SIGNED NONLINEAR NETWORKS

A PASSIVITY AND ELECTRICAL CIRCUIT THEORY APPROACH

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DEPARTMENT OF AEROSPACE ENGINEERING
TECHNION
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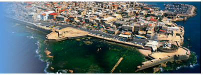
1 – 4 JULY 2019 AKKO, ISRAEL



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The 27th Mediterranean Conference on Control and Automation (MED 2019) will be held on the 1-4 of July 2019 in Akko, Israel. Akko is situated on the Phoenician northern part of the Mediterranean coast of Israel, with an exceptional history and rich cultural heritage, spanning over 4,000 years. It has been designated by UNESCO as a World Heritage site. MED 2019 will include tutorials and workshops, a technical program of presentations, keynote lectures and social events. It offers a great opportunity for academics, researchers and industry working in control and automation to network together, present research progress and address new challenges. The conference will include a wide range of topics on systems, automation, robotics and control including theory, related hardware, software and communication technologies, as well as applications.

TOPICS OF INTEREST

Adaptive Control
Aerospace Control
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Nonlinear Control
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Optimization
Power Systems
Predictive Control
Process Control
Prognostics & Diagnostics
Real-Time Control
Robotics Applications
Robust Control
Sampled-Data Control
Smart Grids & Smart Cities
Societal Challenges
Soft Computing
System Identification
Uncertain Systems
Virtual/Augmented Reality

IMPORTANT DATES

22 January 2019: Invited sessions, tutorial proposals, contributed papers due.

18 April 2019: Notification of acceptance/rejection.

15 May 2019: Final submissions due.

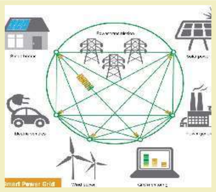
SUBMISSION DETAILS

PAPERS: Papers must be submitted electronically by January 22, 2019. The paper format must follow IEEE paper submission rules: 2 column layout, 10 pt. Times New Roman font. The maximum number of pages per paper is 6. Additional pages, up to a maximum 2, will be charged. All papers will be peer reviewed and acceptance notified by April 18, 2019. Accepted papers are to be uploaded electronically by May 15, 2019. All submissions are via **https://controls.papercept.net**.

INVITED SESSIONS: A statement describing the motivation and relevance of the proposed session, invited paper titles and author names must be submitted electronically for review by January 22, 2019. Authors must submit electronically full versions of the invited papers marked as "Invited Session Paper".

TUTORIALS/WORKSHOPS: Proposals for tutorials or workshops should contain the title of the session, the list of speakers, and extended summaries (2000 words) of the presentations. Proposals for review must be sent to the Tutorial and Workshop Chair by e-mail (giuseppe.notarstefano@unibo.it) by January 22, 2019.

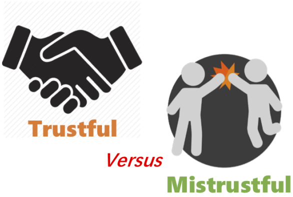
For information visit the website: **https://med19.technion.ac.il** or write to: **medcon19@gmail.com**



Networks of dynamical systems are one of the enabling technologies of the future.

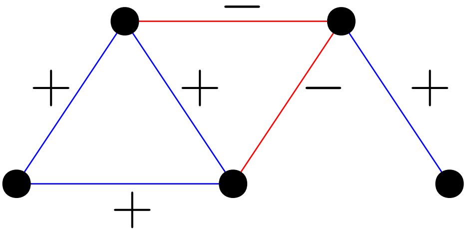
In many *biological* and *social* networks, there are generally **two** kinds of interactions:

- **trustful** versus **distrustful**
- **cooperative** versus **competitive**



COOPERATIVE V. ANTAGONISTIC

Signed Networks provide an abstraction for modeling cooperative or antagonistic interactions.



(+) - *positive* edges
(-) - *negative* edges

$$\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$$

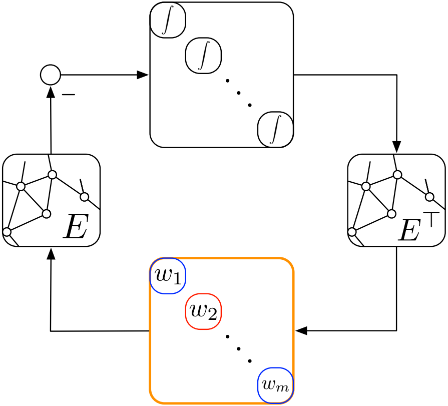
$$\mathcal{W} : \mathcal{E} \to \mathbb{R}$$

$$\mathcal{E}_+ = \{e \in \mathcal{E} \;:\; \mathcal{W}(e) > 0\}$$

$$\mathcal{E}_- = \{e \in \mathcal{E} \;:\; \mathcal{W}(e) < 0\}$$

$$\mathcal{G} = \mathcal{G}_+ \cup \mathcal{G}_-$$

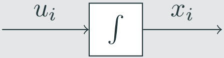
$$\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$$



Edge weights can be positive or negative to indicate the cooperative or antagonistic interactions

