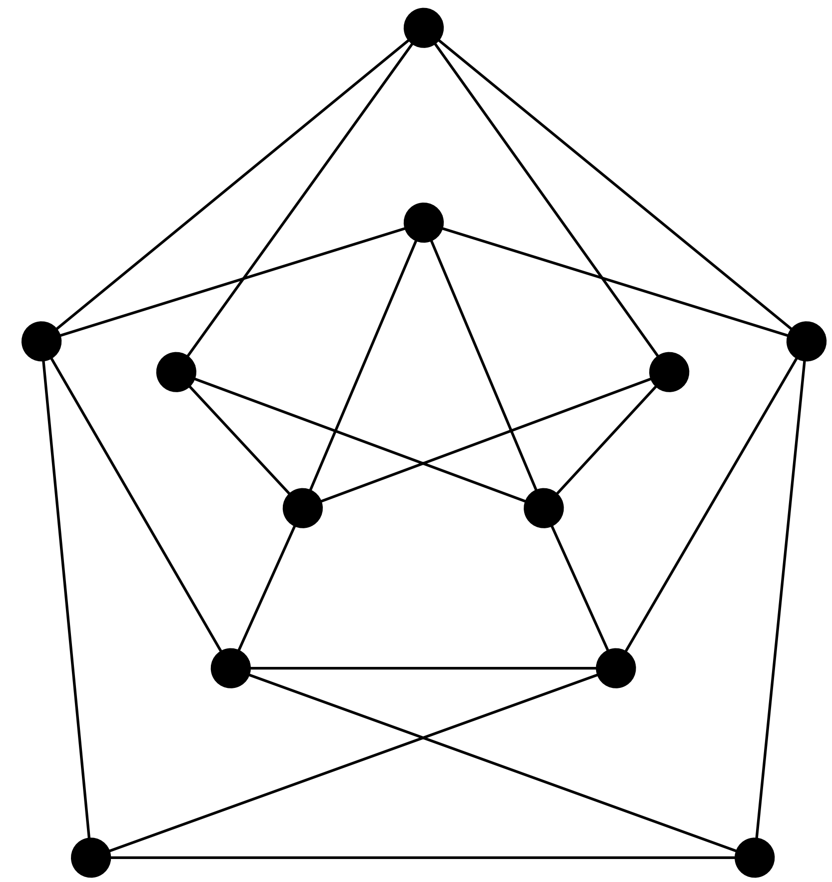
Eulerian Consensus Networks



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Conference on Decision and Control 2012
Maui, Hawaii

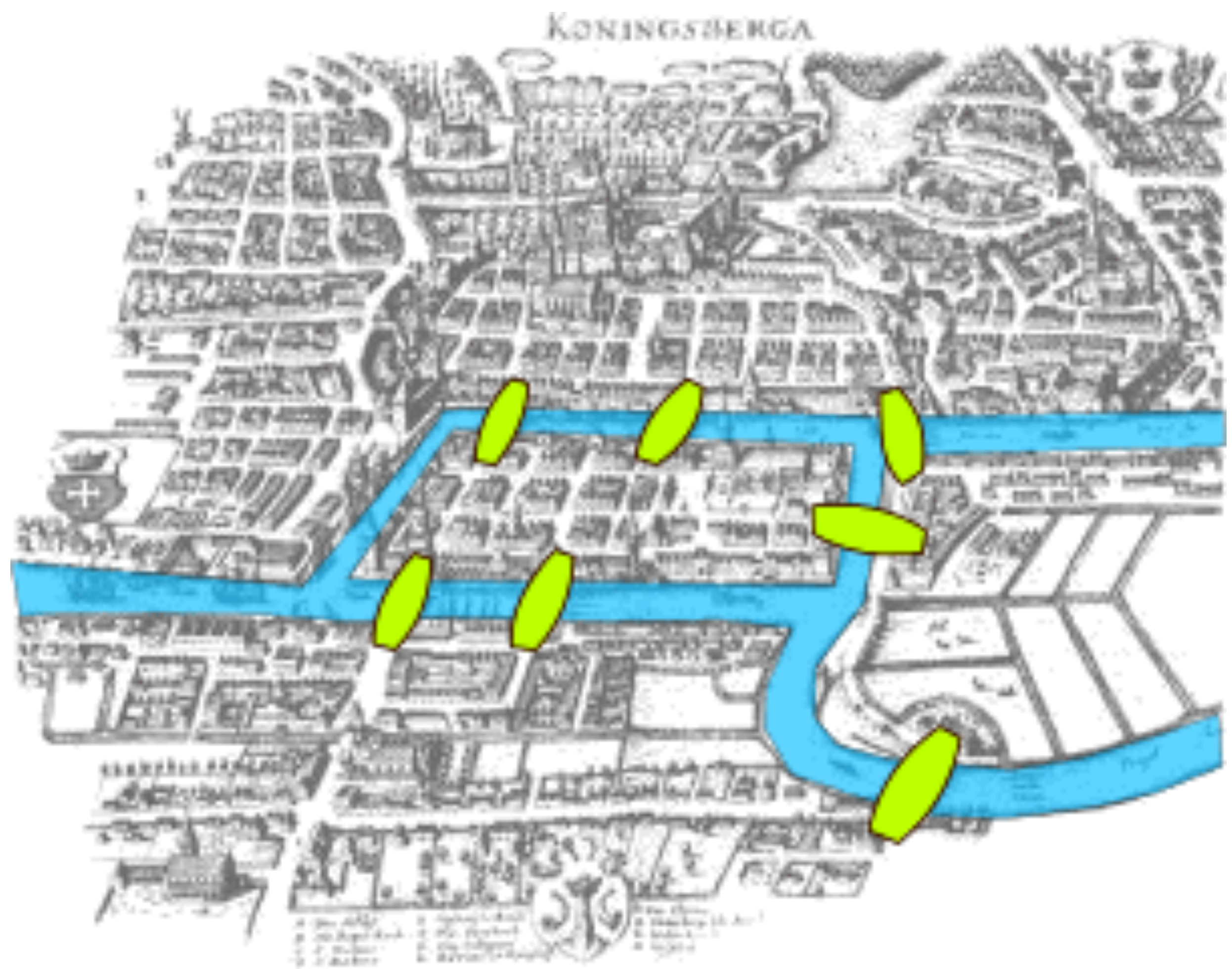
Leonard Euler

1707 - 1783

Euler to Diderot:

Sir, $\frac{a+b^n}{n} = x$ hence, God exists; reply!

The Bridges of Königsberg



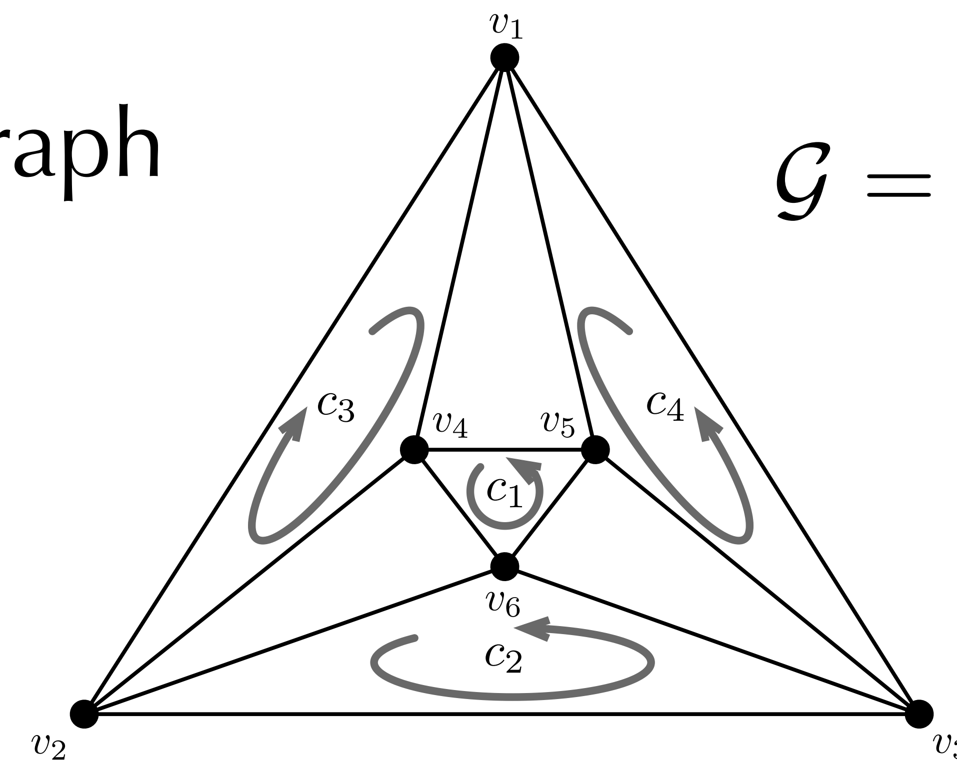
Does there exist a walk that crosses each bridge once and only once?

No!

Eulerian Graphs

An *Eulerian Cycle* is a walk on a graph beginning and ending at the same node that traverses each edge only once.

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$



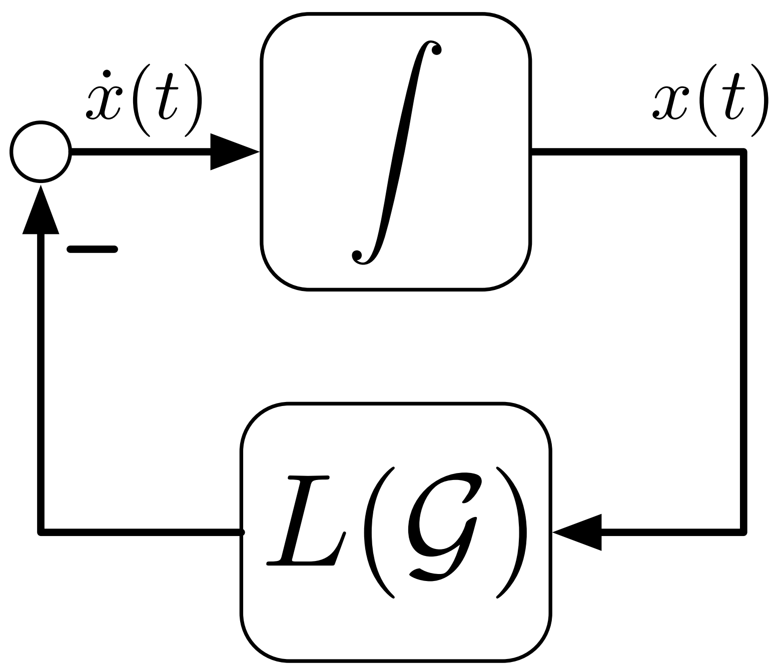
Proposition [Godsil] Given a connected graph, the following are equivalent:

1. $\mathcal{G}$ is an Eulerian graph.
2. The degree of each node is even.
3. $\mathcal{G}$ is the union of edge-disjoint cycles; i.e., $\mathcal{E} = \cup_{i=1}^{k} c_i$ and $c_i \cap c_j = \emptyset, \forall i, j$.

