# MA103 - Class 4

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→ GENERAL REMARKS
- Submissions average and meaning of grades
- class structure and participation
  ↳ - format - level of content

**PROBLEM 1** Def Let $X, Y$ be non-empty sets and let $f: X \to Y$ be a function.
→ we say that $f$ is INJECTIVE when $\forall a,b \in X$, $f(a)=f(b) \Rightarrow a=b$,
OR, equivalently when $\forall a,b \in X$, $a \neq b \Rightarrow f(a) \neq f(b)$.
→ we say that $f$ is SURJECTIVE when $\forall a \in Y\ \exists b \in X$ s.t. $f(b)=a$.

⇝ Let us fix $a,b \in X$ such that $(g\circ f)(a) = (g \circ f)(b)$. Recall that, by definition, $\forall c \in X$, $g\circ f(c) = g(f(c))$. We know that $g$ is injective and that $g(f(a)) = g(f(b))$, therefore we have $f(a) = f(b)$. Since $f$ is also injective, we have $a=b$. Because $a,b$ were arbitrary, we proved $\forall a,b \in X$, $g\circ f(a) = g\circ f(b) \Rightarrow a=b$

⇝ Let us fix $z \in Z$. Since $g$ is surjective, there exists $y \in Y$ s.t. $g(y)=z$. Let us fix such a $y$. Since $f$ is surjective, there exists $x \in X$ such that $f(x)=y$. Let us fix such an $x$. Notice that, by definition,

$$g\circ f(x) = g(f(x)) = g(y) = z.$$

Since $z$ was generic, we proved $\forall b \in Z\ \exists a \in X$, $g\circ f(a) = b$.

~~ COMMON MISTAKES ~~

$\forall z \in Z\ \exists y \in Y\ \ g(y)=z$; $\forall y \in Y\ \exists x \in X\ \ f(x)=y$.
Then $g\circ f(x) = g(f(x)) = g(y) = z$.

Quantifiers do NOT give values to the variables.

**COROLLARY** The relation "having the same cardinality" is transitive. It is an equivalence relation.

**GOLDEN PROBLEM** ① Can you show that $|\mathbb{N}| = |\mathbb{Z}| \overset{\text{difficult}}{=} |\mathbb{Z}\times\mathbb{Z}| = |\mathbb{Q}|$ ?
$= |\mathbb{N}\times\mathbb{N}|$ ⤷ easier, maybe

② Can you show $|(0,1)| = |\mathbb{R}|$ ?

③ Do you think $|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{R}|$ ?

PROBLEM 3 ⇝ By hypothesis, $|A| = m$. By definition, this means that there exists $f: \mathbb{N}_m \to A$ bijective. Let

$$g: \mathbb{N}_{m+1} \longrightarrow A \cup \{x\}$$
$$i \longmapsto \begin{cases} f(i) & i \leq m \\ x & i = m+1 \end{cases}$$

Since $\forall a \in \mathbb{N}_{m+1}$, $a \leq m \vee a = m+1$, this is a well-defined function. Let us show it is bijective.

→ $g$ is surjective: Let $b \in A \cup \{x\}$ (which is, $b \in A \vee b \in \{x\}$). If $b \in A$, $\exists\, i \in \mathbb{N}_m$ s.t. $f(i) = b$. For any such $i$, we have $g(i) = f(i) = b$. If $b \in \{x\}$ we have $g(m+1) = x$.

Not used $x \notin A$ yet

→ $g$ is injective: Let $a, b \in \mathbb{N}_{m+1}$, $a \neq b$. Then either:

- $(a = m+1) \wedge (b \leq m)$: then $g(a) = x$, $g(b) \in A$. Since $x \notin A$, we have $g(a) \neq g(b)$.
- $b = m+1 \wedge a \leq m$. As above
- $a, b \leq m$. Then $f(a) \neq f(b)$. Therefore $g(a) \neq g(b)$.

NEW EXERCISE Let $A, B$ be two finite sets with the same cardinality,

- let $f: A \to B$ be injetive, then $f$ is surjective
- let $g: A \to B$ be surjective, then $f$ is injective.

EXTRA POINT: Does this work if $A, B$ are not finite? Say $A = B = \mathbb{N}$.

EXTRA EXERCISE Let $f: \mathbb{R} \to \mathbb{R}$ such that:

- $\exists!\ a \in \mathbb{R}$ s.t. $f(a) = 2$
- $\forall x, y \in \mathbb{R}$, $f(x+y) = f(x) \cdot f(y) - f(x) - f(y) + 2$

⇝ a) find $a$

b) Can we find $b$ s.t. $f(b) = 0$?

c) Assume we know $f(c)$. What is $f(-c)$?

d) Is $f$ injective?