COORDINATION AND CONTROL OF MULTI-AGENT SYSTEMS

086730

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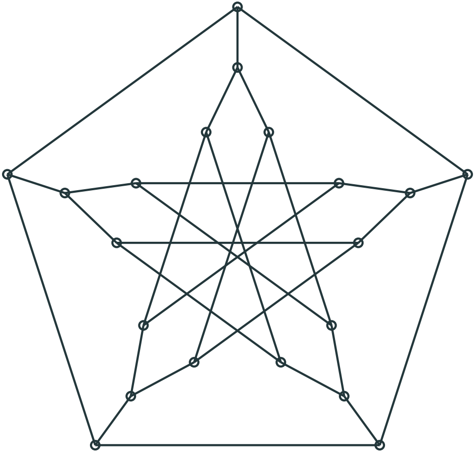
October 25, 2025

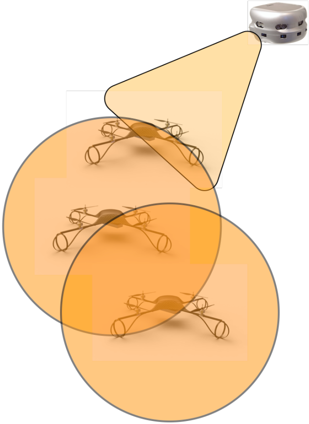
Introduction to Graph Theory



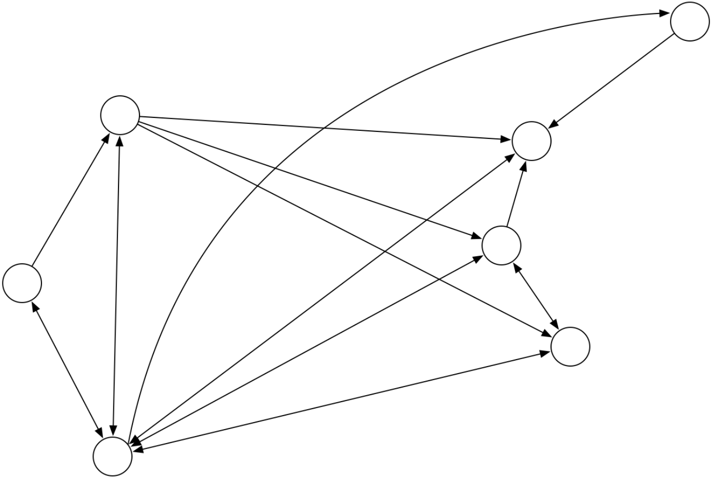
ABSTRACTION USING GRAPHS

$\dot{x}_1(t) = f_1(x_1(t), u_1(t), t)$

$\dot{x}_2(t) = f_2(x_2(t), u_2(t), t)$

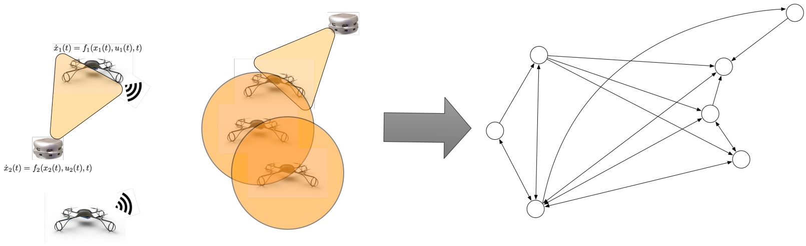


ABSTRACTION USING GRAPHS



- ○ - nodes
- $\rightarrow$ - edges (directed or undirected)

ABSTRACTION USING GRAPHS

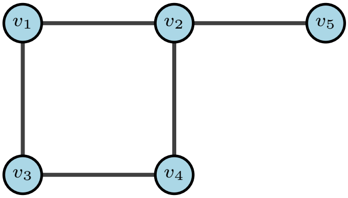


Definition

A graph is an ordered pair comprised of a set of vertices (or nodes), and a set of edges (or links).

- a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- vertex set $\mathcal{V} = \{v_1, \dots, v_n\}$
- edge set $\mathcal{E} \subseteq [\mathcal{V}]^2$ (all 2-element subsets of $\mathcal{V}$)

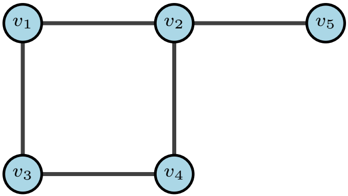
UNDIRECTED GRAPHS



$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

- $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\}$
- $[\mathcal{V}]^2 = \begin{array}{l} \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_4\}, \\ \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_4, v_5\}\} \end{array}$
- $\mathcal{E} = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_2, v_5\}\}$