Eulerian Consensus Networks

the consensus protocol

$$\dot{x}_i(t) = \sum_{i \sim j} x_j(t) - x_i(t)$$



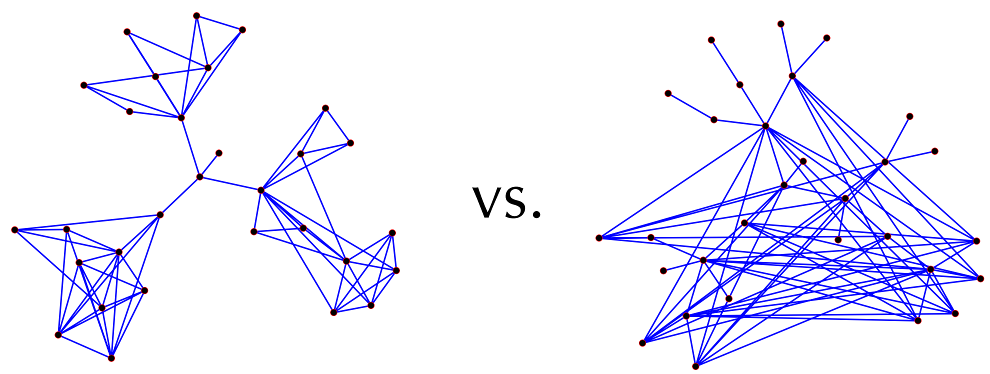
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the graph

$$L(\mathcal{G}) = E(\mathcal{G})E(\mathcal{G})^T$$

Laplacian — Incidence matrix

$\mathcal{G}$ is an Eulerian graph

Performance of Consensus



Are certain information structures more favorable to others?

$$\begin{matrix} \mathcal{H}_2 \\ \mathcal{H}_\infty \\ \vdots \end{matrix} \quad \propto \quad \begin{matrix} \text{cycle lengths} \\ \text{node degree} \\ \vdots \end{matrix}$$

Can notions of *dynamic system performance* be explained in terms of *properties of the graph?*

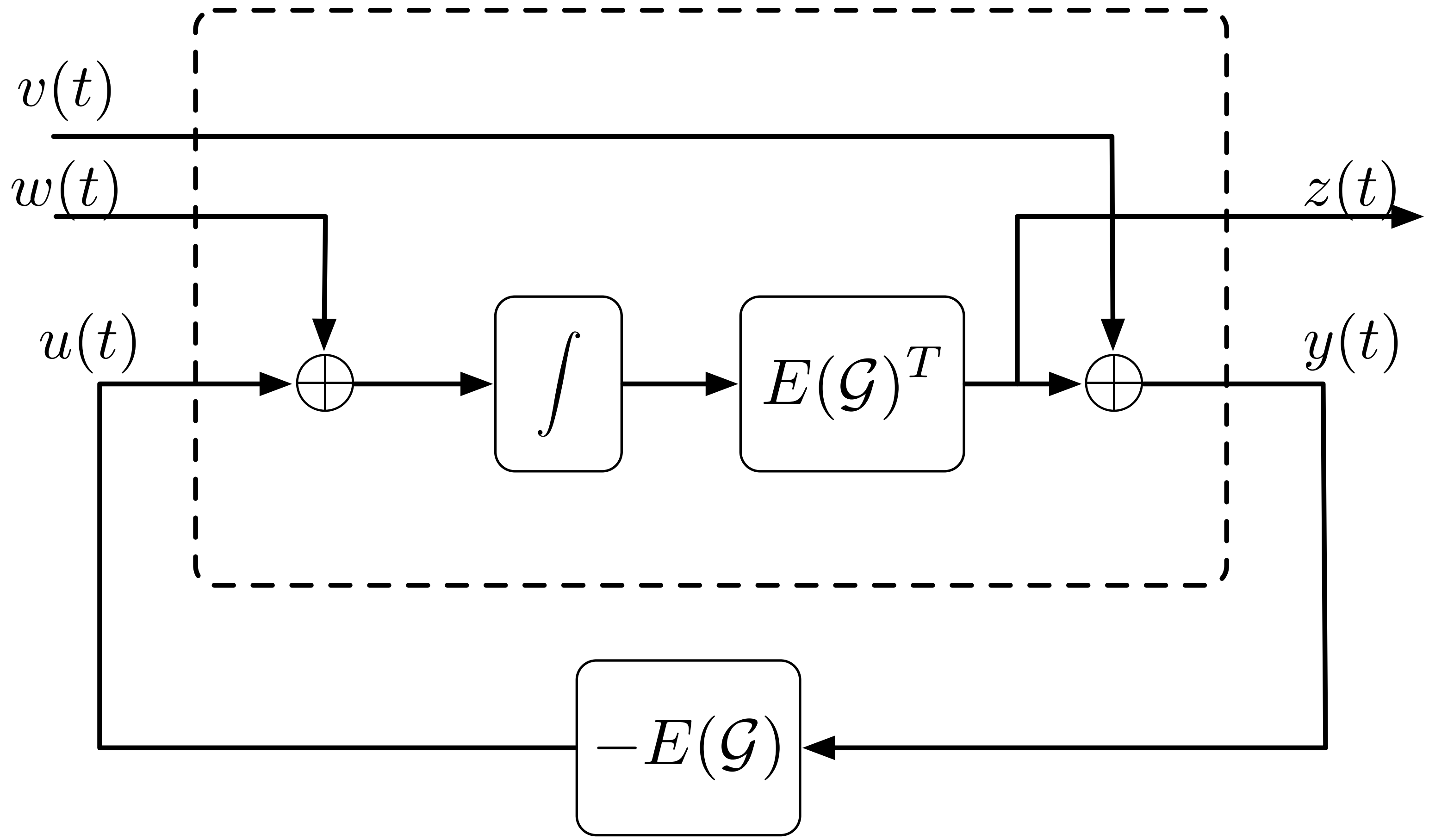
$$\min_{\mathcal{G}} \|\Sigma(\mathcal{G})\|_p$$

How do we *synthesize* good information structures?

Eulerian networks lead to efficient and large-scale design methods

The Consensus Protocol

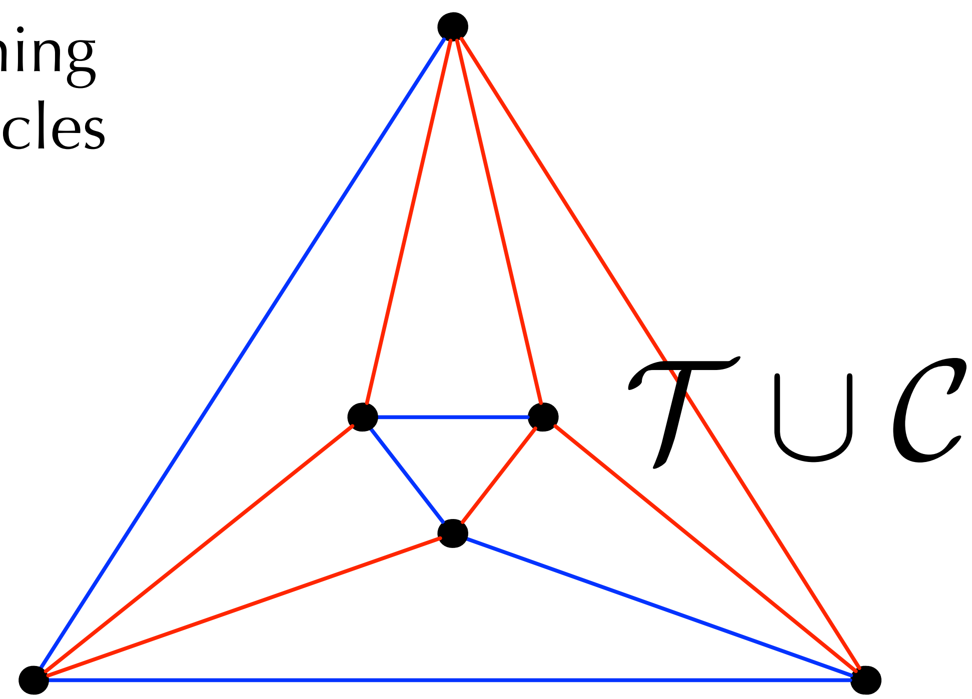
An ‘input-output’ consensus model



How do disturbances and noises affect the performance of the consensus protocol?

Spanning Trees and Cycles

A graph as the union of a spanning tree and edges that complete cycles



a spanning tree

remaining edges "complete cycles"

Edge Laplacian

$$L_e(\mathcal{G}) = E(\mathcal{G})^T E(\mathcal{G})$$

Essential Edge Laplacian

$$L_e(\mathcal{T})\mathcal{R}_{(\mathcal{T},\mathcal{C})}\mathcal{R}^T_{(\mathcal{T},\mathcal{C})}$$

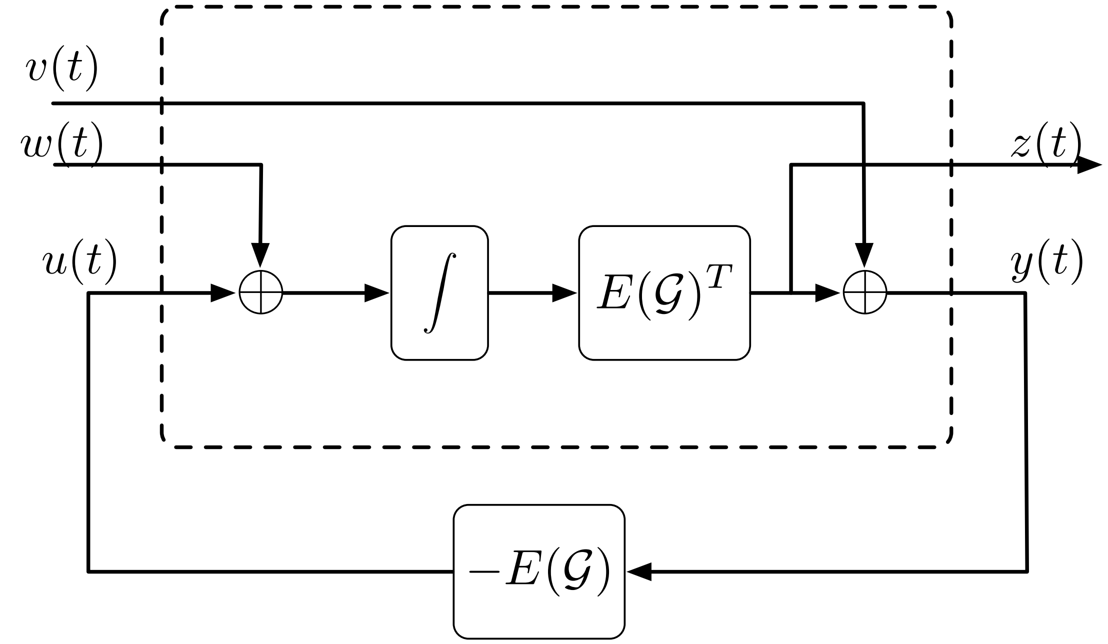
$\mathcal{R}_{(\mathcal{T},\mathcal{C})}$ rows form a basis for the *cut space* of the graph

similarity between edge and graph Laplacians

$L(\mathcal{G}) \rightleftharpoons L_e(\mathcal{G})$

The Edge Agreement Problem

$$\Sigma(\mathcal{G}) : \begin{cases} \dot{x}(t) &= -L(\mathcal{G})x(t) + \begin{bmatrix} I & -E(\mathcal{G}) \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \\ z(t) &= E(\mathcal{G})^T x(t). \end{cases}$$



