A beginner's guide for the formal mathematician

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Introduction

These notes are written as an introductory and accessible guide to formal mathematical writing. In order to understand the content here presented, only patience and a certain resistance to boring text is required. In particular, the only mathematical background required is the common level of knowledge that every high-school student graduate has. This guide was created having in mind first year bachelor students of LSE attending the courses MA102/MA103.

The reason for this guide is to have an accessible standard reference for mistakes commonly seen in assignments of said courses. The need for such a reference was clear to me when I noticed that students often asked me variations of similar questions through the year. During exercise classes, there is often not enough time to stress out the importance of certain subtleties; however, sometimes not understanding these nuisances brings forward further and bigger mistakes. In this guide, I am taking all the time I need to explain why some seemingly small mistakes are indeed important, and how to correct them.

Finally, formalism in mathematics is not only a matter of taste, but it is also relevant because it helps both the reader and the writer to spot mistakes. It is easy to miss a conceptual mistake when the writing is sloppy, while when the text is clear the mistakes have no place to hide. Moreover, a clear and formal text immediately gives a good impression to the reader, and this is important in mathematics as much as everywhere else.

Section A - General etiquette and remarks

In this section we present some general rules about writing mathematics. These rules are not about the mathematical content expressed, and should be interpreted as advice about the form of the text.

Subsection A1 - The Reader

Whenever we are writing, we are writing for someone. This someone might be someone specific (when we write an email or a letter) or someone idealisation of a public. As an example of the latter, I have no idea of who (if anyone) is going to read these notes; however, I have a fairly precise model in mind for the kind of person that might read this text (their knowledge, their objectives etc.).

It is often the case that TAs receive solutions to assignments that are written either having in mind the wrong reader, or no reader at all. Having clear in mind who The Reader of our text is, is the easiest way of improving our mathematical writing. Now try and think who The Reader of your assignments should be.

A first obvious answer is that The Reader is the TA that corrects the assignments. It often happens that students write their solutions as if they were intended for and meant to be read by TAs. However, TAs do not need to read solutions to problems for which they already know the answer, and therefore they should not be the intended public.

Assignments are for the sole benefit of the student, and their objective is to improve the ability of the student to write mathematics. Therefore The Reader should be someone such that writing to The Reader is the best way to develop these abilities in the writer.

Let us be more clear. When you are a BSc student writing the solutions to an assignment, you should think of The Reader as a colleague that is following the same course as you but is having some issues understanding the concepts and CANNOT ask questions. Try to imagine the situation: you are answering a question for an assignment of MA103. If you were writing for the TA, the only obvious answer would be "don't you know how to do this yourself?". However, if you imagine that the same question was asked by a friend of yours that is struggling with a proof, you are more prone to give details and clear steps. Of course in most situations this friend would be able to ask you questions. Since The Reader cannot do that, you have to make an effort to imagine and answer possible questions — in particular, your steps need to be small and clear enough that The Reader does not need to ask 'why is that true?' at any time.

A more detached approach to the same idea is the following. When you are done writing the answers to an assignment, a colleague of yours following the same courses as you should be able to understand everything without asking you any questions.

If you indulge this fantasy of The Reader, it will come to you naturally to write sentences like “recall that the definition of continuous function is [...]” or “now let us reference the Theorem [...] that we saw in the lectures”. Whenever you write a sentence like that in an exam or an assignment, you are making a TA slightly happier.

Finally, you can assume that this friend of yours (The Reader), attended all lectures and has access to the lecture notes, and of course is also a smart person. However, you should imagine that The Reader is also somewhat lazy or prone to misunderstand things, and therefore you have to explain and point out all details.

For future courses (or if you are not a first year student) it is standard practice to imagine that The Reader also followed all the courses that you can expect them to have followed. For example, if you are a graduate student in a mathematical course, you can assume that they followed and understood all the concept of Real Analysis, but maybe not all the theorems of a course in Differential Geometry. Unless you are writing the solutions for an assignment of an advanced course in Differential Geometry (but then why are you reading these notes?).

An implication of this is the good practice of never render trivial an argument. You like The Reader, they are your friend, and you don’t want to make them feel slow in understanding. In particular, you should try to avoid words such as “obviously”, “we can easily see”, “clearly” or similar. This is really a suggestion more than a correction, but it is never pleasant to see “obviously” before a statement that is false or incredibly difficult to prove.

