

# PASSIVITY, MONOTONICITY, AND NETWORK OPTIMIZATION: NEW PERSPECTIVES FOR NETWORK SYSTEMS ANALYSIS

---

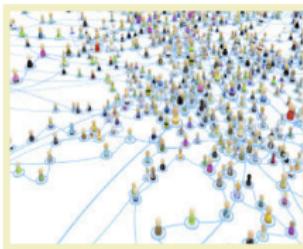
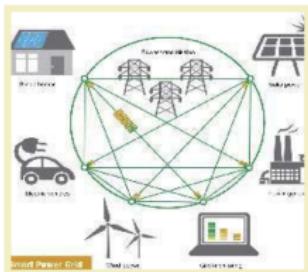
Daniel Zelazo

University of Washington

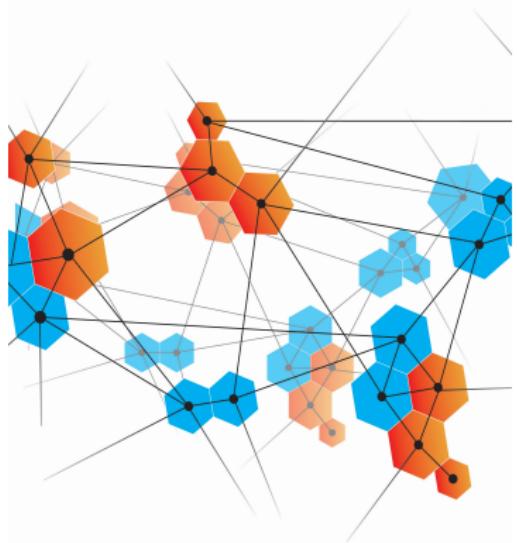
July 25, 2023



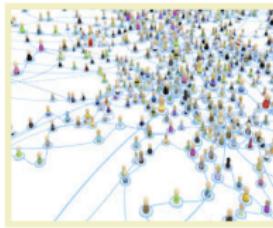
# NETWORKED DYNAMIC SYSTEMS



Networks of dynamical systems are one of the enabling technologies of the future.



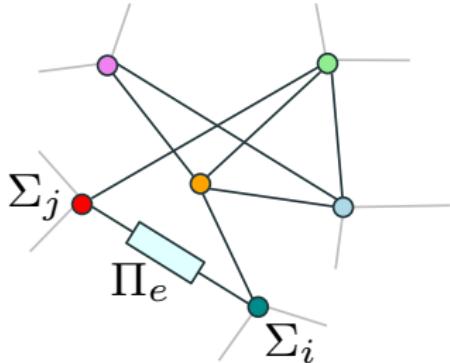
# NETWORKED DYNAMIC SYSTEMS



- ▶ how do we **analyze** these systems
- ▶ how do we **design** these systems



## IN THIS TALK...



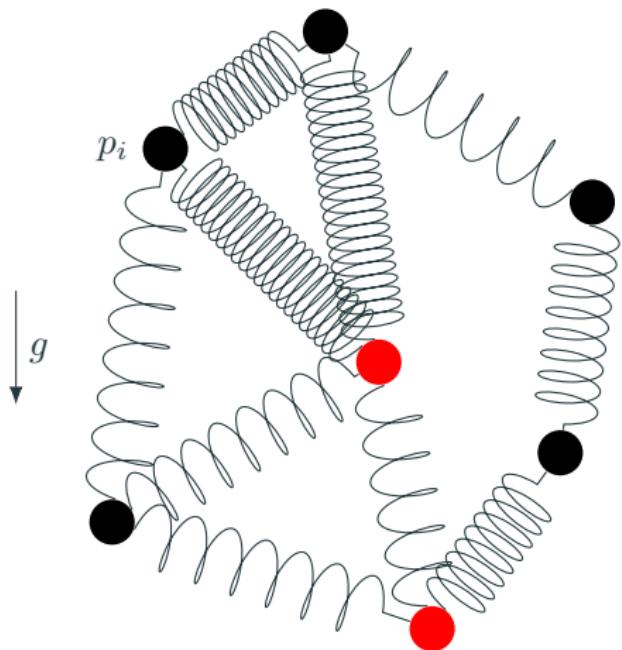
Explore the structure and mechanisms of networked systems to reveal deep connections between properties of dynamical systems and optimization theory.

- ▶ A general model of diffusively coupled networks
- ▶ Characterization of network equilibria via Network Optimization
- ▶ Convergence properties of dynamic networks via passivity theory
- ▶ Passivation, monotonization, and equilibrium independent passive short systems

## A PHYSICS WARM-UP

---

## A MASS-SPRING NETWORK

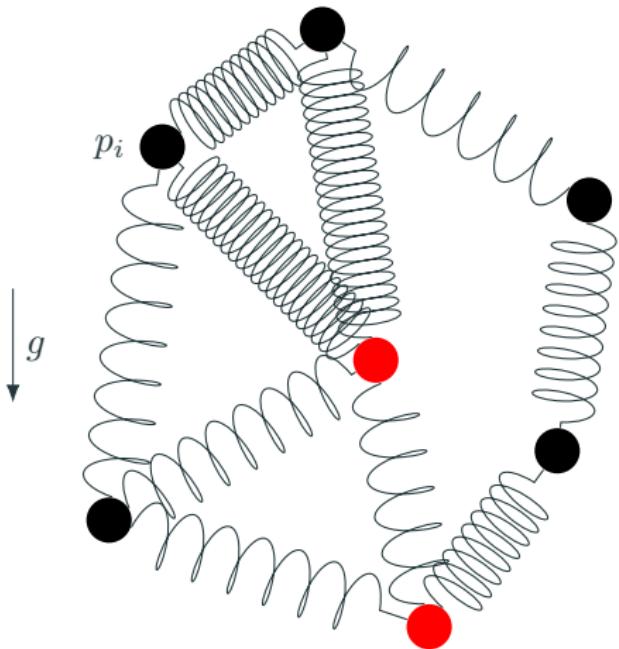


● Free

● Fixed

- ▶ A fixed network of (linear) springs
- ▶ springs connected to masses with position  $p_i \in \mathbb{R}^2$  and mass  $m_i$
- ▶  $r$  masses have a fixed position (**anchors**)

## A MASS-SPRING NETWORK



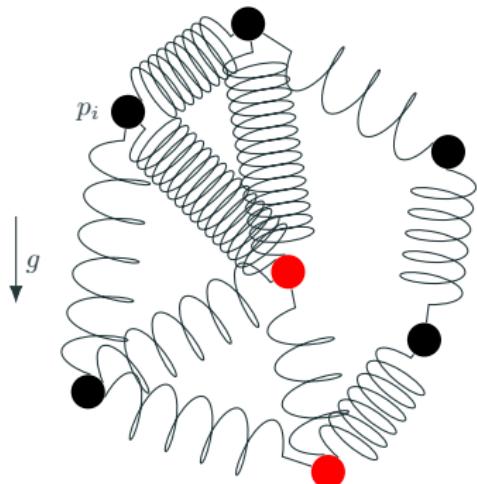
● Free

● Fixed

- ▶ A fixed network of (linear) springs
- ▶ springs connected to masses with position  $p_i \in \mathbb{R}^2$  and mass  $m_i$
- ▶  $r$  masses have a fixed position (**anchors**)

Determine the positions of the free masses that minimize the total potential energy of the mass-spring network.

# A MASS-SPRING NETWORK



► Potential Energy due to gravity

$$m_i g^T p_i$$

► Elastic Potential Energy of springs

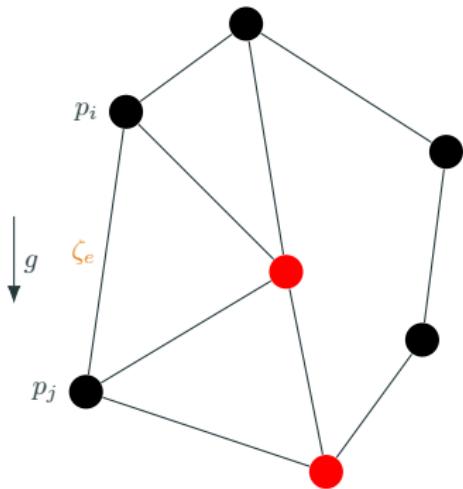
$$\frac{1}{2} k_{ij} (\|p_i - p_j\| - r_{ij})^2$$

an optimization problem (take 1)

$$\min_{p_i} \quad \sum_i m_i g^T p_i + \sum_{i \sim j} \frac{1}{2} k_{ij} (\|p_i - p_j\| - r_{ij})^2$$

$$\text{s.t. } p_i = \mathbf{p}_i^*, i = 1, \dots, r \text{ (fixed nodes)}$$

# A MASS-SPRING NETWORK



- ▶ Potential Energy due to gravity (nodes)

$$m_i g^T p_i, \quad i = 1, \dots, n$$

- ▶ Elastic Potential Energy of springs (edges)

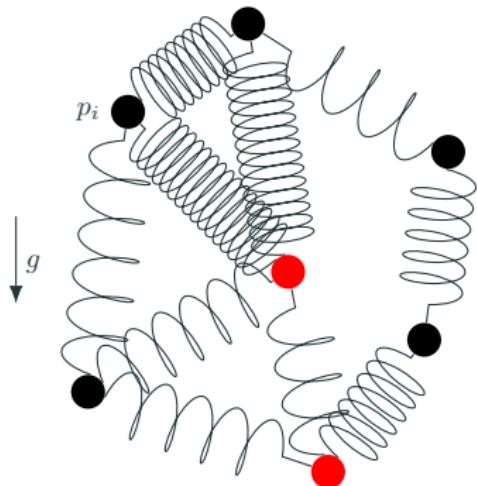
$$\frac{1}{2} k_e (\underbrace{\|p_i - p_j\|}_{\zeta_e} - r_e)^2, \quad e = 1, \dots, m$$

an optimization problem (take 2)

$$\min_{p_i, \zeta_e} \quad \sum_{i=1}^r (m_i g^T p_i + \mathbb{I}_{p_i^*}(p_i)) + \sum_{i=r+1}^n m_i g^T p_i + \sum_e \frac{1}{2} k_e (\| \zeta_e \| - r_e)^2$$

$$\text{s.t. } p_i - p_j = \zeta_e, \quad \forall e = (i, j)$$

# A MASS-SPRING NETWORK



A Convex Program!

an optimization problem (take 2)

$$\min_{p_i, \zeta_e} \quad \sum_i^r (m_i g^T p_i + \mathbb{I}_{\mathbf{p}_i^*}(p_i)) + \sum_{i=r+1}^n m_i g^T p_i + \sum_e \frac{1}{2} k_{ij} (\|\zeta_e\| - r_e)^2$$

$$\text{s.t. } p_i - p_j = \zeta_e, \forall e = (i, j)$$

# A MASS-SPRING NETWORK - THE DYNAMICS

► dynamic model for the masses

► springs couple masses together

$$\Sigma_i : \begin{cases} \begin{bmatrix} \dot{p}_i \\ \ddot{p}_i \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_i + m_i g \\ y_i = \begin{cases} \begin{bmatrix} p_i \\ 0 \end{bmatrix}, & i = 1, \dots, r \text{ (anchors)} \\ \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix}, & i = r+1, \dots, n \end{cases} \end{cases} \quad \Pi_e : \begin{cases} u_i = \sum_{i \sim j} k_{ij} (\|p_i - p_j\| - r_{ij}) \frac{p_j - p_i}{\|p_j - p_i\|} + b_{ij} (\dot{p}_j - \dot{p}_i) \\ = \sum_{i \sim j} \kappa_{ij} (y_i - y_j) \end{cases}$$

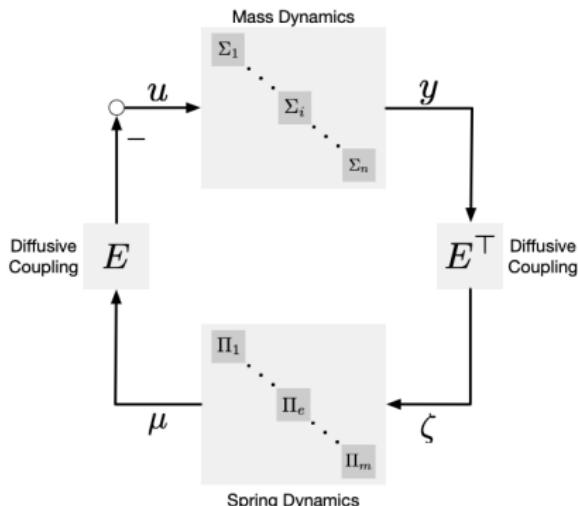
# A MASS-SPRING NETWORK - THE DYNAMICS

► dynamic model for the masses

► springs couple masses together

$$\Sigma_i : \begin{cases} \begin{bmatrix} \dot{p}_i \\ \ddot{p}_i \end{bmatrix} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u_i + m_i g \\ y_i = \begin{cases} \begin{bmatrix} p_i \\ 0 \end{bmatrix}, & i = 1, \dots, r \text{ (anchors)} \\ \begin{bmatrix} p_i \\ \dot{p}_i \end{bmatrix}, & i = r+1, \dots, n \end{cases} \end{cases}$$

$$\Pi_e : \begin{cases} u_i = \sum_{i \sim j} k_{ij} (\|p_i - p_j\| - r_{ij}) \frac{p_j - p_i}{\|p_j - p_i\|} + b_{ij} (\dot{p}_j - \dot{p}_i) \\ = \sum_{i \sim j} \kappa_{ij} (y_i - y_j) \end{cases}$$

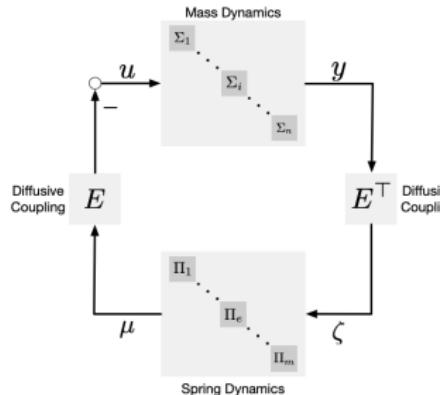


An example of a  
diffusively coupled  
network!

