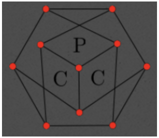
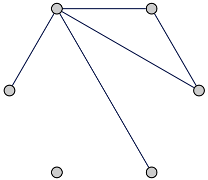
Chromatic profile of $\{C_3, \ldots, C_{2k-1}\}$

J. Böttcher, N. Frankl, **D. Mergoni Cecchelli**, O. Parczyk, J. Skokan

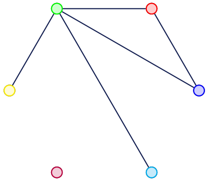


Proper colouring and chromatic number



Introduction

Proper colouring and chromatic number



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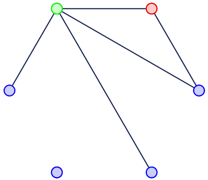
Proper colouring and chromatic number



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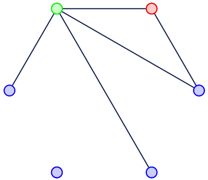

The chromatic number $\chi(G)$ is the number of colours needed to colour G.

Bounding $\chi(G)$ in H-free graphs

Start point

Can we bound $\chi(G)$ if G avoids H?

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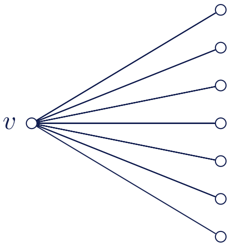

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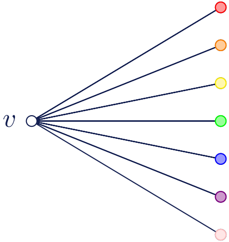

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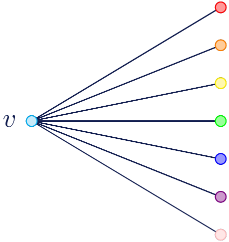

Bounding $\chi(G)$ in H-free graphs

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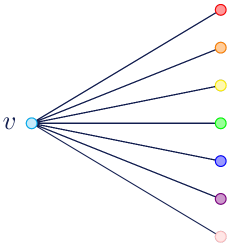
Can we bound $\chi(G)$ if G avoids H?

- $H = K_{1,k}$ ✓ $\chi(G) \leq k$



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Can we bound $\chi(G)$ if G avoids H?

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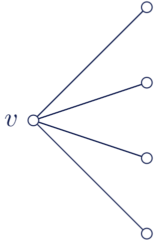

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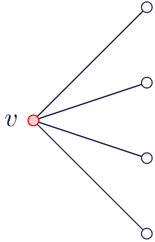

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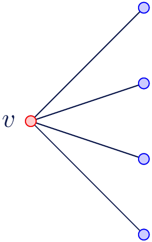

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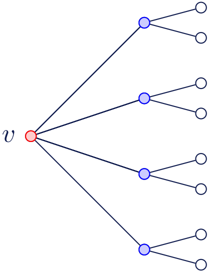

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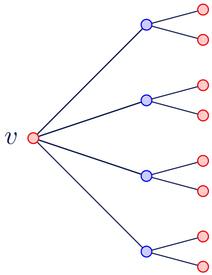

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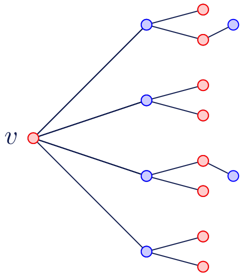

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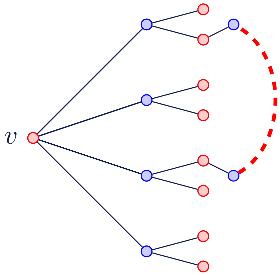

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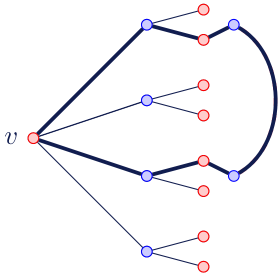

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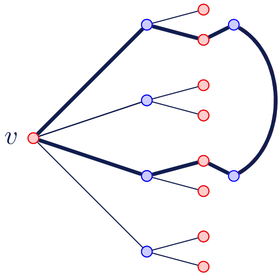

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Can we bound $\chi(G)$ if G avoids H?

- $H = K_{1,k}$ ✓
- $H = \mathcal{C}^{\text{Odd}}$ ✓
- $H = K_3$ ✗

Thm. (Tutte, 1940's). Mycielski, Burling, ...

There are K_3-free graphs of arbitrarily high chromatic number.

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- $H = K_{1,k}$ ✓
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Thm. (Erdős, 1959). Kneser+Lovász, Alon et al., ...

There are graphs with arbitrarily high chromatic number and girth.

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There are graphs with arbitrarily high chromatic number and girth.

One of the first applications of the Probabilistic method.

Bounding $\chi(G)$ in H-free graphs with $\delta(G) > \alpha|G|$

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- $H = K_{1,k}$ ✓
- $H = \mathcal{C}^{\mathsf{Odd}}$ ✓
- $H = K_3$ ✗
- $H = \mathcal{C}^{\mathsf{Odd}}_{\leq 2k-1}$ ✗

Prbl. (Erdős, Simonovits, 1973; insp. Hajnal)

What can we say if G is K_3-free and $\delta(G) \geq \alpha|G|$?

Bounding $\chi(G)$ in H-free graphs with $\delta(G) > \alpha|G|$

Prbl. (Erdős, Simonovits, 1973; insp. Hajnal)

What can we say if G is K_3-free and $\delta(G) \geq \alpha|G|$?

