Product Form of Projection-Based Model Reduction and its Application to Multi-Agent Systems

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Abstract—Orthogonal projection-based reduced order models (PROM) are the output of widely-used model reduction methods. In this work, a novel product form is derived for the reduction error system of these reduced models, and it is shown that any such PROM can be obtained from a sequence of 1-dimensional projection reductions. Investigating the error system product form, we then define interface-invariant PROMs, model order reductions with projection-invariant input and output matrices, and it is shown that for such PROMs the error product systems are strictly proper. Furthermore, exploiting this structure, an analytic $\mathcal{H}_\infty$ reduction error bound is obtained and an $\mathcal{H}_\infty$ bound optimization problem is defined. Interface-invariant reduced models are natural to graph-based model reduction of multi-agent systems where subsets of agents function as the input and output of the system. In the second part of this study, graph contractions are used as a constructive solution approach to the $\mathcal{H}_\infty$ bound optimization problem for multi-agent systems. Edge-based contractions are then utilized in a greedy-edge reduction algorithm and are demonstrated for the model reduction of a first-order Laplacian controlled consensus protocol.

I. Introduction

Model-order reduction is an essential tool for the design and study of large-scale systems introduced by modern technologies. Of particular interest is the study of model reduction for the design, simulation, and implementation of controllers for large-scale systems. For example, optimal controllers for linear systems are often at least the order of the physical system model [9]. In order to implement low-order controllers for large scale systems, model reduction of the design model or full-order controller is commonly performed [13], [17].

A widely-used family of reduced-order models are the *projection-based reduced order models* (PROMs). Well established PROM producing methods, such as truncated balanced realizations, preserve stability, guarantee minimality and provide *a priori* reduction error bounds [23]. These methods, however, may be unfeasible for very large-scale systems due to their computational complexity [3]. As a result, many works aimed at finding sub-optimal efficient solutions, e.g. by alternating projection methods [12], or Krylov-subspace techniques which are computationally efficient and suitable for extremely large-scale systems [2]. Such methods, however, may fail to provide stable and minimal reduced order realizations [15].

An important instance of these large-scale systems are multi-agent systems. A great challenge in the study of multi-agent systems is to find efficient reduction methods that guarantee stability and assure minimality of their reduced order

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realizations, preferably, with optimal or suboptimal reduction errors. For networked multi-agent systems, the interconnection properties of the underlying graph can be related in some cases to the stability, controllability and observability of the system [1]. Therefore, it is desirable that reduced-order models are found that preserve, in some sense, the network structure of the full-order system, which cannot be performed with standard unstructured model-reduction methods. In that direction, several recent studies were performed. In [14], [22], PROMs of consensus-type multi-agent models were considered based on graph-contractions over vertex partitions. In [16], removal of cycle completing edges was suggested for model simplification of the consensus protocol. The reduction of second-order network systems with structure preservation using hierarchical $\mathcal{H}_2$ clustering was demonstrated in [6]. While these methods where limited to first or second-order multi-agent models, in [18], a more general graph-based model reduction was presented that preserves the functional structure of the multi-agent system. A framework for optimal structured model-order reduction of multi-agent systems was recently presented in [28]. Here, a convex relaxation technique was derived for the $\mathcal{H}_2$ model reduction of diffusively coupled second-order networks.

In this work, we reexamine the well known orthogonal PROMs and their realizations. The error system of PROMs is naturally described with an augmented system realization, allowing the PROM reduction error to be evaluated with standard system performance metrics, such as the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms. However, this error system realization usually does not provide any analytic insight, and various transformation techniques were derived to bring it to more useful forms. These include an upper triangular block structure [14], allowing derivation of analytical bounds. In this study we show that any orthogonal PROM error system can be presented as a product of three LTI systems, capturing the reduction effect of the input to state, state to output, and the internal dynamical structure. Investigating this error system product form, an analytic $\mathcal{H}_\infty$ reduction error bound is obtained and an $\mathcal{H}_\infty$ bound optimization problem is defined as a relaxation of the optimal PROM problem. Furthermore, it is shown that any orthogonal PROM can be obtained from a sequence of singleton projections, projections from dimension n to $n-1$.

Of particular interest in this work are *interface invariant* PROMs (IIPROMs). These are reduced models that maintain the input-output structure of the full order model. It is shown that for IIPROMs the error product systems are strictly proper. IIPROMs are natural to multi-agent systems where a subset of agents serve as input and output ports of the network. These I/O ports may be interconnected with an external controller

and it is required, therefore, that any reduced model preserves this interface structure. For this purpose, we propose an edge-based graph contraction method and utilize it in a tree-based greedy-edge heuristic to solve the PROM $\mathcal{H}_\infty$ bound optimization problem. We then apply an this graph contraction algorithm to obtain suboptimal $\mathcal{H}_\infty$ IIPROMs of Laplacian consensus systems.

The remaining sections of this paper are as follows. In Section II, we formulate the optimal orthogonal PROM and IIPROM problems. In Section III-A, the product form of orthogonal PROMs is derived. In Section III, the $\mathcal{H}_\infty$ error bound is derived for PROMs and PROM sequences. Section IV presents model reduction of multi-agent systems by graph contractions and the greedy-edge optimization method. In Section V, these results are demonstrated with some numerical examples of model reduction of a multi-agent system, and Section VI provides concluding remarks.

Notations: The spectrum of a real matrix $A \in \mathbb{R}^{n \times n}$ is the set of eigenvalues $\lambda(A) = \{\lambda_k(A)\}_{k=1}^n$ where $\lambda_k(A) \in \mathbb{C}$ is the kth eigenvalue of A in ascending order, $|\lambda_1| \leq |\lambda_2| \leq \ldots \leq |\lambda_n|$. The corresponding eigenvectors are $\{u_k(A)\}_{k=1}^n$. For a symmetric matrix we have an eigenvalue decomposition $A = P(A)\Lambda(A)P^\top(A)$, where $P(A) = [u_1(A), u_2(A), \ldots, u_n(A)]$ is an orthonormal matrix and $\Lambda(A) = \mathrm{diag}(\lambda(A)) \in \mathbb{R}^{n \times n}$. A symmetric matrix is *positive-definite* if $\lambda_i(A) > 0$ for $i \in [1, n]$ and is denoted as $A \succ 0$. The 2-norm of a matrix A is $\|A\|_2 = \max_i(\sqrt{\lambda_i(A^TA)})$. For two matrices A and B, $\mathrm{diag}(A, B)$ is a block diagonal matrix with A,B on the diagonal. The entries of a matrix A are denoted $[A]_{ij}$. The ith Euclidean basis column vector is denoted as $\mathbf{e}_i$. The Kronecker product is denoted by $\otimes$.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ consists of a vertex set $\mathcal{V}(\mathcal{G})$, an edge set $\mathcal{E}(\mathcal{G}) = \{\epsilon_1, \ldots, \epsilon_{|\mathcal{E}|}\}$ with $\epsilon_k \in \mathcal{V}^2$, and a set of edge weights, $\mathcal{W}(\mathcal{G}) = \{w_1, \ldots, w_{|\mathcal{E}|}\}$ with $w_i \in \mathbb{R}$. The order of the graph is defined as the number of nodes, $|\mathcal{V}|$. Two nodes $u, v \in \mathcal{V}(\mathcal{G})$ are *adjacent* if they are the endpoints of an edge $\{u, v\}$, and we denote this by $u \sim v$. If $\mathcal{G}$ is an *undirected graph* then the head and tail of each edge are arbitrary. A *self-loop* is an edge were the head and tail are the same node, and *duplicate edges* are any pair $\epsilon_i, \epsilon_j \in \mathcal{E}$ with the same head and tail nodes. A *simple graph* does not include self-loops or duplicate edges. A *multi-graph* is a graph that includes duplicate edges.

