



A passivity analysis for nonlinear consensus on balanced digraphs^{☆,☆☆}

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ABSTRACT

This work deals with the output consensus problems for multi-agent systems over balanced digraphs. While passivity-based approaches are widely used for analyzing undirected consensus protocols, we show that they are generally not applicable to the directed linear consensus protocol. To address this limitation, we propose a general approach that enables a passivity-based analysis for network systems with directed couplings. Then, we mitigate the complexity introduced by nonlinearities and directed interconnections by reformulating the general output consensus problem as a convergence analysis on a submanifold. Within this framework, we further focus on the stabilization problem, a specific form of the output consensus problem, and establish a sufficient passivity-based condition for stabilizing multi-agent systems over balanced digraphs. The results are supported by a numerical example.

1. Introduction

Multi-agent systems (MASs) have received extensive attention in both industrial practice and theoretical research, ranging from smart grids, distributed sensing and transportation networks to control, robotics and computer science (Bullo, 2020; Chung et al., 2018; Zhang et al., 2021; Zhao & Zelazo, 2015). From a control perspective, a fundamental challenge in this field is the consensus problem (Scardovi et al., 2009), which aims to coordinate agent dynamics to achieve agreement on a shared state or trajectory.

Consensus analysis requires understanding the fundamental interplay between agent dynamics, information exchange structures, and interaction protocols in MASs (Bürger et al., 2013; Sharf & Zelazo, 2019). Diffusively-coupled networks provide a canonical architecture for studying these relationships (Mesbahi & Egerstedt, 2010). Their inherent structure, composed of symmetric and feedback interconnections, makes passivity theory a natural tool for its analysis (Bai et al., 2011). Passivity theory can simplify the study of complex network systems, as it enables a decoupled treatment of network dynamics and network topology, and is inherently linked to stability and convergence (Bürger et al., 2013). Arcak's seminal work (Arcak, 2007) leveraged passivity to characterize network convergence behavior. This approach was later extended into a comprehensive passivity-based cooperative control framework for single-input single-output (SISO) systems (Bürger et al., 2013). This framework revealed a connection

between the network system steady-states and dual network optimization problems (Rockafellar, 1998).

While the passivity framework has proved very powerful, it relies heavily on the symmetric feedback interconnection of the incidence matrix in diffusively coupled networks. This symmetry requirement confines the framework to systems with undirected interconnections. Replacing one of the incidence matrices in the structure (detailed in Section 2) enables the representation of directed graph topologies but sacrifices the diffusive coupling property due to the loss of symmetry. Moreover, given passive edge controllers, the feedback path in the loop may not preserve passivity for the entire interconnection. These challenges hinder passivity-based analysis for MASs over digraphs, motivating a generalized approach that extends passivity theory's benefits to solving the consensus problem of directed topologies.

On the other hand, since passivity theory enables a separate analysis of system dynamics and the underlying graphs, consensus problems can be categorized by the linearity of the system dynamics and the graph directionality. The problems can be classified, by increasing complexity, as linear dynamics over undirected graphs (e.g., the standard linear consensus protocol), nonlinear dynamics over undirected graphs (e.g., Bürger et al., 2013), linear dynamics over digraphs (e.g., the linear consensus protocol for digraphs), and nonlinear dynamics over digraphs (e.g., Li et al., 2020, 2019). Also, consensus behaviors manifest in two distinct forms: *average consensus*, where agent states converge to the mean of initial conditions, and *regular consensus*, where states

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agree on the same value (not necessarily the average). When applying linear consensus protocols, systems over connected undirected graphs achieve average consensus, whereas systems over digraphs containing a globally reachable node only achieve regular consensus. However, systems with balanced digraphs restore average consensus capability, suggesting balanced digraphs' unique intermediate position between directed and undirected topologies. This motivates the investigation into balanced digraphs in this paper.

This paper focuses on a challenging consensus problem within the above taxonomy: nonlinear dynamics over digraphs. The works (Li et al., 2020, 2019) were related to this topic and developed a passivation approach, but they only considered the case where the controllers are linear static maps and did not provide a general analysis method for network systems with directed coupling. Montenbruck et al. (2017) regarded the agreement space as a submanifold and developed powerful analytical tools to establish connections between passivity properties and stabilization around a submanifold, yielding explicit controller synthesis methods. However, they considered all controllers as a single entity without examining the passivity of each individual agent or controller, so it did not allow for an in-depth investigation of the interplay among the controller dynamics, the agent dynamics, and the underlying digraphs within the MASS.

In this paper, we conduct a passivity analysis for MASs interconnected via balanced digraphs and investigate the relationship between passivity, output consensus and stabilization (a specific form of output consensus) in such systems. Our contributions are as follows. We begin by discussing the difference between the diffusively coupled network and its variant for digraphs. Our analysis uncovers a potential loss of passivity in the feedback path of the variant structure for general digraphs, even under the fundamental linear consensus protocol. Building on these insights, we develop a generalized approach that enables a passivity-based analysis for systems with directed couplings. By reformulating the output agreement problem as convergence to a submanifold, we derive passivity conditions for agents and controllers that stabilize network systems over balanced digraphs.

The remainder of the paper is organized as follows. Section 2 introduces preliminaries on networked systems and submanifold stabilization. Section 3 proposes a general approach for analyzing directed coupling and reformulates the output agreement problem. Section 4 presents a passivity-based analysis for stabilizing networked systems interconnected via balanced digraphs. Numerical examples and concluding remarks are given in Sections 5 and 6.

