The Johansson-Molloy Theorem

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arxiv.org/abs/2109.15215

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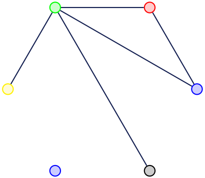
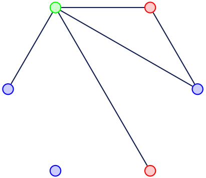
arxiv.org/abs/2111.06214

UCL-LSE Reading group

Graph colouring

Let G be a graph, a colouring of G is a way of assigning a label (colour) to each vertex of G in such a way that the same label is never assigned to adjacent vertices.

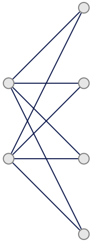




The chromatic number of G (denoted with $\chi(G)$) is the number of colours required to colour G.

List colouring

Let G be a graph, a list for G is a way of assigning for each vertex of G a set of possible colours to use in the colouring.



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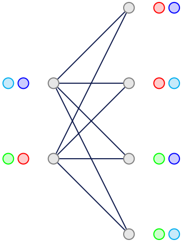

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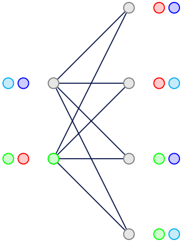

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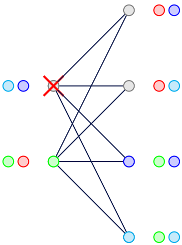

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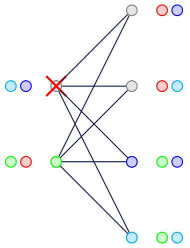

The list chromatic number of G (denoted with $\chi_\ell(G)$) is the minimum k such that any list in which each vertex gets at least k element allows an L-proper colouring. It holds $\chi(G) \leq \chi_\ell(G) \leq \Delta(G) + 1$.

The problem at hand

Thm (Brooks, 1941)

If G is a connected graph, not an odd cycle or a clique, $\chi(G) \leq \Delta(G)$.

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Today we examine what happens when we remove/reduce the number of triangles in the graph.

Why is this interesting?

If the pentagon conjecture is true, then for any 3-regular graph of high girth G we have $\chi(G^3) \leq 5$.

Original result and history

Theorem (Johansson, 1996)

$$\lim_{\Delta \to \infty} \max \left\{ \chi(G) : \Delta(G) = \Delta \text{ and } G \text{ is } \triangle - \text{free} \right\} = O\left(\frac{\Delta}{\ln(\Delta)} \right).$$

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Theorem (Molloy, 2019)

$$\forall\epsilon > 0\ \exists\Delta_\epsilon \in \mathbb{N}\ \forall\triangle\text{−free } G,\ \Delta(G) \geq \Delta_\epsilon \implies \chi(G) \leq (1+\epsilon)\frac{\Delta(G)}{\ln(\Delta(G))}.$$

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Theorem (Bernshteyn, 2019)

Same result, but using Lovász Local Lemma instead of entropy compression.

Results of the papers

Theorem (Hurley & Pirot, 2021+; Martinsson, 2021+)

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Theorem (Hurley & Pirot, 2021+)

Assume $\forall v \in V(G),\ \bar{d}(G[N(v)]) = d \leq \frac{\Delta}{6} - 1$ and L a list such that

$$\forall v \in V(G), \quad |L(v)| \geq \left(1 + \frac{2}{\ln\left(\frac{\Delta(G)}{d+1}\right)}\right) \frac{\deg(v)}{W\left(\frac{\deg(v)}{(d+1)\ln^3\left(\frac{\Delta(G)}{d+1}\right)}\right)}.$$

Then there are at least $\left((d+1)\ln^3\left(\frac{\Delta(G)}{d+1}\right)\right)^{|G|}$ proper L-colourings of G.

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Coro (Hurley & Pirot, 2021+)

Let $\Delta(G) \leq \Delta$ and $\forall v \in V,\ e(G[N(v)]) \leq \frac{\Delta^2}{f}$. Then

$$\chi_\ell(G) \leq (1 + o(1)) \frac{\Delta}{\ln(\min\{\Delta, f\})}.$$

Quick proof of Johansson-Molloy Theorem

Theorem

$$\forall \epsilon > 0\ \exists \Delta_\epsilon \in \mathbb{N}\ \forall \triangle - \text{free } G,\ \Delta(G) \geq \Delta_\epsilon \implies \chi(G) \leq (1+\epsilon)\frac{\Delta(G)}{\ln(\Delta(G))}.$$

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Let G be a $\triangle$-free graph with $\Delta := \Delta(G) \geq \Delta_\epsilon$ TBA. Let $k := \left\lceil \frac{(1+\epsilon)\Delta}{\ln \Delta} \right\rceil$ and for any H, let $\mathscr{C}(H)$ be the set of proper k-colourings of H. Let $\ell := \ln^2(\Delta)$.