II. PROBLEM FORMULATION

An LTI system Σ with realization $\Sigma := (A, B, C, D)$ is the dynamical system,

$$\begin{cases} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{cases}, \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ are the inputs and $y(t) \in \mathbb{R}^{n_y}$ are the outputs. The matrices $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$, $C \in \mathbb{R}^{n_y \times n_x}$ and $D \in \mathbb{R}^{n_y \times n_u}$ are the system parameters. The corresponding transfer function matrix (TFM) representation of Σ is given as

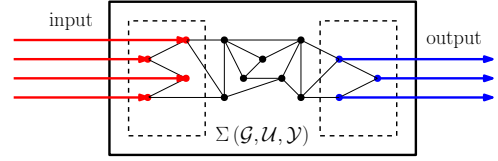$$\hat{\Sigma}(s) = D + C(sI_n - A)^{-1}B. \tag{2}$$

Figure 1: A controlled MAS with interface agents, red nodes identify the set $\mathcal{U}$, blue nodes the set $\mathcal{Y}$.

Hereafter, the notation $\hat{\Sigma}$ will be used, without the explicit dependence on s, to denote the TFM of a system Σ.

A realization $\Sigma := (A, B, C, D)$ is *minimal* if it is controllable and observable. The order of a system is its *McMillan degree*, denoted as $\deg(\Sigma)$, which is the order of any minimal realization of Σ [7].

We consider also *multi-agent systems* (MAS) as a set of n agents that interact with each other over a network described by a graph $\mathcal{G}$. We assume further that in such a network, a subset of the agents may be subject to external control inputs, and a subset may be accessed for measurements. Formally, we denote the *input nodes* by the set $\mathcal{U} \subseteq \mathcal{V}(\mathcal{G})$ with $|\mathcal{U}| = m$, and the set of *output nodes* by the set $\mathcal{Y} \subseteq \mathcal{V}(\mathcal{G})$ with $|\mathcal{Y}| = p$. In the case of linear MAS, we then have the realization

$$\Sigma(\mathcal{G}, \mathcal{U}, \mathcal{Y}) := (A(\mathcal{G}), B(\mathcal{G}, \mathcal{U}), C(\mathcal{G}, \mathcal{Y}), D(\mathcal{U}, \mathcal{Y})), \tag{3}$$

where the matrices $A(\mathcal{G}) \in \mathbb{R}^{n_x \times n_x}$, $B(\mathcal{G}, \mathcal{U}) \in \mathbb{R}^{n_x \times n_u}$, $C(\mathcal{G}, \mathcal{Y}) \in \mathbb{R}^{n_y \times n_x}$ and $D(\mathcal{U}, \mathcal{Y}) \in \mathbb{R}^{n_y \times n_u}$ are the system matrices as a function of the underlying graph structure, with $n_x = d_x \times n$, $n_u = d_u \times m$ and $n_y = d_y \times p$. Here, d_x, d_u, and d_y represent the dimension of the state, input, and output of each agent in the network. This is system is visualized in Figure 1. Note that MAS models of this form include classical setups such as diffusively coupled networks [5] and Laplacian dynamics [21].

A *reduced-order model* of Σ is any system with realization $\Sigma_r := (A_r, B_r, C_r, D_r)$ mapping $u(t) \mapsto y_r(t)$, with $u(t) \in \mathbb{R}^{n_u}$ and $y_r(t) \in \mathbb{R}^{n_y}$, such that $\deg(\Sigma_r) < \deg(\Sigma)$. Reduction error analysis can be performed by constructing an augmented error system,

$$\Sigma_e = \Sigma - \Sigma_r, \tag{4}$$

with realization $\Sigma_e := (A_e, B_e, C_e, D_e)$, where $x_e(t) = \begin{bmatrix} x^\top(t) & x_r^\top(t) \end{bmatrix}^\top$, $y_e(t) = y(t) - y_r(t)$, $A_e = \mathrm{diag}(A, A_r)$, $B_e = \begin{bmatrix} B^\top & B_r^\top \end{bmatrix}^\top$, $C_e = \begin{bmatrix} C & -C_r \end{bmatrix}$ and $D_e = D - D_r$. The reduction error can then be quantified using any system norm $\|\Sigma_e\|$. The two most studied model reduction system norms are the $\mathcal{H}_2$-norm and the $\mathcal{H}_\infty$-norm [8], of which the $\mathcal{H}_\infty$-norm is the focus of this work. The $\mathcal{H}_\infty$-norm of a stable proper system Σ is given as

$$\|\Sigma\|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \bar{\sigma}\left(\hat{\Sigma}(j\omega)\right), \tag{5}$$

where $\bar{\sigma}(M)$ is the largest singular value of the matrix M.

A widely used family of reduction methods are *projection-based reductions*. Given a system with realization $\Sigma :=$

(A, B, C, D) of order n, a *projection-based reduced order model* (PROM) is a system $\Sigma_r := \left(P^\top AV, P^\top B, CV, D\right)$, for any two matrices $P, V \in \mathbb{R}^{n \times r}$ such that $P^\top V = I_r$ [10]. If in addition $P = V$, the PROM is termed *orthogonal*, i.e.,

$$\Sigma_r (\Sigma, P) := \left(P^\top AP, P^\top B, CP, D\right). \tag{6}$$

Hereafter, all PROMs referred to in this study are orthogonal.

In this work we will examine a special class of PROMs that we term *interface-invariant PROMs* . These are reduced models that maintain the input-output structure of the full order model under the projection operation. Such PROMs are required, for example, for the reduction of controlled MAS (3) where the interface agent structure is maintained in the reduced model. We will also show that IIPROMs arise naturally when examining the error system (4) of PROMs. We now define formally the notion of an interface-invariant PROM (IIPROM).

Definition 1 (IIPROM). *Given a system with realization $\Sigma := (A, B, C, D)$, an IIPROM of Σ is any PROM $\Sigma_r (\Sigma, P) := \left(P^\top AP, P^\top B, CP, D\right)$ such that $C = CPP^\top$ and $B = PP^\top B$.*

Note that an IIPROM does not require $PP^\top = I$, e.g., for $C = B^\top = [1\,0\,0]$ we can choose

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

With the above notions in place, we can now formally state the optimal IIPROM problem.

Problem 1 (optimal IIPROM). *Consider a stable proper system of order n with realization $\Sigma := (A, B, C, D)$. Find $P \in \mathbb{R}^{n \times r}$ with $P^\top P = I_r$ such that the PROM $\Sigma_r (\Sigma, P)$ (6) minimizes the $\mathcal{H}_\infty$-norm of the reduction error system (4) and is interface-invariant, i.e.,*

$$\min_{P \in \mathbb{R}^{n \times r}} \|\Sigma_e\|_{\mathcal{H}_\infty} \tag{7}$$
$$s.t. \ \ P^\top P = I_r$$
$$C = CPP^\top$$
$$B = PP^\top B.$$

The constraints $P^\top P = I_r$, $C = CPP^\top$ and $B = PP^\top B$ make Problem 1 non-convex, and there is, in general, no closed-form or computationally efficient solution. In the following section we investigate the error system structure of IIPROMs and derive an IIPROM $\mathcal{H}_\infty$ error upper bound. This bound will then be utilised for obtaining suboptimal solutions of Problem 1.

III. THE PROM $\mathcal{H}_\infty$ BOUND

The error system of PROMs can be described with the augmented system realization (4) of dimension $\dim(x) + \dim(x_r)$. In the following section we show that any orthogonal PROM error system can be presented as a product of three appropriately defined LTI systems, two of dimension $\dim(x_r)$ and the third of dimension $\dim(x)$. This product form is then applied for the derivation of a PROM $\mathcal{H}_\infty$ error bound.

A. The Product Form of Orthogonal PROMs

The following theorem presents a PROM error system product form, and this new error system structure will allow us to derive an $\mathcal{H}_\infty$ bound for the PROM error system.