Notations. The notation $\mathbb{1}_n$ ($\mathbb{0}_n$) denotes the n -dimensional vector of all ones (zeros), and I_n represents the $n \times n$ identity matrix, where the subscript n may be omitted when the dimension is clear from the context. For a set A , its cardinality is denoted by $|A|$. We denote the kernel of a linear transformation $T : X \rightarrow Y$ by $\ker(T)$, and the orthogonal complement of a subspace U by U^\perp .

Fundamental notions from algebraic graph theory are also used in this paper. A directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ comprises of a finite vertex set \mathbb{V} and an edge set $\mathbb{E} \subset \mathbb{V} \times \mathbb{V}$. The incidence matrix $E \in \mathbb{R}^{|\mathbb{V}| \times |\mathbb{E}|}$ is defined as follows. $[E]_{ik} := 1$ if i is the head of edge e_k , $[E]_{ik} := -1$ if i is the tail of edge e_k and $[E]_{ik} := 0$ otherwise. We decompose the incidence matrix into the out-incidence matrix B_o and the in-incidence matrix B_i (Restrepo et al., 2021), i.e., $E = B_o + B_i$, where: $[B_o]_{ik} := 1$ if i is the head of edge $e_k = (i, k)$ and $[B_o]_{ik} := 0$ otherwise; $[B_i]_{ik} := -1$ if i is the tail of edge e_k and $[B_i]_{ik} := 0$ otherwise. The graph Laplacian of undirected graphs is defined as $L = EE^\top$. For digraphs, we define the in-Laplacian matrix $L_i = -B_i E^\top$ and out-Laplacian matrix $L_o = B_o E^\top$.

2. Preliminaries

This section introduces the fundamental concepts from graph theory that are utilized in this paper, and highlights two key structures of network systems: diffusively-coupled networks and their directed variants, along with essential passivity concepts. We then outline the mathematical framework for reformulating output consensus as a convergence analysis on a submanifold.

2.1. Balanced digraphs and globally reachable nodes

A digraph is called *balanced* if the in-degree equals the out-degree for every node.

Lemma 1. For incidence matrix $E \in \mathbb{R}^{|\mathbb{V}| \times |\mathbb{E}|}$ and out-Laplacian $L_o \in \mathbb{R}^{|\mathbb{V}| \times |\mathbb{V}|}$, the following statements are equivalent:

- (i) The digraph is balanced,
- (ii) $E^\top \mathbb{1}_{|\mathbb{V}|} = \mathbb{0}_{|\mathbb{E}|}$, and $E \mathbb{1}_{|\mathbb{E}|} = \mathbb{0}_{|\mathbb{V}|}$,
- (iii) $L_o \mathbb{1}_{|\mathbb{V}|} = \mathbb{0}_{|\mathbb{V}|}$ and $L_o^\top \mathbb{1}_{|\mathbb{V}|} = \mathbb{0}_{|\mathbb{V}|}$.

Proof. (i) \Leftrightarrow (iii): We recommend readers to refer to Lemma 6.4 in Bullo (2020).

(i) \Leftrightarrow (ii): It is sufficient to show the equivalence between statement (i) and $E \mathbb{1}_{|\mathbb{E}|} = \mathbb{0}_{|\mathbb{V}|}$. The incidence matrix can be represented as $E = B_o + B_i$. For row i ($i = 1, \dots, |\mathbb{V}|$) of $B_i(B_o)$, the row-sum of row i is the in(out)-degree of the corresponding node i . Thus, the given digraph is balanced if and only if $E \mathbb{1}_{|\mathbb{E}|} = B_o \mathbb{1}_{|\mathbb{E}|} + B_i \mathbb{1}_{|\mathbb{E}|} = \mathbb{0}_{|\mathbb{V}|}$. \square

In this work, we will use the notion of globally reachable nodes and walks. A *walk* in a directed graph is a sequence of nodes connected by directed edges that point from one node to the next in the sequence.

Definition 1 (Bullo, 2020). A directed graph possesses a *globally reachable node* if one of its nodes can be reached from any other node by traversing a directed walk.

2.2. Network systems and passivity

Consider a population of agents interacting over a network $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, where the vertices \mathbb{V} denote the set of agents and the edges \mathbb{E} represent edge controllers describing the interaction between agents. Each agent $\{\Sigma_i\}_{i \in \mathbb{V}}$ and controller $\{\Pi_k\}_{k \in \mathbb{E}}$ are described by the SISO nonlinear dynamical systems,

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = f_i(x_i(t), u_i(t)), \\ y_i(t) = h_i(x_i(t), u_i(t)), \end{cases} \quad (1)$$

$$\Pi_k : \begin{cases} \dot{\eta}_k(t) = \phi_k(\eta_k(t), \zeta_k(t)) \\ \mu_k(t) = \psi_k(\eta_k(t), \zeta_k(t)) \end{cases} \quad (2)$$

Note that agents and controllers are interconnected in parallel, respectively. Define the stacked inputs of agents $u(t) = [u_1, \dots, u_{|\mathbb{V}|}]^\top$, and similarly for outputs of agents $y(t)$, inputs of controllers $\zeta(t)$ and outputs of controllers $\mu(t)$.

As shown in Fig. 1(a), a network system can be represented by agent dynamics Σ and controller dynamics Π , and two matrices M and N encoding the interconnection of the system. The feedback equations are then completed with the algebraic relations, $\zeta(t) = M y(t)$ and $u(t) = p(t) - N \mu(t)$, where $p(t)$ denotes the exogenous input. In this paper, we only consider the case where $p(t) = 0$.