# A MASS-SPRING NETWORK - THE DYNAMICS

## ► System Equilibrium

$$\begin{cases} 0 = \dot{p}_i \\ 0 = \sum_{i \sim j} k_{ij} (\|p_i - p_j\| - r_{ij}) \frac{p_j - p_i}{\|p_j - p_i\|} + b_{ij}(\dot{p}_j - \dot{p}_i) + m_i g \end{cases}$$

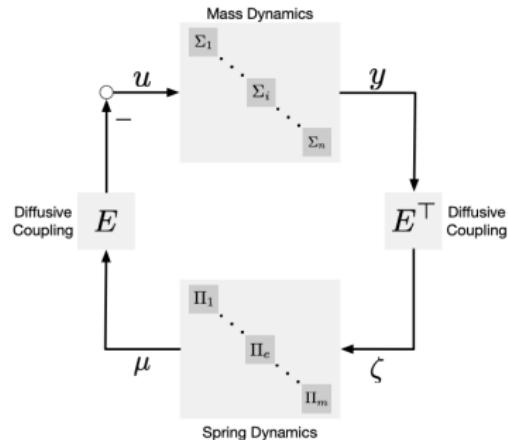


## Minimum Total Potential Energy Principle (MTPE)

Equilibrium configurations extremize the total potential energy. **Stable equilibria** correspond to **minimizers** of the total potential energy.

## Dynamics

### ► Diffusively Coupled Network



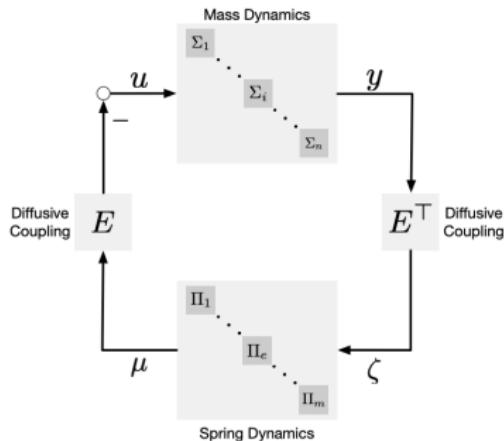
### ► Dissipativity Theory

$$V(x) = \frac{1}{2} \sum_i \|\dot{p}_i\|^2 + \frac{1}{2} \sum_{i \sim j} k_{ij} \|p_i - p_j\|_2^2$$

# LESSONS AND TOOLS

## Dynamics

- Diffusively Coupled Network



## Optimization

- Convex Optimization

$$\min_{p_i, \zeta_e} J(p, \zeta)$$

$$\text{s.t. } p_i - p_j = \zeta_e, \forall e = (i, j)$$

- Optimality Conditions

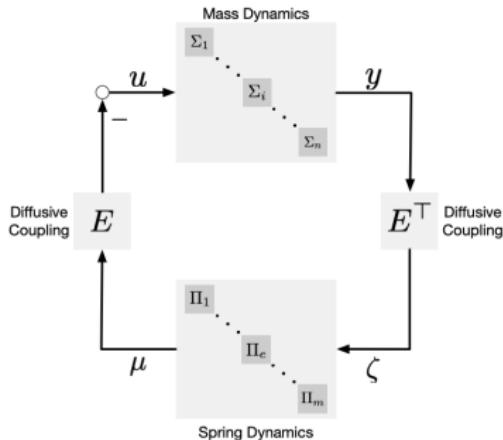
$$0 \in \partial J(p, \zeta)$$

- Dissipativity Theory

$$V(x) = \frac{1}{2} \sum_i \|\dot{p}_i\|^2 + \frac{1}{2} \sum_{i \sim j} k_{ij} \|p_i - p_j\|_2^2$$

## Dynamics

- Diffusively Coupled Network



- Dissipativity Theory

$$V(x) = \frac{1}{2} \sum_i \|\dot{p}_i\|^2 + \frac{1}{2} \sum_{i \sim j} k_{ij} \|p_i - p_j\|_2^2$$

## Optimization

- Convex Optimization

$$\min_{p_i, \zeta_e} J(p, \zeta)$$

$$\text{s.t. } p_i - p_j = \zeta_e, \forall e = (i, j)$$

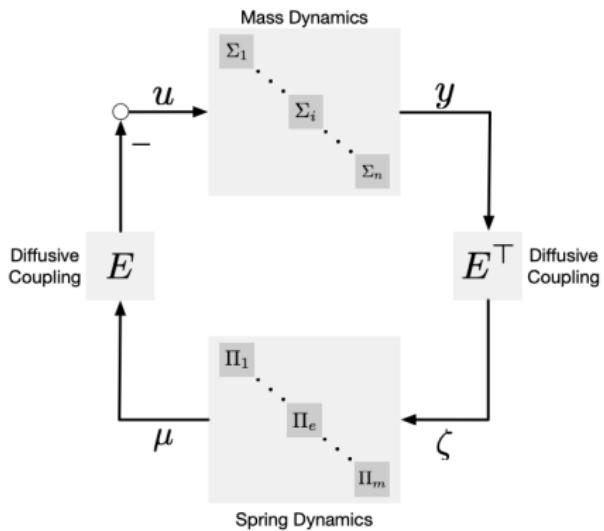
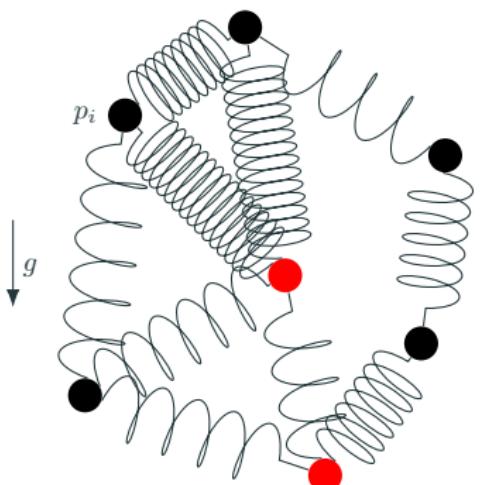
- Optimality Conditions

$\Omega \subset \mathbb{R}^n$

MTPE Principle ensures that the dynamics of the diffusively coupled network solve the optimization problem, and vice versa.

# THE QUESTION

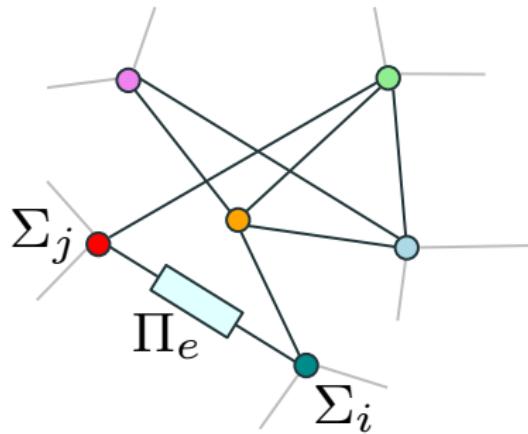
- ▶ What class of systems can be “solved” by examining a related optimization problem?
- ▶ What class of optimization problems can be solved by a dynamical system?



## **DIFFUSIVELY COUPLED NETWORKS**

---

## A NETWORK MODEL



A **network system** is comprised of dynamical systems that interact with each other over an information exchange network (a graph).

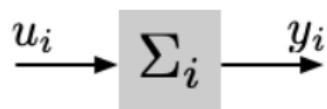
## A NETWORK MODEL

Agent dynamics:

$$u_i \rightarrow \Sigma_i \rightarrow y_i$$
$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}$$

# A NETWORK MODEL

Agent dynamics:



$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}$$

Information Exchange Network:



$$\mathcal{G} = (\mathbb{V}, \mathbb{E})$$

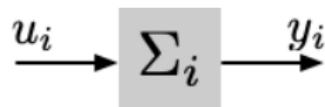
$$E = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$[E]_{ij} = \begin{cases} \pm 1 & (i, j) \in \mathbb{E} \\ 0 & \text{o.w.} \end{cases}$$

$$E^\top \mathbf{1} = 0$$

# A NETWORK MODEL

Agent dynamics:



$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}$$

Information Exchange Network:



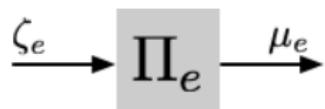
$$\mathcal{G} = (\mathbb{V}, \mathbb{E})$$

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$[E]_{ij} = \begin{cases} \pm 1 & (i, j) \in \mathbb{E} \\ 0 & \text{o.w.} \end{cases}$$

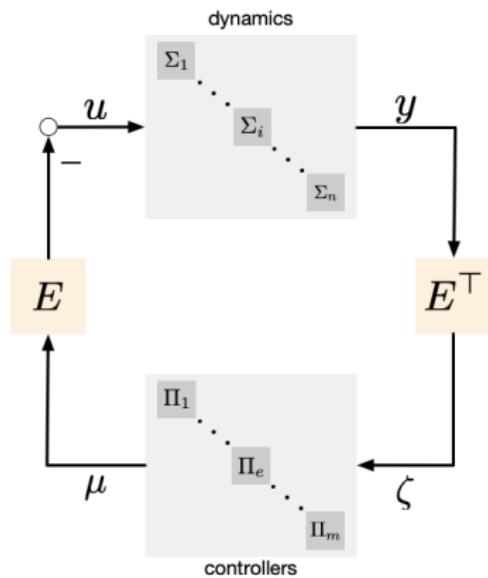
$$E^\top \mathbf{1} = 0$$

Controller dynamics:



$$\Pi_e : \begin{cases} \dot{\eta}_e = \phi_e(\eta_e, \zeta_e) \\ \mu_e = \psi_e(\eta_e, \zeta_e) \end{cases}$$

# DIFFUSIVE COUPLING



► Consensus Dynamics

$$\dot{x}_i = - \sum_{i \sim j} w_{ij} (x_j - x_i)$$

► Kumamoto Model

$$\dot{\theta}_i = -k \sum_{i \sim j} \sin(\theta_i - \theta_j)$$

► Traffic Dynamics

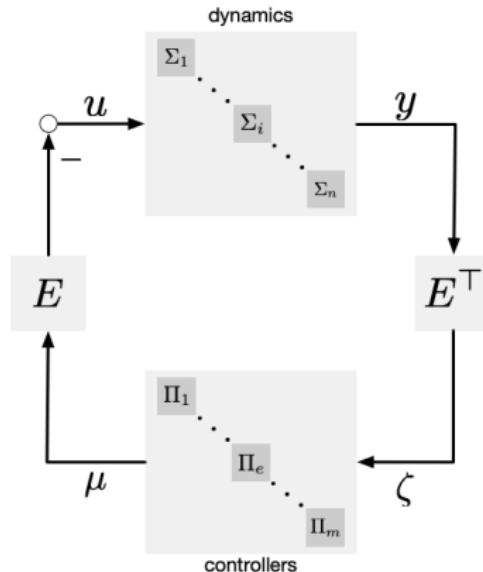
$$\dot{v}_i = \kappa_i \left( V_i^0 - v_i + V_i^1 \sum_{i \sim j} \tanh(p_j - p_i) \right)$$