UNDIRECTED GRAPHS

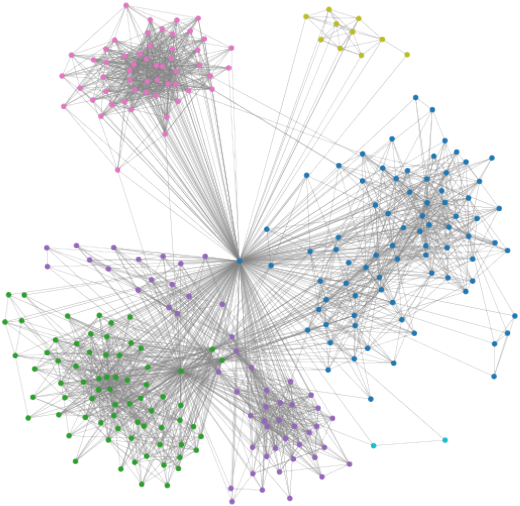


$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

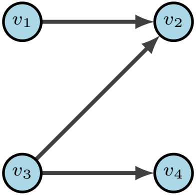
more terminology...

- adjacent nodes : $v_1 \sim v_2 \Leftrightarrow \{v_1, v_2\} \in \mathcal{E} \Leftrightarrow v_1 v_2 \in \mathcal{E}$
- incident nodes : $v_i \in \mathcal{V}$ is *incident* to edge $e_j \in \mathcal{E}$, i.e., $e_j = \{v_i, v_k\} \in \mathcal{E}$ for some $v_k \in \mathcal{V}$
- neighborhood of a node: $\mathcal{N}(v_i) = \mathcal{N}_{v_i} = \{v_j \in \mathcal{V} \ : \ \{v_i, v_j\} \in \mathcal{E}\}$

Example: graphs can model *social interactions*



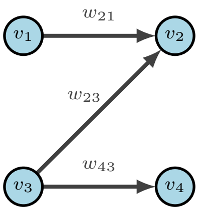
DIRECTED GRAPHS



$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

- $\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5\}$
- $\mathcal{E} = \{(v_1, v_2), (v_3, v_2), (v_3, v_4)\}$
    - edges are ordered pairs with a tail (initial) and head (terminal) node
    - edges are said to have an orientation
- can define (in)- and (out)-Neighborhoods

WEIGHTED GRAPHS



$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

- weights can be assigned to each edge (directed or undirected)
- $\mathcal{W} : \mathcal{E} \to \mathbb{R}$
  - i.e., $\mathcal{W}((v_3, v_4)) = w_{43}$
  - can collect weights into a diagonal matrix

$$W = \begin{bmatrix} \ddots & & \\ & w_{ji} & \\ & & \ddots \end{bmatrix} \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$$

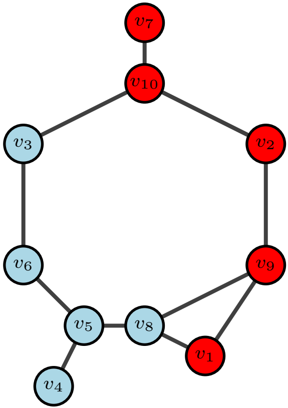
Definition

A (simple) path is a sequence of distinct vertices such that consecutive vertices are adjacent.

Example: path from v_1 to v_7

$$P(v_1, v_7) = v_1 v_9 v_2 v_{10} v_7$$

- path length is the number of edges traversed
- paths are not unique!
  - shortest path



Definition

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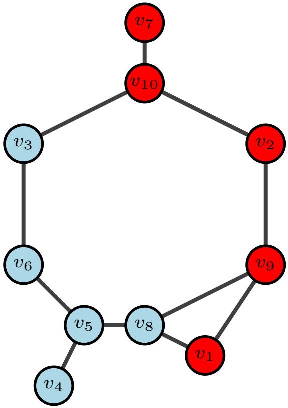
Example: path from v_1 to v_7

$$P(v_1, v_7) = v_1 v_9 v_2 v_{10} v_7$$

Example: *Shortest Path Problem*

Given a graph with two nodes identified as the *start* node and the *terminal* node, find the shortest length path between them.



- Waze and other navigation software
- optimization over graphs (Network Optimization)

Definition

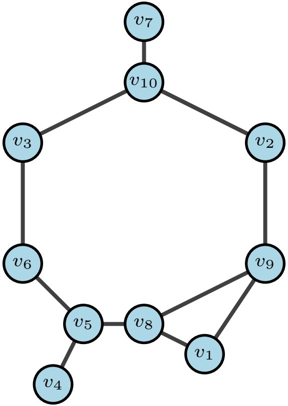
A walk (of length k) is a non-empty alternating sequence $v_0 e_0 v_1 e_1 \cdots e_{k1} v_k$ of vertices and edges in $\mathcal{G}$ such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If $v_0 = v_k$, the walk is closed.

