Isomorphism and Homomorphism

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March 2, 2023

1 Prerequisite

The following document is based on these two questions from a module on Abstract Algebra:

1. Let G and H be groups, and suppose $\theta : G \to H$ is an isomorphism (in the sense of Definition 6.9). Prove that if G is Abelian, then H is Abelian

2. Let V be "the" group from Section 4.1, i.e. a group of order 4 in which each non-identity element has order 2. Let $\theta : V \to \mathbb{Z}_4$ be a bijection. Without using any Cayley tables, show that θ cannot be a group homomorphism.

2 Isomorphism

For the first proof will we use the following definition of isomorphism:

Definition 6.9 Let $(G, *_G)$ and $(H, *_H)$ be groups. We say that G and H are isomorphic (as groups), denoted by $G \cong H$, if there is a bijection $\phi : G \to H$ satisfying $\phi(g_1 *_G g_2) = \phi(g_1) *_H \phi(g_2)\ \forall g_1, g_2 \in G$.

Proof. Suppose G is Abelian, now we have $xy = yx\ \forall x, y \in G$.
We now consider elements $c, d \in H$, since ϕ is an isomorphism from definition 6.9 we see that we have $c = \theta(a)$ and $d = \theta(b)$ for $a, b \in G$. Hence,

$$cd = \phi(a)\phi(b) = \phi(ab) = \phi(ba) = \phi(b)\phi(a) = dc$$

Notice how $\phi(ab) = \phi(ba)$ since G is supposed to be Abelian.
This proves that cd=dc for elements $c, d \in H$ therefore H is also Abelian $\square$

3 Homomorphism

For the second question we will set out to prove against one of the properties of homomorphism which is:

Let $\phi : G \to$ H be a group homomorphism.
(iv) $o(\phi(g))$ divides $o(g)\ \forall g \in G$.

Proof. Consider $g \in G$ to be a non-identity element so that now $o(g) = 2$
Now we need to prove that $o(\theta(g)) \nmid 2$:
Take $g \in G$ to be the element that maps to $\hat{2} \in \mathbb{Z}_4$:
Hence $\theta(g) = \hat{2}$

$$\hat{2} \cdot \hat{2} = \hat{4} = \hat{1}$$
$$\hat{2} \cdot \hat{2} \cdot \hat{2} = \hat{1} \cdot \hat{2} = \hat{2}$$

This shows that $o(\theta(g)) = o(\hat{2}) = 3$
Therefore θ cannot be a group homomorphism since $3 \nmid 2$. Hence we have proved against (iv) because $o(\theta(g)) \nmid o(g)\ \forall g \in G$ $\square$