► Neural Network

$$C\dot{V}_i = f(V_i, h_i) + \sum_{i \sim j} g_{ij} (V_j - V_i)$$

$$\dot{h}_i = g(V_i, h_i)$$

# STEADY-STATE NETWORK SOLUTIONS



What properties must the systems  $\Sigma_i$  and  $\Pi_e$  possess such that  $(\Sigma, \Pi, \mathcal{G})$  admits and converges to a steady-state solution?

$$u(t) \rightarrow u$$

$$y(t) \rightarrow y$$

$$\zeta(t) \rightarrow \zeta$$

$$\mu(t) \rightarrow \mu$$

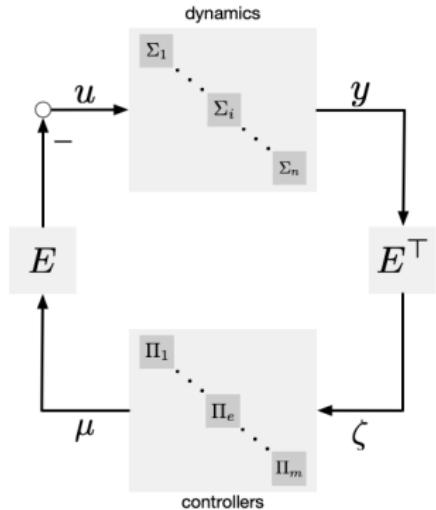
- ▶ Consensus:  $y = \alpha \mathbf{1}$  ( $\zeta = 0$ )
- ▶ Formation:  $\zeta \neq 0$  constant

All signals converge to a **constant** steady-state

## **NETWORK OPTIMIZATION MEETS PASSIVITY THEORY**

---

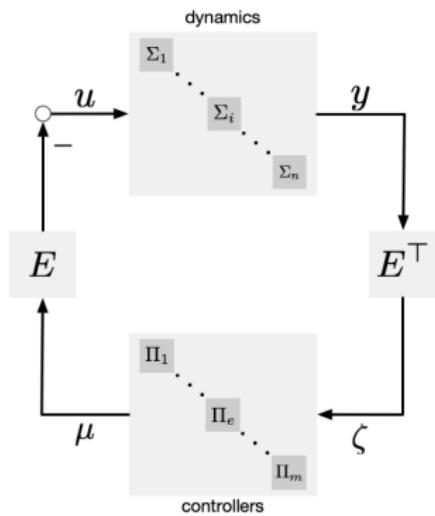
# STEADY-STATE INPUT-OUTPUT MAPS



## Assumption 1

Each agent  $\Sigma_i$  and controller  $\Pi_e$  admit forced steady-state solutions.

# STEADY-STATE INPUT-OUTPUT MAPS



## Assumption 1

Each agent  $\Sigma_i$  and controller  $\Pi_e$  admit forced steady-state solutions.

## Input-Output Maps

The *steady-state input-output map*  $k : \mathcal{U} \rightarrow \mathcal{Y}$  associated with  $\Sigma$  is the set consisting of all steady-state input-output pairs  $(u, y)$  of the system.

$$u_i \xrightarrow{y_i \in k_i(u_i)} y_i$$

$$\zeta_e \xrightarrow{\mu_e \in \gamma_e(\zeta_e)} \mu_e$$

$$u_i \rightarrow \Sigma_i \rightarrow y_i$$

$$\zeta_e \rightarrow \Pi_e \rightarrow \mu_e$$

$$u_i \leftarrow y_i \quad u_i \in k_i^{-1}(y_i)$$

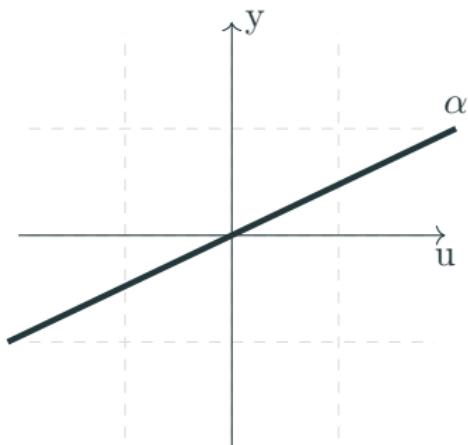
$$\zeta_e \leftarrow \mu_e \quad \zeta_e \in \gamma_e^{-1}(\mu_e)$$

# INPUT-OUTPUT RELATIONS

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\Rightarrow k(u) = \{y \mid \underbrace{(-CA^{-1}B + D)}_{\alpha} u\}$$



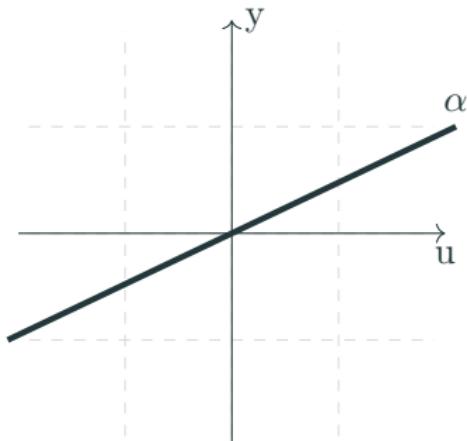
SISO and stable linear system

# INPUT-OUTPUT RELATIONS

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\Rightarrow k(u) = \{y \mid \underbrace{(-CA^{-1}B + D)}_{\alpha} u\}$$

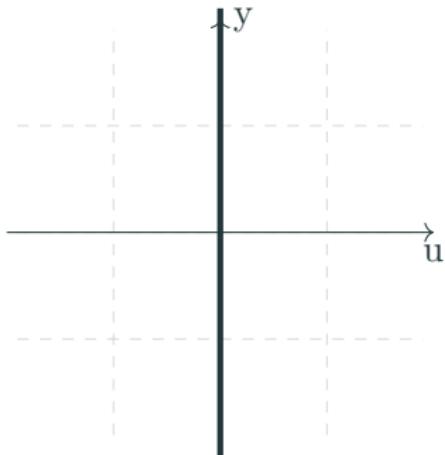


SISO and stable linear system

$$\dot{x} = u$$

$$y = x$$

$$\Rightarrow k = \{(0, y), y \in \mathbb{R}\}$$



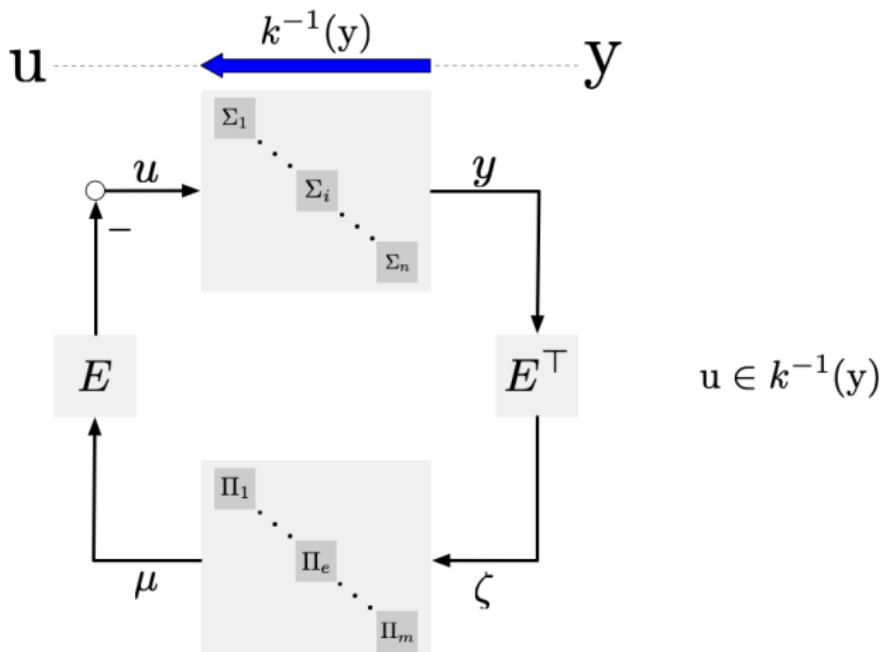
simple integrator

## NETWORK CONSISTENCY EQUATIONS

The network interconnection imposes constraints on allowable steady-states

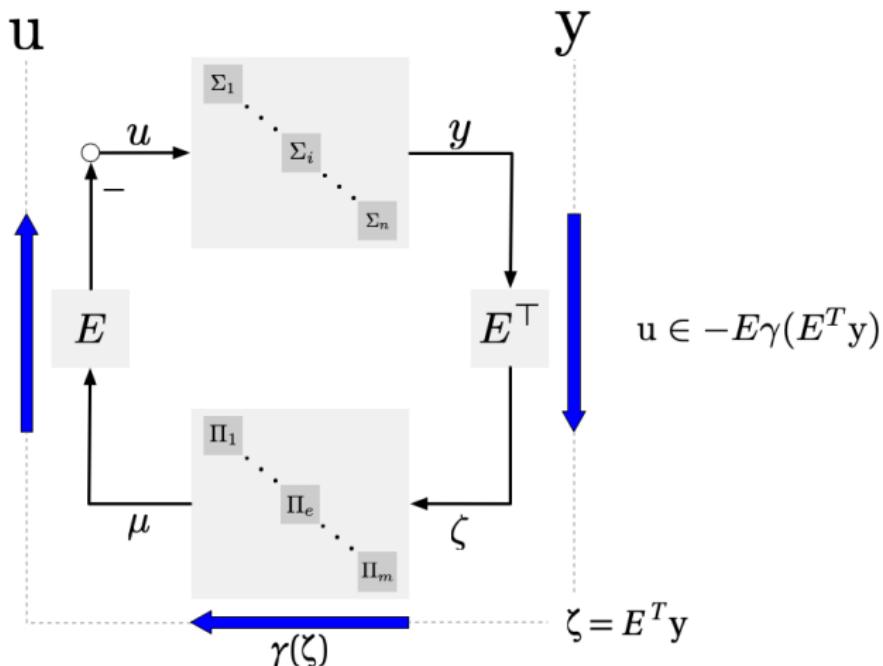
# NETWORK CONSISTENCY EQUATIONS

The network interconnection imposes constraints on allowable steady-states



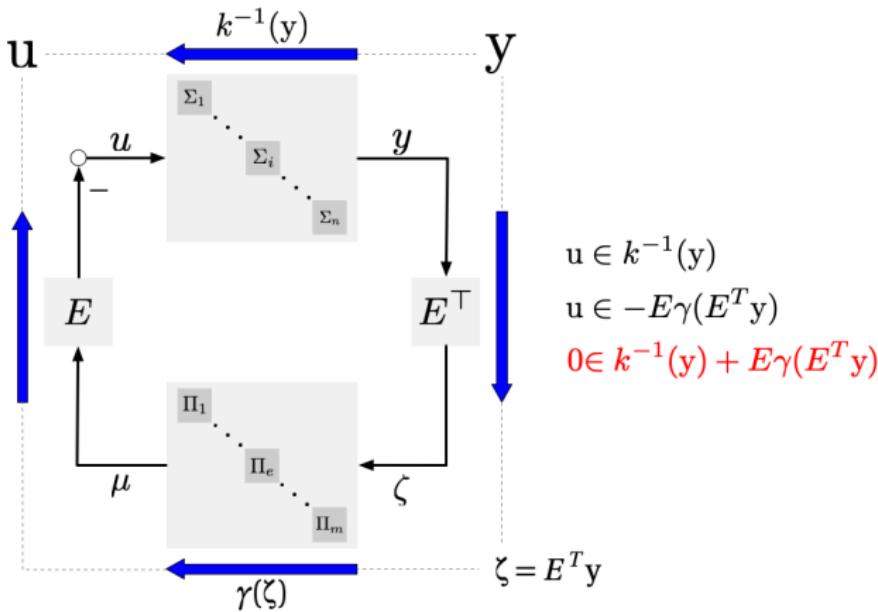
# NETWORK CONSISTENCY EQUATIONS

The network interconnection imposes constraints on allowable steady-states



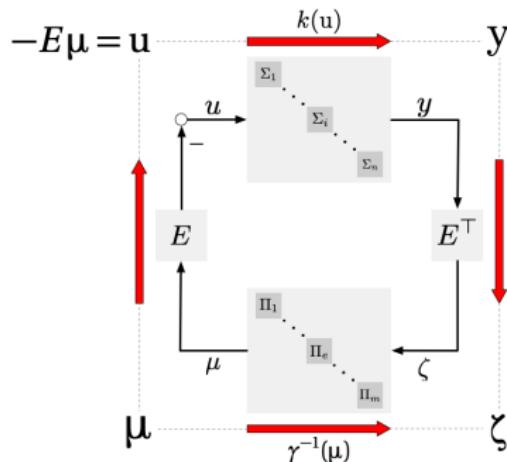
# NETWORK CONSISTENCY EQUATIONS

The network interconnection imposes constraints on allowable steady-states



# NETWORK CONSISTENCY EQUATIONS

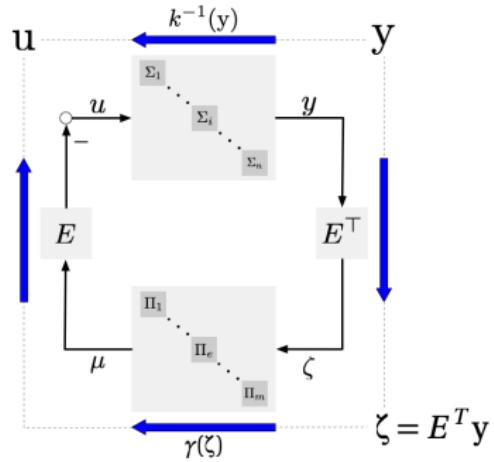
The network interconnection imposes constraints on allowable steady-states