Subsection A2 - The Examiner

If you are doing what we suggested above when you write the solutions to an assignment, the actual reader of your text is going to be quite different from the model of your Reader. Let us call the person that will read your text The Examiner. As you can imagine, our suggestion is that when you are writing your solutions you should not worry about The Examiner, however, it is important to say something about them.

First and foremost: you cannot fool The Examiner, and there is no point in trying. Assignments are done with the explicit purpose of preparing you to the exam, and therefore there is no point in even trying to fool The Examiner. While it is possible that a wrong answer is missed during the correction of an assignment, this will not happen during an exam. It is better to train in the same conditions as the final race.

A second important point is that The Examiner wants you to succeed, and is there to help you. As an example, if you have no idea on how to solve a problem, just say it (or don’t write anything). And if you have some ideas on how to solve a problem, you should try and give an explanation of what your idea is, instead of trying to make it pass as a solution (this also makes sense if you are thinking about The Reader). For example, it is perfectly normal in an assignment to write “My strategy for this problem is to prove XYZ, I was only able to prove X and Y, and I think that Z can be proved by induction; however, I was not able to fill out the details of Z”.

Let us now give an example of a very common way in which students try to fool their Examiner: rewriting the statement as a proof.

Exercise. Show that a natural number n is even if and only if n^2 is even.

Proof. Let us take any natural number n, then we can easily see that there exists a natural number m such that $n = 2m$ if and only if there exists a natural number k such that $n^2 = 2k$. □

Proof. Not totally sure how to prove this claim. The definitions are as follows. We say that a natural number n is even if there exists a natural number k such that $n = 2k$.

I tried to argue by contradiction, but without success; you can find what I tried in the next page. □

Neither of these answers is a solution. However, only the second one looks like an honest attempt to solve the problem.

The risk with indulging with proofs like the first one now is not to be able to recognise what a proper proof is and then miss an important step during an exam.

In the exam, The Examiner will assign more marks to the attempt on the right than on the left. The student on the right makes it clear what they understand and that it is not sufficient, while the student on the left seems not to understand what they are doing well enough to see that they have not proved anything.

Subsection A3 - Use of Symbols

There are two components of a mathematical text: explanatory parts and formulae. The explanatory parts should be written in plain English, while the use of some mathematical symbols should be limited to formulae. Not all symbols are the same, and this is not a strict rule, but it helps to limit the uncontrolled use of symbols in the explanatory parts. Let us consider the following example.

Proposition. *For every natural number $n \geq 4$, it holds that $n! > 2^n$.*

Proof. We want to prove this proposition by induction. For a natural number n, let $P(n)$ be the statement $n! > 2^n$. By the axiom of induction, in order to prove the proposition, it suffices to show the following:

i) The statement $P(4)$ holds,

ii) For every natural number $k \geq 4$, we have $P(k)$ implies $P(k+1)$

We can quickly calculate that $4! = 4\cdot 3\cdot 2\cdot 1 = 24 > 16 = 2^4$ and thus proving the base of this induction. Let us now consider an arbitrary natural number $k \geq 4$, let us assume that $P(k)$ holds, and let us consider the following:

$$\begin{aligned} k+1! &= (k+1)\cdot k! \\ &> (k+1)\cdot 2^k \\ &> 2^{k+1} \end{aligned}$$

Where the first equality holds by definition, the first inequality holds by the inductive hypothesis (because we assumed that $k! > 2^k$) and the last inequality holds because $k \geq 4 > 2$ □

Proof. We want to use induction. $\forall n \in \mathbb{N},\ P(n) =$"$n! > 2^n$". By induction, ists: $P(4) \wedge \forall k \geq 4,\ P(k) \implies P(k+1)$. Note that $4! = 4\cdot 3\cdot 2\cdot 1 = 24 > 16 = 2^4 \therefore P(4)$. Moreover

$$\forall k \geq 4,\ k+1! = (k+1)\cdot k! > (k+1)\cdot 2^k > 2^{k+1}$$

$$\therefore \forall k \geq 4,\ P(k) \implies P(k+1).$$

And we have this respectively by definition, by induction hypothesis (since $k! > 2^k$), and because $k \geq 4 > 2$. □

These two are identical proofs, the only difference is that in the second we used symbols everywhere. Needless to say, the first one looks like an actual proof, while the second one looks like an unreadable chaos of symbols.

