A PASSIVITY ANALYSIS FOR NONLINEAR CONSENSUS ON DIGRAPHS

2025 IEEE CONFERENCE ON DECISION AND CONTROL

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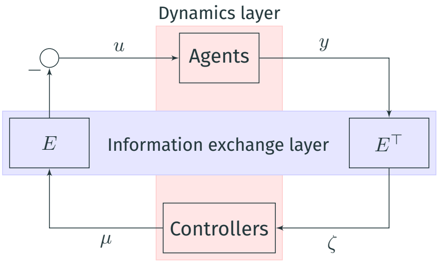
December 12, 2025

THE STEPHEN B. KLEIN FACULTY OF AEROSPACE ENGINEERING

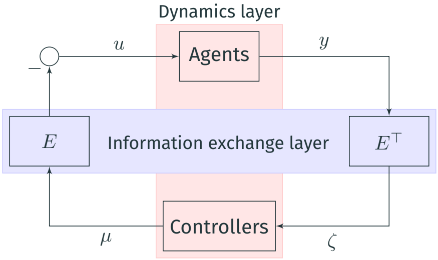
CONNECT LAB
Cooperative Networks and Controls



Diffusively-coupled networks

Why use Passivity Theory?

- Natural and powerful for undirected networks
  - Decouples dynamics and network topologies
  - Convergence, consensus



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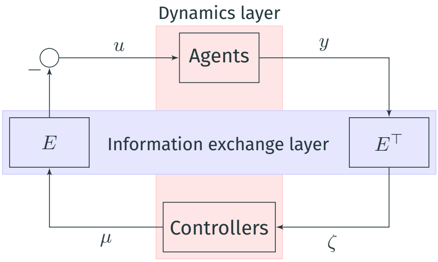

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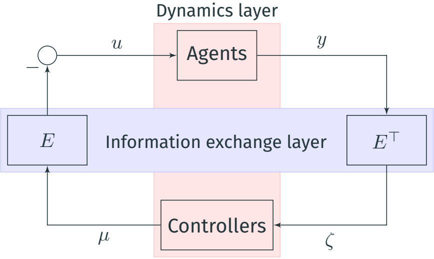

Diffusively-coupled networks

Why use Passivity Theory?

- Natural and powerful for undirected networks
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- **Limitations** in **directed** networks
  - **passivity** $\rightarrow$ **directed networks?**

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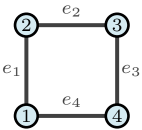
Multi-agent network: A group of SISO agents Σ_i interact over $\mathcal{G}$

$$\Sigma_i : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h_i(x_i, u_i) \end{cases}, i \in [1, n]$$



Undirected Graph $\mathcal{G}$

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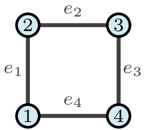

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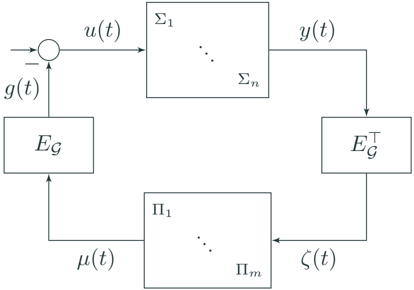

Diffusively-coupled, **Undirected** $(\Sigma, \Pi, \mathcal{G})$

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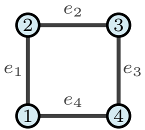

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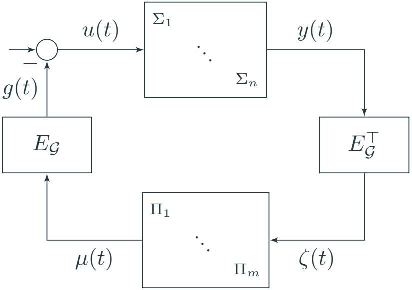

Incidence matrix $E_\mathcal{G} \in \mathbb{R}^{n \times m}$

$$E_\mathcal{G} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Diffusively-coupled, **Undirected** $(\Sigma, \Pi, \mathcal{G})$

Multi-agent network: A group of SISO agents Σ_i interact over $\mathcal{D}$

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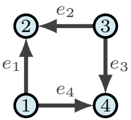