The matrices M and N can be selected to represent different information exchange topologies. A network system described by Fig. 1(a) is called *diffusively coupled* if $M = N^\top$ and the matrix N is set to the incidence matrix E . In this configuration, the topology of the system is characterized by the undirected counterpart of \mathcal{G} , where the edges between the agents represent bidirectional communication links.

When matrices N and M are set to the out-incidence matrix B_o and the transpose of the incidence matrix E^\top , respectively, the resulting structure represents the networked system interconnected by digraphs. In this case, the system can no longer be considered diffusively coupled, as the inherent symmetry of the underlying graph is broken. We adopt the notation $(\Sigma, \Pi, \mathcal{G})_E$ for diffusively-coupled networks and denote the directed case by $(\Sigma, \Pi, \mathcal{G})_{B_o}$.

Now, let us take the system (1) as an example to introduce the definition of passivity (Khalil, 2002).

where $\text{Proj}_{S^\perp}(y(t))$ appears explicitly. This connects the passivity theory to the output consensus problem.

Given these notions, demonstrating convergence to the submanifold S is equivalent to proving that $\text{Proj}_{S^\perp}(y(t))$ approaches zero as $t \rightarrow \infty$. This equivalence arises from the geometric interpretation of the projection operator: as $\text{Proj}_{S^\perp}(y(t))$ tends to zero, the distance between the signal $y(t)$ and its projection onto the submanifold S diminishes, implying convergence to S . The following definition connects output consensus and convergence to a submanifold.

Definition 3. Consider a network system consisting of a group of agents and edge controllers interconnected as in Fig. 1(a). Let $y(t)$ be the output of the system. We say that output $y(t)$ asymptotically converges to the agreement submanifold S , if $\lim_{t \rightarrow \infty} \text{Proj}_{S^\perp}(y(t)) = 0$.

For conciseness, in the following discussion, we adopt the notation $\text{Proj}_{S^\perp}(y)$ in place of $\text{Proj}_{S^\perp}(y(t))$.

3. A passivity analysis for directed coupling

This section begins with a passivity-based analysis of $(\Sigma, \Pi, \mathcal{G})_E$ and $(\Sigma, \Pi, \mathcal{G})_{B_o}$ under the linear consensus protocol, revealing a potential loss of passivity in the feedback path of $(\Sigma, \Pi, \mathcal{G})_{B_o}$. To address this issue, we propose a general approach for analyzing directed information exchange topologies.

3.1. Passivity analysis for the linear consensus protocol

The system $(\Sigma, \Pi, \mathcal{G})_{B_o}$ in Fig. 1(a) might be the most straightforward candidate to analyze directed coupling. However, when applying the basic linear consensus protocol for digraphs to this structure, the passivity of the feedback path (from y to g) cannot be guaranteed even though the edge controllers are output-strictly passive.

Consider the linear consensus protocol for digraphs. Here, we take the agent dynamics Σ^l to be the integrators, and the controller dynamics Π^l to be the linear static map,

$$\Sigma^l : \begin{cases} \dot{x}(t) = u(t), \\ y(t) = x(t), \end{cases} \quad \text{and} \quad \Pi^l : \mu(t) = \zeta(t). \quad (12)$$

Note that the integrator dynamics are passive (Khalil, 2002), and edge controllers are output strictly passive. The closed-loop dynamics then yield $\dot{x}(t) = -L_o(\mathcal{G})x(t)$, and the generated trajectories converge to the agreement space, $S = \text{span}(\mathbb{1})$ if and only if the underlying digraph contains a globally reachable node.

To leverage the benefits of passivity theory for analyzing the diffusively-coupled structure, both the forward and feedback paths should be passive (Khalil, 2002). Consider the system $(\Sigma^l, \Pi^l, \mathcal{G})_{B_o}$, where Σ^l and Π^l are known to be passive. Our objective is to investigate whether the feedback path (from y to g) in Fig. 1(a) is passive. The controllers in this protocol are memoryless functions. Consequently, with input y and output g , the feedback path is passive if $y^\top g \geq 0$ for all y and g (Khalil, 2002). Using the relation $u = -B_o\mu$ and $\mu = \zeta$, it is equivalent to the spectral analysis of the symmetric part of L_o , denoted by $\frac{(L_o + L_o^\top)}{2} = \text{sym}(L_o)$ (Horn & Johnson, 2012). Indeed, if $y^\top g = y^\top (-B_o)E^\top y = y^\top L_o y = y^\top \frac{L_o + L_o^\top}{2} y = y^\top \text{sym}(L_o) y \geq 0$ for all $y \in \mathbb{R}^n$, the feedback path is passive.

Our first result shows that for digraphs with globally reachable nodes, the smallest eigenvalue of $\text{sym}(L_o)$ is non-positive.

Proposition 1. If \mathcal{G} contains a globally reachable node, then the smallest eigenvalue of the symmetric part $\text{sym}(L_o)$ is non-positive.

Proof. Let s_1, \dots, s_n be the singular values of L_o and $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\text{sym}(L_o)$, both arranged in nonincreasing order. The digraph \mathcal{G} containing a globally reachable node implies that the rank

of L_o is $n - 1$ (Bullo, 2020). It follows that $s_{n-1} > s_n = 0$. To establish the relationship between s_j and λ_j , we apply the Fan-Hoffman (Bhatia, 1997, Proposition III.5.1). This proposition implies that $\lambda_j \leq s_j$ for all $j \in [1, n]$. By setting $j = n$, we can deduce that the smallest eigenvalue of $\text{sym}(L_o)$ is non-positive. \square

The following proposition provides a sufficient and necessary condition for $\text{sym}(L_o)$ having a zero eigenvalue.