Theorem 1 (PROM error system product form). *Let $\Sigma := (A, B, C, D)$, and consider a PROM $\Sigma_r(\Sigma, P)$. Then the error reduction system $\Sigma_e(\Sigma, P) = \Sigma_r(\Sigma, P) - \Sigma$ has the TFM*

$$\hat{\Sigma}_e(\Sigma, P) = C\Phi^{-1}Q(Q^\top \Phi^{-1}Q)^{-1}Q^\top \Phi^{-1}B, \tag{8}$$

where $\Phi \triangleq sI_n - A$, and Q is any projection such that $Q^\top P = 0$ and $Q^\top Q = I_{n-r}$. Furthermore, Σ_e can be expressed as the product of three systems,

$$\Sigma_e(\Sigma, P) = \Theta(\Sigma, P)\Delta(\Sigma, P)\Gamma(\Sigma, P), \tag{9}$$

with realizations

$$\Theta(\Sigma, P) := (A_{PP}, A_{PQ}, CP, CQ), \tag{10}$$
$$\Gamma(\Sigma, P) := (A_{PP}, P^\top B, A_{QP}, Q^\top B), \tag{11}$$
$$\Delta(\Sigma, P) := (A, Q, Q^\top, 0_{p \times m}), \tag{12}$$

where $A_{PP} \triangleq P^\top AP$, $A_{PQ} \triangleq P^\top AQ$ and $A_{QP} \triangleq Q^\top AP$.

The proof of Theorem 1 is given in the Appendix.

Investigating the error systems $\Theta(\Sigma, P)$ and $\Gamma(\Sigma, P)$ we observe that if the PROM is an IIPROM, then its realization is strictly-proper.

Corollary 1 (IIPROM error system). *If $\Sigma_r(\Sigma, P) := \left(P^\top AP, P^\top B, CP, D\right)$ is an IIPROM, then $\Theta(\Sigma, P)$ in (10) and $\Gamma(\Sigma, P)$ in (11) are strictly-proper with realizations*

$$\Theta(\Sigma, P) := (A_{PP}, A_{PQ}, CP, 0_{p \times m}), \tag{13}$$

and

$$\Gamma(\Sigma, P) := \left(A_{PP}, P^\top B, A_{QP}, 0_{p \times m}\right). \tag{14}$$

Proof. Since $\Sigma_r(\Sigma, P)$ is an IIPROM, from Definition 1 we have $C = CPP^\top$ and $B = PP^\top B$. From $Q^\top P = 0$ and $Q^\top Q = I_{n-r}$ we get $PP^\top = I_n - Q^\top Q$ such that $C = C\left(I_n - QQ^\top\right)$ and $B = \left(I_n - QQ^\top\right)B$, therefore, $CQ = 0$ and $Q^\top B = 0$ obtaining our desired result. $\quad\square\quad\quad\quad\square$

B. The PROM Error System Bound

The PROM error system is the product of three systems (9), and we will make use of this form to derive an $\mathcal{H}_\infty$ reduction error upper bound.

Proposition 1 (PROM error bound). *Let $\Sigma := (A, B, C, D)$ with A Hurwitz, and consider a PROM $\Sigma_r(\Sigma, P) := \left(P^\top AP, P^\top B, CP, D\right)$. Then the $\mathcal{H}_\infty$ norm of the error reduction system (4) is bounded as*

$$\|\Sigma_e(\Sigma, P)\|_{\mathcal{H}_\infty} \leq b(\Sigma, P),$$

where

$$b(\Sigma, P) = \|\Theta(\Sigma, P)\|_{\mathcal{H}_\infty} \|\Delta(\Sigma, P)\|_{\mathcal{H}_\infty} \|\Gamma(\Sigma, P)\|_{\mathcal{H}_\infty}. \tag{15}$$

Proof. The proof follows directly from the submultiplicative of the $\mathcal{H}_\infty$-norm applied to (9). $\quad\square\quad\quad\quad\square$

For linear systems (1) with a symmetric matrix A, such as the controlled-consensus multi-agent systems studied in Section IV, we can simplify the calculation of the bound (15). The following Lemma, proven in [20, Appendix A] will be used for the derivation of Corollary 2, presenting this simplified bound.

Lemma 1 ([20]). *Let A be symmetric and Hurwitz. Then for any B of appropriate dimension,*

$$\|B^\top (sI - A)^{-1} B\|_{\mathcal{H}_\infty} = \|B^\top A^{-1} B\|_2. \quad (16)$$

Corollary 2. *Let $\Sigma := (A, B, C, D)$ with A symmetric and Hurwitz, and consider a PROM $\Sigma_r(\Sigma, P) := (P^\top AP, P^\top B, CP, D)$. Then the $\mathcal{H}_\infty$ norm of the error reduction system (4) is bounded as*

$$\|\Sigma_e(\Sigma, P)\|_{\mathcal{H}_\infty} \le b(\Sigma, P),$$

where

$$b(\Sigma, P) = \|\Theta(\Sigma, P)\|_{\mathcal{H}_\infty} \|\Gamma(\Sigma, P)\|_{\mathcal{H}_\infty} \|Q^\top A^{-1} Q\|_2. \quad (17)$$

Proof. Applying Lemma 1, we get $\|\Delta(\Sigma, P)\|_{\mathcal{H}_\infty} = \|Q^\top A^{-1} Q\|_2$ and substituting it in (15) we obtain (17). $\square$

The PROM error bound is the product of the $\mathcal{H}_\infty$-norms of the three LTI systems (10)-(12) constructing the error system. We observe that with the unitary transformation $\tilde{\Sigma} := (U^\top AU, U^\top B, CU, D)$ with $U = \begin{bmatrix} P & Q \end{bmatrix}$, the full-order system $\tilde{\Sigma}$ has a realization

$$\tilde{\Sigma} := \left(\left[\begin{array}{c|c} A_{PP} & A_{PQ} \\ \hline A_{QP} & A_{QQ} \end{array} \right], \left[\begin{array}{c} P^T B \\ \hline Q^T B \end{array} \right], \left[\, CP \mid CQ \, \right], D \right),$$

where the PROM is

$$\Sigma_r(\Sigma, P) := (A_{PP}, P^\top B, CP, D),$$

and the block-diagram of $\Sigma_r - \tilde{\Sigma}$ in additive form is shown in Figure 2a. If A_{PQ}, A_{QP}, A_{QQ}, CQ and $Q^T B$ are all zeros, then $\Sigma_r - \Sigma = 0$ (and in this case $\Sigma := (A, B, C, D)$ is not a minimal realization). If the reduction error is not zero, and each of the three product systems implicitly captures the contribution to the reduction error of these parts of $\tilde{\Sigma}$ left out in Σ_r. The map $\Theta(\Sigma, P)$ captures A_{PQ} and CQ, $\Gamma(\Sigma, P)$ captures A_{QP} and $Q^T B$ and $\Delta(\Sigma, P)$ captures A_{QQ} (Figure 2b).

Since there are no closed-form solutions to the optimal IIPROM Problem 1, this structure suggests that minimizing the three reduction error contributions can provide good PROMs. As a first step in obtaining a suboptimal solution, we define the following suboptimal IIPROM problem that attempts to minimize the error reduction upper bound derived in Proposition 1.

Problem 2 (suboptimal IIPROM). *Consider a stable proper system of order n with realization $\Sigma := (A, B, C, D)$. Find $P \in \mathbb{R}^{n \times r}$ with $P^\top P = I_r$ such that the PROM $\Sigma_r(\Sigma, P)$*

(6) *minimizes the reduction error bound (15) and is interface-invariant, i.e.,*

$$\min_{P \in \mathbb{R}^{n \times r}} b(\Sigma, P) \quad (18)$$
$$s.t. \ P^\top P = I_r$$
$$C = CPP^\top$$
$$B = PP^\top B.$$

The following simple example provides a comparison between the solutions for the optimal IIPROM Problem 1 and the optimal IIPROM bound Problem 2.