Directed Graph $\mathcal{D}$ (Digraph)

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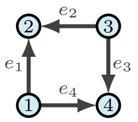

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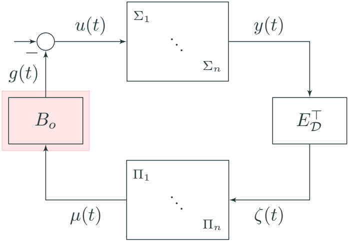

Directed Networks

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$$B_o = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

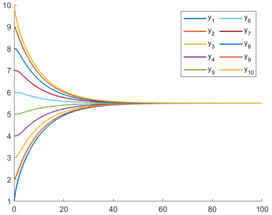
Output consensus problem:

Design distributed Π_k's, such that

$$\lim_{t\to\infty} (y_i(t) - y_j(t)) = 0, \ \forall i, j$$

$$\Leftrightarrow \lim_{t\to\infty} y(t) \in S$$

Agreement space: $S = \mathrm{span}(\mathbb{1})$



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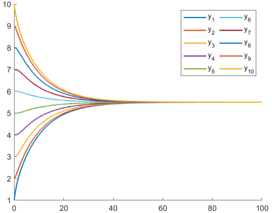
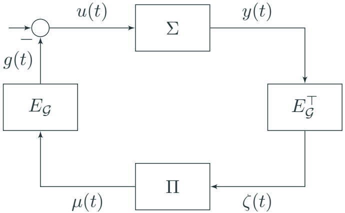


Passivity



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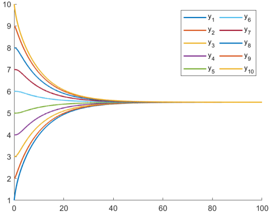

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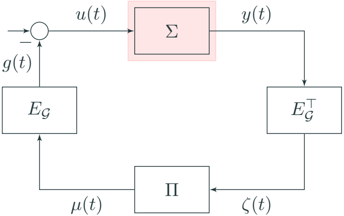

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- Passive Σ:

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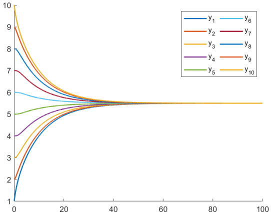

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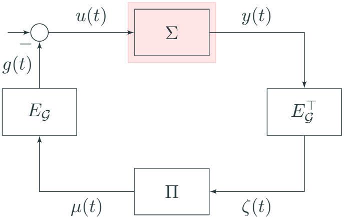

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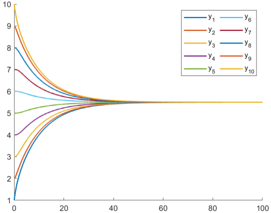

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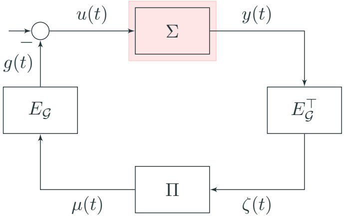

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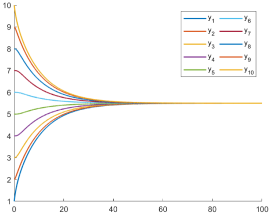

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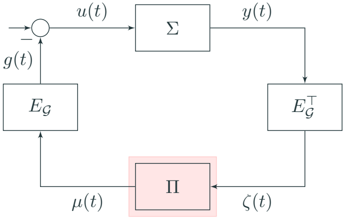

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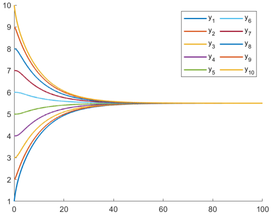

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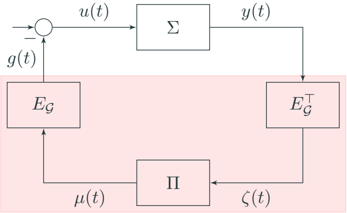

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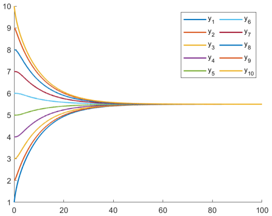

