SIGNED NONLINEAR NETWORKS

A PASSIVITY AND ELECTRICAL CIRCUIT THEORY APPROACH

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January 20, 2020
Technion CST Seminar

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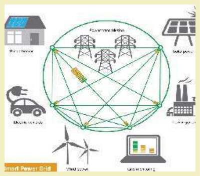
CoNeCt
Cooperative Networks
and Controls Lab

NETWORKED DYNAMIC SYSTEMS



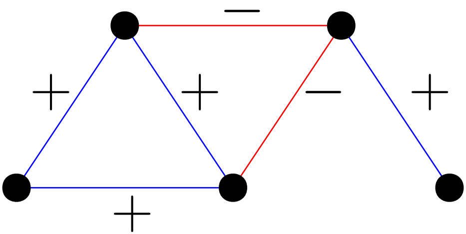
Networks of dynamical systems are one of the enabling technologies of the future.

COOPERATIVE V. ANTAGONISTIC

In many *biological* and *social* networks, there are generally **two** kinds of interactions:

- **trustful** versus **distrustful**
- **cooperative** versus **competitive**

Signed Networks provide an abstraction for modeling cooperative or antagonistic interactions.



(+) - *positive* edges
(-) - *negative* edges

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

$$\mathcal{W} : \mathcal{E} \to \mathbb{R}$$

$$\mathcal{E}_+ = \{e \in \mathcal{E} \ : \ \mathcal{W}(e) > 0\}$$

$$\mathcal{E}_- = \{e \in \mathcal{E} \ : \ \mathcal{W}(e) < 0\}$$

$$\mathcal{G} = \mathcal{G}_+ \cup \mathcal{G}_-$$

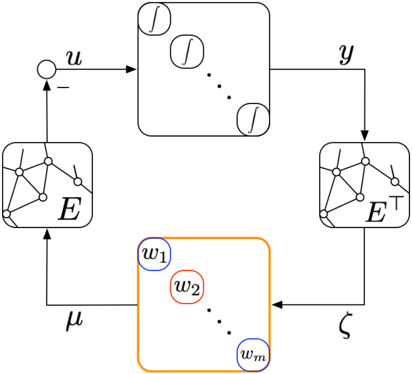
$$\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$$

- A graph can describes how agents interact with each other
- Node i acquires information from its neighbors $\mathcal{N}(i)$

- Node i has a state $x_i(t)$ and neighbor information $I_i(t) = \{x_j(t) | j \in \mathcal{N}(i)\}$
- Provides a naturally distributed dynamics over $\mathcal{G}$

$$\dot{x}_i(t) = f_i(x_i(t), I_i(t))$$

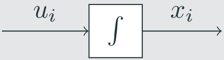
- some of the earlier works in distributed decision-making include: DeGroot ('74), Borkar and Varaiya ('82), Tsitsiklis ('84) ...



Edge weights can be positive or negative to indicate the cooperative or antagonistic interactions

Agent Dynamics

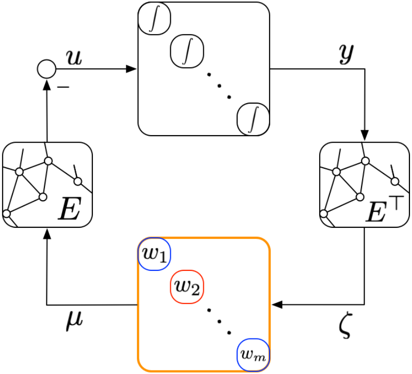
$$\dot{x}_i = u_i$$



Consensus Protocol

$$u_i = \sum_{i \sim j} w_{ij}(x_j - x_i)$$

$$\dot{x} = -L(\mathcal{G})x$$



Consensus Protocol

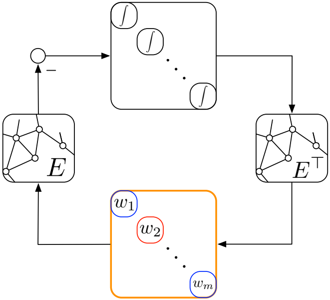
$$u_i = \sum_{i \sim j} w_{ij}(x_j - x_i)$$

$$\dot{x} = -L(\mathcal{G})x$$

Theorem

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a weighted and connected graph with positive edge weights $\mathcal{W}(k) > 0$ for $k = 1, \ldots, |\mathcal{E}|$. Then the consensus dynamics reach agreement, i.e., $\lim_{t \to \infty} x_i(t) = \beta$ for $i = 1, \ldots, |\mathcal{V}|$.

POSITIVE WEIGHTS IS A SUFFICIENT CONDITION



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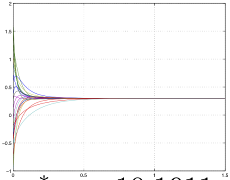
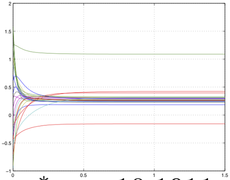


$w^* > -10.1911$



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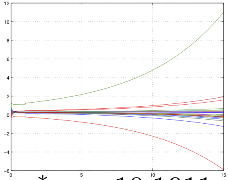
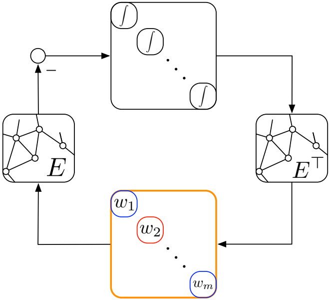
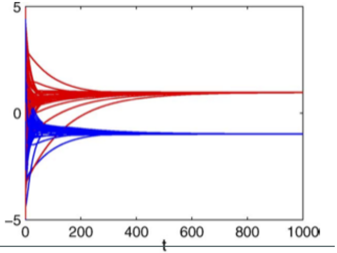


$w^* < -10.1911$

$|\mathcal{V}| = 25,\ |\mathcal{E}| = 98$ w^* is weight of predefined edge

POSITIVE WEIGHTS IS A SUFFICIENT CONDITION





Consensus Protocol

$$u_i = \sum_{i \sim j} w_{ij}(x_j - x_i)$$

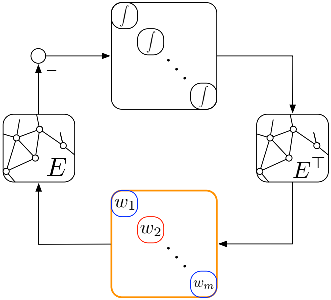
$$\dot{x} = -L(\mathcal{G})x$$

The bipartite consensus problem over signed graphs

- signed graph is structurally balanced

[1] C. Altafini, Consensus Problems on Networks with Antagonistic Interactions, IEEE Transactions on Automatic Control, 58(4):935-946, 2013.

POSITIVE WEIGHTS IS A SUFFICIENT CONDITION



Consensus Protocol

$$u_i = \sum_{i\sim j} w_{ij}(x_j - x_i)$$

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How can we understand dynamic networks over signed graphs?

