A Characterization of Passivizing Input-Output Transformations of Nonlinear MIMO Systems

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***Abstract*—This paper provides a characterization of linear passivizing input-output transformations for MIMO systems with known passivity indices. Building on recent results for SISO systems, we show that any transformation mapping an I/O (ρ,ν)-passive MIMO system to an I/O $(\rho^*,\nu^\star)$-passive system can be expressed as the product of three matrices - two depending on the original and desired passivity indices, and a matrix satisfying a matrix inequality. This parameterization enables formulation of optimal passivation problems that we explore as an application example.**

***Index Terms*—Feedback passivation, passivity theory, nonlinear systems**

I. Introduction

Passive dynamical systems, with their inherent compositional properties, serve as powerful and versatile building blocks for large-scale interconnected systems [1]. These properties enable a streamlined analysis for analyzing the stability and performance of nonlinear systems, and has emerged as a cornerstone analytical tool for studying networked systems [2]–[4], cyber-physical systems [5], and power systems [6].

The passivity of a system can be quantified by so-called *passivity indices* [7]. Passivity indices can be used to characterize the excess or shortage of passivity for a non-linear system. It is often necessary to modify the passivity indices of a system in order to apply techniques from passivity-based control [8] or to ensure that certain interconnections of systems maintain desired passivity properties [9]. A behavioral model approach to the synthesis of passivizing transformations using quadratic forms was considered in [10], while other common methods include combinations of gains, output-feedback, and input-feedthrough [11]–[13]. Passivation methods may play an even more important role as the need for plug-and-play type networks grow [14].

Recent results by [7] showed that any system with finite $\mathcal{L}_2$-gain can be passivized using a (linear) input/output (I/O) transformation, found algebraically. These results were extended in [15] where a geometric approach for the design of a passivizing I/O transformation for SISO systems was proposed that enabled prescribing a desired passivity index. This was achieved using a connection between passivity and cones through the notion of projective quadratic inequalities (PQIs), which can be seen as a specific case of sector bounds [16]. We continued to explore this idea in [17] where we provided a characterization of all linear passivizing I/O transformations of a system with known passivity indices to a system with a prescribed set of indices. These results were limited to SISO systems.

In this work, we extend our results from [17] to deal with MIMO systems with equal input and output dimensions. Our approach leverages the PQI framework to study the action of linear transformations on solution sets of matrix inequalities. We show that any passivizing transformation can be decomposed into three components - two depending on the original and target passivity indices, and a matrix satisfying a special matrix inequality resembling an algebraic Riccati inequality. This parameterization enables the formulation, for example, of optimal passivation problems, analogous to the Youla parameterization for stabilizing controllers [18]. We demonstrate one possible utility for this approach by formulating an optimazation problem that seeks passivizing transformations minimizing some objective function.

The rest of the paper is organized as follows. Section II poses the general feedback passivation problem and summarizing the main result from [17]. Section III provides a characterization of passivizing transformation for MIMO systems. Section IV provides a possible application of the main results.

Notations: We denote the group of all invertible matrices $T\in\mathbb{R}^{d\times d}$ as $GL_d(\mathbb{R})$. Given a linear transformation $S:\mathbb{R}^d\to\mathbb{R}^d$ and a basis $\mathcal{B}$ for $\mathbb{R}^d$, we denote the representing matrix of S in the basis $\mathcal{B}$ as $[S]_\mathcal{B}$. Furthermore, given two bases $\mathcal{B}_1,\mathcal{B}_2$ of $\mathbb{R}^d$, we denote the change-of-base matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ by $I_{\mathcal{B}_1\to\mathcal{B}_2}\in GL_d(\mathbb{R})$. We note that $I^{-1}_{\mathcal{B}_1\to\mathcal{B}_2}=I_{\mathcal{B}_2\to\mathcal{B}_1}$. Moreover, for any linear transformation $S:\mathbb{R}^d\to\mathbb{R}^d$, we have that $[S]_{\mathcal{B}_2}=I_{\mathcal{B}_1\to\mathcal{B}_2}[S]_{\mathcal{B}_1}I_{\mathcal{B}_2\to\mathcal{B}_1}$. We also denote the Kronecker product by $\otimes$, and the $d\times d$ identity matrix as Id_d. Lastly, we denote the unit circle inside $\mathbb{R}^2$ by $\mathbb{S}^1$.

II. Problem Formulation and Background

We consider nonlinear multi-input multi-output (MIMO) dynamical systems given by the state-space representation $\dot{x}=f(x,u)$, $y=h(x,u)$, where $u,y\in\mathbb{R}^m$ are the input and output signals (of same dimension), and $x\in\mathbb{R}^{n_x}$ is the state of the system. We employ the following definition of passivity [19], [20].

