) beat the team immediately after it in the list.

For each of the following questions, you should try to find the answer for *every* integer $n \geq 2$ (The answers might be different for different integers!).

(a) Find one possible outcome for the pulls such that Little Watting cannot claim a moral victory.

(b) Decide whether it is possible that every team can claim a moral victory.

(c) Prove that at least one team can always claim a moral victory.

(d) Is it possible that a team can claim a moral victory but also have less points than any other team in the league?

(e) Is it true that if Little Watting are the only team who can claim a moral victory, then they also win the league?

(f) Is it possible that all teams finish the league with the same number of points?

Some parts of this question are relatively easy; some are not. The only way to find out which is which is to try hard to solve them all. Although you are supposed to find solutions for all n, you may well want to try solving the problems for small values of n (such as $2, 3, 4, 5$) first, in order to get ideas. If you are not sure what to do, either ask me (ideally on the forum so everyone gets the answer) or your class teacher for hints.