Example: possible walk from v_4 to v_6

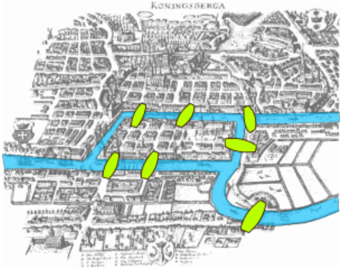
$$v_4 e_{45} v_5 e_{56} v_6 \text{ (length:}2)$$

or

$$v_4 e_{45} v_5 e_{58} v_8 e_{81} v_1 e_{81} v_8 e_{58} v_5 e_{56} v_6 \text{ (length:}6)$$



SEVEN BRIDGES OF KÖNIGSBERG



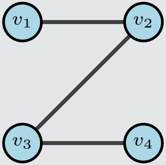
- 7 bridges problem led to the development of graph theory

Is there a *walk* through the city of Königsberg that crosses each bridge once and only once?

Undirected Graphs

Connected Graph

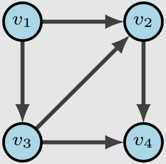
an undirected graph is connected if for every pair of vertices, there exists a path connecting them



Directed Graphs

Strongly Connected Graph

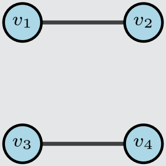
a directed graph is strongly connected if for every pair of vertices, there exists a *directed* path connecting them



Undirected Graphs

Disconnected Graph

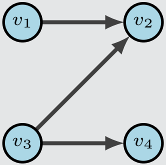
a graph is disconnected if it is not (weakly) connected



Directed Graphs

Weakly Connected Graph

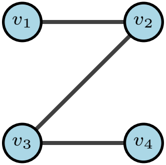
a directed graph is weakly connected if the graph obtained by replacing each directed edge with an undirected edge is connected



Undirected Graphs

- degree of a vertex $v_i \in \mathcal{V}$ is the cardinality of its neighbor set

$$d_{v_i} = |\mathcal{V}(v_i)|$$



$$d_{v_1} = 1$$
$$d_{v_3} = 2$$

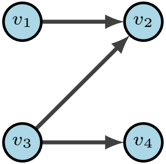
Directed Graphs

- in-degree of a vertex $v_i \in \mathcal{V}$ is the cardinality of its in-neighbor set

$$d_{v_i}^{in} = |\mathcal{V}^{in}(v_i)|$$

- out-degree of a vertex $v_i \in \mathcal{V}$ is the cardinality of its out-neighbor set

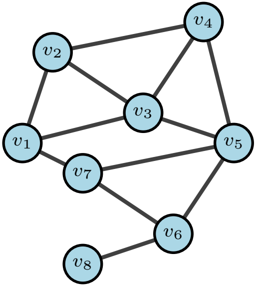
$$d_{v_i}^{out} = |\mathcal{V}^{out}(v_i)|$$



$$d_{v_1}^{in} = 0$$
$$d_{v_2}^{in} = 2$$
$$d_{v_3}^{out} = 2$$

Graphs are a set-theoretic object!

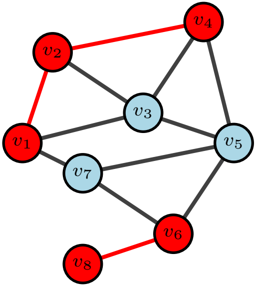
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$
$\mathcal{V} = \{v_1, \ldots, v_8\}$



Graphs are a set-theoretic object!

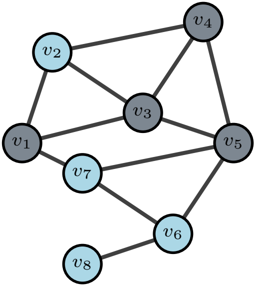
$\mathcal{G} = (\mathcal{V}, \mathcal{E})$
$\mathcal{V} = \{v_1, \ldots, v_8\}$

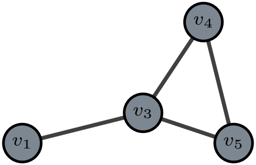
Subgraph
$\mathcal{G}' = (\mathcal{V}', \mathcal{E}') \subset \mathcal{G}$
$\Rightarrow \mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$
$\mathcal{V}' = \{v_1, v_2, v_4, v_6, v_8\}$
$\mathcal{E}' = \{\{v_1, v_2\}, \{v_2, v_4\}, \{v_6, v_8\}\}$



SUBGRAPHS

Graphs are a set-theoretic object!