Conj. (Erdős, Simonovits, 1973)

If G is K_3-free and $\delta(G) > \frac{1}{3}|G|$, then $\chi(G) \leq 3$.

Bounding $\chi(G)$ in H-free graphs with $\delta(G) > \alpha|G|$

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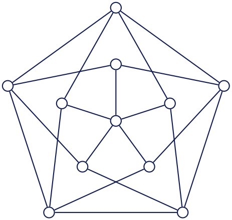

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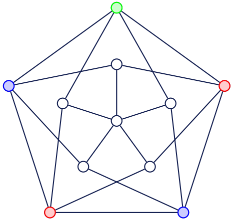

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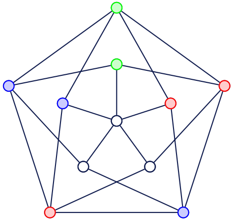

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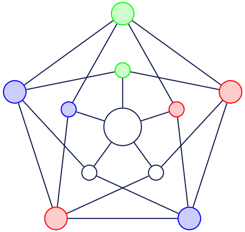
Conj. (Erdős, Simonovits, 1973), Thm. (Häggkvist, 1982)

If G is K_3-free and $\delta(G) > \cancel{\frac{1}{3}}\ {}^{?}\,\frac{10}{29}\,{}^{?}\,|G|$, then $\chi(G) \leq 3$.



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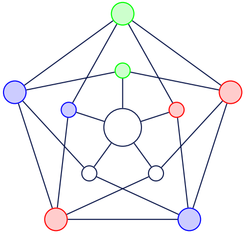
Conj. (Erdős, Simonovits, 1973), Thm. (Häggkvist, 1982)

If G is K_3-free and $\delta(G) > \frac{3}{8}|G|$, then $\chi(G) \leq 3$.



Bounding $\chi(G)$ in H-free graphs with $\delta(G) > \alpha|G|$

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Prbl. (Erdős, Simonovits, 1973)

What can we say if G is K_3-free and $\delta(G) \geq \alpha|G|$?

Thm. (Andrásfai, 1964)

- If G is K_3-free and $\delta(G) > \frac{2}{5}|G|$, then $\chi(G) \leq 2$;

Bounding $\chi(G)$ in H-free graphs with $\delta(G) > \alpha|G|$

Prbl. (Erdős, Simonovits, 1973)

What can we say if G is K_3-free and $\delta(G) \geq \alpha|G|$?

Thm. (Häggkvist, 1982)

- If G is K_3-free and $\delta(G) > \frac{2}{5}|G|$, then $\chi(G) \leq 2$;
- If G is K_3-free and $\delta(G) > \frac{3}{8}^{\,?\,\frac{10}{29}\,?}|G|$, then $\chi(G) \leq 3$;

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Prbl. (Erdős, Simonovits, 1973)

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Thm. (Jin, 1995)

- If G is K_3-free and $\delta(G) > \frac{2}{5}|G|$, then $\chi(G) \leq 2$;
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Prbl. (Erdős, Simonovits, 1973)

What can we say if G is K_3-free and $\delta(G) \geq \alpha|G|$?

Thm. (Thomassen, 2002)

- If G is K_3-free and $\delta(G) > \frac{2}{5}|G|$, then $\chi(G) \leq 2$;
- If G is K_3-free and $\delta(G) > \frac{10}{29}|G|$, then $\chi(G) \leq 3$;
- If G is K_3-free and $\delta(G) > (\frac{1}{3} + \varepsilon)|G|$ then $\chi(G) \leq C_\varepsilon$;

Bounding $\chi(G)$ in H-free graphs with $\delta(G) > \alpha|G|$

Prbl. (Erdős, Simonovits, 1973)

What can we say if G is K_3-free and $\delta(G) \geq \alpha|G|$?

Thm. (Brandt, Thomassé, 2005)

- If G is K_3-free and $\delta(G) > \frac{2}{5}|G|$, then $\chi(G) \leq 2$;
- If G is K_3-free and $\delta(G) > \frac{10}{29}|G|$, then $\chi(G) \leq 3$;
- If G is K_3-free and $\delta(G) > \frac{1}{3}|G|$ then $\chi(G) \leq 4$;

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Thm. (Hajnal graphs)

- If G is K_3-free and $\delta(G) > \frac{2}{5}|G|$, then $\chi(G) \leq 2$;
- If G is K_3-free and $\delta(G) > \frac{10}{29}|G|$, then $\chi(G) \leq 3$;
- If G is K_3-free and $\delta(G) > \frac{1}{3}|G|$ then $\chi(G) \leq 4$;
- $\forall k, \varepsilon > 0$, $\exists$ a K_3-free G with: $\chi(G) \geq k$ and $\delta(G) \geq (\frac{1}{3} - \varepsilon)|G|$.

Bounding $\chi(G)$ in H-free graphs with $\delta(G) > \alpha|G|$

Prbl. (Erdős, Simonovits, 1973)

What can we say if G is K_3-free and $\delta(G) \geq \alpha|G|$?

Summary (VV.AA., '70s - 2005)

Let G be a K_3-free graph on n vertices.

$\delta(G) >$	$\frac{2}{5}n$	$\frac{10}{29}n$	$\frac{1}{3}n$	$(\frac{1}{3} - \varepsilon)n$
$\chi(G) \leq$	2	3	4	∞

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Thm. (Allen, Böttcher, Griffiths, Kohayakawa, Morris, 2013)

If G is H-free and $\delta(G) > (f(H) + \varepsilon)\,|G|$, then $\chi(G) \leq C_{\varepsilon,H}$ (optimal)

Our result

Conj. (Letzter, Snyder, '19; Ebsen, Schacht, '20)

If G is $\{C_3, C_5, \ldots, C_{2k-1}\}$-free and $\delta(G) \geq \frac{1}{2k-1}|G|$, then $\chi(G) \leq 3$.