By symbol, we mean any written piece of ink that was written to communicate a mathematical meaning. There are arguably three categories of symbols: the symbols that should never be used (there are not many of them), the symbols that should only be used in formulae, and the symbols that can be properly used both in formulae and in explanatory parts of the text.

Let us start with the two symbols that I think should never be used: $\therefore$ and $\because$. These are sometimes used in high-school to indicate respectively "therefore" and "since". I see no reason at all to write them since the two symbols $\implies$ and $\impliedby$ are used instead in mathematical formulae, and when we write an explanatory part we should only use English words (I never saw $\therefore$ and $\because$ in a dictionary).

Then there is a quite large (but not huge) category of symbols that are extremely useful in formulae, but have no place in explanatory parts. The examples I can think of are the following: "$\forall$" to say "for all"; "$\exists$" to say "there exists"; "$\implies$" to say "it implies" (and other arrows for implications); "$\wedge$" to say "and"; "$\vee$" to say "or"; "$\neg$" to say "not"; "$\in$" to say "in"; "$>$" to say "larger than"; etc. All these symbols are not in the dictionary, are not words, and therefore have no place in the explanatory part of a text. However, in formulae, they are very helpful!

Finally there are symbols that are perfectly acceptable in explanatory parts of a mathematical text. These are symbols like x, $\mathbb{R}$, α, ϵ, X. These symbols are just names, they are labels for an element, a set, a constant. They have the same right to be in a sentence as any other name like Bob.

Let us now see an example of the proper and improper use of the last two classes of symbols.

Definition. A function f from the set of real numbers to itself is said to be continuous if for every positive ϵ there exists δ positive such that for any x, y real numbers, whenever $|x-y| < \delta$ we have $|f(x) - f(y)| < \epsilon$.

Definition. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if the following holds:

$$\forall \epsilon > 0\ \exists \delta > 0\ \forall x, y \in \mathbb{R},\ |x-y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

One could still argue that the left one is clearer, but in a situation like this, the risk using "plain English" is that The Reader will get confused by the complicated and long sentence. Therefore the main idea is: use “plain English” whenever it is possible and clear to do so. Use symbols with caution. NEVER to just write less, but to improve clarity and have a more concise text when the “plain English” would be too confusing.

Section B - Use of quantifiers

In this section we address some of the most common problems that students encounter when approaching mathematics regarding the use of quantifiers.

Quantifiers are better understood by examples, and are one of two kind: existential (“there exists”) or universal (“for all/every”).

Let us see a first example to clarify the importance of quantifiers. “There exists a natural number n such that $2n = 6$” ($\exists n \in \mathbb{N},\ 2n = 6$). The first part of the statement, the one that says “There exists”/“$\exists$”, is called a quantifier; you should focus on the fact that if we remove the quantifier, the statement has no meaning because the reader does not know what values the variable (in this case n) can take.

The lesson to understand from this introductory subsection is that every mathematical statement with a variable requires a quantifier if you want to say it is true or false, and that it is VERY easy to make mistakes with quantifiers. Also, you should be aware that mistakes with quantifiers can often be tricky, so take your time to understand the concepts of this section.

As we mentioned above, there are two quantifiers: the existential quantifier (we saw one example above) and the universal quantifier. They do not have the same rôle; let us see what happens if we use the universal quantifier instead of the existential one in the example above. We would get the statement: “Every natural number n is such that $2n = 6$” ($\forall n \in \mathbb{N},\ 2n = 6$).

Seen this way, it seems easy to understand what is going on, but let me show you a less easy example. Try to decide whether the following statement uses the universal or the existential quantifier. “Let n be a prime number, then n is odd”. Notice that even if the statement is false, we can still analyse it. At a first glance, it might seem that we are using an existential quantifier (we are asking for just one natural number). However this is probably not the case. If I write down a prime number, and give it the name n, I may well say ‘this n is a prime number, so it is odd’ following the last statement — which is, of course, False: 2 is prime and even.

Subsection B0 - Correct use of quantifiers

As we said, quantifiers are the formal way in which mathematicians refer to two very common ideas: the idea of general behaviour and existence. There are two quantifiers; the universal quantifier is the quantifier “for all” which is indicated in formulae with the symbol “$\forall$”, and the existential quantifier which stands for “it exists” and is indicated in formulae with the symbol “$\exists$”.