$$\zeta \in \gamma^{-1}(\mu)$$

$$\zeta \in E^T k(-E\mu)$$

$$0 \in \gamma^{-1}(\mu) - E^T k(-E\mu)$$



$$u \in k^{-1}(y)$$

$$u \in -E\gamma(E^T y)$$

$$0 \in k^{-1}(y) + E\gamma(E^T y)$$

## SOLUTION OF NETWORK EQUATIONS

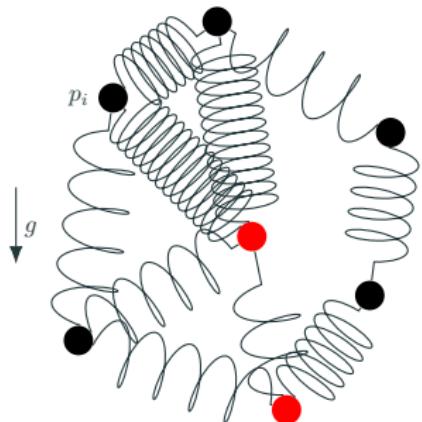
The network system  $(\Sigma, \Pi, \mathcal{G})$  admits a steady-state if and only if there exists a solution to the system of non-linear inclusions

$$0 \in k^{-1}(y) + E\gamma(E^T y)$$

$$0 \in \gamma^{-1}(\mu) - E^T k(-E\mu)$$

- ▶ When do solutions exist?
- ▶ How do we find them?

# A MASS-SPRING NETWORK



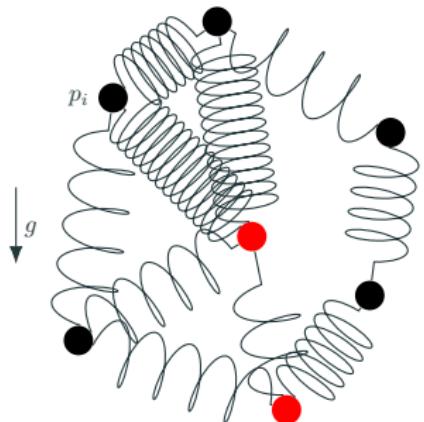
A Convex Program!

Minimum Total Potential Energy Problem

$$\min_{p_i, \zeta_e} \quad \sum_i^r (m_i g^T p_i + \mathbb{I}_{\mathbf{p}_i^*}(p_i)) + \sum_{i=r+1}^n m_i g^T p_i + \sum_e \frac{1}{2} k_{ij} (\|\zeta_e\| - r_e)^2$$

$$\text{s.t. } p_i - p_j = \zeta_e, \forall e = (i, j)$$

## A MASS-SPRING NETWORK

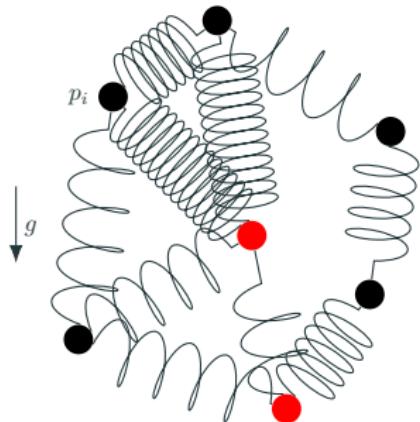


A Convex Program!

Minimum Total Potential Energy Problem

$$\begin{aligned} \min_{p_i, \zeta_e} \quad & \sum_i J_i(p_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t. } & E^T p = \zeta \end{aligned}$$

# A MASS-SPRING NETWORK



A Convex Program!

Minimum Total Potential Energy Problem

$$\min_p \quad J(p) + \Gamma(E^T p)$$

First-order Optimality Condition:

$$0 \in \partial J(p) + E \partial \Gamma(E^T p)$$

## SOLUTION OF NETWORK EQUATIONS

The network system  $(\Sigma, \Pi, \mathcal{G})$  admits a steady-state if and only if there exists a solution to the system of non-linear inclusions

$$\begin{aligned} 0 &\in k^{-1}(y) + E\gamma(E^T y) \\ 0 &\in \gamma^{-1}(\mu) - E^T k(-E\mu) \end{aligned}$$

**RECALL** First-order Optimality Condition:

$$0 \in \partial J(p) + E\partial\Gamma(E^T p)$$

Network equations are the first-order optimality conditions of a corresponding optimization problem!

## SOLUTION OF NETWORK EQUATIONS

The network system  $(\Sigma, \Pi, \mathcal{G})$  admits a steady-state if and only if there exists a solution to the system of non-linear inclusions

$$\begin{aligned} 0 &\in k^{-1}(y) + E\gamma(E^T y) \\ 0 &\in \gamma^{-1}(\mu) - E^T k(-E\mu) \end{aligned}$$

**RECALL** First-order Optimality Condition:

$$0 \in \partial J(p) + E\partial\Gamma(E^T p)$$

Network equations are the first-order optimality conditions of a corresponding optimization problem!

What is it?

# INTEGRAL FUNCTIONS

## Definition

Let  $k_i$  be the input-output relation for system  $\Sigma_i$ . Define the function  $K_i : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\partial K_i(u_i) = k_i(u_i)$  and  $K = \sum_i K_i$ . The function  $K$  is called the **cost function associated with the system  $\Sigma_i$** .

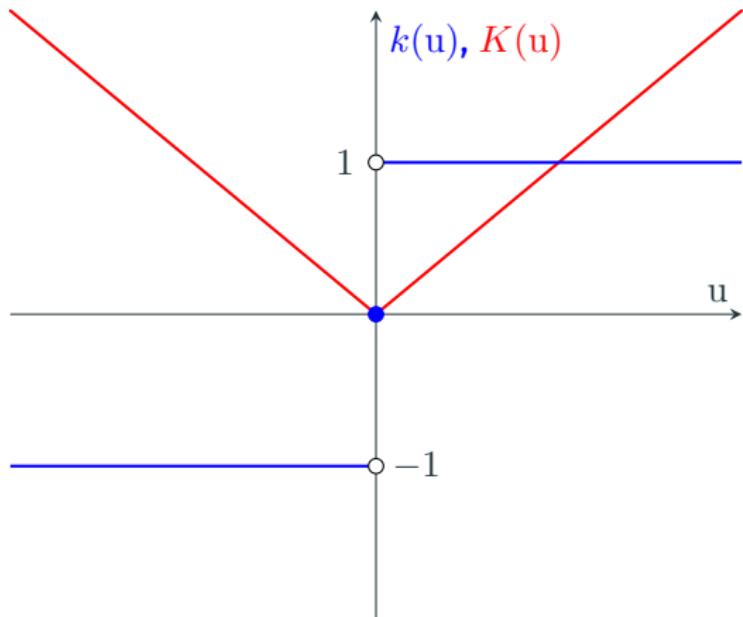
Similarly,

$$\partial K_i^*(y_i) = k_i^{-1}(y_i), \quad K^* = \sum_i K_i^*$$

$$\partial \Gamma_e(\zeta_e) = \gamma_e(\zeta_e), \quad \Gamma = \sum_e \Gamma_e$$

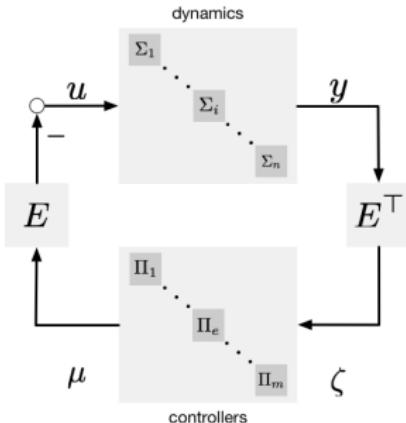
$$\partial \Gamma_e^*(\mu_e) = \gamma_e^{-1}(\mu_e) \quad \Gamma^* = \sum_e \Gamma_e^*$$

# INTEGRAL FUNCTIONS



$$\begin{array}{c} \text{— } K(u) = |u| \\ \text{— } y = k(u) = \operatorname{sgn}(u) \end{array}$$

# NETWORKS AND OPTIMIZATION



Steady-state values  $u, y, \zeta$  and  $\mu$  are the solutions of the following pair of optimization problems<sup>1</sup>:

$$\begin{array}{ll} \min_{y, \zeta} & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t.} & E^T y = \zeta. \end{array} \quad \parallel \quad \begin{array}{ll} \min_{u, \mu} & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ \text{s.t.} & u = -E\mu. \end{array}$$

First-order Optimality Condition

$$0 \in k^{-1}(y) + E\gamma(E^T y)$$

First-order Optimality Condition

$$0 \in \gamma^{-1}(\mu) - E^T k(-E\mu)$$

<sup>1</sup>[Bürger, Z, Allgower, 2014]

## MONOTONE MAPS AND CONVEXITY



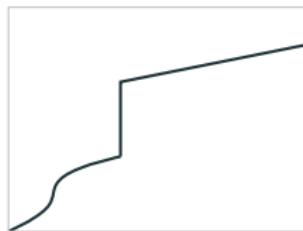
Not Monotone



Monotone but not maximal



Maximal monotone function



Maximal monotone relation

A relation on  $\mathbb{R}$  is **monotone**  
if they are non-decreasing curves in  $\mathbb{R}^2$

# MONOTONE MAPS AND CONVEXITY



Not Monotone



Monotone but not maximal



Maximal monotone function

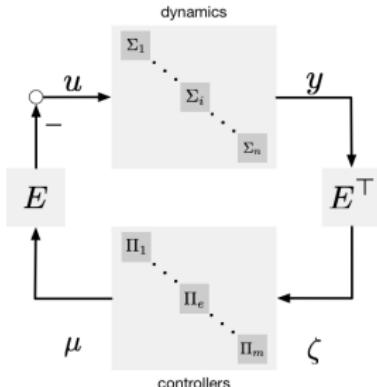


Maximal monotone relation

## Theorem

The subdifferentials of convex functions on  $\mathbb{R}$  are maximally monotone relations from  $\mathbb{R}$  to  $\mathbb{R}^1$ .

# NETWORKS AND OPTIMIZATION



## Theorem<sup>1</sup>

If the input-output maps  $k_i$  and  $\gamma_e$  are maximally monotone, then the steady-state values  $u, y, \zeta$  and  $\mu$  are the solutions of the following pair of convex dual optimization problems:

### Optimal Flow Problem (OFP)

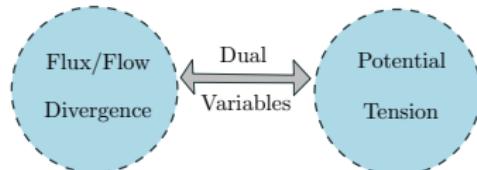
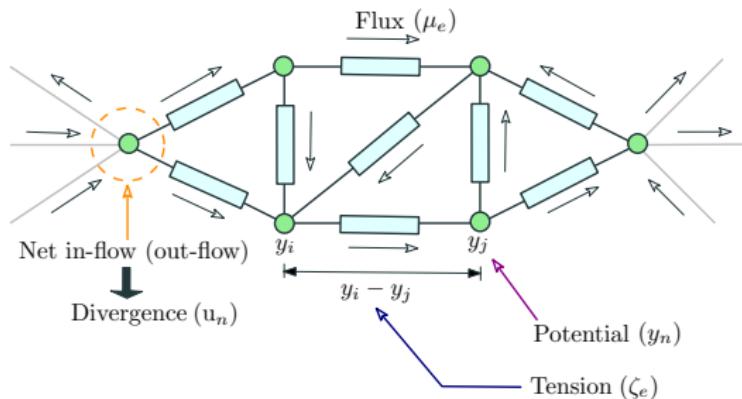
$$\begin{aligned} \min_{y, \zeta} \quad & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t.} \quad & E^T y = \zeta. \end{aligned}$$

### Optimal Potential Problem (OPP)

$$\begin{aligned} \min_{u, \mu} \quad & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ \text{s.t.} \quad & u = -E\mu. \end{aligned}$$