Definition II.1. *Let Σ be a nonlinear MIMO dynamical system with equal input and output dimension. Assume that $u=0,y=0$ is an equilibrium of the system. Let $\rho,\nu\in\mathbb{R}$.*

i) *We say that the system is* output ρ-passive *if there exists a positive semidefinite C^1-smooth function storage function*

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S such that the inequality,

$$\frac{dS(x(t))}{dt} \leq u(t)^\top y(t) - \rho \|y(t)\|^2, \tag{1}$$

holds for any trajectory $(u(t), x(t), y(t))$ of the system.

ii) *We say that the system is* input ν-passive *if there exists a positive semidefinite C^1-smooth function storage function S such that the inequality,*

$$\frac{dS(x(t))}{dt} \leq u(t)^\top y(t) - \nu \|u(t)\|^2, \tag{2}$$

holds for any trajectory $(u(t), x(t), y(t))$ of the system.

iii) *We say that the system is* input-output (ρ, ν)-passive *if $\rho\nu < 1/4$ and there's a positive semidefinite C^1-smooth function storage function S such that,*

$$\frac{dS(x(t))}{dt} \leq u(t)^\top y(t) - \rho \|y(t)\|^2 - \nu \|u(t)\|^2, \tag{3}$$

holds for any trajectory $(u(t), x(t), y(t))$ of the system.

Note than when $\rho, \nu = 0$, the standard notion of passivity is recovered [19]. The common interpretation of the storage function $S(x)$ is that of the potential energy stored inside the system. This implies that the change in the energy stored in the system cannot be greater than the supplied power. The above definition expands this notion to consider both the case of total energy dissipation, and the case of (bounded) total energy gain, by adding either a negative or a positive term to the right-hand side of the inequalities.

The case in which $\rho, \nu > 0$ is commonly referred to as *strict passivity* (or "excess of passivity"), and the case in which $\rho, \nu < 0$ is usually called *passive short* (or "shortage of passivity"). We direct the reader to [20, Appendix Lemma 3] for a discussion on the requirement that $\rho\nu < 1/4$ for the input-output (ρ, ν)-passive case.

It is well-known that through appropriate loop transformations, the passivity indexes of a system can be modified [7], [15]. This is especially useful when there is a desire to apply methods from passivity-based control to passive-short systems. In this direction, we consider a transformed plant $\tilde{\Sigma}$ with new input $\tilde{u}$ and output $\tilde{y}$, which are connected to u, y via

$$\begin{bmatrix} u(t) \\ \tilde{y}(t) \end{bmatrix} = \hat{T} \begin{bmatrix} y(t) \\ \tilde{u}(t) \end{bmatrix} = \begin{bmatrix} \hat{T}_{11} & \hat{T}_{12} \\ \hat{T}_{21} & \hat{T}_{22} \end{bmatrix} \begin{bmatrix} y(t) \\ \tilde{u}(t) \end{bmatrix}, \tag{4}$$

for some invertible matrix $\hat{T}$. This transformation is an aggregation of a constant gain input-feedthrough, constant gain output-feedback, and cascade with a constant gain (see [15] for further details). This is visualized in Figure 1. In this work, it is convenient to work with a different representation of $\hat{T}$, mapping signals $(u(t), y(t))$ to $(\tilde{u}(t), \tilde{y}(t))$, as

$$\begin{bmatrix} \tilde{u}(t) \\ \tilde{y}(t) \end{bmatrix} = T \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} u(t) \\ y(t) \end{bmatrix}. \tag{5}$$

Note this is always possible if $\hat{T}_{21}$ is invertible.

We wish to understand the effect of these I/O transformations (5) on the passivity of the transformed system. Formally, the problem we consider is given below.

Problem II.1. *Let Σ be a dynamical system with equal input and output dimensions, which is I/O (ρ, ν)-passive, and let $\rho_\star, \nu_\star$ be numbers such that $\rho_\star \nu_\star < 1/4$. Characterize all linear I/O transformations of the form* (5) *such that the transformed system $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive.*

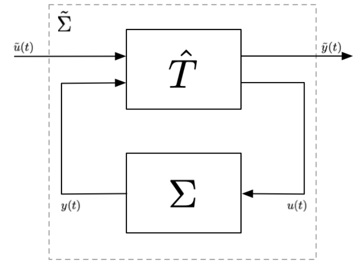

Fig. 1. Feedback passivation of Σ with $\hat{T}$.

In our previous work [17], we solved Problem II.1 for the SISO case (i.e., $m = 1$). The goal of this work is to extend the result to the MIMO case. As in [17] we consider a geometric approach to passivation, introduced in [15]. The main idea is to consider a generalization of the inequalities appearing in Definition II.1, called projective quadratic inequalities. The definition below is a higher-dimensional extension of the scalar definition given in [17].