There is a first important thing to know about quantifiers. Note that in order to say that an element with some property exists, or to say that for every element something happens, we have to specify what the universal set is (the general set in which we are making the choice). To guarantee that this is clear, we have to follow always the same, simple, rule: whenever we have a quantifier, we should always have something of the form: “$\forall x \in X$” or “$\exists x \in X$”, and never just “$\forall x, ...$” or “$\exists x, ...$”. We can see that in this “standard form”, we have two letters. It is important to understand that the first letter (called “variable”) “x” has no particular significance. If $a, x, \star$ have no prior meaning, the writings “$\forall a \in X$”, “$\forall x \in X$” or “$\forall \star \in X$” have all the same meaning. On the other hand, we can easily see that changing the second letter (in this case X, which represents the set) changes the meaning of the statement. Therefore, saying “$\forall x \in \mathbb{R}$” is the same of saying “$\forall a \in \mathbb{R}$” but not the same of saying “$\forall x \in \mathbb{N}$”. The importance of specifying the set is obvious but often forgotten.

All that we said with the example of “$\forall$” can be said with the “$\exists$” quantifier, so for example “$\exists x \in \mathbb{R}$” is the same of saying “$\exists a \in \mathbb{R}$” but not the same of saying “$\exists x \in \mathbb{N}$”.

Note that sometimes we use some alternatives to the “$\in$” symbol, but we should not get distracted by this; for example, sometimes we are going to write $\forall x > 0$ when we should actually say $\forall x \in R^+$. But it is always possible to write a quantifier in the form $\forall x \in X$ or $\exists x \in X$ (up to changing the letters).

If we think about what we said in the previous paragraph, we notice a simple yet often forgotten truth: we cannot use as a variable in a quantifier, a variable that has already a value assigned. What does it means? Let us see an example.

Proposition. *For every n a natural number at least 4, we have $n! > 2^n$.*

Proof. Let $n = 4$; we see that $n! = 4! = 24 > 16 = 2^4 > 2^n$. Now consider that by some easy calculations we have that $\forall n \geq 4,\ n! > 2^n \implies (n+1)! > 2^{n+1}$. □

There are some issues in the previous proof, let us now forget the fact that it is not actually a proof (because we are not proving anything, just saying that the statement is true by some easy calculations), and focus on the use of quantifiers (there are two). There are some issues.

First, we are not using quantifiers in the "standard" form $\forall x \in X$; which is, for example, instead of writing "$\forall n \geq 4$" we should have written "$\forall n \in \{k \in \mathbb{N}, k \geq 4\}$" or "$\forall n \in \mathbb{N}^{\geq 4}$". This is not a real mistake, and is really common practice (mathematicians are lazy too). It is only worth mentioning that you should remember what it means when you write something like that.

The second mistake is more serious. At the beginning of the proof, we say "let $n = 4$", and this means that from now on, the variable (the letter) "n" is a symbol with a specific value. What it means, is that in every future use of the symbol "n" we should read "4" instead of "n". Therefore, the mistake is that after writing "let $n = 4$", when we say "$\forall n \geq 4$" we are actually (formally) writing "$\forall 4 \geq 4$", which does not make sense.

In the same way, we should not use a variable introduced in a quantifier for anything else besides the intended purpose. An example of this could be for example the writing of "$\forall n \in \{n \in \mathbb{N} : n \geq 7\}$" which is an example of bad notation. We should instead write "$\forall n \in \{k \in \mathbb{N} : k \geq 7\}$", provided that we did not use k already for something else.

Let us repeat now what we said earlier: we cannot use as variable in a quantifier a variable (a symbol) that has already a value assigned. For this reason, you should always (without any exception, ever) avoid to write something like "Let a be [...] $\forall a \in A$" or "Let x be [...] $\exists x \in A$" or $\exists x \in \{x[...]\}$.

That said, there is an important distinction between the universal and the existential quantifiers. Let us see the following example.