Introduction

Circuit Analogies

From Linear to Nonlinear Networks

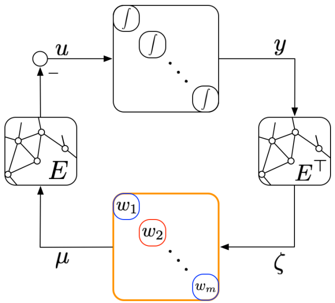
Signed Nonlinear Edges

Convergence Analysis

Conclusion

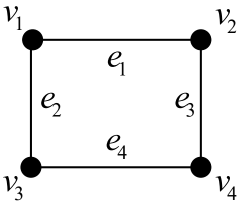
Thanks

Linear Consensus as an RC-Circuit

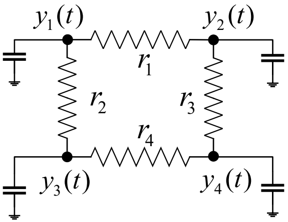


Capacitors $\Leftrightarrow$ Node Dynamics (integrators)

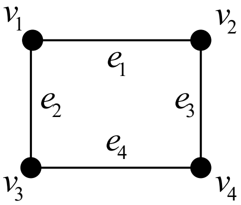
Resistors $\Leftrightarrow$ Edge Dynamics (linear gain)



$\mathcal{G}$



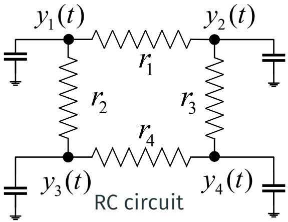
RC circuit



$\mathcal{G}$

Edge Functions:

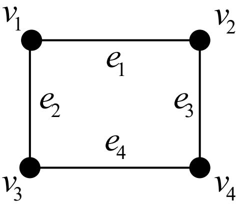
$$\mu_k = w_k \zeta_k = w_k(y_i - y_j)$$



RC circuit

Control : $\boldsymbol{u} = -EWE^T\boldsymbol{y}$

$$w_k = \frac{1}{r_k}$$



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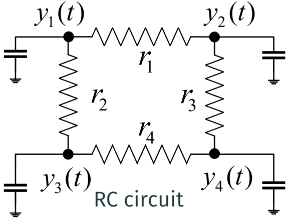
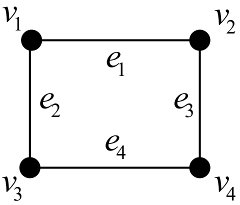


Control : $\boldsymbol{u} = -EWE^T\boldsymbol{y}$

$$w_k = \frac{1}{r_k}$$

In steady-state, $\boldsymbol{u} = 0$, and the network corresponds to a resistive circuit.



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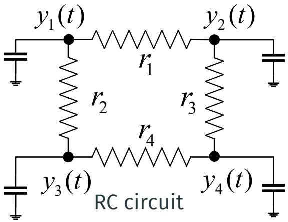


Control : $\boldsymbol{u} = -EWE^T\boldsymbol{y}$

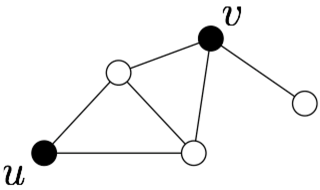
$w_k = \frac{1}{r_k}$

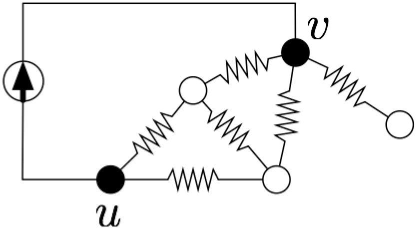
In steady-state, $\boldsymbol{u} = 0$, and the network corresponds to a resistive circuit.

What happens if we add a negative resistor?

EFFECTIVE RESISTANCE

The effective resistance between two nodes u and v is the electrical resistance measured across the nodes when the graph represents a resistive circuit.





Effective Resistance Calculation [Klein and Randić 1993]

$$r_{uv} = [L^{\dagger}(\mathcal{G})]_{uu} + 2[L^{\dagger}(\mathcal{G})]_{uv} + [L^{\dagger}(\mathcal{G})]_{vv}$$

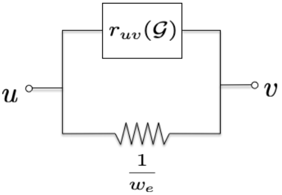
EFFECTIVE RESISTANCE AND SIGNED LINEAR NETWORKS

Theorem **[Z, Bürger CDC2014]**

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$ be a strictly positive network with edge functions $\mu_k = w_k \zeta_k$ (i.e., $w_k > 0$ for all $k \in \mathcal{E}$) and let $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}_> \cup e)$ where $e = (u, v)$ is a negative edge with $\mu_k = w_e \zeta_e$ and $w_e < 0$. Then the signed consensus network reaches agreement if and only if

$$|w_e| \leq r_{uv}^{-1}(\mathcal{G}),$$

where $r_{uv}(\mathcal{G})$ is the effective resistance in $\mathcal{G}$ between nodes u and v.



$$r_{uv}(\bar{\mathcal{G}}) = \frac{\frac{1}{w_e} r_{uv}(\mathcal{G})}{r_{uv}(\mathcal{G}) + \frac{1}{w_e}} \Rightarrow \begin{cases} > 0, & \text{cooperative} \\ < 0, & \text{antagonistic} \\ = \infty, & \text{no interaction} \end{cases}$$

A matched negative edge weights effectively creates an open circuit

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In many applications, nonlinear protocols are used to achieve the desired behavior of the network.

- **Kuramoto model**[1]

$$\mu_k = w_k \sin \zeta_k.$$

- **Finite-time consensus protocol**[2]

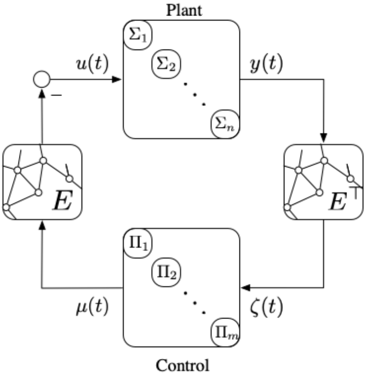
$$\mu_k = w_k \cdot \mathrm{sign}(\zeta_k) \cdot |\zeta_k|^{\alpha_k},\ 0 < \alpha_k < 1$$

Motivation

Generalize the notion of signed networks to the nonlinear case, and analyze its convergence properties.

[1] J. A. Acebrn, L. L. Bonilla, C. J. P. Vicente, et al., The Kuramoto model: A simple paradigm for synchronization phenomena, Reviews of Modern Physics, vol. 77, no. 1, pp. 137-185, 2005.

[2] L. Wang and F. Xiao, Finite-time consensus problems for networks of dynamic agents, IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 950-955, 2010.



The network is denoted by the triple $(\mathcal{G}, \Sigma, \Pi)$.