Our result

For k large enough (≥ 600)

Thm. (Böttcher, Frankl, M., Parczyk, Skokan, '23)

If G is $\{C_3, C_5, \ldots, C_{2k-1}\}$-free and $\delta(G) \geq \frac{1}{2k-1}|G|$, then $\chi(G) \leq 3$.

Idea of the proof

Thm. (Böttcher, Frankl, M., Parczyk, Skokan, '23)

If G is $\{C_3, C_5, \dots, C_{2k-1}\}$-free and $\delta(G) \geq \frac{1}{2k-1}|G|$, then $\chi(G) \leq 3$.

1) Sufficient condition for $\chi(G) \leq 3$;

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2) This can be shown in an auxiliary graph; (Thomassen, 2007)

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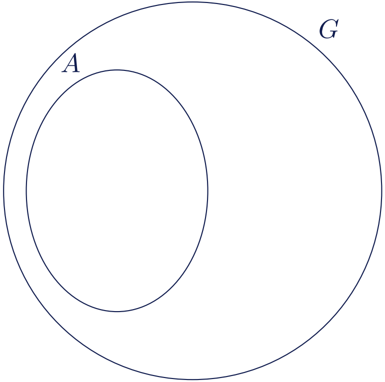
Let G be a $\{C_3, C_5, \ldots, C_{2k-1}\}$-free graph with $\delta(G) \geq \frac{1}{2k-1}|G|$.

2) This property can be shown in an auxiliary (edge-weighted) graph;
3) Work in the auxiliary graph.



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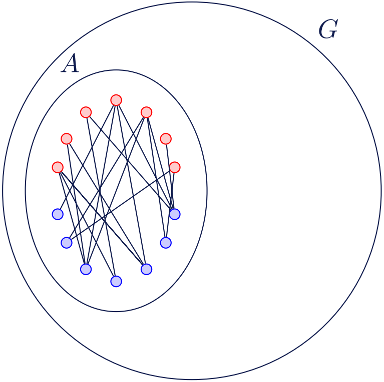

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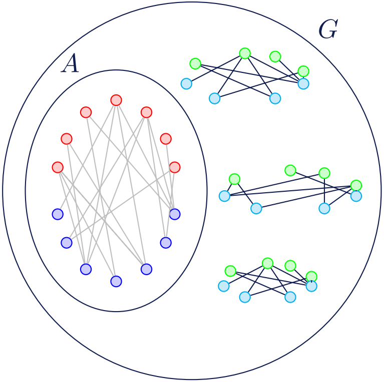

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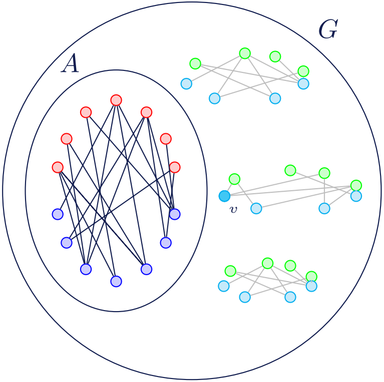

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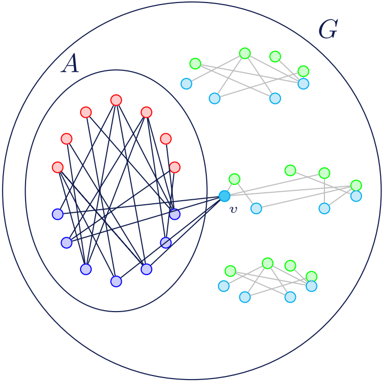

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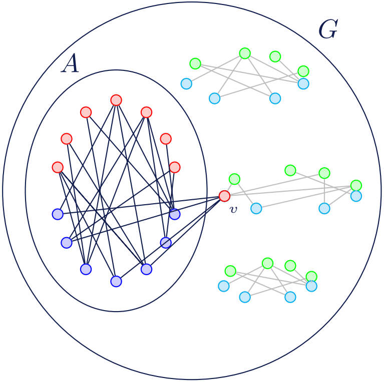

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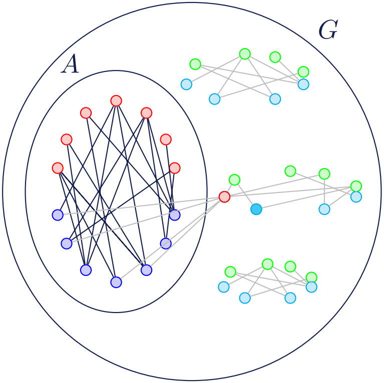

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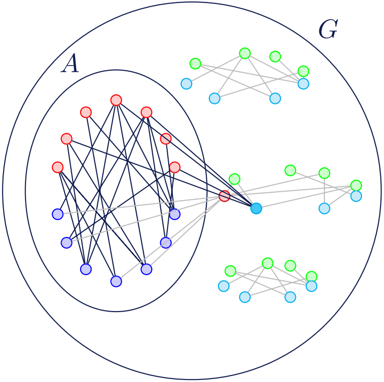

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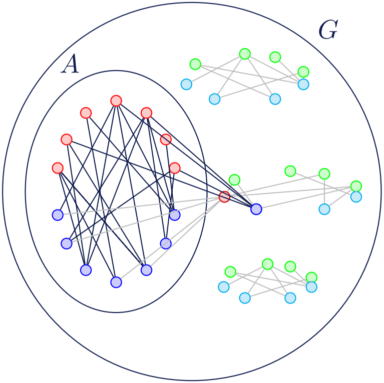