Proposition 2. Let \mathcal{G} contains a globally reachable node. Then L_o and L_o^\top have the same kernel space if and only if $\text{sym}(L_o)$ has a zero eigenvalue.

Proof. We first show the sufficiency. Let $q \neq 0$ be a vector in S . Then we have $\frac{1}{2}L_o q + \frac{1}{2}L_o^\top q = 0 = \frac{1}{2}(L_o + L_o^\top)q = \text{sym}(L_o)q$ and $(q, 0)$ is an eigenpair of $\text{sym}(L_o)$. To prove the necessity, let $v \neq 0$ be the eigenvector w.r.t. the 0 eigenvalue, i.e., $\text{sym}(L_o)v = 0$. It follows that $v^\top \text{sym}(L_o)v = \frac{1}{2}v^\top (L_o + L_o^\top)v = 0$. Since $v^\top L_o v = v^\top L_o^\top v$, we have $v^\top \text{sym}(L_o)v = v^\top L_o v = v^\top L_o^\top v = 0$. $v \neq 0$, so the above equalities are satisfied only when $v \in \ker(L_o)$ and $v \in \ker(L_o^\top)$. The existence of a globally reachable node implies that the dimensions of the kernel space of L_o and L_o^\top are 1, so $\ker(L_o^\top) = \ker(L_o) = S$. \square

This proposition suggests that for a general digraph where L_o and L_o^\top do not have the same kernel space, the smallest eigenvalue of $\text{sym}(L_o)$ is negative. Thus, the feedback path may lose passivity, even though the edge controllers are output strictly passive. The following proposition establishes the equivalence between L_o and L_o^\top having the same kernel space and the digraphs being balanced.

Proposition 3. Let \mathcal{G} contain a globally reachable node. Then, L_o and L_o^\top have the same kernel space if and only if \mathcal{G} is balanced.

Proof. Recall Lemma 1 that a digraph is balanced if and only if $L_o \mathbb{1} = L_o^\top \mathbb{1} = 0$. Now, suppose that L_o and L_o^\top have the same kernel space. According to Bullo (2020, Lemma 6.2) and Bullo (2020, Theorem 6.6), the kernel space of L_o^\top is given by $S = \text{span}(\mathbb{1})$, implying $\mathbb{1} \in \ker(L_o)$ and $\mathbb{1} \in \ker(L_o^\top)$.

Conversely, suppose that \mathcal{G} is balanced. The given conditions imply that both L_o and L_o^\top have one-dimensional kernel spaces, with $\mathbb{1}$ in both $\ker(L_o)$ and $\ker(L_o^\top)$. Consequently, we conclude that $\ker(L_o) = \ker(L_o^\top) = S$. \square

The above results demonstrate that under linear consensus protocol, only the systems on some specified digraphs can preserve passivity. Moreover, the above passivity analysis only considers the case where the edge controllers follow the simplest dynamics. The passivity analysis may be more tricky if the edge controllers are modeled by more complex dynamics. This suggests we need a more general approach for analyzing MASs on digraphs.

Note that the feedback path of $(\Sigma^l, \Pi^l, \mathcal{G})_E$ preserves the passivity of the controllers, mirroring the behavior observed for balanced digraphs. This serves as another example showing the intermediate position of balanced digraphs between undirected graphs and unbalanced digraphs.

3.2. A general approach for directed coupling

Recall that the passivity of a system is preserved after being post-multiplied by a matrix and pre-multiplied by its transpose (Arcak, 2007). Also, the incidence matrix can be represented as $E = B_i + B_o$. Equivalently, the incidence matrix for a directed graph can be expressed as $B_o = E - B_i$. Inspired by these, we use the decomposition idea to design a structure capable of conducting passivity analysis for MASs over digraphs, as illustrated in Fig. 1(b).

To derive the structure depicted in Fig. 1(b), let us begin by examining $(\Sigma, \Pi, \mathcal{G})_{B_o}$. By viewing Fig. 1(a) from a different perspective, where $u = -B_o\mu = -(E - B_i)\mu$, we can decompose its feedback loop into

two distinct branches. The first branch transmits the signals for y to z , and the second branch transmits the signals for y to w . The first branch and the forward path form a feedback connection, as highlighted in the gray box in Fig. 1(b). This sub-structure is the diffusively-coupled network $(\Sigma, \Pi, \mathcal{G}_E)$, and the feedback connection is passive, provided that the agents and edge controllers are passive. Indeed, the first branch preserves the passivity of the edge controllers, and the inner product $z^\top y$ satisfies

$$z^\top y = \mu^\top E^\top y = \mu^\top \zeta \geq \dot{V},$$

where V denotes a continuously differentiable positive semidefinite function known as the storage function. Consequently, by Theorem 6.1 in Khalil (2002), the feedback connection is passive for all input–output pairs.

With this understanding, we can treat w as an external input that carries directed information to the inner-feedback loop (outlined in gray in Fig. 1(b)). Although the passivity of the overall system $(\Sigma, \Pi, \mathcal{G})_{B_o}$ cannot be guaranteed, we can still exploit the passivity properties preserved in the inner-feedback loop and perform analysis on the feedback interconnection with the input–output pair (w, y) .