<sup>1</sup>[Bürger, Z, Allgower, 2014]

# NETWORK OPTIMIZATION



## Optimal Flow Problem<sup>1</sup>

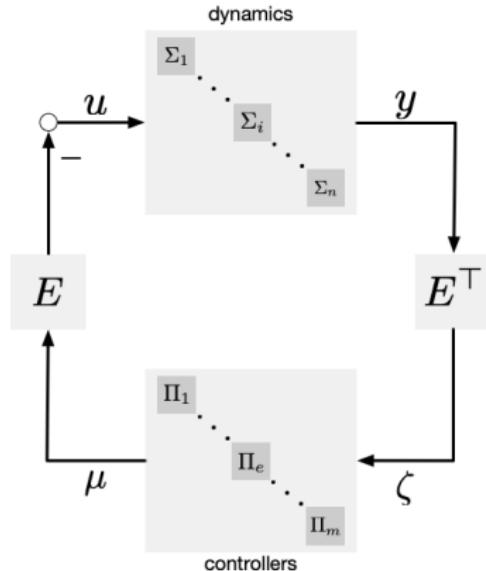
$$\begin{aligned} \min_{u, \mu} \quad & \sum_{n=1}^{|\mathcal{V}|} C_n^{\text{div}}(u_n) + \sum_{e=1}^{|\mathcal{E}|} C_e^{\text{flux}}(\mu_e) \\ s.t. \quad & u + E\mu = 0. \end{aligned}$$

## Optimal Potential Problem<sup>1</sup>

$$\begin{aligned} \min_{y, \zeta} \quad & \sum_{n=1}^{|\mathcal{V}|} C_n^{\text{pot}}(y_n) + \sum_{e=1}^{|\mathcal{E}|} C_e^{\text{ten}}(\zeta_e) \\ s.t. \quad & E^T y = \zeta. \end{aligned}$$

<sup>1</sup>[R. T. Rockafellar, Network Flows and Monotropic Optimizations. John Wiley and Sons, Inc., 1984]

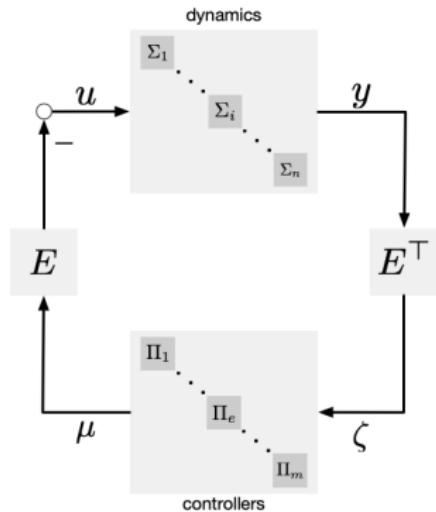
# STEADY-STATE NETWORK SOLUTIONS



Diffusively coupled dynamic networks can be associated to static network optimization problems!

Monotone steady-state maps  $\Leftrightarrow$  Network Duality

# MONOTONE DIFFUSIVE NETWORKS



## Assumption 1

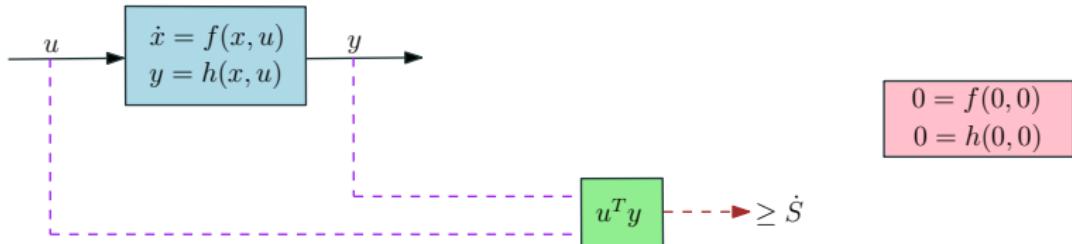
Each agent  $\Sigma_i$  and controller  $\Pi_e$  admit forced steady-state solutions.

## Assumption 2

The input-output maps of each agent,  $k_i$ , and controller,  $\gamma_e$ , are maximally monotone.

Under what conditions does the network actually *converge* to these steady states?

# PASSIVITY FOR DYNAMICAL SYSTEMS



## Definition [Khalil 2002]

A system is passive if there exists a  $C^1$  storage function  $S(x)$  such that

$$u^T y \geq \dot{S} = \frac{\partial S}{\partial x} f(x, u), \quad \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^p$$

Moreover, it is said to be

- ▶ Input-strictly passive if  $\dot{S} \leq u^T y - u^T \phi(u)$  and  $u^T \phi(u) > 0, \forall u \neq 0$
- ▶ Output-strictly passive if  $\dot{S} \leq u^T y - y^T \rho(y)$  and  $y^T \rho(y) > 0, \forall y \neq 0$

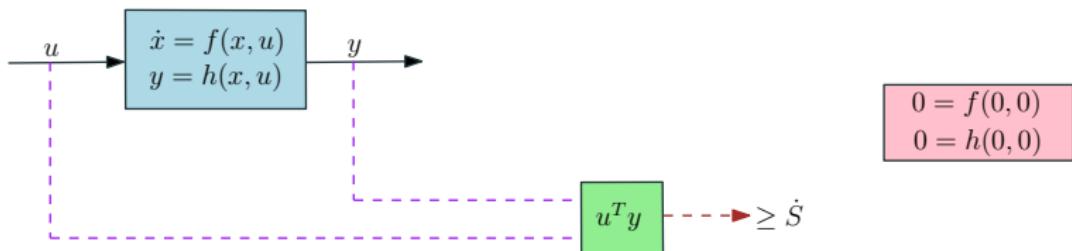
# PASSIVITY FOR DYNAMICAL SYSTEMS

## Definition

Let  $\Sigma$  be a SISO system with a constant input-output steady-state pair  $(u, y)$ . The system is said to be *input-output  $(\rho, \nu)$ -passive* wrt  $(u, y)$  if there exists a storage function  $S(x)$  and numbers  $\rho, \nu \in \mathbb{R}$ , such that  $\rho\nu < 1/4$  and

$$\dot{S} = \frac{\partial S}{\partial x} f(x, u) \leq (y - y)(u - u) - \rho(y - y)^2 - \nu(u - u)^2,$$

for any trajectory  $u, y$ .



## Definition

Let  $\Sigma$  be a SISO system with a constant input-output steady-state pair  $(u, y)$ . The system is said to be *input-output  $(\rho, \nu)$ -passive* wrt  $(u, y)$  if there exists a storage function  $S(x)$  and numbers  $\rho, \nu \in \mathbb{R}$ , such that  $\rho\nu < 1/4$  and

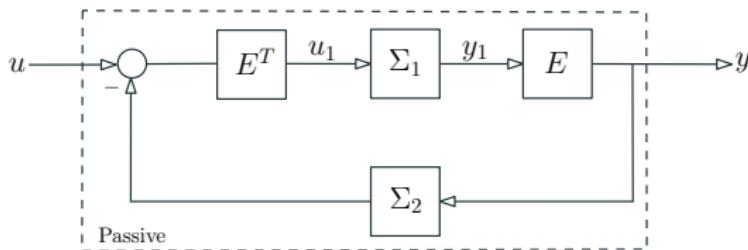
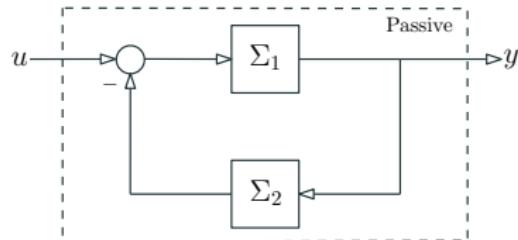
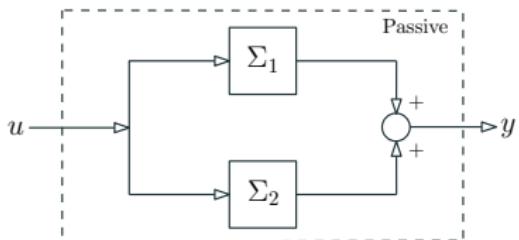
$$\dot{S} = \frac{\partial S}{\partial x} f(x, u) \leq (y - y)(u - u) - \rho(y - y)^2 - \nu(u - u)^2,$$

for any trajectory  $u, y$ .

- ▶  $\rho = \nu = 0 \Rightarrow \text{passivity}$
- ▶  $\rho, \nu > 0 \Rightarrow \text{strict input/output passivity}$
- ▶  $\rho, \nu < 0 \Rightarrow \text{passive short}$

# INTERCONNECTION OF PASSIVE SYSTEMS

- ▶ Parallel Interconnection
- ▶ Negative Feedback Interconnection
- ▶ Symmetric Interconnection



## A CONVERGENCE RESULT

### Theorem<sup>1</sup>

Consider the network system  $(\Sigma, \Pi, \mathcal{G})$  comprised of SISO agents and controllers. Suppose that there are vectors  $u_i, y_i, \zeta_e$  and  $\mu_e$  such that

- i) the systems  $\Sigma_i$  are output strictly-passive with respect to  $u_i$  and  $y_i$ ;
- ii) the systems  $\Pi_e$  are passive with respect to  $\zeta_e$  and  $\mu_e$ ;
- iii) the vectors  $u, y, \zeta$  and  $\mu$  satisfy  $u = -\mathcal{E}\mu$  and  $\zeta = \mathcal{E}^T y$ .

Then the output vector  $y(t)$  converges to  $y$  as  $t \rightarrow \infty$ .

---

<sup>1</sup>[Arcak, 2007], [Bürger, Z, Allgower, 2014]

## Theorem<sup>1</sup>

Consider the network system  $(\Sigma, \Pi, \mathcal{G})$  comprised of SISO agents and controllers. Suppose that there are vectors  $u_i, y_i, \zeta_e$  and  $\mu_e$  such that

- i) the systems  $\Sigma_i$  are output strictly-passive with respect to  $u_i$  and  $y_i$ ;
- ii) the systems  $\Pi_e$  are passive with respect to  $\zeta_e$  and  $\mu_e$ ;
- iii) the vectors  $u, y, \zeta$  and  $\mu$  satisfy  $u = -\mathcal{E}\mu$  and  $\zeta = \mathcal{E}^T y$ .

Then the output vector  $y(t)$  converges to  $y$  as  $t \rightarrow \infty$ .

- requires passivity w.r.t. to specific equilibrium configuration

---

<sup>1</sup>[Arcak, 2007], [Bürger, Z, Allgower, 2014]

# EQUILIBRIUM-INDEPENDENT PASSIVITY (EIP)

## EIP<sup>1</sup>

A SISO system  $\Sigma : u \mapsto y$  is said to be *equilibrium-independent input-output  $(\rho, \nu)$ -passive* if it is input-output  $(\rho, \nu)$ -passive with respect to **any** equilibrium  $(u, k(u))$ .

EIP systems  $(\rho, \nu \geq 0)$  have monotone steady-state input-output maps!

$$\dot{S} \leq (y - y)^T (u - u) \implies \text{**k** monotonically increasing function}$$

<sup>1</sup>[G.H. Hines et al., 2011], [M. Sharf, A. Jain, Z., 2020]

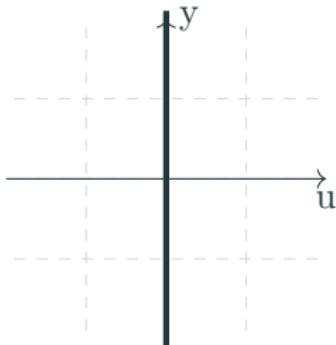
# EQUILIBRIUM-INDEPENDENT PASSIVITY (EIP)

## EIP<sup>1</sup>

A SISO system  $\Sigma : u \mapsto y$  is said to be *equilibrium-independent input-output*  $(\rho, \nu)$ -passive if it is input-output  $(\rho, \nu)$ -passive with respect to **any** equilibrium  $(u, k(u))$ .

EIP systems  $(\rho, \nu \geq 0)$  have monotone steady-state input-output maps!

$$\dot{S} \leq (y - y)^T (u - u) \implies \text{**k** monotonically increasing function}$$



$$\dot{x}(t) = u(t), y(t) = x(t)$$

- ▶ Passive with respect to  $\mathcal{U} = \{0\}$  and any output value  $y \in \mathbb{R}$  with storage function  $S(x) = \frac{1}{2}(x - y)^2$ .
- ▶ The equilibrium input-output map  $k = \{(0, y) : y \in \mathbb{R}\}$  is not a single valued function and hence the integrator is **NOT EIP**.