Proposition. *There is no natural number n such that $n^2 = \sqrt{2}$.*

Proof. We know that $\forall k \in \mathbb{N},\ k^2$ is an element in $\mathbb{N}$. Since $\sqrt{2} \notin \mathbb{N}$ we have that $k^2 \neq \sqrt{2}$. $\square$	*Proof.* Assume for a contradiction that $\exists k \in \mathbb{N}$ such that $k^2 = \sqrt{2}$. Then k^2 would be a natural number and therefore $k^2 \neq \sqrt{2}$ since $\sqrt{2}$ is not a natural number. $\square$

There is the same mistake in the notation of both proofs; but it is a mistake which is way worse in the left case (when we use the universal quantifier). The problem is as follows: when we say that "$\forall k \in \mathbb{N}$" something happens, we are using a "disposable variable", which means that we are just TEMPORARILY using the letter k to indicate a (generic) element of $\mathbb{N}$ that we want to talk about. Since k is not a specific element, but a generic element (an arbitrary element), as soon as we end our statement, the variable k loses all meaning (since it was not used to indicate a specific k). Therefore in the left example, when we say "$k^2 \neq \sqrt{2}$" the variable k has no meaning anymore, since the statement $\forall k \in \mathbb{N}, k^2 \in \mathbb{N}$ is a different statement from $k^2 \neq \sqrt{2}$.

The same holds for the right example, but since we are using an existential quantifier, it is somewhat a minor mistake, and we might do this sometimes. When we use the existential quantifier, it is often implicitly meant that we actually fix an element in the set that satisfies the property. Formally, we should do this on a new line and use a different letter, but often we are too lazy.

The correct way of using quantifiers would be as follows.

Proof. Let us fix an arbitrary $k \in \mathbb{N}$. Since $k \in \mathbb{N}$, we have that k^2 is an element in $\mathbb{N}$ and therefore $k^2 \neq \sqrt{2}$, since $\sqrt{2} \notin \mathbb{N}$. $\square$	*Proof.* Assume for a contradiction that $\exists k \in \mathbb{N}$ such that $k^2 = \sqrt{2}$. Let us fix such a natural number ℓ. Then ℓ^2 would be a natural number and therefore $\ell^2 \neq \sqrt{2}$ since $\sqrt{2}$ is not a natural number. $\square$

Or also,

Proof. We know that $\sqrt{2} \notin \mathbb{N}$; $\forall k \in \mathbb{N},\ k^2 \in \mathbb{N}$ and therefore $\forall a \in \mathbb{N},\ a^2 \neq \sqrt{2}$. $\square$	*Proof.* Let us fix $\ell \in \mathbb{N}$ that contradicts the claim. Then ℓ^2 would be a natural number and hence we have a contradiction since $\sqrt{2}$ is not a natural number. $\square$

Note that in this last example, in the right proof we used a different variable (in this case ℓ) to indicate a generic number. We could have used again k without changing the statement, this is because as soon as the previous mathematical statement ends, the variable k is free, and normally we will do this. However, you have to consider the clarity of the text as well, and sometimes using the same symbol for two different things can be consuming.

Let us now see another (more formal) way to keep track of variables. Even if there is no theoretical reason to do so, mathematicians tend to have a specific taste in choosing variables. For example, it is quite rare to read something like "Let $\alpha \in \mathbb{N}$ be a prime..." (usually we use n or p for natural numbers), even if formally speaking there is no reason why we cannot.

But what happens if we bring this behaviour to an extreme? Is there any reason why we shouldn't write "Let $\mathbb{R} \in \mathbb{N}$ be a prime..." or "Let $+ \in \mathbb{N}$ be a prime..."? After all, $\mathbb{R}$ and $+$ are symbols exactly like n or p.

In order to understand the answer to this question one has to understand that mathematics, even in its abstractness, is a discipline that evolved in time, and not in a smooth way. It is not the case that the theorem and results that we see in the textbooks were proved in the same order we do them, and after 10 weeks of work you know way more about formal writing that many great mathematicians. In particular, formal logic is quite recent (B. Russell was one of the founders of this discipline), and therefore logicians had to adapt to the fact that some symbols were widely used already.

If you were writing mathematics for an alien that understands English but never saw a mathematical text, you should start your text with something like: "In the following text, we are going to use the symbol $=$ to denote equality; we are going to use the symbol $\forall$ to denote the universal quantifier; etc". Which is, you would need to build a dictionary. In this situation, you could use any other symbol instead of $=$, and mathematics would still look the same. However, this is not the case when you write a paper or the solutions for an assignment. In practice, we behave as though every paper starts with an introduction like the one above, and many lecture notes or books actually have an introduction in which this "labelling" is done in a formal way. Let me just add that these tastes change like every other fashion, and therefore if you open an older mathematical text sometimes you can for example see $\mathcal{R}$ to indicate $\mathbb{R}$, and many other "strange" notations.