$$\Sigma_e(\mathcal{G}) : \begin{cases} \dot{x}_{\mathcal{T}}(t) &= -L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T x_{\mathcal{T}}(t) + \\ & \begin{bmatrix} E(\mathcal{T})^T & -L_e(\mathcal{T})R_{(\mathcal{T},\mathcal{C})} \end{bmatrix} \begin{bmatrix} w(t) \\ v(t) \end{bmatrix} \\ z(t) &= x_{\mathcal{T}}(t). \end{cases}$$

stable and minimal realization of consensus protocol

Performance of Consensus

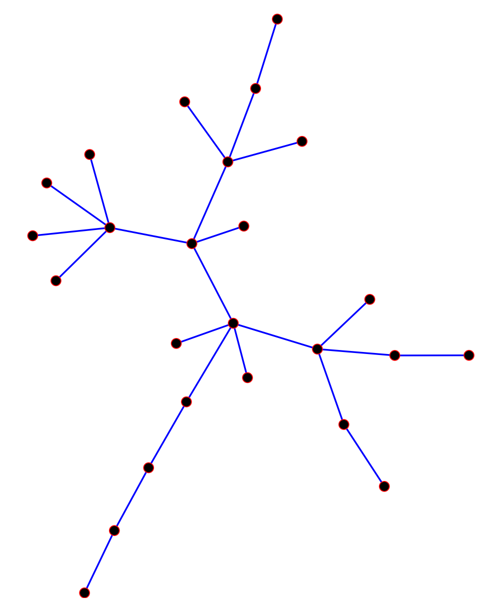
Theorem [Zelazo TAC '11]

$$\|\Sigma_e(\mathcal{G})\|_2^2 = \frac{1}{2}\mathbf{tr}\left[(R_{(\mathcal{T},\mathcal{C})}R_{(\mathcal{T},\mathcal{C})}^T)^{-1}\right] + (n-1)$$

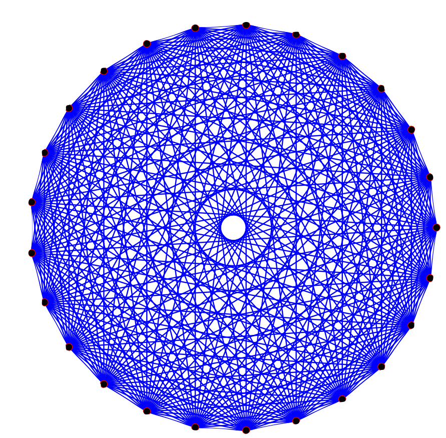
some immediate bounds...

$$\|\Sigma_e(\mathcal{G})\|_2^2 \leq \|\Sigma_e(\mathcal{T})\|_2^2 = \frac{3}{2}(n-1)$$

all trees are the same

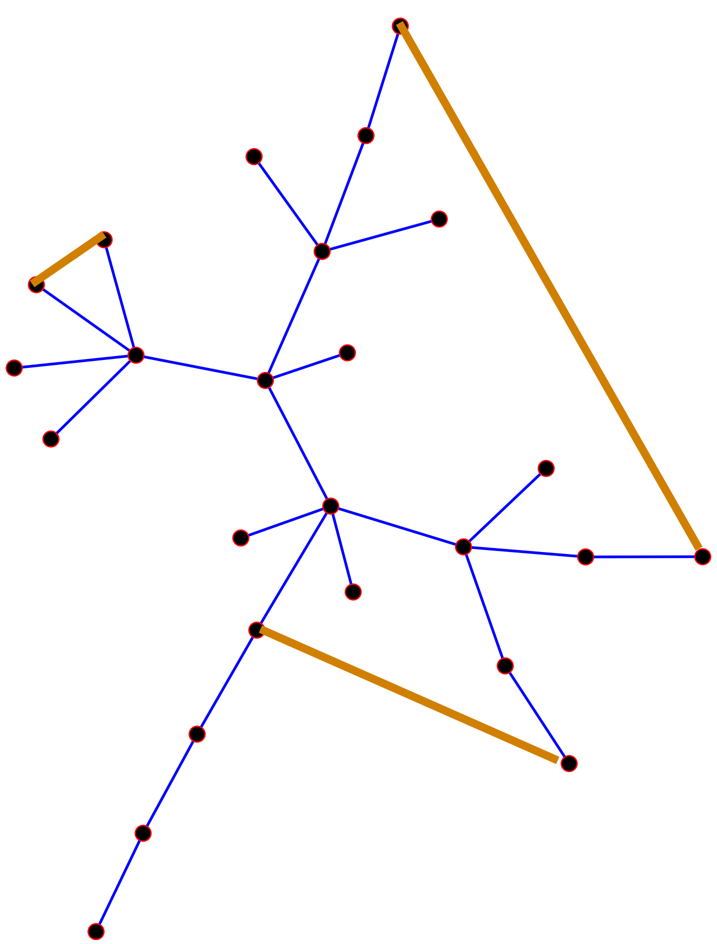


$$\|\Sigma_e(\mathcal{G})\|_2^2 \geq \|\Sigma_e(K_n)\|_2^2 = \frac{n^2-1}{n}$$



Performance and Cycles

Is there a *combinatorial* feature that affects the performance?



Corollary

$$\mathcal{G} = \cup_{i=1}^{k} \mathcal{C}_i \text{ with } \mathcal{C}_i = (\mathcal{V}, c_i),\ c_i \cap c_j = \emptyset$$

$$\|\Sigma_e(\mathcal{T}_k \cup e)\|_2^2 = \|\Sigma_e(\mathcal{T})\|_2^2 - \frac{1}{2}(1 - l(c)^{-1})$$

$$\|\Sigma_e(\mathcal{G})\|_2^2 = \|\Sigma_e(\mathcal{T})\|_2^2 - \frac{1}{2}\left(k - \sum_{i=1}^{k} \frac{1}{|c_i|}\right)$$

performance of Eulerian graph is exactly characterized by its cycle decomposition

Design of Eulerian Graphs

A synthesis problem

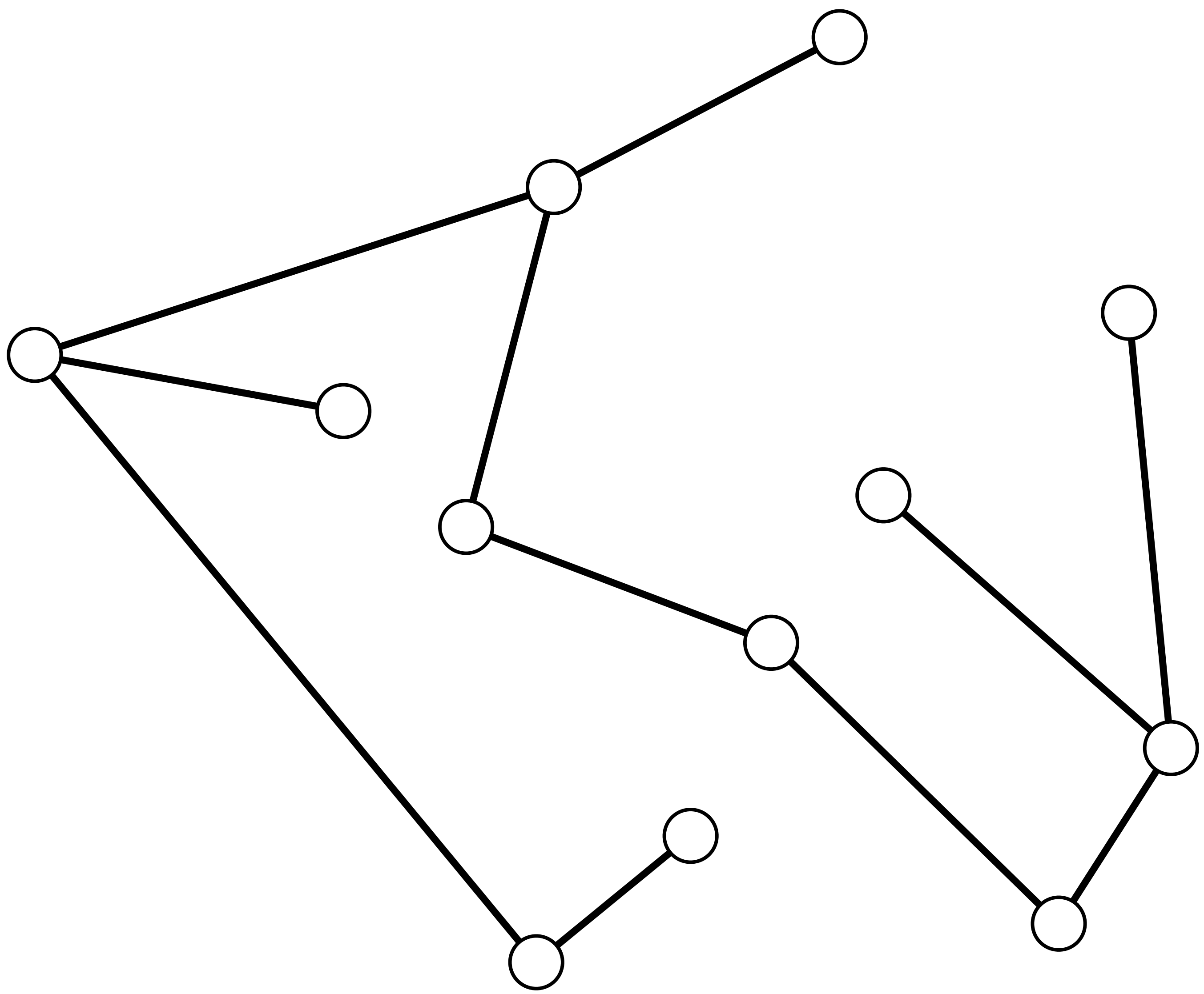
$$\min_{T_{(\mathcal{T},\mathcal{C})}\in\mathbb{R}^{|\mathcal{V}|\times k}} \|\Sigma_e(\mathcal{G})\|_2,$$

l1 - relaxation

$$\begin{aligned}
\min_{M,w_i} \quad & \alpha \textbf{trace}\,[M] + (1-\alpha)\sum_i m_i w_i \\
\text{s.t.} \quad & \begin{bmatrix} M & I \\ I & I + T_{(\mathcal{T},\overline{\mathcal{T}})} W T_{(\mathcal{T},\overline{\mathcal{T}})} \end{bmatrix} \geq 0 \\
& \sum_i w_i = k, \quad 0 \leq w_i \leq 1.
\end{aligned}$$