Node dynamics

$$\Sigma_i : \quad \dot{x}_i(t) = f_i(x_i(t), u_i(t)),$$
$$y_i(t) = h_i(x_i(t), u_i(t)).$$

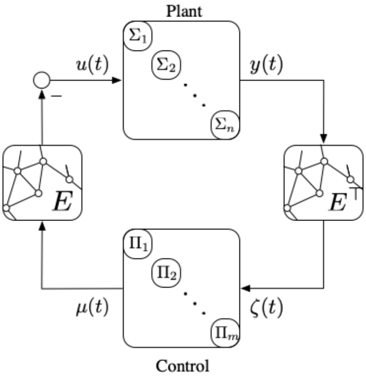
Edge functions

$$\Pi_k : \mu_k(t) = \psi_k(\zeta_k(t)),$$

"Equilibrium Set"

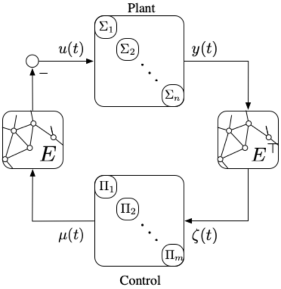
$$I_k = \{\zeta_k \,|\, \psi_k(\zeta_k) = 0\}$$

NETWORK MODEL



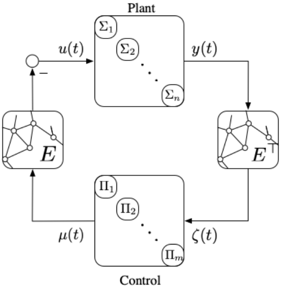
Convergence analysis on these non-linear networks relies on:

- a variant of passivity theory (Maximal Equilibrium Independent Passivity - MEIP)
- the notion of steady-state input output relations



Assumption 1

Each agent Σ_i and controller Π_e admit forced steady-state solutions.



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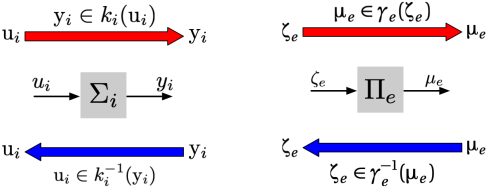
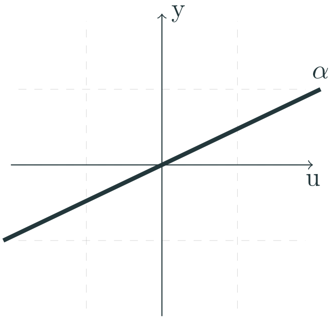
Input-Output Maps

The *steady-state input-output map* $k : \mathcal{U} \to \mathcal{Y}$ associated with Σ is the set consisting of all steady-state input-output pairs (u, y) of the system.



INPUT-OUTPUT RELATIONS

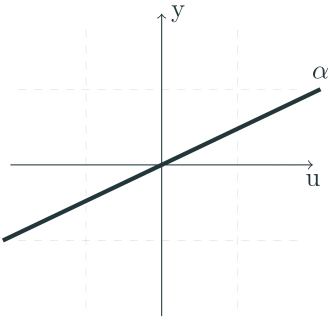
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$\Rightarrow k(\mathrm{u}) = \{\mathrm{y} \mid \underbrace{(-CA^{-1}B + D)}_{\alpha}\,\mathrm{u}\}$$



SISO and stable linear system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$\Rightarrow k(\mathrm{u}) = \{\mathrm{y} \mid \underbrace{(-CA^{-1}B + D)}_{\alpha}\,\mathrm{u}\}$$

$$\dot{x} = u$$
$$y = x$$
$$\Rightarrow k = \{(0, \mathrm{y}),\ \mathrm{y} \in \mathbb{R}\}$$



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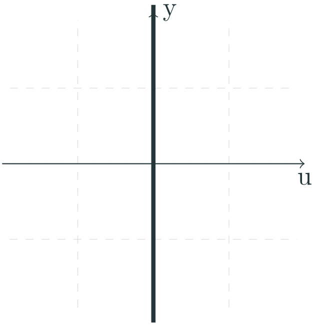
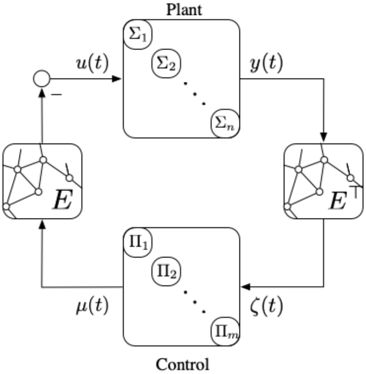


simple integrator



Convergence analysis on these non-linear networks relies on:

- a variant of passivity theory (Maximal Equilibrium Independent Passivity - MEIP)
- the notion of steady-state input output relations

Assumption 1: Feasibility of Consensus

Σ_i is MEIP, and the equilibrium input-output relations of the nodes satisfy $\boldsymbol{k}(\mathbf{0}) \cap \mathcal{N}(E^T) \neq \emptyset$.

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Passivity

A system $\eta = \pi(t, \xi)$, where ξ, η are the system input vector and system output vector, respectively, is

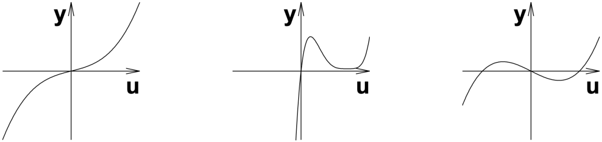
i) passive if $\xi^T \eta \geq 0$;

ii) input strictly passive if there exists $\epsilon > 0$, such that $\xi^T \eta \geq \epsilon \xi^T \xi$;

iii) active if $\xi^T \eta \leq 0$;

iv) input strictly active if there exists $\epsilon > 0$, such that $\xi^T \eta \leq -\epsilon \xi^T \xi$.

In all above cases, the inequality should hold for all (t, ξ).



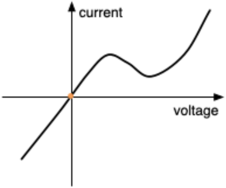
SIGNED NONLINEAR EDGES

Definition

Suppose Π_k is a map from $\mathbb{R}$ to $\mathbb{R}$ with $\psi_k(0) = 0$. Then edge k is termed

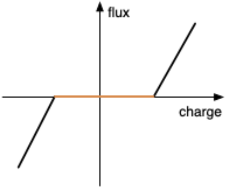
i) (strictly) positive if Π_k is (input strictly) passive;

ii) (strictly) negative if Π_k is (input strictly) active.



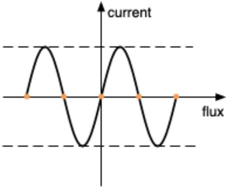
tunnel diode

(strictly) positive



memristor

positive



Josephson junction

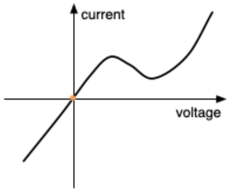
neither positive nor negative

SIGNED NONLINEAR EDGES

Definition: Signed Nonlinear Networks

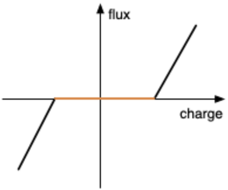
A networked system $(\mathcal{G}, \Sigma, \Pi)$ is a

- positive network if all the edges are positive, and $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$;
- strictly positive network if all the edges are strictly positive, and $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{>})$;
- signed network if not all the edges are positive.