But what is happening from a formal point of view? An easy way to look at it is the following. We can imagine that before starting a mathematical text (before of the introduction where we say what $=$ means) we have a huge box containing all the possible symbols. We call this the box of "free" symbols (or variables). As soon as we start writing our mathematical text, we need to use symbols. We can think of it as taking the symbol from the "free" box and putting it on our working table, the "busy" box.

Notice that we don't need all the symbols for the same amount of time. For example the symbol "$=$" is going to be on the "busy" list all time, as are other symbols like the quantifiers. While other symbols we use less often. The important thing to remember is that whenever we need a new variable, we need to check that it is still in the "free" list before using it, and we need to remember that until it returns there, it is going to be "busy". More of this in the following.

We said already that there are different ways of using a symbol, depending on how long we need it. We mentioned for example that some symbols have a fixed meaning that never changes through the whole mathematical text (like $=$). But even inside the proof of a theorem, there are two ways of using a symbol.

If at some point in a proof we say "Let now x be equal to $\left(1, 2, \sqrt{\frac{2}{3}+2}\right)$" what we mean is that we do not want to write a lot of times a nasty looking vector and therefore FOR THE REST OF THE PROOF (or at least until stated otherwise) the letter x is going to indicate only and uniquely that vector. What we are doing is to fix a symbol to a meaning for a long piece of time. We could also write: "here and in the following, let X be the set of rational numbers", and in this case we are indicating that for the rest of the paper/book/chapter/assignment, the symbol X is going to be a "busy" symbol and should not be used for anything different than the set of rational numbers.

Another way of using symbols is the disposable way, and this is done by using quantifiers. What FORMALLY happens when we use quantifiers is the following: we take a FREE variable, and we use it in a mathematical statement. During the statement, the variable should be treated as BUSY, while once the statement is ended, the variable returns back to its original, FREE, status. Therefore, if we use a quantifier and then we want to continue to use the same symbol to indicate the same object, we should state that clearly as we did above. As examples:

i) "Assume $\exists k \in \mathbb{N}$ such that $k^2 = \sqrt{2}$. Let us fix such a k." This way, we are fixing the value of the variable k.

ii) Instead of writing "$\forall k \in \mathbb{N}$ [...]" (after the mathematical statement we cannot use the variable k to indicate a generic element of $\mathbb{N}$); we can write "let us fix a generic element k of $\mathbb{N}$ [...]".

Keep in mind the fact that quantifiers only temporarily make a variable busy, and that free variables have no meaning, so every time you write a variable in a formula or in an explanation, you need to introduce it first. If you do this, you will avoid one of the most common mistakes.

Subsection B1 - Switching quantifiers

Consider the following sentences:

i) For every door (let us call it D), there is a key (let us call it K) such that K opens D.

ii) There is a key (let us call it K) such that for every door (let us call it D), K opens D.

The only difference between the two statements is the order of the quantifiers, nonetheless, the meaning of the two statements is completely different. This phenomenon thing also happens in mathematics, and it is much more dangerous because it is less easy to notice (especially if one has the habit of writing using a lot of symbols).

The mistake of switching quantifiers is very common. Let us see some rules that help us avoiding it.

i) $\forall a \in A, \forall b \in B, \forall c \in C$ is equivalent to say $\forall c \in C, \forall a \in A, \forall b \in B$. Therefore we often write $\forall a \in A, b \in B, c \in C$.

   What is allowed is to change the order of the SAME quantifier when the same quantifier appears multiple times one near the other (so $\exists a \in A, b \in B$ is also ok, because we are not changing quantifier and the instances are one near the other).

ii) It is NOT allowed to change the order of the same quantifiers if the two instances are not one near the other. For example $\forall a \in A, \exists b \in B, \forall c \in C$ is NOT equivalent to $\forall c \in C, \exists b \in B, \forall a \in A$.

iii) As we already said, it is not allowed to change the order of different quantifiers. Therefore $\forall a \in A, \exists b \in B$ is not equivalent to $\exists b \in B, \forall a \in A$.

