MA210 - Class 1

INTRODUCTION

① ASSIGNMENTS: • best preparation for the exam

• performance in assignments correlated to performance in exams

• OBJECTIVES: • clarity: solutions should be self-explanatory
• content: correct use of the theory

② GRADESCOPE: • Please carefully select the correct pages in your gradescope submissions.

REVIEW of THEORY

① In definition 1.1 it says that if there exists an injection $f: X \to Y$ and a surjection $g: X \to Y$ then there exists a bijection $h: X \to Y$. THIS SHOLD NOT BE OBVIOUS. And it is known as CANTOR-SCHRÖDER-BERNSTEIN theorem.

② PROP: If X and Y are disjoint, $|X \cup Y| = |X| + |Y|$.

This is a special case of:

THM: Let $A_1, A_2, \dots, A_n$ be sets, then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\substack{X \subseteq [n] \\ X \neq \emptyset}} (-1)^{|X|-1} \left|\bigcap_{y \in X} A_y\right|$$

③ PROP: $|A_1 \times A_2 \times \dots \times A_n| = \prod_{i=1}^{n} |A_i|$

NEW EXERCISES

① How many ways are there to assign 5 distinguishable fruits to 8 distinguishable children in such a way that every child gets at most 2 fruits?

$\rightsquigarrow$ Let $\mathcal{F} = \{\text{fruits}\}$ $\mathcal{C} = \{\text{children}\}$. We are counting the functions $f : \mathcal{F} \to C$ such that $\forall z \in \mathcal{C}$, $|f^{-1}(C)| \leq 2$.

Let $G = \{f : \mathcal{F} \to \mathcal{C} : \forall z \in \mathcal{C}, |f^{-1}(z)| \leq 2\}$.

① Since the negation of $\forall z \in \mathcal{C}, |f^{-1}(z)| \leq 2$ is $\exists z \in \mathcal{C}, |f^{-1}(z)| > 2$

we have:

$$|G| = \underbrace{|\{f : \mathcal{F} \to C\}|}_{|\mathcal{C}|^{|\mathcal{F}|} = 8^5} - \underbrace{|\{f : \mathcal{F} \to \mathcal{C} : \exists z \in C \ |f^{-1}(z)| > 2\}|}_{\text{Call this } D}$$

② Compute $|D|$. Notice that $\forall f \in D$ there is EXACTLY one element $z \in \mathcal{C}$ s.t. $|f^{-1}(z)| > 2$. Indeed, we have $|\mathcal{F}| = 5$ and the preimage sets are all disjoint.

For any $f \in D$ we are in exactly one of the following:

CASE 1: $\exists z \in \mathcal{C}$, $|f^{-1}(z)| = 5$. (which is, $f^{-1}(z) = \mathcal{F}$)

There are 8 functions that satisfy this case: each of these functions is uniquely identified by the image $f(\mathcal{F})$.

Case 2: $\exists z \in C$, $|f^{-1}(z)| = 4$.

There are $\underbrace{8}_{\substack{\text{choice of the} \\ \text{lucky child}}} \cdot \underbrace{\binom{5}{4}}_{\substack{\text{choice of} \\ \text{which fruits to} \\ \text{give to the lucky} \\ \text{child}}} \cdot \underbrace{7}_{\substack{\text{choices for the} \\ \text{remaining fruit}}}$ choices in this case

case 3: $\exists z \in C$, $|f^{-1}(z)| = 3$. There are

$$\underbrace{8}_{\text{choice of the lucky child}} \cdot \underbrace{\binom{5}{3}}_{\text{choice of which fruits to give to the lucky child}} \cdot \underbrace{7^2}_{\text{choices for the remaining fruits}} \quad \text{choices in this case}$$

③ We have $|G| = |\{f : F \to C\}| - |D|$

$$= 8^5 - 8\left(1 + 7\binom{5}{4} + 7^2\binom{5}{3}\right)$$

② Understand the reason why

$$\sum_{k=0}^{m} 2^k \binom{m}{k} = 3^m$$

mod $3^m = \{1,2,3\}^m$.

Suppose we want to count $\{1,2,3\}^m$ in a less traditional way.

$$\{1,2,3\}^m = \bigcup_{j=0}^{m} \{x \in \{1,2,3\}^m \mid x \text{ contains } j \text{ copies of the digit } 3\}.$$

Because the RHS sets are disjoint,

$$|\{1,2,3\}^m| = \sum_{j=0}^{m} |\{x \in \{1,2,3\}^m \mid x \text{ contains } j \text{ copies of } \text{`}3\text{'}\}|$$

How many $x \in \{1,2,3\}^m$ have exactly j copies of 3?

$$\underbrace{\binom{m}{j}}_{\text{possible choices for the positions of the "3"s}} \underbrace{2^{m-j}}_{\text{possible choices for the remaining } m-j \text{ digits.}}$$

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This gives us

$$3^n = |\{1,2,3\}^n| = \sum_{j=0}^{n} |\{x \in \{1,2,3\}^n \mid x \text{ contains } j \text{ copies of '3'}\}|$$

$$= \sum_{j=0}^{n} \binom{n}{j} 2^{n-j} = \sum_{j=0}^{n} \binom{n}{n-j} 2^{n-j} = \sum_{k=0}^{n} \binom{n}{k} 2^{k}$$