Example 1. *Consider the SISO system $\Sigma := (A, B, C, D)$ where*

$$A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D = 0.$$

The corresponding TF is

$$\hat{\Sigma} = \frac{s^2 + 3s + 1}{s^3 + 5s^2 + 6s + 1}$$

and $\|\Sigma\|_{\mathcal{H}_\infty} = 1$. We observe that all matrices $P \in \mathbb{R}^{3 \times 2}$ complying with $P^\top P = I_2$, $CPP^\top = C$ and $PP^\top B = B$, can be parameterized by a scalar $\alpha \in [-1, 1]$ in the following form,

$$P(\alpha) = \begin{bmatrix} 1 & 0 \\ 0 & \alpha \\ 0 & \beta(\alpha) \end{bmatrix},$$

where $\beta(\alpha) = \sqrt{1 - \alpha^2}$. All matrices $Q \in \mathbb{R}^{3 \times 1}$ such that $P^\top Q = 0$ are parametrized by

$$Q(\alpha) = \begin{bmatrix} 0 & -\beta(\alpha) & \alpha \end{bmatrix}^\top.$$

All IIPROMs $\Sigma_r(\Sigma, P)$ are then parameterized also by α, such that the matrices of the product form system realizations of Theorem 1 are

$$A_{PP}(\alpha) = \begin{bmatrix} -2 & \alpha \\ \alpha & -\alpha^2 - (\beta(\alpha) - \alpha)^2 \end{bmatrix}$$

and

$$A_{PQ}(\alpha) = \begin{bmatrix} -\beta(\alpha) \\ 2\alpha^2 + \alpha\beta(\alpha) - 1 \end{bmatrix} = A_{QP}(\alpha)^\top.$$

The PROM TF is

$$\hat{\Sigma}_r = \frac{s + \gamma^2}{s^2 + (\gamma^2 + 2) s + 2\gamma^2 - \alpha^2},$$

and the product systems (10)-(12) TFs are

$$\hat{\Theta} = \hat{\Gamma} = \frac{-\beta s + \alpha^3 - \alpha^2 \beta + \alpha\beta^2 - \beta^3}{s^2 + (\alpha^2 + \gamma^2 + 2) s + \alpha^2 + 2\gamma^2}$$

and

$$\hat{\Delta} = \frac{(\alpha^2 + \beta^2) s^2 + (2\gamma^2 + 2\alpha^2 + \beta^2) s + 2\gamma^2 + \alpha^2)}{s^3 + 5s^2 + 6s + 1},$$

where $\gamma \triangleq \sqrt{\alpha^2 + (\alpha - \beta)^2}$.

The PROM reduction error and reduction error bound are plotted in Figure 3. It is observed that the solution of the
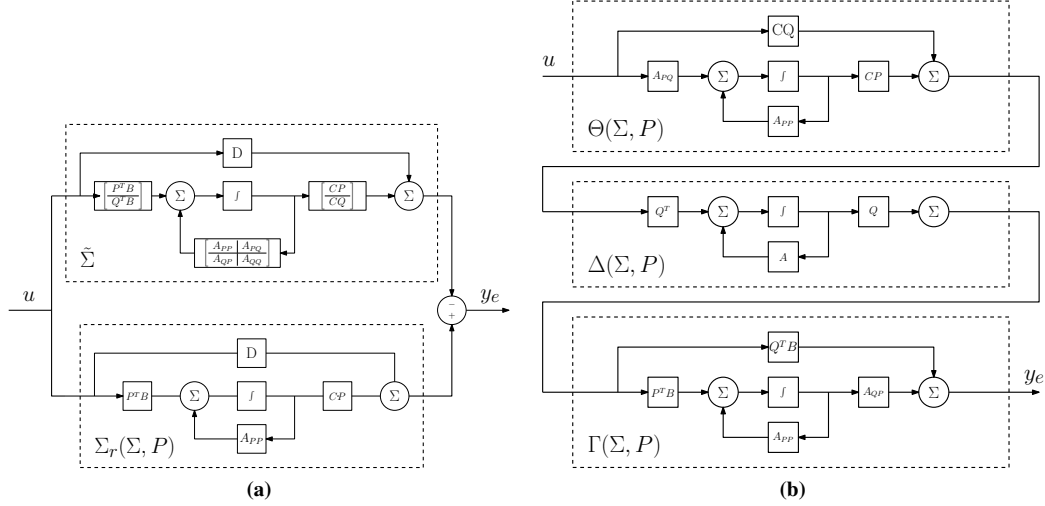
Figure 2: The PROM error system block diagram in (a) additive form and (b) product form (1).
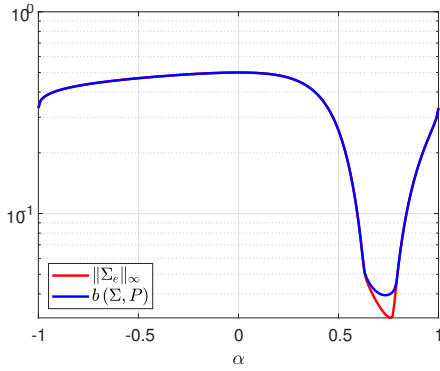

Figure 3: The IIPROM reduction error and bound of the second order system as a function of the projection parameter α.

optimal IIPROM problem (Problem 1) is $\min_\alpha \|\Sigma - \Sigma_r\|_\infty = 0.03$ *obtained for* $\alpha^* = 0.76$. *The solution of the optimal IIPROM bound problem (Problem 2) is* $\min_\alpha b(\Sigma, P) = 0.039$ *obtained for* $\alpha^{**} = 0.73$. *We observe that the optimal IIPROM error is close to the sub-optimal bound.*

C. The PROM Sequence Bound

In the following subsection we present a Lemma showing that any projection from $\mathbb{R}^n$ to $\mathbb{R}^r$ can be obtained from a sequence of $n - r$ projections, each reducing the dimension by one. We denote such projections as *singleton projections*. This sequential projection representation is then utilized to obtain a sequential PROM bound that is useful for obtaining sub-optimal solutions to Problem 2.

Lemma 2. *Let* $P \in \mathbb{R}^{n \times r}$ *with* $P^\top P = I_r$ *be a projection for* $r < n$. *Then there exists a sequence* $\{P_{(k)}\}_{k=1}^{n-r}$ *with* $P_{(k)} \in \mathbb{R}^{n-k+1 \times n-k}$, $P_{(k)}^\top P_{(k)} = I_{n-k}$ *such that* $P = \Pi_{k=1}^{n-r} P_{(k)}$.

Proof. Let $P \in \mathbb{R}^{n \times r}$, $P^\top P = I_r$, then there exists $Q \in \mathbb{R}^{n \times n-r}$ with $Q^\top Q = I_{n-r}$ such that $P^\top Q = 0$. Construct $P_{(1)} = [P, q_1, q_2, \ldots, q_{n-r-1}]$, where q_i is the ith column of

Q, and $P_{(k)} = [I_{n-k}, 0_{n-k \times 1}]^\top$ for $k \in [2, n-r]$. Since $P^\top P = I_r$ and $P^\top Q = 0$ we have $P_{(1)}^\top P_{(1)} = I_{n-1}$, and for $k \in [2, n-r]$ it is trivial that $P_{(k)}^\top P_{(k)} = I_{n-k}$. From the construction we get

$$\left(\prod_{k=1}^{n-r} P_{(k)} \right) = P_{(1)} \left(\prod_{i=1}^{n-r-1} P_{(1+i)} \right)$$
$$= P_{(1)} \left(\prod_{i=1}^{n-r-1} [I_{n-i}, 0_{n-1 \times 1}]^\top \right)$$
$$= P.$$

□   □

Note that infinite other sequences $P = \Pi_{k=1}^{n-r} \tilde{P}_{(k)}$ can be produced by the transformations $\tilde{P}_{(1)} = P_{(1)} U_{(1)}$, $\tilde{P}_{(k)} = U_{(k-1)}^\top P_{(k)} U_{(k)}$ for $k \in [2, n-r-1]$, and $\tilde{P}_{(n-r)} = U_{(n-r-1)}^\top P_{(n-r)}$, where $U_{(k)} \in \mathbb{R}^{n-k \times n-k}$ is an orthogonal matrix.

By expressing a PROM as a sequence of singleton projections, we obtain the following PROM sequence bound.