The only suggestion regarding quantifiers is more difficult than it sounds: try to be sure of what you want to write. If you want to say that every door has a key, what you mean is that if you fix a door, any door, THEN you can select a key that opens it. Therefore the selection of the door comes before the selection of the key. Anyway, it is easy to over-complicate this particular issue, but it is important to remember that the order is important when dealing with quantifiers. Let us do a last mathematical example.

i) $\forall n \in \mathbb{N}, \exists m \in \mathbb{N}$ such that $m > n$.

ii) $\exists m \in \mathbb{N}, \forall n \in \mathbb{N}$ we have $m > n$.

One of the statements is true, one is not. Which one? What do these statements mean? Try to write down some statements like this and try to familiarise yourself with this kind of behaviour.

Subsection B2 - Positioning of quantifiers

As we said, whenever we have a mathematical statement, all the variables must be quantified, and all the quantifiers go at the beginning of the statement. We are going to see some of the mistakes that can occur when we break this very general rule.

Subsubsection B2A - Quantifiers at the end of the statement

It is practice of someone to put some "for all" quantifiers at the end of a statement. This is an extremely bad practice, because of what we said in the previous subsection, which is that the order of quantifiers is extremely important and it is not something that can be messed up without problems. The general rule that always holds is the following: whenever we have a proposition with quantifiers, ALL the quantifiers go at the beginning of the statement. Let us see the following example.

There exists a real number x such that for any positive integer n we have $\frac{1}{n} \leq x$.	There exists a real number x such that $\frac{1}{n} \leq x$ for any positive integer n.

These two sentences might look the same, and most people would interpret them in the same way. However, the right one is a major example of bad mathematical practice. Let us write again the above statements in a more concise and less elegant way to stress the order of quantifiers.

$\exists x \in \mathbb{R}, \forall n \in \mathbb{N}^+,\ \frac{1}{n} \leq x$.	$\exists x \in \mathbb{R}, \frac{1}{n} \leq x, \forall n \in \mathbb{N}^+$.

The right notation is again an example of bad notation, and the reason is because it is not clear whether the "for all" should come before or after the "there exists". As we know, the difference between the two options is quite important. Let us do a last example that, unfortunately, was taken from the lecture notes of a mathematical course.

A sequence (x_n) in $\mathbb{R}$ is said to be bounded if there exists a real number $M > 0$ such that for all $n \in \mathbb{N}$, we have $\lvert x_n \rvert \leq M$.	A sequence (x_n) in $\mathbb{R}$ is said to be bounded if there exists a real number $M > 0$ such that $\lvert x_n \rvert \leq M$ for all $n \in \mathbb{N}$.

Can you spot the mistake? The correct form is the left one, where the quantifiers are at the beginning of the mathematical statement that defines the concept of bounded sequence.

This is a mistake commonly done, but easily fixed. Just remember that quantifiers ALWAYS go at the beginning of a statement, both when it is written in formulae and when it is written in plain English.

Subsubsection B2B - Quantifiers after the variable

Another common mistake is to put the first instance of a variable before it is quantified.

A relevant example is the following, that was written by a student in the solutions of an assignment. This was the first sentence in the answer to the exercise.

$P(n)$ is the statement that for every $n \in \mathbb{N}$, $n^3 + 5n$ is a multiple of 6.

The problem here is again that the quantifier is in the wrong position. If we read the statement from left to right (as we always do both in English and in mathematics) we encounter the variable n before encountering its quantifier. Therefore when a reader encounters the "n", they don't know what to do.

Therefore this practice should always be avoided. There is another mistake in the statement. Indeed, the statement $P(n)$ actually does not depend on n at all, because it simply means that for every natural number a certain property holds. This is maybe a more subtle mistake, but it is still very important. See if you can appreciate it by looking at the right way of writing this statement.

$P(n)$ is the statement that for every $n \in \mathbb{N}$, $n^3 + 5n$ is a multiple of 6.	For each $n \in \mathbb{N}$, let $P(n)$ be the statement that $n^3 + 5n$ is a multiple of 6. (We want to prove that for any $n \in \mathbb{N}$, $P(n)$ holds.)

We are going to discuss this again in the next subsection, but the general idea is that we have to be as formal as if we were writing for a machine. We always have to keep in mind that the reader does not know what we are thinking and therefore it is our job to make it as clear as possible. The reader is going to read from left to right and from top to bottom, and it should never be the case that the reader encounters something that they do not know.