MA210 - Class 7

THM Let G be a graph on $n \geq 1$ vertices. The following are equivalent (AND characterise trees).

a) G is connected and has no cycle
b) G is connected and has $n-1$ edges
c) G has $n-1$ edges and no cycle
d) $\forall u,v \in V$ there is a unique uv-path.

$\leadsto$ a) $\Rightarrow$ b), c) By induction, using the existence of leaves

b) $\Rightarrow$ c), a) Remove all edges in cycles. Now we have a), which implies we removed no edge at all.

c) $\Rightarrow$ a), b) Divide the graph in components.

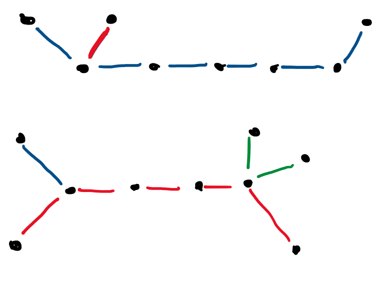
d) $\Rightarrow$ a) Trivial

a) $\Rightarrow$ d) Assume there are $u,v \in V$ and P,Q two distinct paths between them. Then P,Q is a CLOSED walk and hence it contains a cycle.

EXE Let T be a tree with $2k$ odd-degree vertices. Prove that there are k paths s.t.

i) No two of them have common edges
ii) They cover all edges of T.

EXA



$\rightsquigarrow$ Let $\{v_1, \dots, v_{2k}\}$ be the vertices of odd degree.
We choose a first pair (say v_1, v_2), and we remove the path between v_1 and v_2.
Then we get a graph with $2k-2$ vertices of odd degree.
We repeat the procedure until we remain with a graph with NO odd vertex.
Since the only graphs with all vertices of even degree are unions of cycles, we are done
(T has no cycles so we are left with no edge).

ALTERNATIVE: Prove by induction $P(k)$: "every FOREST with $2k$ odd-degree vertices can be decomposed into k edge-disjoint paths".

EXE Consider the complete graph K_n on $\{1, \dots, n\}$ with weights $w(ij) = i+j$.
Find the minimum weight of a spanning tree.

$\rightsquigarrow$ KRUSKAL.
CLAIM The minimal weight is $\sum_{i=3}^{n+1} i = \frac{(n+4)(n-1)}{2}$

CLAIM At the step i, the algorithm chooses $1\,(i+1)$.
$\rightsquigarrow$ By induction.
At step k, we choose an edge of weight $k+2$, so at step $k+1$ we must choose an edge of weight at least $k+2$.