Definition II.2 (PQI)**.** *A d-dimensional* projective quadratic inequality *(PQI) is an inequality in the variables $\xi, \chi \in \mathbb{R}^d$ of the form*

$$0 \leq a\|\xi\|^2 + b\xi^T\chi + c\|\chi\|^2 := \mathbf{f}_{(a,b,c)}(\xi, \chi), \tag{6}$$

for some numbers a, b, c, not all zero. The inequality is called non-trivial *if $b^2 - 4ac > 0$. The associated solution set of PQI is the set of all points $(\xi, \chi) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfying the inequality.*

Quadratic representations like this have been used, for example, to formulate passivity for sector bounded operators [16]. The relation of PQIs to input-output (ρ, ν)-passivity (3) can be seen by replacing ξ, χ, a, b, c with $u, y, -\nu, 1, -\rho$ respectively. For this reason, we denote $\mathbf{f}_{(a,b,c)}(\xi, \chi)$ as $\varphi_{\rho,\nu}(\xi, \chi)$, and the corresponding solution set as $C_{\rho,\nu}$. More explicitly, we define $\varphi_{\rho,\nu}$ as

$$\varphi_{\rho,\nu}(\xi, \chi) = -\nu\|\xi\|^2 + \xi\chi - \rho\|\chi\|^2,$$

and $C_{\rho,\nu,d}$ as $\{(\xi, \chi) \in \mathbb{R}^d \times \mathbb{R}^d : \ \varphi_{\rho,\nu}(\xi, \chi) \geq 0\}$. When $d = 1$ we simply write $C_{\rho,\nu}$.

We now relate the collection of solution sets of d-dimensional PQI's to the group of $2d \times 2d$ invertible matrices, $GL_{2d}(\mathbb{R})$. Let A be the solution set of a PQI, $A = \{(\xi, \chi) : 0 \leq \mathbf{f}_{(a,b,c)}(\xi, \chi)\}$. For any invertible matrix $T \in GL_{2d}(\mathbb{R})$, the solution set of the transformed PQI is given by $T(A)$, the image of A under T. Furthermore, one can show that an I/O transformation maps an I/O (ρ, ν)-passive system to an I/O $(\rho_\star, \nu_\star)$-passive system if and only if it maps the PQI $0 \leq \varphi_{\rho,\nu}(\xi, \chi)$ to the d-dimensional PQI $0 \leq \varphi_{\rho_\star,\nu_\star}(\xi, \chi)$ (or to a stricter inequality). For more discussion, see [15].

For SISO systems, the solution to Problem II.1 is built on the relationship between cones in $\mathbb{R}^2$ and the solution of 1-

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dimensional PQI's. We review the main ideas from [17] here which will be needed for the extension to the MIMO case.

Definition II.3. *A* symmetric section S *on the unit circle* $\mathbb{S}^1 \subseteq \mathbb{R}^2$ *is defined as the union of two closed disjoint sections that are opposite to each other, i.e.,* $S = B \cup (-B)$*, where* B *is a closed section of angle* $< \pi$*. A* symmetric double cone *is defined as* $A = \{\lambda s : \ \lambda > 0, s \in S\}$ *for some symmetric section* S*.*

An example of a symmetric section and the associated symmetric double-cone can be seen in Fig. 2. These are of interest due to their relationship with PQIs.

Theorem II.1 ([15])**.** *The solution set of any non-trivial* 1*-dimensional PQI is a symmetric double cone. Moreover, any symmetric double-cone is the solution set of some non-trivial* 1*-dimensional PQI, which is unique up to a multiplicative positive constant.*

From this result, it can be shown that a map transforms an I/O (ρ,ν)-passive system to an I/O $(\rho_\star,\nu_\star)$-passive system if and only if it sends $C_{\rho,\nu}$ into $C_{\rho_\star,\nu_\star}$, which we denote by $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$.

Theorem II.2 ([15])**.** *Let* $\rho,\nu,\rho_\star,\nu_\star$ *be any numbers such that* $\rho\nu, \rho_\star\nu_\star < 1/4$*. Let* (ξ_1,χ_1) *and* (ξ_2,χ_2) *be two non-colinear solutions to* $\varphi_{\rho,\nu}(\xi,\chi) = 0$*. Moreover, let* (ξ_3,χ_3) *and* (ξ_4,χ_4) *be two non-colinear solutions to* $\varphi_{\rho_\star,\nu_\star}(\xi,\chi) = 0$*. Define*

$$T_1 = \begin{bmatrix} \xi_3 & \xi_4 \\ \chi_3 & \chi_4 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1}, T_2 = \begin{bmatrix} \xi_3 & -\xi_4 \\ \chi_3 & -\chi_4 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1}.$$

Let α_1 *be equal to* 1 *if* $\varphi_{\rho,\nu}(\xi_1+\xi_2,\chi_1+\chi_2) \geq 0$ *and zero otherwise. Moreover, let* α_2 *be equal to* 1 *if* $\varphi_{\rho_\star,\nu_\star}(\xi_3+\xi_4,\chi_3+\chi_4) \geq 0$ *and zero otherwise.*

i) *If* $\alpha_1 = \alpha_2$*, then* T_1 *is* $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$*.*
ii) *If* $\alpha_1 \neq \alpha_2$*, then* T_2 *is* $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$*.*

Theorem II.2 can be used to show that all maps from an arbitrary cone into another arbitrary cone can be built using maps from $C_{0,0}$ into itself.