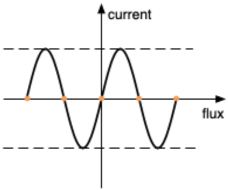
tunnel diode

(strictly) positive



memristor

positive



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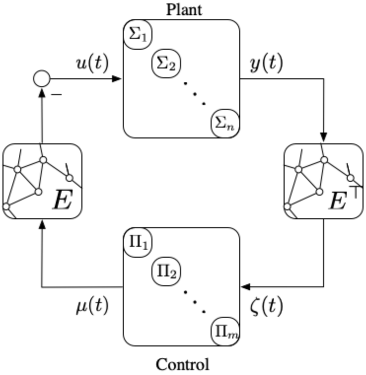

Node dynamics

$$\Sigma_i : \quad \begin{aligned} \dot{x}_i(t) &= f_i(x_i(t), u_i(t)), \\ y_i(t) &= h_i(x_i(t), u_i(t)). \end{aligned}$$

"Signed" Edge functions

$$\Pi_k : \mu_k(t) = \psi_k(\zeta_k(t)),$$

Assumption 1: Feasibility of Consensus

Σ_i is MEIP, and the equilibrium input-output relations of the nodes satisfy $\boldsymbol{k}(\mathbf{0}) \cap \mathcal{N}(E^T) \neq \emptyset$.

Theorem

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. Then $\lim_{t\to\infty} \zeta(t) = \tilde{\zeta}$ exists, and $\tilde{\zeta} \in \boldsymbol{I}$.

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PROOF

Consider the Lyapunov function $V(\boldsymbol{x}(t)) = \sum_{i=1}^{|\mathcal{V}|} S_i(x_i(t))$

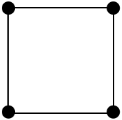
$$\dot{V} = \sum_{i=1}^{|\mathcal{V}|} \dot{S}_i \leq (\boldsymbol{u}(t) - \mathbf{0})^T (\boldsymbol{y}(t) - \tilde{\mathbf{y}}) = -\boldsymbol{\zeta}(t)^T \boldsymbol{\mu}(t) \leq 0.$$

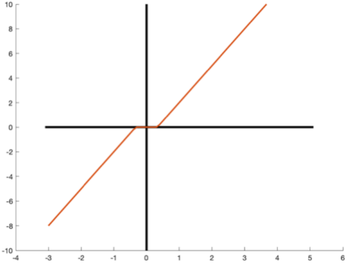
By using LaSalle's invariance principle, $\lim_{t\to\infty} \boldsymbol{\zeta}(t)^T \boldsymbol{\mu}(t) = \mathbf{0}$, meaning $\lim_{t\to\infty} \boldsymbol{\mu}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \boldsymbol{\zeta}(t) \in \boldsymbol{I}$. As a result, $\lim_{t\to\infty} \boldsymbol{u}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \boldsymbol{y}(t) = \tilde{\mathbf{y}} \in \boldsymbol{k}(\mathbf{0})$, therefore $\lim_{t\to\infty} \boldsymbol{\zeta}(t) = \tilde{\boldsymbol{\zeta}} = E^T \tilde{\mathbf{y}}$ exists.

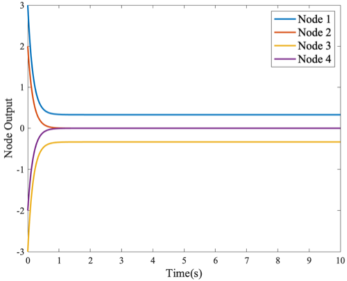
Theorem

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. Then $\lim_{t\to\infty} \zeta(t) = \tilde{\zeta}$ exists, and $\tilde{\zeta} \in \boldsymbol{I}$.

$$\mu_k = \operatorname{sign}(\zeta_k) \max\{3|\zeta_k| - 1, 0\}$$



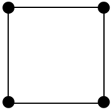




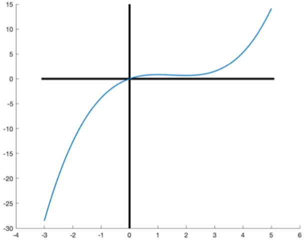
STRICTLY POSITIVE NETWORKS

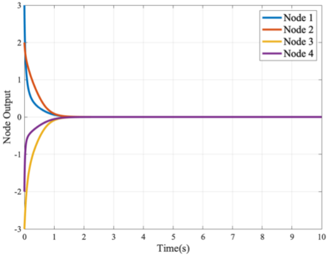
Corollary

Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$ and suppose Assumption 1 holds. Then $\lim_{t\to\infty} \zeta(t) = \mathbf{0}$, and $\lim_{t\to\infty} \boldsymbol{y}(t) = \beta\mathbf{1}, \; \beta \in \mathbb{R}$.



$$\mu_k = \frac{1}{3}\zeta_k^3 - \frac{3}{2}\zeta_k^2 + 2\zeta_k$$





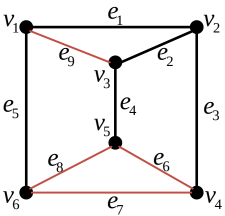
Corollary

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. If there exists a connected subgraph spanning all nodes and strictly positive edges, then

$$\lim_{t\to\infty} \boldsymbol{\zeta}(t) = \mathbf{0} \text{ and } \lim_{t\to\infty} \boldsymbol{y}(t) = \beta\mathbf{1},\ \beta \in \mathbb{R}.$$

A strictly positive spanning subgraph
is needed to guarantee consensus!

AN EXAMPLE



Edges e_k $(k = 1, \ldots, 5)$ form a spanning tree.

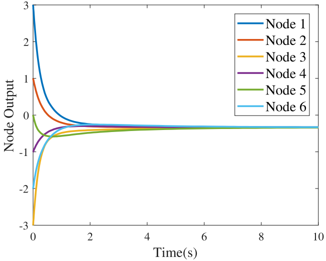
Two possible edge functions:

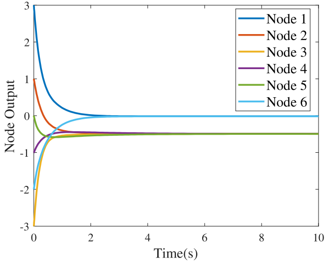
$$\mu_k(t) = \zeta_k(t), \tag{1}$$

$$\mu_k(t) = \operatorname{sign}(\zeta_k(t)) \cdot \max\{\zeta_k(t) - 1, 0\}, \tag{2}$$

Two scenarios:

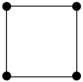
- Edges e_k $(k = 1, \ldots, 5)$ use (1), and edges e_k $(k = 6, \ldots, 9)$ use (2).
- Edges e_k $(k = 2, \ldots, 5)$ use (1), and edges e_k $(k = 1, 6, \ldots, 9)$ use (2).