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Claim

ISTS that for any $H \subseteq G$ induced, $\forall\ v \in V(H)$ we have $\frac{|\mathscr{C}(H)|}{|\mathscr{C}(H \setminus v)|} \geq \ell$.

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Proof: By induction. The case $H = \{v\}$ being trivial.

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   If $1 \ll t \ll \frac{\ell}{\ln(\Delta)}$ Markov's gives $|\{u \in N(v) : |L_c(u) \leq t|\}| \leq o(k)$ whp.

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3. $|\{u \in N(v) : |L_c(u) \leq t|\}| \leq o(k)$ whp.
4. $L_c(u)$ is determined by $c|_{H\setminus\{v,N(v)\}}$ since H is $\triangle$-free.

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3. $|\{u \in N(v) : |L_c(u) \leq t|\}| \leq o(k)$ whp.
4. $L_c(u)$ is determined by $c\big|_{H\backslash\{v,N(v)\}}$ since H is $\triangle$-free.
5. Conditioning of $c\big|_{H\backslash\{v,N(v)\}} = c_0$ each $u \in N(v)$ is coloured indep.

Lemma

Let $L_1, \ldots, L_d \subseteq [k]$ for $d \leq \Delta$, s.t. all but $o(k)$ of the L_i have $|L_i| > t$. Let $X_i \in_u L_i$ be taken independently. Let $X := [k] \setminus \{X_1, \ldots, X_d\}$, then whp $|X| \gg \ell$.

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Proof: Let $B := \{i : |L_i| \leq t\}$. Fix $\{X_i\}_{i \in B}$ and condition on this event.

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Proof: Let $B := \{i : |L_i| \leq t\}$. Fix $\{X_i\}_{i \in B}$ and condition on this event.

$$\mathbb{E}_{\mathscr{C}(H \setminus v) | \{X_i\}_{i \in B}} [|X|] \overset{7}{=} \sum_{j \in [k] \setminus B} \prod_{\substack{L_i \ni j \\ |L_i| > t}} \left(1 - \frac{1}{|L_i|}\right)$$

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$$\geq (k - |B|) \left(\prod_{j \in [k]\setminus B} \prod_{\substack{L_i \ni j \\ |L_i| > t}} \left(1 - \frac{1}{|L_i|}\right)\right)^{\frac{1}{k-|B|}}$$

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<u>*Proof*</u>: Let $B := \{i : |L_i| \leq t\}$. Fix $\{X_i\}_{i \in B}$ and condition on this event.

$$\begin{aligned}
\mathbb{E}_{\mathscr{C}(H\setminus v)|\{X_i\}_{i\in B}}[|X|] &\geq (k - |B|)\left(\prod_{j\in[k]\setminus B}\ \prod_{\substack{L_i\ni j\\ |L_i|>t}}\left(1-\frac{1}{|L_i|}\right)\right)^{\frac{1}{k-|B|}}\\
&\geq \prod_{i:|L_i|>t}\ \prod_{j\in L_i\setminus B}\left(1-\frac{1}{|L_i|}\right)
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&\geq \prod_{i : |L_i| > t} \prod_{j \in L_i \setminus B} \left(1 - \frac{1}{|L_i|}\right) \\
&\geq ((1 - \tfrac{1}{t})^t)^{\Delta}
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$$\begin{aligned}\mathbb{E}_{\mathscr{C}(H \setminus v) | \{X_i\}_{i \in B}} [|X|] &\geq (k - |B|) \left(\prod_{j \in [k] \setminus B} \prod_{\substack{L_i \ni j \\ |L_i| > t}} \left(1 - \frac{1}{|L_i|}\right)\right)^{\frac{1}{k-|B|}} \\ &\geq ((1 - \tfrac{1}{t})^t)^{\Delta} \\ &\geq \exp\left(-\left(1 + O\left(\tfrac{1}{t}\right)\right)\Delta\right)\end{aligned}$$

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Lemma

Let $L_1, \dots, L_d \subseteq [k]$ for $d \le \Delta$, s.t. all but $o(k)$ of the L_i have $|L_i| > t$. Let $X_i \in_u L_i$ be taken independently. Let $X := [k] \setminus \{X_1, \dots, X_d\}$, then whp $|X| \gg \ell$.