Proposition II.1 ([17])**.** *Let* $\rho,\nu,\rho_\star,\nu_\star$ *be any four numbers such that* $\rho\nu, \rho_\star\nu_\star < 1/4$*, and let* T *be any matrix* $C_{\rho,\nu} \hookrightarrow C_{\rho_\star,\nu_\star}$*. Let* $S_{\rho,\nu}, S_{\rho_\star,\nu_\star}$ *be the invertible matrices* $C_{0,0} \hookrightarrow C_{\rho,\nu}, C_{0,0} \hookrightarrow C_{\rho_\star,\nu_\star}$ *respectively, as built using Theorem II.2. Then there exists a matrix* Q*, which is* $C_{0,0} \hookrightarrow C_{0,0}$*, such that* $T = S_{\rho_\star,\nu_\star} Q S_{\rho,\nu}^{-1}$ *holds.*

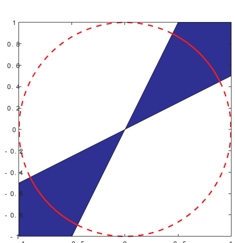

Fig. 2. A double cone (in blue), and the associated symmetric section (in solid red). The parts of $\mathbb{S}^1$ outside the symmetric section are presented by the dashed red line.

Proposition II.1 depends on construction of matrices $S_{\mu,\tau}$, and matrices mapping $C_{0,0}$ into itself. We summarize how to do this below.

Proposition II.2 ([17])**.** *A matrix* $T \in GL_2(\mathbb{R})$ *sends* $C_{0,0}$ *into itself if and only if all of the entries of* T *have the same sign, i.e.,* $T_{ij}T_{kl} \geq 0$ *for every* $i,j,k,l \in \{1,2\}$*.*

For the matrices $S_{\mu,\tau}$:

Proposition II.3 ([17])**.** *, let* μ,τ *be any two numbers such that* $\mu\tau < 1/4$*. Recall that* $S_{\mu,\tau}$ *is a map* $C_{0,0} \hookrightarrow C_{\mu,\tau}$*, as constructed in Theorem II.2. Define* $R = \sqrt{1-4\tau\mu}$*.*

i) *If* $\tau < 0$*, we can choose* $S_{\mu,\tau} = \frac{1}{2\tau}\left[\begin{smallmatrix} -1-R & 1-R \\ -2\tau & 2\tau \end{smallmatrix}\right]$*.*
ii) *If* $\tau > 0$*, we can choose* $S_{\mu,\tau} = \frac{1}{2\tau}\left[\begin{smallmatrix} 1+R & 1-R \\ 2\tau & 2\tau \end{smallmatrix}\right]$*.*
iii) *If* $\tau = 0$*, we can choose* $S_{\mu,\tau} = \left[\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right]$*.*

We can now state the main result from [17].

Theorem II.3 ([17])**.** *Let* Σ *be a SISO I/O* (ρ,ν)*-passive system, and let* $T \in GL_2(\mathbb{R})$ *be an invertible matrix transformation of the form* (5)*. The transformed system* $\tilde{\Sigma}$ *is I/O* $(\rho_\star,\nu_\star)$*-passive if and only if there exists a matrix* $M \in GL_2(\mathbb{R})$ *such that* $M_{ij} \geq 0$ *for all* $i,j \in \{1,2\}$ *and some* $\theta \in \{\pm 1\}$ *with* $T = \theta S_{\rho_\star,\nu_\star} M S_{\rho,\nu}^{-1}$*, where* $S_{\rho,\nu}, S_{\rho_\star,\nu_\star}$ *are given in Proposition II.3. In other words, the transformed system* $\tilde{\Sigma}$ *is I/O* $(\rho_\star,\nu_\star)$*-passive if and only if all of the entries of the matrix* $S_{\rho_\star,\nu_\star}^{-1} T S_{\rho,\nu}$ *have the same sign.*

III. A Characterization of Linear Passivizing Transformations for MIMO Systems

Our result from [17] gave an explicit description of all linear I/O transformations mapping I/O (ρ,ν)-passive SISO systems to I/O $(\rho_\star,\nu_\star)$-passive SISO systems. One could try and generalize this idea to MIMO systems, but a few problems arise. The cornerstone in the characterization for SISO systems was Theorem II.2, whose proof uses the fact that for SISO systems, the solution sets of PQIs are two-dimensional, and their boundary is the union of two straight lines [15]. For $d \times d$ MIMO systems, the solution set of a PQI lies in $\mathbb{R}^{2d}$, and its boundary, in general, is of dimension $2d-1$ (almost everywhere). Thus, a geometric framework for MIMO systems cannot be applied easily.