To differentiate between the structures represented in Figs. 1(a) and 1(b), we introduce the notation $(\Sigma, \Pi, \mathcal{G}, w)$ to denote the system depicted in Fig. 1(b). Now, we can define new interconnection relations for $(\Sigma, \Pi, \mathcal{G}, w)$. Let $w(t) = B_o \mu(t)$ and $z(t) = E \mu(t)$. It follows that,

$$u(t) = w(t) - z(t) = -B_o \mu(t) \quad (13)$$

$$\zeta(t) = E^\top y(t), \quad (14)$$

where the structure reduces to $(\Sigma, \Pi, \mathcal{G}_E)$ when $w(t) = 0$. This decomposition serves as a general approach to handling directed coupling and allows us to analyze the system's behavior using passivity theory.

Recall that the output consensus problem can be transformed to an equivalent problem of analyzing convergence to a submanifold. To apply the idea of passivity relations (10), define two relations, i.e., the agent relation H_a ,

$$(u(t), y(t)) \in H_a \subset (\mathcal{L}^2, \mathcal{L}_S^2),$$

and the controller relation H_c ,

$$(y(t), z(t)) \in H_c \subset (\mathcal{L}_S^2, \mathcal{L}^2).$$

We can now define the problem that we will consider.

Problem 1. Consider the network system $(\Sigma, \Pi, \mathcal{G}, w)$. Under what passivity conditions on H_a and H_c does the output of the system converge to the agreement submanifold?

4. Stabilization of network systems over balanced digraphs

This section focuses on a particular type of digraph, denoted by \mathcal{G}_b , which is characterized by being balanced and having a globally reachable node. We consider the stabilization problem, which is a specific form of Problem 1, where the objective is to ensure that the outputs of all agents converge to zero, i.e., $\lim_{t \rightarrow \infty} y(t) = 0 \in \text{span}(\mathbb{1})$. Addressing this problem provides a foundation for solving the more general output consensus problem for networked systems over general digraphs (i.e., Problem 1).

Assume that for $i \in \mathbb{V}$, the agents follow the dynamics,

$$\Sigma_i^o : \begin{cases} \dot{x}_i(t) = f_i(x_i(t), u_i(t)), \\ y_i(t) = h_i(x_i(t)), \end{cases} \quad (15)$$

where f_i and h_i are continuously differentiable functions. For the system (15), we say x_0 is *asymptotically reachable from {0}* if there exists an input $u_{x_0} : (-\infty, 0] \rightarrow \mathbb{R}^{|\mathbb{V}|}$ such that, when applying u_{x_0} to the state equation of (15) for $t \in (-\infty, 0]$, we have $x(0) = x_0$ and $x(t) \rightarrow 0$ as $t \rightarrow -\infty$.

Consider the stablization problem of a network system $(\Sigma^o, \Pi, \mathcal{G}_b, w)$. We first establish the passivity-like inequalities of H_a and H_c by exploiting the inherent passivity properties of the individual agents and controllers. Then, we derive a sufficient condition to guarantee stablization of the system.

The following result provides a passivity-like inequality for the agent relation H_a .

Proposition 4. Consider a group of $|\mathbb{V}|$ SISO agents (15) interconnected over a digraph \mathcal{G}_b . Assume that each agent Σ_i^o , for $i \in \{1, \dots, |\mathbb{V}|\}$, is OP- ϵ_i and with initial conditions that are asymptotically reachable from {0}. Let $\epsilon = \min_i(\epsilon_i)$. Then, it follows that

$$u^\top \text{Proj}_{S^\perp}(y) \geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i(x_i) - \|u\|_2 \|y\|_2 + \epsilon \|\text{Proj}_{S^\perp}(y)\|_2^2, \quad (16)$$

and the passivity relation satisfies,

$$\langle u^\tau, \text{Proj}_{S^\perp}(y^\tau) \rangle \geq -\frac{\max(D_o)}{\epsilon} \|u^\tau\|_{\mathcal{L}_2}^2 + \epsilon \|\text{Proj}_{S^\perp}(y^\tau)\|_{\mathcal{L}_2}^2, \quad (17)$$

where $Q_i(x_i)$ is the storage function, and $\max(D_o)$ denotes the maximal out-degree of \mathcal{G}_b , respectively.

Proof. We start by summing up the passivity inequalities of all the agents, i.e.,

$$u^\top y \geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i + \sum_{i=1}^{|\mathbb{V}|} \epsilon_i y_i^2 \geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i + \epsilon \|y\|_2^2. \quad (18)$$

Then, consider $u^\top \text{Proj}_{S^\perp}(y)$,

$$\begin{aligned} u^\top \text{Proj}_{S^\perp}(y) &= u^\top y - \frac{1}{|\mathbb{V}|} u^\top \mathbb{1} \mathbb{1}^\top y \\ &\geq \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i + \epsilon \|y\|_2^2 - \frac{1}{|\mathbb{V}|} u^\top \mathbb{1} \mathbb{1}^\top y. \end{aligned} \quad (19)$$

For $-\frac{1}{|\mathbb{V}|} u^\top \mathbb{1} \mathbb{1}^\top y$, by the Cauchy–Schwarz inequality,

$$\begin{aligned} |-\frac{1}{|\mathbb{V}|} u^\top \mathbb{1} \mathbb{1}^\top y| &= \frac{1}{|\mathbb{V}|} |u^\top \mathbb{1} \mathbb{1}^\top y| = \frac{1}{|\mathbb{V}|} |u^\top \mathbb{1}| |\mathbb{1}^\top y| \\ &\leq \frac{1}{|\mathbb{V}|} \|u\|_2 \|\mathbb{1}\|_2 \|y\|_2 \|\mathbb{1}\|_2 = \|u\|_2 \|y\|_2. \end{aligned}$$