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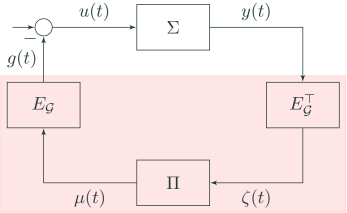

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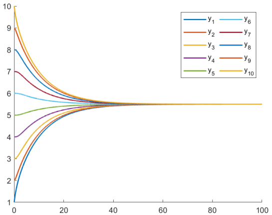

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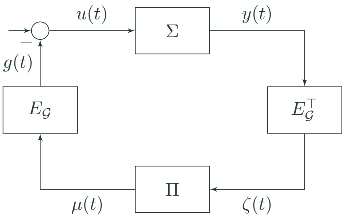

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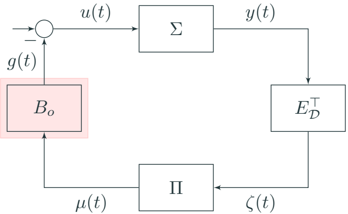
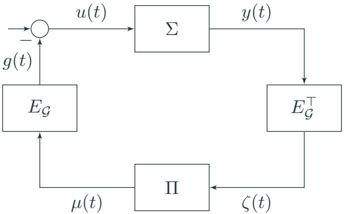
- **Passivity Analysis** $\checkmark$



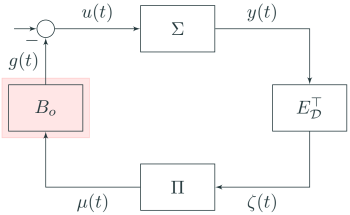
Directed Networks



Undirected $(\Sigma, \Pi, \mathcal{G})$
Passivity analysis ✓

Passivity for Directed Networks

LIMITATION OF PASSIVITY FOR DIRECTED NETWORKS



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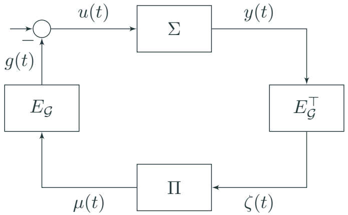
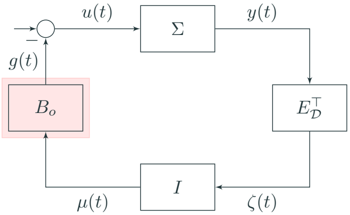


Undirected $(\Sigma, \Pi, \mathcal{G})$

Passivity analysis ✓

Passivity for Directed Networks

- $B_o \Pi E_{\mathcal{G}}^\top$: hard to conclude passivity



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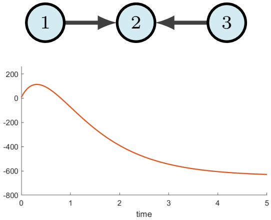
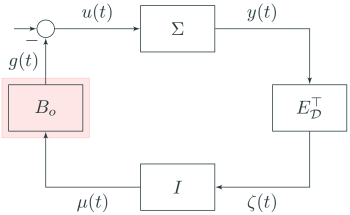


$\int_0^t g(t)^\top y(t)\mathrm{d}t \ngeq 0, \forall t$

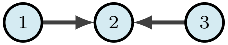
Passivity for Directed Networks

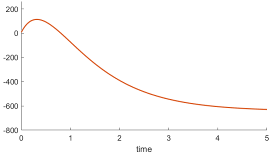
- $B_o \Pi E_{\mathcal{G}}^{\top}$: hard to conclude passivity
- $B_o I E_{\mathcal{G}}^{\top}$: linear consensus protocol for directed networks

LIMITATION OF PASSIVITY FOR DIRECTED NETWORKS



Directed Networks



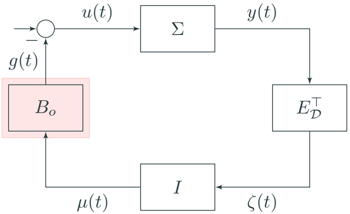


$\int_0^t g(t)^\top y(t)\mathrm{d}t \not\geq 0, \forall t$

Passivity for Directed Networks

- $B_o \Pi E_{\mathcal{G}}^{\top}$: hard to conclude passivity
- $B_o I E_{\mathcal{G}}^{\top}$: linear consensus protocol for directed networks
  - not passive in general