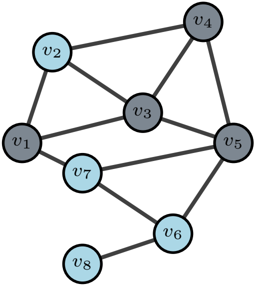


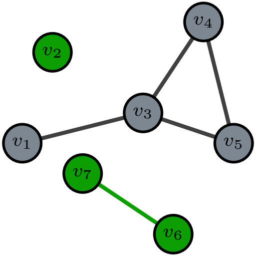
$$\mathcal{G}_{\mathcal{S}} = (\mathcal{S}, \mathcal{E}_{\mathcal{S}}) \subseteq \mathcal{G}$$
$$\mathcal{E}_{\mathcal{S}} = \{\{v_i, v_j\} \in \mathcal{E} \,|\, v_i, v_j \in \mathcal{S}\}$$

Generate a subgraph that is induced by a set of nodes $\mathcal{S} = \{v_1, v_3, v_4, v_5\}$

SUBGRAPHS

Graphs are a set-theoretic object!





Generate a subgraph that is induced by a set of nodes

$\mathcal{S} = \{v_1, v_3, v_4, v_5\}$

$\mathcal{G}_{\mathcal{S}} = (\mathcal{S}, \mathcal{E}_{\mathcal{S}}) \subseteq \mathcal{G}$

$\mathcal{E}_{\mathcal{S}} = \{\{v_i, v_j\} \in \mathcal{E} \mid v_i, v_j \in \mathcal{S}\}$

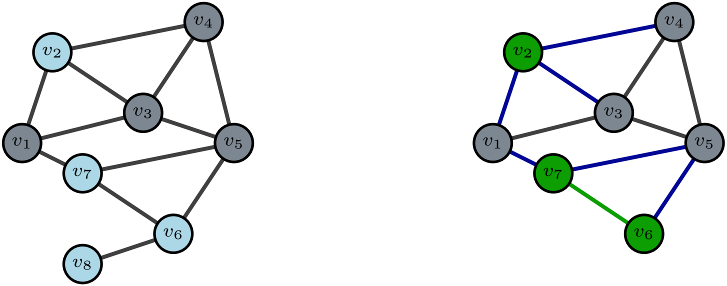
Boundary of a subgraph

$\partial\mathcal{G}_{\mathcal{S}} = (\partial\mathcal{S}, \mathcal{E}_{\partial\mathcal{S}})$

$\partial\mathcal{S} = \{v_i \in \mathcal{V} \mid v_i \notin \mathcal{S},\, \exists v_j \in \mathcal{S}\ \text{s.t.} \{v_i, v_j\} \in \mathcal{E}\} = \{v_2, v_7, v_7\}$

$\mathcal{E}_{\partial\mathcal{S}} = \{\{v_i, v_j\} \in \mathcal{E} \mid v_i, v_j \in \partial\mathcal{S}\}$

Graphs are a set-theoretic object!



Generate a subgraph that is induced by a set of nodes

$\mathcal{S} = \{v_1, v_3, v_4, v_5\}$

$\mathcal{G}_{\mathcal{S}} = (\mathcal{S}, \mathcal{E}_{\mathcal{S}}) \subseteq \mathcal{G}$

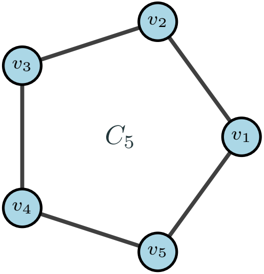
$\mathcal{E}_{\mathcal{S}} = \{\{v_i, v_j\} \in \mathcal{E} \mid v_i, v_j \in \mathcal{S}\}$

Closure of a subgraph

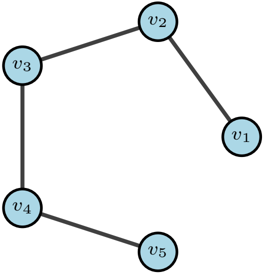
$\mathrm{cl}\mathcal{G}_{\mathcal{S}} = \mathcal{G}_{S \cup \partial S}$

Trees and Cycles

A cycle is a connected graph where each node has degree 2

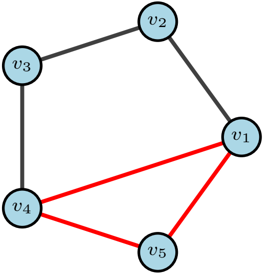


A tree is a connected graph containing no cycles (acyclic)