To deal with the MIMO case, we use a similar idea, studying the action of the collection of $2d \times 2d$ invertible linear transformations, $GL_{2d}(\mathbb{R})$, on the collection of d-dimensional PQIs. As before, we use the notion of solution sets. We recall that we denoted the solution set of the d-dimensional PQI $\varphi_{\rho,\nu}(\xi,\chi) \geq 0$ by $C_{\rho,\nu,d}$. The map T maps one d-dimensional PQI to another if and only if it maps the associated solution sets to another. We start with the following proposition:

Proposition III.1. *Let* $\rho,\nu,\rho_\star,\nu_\star$ *be any real numbers, and let* $S \in GL_2(\mathbb{R})$ *be any matrix mapping the 1-dimensional PQI* $0 \leq -\nu\xi^2 + \xi\chi - \rho\chi^2$ *to the 1-dimensional PQI* $0 \leq -\nu_\star\xi^2 + \xi\chi - \rho_\star\chi^2$*. Then* $S \otimes \mathrm{Id}_d$ *maps the* d*-dimensional PQI* $-\nu\|\xi\|^2 + \xi^\top\chi - \rho\|\chi\|^2$ *to the* d*-dimensional PQI* $-\nu_\star\|\xi\|^2 + \xi^\top\chi - \rho_\star\|\chi\|^2$*.*

Thus, the MIMO analogue of the transformations $S_{\rho,\nu}$ are $S_{\rho,\nu} \otimes \mathrm{Id}_d$. We now prove the proposition.

Proof. We define $A = \left[\begin{smallmatrix} -\rho & 0.5 \\ 0.5 & -\nu \end{smallmatrix}\right], B = \left[\begin{smallmatrix} -\rho_\star & 0.5 \\ 0.5 & -\nu_\star \end{smallmatrix}\right]$. The 1-dimensional PQI $0 \leq -\nu_\star\xi^2 + \xi\chi - \rho_\star\chi^2$ can be written as $\Xi^\top A \Xi \geq 0$, where $\Xi = [\chi, \xi]^\top \in \mathbb{R}^2$, and the 1-dimensional PQI $0 \leq -\nu_\star\xi^2 + \xi\chi - \rho_\star\chi^2$ is written as $\Xi^\top B \Xi \geq 0$. By setting $\tilde{\Xi} = S\Xi$, we see that S maps the first 1-dimensional PQI to the second if and only if $(S^{-1})^\top A S^{-1} = B$, and the latter condition implies

$$((S \otimes \mathrm{Id}_d)^{-1})^\top (A \otimes \mathrm{Id}_d)(S \otimes \mathrm{Id}_d)^{-1} = B \otimes \mathrm{Id}_d.$$

The proof in now complete, as we note the d-dimensional PQIs can be written as $\Xi_d^\top (A \otimes \mathrm{Id}_d)\Xi_d \geq 0$ and $\Xi_d^\top (B \otimes \mathrm{Id}_d)\Xi_d \geq 0$, where $\Xi_d = [\xi^\top, \chi^\top]^\top \in \mathbb{R}^{2d}$. □

Remark III.1. *Proposition III.1 does* not *claim that* all *maps between d-dimensional PQIs stem from maps between 1-dimensional PQIs using the Kronecker product, but that examples of such maps can be built using the Kronecker products. Thus, Proposition III.1 provides a practical construction of such maps. We show in the following results that this is sufficient to characterize all linear transformations from a given passivity index to a desired passivity index.*

We now search for a MIMO analogue for the second component we had, namely non-negative matrices. Before, non-negative matrices stemmed from maps $C_{0,0} \hookrightarrow C_{0,0}$.

Proposition III.2. *An invertible matrix $T \in GL_{2d}(\mathbb{R})$ maps $C_{0,0,d}$ into itself if and only if there exists some $\lambda > 0$ such that $T^\top JT - \lambda J \geq 0$, where $J = \begin{bmatrix} 0 & 0.5\mathrm{Id}_d \\ 0.5\mathrm{Id}_d & 0 \end{bmatrix}$.*

Proof. As before, we denote the stacked variable vector as $\Xi_d = [\xi^\top, \chi^\top]^\top \in \mathbb{R}^{2d}$. The set $C_{0,0,d}$ is the collection of all vectors Ξ_d satisfying $\Xi_d^\top J \Xi_d \geq 0$. The image of $C_{0,0,d}$ under T consists of all vectors $\tilde{\Xi}_d$ such that $\tilde{\Xi}_d^\top (T^{-1})^\top J T^{-1} \tilde{\Xi}_d \geq 0$. Thus, T maps $C_{0,0,d}$ inside itself if and only if the following implication holds:

$$\tilde{\Xi}_d^\top (T^{-1})^\top J T^{-1} \tilde{\Xi}_d \geq 0 \implies \tilde{\Xi}_d^\top J \tilde{\Xi}_d \geq 0, \forall\, \tilde{\Xi}_d \in \mathbb{R}^{2d}.$$

By the S-lemma, or S-procedure, [21, Appendix B], the above implication is equivalent to the existence of some $\mu > 0$ such that $(T^{-1})^\top J T^{-1} - \mu J \leq 0$. By multiplying the inequality by $T^\top$ on the left and by $\mu^{-1}T$ on the right, the inequality is equivalent to $T^\top JT - \lambda J \geq 0$, where $\lambda = \mu^{-1} > 0$. □

