

On Certain Properties of Convex Functions

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Abstract—This note deals with certain properties of convex functions. We provide results on the convexity of the set of minima of these functions, the behaviour of their subgradient set under restriction, and optimization of these functions over an affine subspace.

I. INTRODUCTION

This paper deals with certain properties of convex function, most of them well-known but generally unmentioned in the literature. This work employs subgradient calculus, calculating the subgradient space after restricting to a subspace, or optimizing on a “moving” affine subspace, and also deals with the collection of minima of the convex function.

II. THE LEMMAS

This paper contains the proof of the three following lemmas:

Lemma 1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a convex function, and let $S : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be some linear operator. Fix some $\zeta \in \text{Im}(S)$ and define a map $g : \{y : Sy = \zeta\} \rightarrow \mathbb{R}$ by $g(x) = f(x)$. Then

- 1) g is a convex function on the smaller space $\text{Im}(S^T)$,
- 2) the subdifferential of g at x is given by

$$\partial g(x) = \text{Proj}_{\text{Im}(S)}(\partial f(x)),$$

where Proj_W is the orthogonal projection on the subspace W .

Lemma 2. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function, and let $S : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be some linear operator. Define a map $h : \text{Im}(S^T) \rightarrow \mathbb{R}$ by

$$h(x) = \min_{r: S^T r = x} f(r),$$

assuming that the minimum is always achieved. Then

- 1) h is a convex function,
- 2) if f is strictly convex, then h is strictly convex.

Lemma 3. Let $C \subseteq \mathbb{R}^n$ be a convex set and let $f : C \rightarrow \mathbb{R}$ be convex. Suppose that f achieves its minimum m in C , and let $M = \{x : f(x) = m\}$ be the set of f ’s minima. Then M is convex.

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PROOFS

We start by proving lemma 2:

Proof. We denote $V = \text{Im}(S^T)$ for simplicity. We take some $x, y \in V$ and $t \in [0, 1]$. Our goal is to show that

$$h(tx + (1-t)y) \leq th(x) + (1-t)h(y)$$

pick $r_x, r_y \in \mathbb{R}^d$ such that $g(x) = f(r_x)$ and $g(y) = f(r_y)$ (these exist by assumption that the minimum is always achieved). Then on one hand, we have $S^T(tr_x + (1-t)r_y) = tx + (1-t)y$ by linearity, so $h(tx + (1-t)y) \leq f(tr_x + (1-t)r_y)$. On the other hand, by convexity:

$$f(tr_x + (1-t)r_y) \leq tf(r_x) + (1-t)f(r_y) = th(x) + (1-t)h(y) \quad (1)$$

so we get the wanted inequality by chaining the two inequalities.

As for strict convexity, we should note that if $x \neq y$ then $r_x \neq r_y$, so the inequality 1 becomes strict. This completes the proof of the lemma. \square

We now prove lemma 3:

Proof. Let $x, y \in M$ and let $t \in [0, 1]$. We need to show that $tx + (1-t)y \in M$. Indeed, because f is convex,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) = tm + (1-t)m = m$$

but on the other hand, $f(tx + (1-t)y)$ cannot be smaller than m , as m is the minimum of f . Thus $f(tx + (1-t)y) = m$ and thus $tx + (1-t)y \in C$. \square

Lastly, we prove lemma 1, which is the “toughest” of the three:

Proof. We denote $X = \{y : Sy = \zeta\}$, $V = \ker S$ and $U = \text{Im}(S^T)$. We know that $\text{Im}(S^T) = \ker(S)^\perp$, so we can identify \mathbb{R}^n as a direct sum of U and V . Thus we get a function $F : U \times V \rightarrow \mathbb{R}$ defined by $F(u, v) = f(u + v)$.

In [1], one shows that if $\chi : \mathbb{R} \rightarrow \mathbb{R}$ is convex, then $\partial\chi(t) = [\chi'_-, (t)\chi'_+(t)]$, where χ'_\pm are the one-sided derivatives. Furthermore, we know (again by [1]) that if $\rho : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, and we fix $v_0, v_1 \in V$ and define $\chi(t) = \rho(v_0 + tv_1)$, then χ ’s one-sided derivatives are given via:

$$\begin{aligned} \chi'_+(t) &= \max_{\alpha \in \partial\rho(v_0 + tv_1)} \alpha^T v_1 \\ \chi'_-(t) &= \min_{\alpha \in \partial\rho(v_0 + tv_1)} \alpha^T v_1 \end{aligned} \quad (2)$$

These facts, together with the fact that $\partial\rho(v_0 + tv_1)$ is convex, imply that $\partial\chi(t) = v_1^T \cdot \partial\rho(v_0 + tv_1)$

Now, we can finally begin our proof. By assumption, there’s some $y \in \mathbb{R}$ such that $Sy = \zeta$. We can decompose y as $u_0 + v_0$ for some $u_0 \in \text{Im}(S^T)$ and $v_0 \in \ker(S)$. The set X is equal to $u_0 + V$. Thus, we map g can be described as $g(v) = F(u_0, v)$.

Take some $v_0, v_1 \in V$. Restricting g to the line $\{v_0 + tv_1 : t \in \mathbb{R}\}$ is identical to restricting f to the line $\{(u_0, v_0 + tv_1) : t \in \mathbb{R}\}$. Thus they yield the same subdifferential sets at $t = 0$. By above, we get that:

$$v_1^T \cdot \partial g(v_0) = v_1^T \cdot \partial F(u_0, v_0) = v_1^T \cdot \text{Proj}_V(\partial F(u_0, v_0))$$

meaning that the sets $\partial g(v_0)$ and $\text{Proj}_V(\partial F(u_0, v_0))$ look the same when hit by a linear functional on V . However, both of these sets are both convex and closed (see [1]). Thus, the separating hyperplane theorem (see [2]) implies that they are equal. Reading what V and F are, we get that for any $x \in \text{Im}(S)$,

$$\partial g(x) = \text{Proj}_{\text{Im}(S)}(\partial f(x))$$

which completes the proof. \square

REFERENCES

- [1] R. T. Rockafellar, *Convex Analysis*. Princeton Landmarks in Mathematics and Physics, Princeton University Press, 1997.
- [2] J. Conway, *A Course in Functional Analysis*. Springer, 2nd ed., 1994. pp. 110–111.