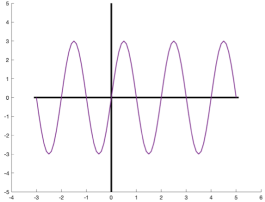


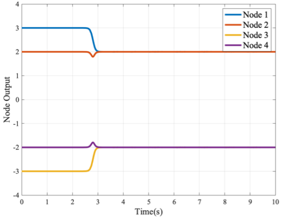


AN EXAMPLE: NON-POSITIVE NETWORKS

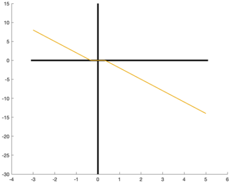
$$\mu_k = 3\sin(\pi\zeta_k)$$

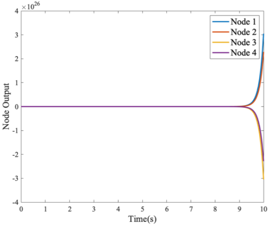


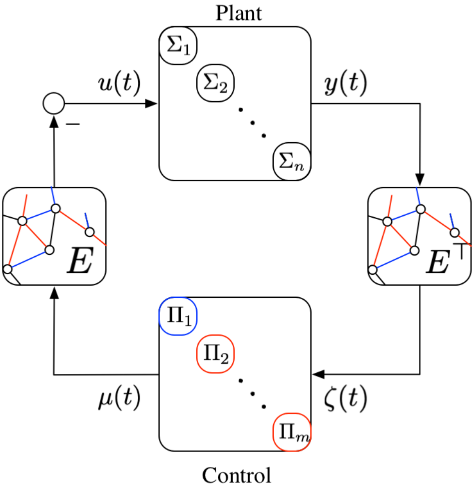




$$\mu_k = -\mathrm{sign}(\zeta_k)\max\{3|\zeta_k| - 1, 0\}$$

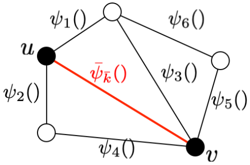






How do we analyze nonlinear networks with positive and negative edges?

Generalizations for non-linear resistors



Equivalent (non-linear) Edge Functions

Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E} \cup \bar{k})$, with $\bar{k} = (u, v)$.

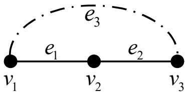
For each given $\zeta_{\bar{k}}$, if there exists a unique $(\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{\mu}})$ and some $\boldsymbol{y} \in \mathbb{R}^{|\mathcal{V}|}$ such that

$$\begin{cases} E(\bar{\mathcal{G}})^T \boldsymbol{y} & = \bar{\boldsymbol{\zeta}} \quad \text{KVL} \\ \boldsymbol{\mu} & = \Psi(\boldsymbol{\zeta}) \\ E(\bar{\mathcal{G}})\bar{\boldsymbol{\mu}} & = \mathbf{0} \quad \text{KCL} \end{cases},$$

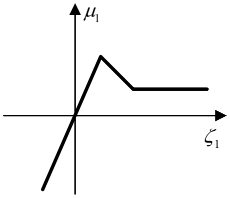
then the flow $\mu_{\bar{k}}$ on the edge $\bar{k}$ can be represented as a function of $\zeta_{\bar{k}}$, which we denote as $\mu_{\bar{k}} = -\bar{\psi}_{\bar{k}}(\zeta_{\bar{k}})$.

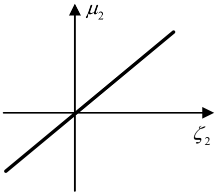
$\bar{\psi}_{\bar{k}}(\cdot) : \mathbb{R} \to \mathbb{R}$ is termed the equivalent edge function between nodes u and v.

DOES EQUIVALENT EDGE FUNCTION ALWAYS EXIST?



$$\bar{E} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$





$$\mu_1 = \begin{cases} 2\zeta_1, & \zeta_1 \in (-\infty, 1]; \\ 3 - \zeta_1, & \zeta_1 \in (1, 2]; \\ 1, & \zeta_1 \in (2, \infty). \end{cases}$$

$$\mu_2 = \zeta_2.$$

For each ζ_3, there should exist a unique μ_3 such that

$$\begin{cases} \bar{E}^T \boldsymbol{y} = \bar{\boldsymbol{\zeta}} \\ \boldsymbol{\mu} = \Psi(\boldsymbol{\zeta}) \\ \bar{E}\bar{\boldsymbol{\mu}} = \mathbf{0} \end{cases}$$

Suppose $\zeta_3 = 3$,

- if $\zeta_1 = 1$, then $\zeta_2 = 2$, $\mu_1 = \mu_2 = 2$, $\mu_3 = -2$;
- if $\zeta_1 = 2$, then $\zeta_2 = 1$, $\mu_1 = \mu_2 = 1$, $\mu_3 = -1$.

Not unique!

EXISTENCE OF EQUIVALENT EDGE FUNCTIONS

Lemma [L. O. Chua (1969)]

Consider a circuit containing only resistors and independent voltage sources. If the current-voltage function of each resistor is strictly monotonically increasing with the current tending to $\pm\infty$ as the voltage tends to $\pm\infty$, and if the circuit contains no cycles of voltage sources, then for each pair of the voltage source values, the current and voltage drop on each resistor, as well as the current on each source, are unique.

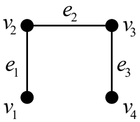
Proposition

Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$. Identify $p, q \in \mathcal{V}$ as two terminals of interest. If for each $k \in \mathcal{E}_>$, its edge function $\mu_k(t) = \psi_k(\zeta_k(t))$ is strictly monotonically increasing, and $\mu_k(t) \to \pm\infty$ as $\zeta_k(t) \to \pm\infty$, then the equivalent edge function of the two-terminal network between nodes p and q exists.

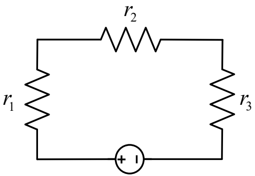
Algorithm 1 Computation of Equivalent Edge Functions

1: Obtain the corresponding resistive circuit, and add a voltage source between p and q;
2: Obtain a finite set $\{\zeta_{pq}\}$ by sampling the interval $[-N, N]$;
3: **for all** ζ_{pq} **do**
4: Set ζ_{pq} as the value of voltage source;
5: Calculate the flow on the voltage source $\bar{\mu}_{pq}$ while satisfying KVL and KCL;
6: **end for**
7: Approximate the equivalent edge function by interpolation based on $\{(\zeta_{pq}, \bar{\mu}_{pq})\}$.

EQUIVALENT EDGE FUNCTION

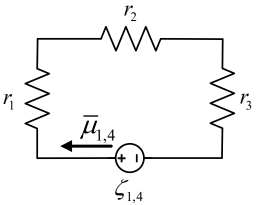


Original underlying graph.