LIMITATION OF PASSIVITY FOR DIRECTED NETWORKS



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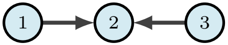

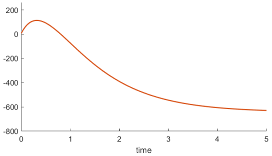

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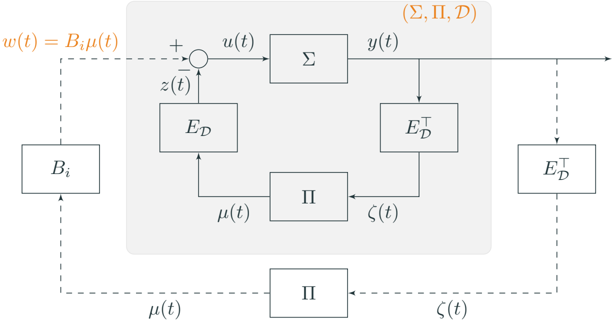
Passivity for Directed Networks

- $B_o \Pi E_{\mathcal{G}}^\top$: hard to conclude passivity
- $B_o I E_{\mathcal{G}}^\top$: linear consensus protocol for directed networks
    - not passive in general
    - Passive only when the underlying graph is **balanced**

[1] Yue, F.-Y. and Zelazo, D. "A passivity analysis for nonlinear consensus on balanced digraphs." EJC (ECC25 SI).

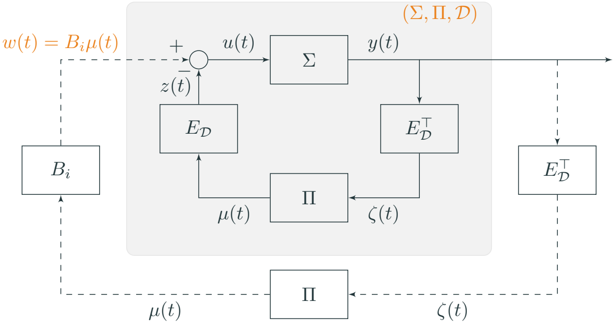
$E_{\mathcal{D}} = B_o + B_i$



Directed Networks $(\Sigma, \Pi, \mathcal{D}, w)$

- "External" input: $w(t) = B_i\mu(t)$
- Agent input: $u(t) = -B_o\mu(t)$
- Controller input: $\zeta(t) = E_{\mathcal{D}}^\top y(t)$

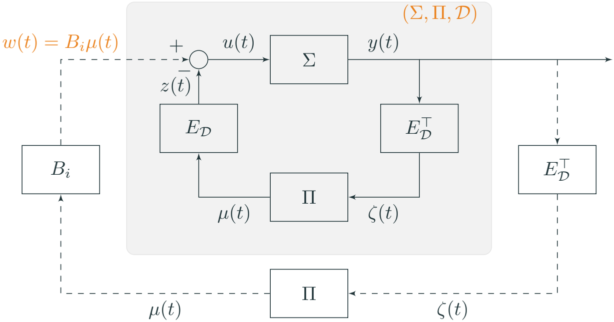
LOOP DECOMPOSITION AND OUTPUT CONSENSUS



Directed Networks $(\Sigma, \Pi, \mathcal{D}, w)$

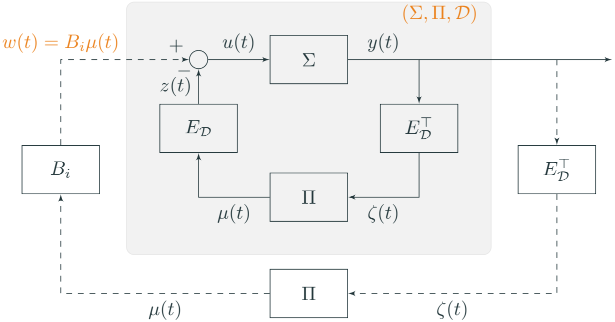
- Output Consensus: $\lim_{t\to\infty} y(t) \in S = \mathrm{span}(\mathbb{1})$

