MA210 Discrete Mathematics

Notes 3

13 and 20 February 2023

Introduction to Graph Theory: Walks, Paths, Tours, and Cycles

Basic Definitions

A graph is a mathematical model for any situation where we have entities, any pair of which either is or is not in a certain relation.

For a set S, we use $\binom{S}{2}$ to denote the subsets of S with exactly 2 elements; so $\binom{S}{2} = \{\, A \subseteq S : |A| = 2 \,\}$. Note that this means that $\left|\binom{S}{2}\right| = \binom{|S|}{2}$.

Definition 3.1. *A* graph $G = (V(G), E(G))$ *is a set* $V(G)$ *of* vertices*, together with a set* $E(G)$ *of* edges*, where* $E(G)$ *is a subset of* $\binom{V(G)}{2}$.

An edge consists of two vertices, its *endpoints*. In general, we denote an edge $\{u, v\}$ by uv. If $uv \in E(G)$, then we say that u and v are *adjacent* or *neighbours* (in G).

In this course we always assume that $V(G)$ is finite. This, of course, means that $E(G)$ is also finite. If there is no danger of confusion, we write $G = (V, E)$ instead of $G = (V(G), E(G))$.

If $u \in V(G)$ and $uv \in E(G)$, then we say that the vertex u and the edge uv are *incident*.

If $u \in V(G)$, then the *neighbourhood* of u, denoted by $N_G(u)$ or $N(u)$, is the set of vertices in $V(G)$ adjacent to u: $N(u) = \{v \in V(G) \mid uv \in E(G)\}$. The *degree of* u, denoted by $d_G(u)$ or $d(u)$, is the number of vertices in $V(G)$ adjacent to u. Note that $d(u) = |N(u)|$.

We say that a graph is *k-regular* if every vertex has degree k.

The *degree multiset* of a graph G is the collection of the degrees of all vertices in G, (usually) written in non-increasing order.

Isomorphism of graphs

Two graphs G and H are *identical* if $V(G) = V(H)$ and $E(G) = E(H)$.

Authors: P. Allen, J. van den Heuvel, J. Skokan, R. Kirsch

Definition 3.2 (isomorphism). *Two graphs G and H are* isomorphic *if there exists a bijection $\phi : V(G) \to V(H)$ such that for every $u, v \in V(G)$ we have:*

$$uv \in E(G) \quad \textit{if and only if} \quad \phi(u)\phi(v) \in E(H).$$

Graph isomorphism is easily seen to be an equivalence relation (reflexive, symmetric, transitive) on the collection of all graphs. Hence, the collection of all graphs is partitioned into equivalence classes which we usually call *unlabelled graphs*. We should not worry too much about the formalities here. Officially, an 'unlabelled graph', by this definition, means a big set of graphs with all kinds of different vertex labellings, which are all isomorphic. But we will simply draw a picture of a graph without labelling the vertices; this is what you should think of. The connection is this: given the unlabelled picture, the isomorphism class is the set of all labelled graph we can get by writing labels on the vertices of our picture.

Definition 3.3 (complete graph, clique). *The* complete graph K_n *on n vertices is a graph with n vertices in which every two vertices are adjacent.*

If a subgraph H of a graph G is a complete graph (i.e. all the vertices of H are also vertices of G, and all the edges of H are also edges of G), we often say that H is a clique *in G. We also say that $X \subseteq V(G)$ is a clique in G if every pair of vertices in X is an edge of G.*

Walks, paths, tours, cycles

Definition 3.4 (path, cycle). *A* path *is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in that ordering. We use P_n to denote an (unlabelled) path on n vertices. In other words, P_n represents all graphs isomorphic to the labelled graph on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$ and edge set $\{v_1v_2,\ v_2v_3,\ \ldots,\ v_{n-1}v_n\}$.*

In a similar fashion, we have: a cycle *is a graph on at least 3 vertices whose vertices can be ordered into a circle so that two vertices are adjacent if and only if they are consecutive in that circular ordering. We use C_n to denote an (unlabelled) cycle on n vertices. So, C_n represents all graphs isomorphic to the labelled graph on n vertices with vertex set $\{v_1, v_2, \ldots, v_n\}$ (where $n \geq 3$) and edge set $\{v_1v_2,\ v_2v_3,\ \ldots,\ v_{n-1}v_n,\ v_nv_1\}$.*

Definition 3.5 (walk, trail). *A* walk *in a graph G is a sequence of vertices $v_1, v_2, \ldots, v_m$ such that v_i is adjacent to v_{i+1} for all $i = 1, 2, \ldots, m-1$. The* length *of the walk is equal to $m - 1$.*

If the edges v_iv_{i+1}, $i = 1, \ldots, m-1$, of a walk are all distinct, then we talk about a trail.

Definition 3.6 (closed, tour). *A walk in a graph is* closed *if it has positive length and the first and the last vertex are the same.*

We define a closed trail *in a similar way: it is a trail on at least 3 vertices in which the first and the last vertex are the same. A closed trail is also called a* tour.