Eulerian Graph Construction

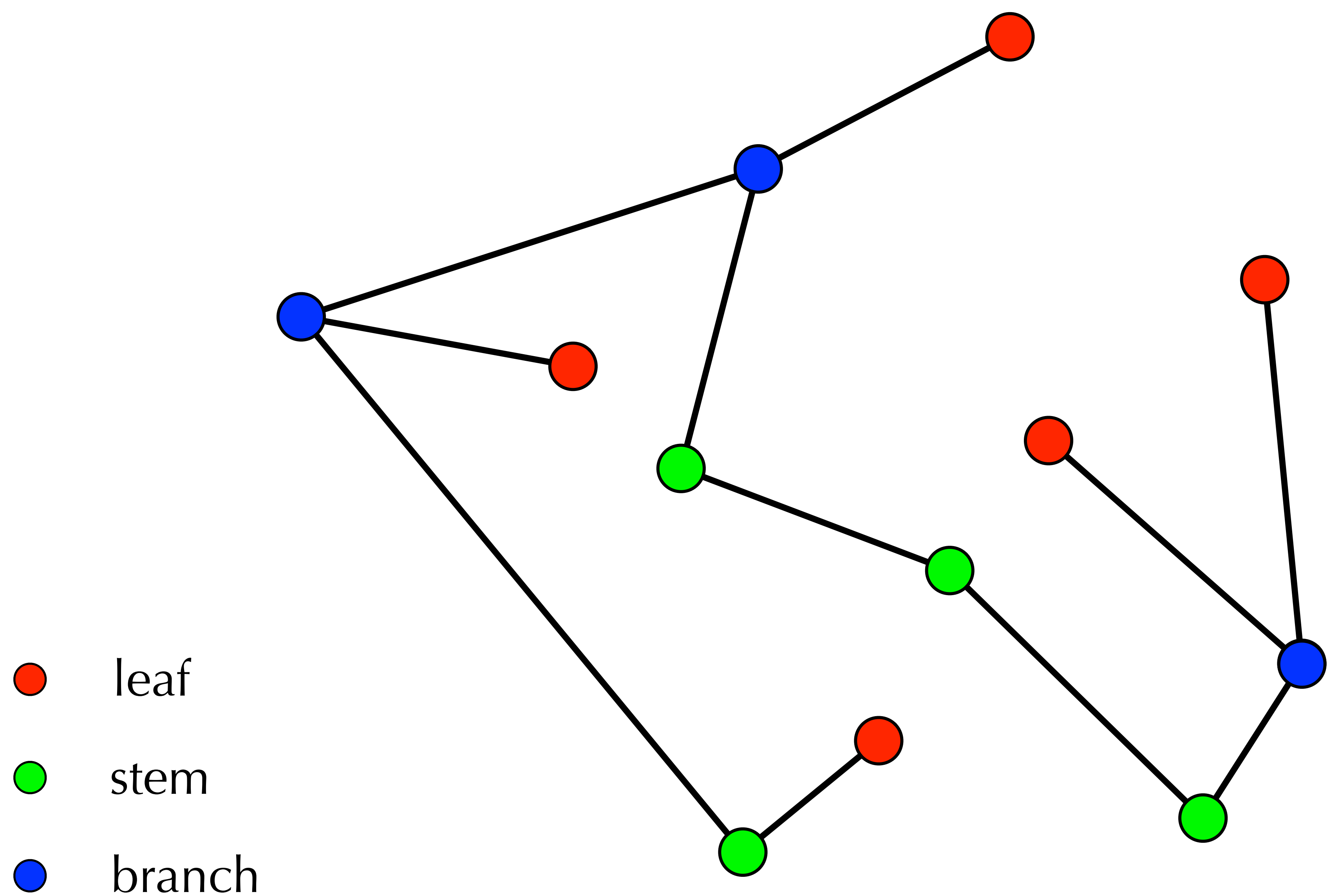
start with a tree



$$\|\Sigma(\mathcal{T})\|_2^2 = \frac{3}{2}(n-1) = \frac{33}{2}$$

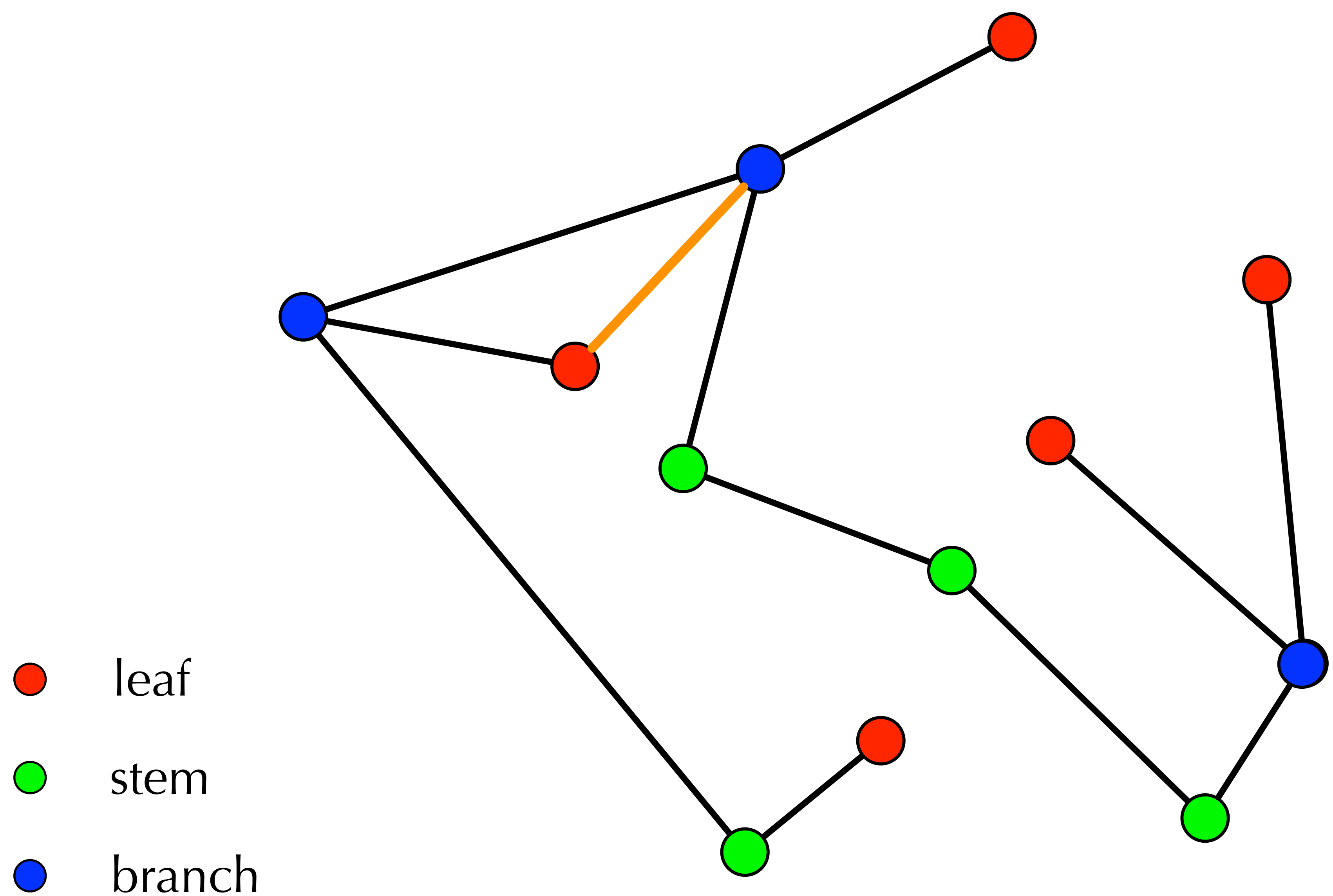
Eulerian Graph Construction

3 node classifications



Eulerian Graph Construction

connect leaves to branches



Eulerian Graph Construction

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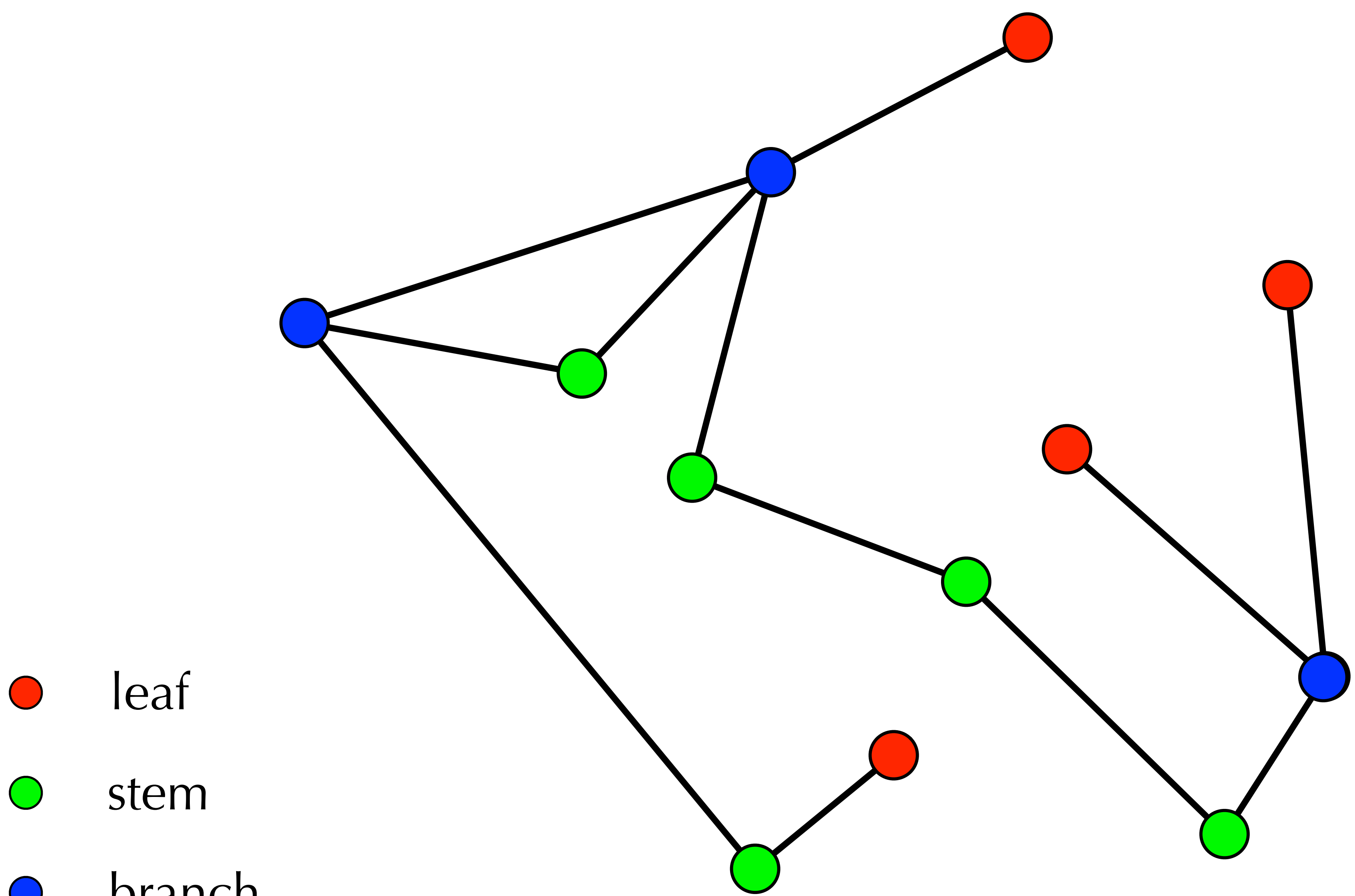