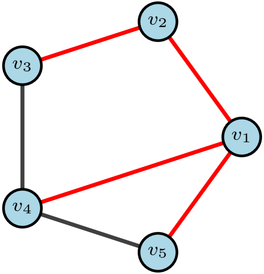


Trees and Cycles

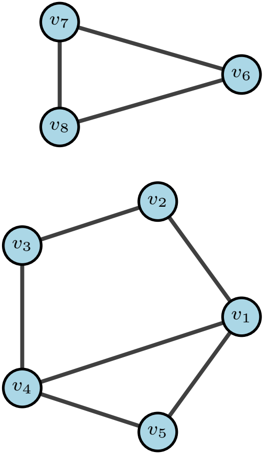
A graph contains cycles if there is a subgraph that is a cycle



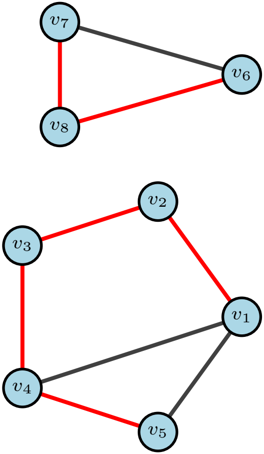
A spanning tree of a connected graph is a subgraph that is a tree



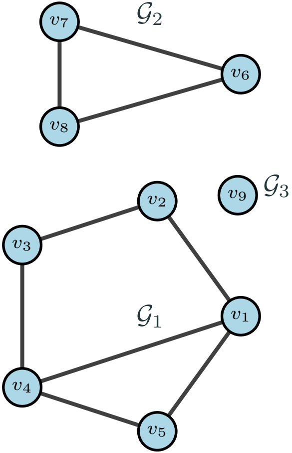
Forests



A spanning forest is a maximal acyclic subgraph



Connected Components



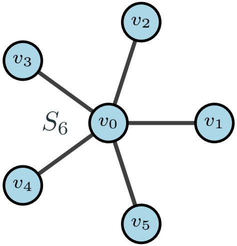
A connected component is a connected subgraph of $\mathcal{G}$

$$\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$$

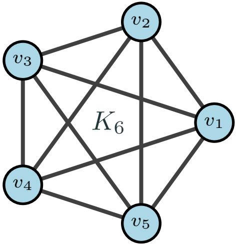
$\mathcal{G}$ has 3 connected components

SPECIAL GRAPH CLASSES

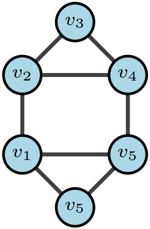
Star Graph



Complete Graph

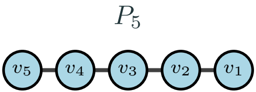


k-Regular Graph

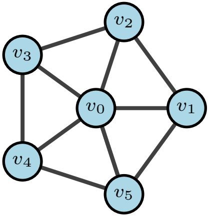


3-regular

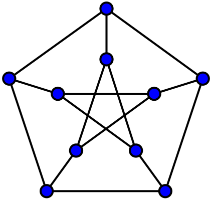
Path Graph



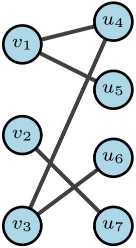
Wheel Graph



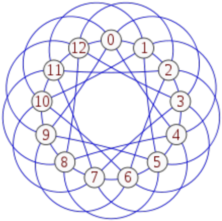
Peterson Graph



Bipartite Graph

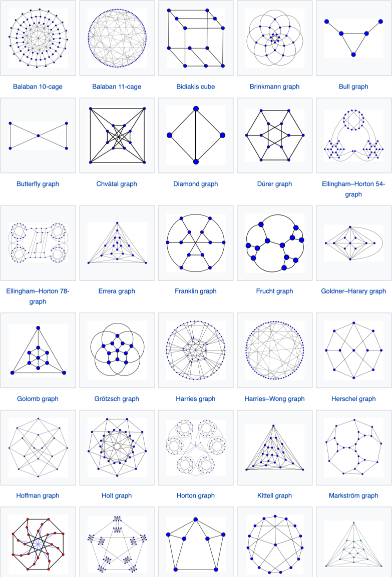


Payley Graph



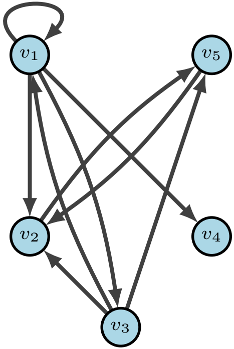
SPECIAL GRAPH CLASSES

so many named graphs!