LOOP DECOMPOSITION AND OUTPUT CONSENSUS



Directed Networks $(\Sigma, \Pi, \mathcal{D}, w)$

- Output Consensus: $\lim_{t\to\infty} y(t) \in S = \mathrm{span}(\mathbb{1}) \quad \Rightarrow \lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0$



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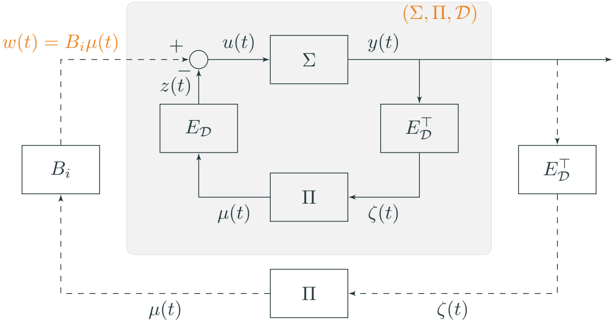

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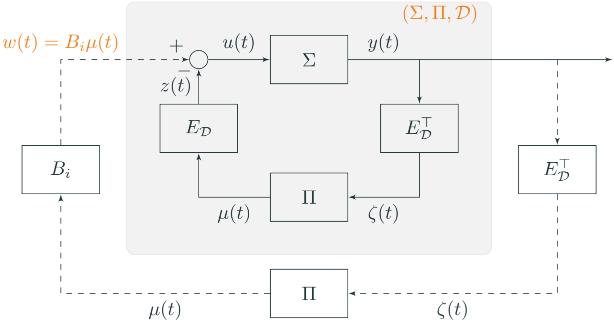

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- Passivity to **Passivity relations**
  - forward path $u^\top \mathrm{Proj}_{S^\perp}(y) \geq \dot{Q} - a\|\mu\|_2^2 - b\| \mathrm{Proj}_{S^\perp}(y)\|_2^2$
  - feedback path $z^\top \mathrm{Proj}_{S^\perp}(y) \geq \dot{W} + c\|\mu\|_2^2 + d\| \mathrm{Proj}_{S^\perp}(y)\|_2^2$

$$\Lambda : \ \dot{x}(t) = f(x(t), u(t)), y(t) = h(x(t)), \quad f : (\mathbb{R}^n, \mathbb{R}^p) \to \mathbb{R}^n, \ h : \mathbb{R}^n \to \mathbb{R}^p$$

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Recall **Passivity**

- Storage Function $V: \mathbb{R}^n \to \mathbb{R}$
  $(1)\ V(x) \geq 0$; $(2)\ V(0) = 0$

(Input-output) Passive: $u^\top(t)y(t) \geq \dot{V}(x(t)) + \delta\|u(t)\|_2^2 + \varepsilon\|y(t)\|_2^2, \ \delta\varepsilon < \frac{1}{4}, \ \forall t$

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PASSIVITY W.R.T. AGREEMENT SPACE

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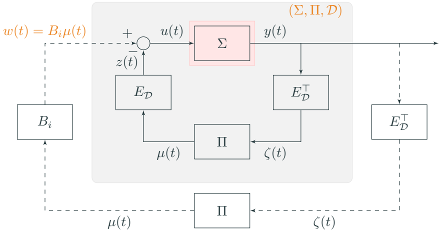
Definition: Passivity w.r.t. S

The system Λ is said to be S**-passive**, if there exist a **constrained** storage function Q and $\varepsilon\delta < \frac{1}{4}$, such that,

$$u^\top(t)\, \mathrm{Proj}_{S^\perp}(y(t)) \geq \dot{Q}(x(t)) + \varepsilon \|\mathrm{Proj}_{S^\perp}(y(t))\|_2^2 + \delta \|u(t)\|_2^2, \ \forall t$$