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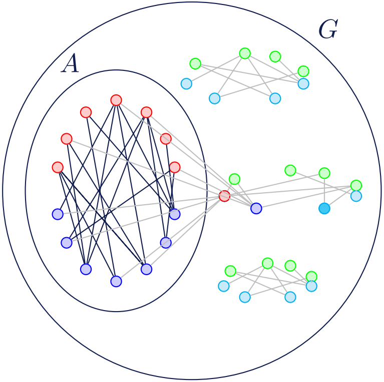

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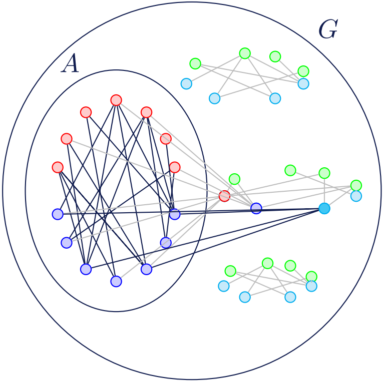

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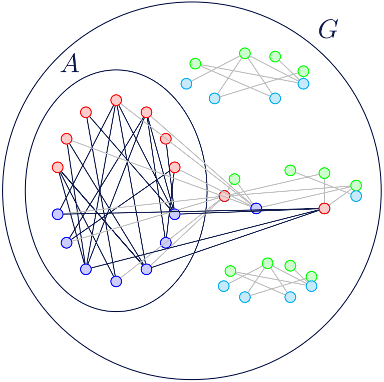

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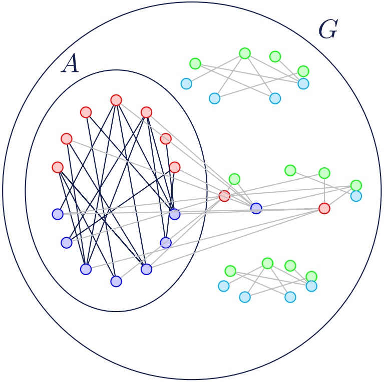

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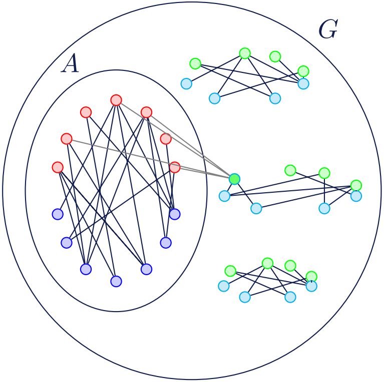

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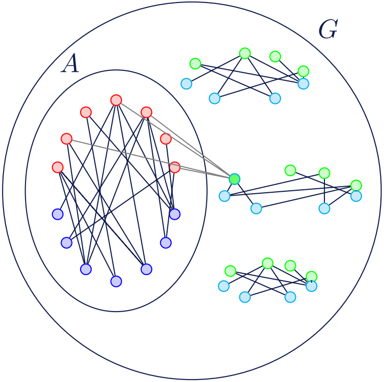

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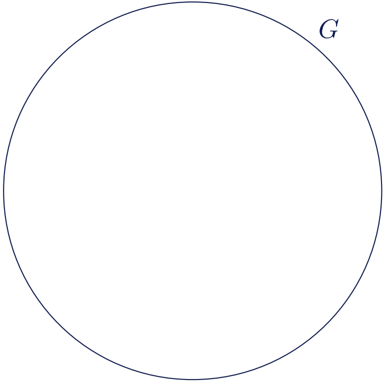

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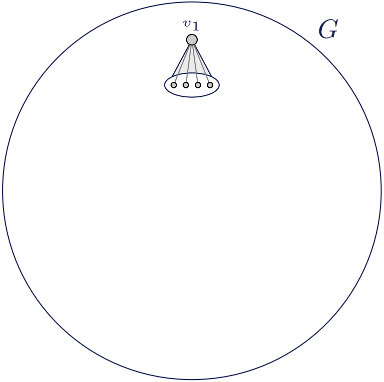

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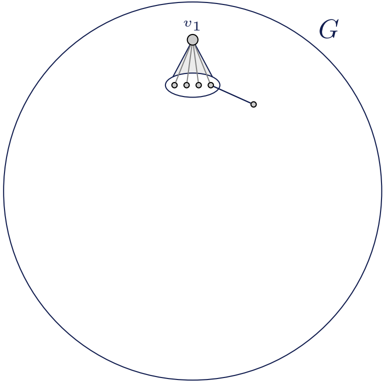

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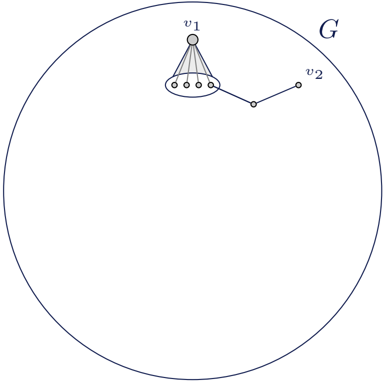

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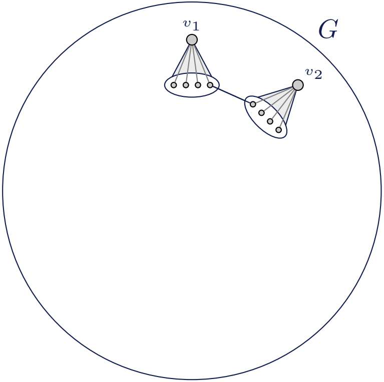

