Reading Course

STRENGTHENING RÖDL'S THEOREM

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LSE

Rödl's Theorem – General Situation and Statement

Q Let H be a fixed graph; what can we say about a graph G that does not contain H as an induced subgraph (H-free)?

EXA $K_{m,m}$ is Δ-free → there are m^2 edges and $m(m-1)$ non-edges, → There is a LARGE independent set.

Def G a graph, $X \subseteq V(G)$ is : – weakly ε-restricted if $e(G[X]) \leq \varepsilon |X|^2$ or $e(G[X]) \geq (1-\varepsilon)|X|^2$

– ε-restricted if $\Delta(G[X]) \leq \varepsilon |X|$ or $\delta(G[X]) \geq (1-\varepsilon)|X|$

EXA $K_{1,m}$ is weakly $\frac{1}{m}$-restricted but it is not $\frac{1}{2}$-restricted.

Rem X ε-restricted $\Rightarrow$ X weakly ε-restricted $\Rightarrow$ $\exists X' \subseteq X$; $|X'| \geq \frac{|X|}{2}$ with X' 2ε-restricted.

→ Say $e(G[X]) \leq \varepsilon |X|^2$ and take $X' = \{x \in X \mid d_{G[X]}(x) \leq \sqrt{2\varepsilon}|X|\}$. We have $|X'| \geq (1-\frac{\varepsilon}{\sqrt{2}})|X|$

Then $\forall x \in X'$, $d_{G[X']}(x) \leq d_{G[X]}(x) = \sqrt{2\varepsilon}|X| \leq \frac{2\varepsilon}{\sqrt{2}-\varepsilon}|X'|$

THM (RÖDL) Let G be a H-free graph. We can find an (weakly) ε-restricted set $X \subseteq V(G)$ of size at least $\delta|G|$.

$(\forall H\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall G\ H\text{-free}\ldots)$

→ If we want an ε-restricted set, let $\varepsilon' = \frac{\varepsilon}{2}$; we can apply the weak version of the theorem with ε' and then use the remark.

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Rödl's Theorem – Remarks

THM (RÖDL) Let H be a graph and $\varepsilon > 0$; there exists $\delta > 0$ such that every H-free graph G we can find an ε-restricted set $X \subseteq V(G)$ of size at least $\delta|G|$.

CORO Every H-free graph G can be partitioned into at most N weakly ε-restricted subsets. $(\forall H\ \forall \varepsilon > 0\ \exists N \ldots)$

→ Apply RÖDL'S THM with $\frac{\varepsilon}{2}$ repeatedly; then use the bigger sets to take care of the last vertices

→ THM (NIKIFOROV) If G has $< \zeta|G|^{|H|}$ induced copies of H, then we can find an equipartition of G into at most N weakly ε-restricted sets.

THM ~~CORO~~ Every H-free graph G can be partitioned into at most N ~~weakly~~ ε-restricted subsets. $(\forall H\ \forall \varepsilon > 0\ \exists N \ldots)$

→ Some of the sets might NEED to be of cardinality 2.

EXA $K_{1,m}$ with $\varepsilon = \frac{1}{4}$.

→ ε-restricted does NOT behave well w.r.t. adding vertices.

→ The proof does NOT use REGULARITY.

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Rödl's Theorem – Proofs and Consequences

THM (RÖDL) Let H be a graph and $\varepsilon > 0$; there exists $\delta > 0$ such that in every H-free graph G we can find an (weakly) ε-restricted set $X \subseteq V(G)$ of size at least $\delta|G|$.

THM (RÖDL) If G is large and if for every $X \subseteq V(G)$ with $|X| \geq \delta|G|$ we have $e(G[X]) \in [\varepsilon|X|^2, (1-\varepsilon)|X|^2]$, then G containg all the graphs with k vertices as induced subgraphs. $\forall k \in \mathbb{N}, \forall \varepsilon > 0, \exists \delta > 0$

① RÖDL **LEMMA** If G large enough can be k-equi-partitioned so that all the pairs are ε-regular and with density in $(\beta, 1-\beta)$, then G contains induced copies of all graphs on k vertices. $(\forall \beta, \forall k, \exists \varepsilon)$

+ REGULARITY LEMMA

⟿ Use REGULARITY + TURAN to find $M = R(k,k,k)$ clusters in G, all pairs ε-regular.