Proposition 2 (Singleton PROM sequence bound)**.** *Let* Σ *be an LTI system with realization* (A, B, C, D) *with* A *Hurwitz, and consider the PROM,* $\Sigma_r := (P^\top A P, P^\top B, C P, D)$, *and let* $\{P_{(k)}\}_{k=1}^{n-r}$ *be a sequence with* $P_{(k)} \in \mathbb{R}^{n-k+1 \times n-k}$, $P_{(k)}^\top P_{(k)} = I_{n-k}$ *such that* $P = \Pi_{k=1}^{n-r} P_{(k)}$. *Then the* $\mathcal{H}_\infty$ *norm of the error reduction system* Σ_e (4) *is bounded by*

$$\|\Sigma_e\|_{\mathcal{H}_\infty} \leq \sum_{k=1}^{n-1} b\left(\Sigma_{(k-1)}, P_{(k)}\right), \qquad (19)$$

with $b(\Sigma_{(k-1)}, P_{(k)})$ *given in* (15)*, and*

$$\Sigma_{(k)} := \left(P_{(k)}^\top A_{(k-1)} P_{(k)}, P_{(k)}^\top B_{(k-1)}, C_{(k-1)} P_{(k)}, D \right) \quad (20)$$

with $\Sigma_{(0)} := (A, B, C, D)$.

Proof. We express $\Sigma_e(s) = \Sigma_r - \Sigma$ as the telescoping sum $\sum_{k=1}^{n-r}\left(\Sigma_{(k)} - \Sigma_{(k-1)}\right)$ with $\Sigma_{(0)} = \Sigma$ and $\Sigma_{(n-r)} = \Sigma_r$, such that

$$\|\Sigma_e(s)\|_{\mathcal{H}_\infty} = \|\Sigma_r - \Sigma\|_{\mathcal{H}_\infty}$$
$$= \|\sum_{k=1}^{n-r}\left(\Sigma_{(k)} - \Sigma_{(k-1)}\right)\|_{\mathcal{H}_\infty}$$

and from the triangle inequality we get

$$\|\Sigma_e(s)\|_{\mathcal{H}_\infty} \le \sum_{k=1}^{n-r}\|\Sigma_{(k)} - \Sigma_{(k-1)}\|_{\mathcal{H}_\infty}.$$

The system $\Sigma_{(k)}$ has realization

$$\Sigma_{(k)} := \left(A_{(k)}, B_{(k)}, C_{(k)}, D\right)$$
$$= \left(P_{(k)}^\top A_{(k-1)} P_{(k)}, P_{(k)}^\top B_{(k-1)}, C_{(k-1)} P_{(k)}, D\right),$$

which is an IIPROM of $\Sigma_{(k-1)}$, therefore, from Theorem 1,

$$\|\Sigma_{(k)} - \Sigma_{(k-1)}\|_{\mathcal{H}_\infty} \le b\left(\Sigma_{(k-1)}, P_{(k)}\right),$$

and we obtain that

$$\|\Sigma_e(s)\|_{\mathcal{H}_\infty} \le \sum_{k=1}^{n-r} b\left(\Sigma_{(k-1)}, P_{(k)}\right).$$

□   □

In the following section, we will utilize graph contractions for obtaining suboptimal solutions of Problem 2 for multi-agent systems, and therefore, also Problem 1.

IV. MODEL REDUCTION OF MULTI-AGENT SYSTEMS BY GRAPH CONTRACTIONS

Multi-agent systems may be of extremely large scale, and designing and implementing full order controllers for such systems is not feasible without applying model reduction on the design model or the full-order controller. The general statement of Problem 2 does not suggest any constructive way to find the optimal PROM bound for MAS. However, it is expected that an optimal solution will have some functional dependency on the MAS structure. Vertex partitions have been widely used in graph theory, e.g., for graph clustering [25] and in the study of network communities [24]. Vertex partitions have been also used for constructing projection-based model reductions of multi-agent systems as the consensus protocol [22] and bidirectional networks [14].

It has been observed in these previous studies that partition-based PROMs maintain an MAS structue, i.e., the PROM $\Sigma_r(\Sigma, P)$ is an MAS (3) defined over a reduced order graph $\mathcal{G}_r, \Sigma(\mathcal{G}_r, \mathcal{U}, \mathcal{Y})$.

In this work we introduce the notion of edge-induced PROMs. These are PROMs which are constructed over edge-induced partitions of the graph. This graph-based model reduction method allows us to derive sub-optimal but efficient IIPROMs of MAS. These algorithms give good in practice results as demonstrated in Section V; however, an analytic result quantifying their sub-optimality is yet to be derived. We first define several combinatorial graph operations that will be used in this section.

An *r-partition*, π, of a vertex set $\mathcal{V}$ is the partition $\{C_i\}_{i=1}^r$ of $\mathcal{V}$ to r cells such that $\cup_{i=1}^r C_i = \mathcal{V}$ and $|C_i \cap C_j| = 0$ for $i \ne j$. An r-partition $\pi = \{C_i\}_{i=1}^r$ is $\mathcal{I}$-invariant for $\mathcal{I} \subseteq \mathcal{V}$ if $\forall i \in [1, r]$, one has $|C_i \cap \mathcal{I}| \le 1$, i.e., no partition cell contains more than one node in $\mathcal{I}$. The r-partition is *strongly $\mathcal{I}$-invariant* if all partition cells containing nodes in $\mathcal{I}$ are singletons, i.e., $|C_i| = 1$ whenever $C_i \subset \mathcal{I}$. We denote the set of all strongly $\mathcal{I}$-invariant r-partitions as $S_r(\mathcal{V})$. Figure 4 illustrates these definitions.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and an $n - r$ edge subset $\mathcal{E}_S \subseteq \mathcal{E}$, an *edge-induced partition* $\pi(\mathcal{E}_S)$ is an r-partition of $\mathcal{V}$ constructed as follows [19]: (i) A graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_S)$ is created from the vertices of $\mathcal{G}$ and the edge-subset $\mathcal{E}_S$, (ii) the connected components of $\mathcal{G}(\mathcal{V}, \mathcal{E}_S)$ are found, (iii) the vertices of each component is registered as a partition cell, and the set of all components cells constitutes the partition $\pi(\mathcal{E}_S)$ of $\mathcal{V}$ (Figure 5).

Given an r-partition $\pi = \{C_i\}_{i=1}^r$, we define $M(\pi) \in \mathbb{R}^{n \times r}$, the *partition characteristic matrix* (PCM) with entries $[M(\pi)]_{ij} = 1$ if $i \in C_j$, and 0 otherwise. The corresponding *partition projection matrix* (PPM) is $P(\pi) \triangleq M(\pi) D^{-\frac{1}{2}}(\pi)$ where $D(\pi) \triangleq M^\top(\pi) M(\pi)$

For multi-agent systems we restrict the projection to be PPMs, and the partition IIPROM bound problem is introduced:

Problem 3 (optimal partition IIPROM bound). *Consider a stable MAS $\Sigma(\mathcal{G}, \mathcal{U}, \mathcal{Y})$ with n agents, each with local state of dimension d_x, and an interface set $\mathcal{I} = \mathcal{U} \cup \mathcal{Y}$. Find an $\mathcal{I}$-invariant r-partition π such that the PROM $\Sigma_r(\Sigma, P(\pi))$ minimizes the reduction error bound $b\left(\Sigma(\mathcal{G}, \mathcal{U}, \mathcal{Y}), I_{d_x} \otimes P(\pi)\right)$ given in* (17).

Finding a solution to Problem 3 may be numerically intractable for a moderate number of nodes, as the number of
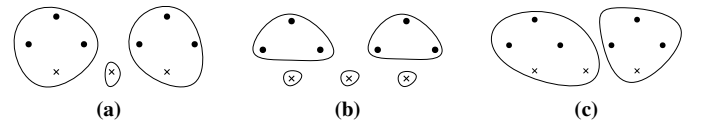

(a)   **(b)**   **(c)**

Figure 4: A set of 9 nodes, with subset $\mathcal{I}$ of three nodes (marked with x), (a) is an $\mathcal{I}$-invariant 3-partition, (b) is a strongly $\mathcal{I}$-invariant 5-partition, and (c) is a 2-partition which is not $\mathcal{I}$-invariant.
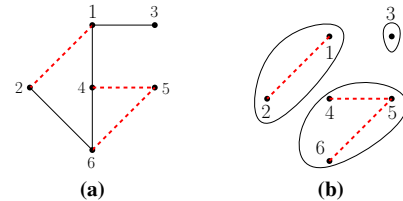

(a)   **(b)**

Figure 5: An edge-induced partition of a graph of order r, (a) a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with a selected edge subset $\mathcal{E}_S = \{\{1, 2\}, \{4, 5\}, \{5, 6\}\}$ (dashed red), (b) the graph $\mathcal{G}(\mathcal{V}, \mathcal{E}_S)$ and its connected components inducing the partition $\pi(\mathcal{E}_S) = \{\{1, 2\}, \{3\}, \{4, 5, 6\}\}$ on $\mathcal{V}(\mathcal{G})$.

r-partitions is the Stirling number of the second kind,

$$S\left(n,r\right) = \sum_{k=1}^{r} \left(-1\right)^{r-k} \frac{k^n}{k!\left(r-k\right)!},$$

which for $r \ll n$ is asymptotically $S\left(n,r\right) \sim \frac{r^n}{r!}$ [27, p.18].