Proof: Let $B := \{i : |L_i| \le t\}$. Fix $\{X_i\}_{i \in B}$ and condition on this event.

$$\mathbb{E}_{\mathscr{C}(H \setminus v) | \{X_i\}_{i \in B}} [|X|] \ge (k - |B|) \left(\prod_{j \in [k] \setminus B} \prod_{\substack{L_i \ni j \\ |L_i| > t}} \left(1 - \frac{1}{|L_i|} \right) \right)^{\frac{1}{k - |B|}}$$

$$\ge (k - |B|) \exp \left(- \frac{(1 + o(1))\Delta}{k - |B|} \right)$$

Quick proof of Johansson-Molloy Theorem

Lemma

Let $L_1, \ldots, L_d \subseteq [k]$ for $d \leq \Delta$, s.t. all but $o(k)$ of the L_i have $|L_i| > t$. Let $X_i \in_u L_i$ be taken independently. Let $X := [k] \setminus \{X_1, \ldots, X_d\}$, then whp $|X| \gg \ell$.

Proof: Let $B := \{i : |L_i| \leq t\}$. Fix $\{X_i\}_{i \in B}$ and condition on this event.

$$\begin{aligned}
\mathbb{E}_{\mathscr{C}(H\setminus v)|\{X_i\}_{i\in B}}[|X|] &\geq (k - |B|)\exp\left(-\frac{(1+o(1))\Delta}{k-|B|}\right) \\
&\geq \Theta\left(\frac{\Delta}{\ln \Delta}\right)\exp\left(-\frac{1+o(1)}{(1+\epsilon)}\ln\Delta\right)
\end{aligned}$$

Quick proof of Johansson-Molloy Theorem

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Let $L_1, \ldots, L_d \subseteq [k]$ for $d \leq \Delta$, s.t. all but $o(k)$ of the L_i have $|L_i| > t$. Let $X_i \in_u L_i$ be taken independently. Let $X := [k] \setminus \{X_1, \ldots, X_d\}$, then whp $|X| \gg \ell$.

<u>*Proof*</u>: Let $B := \{i : |L_i| \leq t\}$. Fix $\{X_i\}_{i \in B}$ and condition on this event.

$$\begin{aligned}\mathbb{E}_{\mathscr{C}(H \setminus v) | \{X_i\}_{i \in B}}[|X|] &\geq \Theta\left(\frac{\Delta}{\ln \Delta}\right) \exp\left(-\frac{1+o(1)}{(1+\epsilon)} \ln \Delta\right) \\ &\geq \Delta^{\frac{\epsilon}{(1+\epsilon)} - o(1)}\end{aligned}$$

Quick proof of Johansson-Molloy Theorem

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Let $L_1, \ldots, L_d \subseteq [k]$ for $d \leq \Delta$, s.t. all but $o(k)$ of the L_i have $|L_i| > t$. Let $X_i \in_u L_i$ be taken independently. Let $X := [k] \setminus \{X_1, \ldots, X_d\}$, then whp $|X| \gg \ell$.

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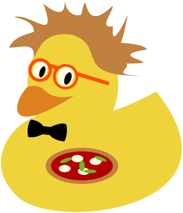
Analysis of the extension

Theorem (Hurley & Pirot, 2021+)

Assume $\forall v \in V(G),\ \bar{d}(G[N(v)]) = d \leq \frac{\Delta}{6} - 1$ and L a list such that

$$\forall v \in V(G), \quad |L(v)| \geq \left(1 + \frac{2}{\ln\left(\frac{\Delta(G)}{d+1}\right)}\right) \frac{\deg(v)}{W\left(\frac{\deg(v)}{(d+1)\ln^3\left(\frac{\Delta(G)}{d+1}\right)}\right)}.$$

Then there are at least $\left((d+1)\ln^3\left(\frac{\Delta(G)}{d+1}\right)\right)^{|G|}$ proper L-colourings of G.




Thank
You!!