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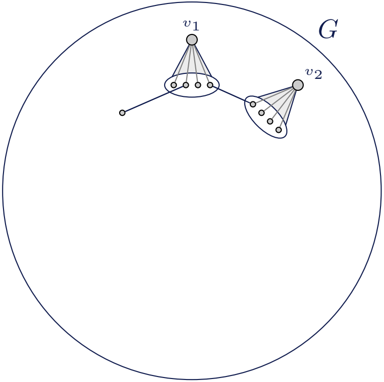

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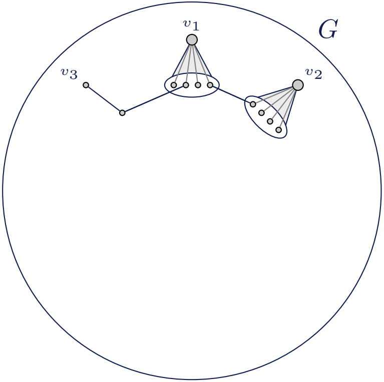

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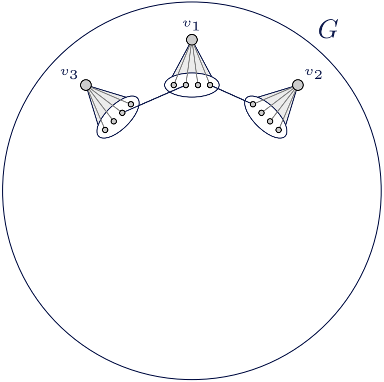

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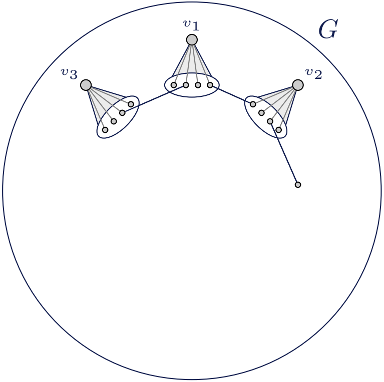

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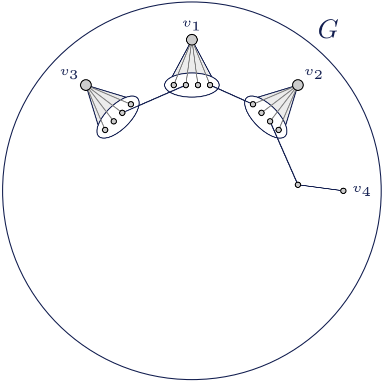

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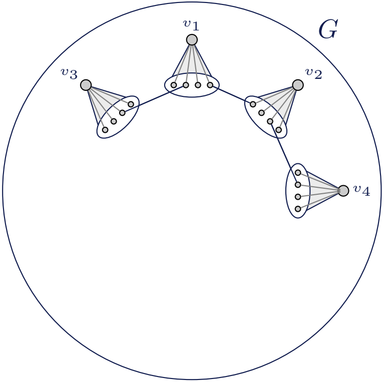

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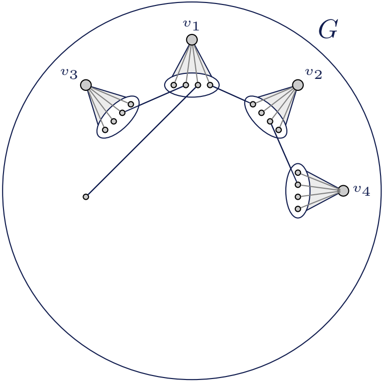

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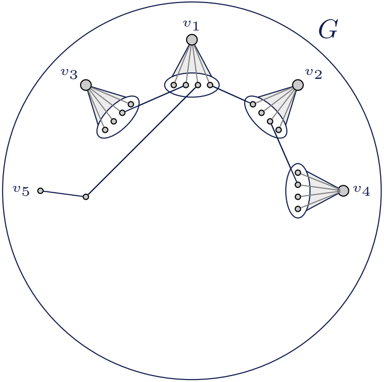

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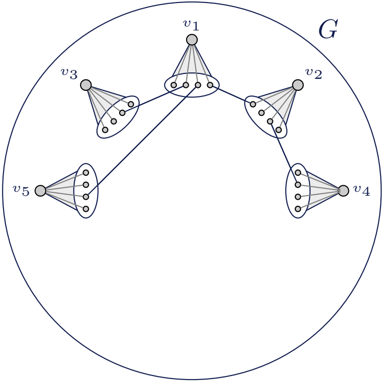

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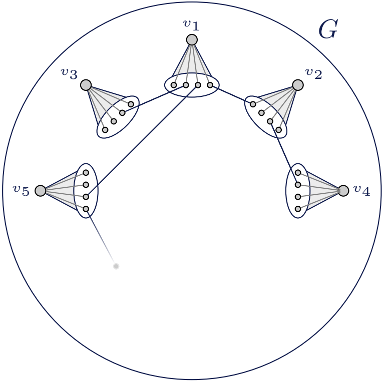

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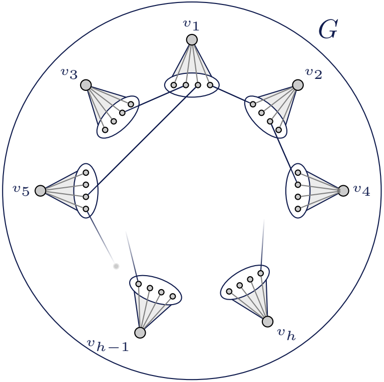