All square matrices have a (directed) graph representation!

$$M = \begin{bmatrix} 3 & 3 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 10 \\ 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \Leftrightarrow$$



For a matrix $M \in \mathbb{R}^{n \times n}$

$\mathcal{G}(M) = (\mathcal{V}(M), \mathcal{E}(M))$

$|\mathcal{V}(M)| = n$

$e = (v_i, v_j) \in \mathcal{E}(M) \Leftrightarrow [M]_{ij} \neq 0$

Definition

A matrix $M \in \mathbb{R}^{n \times n}$ is said to be irreducible if there does not exist a permutation matrix P and an integer r such that

$$P^T M P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

with $B \in \mathbb{R}^{r \times r}$, $C \in \mathbb{R}^{r \times n-r}$, and $D \in \mathbb{R}^{n-r \times n-r}$.

Definition

A matrix $M \in \mathbb{R}^{n\times n}$ is said to be irreducible if there does not exist a permutation matrix P and an integer r such that

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What is a permutation matrix?

Definition

A permutation matrix P is a square matrix or order n such that each row and column contains one element equal to 1, with remaining elements equal to 0. Furthermore, permutation matrices satisfy the property $P^T = P^{-1}$.

Example:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

Definition

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Which matrix is irreducible?

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{or} \quad M = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

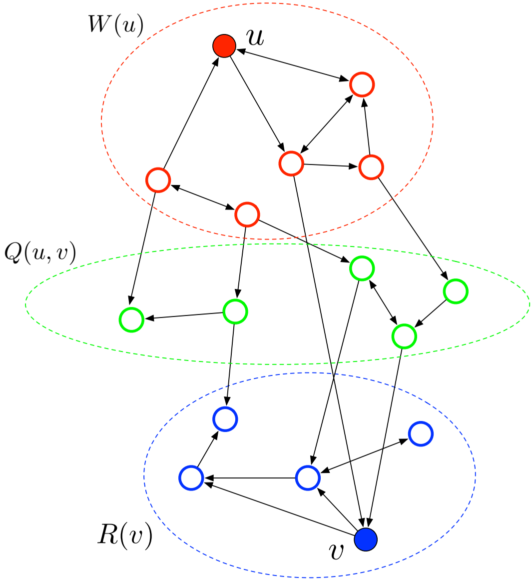
what is the permutation matrix?

Theorem

Let $M \in \mathbb{R}^{n\times n}$. The following statements are equivalent:

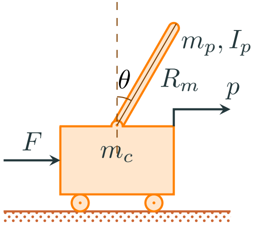
i) M is irreducible.

ii) The digraph associated with M, $\mathcal{G}(M)$, is strongly connected.


$W(u)$
u
$Q(u,v)$
$R(v)$
v

Structured Linear Systems

A *structured linear system* is a description of a dynamic system that considers only the interaction and influence between system states, control, and outputs independent on any realization of parameter values.



$$\begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \ddot{p} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-m_p g}{m_c} & \frac{-K_1^2}{R_m m_c} & 0 \\ 0 & \frac{(m_p+m_c)g}{m_c I_p} & \frac{K_1^2}{R_m m_c I_p} & 0 \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{K_1}{R_m m_c} \\ \frac{-K_1}{R_m m_c I_p} \end{bmatrix} F$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix}$$

Structured Linear Systems

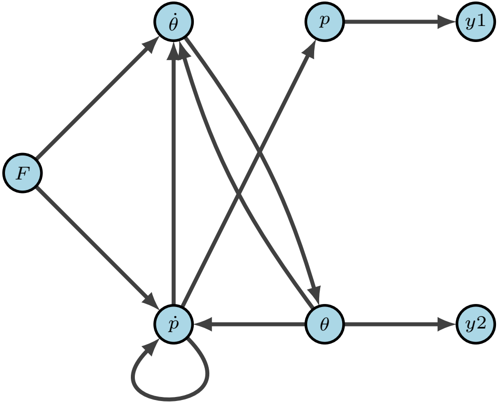
A *structured linear system* is a description of a dynamic system that considers only the interaction and influence between system states, control, and outputs independent on any realization of parameter values.

- Express dynamics as a matrix

$$\begin{bmatrix} \dot{\mathbf{x}} \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}}_{M} \begin{bmatrix} \mathbf{x} \\ F \end{bmatrix}$$

- Define the digraph associated with M

$$\mathcal{V} = \{F, p, \dot{p}, \theta, \dot{\theta}, y_1, y_2\}$$



the strucure of a system

- system states and controls are either related (non-zero entry in state-space) or not (0-entry)
- values of parameters are neglected

$$\begin{bmatrix} \dot{p} \\ \dot{\theta} \\ \ddot{p} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star \\ 0 & \star & \star & 0 \\ 0 & \star & \star & 0 \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \star \\ \star \end{bmatrix} F$$

Definition

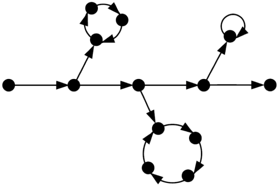
A system (A, B) is structurally controllable if there exists a system structurally equivalent to (A, B) which is controllable in the usual sense.