Remark III.2. *The inequality $T^\top JT - \lambda J \geq 0$ can be seen as a generalized version of an algebraic Riccati inequality. Indeed, the algebraic Riccati inequality is given as $A^\top P + PA - PXP + Q \leq 0$, where X, Q are positive-definite matrices, and P is a symmetric matrix variable [22]. Choosing $Q = \lambda J, A = 0$ and $X = J$, and not restricting the matrix P to be symmetric, results in the inequality $\lambda J - P^\top JP \leq 0$.[1] As Q, J are not positive definite, this is a generalized version of an algebraic Riccati inequality.*

Combining Propositions III.1 and III.2, we conclude with the following theorem:

[1]We use $P^\top XP$ instead of PXP to guarantee the product is symmetric.

Theorem III.1. *Let Σ be an I/O (ρ, ν)-passive system with input and output dimension equal to d, and let $T \in GL_{2d}(\mathbb{R})$ be an invertible matrix inducing an I/O transformation of the form* (5)*. The transformed system $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive if and only if there exists a matrix $M \in GL_{2d}(\mathbb{R})$ and some positive $\lambda > 0$ such that:*

$$T = (S_{\rho_\star,\nu_\star} \otimes \mathrm{Id}_d) M (S_{\rho,\nu}^{-1} \otimes \mathrm{Id}_d), \quad M^\top JM - \lambda J \geq 0,$$

where $J = \begin{bmatrix} 0 & 0.5\mathrm{Id}_d \\ 0.5\mathrm{Id}_d & 0 \end{bmatrix}$, i.e., $\tilde{\Sigma}$ is I/O $(\rho_\star, \nu_\star)$-passive if and only if there exists $\lambda > 0$ such that $X = (S_{\rho_\star,\nu_\star}^{-1} \otimes \mathrm{Id}_d) T (S_{\rho,\nu} \otimes \mathrm{Id}_d)$ satisfies $X^\top JX - \lambda J \geq 0$.

Proof. The proof is nearly identical to that of Proposition II.1, where we replace the sets $C_{\rho,\nu}, C_{\rho_\star,\nu_\star}$ by the corresponding d-dimensional PQIs. □

The theorem can be seen as a generalization of Theorem II.3, as one can verify that for $d = 1$, $X^\top JX - \lambda J \geq 0$ for some $\lambda > 0$ if and only if all of X-s entries possess the same sign.

Proposition III.3. *Let $X \in GL_2(\mathbb{R})$, and let $J = \left[\begin{smallmatrix} 0 & 0.5 \\ 0.5 & 0 \end{smallmatrix}\right]$. There exists some $\lambda > 0$ such that $X^\top JX - \lambda J \geq 0$ if and only if all of the entries of X possess the same sign.*

Proof. Write $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. The matrix $X^\top JX - \lambda J$ can be computed as:

$$X^\top JX - \lambda J = 0.5 \left[\begin{smallmatrix} 2ac & ad+bc-\lambda \\ ad+bc-\lambda & 2bd \end{smallmatrix}\right].$$

By Sylvester's criterion, $X^\top JX - \lambda J$ is positive semi-definite if and only if all of its principal minors are non-negative, i.e. $ac \geq 0$, $bd \geq 0$ and $\det(X^\top JX - \lambda J) \geq 0$. From the first two inequalities we conclude that a, c possess the same sign, and the same holds for b, d. By switching between $X, -X$, we may assume without loss of generality that a, c are non-negative. If b, d are also non-negative, the proof is complete. Thus, it's enough to show that if b, d are non-positive (and not both zero), then for any $\lambda > 0$, $2\det(X^\top JX - \lambda J) < 0$. By definition, we have,

$$2\det(X^\top JX - \lambda J) = -(ad + bc - \lambda)^2 + 4abcd.$$

Moreover, if b, d are non-positive then $abcd \leq 0$. If $abcd < 0$, then $4\det(X^\top JX - \lambda J)$ must be negative. Otherwise, the determinant can be non-negative only at $\lambda = ad + bc$, but because $a, c \geq 0$, if $b, d \leq 0$ then $ad + bc$ is non-positive. In particular, if $b, d \leq 0$ then the determinant is negative for all $\lambda > 0$. Thus, $\det(X^\top JX - \lambda J) \geq 0$ is equivalent to $a, b, c, d \geq 0$. This concludes the proof. □

Remark III.3. *The above discussion provides a complete characterization of all transformations that map a system with given passivity indices to a system that has at least a prescribed amount of excess of passivity. One could ask what transformations T yield the exact required excess of passivity, assuming that the original passivity indices where sharp. A geometric argument shows that this occurs if and only if both the matrix M, defined in the decomposition of Theorem III.1, as well as its inverse M^{-1}, satisfy the generalized Ricatti inequality. For the specific case of one dimension, an* *easy computation shows that the corresponding version of Proposition III.3 would add the requirement that the matrix X is diagonal.*

It is also of note that the above results can be easily adapted to handle other notions of passivity, such as equilibrium independent passivity (EIP) [23], by choosing different values of ξ and χ to capture the specific definition. Here, we consider finding passivizing transformations for a collection of steady-state equilibrium pairs. The result below is a direct consequence of Proposition III.1.