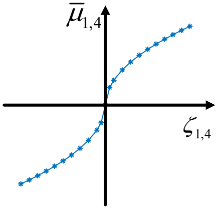


Obtain the corresponding resistive circuit, and add a voltage source.

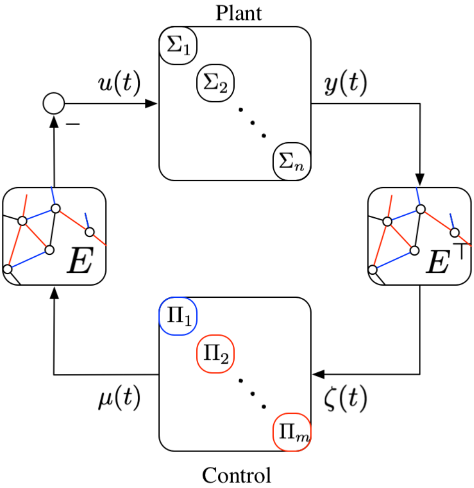
Obtain a finite set $\{\zeta_{1,4}\}$ by sampling the interval $[-N, N]$.



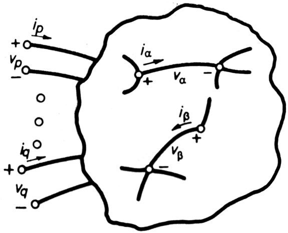
Calculating $\overline{\mu}_{1,4}$ corresponding to each $\zeta_{1,4}$



Approximate the equivalent edge function by interpolation based on $\{(\zeta_{1,4}, \overline{\mu}_{1,4})\}$.



How do we analyze networks with positive and negative edges?



Telegen's Theorem states that the instantaneous powers in all elements in a circuit sum to the power injected into the network,

$$\sum_p i_p v_p = \sum_\alpha i_\alpha v_\alpha$$

Proposition

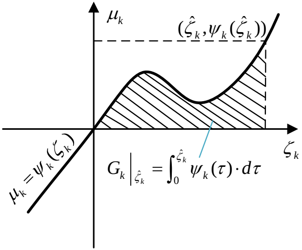
Arbitary interconnections of inductors and capacitors with passive resistors verify the energy-balance inequality

$$\int_0^t i_p^T(\tau) v_p(\tau) d\tau \geq E[\varphi_L(t), q_C(t)] - E[\varphi_L(0), q(0)].$$

COCONTENT FUNCTION

The cocontent[1] of a nonlinear resistor (edge k) when its tension ζ_k is specified, is defined as

$$G_k|_{\zeta_k} = \int_0^{\zeta_k} \psi_k(\tau) \cdot d\tau.$$



Cocontent is area under curve. Content ($G_k^*|_{\mu_k}$) is area above curve.

Total Power: $G_k|_{\zeta_k} + G_k^*|_{\mu_k} = \mu_k \zeta_k$

[1] M. Parodi and M. Storace, Linear and Nonlinear Circuits: Basic and Advanced Concepts. Cham, Switzerland: Springer International Publishing, 2018.

COCONTENT FUNCTION

The cocontent[1] of a nonlinear resistor (edge k) when its tension ζ_k is specified, is defined as

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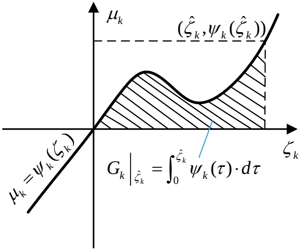
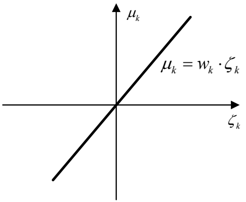


Cocontent of the network denoted as $\mathbf{G} = \sum_{k=1}^{|\mathcal{E}|} G_k|_{\zeta_k}$.

[1] M. Parodi and M. Storace, *Linear and Nonlinear Circuits: Basic and Advanced Concepts*. Cham, Switzerland: Springer International Publishing, 2018.

Example 1: A resistor with constant resistance.



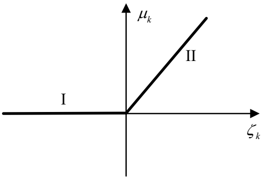
Edge function:

$$\mu_k = w_k \cdot \zeta_k.$$

Cocontent function:

$$G_k|_{\zeta_k} = \int_0^{\zeta_k} w_k \cdot \tau d\tau$$
$$= \frac{1}{2} w_k \zeta_k^2.$$

Example 2: An ideal diode.

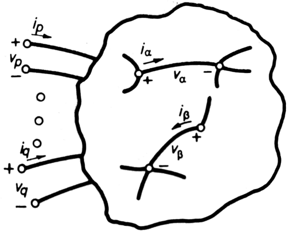


Edge function:

$$\mu_k = \begin{cases} 0, & \text{if } \zeta_k < 0. \\ w_k \cdot \zeta_k, & \text{if } \zeta_k \geq 0. \end{cases}$$

Cocontent function:

$$G_k|_{\zeta_k} = \begin{cases} 0, & \text{if } \zeta_k < 0. \\ \frac{1}{2} w_k \zeta_k^2, & \text{if } \zeta_k \geq 0. \end{cases}$$



Telegen's Theorem states that the instantaneous powers in all elements in a circuit sum to the power injected into the network,

$$\sum_p i_p v_p = \sum_\alpha i_\alpha v_\alpha$$

Proposition

Arbitary interconnections of passive capacitors with convex energy function $E_C(q_C)$, voltage-controlled resistors, and sources, satisfy the power-balance inequality

$$\int_0^t i_p^T(\tau) \frac{dv_p}{d\tau}(\tau) d\tau \geq \mathbf{G}.$$

[1] D. Jeltsema, R. Ortega, and J. Scherpen, *On Passivity and Power-Balance Inequalities of Nonlinear RLC Circuits*, IEEE Tran. Circuits and Systems 50(9):1174-1179, 2003.

Lemma

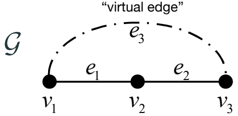
Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$, and the augmented graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}})$ obtained by adding the virtual edge $\bar{k}$ ($\bar{\mathcal{E}} = \mathcal{E}_> \cup \{\bar{k}\}$). Suppose for each $k \in \mathcal{E}_>$, $\mu_k = \psi_k(\zeta_k)$ is monotonically increasing. For any fixed $\zeta_{\bar{k}}$, if there exists $\bar{\boldsymbol{\zeta}}^0, \bar{\boldsymbol{\mu}}^0, \boldsymbol{y}^0$, such that

$$\begin{cases} E(\bar{\mathcal{G}})^T \boldsymbol{y}^0 & = \bar{\boldsymbol{\zeta}}^0 \quad \text{KVL} \\ \boldsymbol{\mu}^0 & = \Psi(\boldsymbol{\zeta}^0) \\ E(\bar{\mathcal{G}})\bar{\boldsymbol{\mu}}^0 & = \mathbf{0} \quad \text{KCL}, \end{cases}$$

then $\sum_{k=1}^{|\mathcal{E}|} G_k|_{\zeta_k}$ reaches its minimum at $(\bar{\boldsymbol{\zeta}}^0, \bar{\boldsymbol{\mu}}^0, \boldsymbol{y}^0)$.