Theorem [Lin '74]

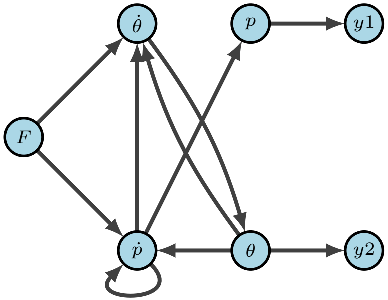
The following statements for a structured system (A, B) are equivalent:

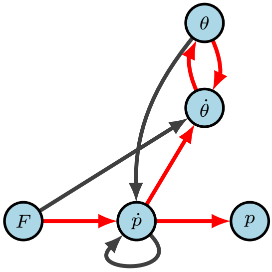
i) (A, B) is structurally controllable

ii) In the graph $\mathcal{G}(A, B)$, there exists a disjoint union of cacti that covers all the state vertices.



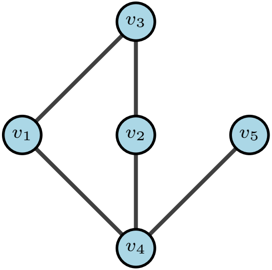
a cactus graph with 3 buds





the graph of the system contains a cactus! the system is structurally controllable!

Graphs and their properties can be studied using matrices and constructs from linear algebra

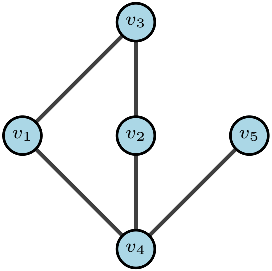


Degree Matrix: $\Delta(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
A diagonal matrix with the degree of each node on the diagonal

$$[\Delta(\mathcal{G})]_{ij} = \begin{cases} d(v_i), & i = j \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta(\mathcal{G}) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Graphs and their properties can be studied using matrices and constructs from linear algebra



Adjacency Matrix: $A(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$
A symmetric matrix encoding the adjacency relationship of nodes in the graph

$$[A(\mathcal{G})]_{ij} = \begin{cases} 1, & i \sim j \\ 0, & \text{otherwise} \end{cases}$$

$$A(\mathcal{G}) = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Lemma

Let $\mathcal{G}$ be a graph with adjacency matrix $A(\mathcal{G})$. The number of walks from node v_i to v_j of length r is $[A(\mathcal{G})^r]_{ij}$.

Proof:

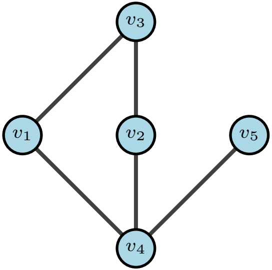
Homework

Corollary

Let $\mathcal{G}$ be an undirected graph with e edges, t triangles, and adjacency matrix $A(\mathcal{G})$. Then

i) $\operatorname{tr} A(\mathcal{G}) = 0$
ii) $\operatorname{tr} A(\mathcal{G})^2 = 2e$
iii) $\operatorname{tr} A(\mathcal{G})^3 = 6t$

Graphs and their properties can be studied using matrices and constructs from linear algebra

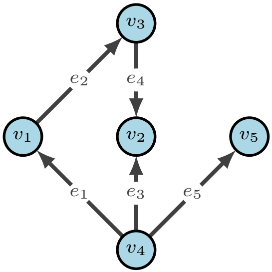


Incidence Matrix: $E(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$
A matrix encoding the incidence relation between nodes and edges

$$[E(\mathcal{G})]_{ij} = \begin{cases} 1, & v_i \text{ is tail of edge } e_j \\ -1, & v_i \text{ is head of edge } e_j \\ 0, & \text{otherwise} \end{cases}$$

$$E(\mathcal{G}) = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Graphs and their properties can be studied using matrices and constructs from linear algebra



assign an arbitrary orientation to each edge

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Theorem

Let $\mathcal{G}$ be a graph with n vertices, c connected components, and an arbitrary orientation assigned to each edge. Then $\operatorname{rank} E(\mathcal{G}) = n - c$.