<sup>1</sup>[G.H. Hines et al., 2011], [M. Sharf, A. Jain, Z., 2020]

## MAXIMALLY EQUILIBRIUM-INDEPENDENT PASSIVITY (MEIP)

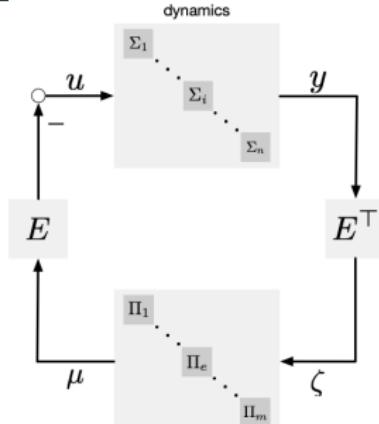
### MEIP<sup>1</sup>

A dynamical SISO system  $\Sigma$  is *maximal equilibrium independent passive* if the following conditions hold:

- ▶ The system  $\Sigma$  is passive with respect to any steady-state  $(u, y) \in k$ .
- ▶ The relation  $k$  is maximally monotone.

---

<sup>1</sup>[M. Bürger et al., 2014]



## Assumption 1

Each agent  $\Sigma_i$  and controller  $\Pi_e$  admit forced steady-state solutions.

## Assumption 2

The agent dynamics  $\Sigma_i$  are output-strictly MEIP and the controllers are MEIP.

## Theorem<sup>1</sup>

Assume Assumptions 1 and 2 hold. Then the signals  $u(t), y(t), \zeta(t), \mu(t)$  converge to the solutions of the following pair of convex dual optimization problems:

### Optimal Flow Problem (OFP)

$$\begin{aligned} \min_{y, \zeta} \quad & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ \text{s.t.} \quad & E^T y = \zeta. \end{aligned}$$

### Optimal Potential Problem (OPP)

$$\begin{aligned} \min_{u, \mu} \quad & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ \text{s.t.} \quad & u = -E\mu. \end{aligned}$$

<sup>1</sup>[Bürger, Z, Allgower, 2014]

## **NEW PERSPECTIVES ON PASSIVATION**

---

# MONOTONICITY AND ITS ROLE IN SYSTEMS THEORY

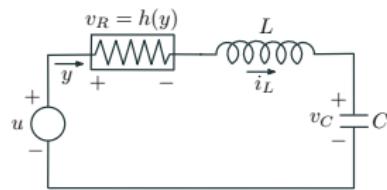
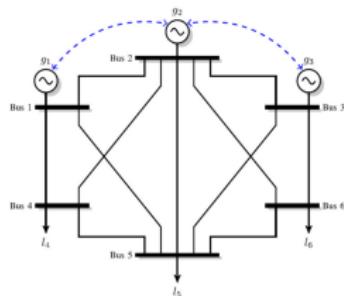
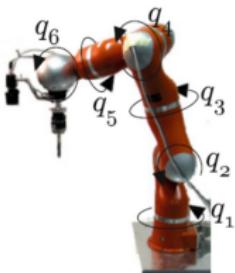


What else can we say about MEIP systems?

# PASSIVITY-SHORT SYSTEMS

In practice, systems are usually passivity-short (or non-passive)!

- ▶ Generator (always generates energy) [R. Harvey , 2016]
- ▶ Oscillating systems with small or nonexistent damping [R. Harvey, 2017]
- ▶ Dynamics of robot system from torque to position [D. Babu, 2018]
- ▶ Power-system network (turbine-governor dynamics) [S. Trip, 2018]
- ▶ Electrical circuits with nonlinear components
- ▶ More general as include non-minimum phase systems and systems with relative degree greater than 1 [Z. Qu, 2014]



$$h(\cdot) \in [\sigma, \infty] \text{ with } \sigma < 0$$

# PASSIVITY SHORT SYSTEMS AND THE NETWORK FRAMEWORK

Passive short systems can destroy  
the developed network optimization framework!

System Type	Relations	Integral Function
MEIP	$k, k^{-1}$ max. monotone	$K(u), K^*(y)$ are convex
Input PS	$k$ is not monotone	$K(u)$ is non-convex
Output PS	$k^{-1}$ is not monotone	$K^*(y)$ is non-convex
Input-output PS	$k, k^{-1}$ are not monotone	May not exist

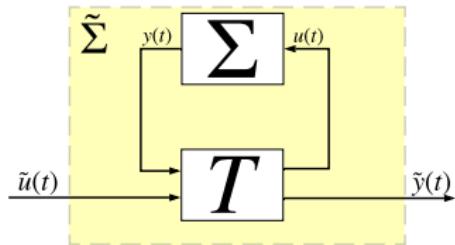
## Optimal Flow Problem (OFP)

$$\begin{aligned} \min_{y, \zeta} \quad & \sum_i K_i^*(y_i) + \sum_e \Gamma_e(\zeta_e) \\ s.t. \quad & E^T y = \zeta. \end{aligned}$$

## Optimal Potential Problem (OPP)

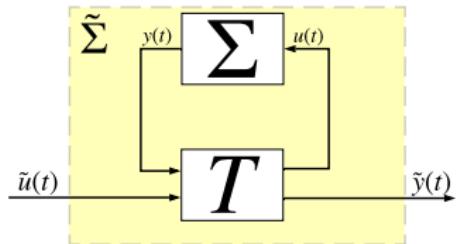
$$\begin{aligned} \min_{u, \mu} \quad & \sum_i K_i(u_i) + \sum_e \Gamma_e^*(\mu_e) \\ s.t. \quad & u = -E\mu. \end{aligned}$$

## FEEDBACK PASSIVATION

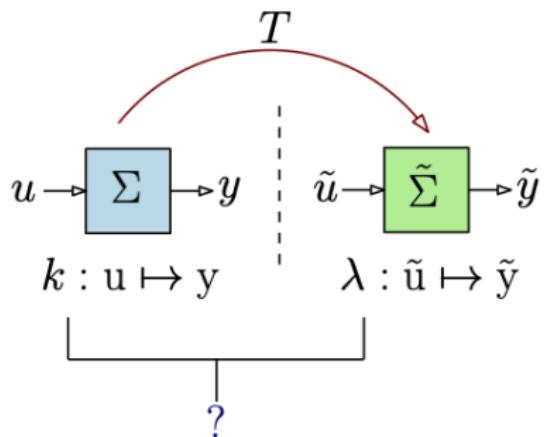


For a passive-short system  $\Sigma : u \mapsto y$ , we aim to find a map  $T$  such that the closed-loop system  $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$  is passive. This is known as **feedback passivation**.

## FEEDBACK PASSIVATION



For a passive-short system  $\Sigma : u \mapsto y$ , we aim to find a map  $T$  such that the closed-loop system  $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$  is passive. This is known as **feedback passivation**.



how does feedback passivation affect the steady-state input/output maps?

## AN EXAMPLE

an example

$$\dot{x} = -x + \sqrt[3]{x} + u$$

$$y = \sqrt[3]{x}$$

$$\bar{u} = k^{-1}(\bar{y}) = \bar{y}^3 - \bar{y}$$

not a monotone input-output relation!

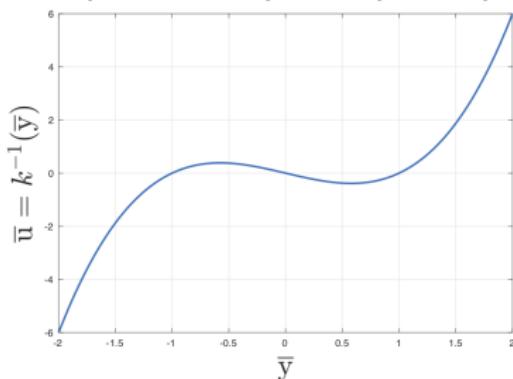
System is output passivity-short

$$S(x) = \frac{3}{4}x^{4/3} - \bar{y}x + \frac{1}{4}\bar{y}$$

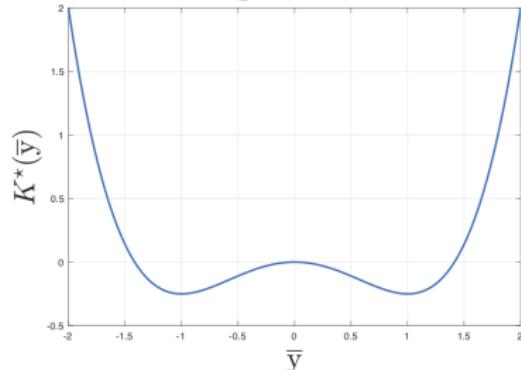
$$\dot{S} \leq (y - \bar{y})(u - \bar{u}) + (y - \bar{y})^2$$

(passivity index  $\rho = -1$ )

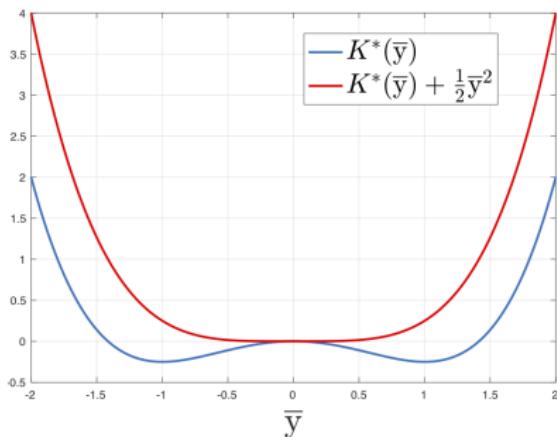
equilibrium input-output map



integral function



## AN EXAMPLE



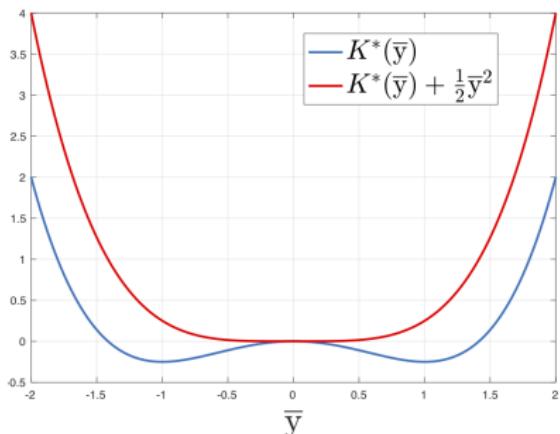
what is the system interpretation of  
a "convexified" integral function?

$$K^*(\bar{y}) = \frac{1}{4}\bar{y}^4 - \frac{1}{2}\bar{y}^2$$

$$\tilde{K}^*(\bar{y}) = K^*(\bar{y}) + \frac{1}{2}\bar{y}^2$$

*(Tikhonov regularization term)*

## AN EXAMPLE



what is the system interpretation of a “convexified” integral function?

$$K^*(\bar{y}) = \frac{1}{4}\bar{y}^4 - \frac{1}{2}\bar{y}^2$$

$$\tilde{K}^*(\bar{y}) = K^*(\bar{y}) + \frac{1}{2}\bar{y}^2$$

(Tikhonov regularization term)

what system yields this steady-state I/O map?

$$\partial \tilde{K}^*(\bar{y}) = \partial K^*(\bar{y}) + \bar{y}$$

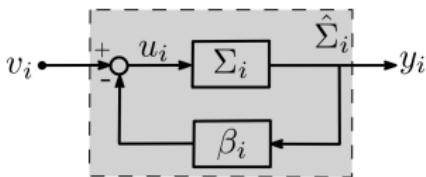
$$\begin{aligned}\tilde{k}^{-1}(\bar{y}) &= k^{-1}(\bar{y}) + \bar{y} \\ &= \bar{y}^3 - \bar{y} + \bar{y} = \bar{y}^3\end{aligned}$$

a monotone function!

$$\dot{x} = -x + \sqrt[3]{x} - \underbrace{\sqrt[3]{x}}_u + v = -x + v$$

$$y = \sqrt[3]{x}$$

## AN EXAMPLE



regularization is realized by output feedback!

$$u = v - y$$

$$\Rightarrow \dot{x} = -x + v$$

$$\Rightarrow \bar{v} = \tilde{k}^{-1}(\bar{y}) = \bar{y}^3$$

(maximally monotone!)