Eulerian Graph Construction

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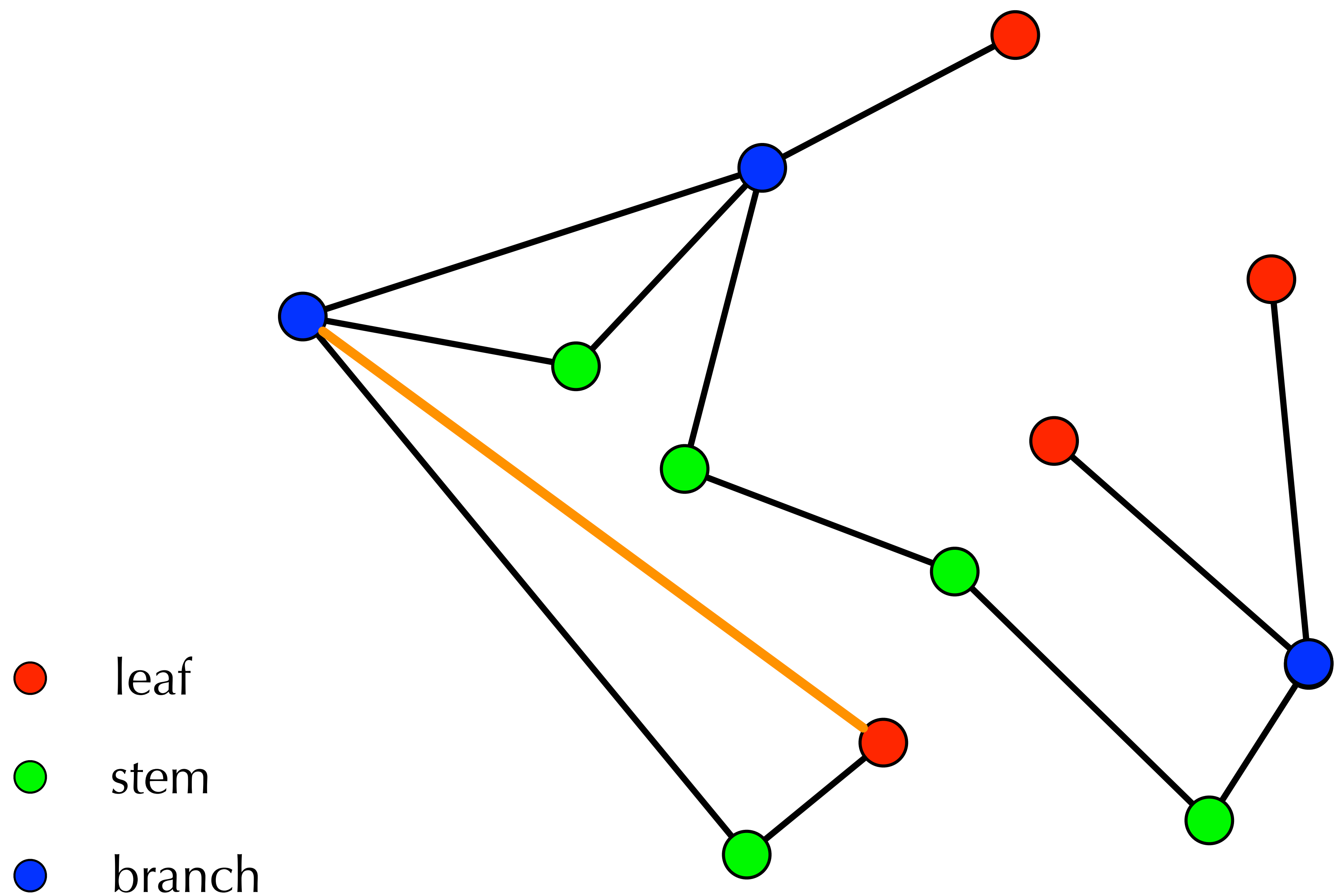
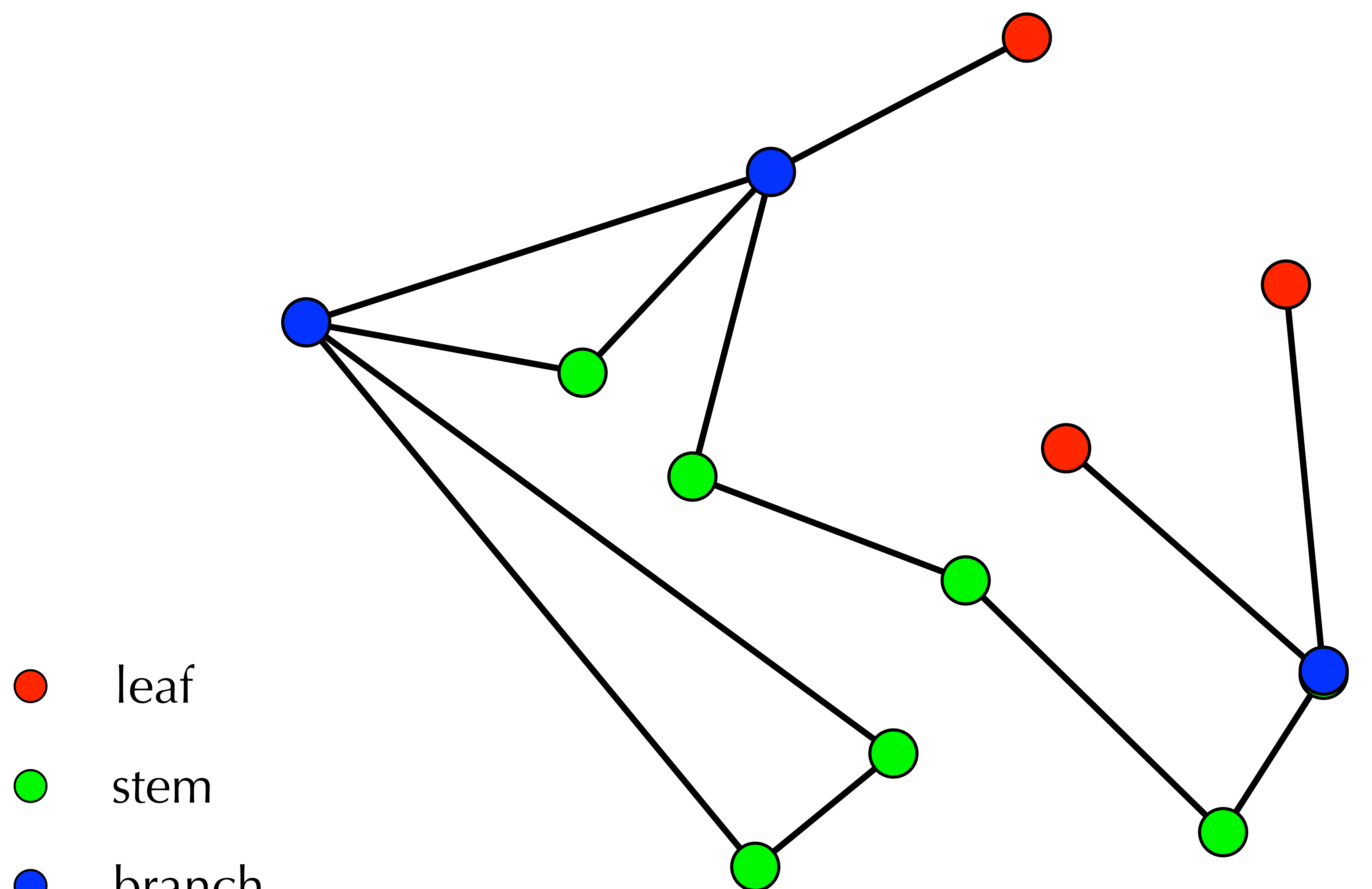