consequence of Tellegen’s Theorem and Minimum Heat Theorem

MINIMUM OF COCONTENT FUNCTION



$$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \bar{E} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Edge Functions

$$\begin{aligned} \mu_1 &= \tfrac{1}{2}\zeta_1 \\ \mu_2 &= \zeta_2 \end{aligned}$$

Network Co-Content

$$\begin{aligned} \mathbf{G} &= \int_0^{\zeta_1} \tfrac{1}{2}\tau \cdot d\tau + \int_0^{\zeta_2} \tau \cdot d\tau \\ &= \tfrac{1}{4}\zeta_1^2 + \tfrac{1}{2}\zeta_2^2 \end{aligned}$$

Example 1

ζ_1	$= 2$	potential drop between v1-v2
ζ_2	$= 1$	potential drop between v2-v3
ζ_3	$= 3$	potential drop between v1-v3

network equations

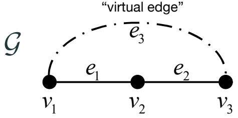
$$\bar{E}^T y = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \bar{\zeta}$$

$$\mu = \Psi(\zeta) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{E}\bar{\mu} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{G} = 1.5$$

MINIMUM OF COCONTENT FUNCTION



$$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \bar{E} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Edge Functions

$$\begin{aligned} \mu_1 &= \tfrac{1}{2}\zeta_1 \\ \mu_2 &= \zeta_2 \end{aligned}$$

Network Co-Content

$$\begin{aligned} \mathbf{G} &= \int_0^{\zeta_1} \tfrac{1}{2}\tau \cdot d\tau + \int_0^{\zeta_2} \tau \cdot d\tau \\ &= \tfrac{1}{4}\zeta_1^2 + \tfrac{1}{2}\zeta_2^2 \end{aligned}$$

Example 2

ζ_1	$= 1$	potential drop between v1-v2
ζ_2	$= 2$	potential drop between v2-v3
ζ_3	$= 3$	potential drop between v1-v3

network equations

$$\bar{E}^T y = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\bar{\mu} = \Psi(\zeta) = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

$$\bar{E}\bar{\mu} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 2 \\ \star \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\mathbf{G} \quad = 2.25 > 1.5}$$

Proposition: Minimum of Cocontent

Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$, and all edge functions are monotonically increasing. Identify $p, q \in \mathcal{V}$ as two terminals of interest, and the tension between nodes p and q is specified as ζ_{pq}. If the equivalent edge function between nodes p and q exists, then the minimum cocontent of the network system $(\mathcal{G}, \Sigma, \Pi)$ is the cocontent of the equivalent edge function between nodes p and q, denoted as $\min \mathbf{G}|_{\zeta_{pq}} = G_{pq}|_{\zeta_{pq}}$.

Establishes connection between
equivalent edge functions and cocontent

Nonlinear integrator dynamics:

$$\Sigma_i: \quad \dot{x}_i(t) = \gamma_i(u_i(t)), \quad y_i(t) = x_i(t), \quad i \in \mathcal{V}.$$

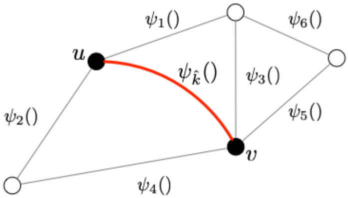
where $\gamma_i(\cdot)$ satisfies $u_i \cdot \gamma_i(u_i) \geq 0$, and equality holds if and only if $u_i = 0$.

Theorem

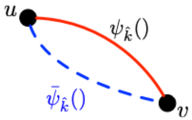
Consider a signed network system $(\mathcal{G}, \Sigma, \Pi)$ of nonlinear integrators with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose there is only one non-strictly positive edge $\hat{k}$ in $\mathcal{E}$, with edge function $\mu_{\hat{k}}(t) = \psi_{\hat{k}}(\zeta_{\hat{k}}(t))$, and $\psi_{\hat{k}}(0) = 0$. Furthermore, $\forall\, k \in \mathcal{E}_>$, $\psi_k(\cdot)$ is monotonically increasing. Identify $p, q \in \mathcal{V}$, which are connected by edge $\hat{k}$, as the two terminals of the strictly positive subnetwork system $(\mathcal{G}_>, \Sigma, \bar{\Pi})$. If the equivalent edge function $\bar{\mu}_{pq}(t) = \bar{\psi}_{pq}(\zeta_{\hat{k}}(t))$ between p and q in $(\mathcal{G}_>, \Sigma, \bar{\Pi})$ exists, and

$$(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) \geq 0$$

holds for any $\zeta_{\hat{k}}(t) \in \mathbb{R}$, then $\lim_{t\to\infty} \boldsymbol{u}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t) = 0$.



$$\Leftrightarrow$$



$$(\mu_{\hat{k}}(t) + \bar{\mu}_{\hat{k}}(t)) \cdot \zeta_{\hat{k}}(t) \geq 0$$

parallel edges must be positive!

Proof

Let $V(t) := G_{\hat{k}}|_{\zeta_{\hat{k}}(t)} + \mathbf{G}_{>}|_{\zeta_{\hat{k}}(t)}$ be the cocontent of the network system $(\mathcal{G}, \Sigma, \Pi)$, where $G_{\hat{k}}|_{\zeta_{\hat{k}}(t)}$ is the cocontent of edge $\hat{k}$, and $\mathbf{G}_{>}|_{\zeta_{\hat{k}}(t)}$ is the cocontent of the subnetwork system $(\mathcal{G}_{>}, \Sigma, \bar{\Pi})$ for a fixed value of $\zeta_{\hat{k}}(t)$. Then

$$V(t) \overset{(a)}{\geq} G_{\hat{k}}|_{\zeta_{\hat{k}}(t)} + G_{pq}|_{\zeta_{\hat{k}}(t)} \overset{(b)}{\geq} 0,$$

where (b) follows from $(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) \geq 0$.

As $\dot{G}_k|_{\zeta(t)}(t) = \mu_k(t)\dot{\zeta}_k(t), \ \forall\, k \in \mathcal{E}$, then

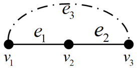
$$\dot{V}(t) = \boldsymbol{\mu}(t)^T \dot{\boldsymbol{\zeta}}(t) = -\boldsymbol{u}(t)^T \dot{\boldsymbol{y}}(t) = -\sum_{i=1}^{|\mathcal{V}|} u_i(t) \cdot \gamma_i(u_i(t)) \leq 0.$$

From LaSalle's invariance principle, $\lim_{t\to\infty} \boldsymbol{u}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t) = 0$ follows the definition of equivalent edge function.