Proposition III.4. *Consider a MIMO system Σ with input- and output-dimension equal to d. Let $\{(\mathrm{u}_i, \mathrm{y}_i)\}$ be a collection of steady-state I/O pairs of Σ, and let $(\rho_i, \nu_i), (\rho_i^\star, \nu_i^\star)$ be real numbers such that for each i, $\rho_i\nu_i, \rho_i^\star\nu_i^\star < 1/4$. Suppose that for each i, the system Σ is I/O (ρ_i, ν_i)-passive with respect to the steady-state I/O pair $(\mathrm{u}_i, \mathrm{y}_i)$. Consider a general I/O transformation T of the form* (5)*, and consider the new system $\tilde{\Sigma}$ and the new steady-state pairs $\{T(\mathrm{u}_i, \mathrm{y}_i)\}$. $\tilde{\Sigma}$ is I/O $(\rho_i^\star, \nu_i^\star)$-passive with respect to $T(\mathrm{u}_i, \mathrm{y}_i)$, for all i, if and only if there exists matrices $T_i \in GL_{2d}(\mathbb{R})$, and numbers $\lambda_i \geq 0$ such that the following set of constraints holds:*

$$\begin{cases} T = (S_{\rho_i^\star,\nu_i^\star} \otimes \mathrm{Id}_d) T_i (S_{\rho_i,\nu_i}^{-1} \otimes \mathrm{Id}_d) \\ T_i^\top J T_i - \lambda_i J \geq 0. \end{cases} \tag{7}$$

As before, the proof of the proposition follows immediately from Proposition III.1. We again note that when we wish to passivize the system Σ with respect to all equilibria (i.e., $\rho_i^\star, \nu_i^\star = 0$), we get the following set of equations and inequalities:

$$T = T_i (S_{\rho_i,\nu_i}^{-1} \otimes \mathrm{Id}_d), \text{ and } T_i^\top J T_i - \lambda_i J \geq 0.$$

IV. Application: Optimal Passivizing Transformations

In this section, we consider a possible application of the achieved characterization for synthesis. In the previous sections of the paper, we characterized all transformations that passivize a given system Σ. Thus, it is natural to ask questions such as “which passivizing transformation minimizes (or maximizes) a given quantity?” One class of quantities of interest can be system-theoretic properties of the transformed system, e.g. the $\mathcal{L}_2$-gain or tracking error for a given input and a desired output. Another class of interesting quantities to optimize consists of properties of T. These include, for example, the distance of the transformation T from a nominal transformation T_0, e.g. the identity, which is related to the control effort applied on the system. One could also consider “mixed” quantities, e.g. the distance in the $\mathcal{H}_\infty$-norm between the original system and the transformed system.

Generally, one could consider a transformation T that maps a given I/O (ρ, ν)-system to an I/O $(\rho_\star, \nu_\star)$ system. The quantity we wish to minimize can be written as a function $\Phi(T)$ of T. The associated optimization problem reads:

$$\min_T \quad \Phi(T) \quad \text{s.t.} \quad T \text{ maps I/O } (\rho,\nu) \text{ to I/O } (\rho_\star,\nu_\star).$$

One could use Theorem III.1 to restate the optimization problem in a tractable form:

$$\begin{aligned} \min_{T,\lambda,M} \quad & \Phi(T) \\ \text{s.t.} \quad & M = (S_{\rho^\star,\nu^\star} \otimes \mathrm{Id}_d)^{-1} T (S_{\rho,\nu} \otimes \mathrm{Id}_d) \\ & M^\top J M - \lambda J \geq 0 \\ & \lambda \geq 0, \end{aligned} \tag{8}$$

where $J = \left[\begin{smallmatrix} 0 & 1/2 \\ 1/2 & 0 \end{smallmatrix}\right]$. This optimization problem can be easily defined for any cost function Φ, whether it is explicitly defined using a formula involving T, or implicitly defined by a characteristic of the transformed system $\tilde{\Sigma}$. However, solving the optimization problem can be hard. First, the function Φ might not be explicitly given, or non-convex. Second, even if the function Φ was convex, the constraint $M^\top J M - \lambda J \geq 0$ is non-convex, as the matrix J is not positive semi-definite. We should note, however, that the latter problem can be easily remedied for SISO systems. Indeed, by Proposition III.3, the constraint can be replaced by:

i) $T = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$
ii) a, b, c, d have the same sign.

This constraint is still non-convex, but can be convexified by separating the problem into two sub-problems, one with the constraint $a, b, c, d \geq 0$, and one with the constraint $a, b, c, d \leq 0$.