So, we arrive at

$$-\frac{1}{|\mathbb{V}|} u^\top \mathbb{1} \mathbb{1}^\top y \geq -\|u\|_2 \|y\|_2. \quad (20)$$

Before considering $\|\text{Proj}_{S^\perp}(y)\|_2^2$, recall that $I - \frac{1}{|\mathbb{V}|} \mathbb{1} \mathbb{1}^\top$ is a projection matrix with eigenvalues $\{0, 1^{(|\mathbb{V}|-1)}\}$. Using the properties of the Rayleigh quotient (Horn & Johnson, 2012, Theorem 4.2.2), we get

$$\|\text{Proj}_{S^\perp}(y)\|_2^2 = y^\top (I - \frac{1}{|\mathbb{V}|} \mathbb{1} \mathbb{1}^\top) y \leq y^\top y. \quad (21)$$

Plugging (21) and (20) into (19), we obtain (16).

To show (17), we first rearrange (18),

$$\begin{aligned} \sum_{i=1}^{|\mathbb{V}|} \dot{Q}_i &\leq u^\top y - \epsilon y^\top y = -\frac{1}{2\epsilon} (u - \epsilon y)^\top (u - \epsilon y) \\ &\quad + \frac{1}{2\epsilon} u^\top u - \frac{\epsilon}{2} y^\top y \leq \frac{1}{2\epsilon} u^\top u - \frac{\epsilon}{2} y^\top y. \end{aligned}$$

Integrating both sides over $(-\infty, \tau]$ yields,

$$\begin{aligned} \int_{-\infty}^{\tau} y^\top(t) y(t) dt &\leq \frac{1}{\epsilon^2} \int_{-\infty}^{\tau} u^\top(t) u(t) dt - \frac{2}{\epsilon} \sum_{i=1}^{|\mathbb{V}|} Q_i(x(\tau)) \\ &\quad + \frac{2}{\epsilon} \lim_{t \rightarrow \infty} \sum_{i=1}^{|\mathbb{V}|} Q_i(x(-t)). \end{aligned}$$

Employing $\lim_{t \rightarrow \infty} \sum_{i=1}^{|\mathbb{V}|} Q_i(x(-t)) = 0$ (i.e., the asymptotical reachability of initial conditions) and the non-negative nature of $Q(x)$, it follows that,

$$\|y^\tau\|_{\mathcal{L}_2} \leq \frac{1}{\epsilon} \|u^\tau\|_{\mathcal{L}_2}. \quad (22)$$

Now, by integrating $-\|u\|_2\|y\|_2$ over $(-\infty, \tau]$ and using Cauchy-Schwarz, we have that

$$\left| - \int_{-\infty}^{\tau} \|u(t)\|_2 \|y(t)\|_2 dt \right| \leq \|u^\tau\|_{\mathcal{L}_2} \|y^\tau\|_{\mathcal{L}_2}. \quad (23)$$

With inequalities (22) and (23), we obtain that

$$- \int_{-\infty}^{\tau} \|u(t)\|_2 \|y(t)\|_2 dt \geq -\|u^\tau\|_{\mathcal{L}_2} \|y^\tau\|_{\mathcal{L}_2} \geq -\frac{1}{\epsilon} \|u^\tau\|_{\mathcal{L}_2}^2. \quad (24)$$

Next, we establish the relationship between u and μ . Since $u = -B_o\mu$ (see (13)), the following relation holds,

$$-\|u^\tau\|_{\mathcal{L}_2}^2 = - \int_{-\infty}^{\tau} \mu^\top(t) B_o^\top B_o \mu(t) dt \geq -\max(D_o) \|\mu^\tau\|_{\mathcal{L}_2}^2, \quad (25)$$

where the last inequality is by the properties of the Geršgorin Disks Theorem (Bullo, 2020, Theorem 2.8) and Rayleigh quotient (Horn & Johnson, 2012, Theorem 4.2.2). Indeed, observe that the entries of $B_o^\top B_o$ are either 0 or 1, with diagonal elements equal to 1. Furthermore, the largest row sum of this matrix is given by $\max(D_o)$. This implies that the maximal eigenvalue of $B_o^\top B_o$ is less than or equal to $\max(D_o)$.

Now, integrate both sides of (16) over $(-\infty, \tau]$ for $\tau \in [0, \infty)$,

$$\begin{aligned} \langle u^\tau, \text{Proj}_{S^\perp}(y^\tau) \rangle &\geq \sum_{i=1}^{|\mathbb{V}|} Q_i(x_i(\tau)) - \lim_{t \rightarrow -\infty} \sum_{i=1}^{|\mathbb{V}|} Q_i(x_i(t)) \\ &\quad - \int_{-\infty}^{\tau} \|u\|_2 \|y\|_2 dt + \epsilon \|\text{Proj}_{S^\perp}(y^\tau)\|_{\mathcal{L}_2}^2. \end{aligned} \quad (26)$$

Recall that $\lim_{t \rightarrow -\infty} \sum_{i=1}^{|\mathbb{V}|} Q_i(x_i(t)) = 0$ and $Q_i(x_i(t))$ are non-negative. Plugging (24) and (25) into (26), we get Eq. (17). \square

The following proposition is for the controller relation H_c .