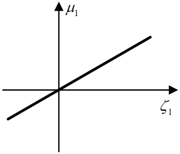
Main Point

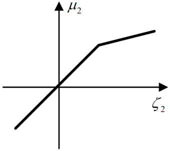
Cocontent function works as a Lyapunov Function

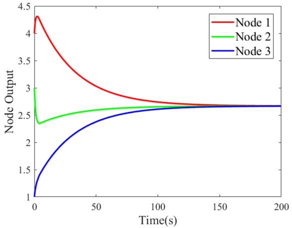
CONVERGENCE ANALYSIS OF NONLINEAR INTEGRATORS



$$\bar{E} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

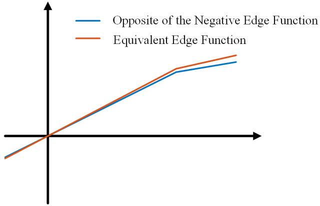


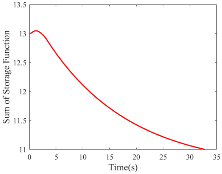


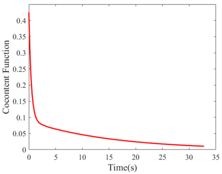


$$\mu_1 = 0.5\zeta_1 \qquad \mu_2 = \begin{cases} \zeta_2, & \text{when} \quad \zeta_2 < 1; \\ 0.2\zeta_2 + 0.8, & \text{when} \quad \zeta_2 \geq 1. \end{cases}$$

Consensus!







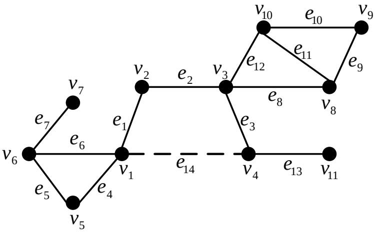
Derivative of storage function is indefinite!

Corollary: Agreement

Consider a signed network system $(\mathcal{G}, \Sigma, \Pi)$ of nonlinear integrators with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose there is only one non-strictly positive edge $\hat{k}$ in $\mathcal{E}$, with edge function $\mu_{\hat{k}}(t) = \psi_{\hat{k}}(\zeta_{\hat{k}}(t))$, and $\psi_{\hat{k}}(0) = 0$. Furthermore, $\forall\, k \in \mathcal{E}_>$, $\psi_k(\cdot)$ is monotonically increasing. Identify $p, q \in \mathcal{V}$, which are connected by edge $\hat{k}$, as the two terminals of the strictly positive subnetwork system $(\mathcal{G}_>, \Sigma, \bar{\Pi})$. If the equivalent edge function $\bar{\mu}_{pq}(t) = \bar{\psi}_{pq}(\zeta_{\hat{k}}(t))$ between p and q in $(\mathcal{G}_>, \Sigma, \bar{\Pi})$ exists, and

$$(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) \geq 0$$

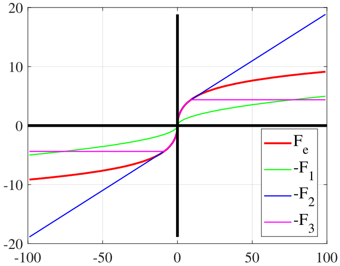
holds for any $\zeta_{\hat{k}}(t) \in \mathbb{R}$, and $(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) = 0$ if and only if $\zeta_{\hat{k}}(t) = 0$, then $\lim_{t\to\infty} \boldsymbol{\zeta}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \mathbf{y}(t) = \beta\mathbf{1},\ \beta \in \mathbb{R}$.

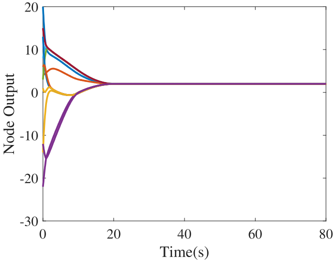


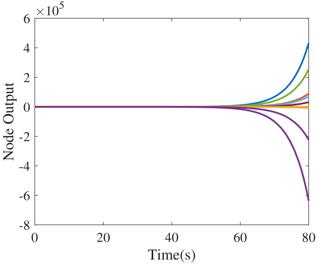
- The original network consists of 11 nodes and 13 edges, and e_{14} is introduced by an attacker.
- The original 13 edges are strictly positive, with edge functions described by

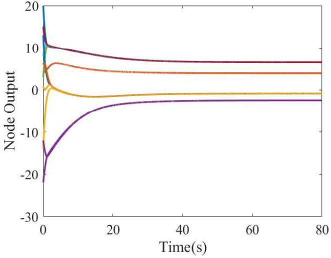
$$\mu_k(t) = w_k \cdot \operatorname{sign}(\zeta_k(t)) \cdot |\zeta_k(t)|^{\alpha_k},$$

  and $\boldsymbol{w} = (3, 2, 4, 1, 2, 1, 3, 2, 2, 1, 1, 1, 2)^T$,
  $\boldsymbol{\alpha} = (0.4, 0.5, 0.2, 0.8, 0.4, 0.4, 0.5, 0.5, 0.5, 0.6, 0.8, 0.2, 0.5)^T$.
- Only one cycle contains e_{14}, i.e., the cycle consisting of nodes v_1, v_2, v_3 and v_4, and edges e_1, e_2, e_3 and e_{14}.
- We consider three strictly negative candidate functions for e_{14}.


20
10
0
-10
-20
-100
-50
0
50
100
F_e
-F_1
-F_2
-F_3


Node Output
Time(s)
20
10
0
-10
-20
-30
0
20
40
60
80


×10^5
Node Output
Time(s)
6
4
2
0
-2
-4
-6
-8
0
20
40
60
80


Node Output
Time(s)
20
10
0
-10
-20
-30
0
20
40
60
80

CONCLUSION

- We generalize the definition of signed linear networks to graphs with nonlinear functions on the edges.
- Convergence analysis for positive and strictly positive networks. Spanning subgraphs of strictly positive edges are required for consensus.
- For networks comprised of nonlinear integrator agents, we show a connection to notions from electrical circuit theory and the equivalent circuit model to derive convergence results for networks with non-positive edges. We also propose an algorithm for constructing equivalent edge functions.

Future research directions include:

- Convergence analysis of general MEIP nodes in a signed network of more than one non-strictly positive edges.
- Real-world applications.

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Signed Linear Networks

D. Zelazo and M. Bürger, *On the definiteness of the weighted Laplacian and its connection to effective resistance*, IEEE Conference on Decision and Control, 2014.

D. Zelazo and M. Bürger, *On the robustness of uncertain consensus networks*, IEEE Control of Network Systems, 4(2):170–178, 2017.

Signed Nonlinear Networks

H. Chen, D. Zelazo, X. Wang, L. Shen, *Convergence analysis of signed nonlinear networks*, submitted to IEEE Control of Network Systems, 2018 (arXiv preprint, arXiv:1810:11274).

THANK YOU!

Acknowledgements

Dr. Mathias Bürger

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at Robert Bosch GmbH

הקרן הלאומית למדע
ISRAEL SCIENCE FOUNDATION

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