Given a subset of edges $\mathcal{E}_c \subseteq \mathcal{E}$, an *edge-induced partition* $\pi\left(\mathcal{E}_c\right)$ can be constructed as described above. Here we utilize the edge-induced partition to derive a greedy-edge IIPROM bound (GEIB) algorithm (Algorithm 1). The input to the algorithm is an MAS $\Sigma\left(\mathcal{G}, \mathcal{U}, \mathcal{Y}\right)$ (3), the required reduction order r and a subset of candidate edges $\mathcal{E}_c \subseteq \mathcal{E}$ (assuming there are more than $n-r$ edges in $\mathcal{E}_c$). The first step of the algorithm is to check if each of the edges is strongly $\mathcal{I}$-invariant by examining if both end nodes of an edge are not in $\mathcal{I}$. If an edge is found to be strongly $\mathcal{I}$-invariant, a PPM in constructed with its induced edge partition and the PROM error bound (1) is calculated. The PPM of the induced-edge partition of the $n-r$ edges with lowest bound is then used to obtain a PROM and serves as a suboptimal solution to Problem 3.

Algorithm 1 Greedy-edge IIPROM bound Algorithm

Input: An MAS $\Sigma\left(\mathcal{G}, \mathcal{U}, \mathcal{Y}\right)$ of order n with realization $\Sigma :=$ $\left(A\left(\mathcal{G}\right), B\left(\mathcal{G}, \mathcal{U}\right), C\left(\mathcal{G}, \mathcal{Y}\right), D\left(\mathcal{U}, \mathcal{Y}\right)\right)$, interface set $\mathcal{I} = \mathcal{U} \cup \mathcal{Y}$, edge subset $\mathcal{E}_c \subseteq \mathcal{E}$, reduction order r.
  1) **For** each $\{u, v\} \in \mathcal{E}_c$
     - **if** $u \in \mathcal{I}$ or $v \in \mathcal{I}$, skip to next edge,
     - **else** construct the edge-induced partition $\pi\left(\{u, v\}\right)$ and its PPM $P\left(\pi\right)$, and calculate $b\left(\Sigma, P\left(\pi\right)\right)$.
  2) **Find** $n-r$ edges $\mathcal{E}^* \subseteq \mathcal{E}_c$ with the lowest bound values.
  3) **Construct** the edge-induced r-partition $\pi\left(\mathcal{E}^*\right)$.
Output: $\Sigma_r\left(\Sigma, P\left(\pi\left(\mathcal{E}^*\right)\right)\right)$ (6)
Notation: $\Sigma_r = GEIB\left(\Sigma\left(\mathcal{G}, \mathcal{U}, \mathcal{Y}\right), \mathcal{E}_c, r\right)$

The greedy-edge IIPROM bound algorithm does not specify the method to choose the edge subset $\mathcal{E}_c$. Trees are the building blocks of any connected graph. A basic graph-theoretic principle is that a spanning tree of a connected graph $\mathcal{G} = \left(\mathcal{V}, \mathcal{E}\right)$ of order n is a subgraph with a minimal set of $n-1$ edges connecting all vertices. Furthermore, trees and cycle-completing edges are strongly related to the performance of networked systems [29]. With this intuition, we derive the tree-based IIPROM bound (TBIB) algorithm (Algorithm 2), where given an MAS $\Sigma\left(\mathcal{G}, \mathcal{U}, \mathcal{Y}\right)$ (3), a spanning-tree is found. The edges of the tree are then used as the subset $\mathcal{E}_c$ when applying the GEIB algorithm.

We have derived the TBIB algorithm utilizing the PROM error bound as a general framework for model reduction of MAS. In the following case study section, we examine the performance of the algorithm, and it will be shown to provide excellent results for the reduction of large-scale MAS.

V. CASE STUDIES

In this section we present some numerical examples illustrating the results of this work.

Algorithm 2 Tree-based IIPROM bound Algorithm

Input: An MAS $\Sigma\left(\mathcal{G}, \mathcal{U}, \mathcal{Y}\right)$ of order n with realization $\Sigma :=$ $\left(A\left(\mathcal{G}\right), B\left(\mathcal{G}, \mathcal{U}\right), C\left(\mathcal{G}, \mathcal{Y}\right), D\left(\mathcal{U}, \mathcal{Y}\right)\right)$, interface set $\mathcal{I} = \mathcal{U} \cup \mathcal{Y}$, reduction order r.
  1) **Find** a spanning tree $\mathcal{T}$ of $\mathcal{G}$.
  2) **Perform** greedy edge algorithm

$$\Sigma_r = GEIB\left(\Sigma\left(\mathcal{G}, \mathcal{U}, \mathcal{Y}\right), \mathcal{E}\left(\mathcal{T}\right), r\right).$$

Output: Σ_r
Notation: $\Sigma_r = TBIB\left(\Sigma\left(\mathcal{G}, \mathcal{U}, \mathcal{Y}\right), r\right)$

The Laplacian matrix plays a key role in networked an multi-agent systems. For a multi-graph, i.e., a graph that may include duplicate edges and self loops, the Laplacian matrix $L\left(\mathcal{G}\right) \in \mathbb{R}^{|\mathcal{V}| \times |\mathcal{V}|}$ is defined as [11]

$$\left[L(\mathcal{G})\right]_{uv} = \begin{cases} \sum_{v \in \{\mathcal{N}_u \cup u\}} w\left(\{u, v\}\right) & u = v \\ -w\left(\{u, v\}\right) & u \sim v \\ 0 & o.w. \end{cases} \quad (21)$$

where $w(\{u, v\})$ denotes the weight of the edge $\{u, v\}$.

As a test case model of an interface MAS of the form (3), we consider the *Laplacian controlled consensus model (LCC)* (22), a generalisation of the Laplacian Consensus protocol $\dot{x} = -L\left(\mathcal{G}\right)x$, a benchmark model of an uncontrolled multi-agent system [21],

$$\begin{cases} \dot{x} &= -L\left(\mathcal{G}\right)x + B\left(\mathcal{U}\right)P \\ y &= C\left(\mathcal{Y}\right)x + D\left(\mathcal{U}, \mathcal{Y}\right)u \end{cases}. \quad (22)$$

In the following subsections we will demonstrate the effectiveness of the graph-based model reduction Algorithms derived in Section IV for the reduction of small and large-scale LCC models.

A. IIPROM of a Small-Scale Laplacian Consensus Model

As a first case study we consider the reduction of an LCC model (22) associated to a small-scale graph $\mathcal{G}$ with 20 nodes and 23 edges with 2 interface nodes (Figure 6a). We perform an edge-based IIPROM for each of the $\mathcal{I}$-invariant singleton partitions induced by the edges of $\mathcal{G}$. In this case, only the partitions induced by edge 1 are not $\mathcal{I}$-invariant since both its end nodes are interface nodes. For each of these IIPROMs, the $\mathcal{H}_\infty$-norm of the error system (1) and the $\mathcal{H}_\infty$ error bound (1) are calculated and normalized by $\|\Sigma\|_{\mathcal{H}_\infty}$ (Figure 6b). We observe that the choice of edge has a great effect on the reduction error magnitude, some edges produce reduction errors smaller than 1% of $\|\Sigma\|_{\mathcal{H}_\infty}$, while other edges induce IIPROM errors that are 10% of $\|\Sigma\|_{\mathcal{H}_\infty}$. It is also observed that the error bound is relatively tight for small reduction errors, while for larger errors the bound may differ more than an order of magnitude from the error.