Finally, if, in addition, all the vertices of a tour, except the first and the last, are distinct, then we have a *cycle* again.

Closed walks, tours and cycles are often denoted by $v_1, v_2, \ldots, v_m, v_1$. The *length* of such a closed walk, tour or cycle is equal to m.

Basic graph results

Lemma 3.7 (Hand-shaking Lemma)**.** *For any finite graph G, we have*

$$\sum_{u \in V(G)} d(u) \;=\; 2\big|E(G)\big|\,.$$

Proposition 3.8. *In any finite graph, the number of vertices of odd degree is even.*

Example 3.9. How many edges does a k-regular graph on n vertices have?

There are many ways in which we can represent a graph G.

Definition 3.10 (subgraph)**.** *A graph H is a* subgraph *of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*

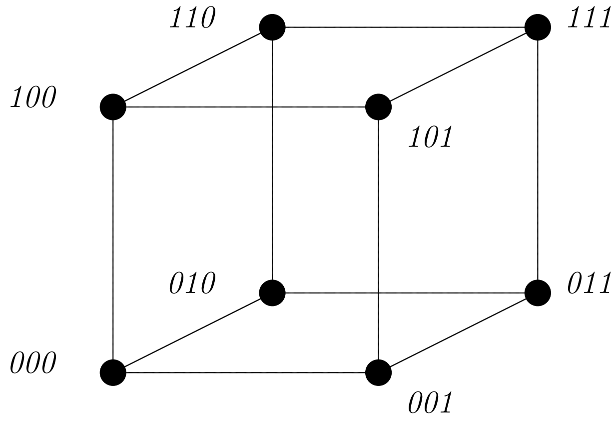
Slightly abusing notation, we often say 'H is a subgraph of G', or 'G contains H' when we mean 'H is isomorphic to a subgraph of G'.

If P_n is isomorphic to a subgraph of G, that means there are distinct vertices $v_1, \ldots, v_n$ of G such that $v_i v_{i+1}$ is an edge of G for each $1 \le i \le n-1$. We will often write, informally, "Let $v_1, \ldots, v_n$ be a path in G".

A cycle with m vertices (and so m edges too) has *length* m. Unfortunately some authors think the length of a path is the number of edges it contains, and others think it is the number of vertices. I won't talk about 'the length of a path' in this course; I'll stick to 'an n-vertex path' or similar. Furthermore, some authors may think P_2 is a 2-edge path (which for us is P_3). Be careful if you read textbooks!

A path has two *endpoints*, namely the first and last vertices in the path; the only ones whose degree is less than 2. We will often talk about 'a path from u to v' meaning a path whose endpoints are u and v.

Definition 3.11. *For $n \ge 1$, we define the* hypercube Q_n *as follows: $V(Q_n)$ is the set of all $0,1$-sequences of length n. Two sequences are adjacent if they differ in exactly one position.*



The hypercube Q_3

Example 3.12. Let H be some fixed graph. Suppose that G_1 and G_2 are two graphs such that G_1 contains H and G_2 does not contain H. Explain why G_1 and G_2 are not isomorphic.

Connectivity

Definition 3.13 (connected)**.** *We say that vertices u, v are* connected *if there is a walk $v_1, v_2, \ldots, v_m$ such that $v_1 = u$ and $v_m = v$.*

We say that a graph G is connected *if for every pair of vertices u, v in G, the vertices u, v are connected.*

Example 3.14. If u and v are connected in a graph G, then there is a path in G whose endpoints are u and v.

Example 3.15. If $n \geq 1$, determine whether or not the hypercube Q_n is connected.

Definition 3.16 (component)**.** *A* component *of a graph G is a maximal connected subgraph.*

In other words, a subgraph H of G is a *component* if (a) H is connected, and (b) if H' is a connected subgraph of G such that H is a subgraph of H', then we must have $H = H'$.

Hamilton cycles, Euler tours

Definition 3.17 (Hamilton cycle)**.** *A* Hamilton cycle *is a cycle in which every vertex of a graph appears exactly once. A graph which contains a Hamilton cycle is called a* Hamiltonian graph*.*

Definition 3.18 (Euler trail, Euler tour)**.** *An* Euler trail *in a graph G is a trail in which every edge of G appears exactly once.*

An Euler tour *is a closed Euler trail (i.e. a a trail in which every edge of G appears exactly once and in which the first and last vertex are the same). A graph G is* Eulerian *if G contains an Euler tour.*

Theorem 3.19 (Euler, 1736; Hierholzer 1873). *Let G be a graph without vertices of degree zero.*

(a) Then G has an Euler tour if and only if G is connected and every vertex of G has even degree.

(b) And G has an Euler trail if and only if G is connected and there are at most two vertices with odd degree.

Example 3.20. For which $n \geq 1$ is the hypercube Q_n Eulerian?