(i,j) in T_1 if $d(C_i, C_j) \leq \frac{1}{2}\beta$

(i,j) in T_2 if $d(C_i, C_j) \in (\frac{1}{2}\beta, 1-\frac{1}{2}\beta)$

(i,j) in T_3 if $d(C_i, C_j) \geq 1-\frac{1}{2}\beta$

Since $M = R(k,k,k)$ we find $I \subseteq [M]$ with $|I| = k$ and $I^2 \subseteq T_i$

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Rödl's Theorem – Proofs and Consequences

THM (RÖDL) Let H be a graph and $\varepsilon > 0$; there exists $\delta > 0$ such that every H-free graph G we can find an ε-(weakly) restricted set $X \subseteq V(G)$ of size at least $\delta|G|$.

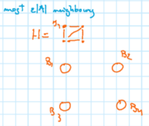
② FOX, SUDAKOV **DEF** Let H be a graph; given $\varepsilon_1, \varepsilon_2 > 0$ define $\delta_H(\varepsilon_1, \varepsilon_2)$ as the largest δ s.t. every H-free graph G has an induced graph of size $\delta|G|$ with edge density at most ε_1 or at least $1-\varepsilon_2$.

THM (RÖDL) $\forall H, \forall \varepsilon > 0, \quad \delta_H(\varepsilon, \varepsilon) > 0$

THM (FOX, SUDAKOV) $\forall H, \forall \varepsilon > 0, \quad \delta_H(\varepsilon, \varepsilon) > 2^{-O(\log(\frac{1}{\varepsilon})^2)}$

LEMMA (ERDŐS, HAJNAL) Let $\varepsilon \in (0, \frac{1}{2})$, and let G be an H-free graph.

There are $A, B \subseteq V(G)$ disjoint with $|A|, |B| \geq \frac{\varepsilon^{|H|-1}}{|H|}|G|$ such that either every vertex in B has at most $\varepsilon|A|$ neighbours in A, or every vertex in B has at least $(1-\varepsilon)|A|$ neighbours in A.



⟿ $V = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_{|H|}$ in an equitable way; let $V(H) = \{y_1, \ldots, y_{|H|}\}$.

CASE A $\exists x_1 \in B_1$ s.t. $\forall i > 1$ • $|N(x_1) \cap B_i| > \varepsilon|B_i|$ if $y_1 y_i \in E(H)$

• $|N(x_i) \cap B_i| < (1-\varepsilon)|B_i|$ if $y_1 y_i \notin E(H)$

CASE B $\forall x \in B_1, \exists j(x)$ s.t. $|N(x) \cap B_j| < \varepsilon|B_j|$ and $y_1 y_j \in E(H)$

or $|N(x) \cap B_j| > (1-\varepsilon)|B_j|$ and $y_1 y_j \notin E(H)$

We can now take $A = B_j$ and $B = \{x \in B_1 : j(x) = j\}$ if we choose j carefully.

LEMMA Let $\varepsilon_1, \varepsilon_2 > 0$ with $\varepsilon_1 + \varepsilon_2 < 1$, and let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Let $k := |H|$;

$$\delta_H(\varepsilon_1, \varepsilon_2) \geq \left(\frac{\varepsilon}{4}\right)^k k^{-1} \min\left(\delta_H\left(\frac{3\varepsilon_1}{2}, \varepsilon_2\right), \delta_H\left(\varepsilon_1, \frac{3\varepsilon_2}{2}\right)\right).$$

• Apply the lemma with $\frac{\varepsilon}{4}$ and get A, B.

• Take $B' \subseteq B$ with $|B'| \geq \delta(\frac{3}{2}\varepsilon_1, \varepsilon_2)|B|$ and edge density $\leq \frac{3}{2}\varepsilon_1$ or $\geq 1-\varepsilon_2$.

• Let $A_1 = \{v \in A : |N(v) \cap B'| \leq \frac{\varepsilon}{2}|B'|\}$. We get $|A_1| \geq |A|/2$.

• Let $A' \subseteq A_1$ with edge density $\leq \frac{3}{2}\varepsilon_1$ (or $\geq (1-\varepsilon_2)$).

• Consider $A' \cup B'$.

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Rödl's Theorem – Fundamental tool

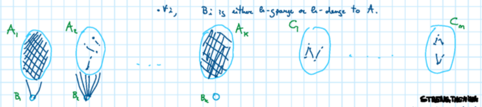
THM $\forall H, \forall \varepsilon>0\ \exists N$ s.t. every H-free graph can be partitioned into at most N ε-restricted subsets.