### Theorem<sup>1</sup>

Consider the passive-short SISO dynamical system  $\Sigma : u \mapsto y$  with I/O steady-state map  $k$  and output passivity index  $\rho < 0$ . Then for any  $\beta > |\rho|$ , the feedback

$$u = v - \beta y$$

renders the system  $\tilde{\Sigma} : v \mapsto y$  output-strictly maximally monotone EIP with steady-state input map  $\tilde{k}$  satisfying

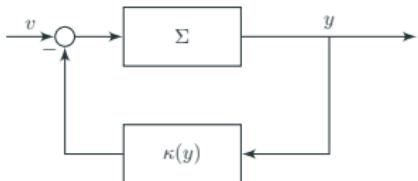
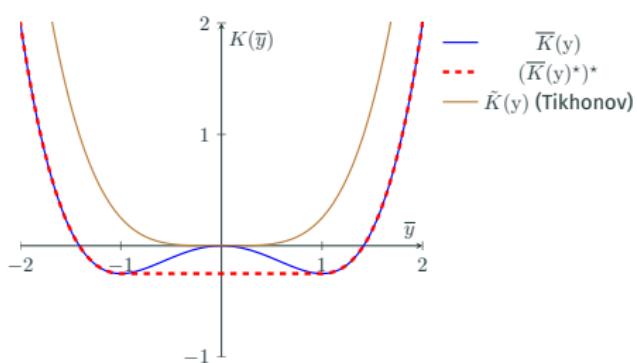
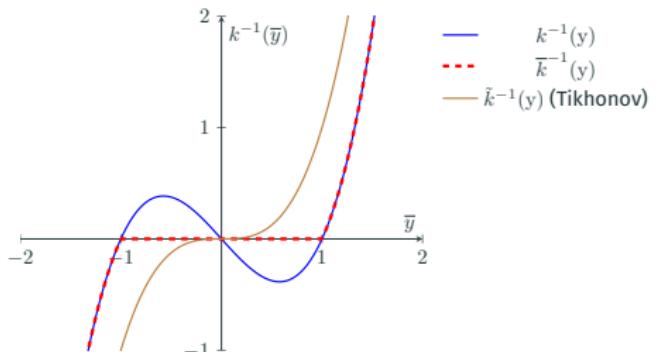
$$\tilde{k}^{-1}(\bar{y}) = k^{-1}(\bar{y}) + \beta \bar{y}.$$

---

<sup>1</sup>[Jain, Sharf, Z, 2018]

# MONOTONIZATION AND CONVEXIFICATION

A “better” convexification leads to different feedback passivation!



the feedback

$$\kappa(y) = \begin{cases} 0, & |x| = |y^3| > 1 \\ y^3 - y, & |x| = |y^3| \leq 1 \end{cases}$$

the closed-loop

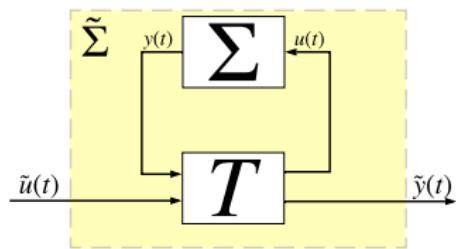
$$\begin{aligned} \dot{x} &= \begin{cases} -x + \sqrt[3]{x} + v, & |x| > 1 \\ v, & |x| \leq 1 \end{cases} \\ y &= \sqrt[3]{x}. \end{aligned}$$

## MONOTONIZATION OF I/O RELATIONS

Is it possible to find a linear transformation  $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$  for a non-monotone I/O map  $k : u \mapsto y$  such that  $\tilde{k} : \tilde{u} \mapsto \tilde{y}$  is monotone?

# MONOTONIZATION OF I/O RELATIONS

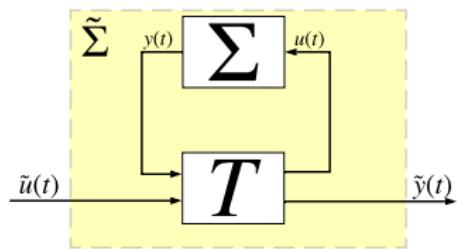
Is it possible to find a linear transformation  $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$  for a non-monotone I/O map  $k : u \mapsto y$  such that  $\tilde{k} : \tilde{u} \mapsto \tilde{y}$  is monotone?



For a passive-short system  $\Sigma : u \mapsto y$ , we aim to find a map  $T$  such that the closed-loop system  $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$  is passive. This is known as **feedback passivation**.

# MONOTONIZATION OF I/O RELATIONS

Is it possible to find a linear transformation  $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$  for a non-monotone I/O map  $k : u \mapsto y$  such that  $\tilde{k} : \tilde{u} \mapsto \tilde{y}$  is monotone?



For a passive-short system  $\Sigma : u \mapsto y$ , we aim to find a map  $T$  such that the closed-loop system  $\tilde{\Sigma} : \tilde{u} \mapsto \tilde{y}$  is passive. This is known as **feedback passivation**.

Are these  $T$  maps the same?

## A GEOMETRIC APPROACH

For an EI-IOP( $\rho, \nu$ ) system, for any two points  $(u_1, y_1), (u_2, y_2) \in k$ , the following inequality holds:

$$0 \leq -\rho(y_1 - y_2)^2 + (u_1 - u_2)(y_1 - y_2) - \nu(u_1 - u_2)^2.$$

### Projective Quadratic Inequalities and EI-IOP

A *projective quadratic inequality (PQI)* is an inequality with variables  $\xi, \chi \in \mathbb{R}$  of the form

$$0 \leq a\xi^2 + b\xi\chi + c\chi^2 = F(\xi, \chi),$$

for some numbers  $a, b, c$ , not all zero. The inequality is called *non-trivial* if  $b^2 - 4ac > 0$ . The associated solution set  $\mathcal{A}$  of the PQI is the set of all points  $(\xi, \chi) \in \mathbb{R}^2$  satisfying the inequality.

- ▶ passivity inequality is a PQI:  $\xi = u_1 - u_2, \chi = y_1 - y_2$
- ▶ monotonicity is a PQI:  $0 \leq (u_1 - u_2)(y_1 - y_2)$  with  $a = c = 0$  and  $b = 1$

## A GEOMETRIC APPROACH

$$0 \leq a\xi^2 + b\xi\chi + c\chi^2 = F(\xi, \chi)$$

A Recap:

- $F(u_1 - u_2, y_1 - y_2) \geq 0$  is a PQI for a EI-IOP( $\rho, \nu$ ) system

## A GEOMETRIC APPROACH

$$0 \leq a\xi^2 + b\xi\chi + c\chi^2 = F(\xi, \chi)$$

A Recap:

- ▶  $F(u_1 - u_2, y_1 - y_2) \geq 0$  is a PQI for a EI-IOP( $\rho, \nu$ ) system
- ▶ For the linear map  $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$ ,

$$F(\tilde{u}_1 - \tilde{u}_2, \tilde{y}_1 - \tilde{y}_2) \geq 0$$

is also a PQI for a EI-IOP( $\tilde{\rho}, \tilde{\nu}$ ) system

## A GEOMETRIC APPROACH

$$0 \leq a\xi^2 + b\xi\chi + c\chi^2 = F(\xi, \chi)$$

A Recap:

- ▶  $F(u_1 - u_2, y_1 - y_2) \geq 0$  is a PQI for a EI-IOP( $\rho, \nu$ ) system
- ▶ For the linear map  $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$ ,

$$F(\tilde{u}_1 - \tilde{u}_2, \tilde{y}_1 - \tilde{y}_2) \geq 0$$

is also a PQI for a EI-IOP( $\tilde{\rho}, \tilde{\nu}$ ) system

- ▶  $F(\tilde{u}_1 - \tilde{u}_2, \tilde{y}_1 - \tilde{y}_2) = (\tilde{u}_1 - \tilde{u}_2)(\tilde{y}_1 - \tilde{y}_2)$  corresponds to monotonicity

## A GEOMETRIC APPROACH

$$0 \leq a\xi^2 + b\xi\chi + c\chi^2 = F(\xi, \chi)$$

A Recap:

- ▶  $F(u_1 - u_2, y_1 - y_2) \geq 0$  is a PQI for a EI-IOP( $\rho, \nu$ ) system
- ▶ For the linear map  $T : (u, y) \mapsto (\tilde{u}, \tilde{y})$ ,

$$F(\tilde{u}_1 - \tilde{u}_2, \tilde{y}_1 - \tilde{y}_2) \geq 0$$

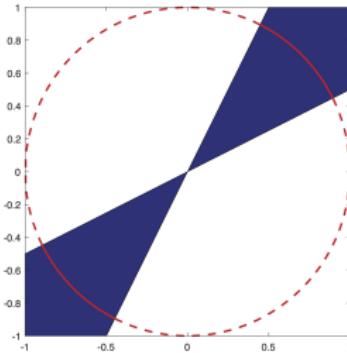
is also a PQI for a EI-IOP( $\tilde{\rho}, \tilde{\nu}$ ) system

- ▶  $F(\tilde{u}_1 - \tilde{u}_2, \tilde{y}_1 - \tilde{y}_2) = (\tilde{u}_1 - \tilde{u}_2)(\tilde{y}_1 - \tilde{y}_2)$  corresponds to monotonicity

Study the effect of the map  $T$  on the solution sets of the PQIs,  $T(\mathcal{A})$

# A GEOMETRIC APPROACH

The solution set of any non-trivial PQI is a symmetric double-cone. Moreover, any symmetric double-cone is the solution set of some non-trivial PQI.



## Theorem<sup>1</sup>

Let  $(\xi_1, \chi_1), (\xi_2, \chi_2)$  be non-colinear solutions of  $a_1\xi^2 + \xi\chi + c_1\chi^2 = 0$ , and  $(\tilde{\xi}_1, \tilde{\chi}_1), (\tilde{\xi}_2, \tilde{\chi}_2)$  be non-colinear solutions of  $a_2\xi^2 + \xi\chi + c_2\chi^2 = 0$ .

Define

$$T_1 = \begin{bmatrix} \tilde{\xi}_1 & \tilde{\xi}_2 \\ \tilde{\chi}_1 & \tilde{\chi}_2 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1}, T_2 = \begin{bmatrix} \tilde{\xi}_1 & -\tilde{\xi}_2 \\ \tilde{\chi}_1 & -\tilde{\chi}_2 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1}.$$

Then one of  $T_1, T_2$  transforms the PQI  $a_1\xi^2 + \xi\chi + c_1\chi^2 \geq 0$  to the PQI  $\tau a_2\xi^2 + \tau\xi\chi + \tau c_2\chi^2 \geq 0$  for some  $\tau > 0$ .

<sup>1</sup>[Sharf, Jain, Z, 2021]

## EXAMPLE

Consider the system

$$\Sigma : \dot{x} = -\sqrt[3]{x} + .5x + .5u, y = .5x - .5u$$

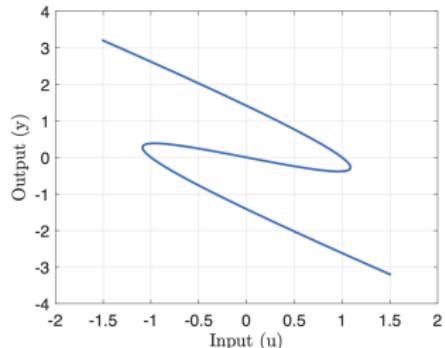
Using  $S(x) = \frac{1}{6}(x - x)^2$  we have

$$\dot{S}(x) \leq (u - u)(y - y) + \frac{1}{3}(u - u)^2 + \frac{2}{3}(y - y)^2$$

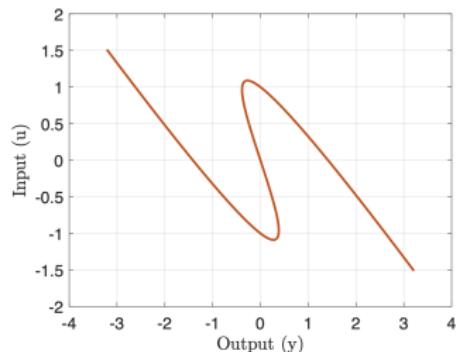
System is EI-IOP( $\rho, \nu$ ) with

$$\rho = -2/3, \nu = -1/3$$

Passive-short system with  
non-monotone input-output  
relations (not even a function!)