Bipartite graphs

Definition 3.21. *A graph G is* bipartite *if we can write $V(G) = A \cup B$ with $A \cap B = \varnothing$, and all edges of G have one endpoint in A and the other endpoint in B.*

Theorem 3.22. *A graph G is bipartite if and only if G contains no odd cycles.*

Additional reading and exercises

Biggs, *Discrete Mathematics*

Reading: Sections 15.1 – 15.4; 17.1.

Exercises: Section 15.1: 1 – 4; Section 15.2: 1 – 3; Section 15.3: 1 – 5; Section 15.4: 1 – 5; Section 15.8: 1 – 10, 12 – 14, 16 – 19, 21; Section 17.1: 1 – 3.

Cameron, *Combinatorics*

Reading: Sections 11.1, 11.4, 11.5.

Exercises: Section 11.13: 1, 2, 6, 7.

Exercises

1. Let $V = \{1, 2, \ldots, n\}$. How many different graphs with vertex set V are there?

2. Suppose that we have a graph with at least two vertices. Show that it is not possible that all vertices have different degrees.

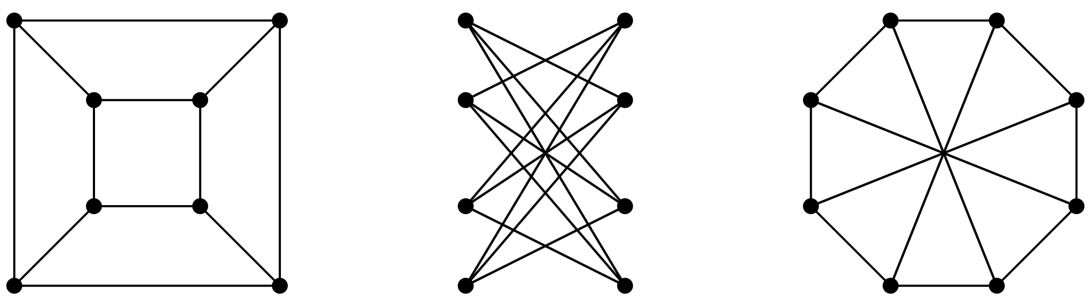
3. There are four married couples at a party. Various people shake hands, but of course no one shakes hands with his/her own wife or husband. At the end of the party, the host asks everybody else how many hands they shook and he receives seven different answers.

How many hands did the wife of the host shake?

4. How many *non-isomorphic* graphs with four vertices are there?

(Hint: the answer is not the same as the answer in Question 1 for $n = 4$.)

5. Determine which pairs of graphs below are isomorphic. Justify your answer!



6. (a) Prove that two isomorphic graphs must have the same degree sequence.

(b) Is it true that every two graphs with the same degree sequence are isomorphic? Justify your answer!

7. How many non-isomorphic 3-regular graphs with 6 vertices are there?

And how many with 7 vertices?

8. Prove the following statements.

(a) If there is a walk between two vertices x and y in some graph G, then there is also a path between x and y in G.

(b) If G has a walk between vertices x and y and a walk between vertices y and z, then G also has a walk between x and z.

(c) If G has a path between vertices x and y and a path between vertices y and z, then G also has a path between x and z.

9. Let G be a graph and define the relation R on $V(G)$ as follows: for $u, v \in V(G)$ we have uRv if and only if there is a walk $v_1, v_2, \ldots, v_m$ such that $v_1 = u$ and $v_m = v$.

(a) Show that R is an equivalence relation on $V(G)$.

(b) Show that the equivalence classes of R are exactly the vertex sets of the components of G.

10. Suppose that G is a connected graph and C is a cycle in G. Let e be an edge of the cycle C.

Prove that if we remove e from G, then the resulting graph is still connected.

11. Let G be a graph in which every vertex has even degree, and let v be a vertex of degree at least two in G.

(a) Prove that G contains a tour such that v is on the tour.

(b) Prove that G contains a cycle such that v is on the cycle.

12. Prove that every closed walk with an odd number of edges contains an odd cycle.

13. For natural numbers m, n, the *complete bipartite graph* $K_{m,n}$ is defined by taking two disjoint sets, V_1 of size m and V_2 of size n, and putting an edge between u and v whenever $u \in V_1$ and $v \in V_2$.

(a) How many edges does $K_{m,n}$ have?

(b) What is the degree sequence of $K_{m,n}$?

(c) For what $m, n \in \mathbb{N}$ is $K_{m,n}$ connected?

(d) For what $m, n \in \mathbb{N}$ is $K_{m,n}$ Eulerian?

(e) For what $m, n \in \mathbb{N}$ does $K_{m,n}$ have a Hamilton cycle?

Make sure you justify all your answers!

14. The definition of the hypercube Q_n, $n \in \mathbb{N}$, is given in Definition 3.11.

(a) How many edges does Q_n have?

(b) What is the degree sequence of Q_n?

(c) For what $n \in \mathbb{N}$ is Q_n bipartite?

(d) For what $n \in \mathbb{N}$ is Q_n connected?

(e) For what $n \in \mathbb{N}$ is Q_n Eulerian?

(f) For what $n \in \mathbb{N}$ does Q_n have a Hamilton cycle?

Justify all your answers!