- Integrator-like agents

$$\Sigma : \begin{cases} \dot{x} = u \\ y = h(x) \end{cases}$$

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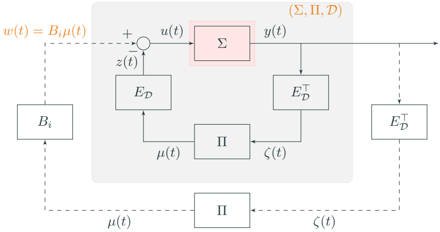

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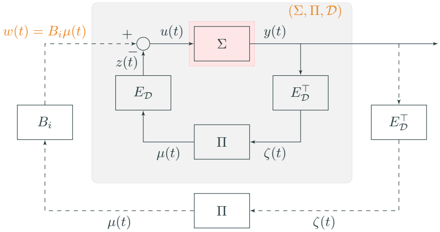

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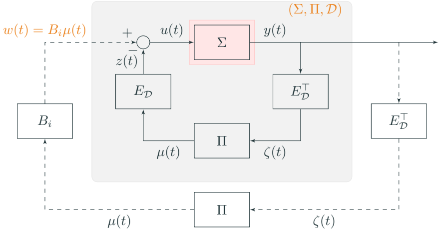

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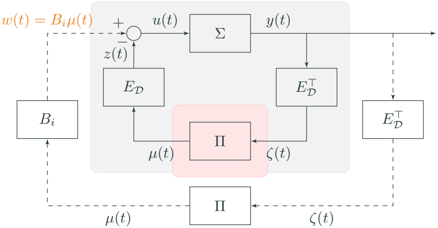
Proposition: Forward Path S-Passivity

Assume that h_i's are continuously differentiable, monotonically passive, and have bounded derivatives. Let $M = \max(1, |1-m|)$. Then,

$$u^\top \mathrm{Proj}_{S^\perp}(y) \geq \dot{Q}(x) - \frac{M}{2}K\|\mu\|_2^2 - \frac{M}{2}\|\mathrm{Proj}_{S^\perp}(y)\|_2^2,$$

with the S-constrained storage function $Q(x)$.



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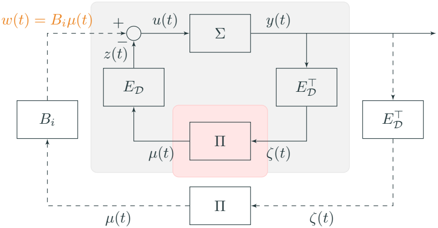

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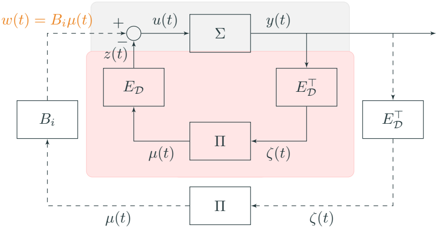
Input-output passive Π + Symmetric $E_{\mathcal{D}}\Pi E_{\mathcal{D}}^\top$:

- $z^\top y = \mu^\top \zeta \geq \dot{W} + \alpha\|\mu\|_2^2 + \gamma\|\zeta\|_2^2, \;\; \alpha, \gamma > 0, \;\; \alpha\gamma < \frac{1}{4}$



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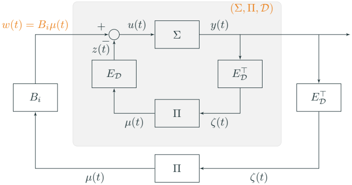
Proposition: Inner-Feedback Path Passivity Relation

Assume that the controllers Π are IOP-(γ, α). Then, it follows that,

$$\mu^\top \zeta = \mu^\top E^\top y = z^\top \operatorname{Proj}_{S^\perp}(y) \geq \dot{W} + \alpha\|\mu\|_2^2 + \gamma\lambda_2\|\operatorname{Proj}_{S^\perp}(y)\|_2^2$$

where λ_2 denotes the second smallest eigenvalue of $L = E_{\mathcal{D}}E_{\mathcal{D}}^\top$.