(a)  $k.$



(b)  $k^{-1}$

## EXAMPLE

Consider the system

$$\Sigma : \dot{x} = -\sqrt[3]{x} + .5x + .5u, y = .5x - .5u$$

Using  $S(x) = \frac{1}{6}(x - \bar{x})^2$  we have

$$\dot{S}(x) \leq (u - \bar{u})(y - \bar{y}) + \frac{1}{3}(u - \bar{u})^2 + \frac{2}{3}(y - \bar{y})^2$$

System is EI-IOP( $\rho, \nu$ ) with  $\rho = -2/3, \nu = -1/3$

Corresponding PQI:

$$0 \leq \frac{1}{3}\xi^2 + \xi\chi + \frac{2}{3}\chi^2$$

Find a linear map  $T$  that monotonizes the input-output relations, i.e., leads to the PQI

$$\tilde{\xi}\tilde{\chi} \geq 0$$

## EXAMPLE

non-colinear solutions to  
PQI

$$\tilde{\xi}\tilde{\chi} = 0$$

$$\begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\chi}_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \tilde{\xi}_2 \\ \tilde{\chi}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

non-colinear solutions to original  
PQI

$$0 = \frac{1}{3}\xi^2 + \xi\chi + \frac{2}{3}\chi^2$$

$$\begin{bmatrix} \xi_1 \\ \chi_1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} \xi_2 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The map

$$T_1 = \begin{bmatrix} \tilde{\xi}_1 & \tilde{\xi}_2 \\ \tilde{\chi}_1 & \tilde{\chi}_2 \end{bmatrix} \begin{bmatrix} \xi_1 & \xi_2 \\ \chi_1 & \chi_2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

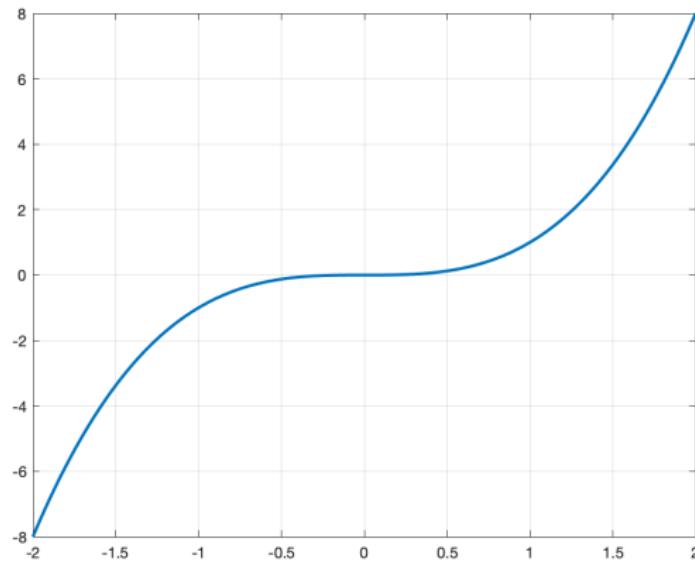
can be used to monotonize the relation! Indeed, for  $(\xi, \chi) = T_1^{-1}(\tilde{\xi}, \tilde{\chi})$

$$\begin{aligned} 0 &\leq \frac{1}{3}\xi^2 + \xi\chi + \frac{2}{3}\chi^2 \\ &= \frac{1}{3}(2\tilde{\xi} - \tilde{\chi})^2 + (2\tilde{\xi} - \tilde{\chi})(-\tilde{\xi} + \tilde{\chi}) + \frac{2}{3}(-\tilde{\xi} + \tilde{\chi})^2 = \frac{1}{3}\tilde{\xi}\tilde{\chi}, \end{aligned}$$

## EXAMPLE

Steady-state input-output maps under  $T_1$ ,

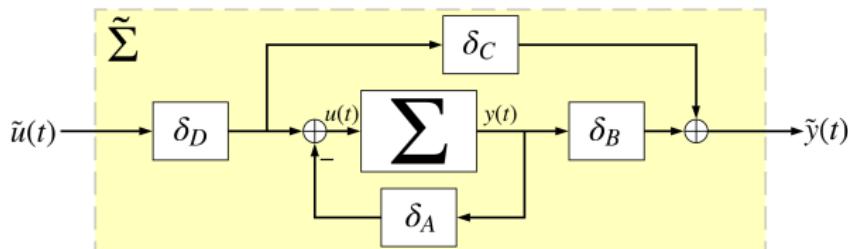
$$\begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix} = T_1 \begin{bmatrix} u \\ y \end{bmatrix}$$



# MONOTONIZATION TO PASSIVATION

## Theorem<sup>1</sup>

Let  $\Sigma$  be EI-IOP( $\rho, \nu$ ). If the map  $T$  monotonizes the input-output relation  $k$ , then it passivizes the system  $\Sigma$ .



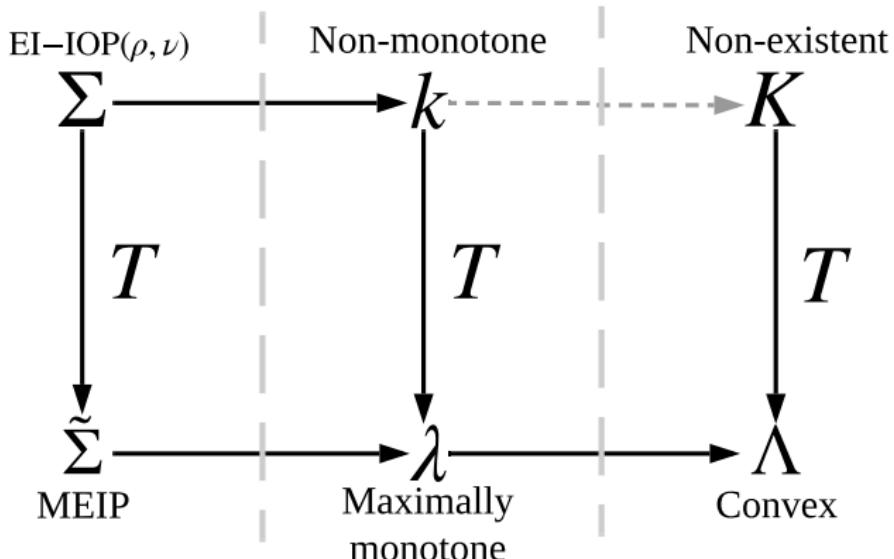
$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underbrace{\begin{bmatrix} \delta_D & 0 \\ 0 & 1 \end{bmatrix}}_{L_D} \underbrace{\begin{bmatrix} 1 & 0 \\ \delta_C & 1 \end{bmatrix}}_{L_C} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \delta_B \end{bmatrix}}_{L_B} \underbrace{\begin{bmatrix} 1 & \delta_A \\ 0 & 1 \end{bmatrix}}_{L_A},$$

<sup>1</sup>[Sharf, Jain, Z, 2020]

# MONOTIZATION AND PASSIVATION

Elementary Transformation	Relation between I/O of $\Sigma$ and $\tilde{\Sigma}$	Effect on Steady-State Relations	Realization	Effect on Integral Functions
$L_A = \begin{bmatrix} 1 & \delta_A \\ 0 & 1 \end{bmatrix}$	$\tilde{u} = u + \delta_A y$ $\tilde{y} = y$	$\lambda_A^{-1}(\tilde{y}) = k^{-1}(\tilde{y}) + \delta_A \tilde{y}$	output-feedback	$\Lambda^*(y) = K^*(y) + \frac{1}{2}\delta_A y^2$
$L_B = \begin{bmatrix} 1 & 0 \\ 0 & \delta_B \end{bmatrix}$	$\tilde{u} = u$ $\tilde{y} = \delta_B y$	$\lambda_B(u) = \delta_B k(u)$ or $\lambda_B^{-1}(\tilde{y}) = k^{-1}(\frac{1}{\delta_B} \tilde{y})$	post-gain	$\Lambda^*(y) = \frac{1}{\delta_B} K^*(\frac{1}{\delta_B} y)$ or $\Lambda(u) = \delta_B K(u)$
$L_C = \begin{bmatrix} 1 & 0 \\ \delta_C & 1 \end{bmatrix}$	$\tilde{u} = u$ $\tilde{y} = y + \delta_C u$	$\lambda_C(\tilde{u}) = k(\tilde{u}) + \delta_C \tilde{u}$	input-feedthrough	$\Lambda(u) = K(u) + \frac{1}{2}\delta_C u^2$
$L_D = \begin{bmatrix} \delta_D & 0 \\ 0 & 1 \end{bmatrix}$	$\tilde{u} = \delta_D u$ $\tilde{y} = y$	$\lambda_D^{-1}(y) = \delta_D k^{-1}(y)$ or $\lambda_D(\tilde{u}) = k(\frac{1}{\delta_D} \tilde{u})$	pre-gain	$\Lambda^*(y) = \delta_D K^*(y)$ or $\Lambda(u) = \frac{1}{\delta_D} K(\frac{1}{\delta_D} u)$

# PASSIVATION, MONOTONIZATION AND CONVEXIFICATION

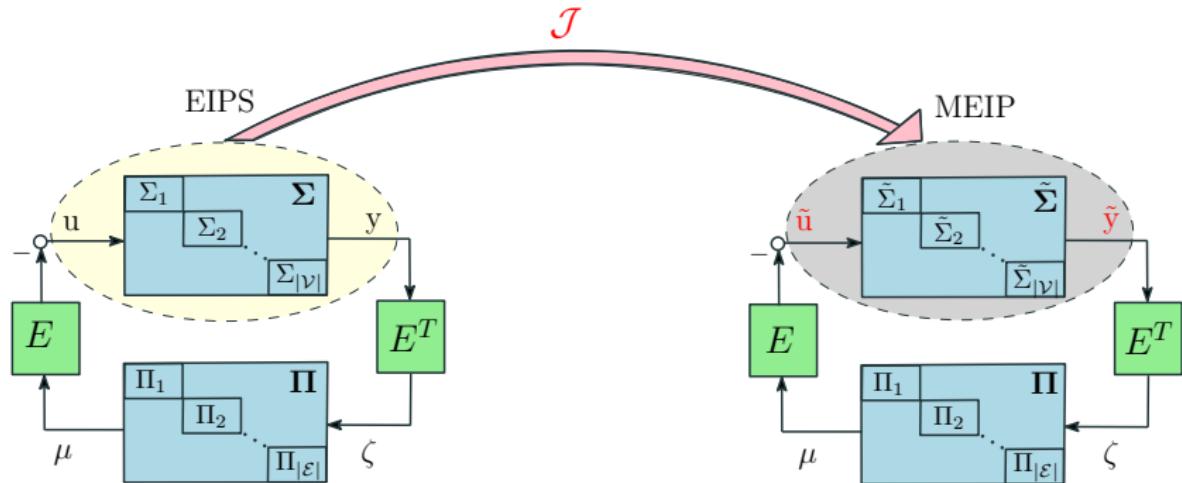


Passivation  
(system)

Monotonization  
(I/O Maps)

Convexification  
(integral functions)

# PASSIVATION OF DIFFUSIVELY-COUPLED NETWORKS OF EIPS SYSTEMS



- Without loss of generality assume that the systems at nodes are EIPS (applicable if some of the systems are EIPS)
- Loop Transformation results in a pair of **regularized** network optimization problems

$$\mathcal{J} = \text{diag}(T_i)$$

## **CONCLUDING REMARKS**

---

## A MONOTONE VIEW



New perspectives on networks and passivity

- ▶ networks of EIP agents can be understood through solutions of a pair of static dual optimization problems
- ▶ passivity and monotonicity of input-output maps are essential
- ▶ passivation means monotonization - monotonization means convexification

# ACKNOWLEDGEMENTS



German-Israeli  
Foundation for Scientific  
Research and Development



Dr. Anoop Jain  
IIT-Jodhpur



Dr. Miel Sharf



Dr. Mathias Bürger



Prof. Dr.-Ing.  
Frank Allgöwer

## QUESTIONS?

## SELECTED PUBLICATIONS

- ▶ M. Bürger, D. Zelazo and F. Allgower, "Duality and network theory in passivity-based cooperative control," *Automatica*, 50(8): 2051-2061, 2014.
- ▶ M. Sharf and D. Zelazo, "Analysis and Synthesis of MIMO Multi-Agent Systems Using Network Optimization," *IEEE Transactions on Automatic Control*, 64(11):1558-2523, 2019.
- ▶ M. Sharf and D. Zelazo, "A Network Optimization Approach to Cooperative Control Synthesis," *IEEE Control Systems Letters*, 1(1):86-91, 2017.
- ▶ A. Jain, M. Sharf and D. Zelazo, "Regularization and Feedback Passivation in Cooperative Control of Passivity-Short Systems : A Network Optimization Perspective", *IEEE Control Systems Letters*, (2):4:731-736, 2018.
- ▶ M. Sharf, A. Jain and D. Zelazo, "A Geometric Method for Passivation and Cooperative Control of Equilibrium-Independent Passivity-Short Systems", *IEEE Transactions on Automatic Control*, 66(12):5877-5892, 2021.
- ▶ M. Sharf and D. Zelazo, "Network Identification for Diffusively-Coupled Networks with Minimal Time Complexityt, *IEEE Transactions on Control of Network Systems*, 2023 (early access).
- ▶ M. Sharf and D. Zelazo, "A Characterization of All Passivizing Input-Output Transformations of a Passive-Short System," arXiv preprint, 2020.

**WE ARE HIRING!**



**DEPARTMENT OF  
AEROSPACE ENGINEERING**

**TECHNION**

Israel Institute  
of Technology

[https://aerospace.technion.ac.il/  
department/faculty-positions/](https://aerospace.technion.ac.il/department/faculty-positions/)



- ▶ GNC
- ▶ Robotics
- ▶ Networked and Multi-agent Systems