DEF For $A, B \subseteq V(G)$ disjoint and $\varepsilon>0$, we say B is ε-sparse to A if every vertex in B has at most $\varepsilon|A|$ neighbours in A; it is ε-dense if it is ε-sparse in $\overline{G}$.

LEMMA (ERDÖS-HAJNAL) $\forall \varepsilon>0\ \forall$ H-free G, $\exists A, B \subseteq V(G)$ with $|A|, |B| \geq \frac{\varepsilon^{|H|-1}}{|H|}|G|$ and B is ε-sparse or ε-dense to A.

LEMMA For every graph H and $\varepsilon, \eta, \theta>0$ there exists N such that if G is H-free, then we can partition $V(G)$ into $A_1, \dots, A_k; B_1, \dots, B_k; C_1, \dots, C_m$ such that:

- $m \leq N$; $k \leq |H|^2$
- $A_1, \dots, A_k$ and $C_1, \dots, C_m$ are ε-restricted,
- $\forall i$, $|B_i| \leq \eta|A_i|$,
- $\forall i$, B_i is either θ-sparse or θ-dense to A.



Rödl's Theorem – Proofs and Consequences

THM (RÖDL) Let H be a graph and $\varepsilon>0$; there exists $\delta>0$ such that every H-free graph G we can find an ε-(weakly) restricted set $X \subseteq V(G)$ of size at least $\delta|G|$.

THM (BOUSQUET, LAGOUTTE, THOMASSÉ) For every $k \geq 3$, $\{P_k, \overline{P_k}\}$ has the Erdős-Hajnal property.

LEMMA For $k \geq 2$ exists $\varepsilon_k>0$ and c_k s.t. every connected n-graph G satisfies one of the following:

- $\exists v \in V(G)$, $d(v) > \varepsilon_k n$,
- $\forall v \in V(G)$, G containing an induced P_k starting at v,
- $\overline{G}$ containing a $K_{c_k n, c_k n}$ (not necess. induced).

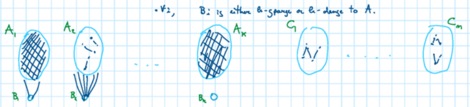
⇝ ① Use Rödl and obtain a weakly ε-restricted set of size δn (wlog it is ε-stable).
② Let S be s.t. $\Delta(G[S]) \leq 4\varepsilon S$ and $|S| \geq \frac{\delta n}{2}$.
③ We want to find c s.t. $\overline{G}$ containing $K_{c\delta n, c\delta n}$.
- If $G[S]$ has many small components we are done,
- Otherwise we can use the lemma.

Rödl's Theorem – Fundamental tool

THM $\forall H, \forall \varepsilon>0\ \exists N$ s.t. every H-free graph can be partitioned into at most N ε-restricted subsets.

LEMMA For every graph H and $\varepsilon, \eta, \theta>0$ there exists N such that if G is H-free, then we can partition $V(G)$ into $A_1, \dots, A_k; B_1, \dots, B_k; C_1, \dots, C_m$ such that:

- $m \leq N$; $k \leq |H|^2$
- $A_1, \dots, A_k$ and $C_1, \dots, C_m$ are ε-restricted,
- $\forall i$, $|B_i| \leq \eta|A_i|$,
- $\forall i$, B_i is either θ-sparse or θ-dense to A.



LEMMA $\forall \varepsilon>0$ let B be ε-sparse to A. $\forall p \in \left[\frac{\log(2|B|)}{\varepsilon}, \frac{|A|}{12}\right]$ there is $P \subseteq A$ of cardinality p s.t. P is 2ε-sparse to B and B is 12ε-sparse to P.

⇝ • Let $Q = \{v \in A : |N(v) \cap B| \leq 2\varepsilon|B|\}$. $|Q| \geq |A|/2$.
- Take $2p$ vertices $\{u_1, \dots, u_{2p}\}$ indep. unif. at random in Q.
- with positive probability $|\{u_1, \dots, u_{2p}\}| \geq p$ and $\forall v \in B$, v does not have $12\varepsilon p$ neigh. in $\{u_1, \dots, u_{2p}\}$.

$\left(\forall v \in B,\quad \varepsilon(v) = \mathbb{E}\left[|\{I \subseteq \{u_1, \dots, u_{2p}\} : |I| \geq 12\varepsilon p \text{ and } N(v) \subseteq I\}|\right] \leq \frac{1}{2|B|}\right).$

Rödl's Theorem – Fundamental tool

THM $\forall H, \forall \varepsilon>0\ \exists N$ s.t. every H-free graph can be partitioned into at most N ε-restricted subsets.