Remark IV.1. *Returning to the MIMO case, one can prove that the matrix $M = \left[\begin{smallmatrix} \alpha\mathrm{Id}_d & \beta\mathrm{Id}_d \\ \gamma\mathrm{Id}_d & \delta\mathrm{Id}_d \end{smallmatrix}\right] \in \mathbb{R}^{2d\times 2d}$ satisfies the inequality $M^\top JM - \lambda J \geq 0$ for some $\lambda > 0$ whenever $\alpha, \beta, \gamma, \delta$ have the same sign, similarly to Proposition III.3. Thus, one can consider a tractable relaxation of the optimization problem* (8) *by similarly replacing the constraints $M^\top JM - \lambda J \geq 0, \lambda \geq 0$ by the constraint $M = \left[\begin{smallmatrix} \alpha\mathrm{Id}_d & \beta\mathrm{Id}_d \\ \gamma\mathrm{Id}_d & \delta\mathrm{Id}_d \end{smallmatrix}\right] \in \mathbb{R}^{2d\times 2d}$ and demanding that $\alpha, \beta, \gamma, \delta$ have the same sign. This will result in a tractable solution transformation T, which might be suboptimal in terms of the cost function Φ.*

We now give examples of a tractable optimization problem. The example is stated for SISO systems to aid in the exposition, but the methods employed rely on the general MIMO formulation developed in Section III.

Example IV.1. *Consider the problem of transforming an SISO I/O (ρ, ν)-passive system to an I/O $(\rho_\star, \nu_\star)$-passive system. We wish to find such a transformation which is closest to a given transformation T_0, i.e., minimizes the operator norm $\|T - T_0\|$. By the discussion above, we can write* (8) *with $\Phi(T) = \|T - T_0\|$. However, minimizing $\|T - T_0\|$ directly is hard. Instead, we introduce a new variable γ and demand that $(T - T_0)(T - T_0)^\top \leq \gamma\mathrm{Id}_2$, so that minimizing γ gives the desired result (and the operator norm $\|T - T_0\|$ is given by $\sqrt{\gamma}$). One can rewrite the last inequality as a linear matrix inequality using Schur's complements, giving the following equivalent* *optimization problem:*

$$\begin{aligned}\min_{T,M,\lambda,\gamma} \quad & \gamma \\ \text{s.t.} \quad & M = (S_{\rho^\star,\nu^\star}\otimes \mathrm{Id}_d)^{-1}T(S_{\rho,\nu}\otimes \mathrm{Id}_d) \\ & M^\top J M - \lambda J \geq 0 \\ & \begin{bmatrix} \mathrm{Id}_2 & T-T_0 \\ T^\top - T_0^\top & \gamma \mathrm{Id}_2 \end{bmatrix} \geq 0 \\ & \lambda \geq 0. \end{aligned} \tag{9}$$

where the matrix inequalities are understood as LMIs (rather than elementwise inequalities).

As a concrete example, we take $(\rho,\nu) = (0,-1)$ *and* $(\rho_\star,\nu_\star) = (1,0)$. *We thus seek a transformation of the form* (5) *mapping an input passive-short SISO system with parameter* $\nu = -1$ *to an output strictly-passive system with* $\rho = 1$. *Classically, one would first use feed-through to passivize the system, and then implement a feedback to increase its output passivity. This results in the transformation* $T = \left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right] = \left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]$. *We wish to find such a transformation which is closest to the identity transformation, i.e., minimizes the operator norm* $\|T - \mathrm{Id}_2\|$. *Using Proposition III.3,* (9) *is recast as:*

$$\begin{aligned}\min_{T,\gamma,a,b,c,d} \quad & \gamma \\ \text{s.t.} \quad & a,b,c,d \text{ have the same sign} \\ & T = \begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}, \\ & \begin{bmatrix} \mathrm{Id}_2 & T-\mathrm{Id}_2 \\ T^\top - \mathrm{Id}_2 & \gamma \mathrm{Id}_2 \end{bmatrix} \geq 0. \end{aligned}$$

This problem is non-convex, but it can be written as the minimum of cone programming problems, one with the constraint $a,b,c,d \geq 0$, *and one with the constraints* $a,b,c,d \leq 0$. *We can thus solve the problem by computer and find the optimal transformation* $T = \left[\begin{smallmatrix}1.5&0.5\\0.5&0.5\end{smallmatrix}\right]$, *which corresponds to* $a = 1, b = c = 0, d = 0.5$. *The operator norm in this case is* $\sqrt{\gamma} = \frac{1}{\sqrt{2}} \approx 0.707$. *As a comparison, the operator norm* $\|T - I\|$ *for* $T = \left[\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right]$ *is* $\frac{\sqrt{3+\sqrt{5}}}{\sqrt{2}} \approx 1.618$.

V. Conclusions

This work extended our recent results from [17] which provided a characterization of all linear passivizing transformations of a plant with given passivity indices. We showed how this may also be performed for MIMO systems. The main novelty compared to the SISO case is that the non-negative matrix is replaced by a matrix satisfying a special matrix inequality. Future work can try and better characterize the collection of matrices satisfying this inequality, giving a more explicit characterization for the MIMO case.

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