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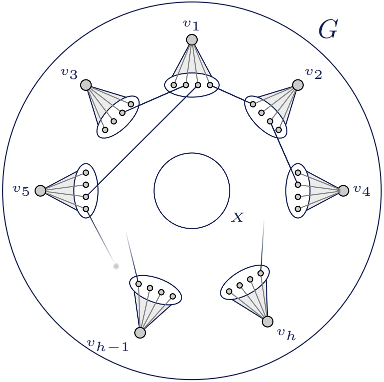

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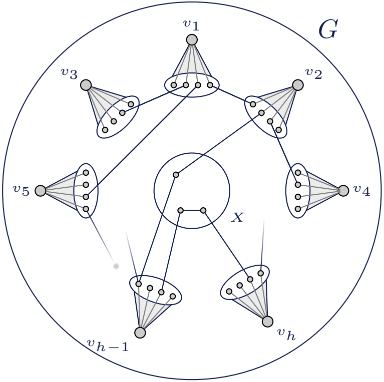

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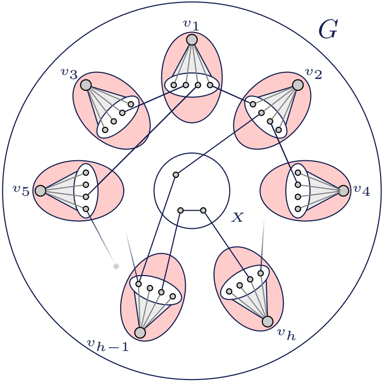

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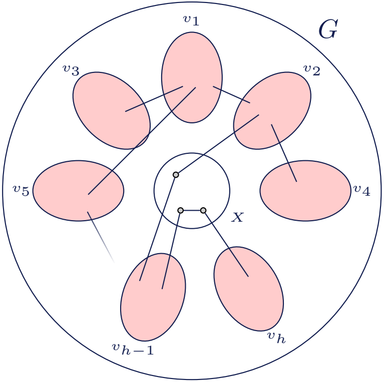

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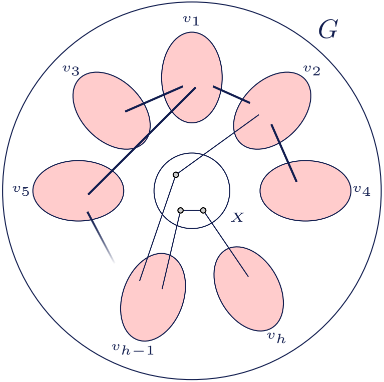

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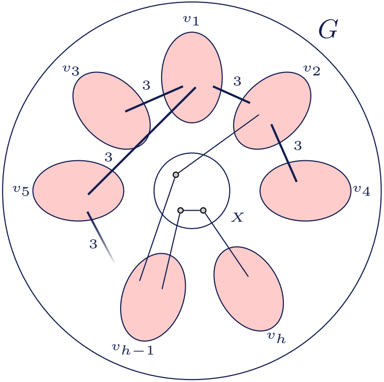

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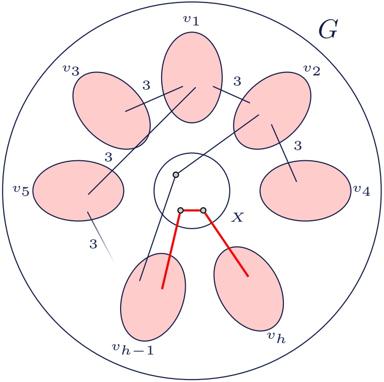

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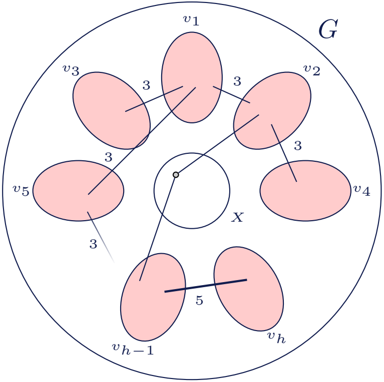

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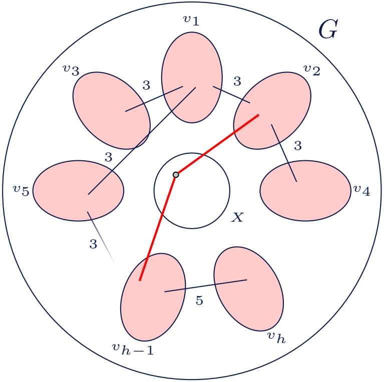

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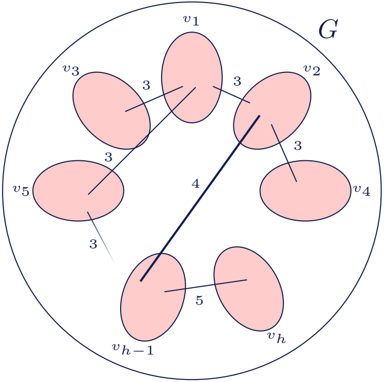

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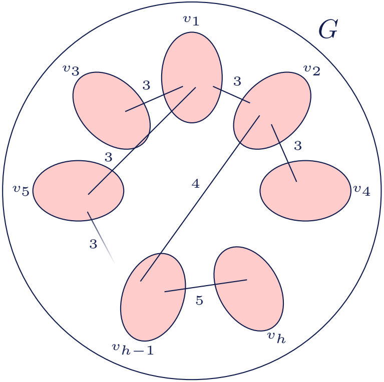