We then demonstrate the tree-based IIPROM bound algorithm (Algorithm 2) on the LCC. The spanning tree edges are found and Algorithm 2 is then performed for $r \in [10, 19]$. Figure 6c plots the reduction error $\|\Sigma_e\|_{\mathcal{H}_\infty}$, and the error bound $b_r\left(\mathcal{G}, U_r\right)$ (15), as a function of the normalized reduction depth $\frac{n-r}{n}$. We observe that the log reduction error
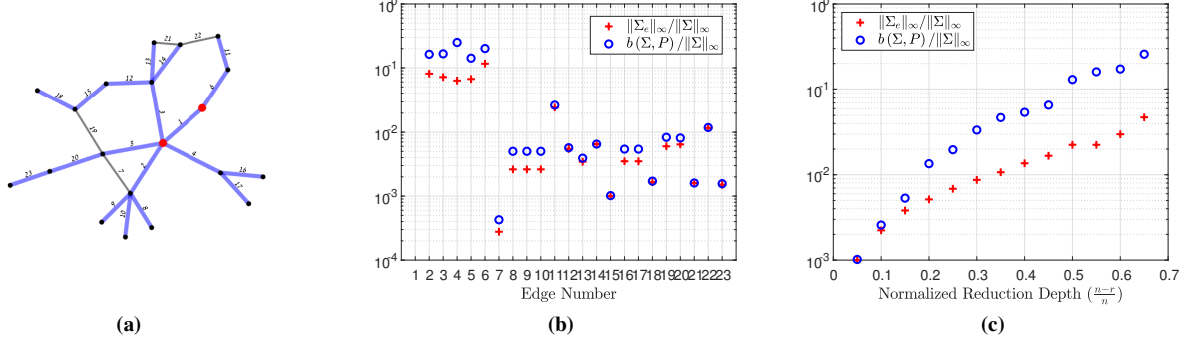
Figure 6: A small-scale edge-based IIPROM case study (a) The graph associated to the Laplacian consensus in this example, interface nodes marked red, labels on edges indicate the edge numbers, a spanning tree is highlighted in blue and (b) The reduction error and reduction error bound of each edge-based IIPROM of the Laplacian consensus associated to the graph in Figure 6a. Notice that no values are assigned to edge 1 since it does not induce an interface invariant PROM. (c) The reduction error and reduction error bound resulting from applying the tree-based IIPROM bound algorithm to the the Laplacian consensus associated to the graph.

has a quasi-linear trend as a function of $\frac{n-r}{n}$, and that the bound is tight for low reduction depths and differs as the reduction increases; however, the bound follows the same trend as the error, therefore, minimizing the bound is consistent with minimizing the error in this case.

B. IIPROM of A Large-Scale Small-World Laplacian Consensus Model

As a large-scale case study, a small-world graph is created with the Watts-Strogatz random rewiring procedure with $k = 5$ and $\beta = 0.15$ [26] starting from a 5-regular graph of order 100 with 5 interface nodes $\mathcal{U} = \mathcal{Y} = \{1, \ldots, 5\}$ (Figure 7). An LCC is constructed over this graph and the tree-based IIPROM bound algorithm (Algorithm 2) is then performed. Figure 8 plots the reduction error $\|\Sigma_e\|_{\mathcal{H}_\infty}$, and the error bound $b_r(\mathcal{G}, U_r)$ (15) (normalized by $\|\Sigma\|_{\mathcal{H}_\infty}$), as a function of the normalized reduction depth $\frac{n-r}{n}$. As a comparison to the reduction error and bound results, we calculate the empirical mean-IIPROM $\mu_P(\Sigma)$, the mean reduction error $\|\Sigma_e\|_{\mathcal{H}_\infty}$ of $N = 50$ randomly selected IIPROMs, and the empirical mean edge-based IIPROM $\mu_\varepsilon(\Sigma)$, the mean reduction error $\|\Sigma_e\|_{\mathcal{H}_\infty}$ of $N = 50$ randomly selected edge-based PPM IIPROMs.

We observe that bound is tight for the entire reduction depth range. Furthermore, the reduction with the proposed tree-based method is several orders of magnitude lower than the empirical mean-IIPROM, and for lower reduction depth is significantly better than the empirical mean edge-based IIPROM. As expected, for high reduction depth, the tree-based method converges to the empirical mean edge-based IIPROM.

VI. CONCLUSIONS

This work derived a unique product form of the error system of orthogonal projection-based reduced models. This product form is then used to derive an $\mathcal{H}_\infty$ error bound for the PROM error system. A suboptimal bound optimization solution for
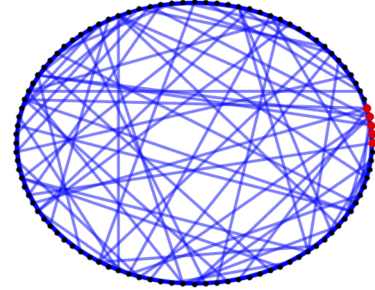

Figure 7: The Laplacian consensus in this example is associated to a small-world graph with 5 interface nodes (marked red).
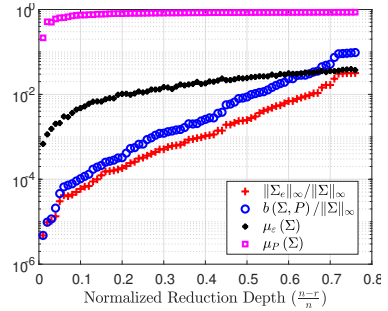

Figure 8: The reduction error and reduction error bound of the Laplacian consensus associated to the graph in Figure 7.

multi-agent systems is obtained with a graph-based spanning tree algorithm. Applying this algorithm on a Laplacian consensus model constructed with a large-scale small-world network, demonstrates the utility of the method to large scale multi-agent systems. In the examined case studies, the bound obtained with the tree-base algorithm is tight, therefore, the optimal PROM bound for those cases is close to the optimal reduction error PROM. It is an open research question to explain why the bound tight. The same technique presented

can be applied to various multi-agent systems other than the consensus models. The derived reduction error product form and bound can be the basis for additional optimization methods, such as convex relaxations.

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APPENDIX

In this Appendix, we present the proof of Theorem 1. The proof is based on the matrix inverse lifting lemma along with a projection inversion corollary that we present here.

Definition 2 (Projected Schur complement). *Let $M \in \mathbb{C}^{n \times n}$ and let P and Q be projections such that $P^\top P = I_r$, $Q^\top Q = I_{n-r}$ and $Q^\top P = 0$. Then for $P^\top M P$ invertible, we define the* projected Schur complement *of S by P as*

$$S(M, P) \triangleq M_{QQ} - M_{QP} M_{PP}^{-1} M_{PQ}, \qquad (23)$$

and the projected Schur complement of S by Q as

$$S(M, Q) \triangleq M_{PP} - M_{PQ} M_{QQ}^{-1} M_{QP}, \qquad (24)$$

where $M_{PP} \triangleq P^\top M P$, $M_{PQ} \triangleq P^\top M Q$, $M_{QP} \triangleq Q^\top M P$ and $M_{QQ} \triangleq Q^\top M Q$.

With the definition of the projected Schur complement we can derive the following corollary of the matrix inversion Lemma [4].

