MA102 Mathematical Proof and Analysis
and
MA103 Introduction to Abstract Mathematics

Exercises 4

- Before you start these exercises, make sure you have in front of you the *precise* definitions you need, e.g., exactly what it means for a set to have m elements.

1 Prove that the following statements about functions $f : X \to Y$ and $g : Y \to Z$ are true.

(a) If f and g are injective, then $g \circ f$ is injective.

(b) If f and g are surjective, then $g \circ f$ is surjective.

(c) If f and g are bijective, then $g \circ f$ is bijective.

2 In each of the following cases, find a value for m such that a bijection $f : \mathbb{N}_m \longrightarrow X$ exists, and give a formula for such a bijection:

(a) $X = \{12, 15, 18, 21, 24, 27, 30, 33\}$;

(b) $X = \{x \in \mathbb{Z} \mid -4 \leq x \leq 2\}$;

(c) $X = \{x \in \mathbb{Z} \mid x^2 \leq 26\}$.

3 Let A be a finite set with m elements, for some $m \in \mathbb{N}$. And suppose x is an object that is not a member of A.

Prove, **using the definition of cardinality**, that $A \cup \{x\}$ has $m + 1$ elements.

(You cannot simply say 'A has m elements and so when I add one more I have $m + 1$.' You need to write down a bijection from $\mathbb{N}_{m+1}$ to $A \cup \{x\}$.)

4 Let T be a set of 11 different natural numbers. Show that there are two elements $t_1, t_2 \in T$, $t_1 \neq t_2$, such that $t_2 - t_1$ is divisible by 10.

5 Explain what is wrong with the following proof of the (False) statement:

If S is a relation on a set X that is both symmetric and transitive, then S is an equivalence relation.

Proof: Suppose S is a symmetric, transitive relation on a set X, and let a be any element of X. Now, if aSb, then bSa (since S is symmetric), and so aSa (since S is transitive). Therefore S is reflexive, as well as being symmetric and transitive. So S is an equivalence relation. $\square$