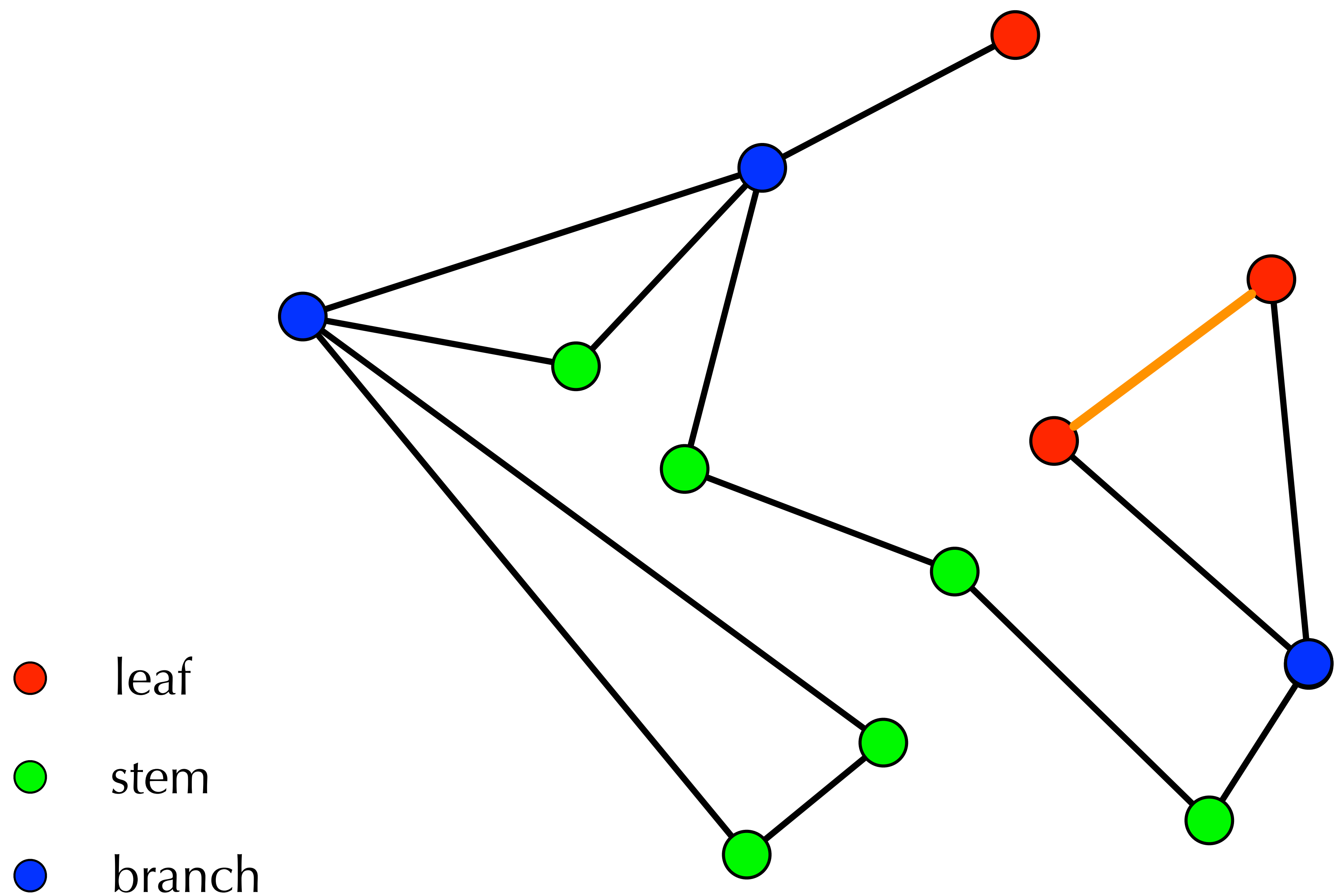
Eulerian Graph Construction

connect leaves to branches



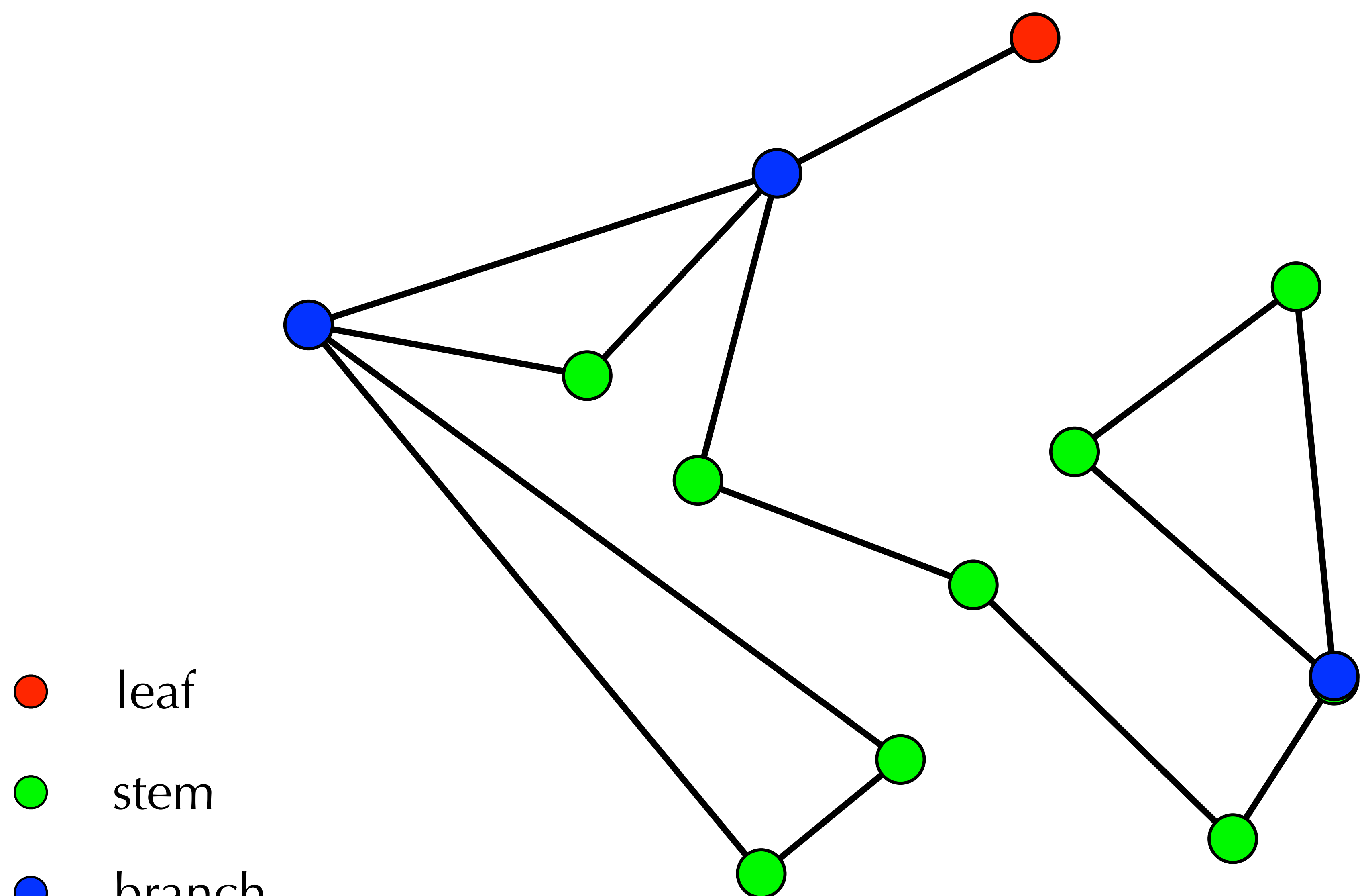
Eulerian Graph Construction

connect leaves to branches



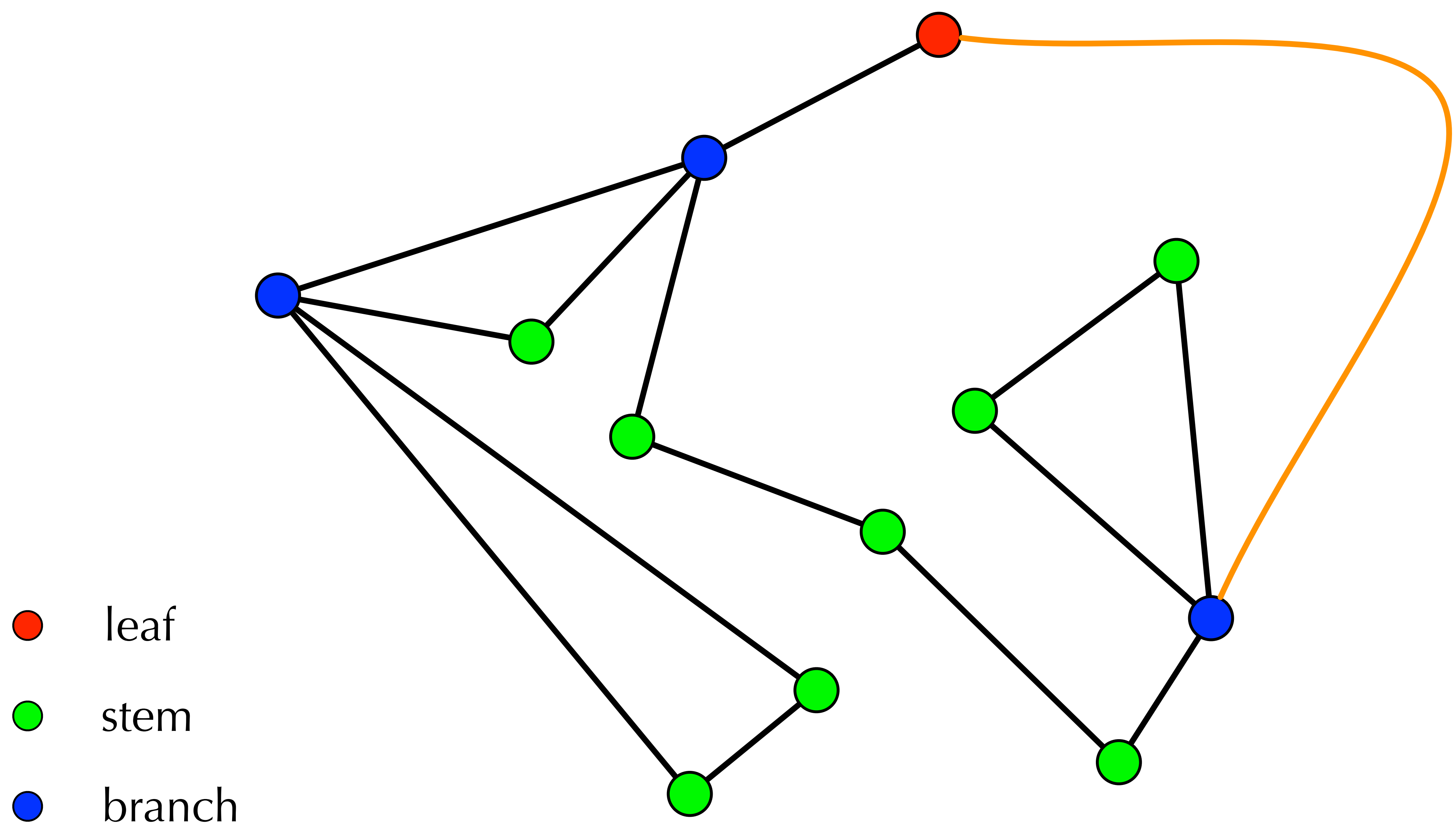
Eulerian Graph Construction

connect leaves to branches



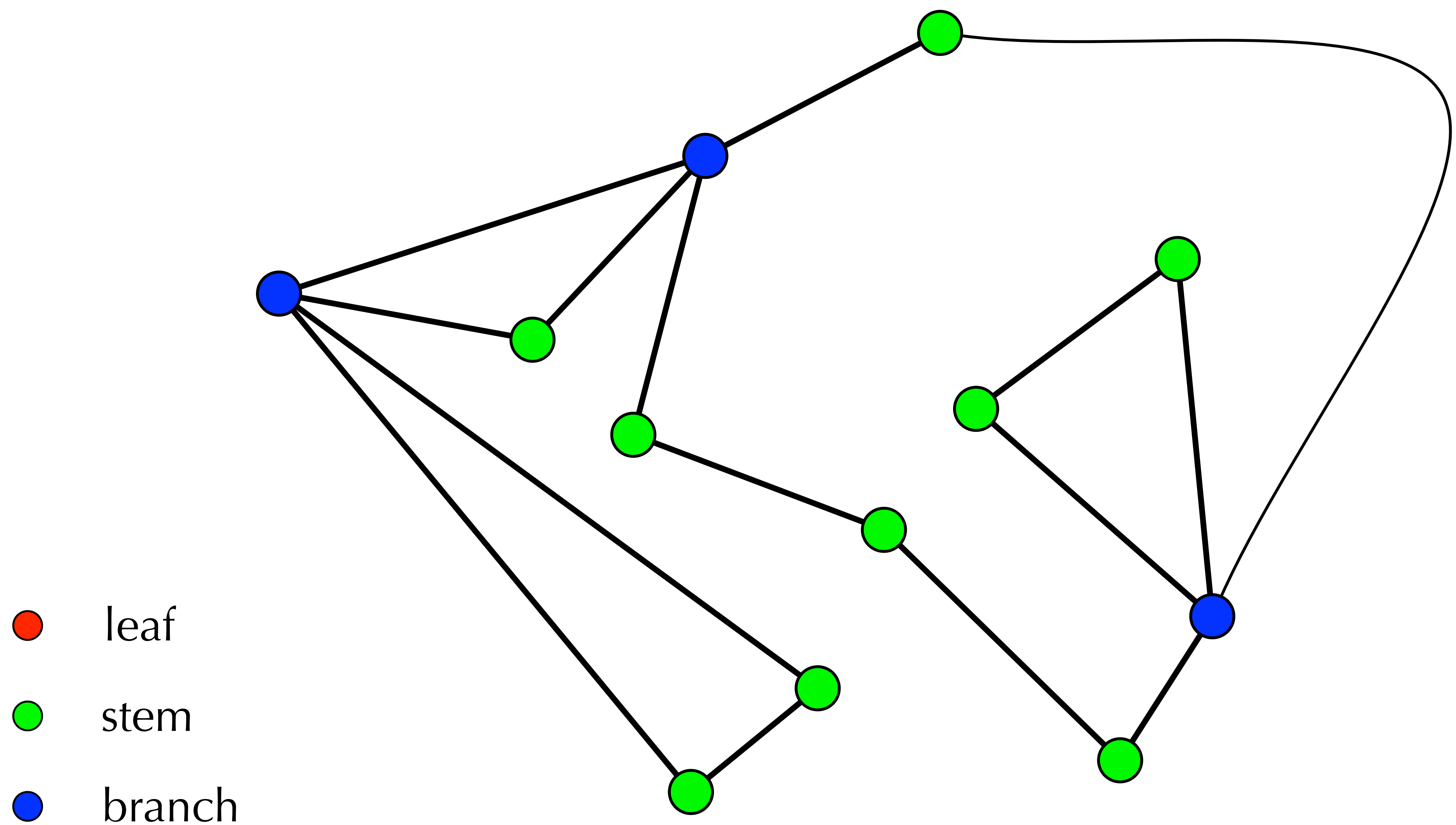
Eulerian Graph Construction

connect leaves to branches



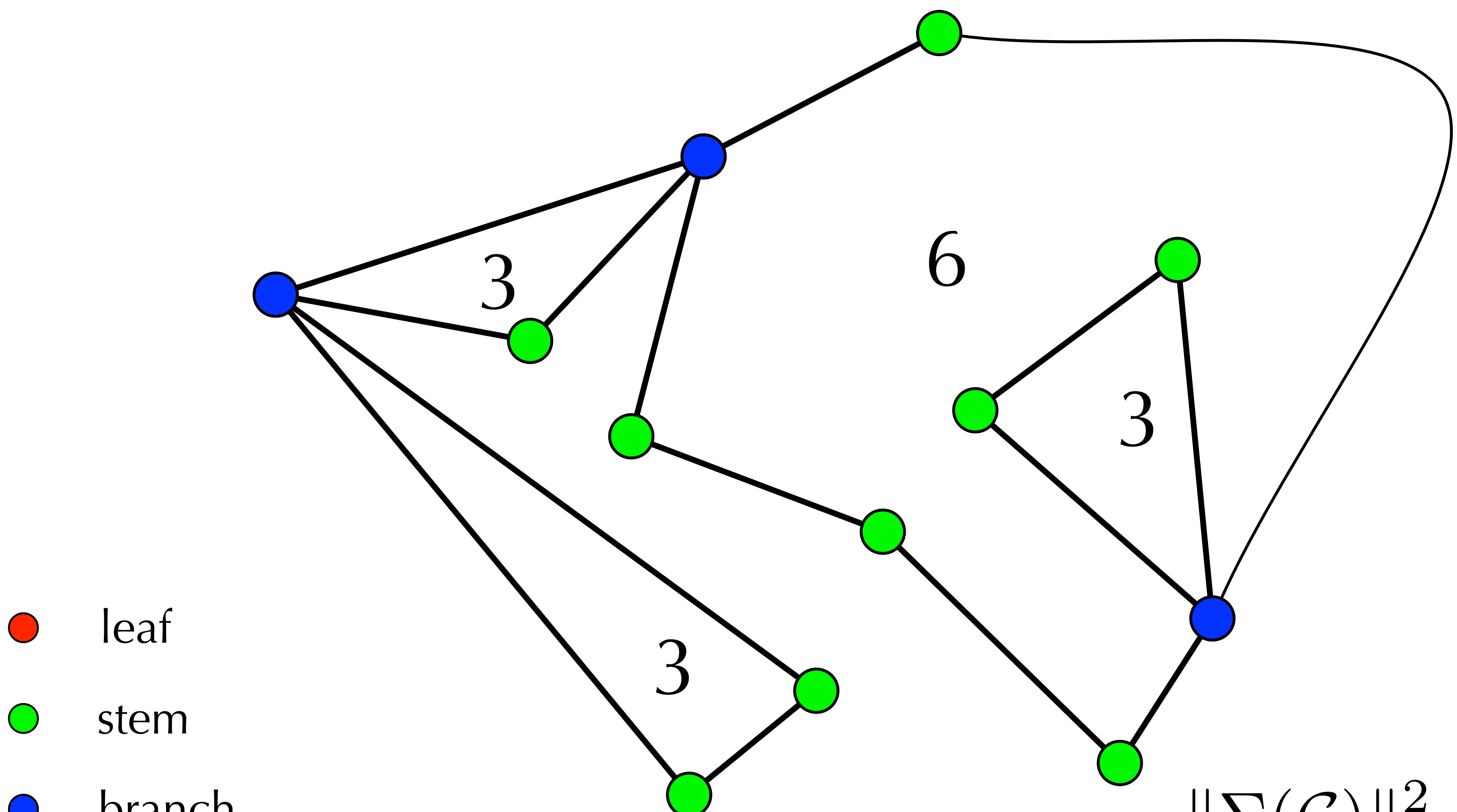