LEMMA For every graph H and $\varepsilon, \eta, \theta>0$ there exists N such that if G is H-free, then we can partition $V(G)$ into $A_1, \dots, A_k; B_1, \dots, B_k; C_1, \dots, C_m$ such that:

- $m \leq N$; $k \leq |H|^2$
- $A_1, \dots, A_k$ and $C_1, \dots, C_m$ are ε-restricted,
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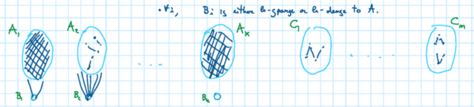
- $\forall i$, B_i is either θ-sparse or θ-dense to A.



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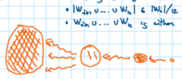
DEF Let G be a graph, $k \in \mathbb{N}$ and $\varepsilon>0$. A (k, ε)-PATH-PARTITION of G is a partition $(W_0, W_1, \dots, W_k)$ such that $\forall i = 0, \dots, k-1$ we have:

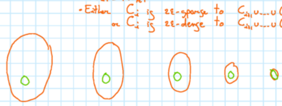
- W_i is ε-restricted
- $|W_{i+1} \cup \dots \cup W_k| \leq |W_i|/12$
- $W_{i+1} \cup \dots \cup W_k$ is either $\frac{\varepsilon}{12}$-sparse or $\frac{\varepsilon}{12}$-dense to W_i.



LEMMA Let $\varepsilon>0$; if G admits a $(\underbrace{\lceil \tfrac{2}{\varepsilon} \rceil}_{=k}, \frac{\varepsilon}{2})$-path-partition then G can be partitioned into $10'000\,\varepsilon^{-2}$ ε-restricted subsets.

⇝ → For $i = 0, \dots, k$ we find $C_i \subseteq W_i$ s.t.

- $|C_i| = |W_k|$
- Either C_i is 2ε-sparse to $C_{i+1} \cup \dots \cup C_k$ and $C_{i+1} \cup \dots \cup C_k$ is 12ε sparse to C_i
  or C_i is 2ε-dense to $C_{i+1} \cup \dots \cup C_k$ and $C_{i+1} \cup \dots \cup C_k$ is 12ε dense to C_i



→ We find C_k, then $C_{k-1}, \dots$ Assume we have $C_{i+1}, \dots, C_k$; then we can use the LEMMA to find C_i since $C_{i+1} \cup \dots \cup C_k$ behaves well to W_i.

→ Select $I \subseteq \{0, \dots, k\}$ s.t. C_i is always ε-sparse or always ε-dense to $C_{i+1} \cup \dots \cup C_k$.
Let $C = \bigcup_{i \in I \cup \{k\}} C_i$. Then C is ε-restricted.

RÖDL'S THEOREM – FUNDAMENTAL TOOL

THM $\forall H, \forall \varepsilon > 0\ \exists N$ s.t. every H-free graph can be partitioned into at most N ε-restricted subsets.

LEMMA For every graph H and $\varepsilon, \eta, \theta > 0$ there exists N such that if G is H-free, then we can partition $V(G)$ into $A_1, \dots, A_k; B_1, \dots, B_k; C_1, \dots, C_m$ such that:

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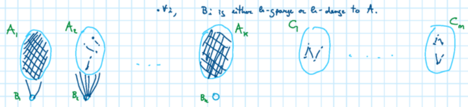

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DEF Let G be a graph, $k \in \mathbb{N}$ and $\varepsilon > 0$. A (k, ε, η)-PATH-PARTITION of G is a partition $(W_0, W_1, \dots, W_k)$ such that $\forall i = 0, \dots, k-1$ we have:

- W_i is ε-restricted
- $|W_{i+1} \cup \dots \cup W_k| \leq \eta |W_i|$
- $W_{i+1} \cup \dots \cup W_k$ is either $\frac{\varepsilon}{12}$-sparse or $\frac{\varepsilon}{12}$ dense to W_i.

LEMMA Let $\varepsilon > 0$ and H be fixed. If G is an H-free graph that admits a $\left(k, \frac{\varepsilon}{|H|^{\lceil k \rceil}}, \frac{1}{24|H|^{\lceil k \rceil}}\right)$- path partition with $k \leq \lceil \frac{3}{\varepsilon} \rceil$, then $V(G)$ can be partitioned into $O(\varepsilon^{-1})$ ε-restricted subsets.