Proposition 5. Consider a group of $|\mathbb{E}|$ SISO edge controllers (2). Assume that for all $k \in \{1, \dots, |\mathbb{E}|\}$, the controllers Π_k are OP- a_i . Then, it follows that,

$$z^\top \text{Proj}_{S^\perp}(y) \geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k(\eta_k) + \alpha \|\mu\|_2^2, \quad (27)$$

and the passivity relation satisfies,

$$\langle z^\tau, \text{Proj}_{S^\perp}(y^\tau) \rangle \geq - \lim_{t \rightarrow -\infty} \sum_{k=1}^{|\mathbb{E}|} W_k(\eta_k(t)) + \alpha \|\mu^\tau\|_{\mathcal{L}_2}^2, \quad (28)$$

where $W_k(\eta_k)$ are the storage functions and $\alpha = \min_k(\alpha_k)$.

Proof. Recall that for the incidence matrix of a balanced digraph, the relation $E^\top \mathbb{1}_n = \mathbb{0}_m$ and $E \mathbb{1}_m = \mathbb{0}_n$ hold. Thus,

$$\begin{aligned} E^\top y &= (I_m - \frac{1}{n} \mathbb{1}_m \mathbb{1}_m^\top)(E^\top y) = \text{Proj}_{S^\perp}(E^\top y) \\ &= E^\top(I_n - \frac{1}{n} \mathbb{1}_n \mathbb{1}_n^\top)y = E^\top \text{Proj}_{S^\perp}(y). \end{aligned}$$

With this relation, summing up the passivity inequalities of all the controllers yields,

$$\begin{aligned} \mu^\top \zeta &= \mu^\top E^\top y = z^\top \text{Proj}_{S^\perp}(y) \\ &\geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k + \sum_{k=1}^{|\mathbb{E}|} \alpha_k \mu_k^2 \geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k + \alpha \|\mu\|_2^2. \end{aligned} \quad (29)$$

This demonstrates (27). Now, integrate both sides of (27) over $(-\infty, \tau]$ for $\tau \in [0, \infty)$, and we obtain (28). \square

Recall that achieving output agreement suggests $\lim_{t \rightarrow \infty} \text{Proj}_{S^\perp}(y(t)) = 0$. Now, we are ready to present the main result of this section.

Theorem 1. Consider a network system $(\Sigma^o, \Pi, \mathcal{G}_b, w)$. Suppose the conditions of Propositions 4 and 5 are met. If $\alpha \geq \frac{\max(D_o)}{\epsilon}$ where $\max(D_o)$ denotes the maximal out-degree of \mathcal{G}_b , then the system is stabilized.

Proof. We start by showing that the trajectories of $(\Sigma^o, \Pi, \mathcal{G}_b, w)$ are bounded. From Propositions 4 and 5, both the forward path (from u

to y) and the feedback path (from y to z) of Fig. 1(b) are passive. Consequently, it satisfies a global dissipation inequality, where the rate of change of the storage function is bounded by the supply rate. Since the storage function can be chosen to be radially unbounded (i.e., a quadratic storage function), the trajectories must be bounded.

Recall that $w = z + u$. Sum up (17) and (28),

$$\epsilon \|\text{Proj}_{S^\perp}(y^\tau)\|_{\mathcal{L}_2}^2 \leq \langle w^\tau, \text{Proj}_{S^\perp}(y^\tau) \rangle + \lim_{t \rightarrow -\infty} \sum_{k=1}^{|\mathbb{E}|} W_k(\eta_k(t)), \quad (30)$$

and the pointwise form yields,

$$w^\top \text{Proj}_{S^\perp}(y) \geq \sum_{k=1}^{|\mathbb{E}|} \dot{W}_k(\eta_k) + \epsilon \|\text{Proj}_{S^\perp}(y)\|_{\mathcal{L}_2}^2.$$

Then, following an approach analogous to the proof of Lemma 6.5 in Khalil (2002), it can be shown that $\text{Proj}_{S^\perp}(y)$ is bounded for bounded w , as described by,

$$\|\text{Proj}_{S^\perp}(y^\tau)\|_{\mathcal{L}_2} \leq \frac{1}{\epsilon} \|w^\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2}{\epsilon} \lim_{t \rightarrow -\infty} \sum_{k=1}^{|\mathbb{E}|} W_k(\eta_k(t))} \quad (31)$$

Now, let $s(t) = \text{Proj}_{S^\perp}^\top(y(t)) \text{Proj}_{S^\perp}(y(t))$. From (30), due to the boundedness of trajectories and (31), we obtain that $\lim_{t \rightarrow \infty} \int_0^t s(t) dt$ exists and is finite.

Next, we demonstrate that $s(t)$ is uniformly continuous. Given the dynamics of Σ^o , the derivative of y_i can be expressed as $\dot{y}_i = \frac{\partial h_i}{\partial x_i} f_i(x_i, u_i)$. Since the trajectories are bounded, $\frac{\partial h_i}{\partial x_i}$ and $f_i(x_i, u_i)$ are also bounded, implying the boundedness of \dot{y}_i for all $i \in [1, |\mathbb{V}|]$. Consequently, $\frac{d}{dy} s(t) = 2y^\top(I - \frac{1}{|\mathbb{V}|} \mathbb{1}\mathbb{1}^\top)\dot{y}$ is bounded. Thus, $s(t)$ is uniformly continuous.