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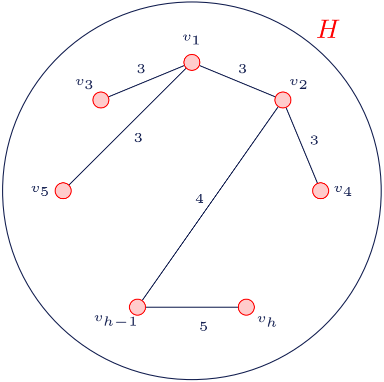

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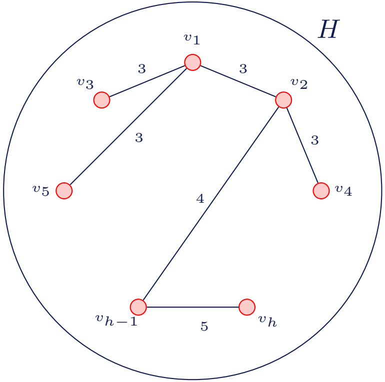

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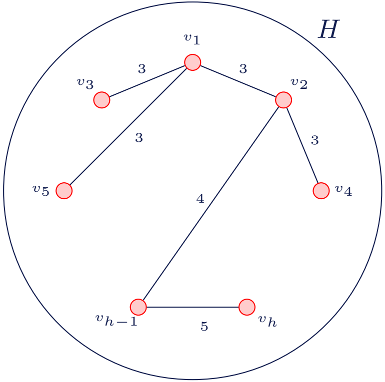

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   Want: Good partition of $V(H)$.

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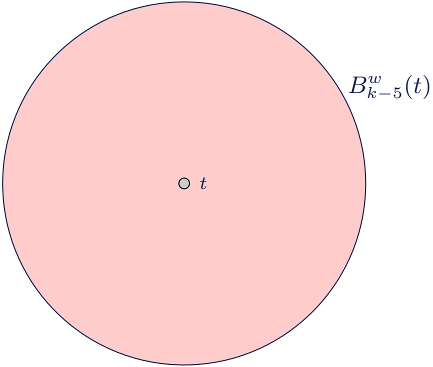

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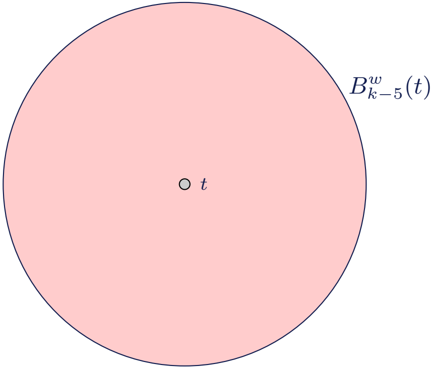

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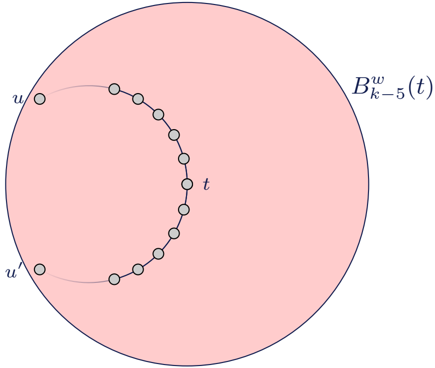

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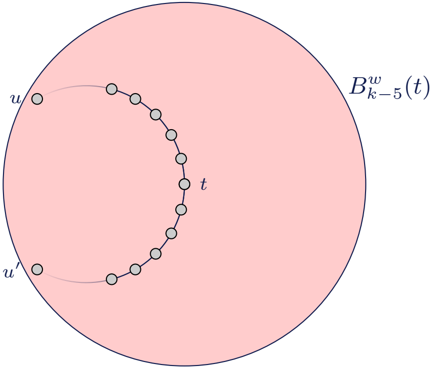

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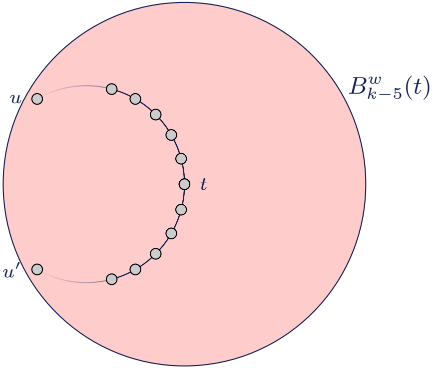

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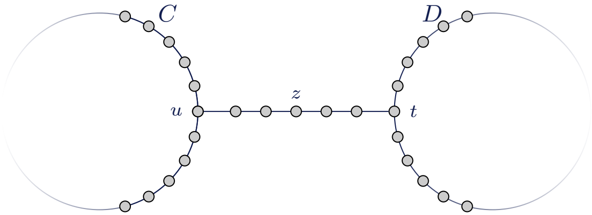

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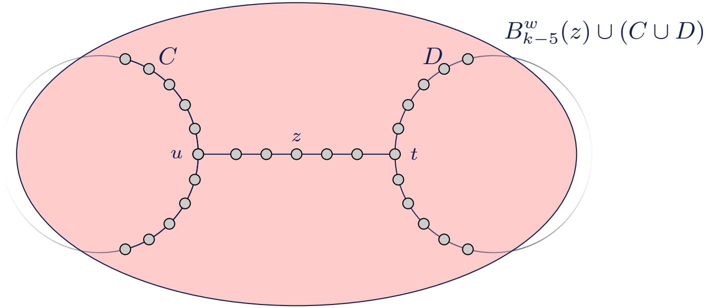