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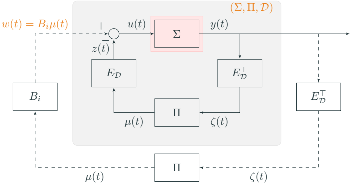

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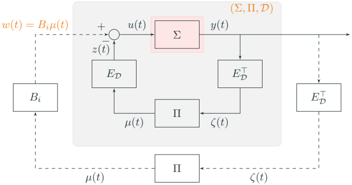

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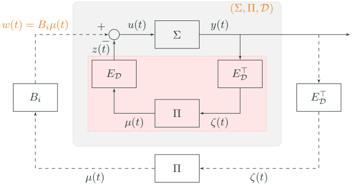
COMPENSATION THEOREM: PASSIVITY TO OUTPUT CONSENSUS



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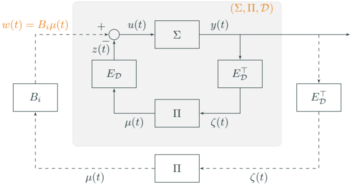

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- **Compensation + Barbalat's Lemma** $\quad \Rightarrow \lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0$

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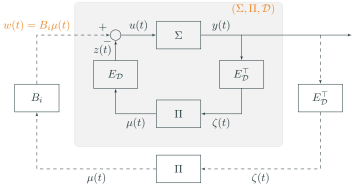

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$$u^{\top} \mathrm{Proj}_{S^{\perp}}(y) \geq \dot{V} - \tfrac{1}{2}K\|\mu\|_2^2 - \tfrac{1}{2}\|y\|_2^2$$

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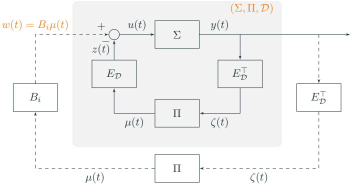

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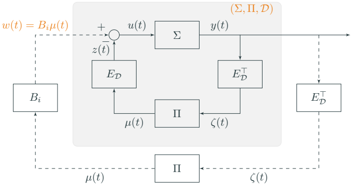
COMPENSATION THEOREM: PASSIVITY TO OUTPUT CONSENSUS



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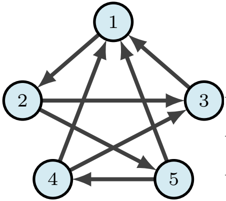
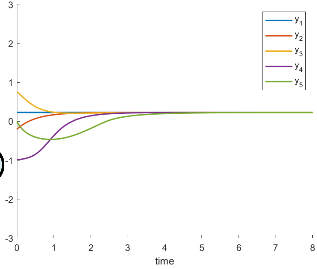
- **Compensation + Barbalat's Lemma** $\quad \Rightarrow \lim_{t\to\infty} \mathrm{Proj}_{S^\perp}(y(t)) = 0$
- Output strict passivity + balanced digraphs $\quad \Rightarrow$ Stabilization

- Systems

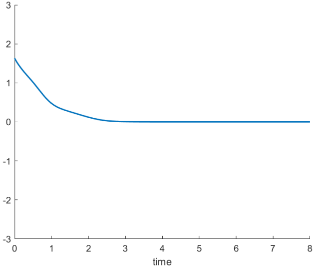
  $\Sigma^o : \dot{x}(t) = u(t), \;\; y(t) = [x_1(t), x_2(t), \tanh(x_3(t)), \tanh(x_4(t)), \frac{x_5(t)}{1+|x_5(t)|}]^\top$

  $\Pi : \mu(t) = 2\zeta(t)$
- Parameters
  - Constrained storage function: $Q(x) = \frac{1}{2}h^\top(x)(I - \frac{1}{|\mathcal{V}|}\mathbb{1}\mathbb{1}^\top)h(x)$
  - Algebraic connectivity: $\lambda_2 = 3$
  - Maximal out-degree: $\max(D_o) = 2$





Outputs of agents



Evolution of $Q(x(t))$

Summary:

- Limitations of the Compensation idea
- Compensation Theorem: Constrained storage functions, passivity w.r.t. agreement space.
- A passivity-based analysis for integrator-like agents that interact over digraphs.

Future work:

- Formal definition for Passivity w.r.t Submanifolds

Thank-You!

THE STEPHEN B. KLEIN FACULTY OF AEROSPACE ENGINEERING

CONNECT LAB
Cooperative Networks and Controls