Now, having satisfied the conditions for applying Barbalat's Lemma (Khalil, 2002, Lemma 8.2) to $s(t)$, we conclude that $\text{Proj}_{S^\perp}(y(t)) \rightarrow 0$ as $t \rightarrow \infty$, which implies asymptotic output agreement. Consequently, $E^\top y(t) = \zeta(t) = \mathbb{0}$. Since both the agents and controllers are output strictly passive, their input-output trajectories pass through the origin, yielding $\zeta(t) = \mu(t) = u(t) = y(t) = \mathbb{0}$. Thus, the network system is stabilized. \square

Theorem 1 presents a passivity-based analysis of network systems with balanced digraphs, establishing a sufficient condition for output consensus (stabilization) in terms of the passivity index of the edge controller dynamics. The theorem provides a lower bound on the passivity index, which has an insightful physical interpretation: it is the ratio between the maximal out-degree of the underlying digraph and the minimal passivity index of the agents. This result guarantees that the system is stabilized, but does not necessarily ensure average consensus. Theorem 1 also has some limitations that should be acknowledged. First, the provided condition is only sufficient for stabilization, and a tighter lower bound on the passivity index may exist. Second, the theorem is applicable only to balanced digraphs that contain a globally reachable node, and it requires the agents to be output strictly passive with their outputs determined solely by their states.

5. Case study: Neural network

In this section, we consider a continuous neural network on n neurons (Scardovi et al., 2009; Sharf & Zelazo, 2018),

$$\dot{x}_i = -a_i x_i + b \sum_{j \neq i} (\tanh(x_j) - \tanh(x_i)) + w_i, \quad (32)$$

where x_i and $\frac{1}{a_i} > 0$ denote the voltage on the i th neuron and the self-correlation time of the neuron, respectively, b is the coupling coefficient and w_i is the exogenous input of the neuron. Note that the agents can be modeled by $\dot{x}_i = -a_i x_i + u_i$; $y_i = \tanh(x_i)$ and the edge controllers follow the linear consensus protocol, i.e., $\mu_k = b \zeta_k$. The agents are OP- a_i and the controllers are OP- b .

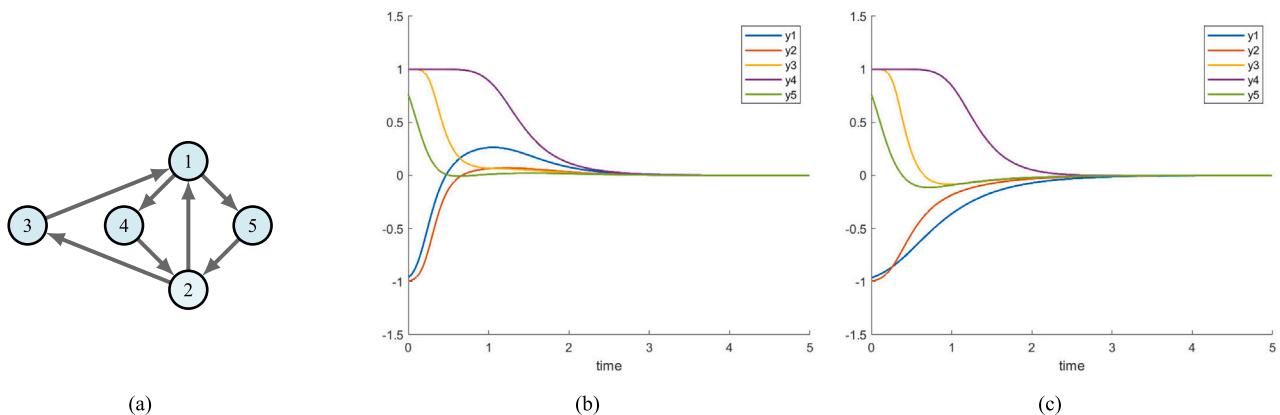


Fig. 2. The underlying graph and trajectories for the outputs. (a) A balanced digraph. (b) Outputs for the system with edge controllers $\mu_k = \frac{4}{3} \max(\zeta_k, 0)$. (c) Outputs for systems with nonlinear edge controllers $\mu_k = \frac{4}{3} \max(\zeta_k, 0)$.

We run the system with 5 neurons. The underlying graph of the system, as shown in Fig. 2(a), is balanced and contains a globally reachable node. The maximum out-degree of the graph is 2. In this example, the value $w_i = 0$ and values a_i , are chosen randomly, $a = [1.66, 3.22, 4.62, 1.5, 2.56]$. The initial conditions are set to $x(0) = [-2; -3; 6; 10; 1]$. According to Theorem 1, if $b \geq \frac{\max(D_o)}{\min(a_i)} = \frac{4}{3}$, the system is stabilized. This is verified by the simulation result, as shown in Fig. 2(b).

Furthermore, the models of the edge controllers can be chosen to be other nonlinear OP- $\frac{4}{3}$ systems, for example, $\mu_k = \frac{4}{3} \max(\zeta_k, 0)$. Fig. 2(c) shows the outputs of the system, and the system is stabilized. In both cases, the system states converge to regular consensus.

6. Concluding remarks

In this work, we first propose a general approach capable of conducting passivity analysis for network systems with directed couplings. Then, we transform the consensus problem into a convergence analysis on a submanifold. Finally, we provide a passivity-based analysis for network systems over balanced digraphs, which serves as a sufficient condition for stabilization (a specific instance of output consensus). In future work, we will explore the more general output consensus problem for networked systems over general digraphs.

CRediT authorship contribution statement

Feng-Yu Yue: Conceptualization, Data curation, Formal analysis, Methodology, Software, Writing – original draft. **Daniel Zelazo:** Conceptualization, Formal analysis, Funding acquisition, Methodology, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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