Eulerian Graph Construction

connect leaves to branches



Eulerian Graph Construction

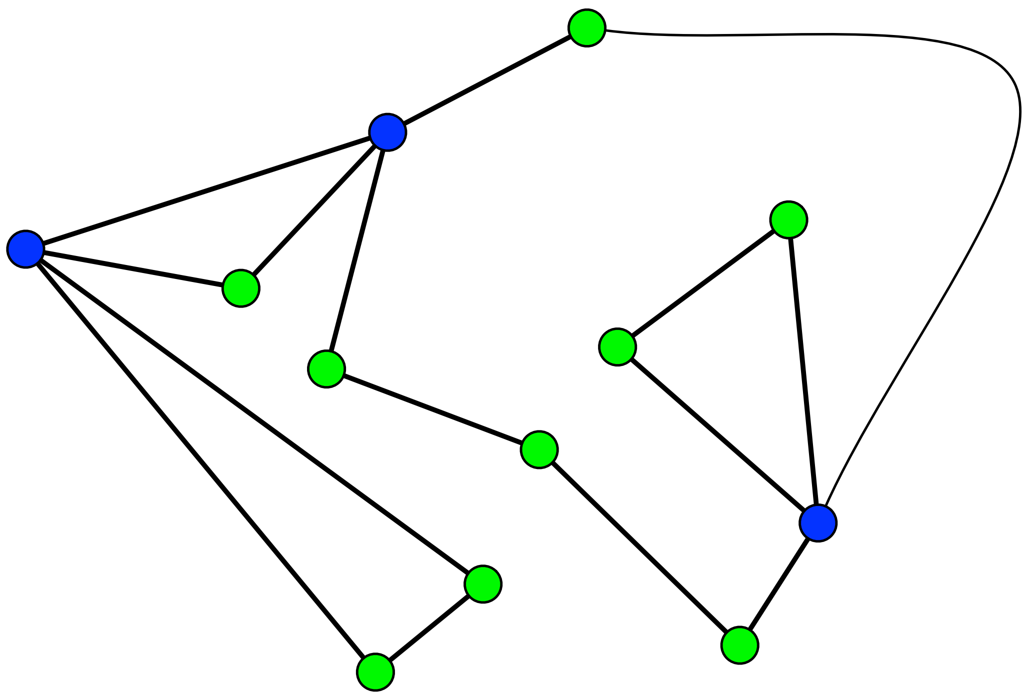
result is a Eulerian graph



$$\|\Sigma(\mathcal{G})\|_2^2 = 15.083$$

Eulerian Graph Construction

not the "best" Eulerian graph!

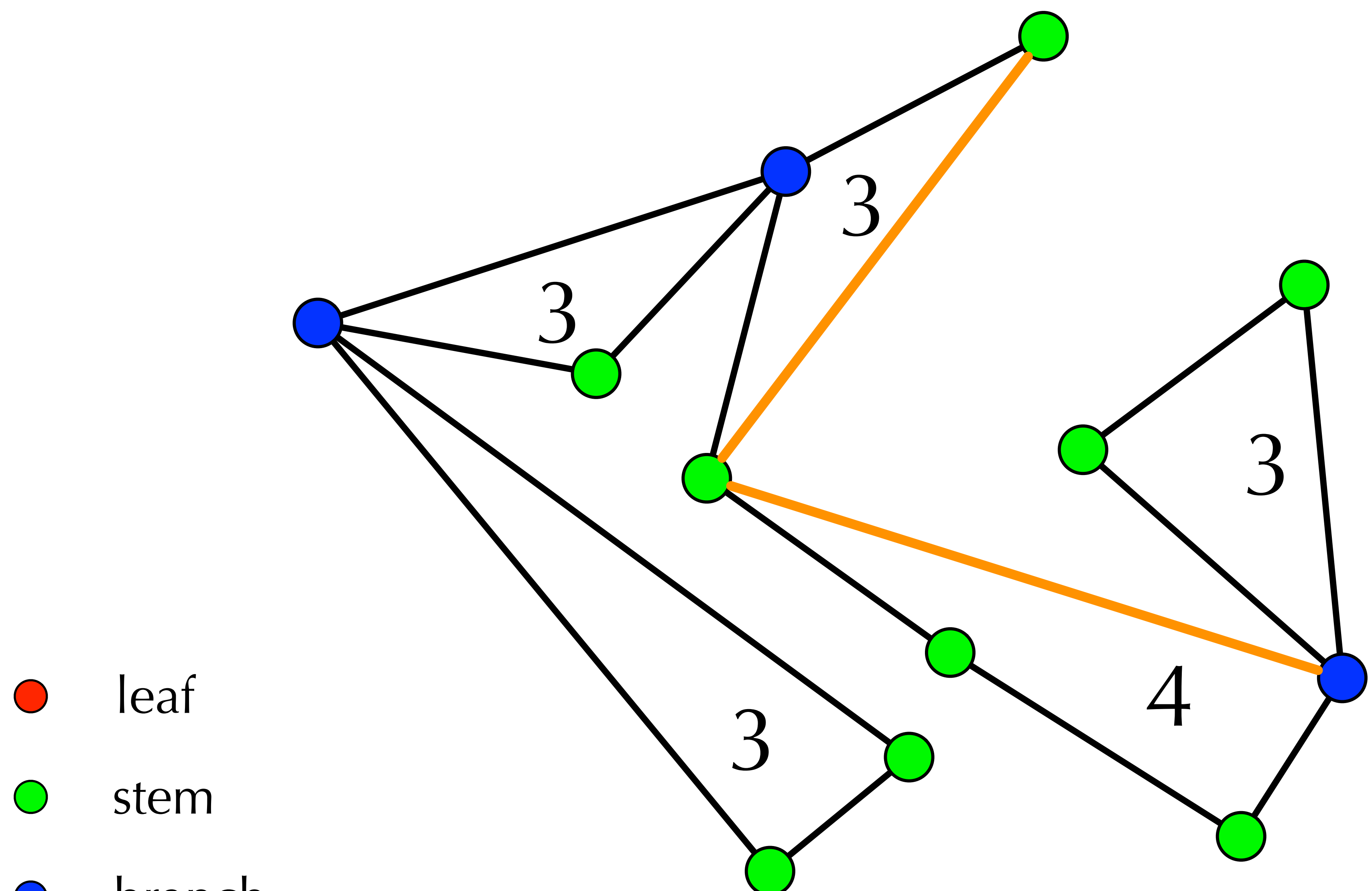


$$1 - \frac{1}{|c_1| + |c_2| - 1} < 2 - \frac{1}{|c_1|} - \frac{1}{|c_2|}$$

multiple short cycles are better than fewer long cycles!