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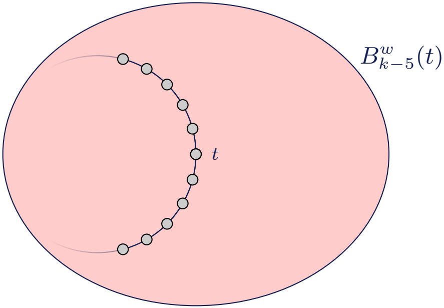

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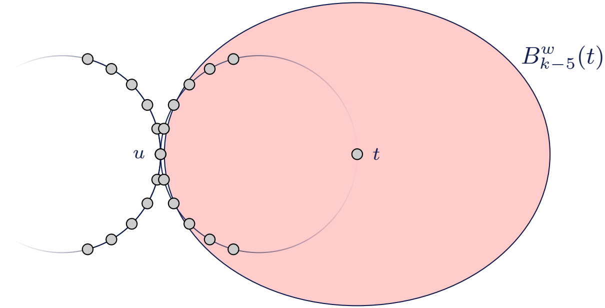

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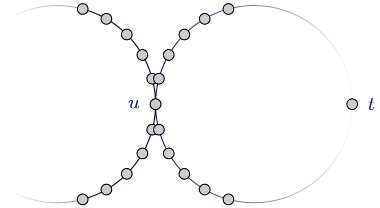

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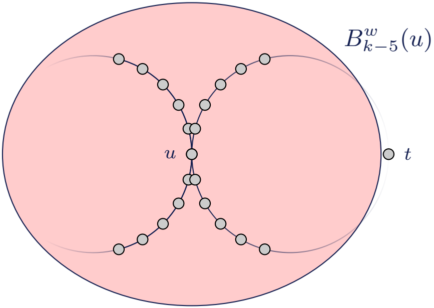

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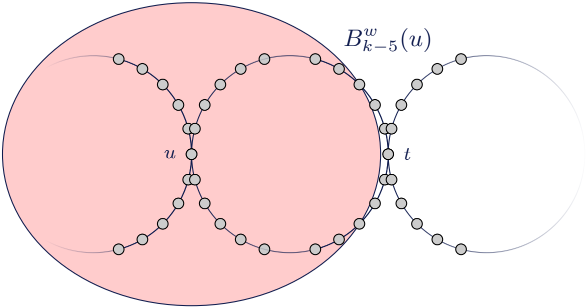

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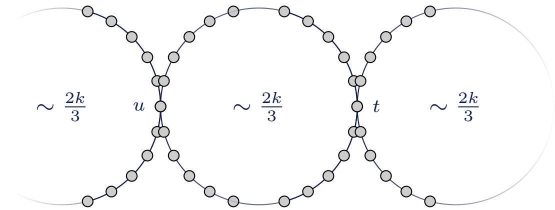

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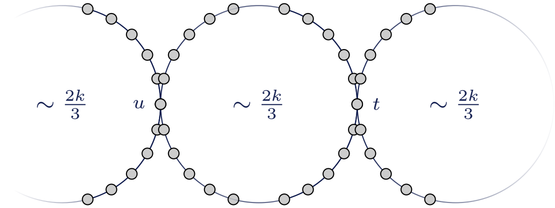

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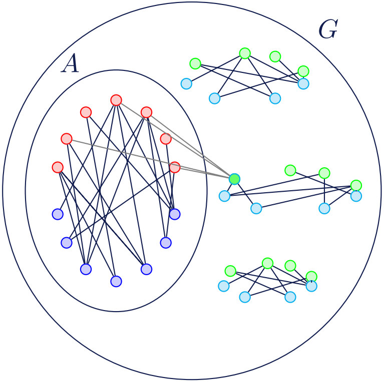

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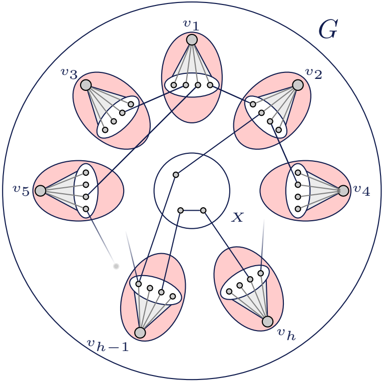

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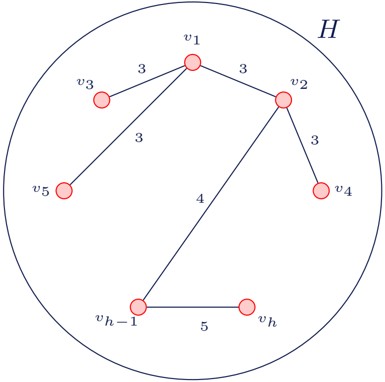

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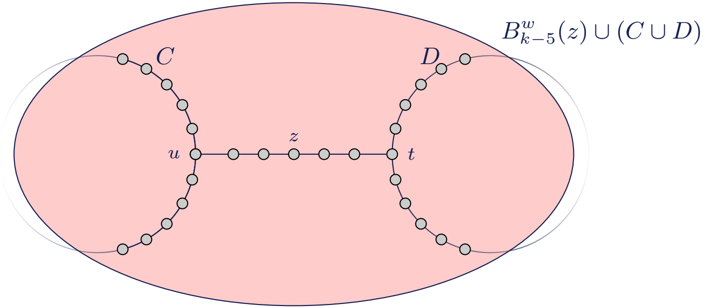

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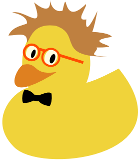

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