Euclid’s Theorem

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Abstract

We present in this short note a proof of Euclid’s Theorem.

1 Introduction

One of the most fundamental concepts in Mathematics is the concept of prime number. In his seminal work [1], Euclid proved that there are infinitely such prime numbers. Another cool result can be found in [2]. The goal of this work is to present a simple proof of this important result.

2 Definitions and statement

We start with the definition of prime number.

Definition 2.1. Let $p \in \mathbb{N}$ be a positive integer larger or equal to 2. We say that p is a prime number if the only positive integer divisors of p are 1 and p. More formally, we define the set $\mathcal{P}$ of prime numbers as:

$$\mathcal{P} = \{p \in \mathbb{N} : p \geq 2 \wedge \forall d \in \mathbb{N}, d|p \implies d \in \{1, p\}\}.$$

We are now ready to introduce the main statement of this note.

Theorem 2.2 (Euclid’s Theorem)**.** *The set of prime number is infinite.*

3 Proof of the main result

In order to prove our main result, we first need to state an important lemma, which we will assume here for sake of brevity.

Lemma 3.1. *All positive integers are either prime, or divided by a prime. Which is to say* $\forall n \in \mathbb{N},\ n \in \mathcal{P} \vee \exists p \in \mathcal{P}$ *s.t.* $p|n$.

We are now ready to prove our main result: Theorem 2.2 as follows.

Proof of Theorem 2.2. Assume for sake of contradiction that the set $\mathcal{P}$ has finite cardinality. In particular, for some $n \in \mathbb{N}$ we can write $\mathcal{P} = \{p_1, \ldots, p_n\}$. Consider now the positive integer $p = 1 + \prod_{i=1}^{n} p_i$. By Lemma 3.1, either p is prime, which is absurd because $p \notin \mathcal{P}$ (p is strictly larger than any element in $\mathcal{P}$), or p has a prime divisor, which is also absurd because all prime numbers divide $p - 1$, and therefore cannot divide p. $\square$

Remark. Therefore there are infinite prime numbers, for more details, see Definition 2.1.

References

[1] Euclid. *The Elements of Euclid.* Ed. by Sir Thomas L. Heath. 2nd. New York: Dover Publications Inc., 1956. URL: `https://archive.org/details/euclid_heath_2nd_ed`.

[2] Eng Keat Hng and Domenico Mergoni Cecchelli. "Density of small diameter subgraphs in K_r-free graphs". In: *arXiv preprint arXiv:2207.14297* (2022).