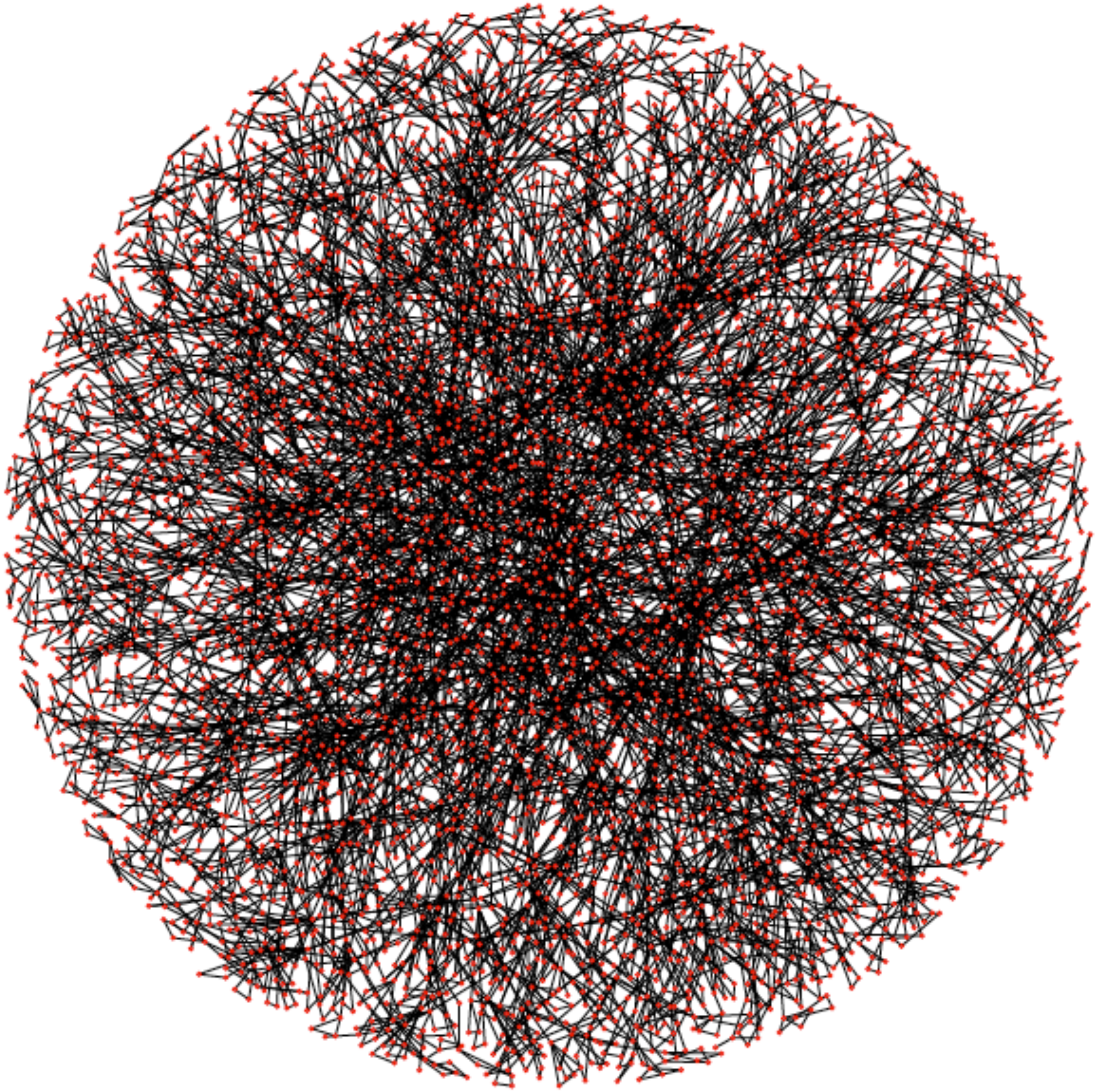
Eulerian Graph Construction

performance "enhancing" algorithm



$$\|\Sigma(\mathcal{G})\|_2^2 = 14.7916$$

Eulerian Graph Construction



5000 nodes

$\|\Sigma(\mathcal{T})\|_2^2 = 7498.5$

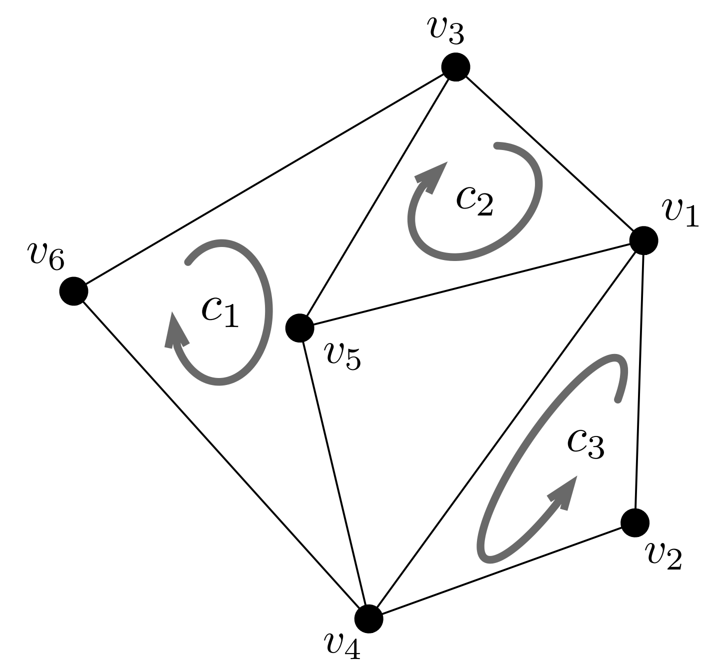
$\|\Sigma(\hat{\mathcal{G}})\|_2^2 = 6535.75$

2329 cycles added

Concluding Remarks

role of cycles in consensus networks

✳ internal feedback
✳ performance



large-scale design

✳efficient algorithm
✳known performance

future works

✳ additional performance metrics
✳ distributed architectures

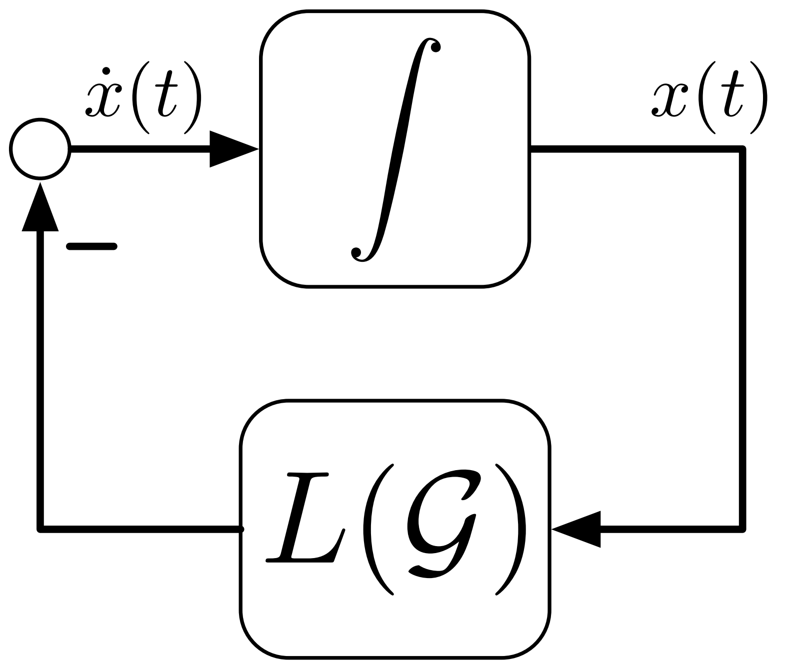
Concluding Remarks

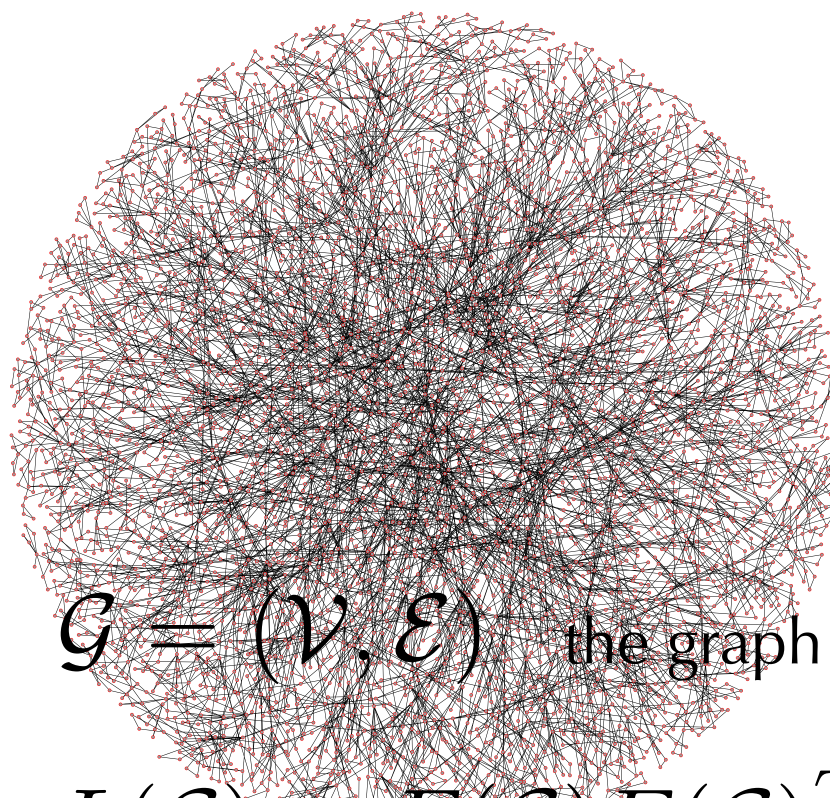
Questions?

Mahalo!

The Consensus Protocol

$$\dot{x}_i(t) = \sum_{i \sim j} x_j(t) - x_i(t)$$



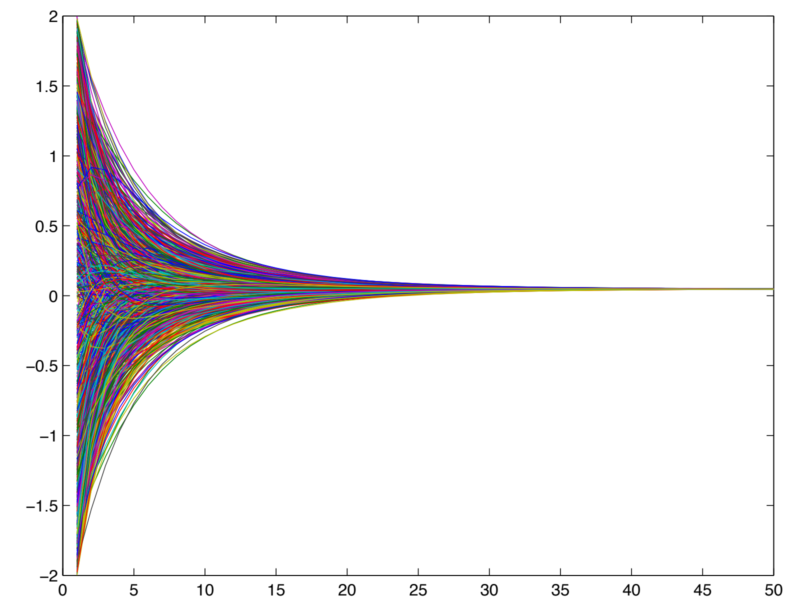


$\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the graph

$$L(\mathcal{G}) = E(\mathcal{G})E(\mathcal{G})^T$$

Laplacian

Incidence matrix



“distributed averaging”