Lemma 3. *Let $M \in \mathbb{C}^{n \times n}$ and M^{-1} be a matrix and its inverse with corresponding block structures*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix},$$

then

$$W = (M_{11} - M_{12} M_{22}^{-1} M_{21})^{-1}, \qquad (25)$$

and

$$X = -M_{11}^{-1} M_{12} Z. \qquad (26)$$

Corollary 3. *Consider a matrix $M \in \mathbb{C}^{n \times n}$, and let P and Q be matrices such that $P^\top P = I_r$, $Q^\top Q = I_{n-r}$ and $Q^\top P = 0$. Then*

$$P^\top M^{-1} P = S^{-1}(M, Q), \qquad (27)$$

and

$$P^\top M^{-1} Q = -M_{PP}^{-1} M_{PQ} (Q^\top M^{-1} Q). \qquad (28)$$

Proof. Apply the matrix inversion Lemma to the matrix $\tilde{M} = [PQ]^\top M [PQ]$ and obtain $P^\top M^{-1} P = S^{-1}(M, Q)$ and $P^\top M^{-1} Q = -M_{PP}^{-1} M_{PQ} (Q^\top M^{-1} Q)$. □ □

We denote a *matrix lifting* $f_P : \mathbb{R}^{r \times r} \to \mathbb{R}^{n \times n}$ as the function $f_P(M) \triangleq P M P^\top$. The following lemma extends the matrix inversion lemma to the matrix lifting of $(P^\top M P)^{-1}$.

Lemma 4 (Matrix Inverse Lifting). *Consider a matrix $M \in \mathbb{C}^{n \times n}$ and let P and Q be matrices such that $P^\top P = I_r$, $Q^\top Q = I_{n-r}$ and $Q^\top P = 0$. Then*

$$P(P^\top M P)^{-1} P^\top = \Upsilon(M, Q), \qquad (29)$$

where we define $\Upsilon(M,Q)$ *as,*

$$\Upsilon(M,Q) \triangleq M^{-1} - M^{-1}Q\left(Q^\top M^{-1}Q\right)^{-1}Q^\top M^{-1}.$$

Proof. We have $I_n = PP^\top + QQ^\top$ such that

$$\begin{aligned}
\Upsilon(M,Q) &= \left(PP^\top + QQ^\top\right)\Upsilon(M,Q)\left(PP^\top + QQ^\top\right) \\
&= PP^\top\Upsilon(M,Q)PP^\top + PP^\top\Upsilon(M,Q)QQ^\top \\
&\quad + QQ^\top\Upsilon(M,Q)PP^\top + QQ^\top\Upsilon(M,Q)QQ^\top.
\end{aligned}$$

Evaluating the four expressions in the sum we get

$$P^\top\Upsilon(M,Q)Q =$$
$$P^\top M^{-1}Q - P^\top M^{-1}Q\left(Q^\top M^{-1}Q\right)^{-1}Q^\top M^{-1}Q = 0,$$
$$Q^\top\Upsilon(M,Q)P =$$
$$Q^\top M^{-1}P - Q^\top M^{-1}Q\left(Q^\top M^{-1}Q\right)^{-1}Q^\top M^{-1}P = 0,$$
$$Q^\top\Upsilon(M,Q)Q =$$
$$Q^\top M^{-1}Q - Q^\top M^{-1}Q\left(Q^\top M^{-1}Q\right)^{-1}Q^\top M^{-1}Q = 0,$$
$$P^\top\Upsilon(M,Q)P =$$
$$P^\top M^{-1}P - P^\top M^{-1}Q\left(Q^\top M^{-1}Q\right)^{-1}Q^\top M^{-1}P.$$

We observe that the last term $P^\top\Upsilon(M,Q)P$ is the projected Schur complement $S\left(M^{-1},Q\right)$ (Def. 2). From the projection inversion corollary (Corollary 3) we then obtain

$$P^\top\Upsilon(M,Q)P = \left(P^\top M P\right)^{-1},$$

and therefore

$$\Upsilon(M,Q) = P\left(P^\top M P\right)^{-1}P^\top.$$

$\square$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now prepared to proceed with the proof of Theorem 1.

Proof. We begin by proving the first part of the theorem, i.e.,

$$\hat{\Sigma}_e(\Sigma,P) = C\Phi^{-1}Q(Q^\top\Phi^{-1}Q)^{-1}Q^\top\Phi^{-1}B. \qquad (30)$$

The error system TFM is

$$\begin{aligned}
\hat{\Sigma}_e &= \hat{\Sigma}_r - \hat{\Sigma} \\
&= CP\left(sI_r - P^\top AP\right)^{-1}P^\top B - C\Phi^{-1}B \\
&= CP\left(P^\top\Phi P\right)^{-1}P^\top B - C\Phi^{-1}B \\
&= C\left(P\left(P^\top\Phi P\right)^{-1}P^\top - \Phi^{-1}\right)B.
\end{aligned}$$

We now employ the matrix inverse lifting lemma (Lemma 4),

$$P(P^\top\Phi P)^{-1}P^\top - \Phi^{-1} = \Phi^{-1}Q(Q^\top\Phi^{-1}Q)^{-1}Q^\top\Phi^{-1},$$

thus leading to the expression (30). Next, we prove the second part of the theorem, i.e.,

$$\Sigma_e(\Sigma,P) = \Theta(\Sigma,P)\Delta(\Sigma,P)\Gamma(\Sigma,P), \qquad (31)$$

with the three realizations

$$\begin{aligned}
\Theta(\Sigma,P) &:= (A_{PP}, A_{PQ}, CP, CQ) \\
\Gamma(\Sigma,P) &:= (A_{PP}, P^\top B, A_{QP}, Q^\top B) \\
\Delta(\Sigma,P) &:= (A, Q, Q^\top, 0_{p\times m}).
\end{aligned}$$

With $I = \left(Q^\top\Phi^{-1}Q\right)\left(Q^\top\Phi^{-1}Q\right)^{-1}$ we get

$$C\Phi^{-1}Q\Xi(\Phi,Q)Q^\top\Phi^{-1}B =$$
$$\hat{\Theta}(\Sigma,P)\hat{\Delta}(\Sigma,P)\hat{\Gamma}(\Sigma,P)$$

where $\Xi(\Phi,Q) \triangleq \left(Q^\top\Phi^{-1}Q\right)^{-1}$ and

$$\begin{aligned}
\hat{\Theta}(\Sigma,P) &\triangleq C\Phi^{-1}Q\Xi(\Phi,Q), \\
\hat{\Delta}(\Sigma,P) &\triangleq Q^\top\Phi^{-1}Q, \\
\hat{\Gamma}(\Sigma,P) &\triangleq \Xi(\Phi,Q)Q^\top\Phi^{-1}B.
\end{aligned}$$

We have $I = PP^\top + QQ^\top$ such that

$$\begin{aligned}
\Phi^{-1}Q\left(Q^\top\Phi^{-1}Q\right)^{-1} &= \left(PP^\top + QQ^\top\right)\Phi^{-1}Q\Xi(\Phi,Q) \\
&= Q + P\left(P^\top\Phi^{-1}Q\right)\Xi(\Phi,Q).
\end{aligned}$$

From the projection inversion corollary (Corollary 3), we have

$$\begin{aligned}
P^\top\Phi^{-1}Q &= -\Phi_{PP}^{-1}\Phi_{PQ}\left(Q^\top\Phi^{-1}Q\right) \\
&= \left(sI_r - P^\top AP\right)^{-1}P^\top AQ\left(Q^\top\Phi^{-1}Q\right)
\end{aligned}$$

and

$$\Phi^{-1}Q\Xi(\Phi,Q) = Q + P\left(sI_r - P^\top AP\right)^{-1}P^\top AQ, \quad (32)$$

such that

$$\hat{\Theta}(\Sigma,P) = CP\left(sI_r - P^\top AP\right)^{-1}P^\top AQ + CQ,$$

which is the TFM with realization $\Theta(\Sigma,P) := (A_{PP}, A_{PQ}, CP, CQ)$. Similarly we get

$$\begin{aligned}
\hat{\Gamma}(\Sigma,P) &= \left[Q^\top\Phi^{-1}Q\right]^{-1}Q^\top\Phi^{-1}B \\
&= Q^\top AP\left(sI_r - P^\top AP\right)^{-1}P^\top B + Q^\top B,
\end{aligned}$$

and the realization $\Gamma(\Sigma,P) := \left(A_{PP}, P^\top B, A_{QP}, Q^\top B\right)$. Finaly $\hat{\Delta}(\Sigma,P) = Q\Phi^{-1}Q$ has realzation $\Delta(\Sigma,P) := \left(A, Q, Q^\top, 0_{p\times m}\right)$. $\square$ $\qquad\qquad\square$