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- with appropriate relabelling of nodes/edges, can write

$$E(\mathcal{G}) = \begin{bmatrix} E(\mathcal{G}_1) & & \\ & \ddots & \\ & & E(\mathcal{G}_c) \end{bmatrix}$$

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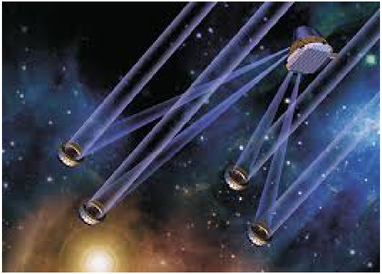
- let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a connected graph. Show that $\operatorname{rank} E(\mathcal{H}) = |\mathcal{V}| - 1$

Interferometry is a technique used for imaging in deep space. Rather than using 1 large (and expensive!) telescope, a team of smaller (and cheaper!) sensors can achieve the same goal. This requires high accuracy and precision of relative spacing between satellites.



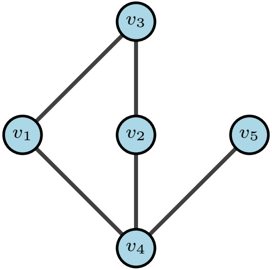
$$\dot{x} = f(x_i, u_i)$$

$$y = \begin{bmatrix} \vdots \\ x_i - x_j \\ \vdots \end{bmatrix}$$

For the sensing graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, each edge $e_i = (v_i, v_j) \in \mathcal{E}$ encodes the relative measurement $x_i - x_j$

$$\boxed{y = E(\mathcal{G})^T x}$$

Graphs and their properties can be studied using matrices and constructs from linear algebra



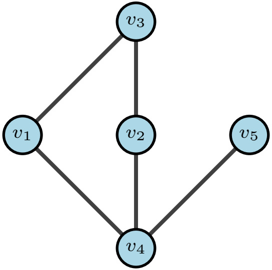
Combinatorial Graph Laplacian: $L(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$

A symmetric matrix

$$[L(\mathcal{G})]_{ij} = \begin{cases} d(v_i), & i = j \\ -1, & \{i,j\} \in \mathcal{E} \end{cases}$$

$$L(\mathcal{G}) = \begin{bmatrix} 2 & 0 & -1 & -1 & 0 \\ 0 & 2 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Graphs and their properties can be studied using matrices and constructs from linear algebra



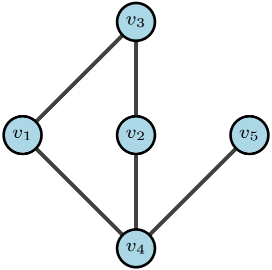
Combinatorial Graph Laplacian: $L(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$

Constructions

$$\begin{aligned} L(\mathcal{G}) &= \Delta(\mathcal{G}) - A(\mathcal{G}) \\ &= E(\mathcal{G})E(\mathcal{G})^T \end{aligned}$$

- using incidence matrix, construction is independent of the edge orientation!

Graphs and their properties can be studied using matrices and constructs from linear algebra



Combinatorial Graph Laplacian: $L(\mathcal{G}) \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{V}|}$

- $\operatorname{rank} L(\mathcal{G}) = |\mathcal{V}| - 1 \Leftrightarrow \mathcal{G}$ is connected
- $\mathcal{G}$ is connected, then 0 is a simple eigenvalue and $L(\mathcal{G})\mathbb{1} = 0$
- $L(\mathcal{G})$ is a positive semi-definite matrix

$$x^T L(\mathcal{G}) x \geq 0 \, \forall x \in \mathbb{R}^{|\mathcal{V}|}$$

- ordered eigenvalues

$$0 = \lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \cdots \leq \lambda_{|\mathcal{V}|}(\mathcal{G})$$

- Algebraic Connectivity (Fiedler Eigenvalue) : $\lambda_2(\mathcal{G})$

Theorem

For a graph $\mathcal{G}$, the following statements are equivalent:

i) $\mathcal{G}$ is connected

ii) $\lambda_2(\mathcal{G}) > 0$.

Theorem

Let $\tau(\mathcal{G})$ be the number of spanning trees in $\mathcal{G}$. Then

$$\tau(\mathcal{G}) = \det L(\mathcal{G})_{(ij)}.$$

- For a matrix $M \in \mathbb{R}^{n\times n}$, $M_{(ij)} \in \mathbb{R}^{n-1\times n-1}$ is obtained by deleting the ith row and jth column of M

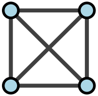
$$M = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \Rightarrow M_{(23)} = \begin{bmatrix} 1 & 2 & 4 \\ 9 & 10 & 12 \\ 13 & 14 & 16 \end{bmatrix}$$

- $\det M_{(ij)}$ is called the ij-minor of M

Theorem

Let $\tau(\mathcal{G})$ be the number of spanning trees in $\mathcal{G}$. Then

$$\tau(\mathcal{G}) = \det L(\mathcal{G})_{(ij)}.$$



$$L(\mathcal{G}) = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix} \quad \tau(\mathcal{G}) = 16$$