Agent Dynamics

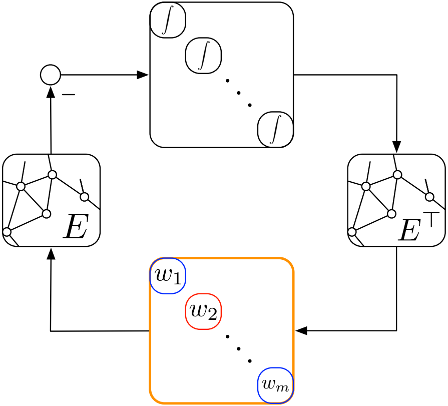
$$\dot{x}_i = u_i$$



Consensus Protocol

$$u_i = \sum_{i \sim j} w_{ij}(x_j - x_i)$$

$$\dot{x} = -L(\mathcal{G})x$$



Consensus Protocol

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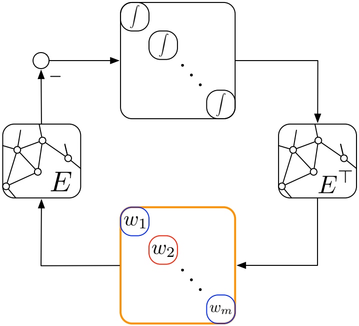
Theorem

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$ be a weighted and connected graph with positive edge weights $\mathcal{W}(k) > 0$ for $k = 1, \ldots, |\mathcal{E}|$. Then the consensus dynamics reach agreement, i.e., $\lim_{t \to \infty} x_i(t) = \beta$ for $i = 1, \ldots, |\mathcal{V}|$.

AGREEMENT IS A SUFFICIENT CONDITION



Consensus Protocol

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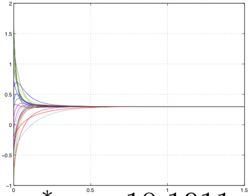
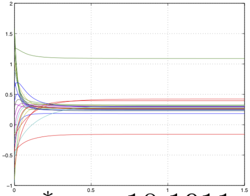


$w^* > -10.1911$



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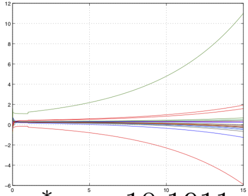
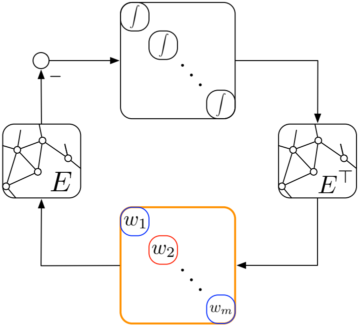


$w^* < -10.1911$

$|\mathcal{V}| = 25$, $|\mathcal{E}| = 98$ w^* is weight of predefined edge

AGREEMENT IS A SUFFICIENT CONDITION



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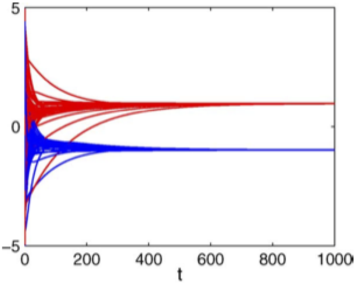


[1] C. Altafini, Consensus Problems on Networks with Antagonistic Interactions, IEEE Transactions on Automatic Control, 58(4):935-946, 2013.

In many applications, nonlinear protocols are used to achieve the desired behavior of the network.

- **Kuramoto model**[1]

$$\mu_k = w_k \sin \zeta_k .$$

- **Finite-time consensus protocol**[2]

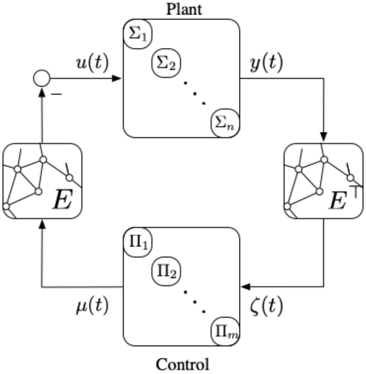
$$\mu_k = w_k \cdot \mathrm{sign}(\zeta_k) \cdot |\zeta_k|^{\alpha_k},\ 0 < \alpha_k < 1$$

Motivation

Generalize the notion of signed networks to the nonlinear case, and analyze its convergence properties.

[1] J. A. Acebrn, L. L. Bonilla, C. J. P. Vicente, et al., The Kuramoto model: A simple paradigm for synchronization phenomena, Reviews of Modern Physics, vol. 77, no. 1, pp. 137-185, 2005.

[2] L. Wang and F. Xiao, Finite-time consensus problems for networks of dynamic agents, IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 950-955, 2010.



The network is denoted by the triple $(\mathcal{G}, \Sigma, \Pi)$.

Node dynamics

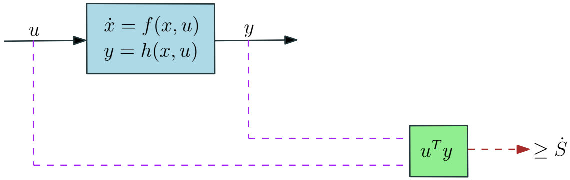
$$\Sigma_i : \quad \dot{x}_i(t) = f_i(x_i(t), u_i(t)),$$
$$y_i(t) = h_i(x_i(t), u_i(t)).$$

Edge functions

$$\Pi_k : \mu_k(t) = \psi_k(\zeta_k(t)),$$

"Equilibrium Set"

$$I_k = \{\zeta_k \,|\, \psi_k(\zeta_k) = 0\}$$



$$\begin{aligned} 0 &= f(0,0) \\ 0 &= h(0,0) \end{aligned}$$

Definition [Khalil 2002]

A system is passive if there exists a C^1 storage function $S(x)$ such that

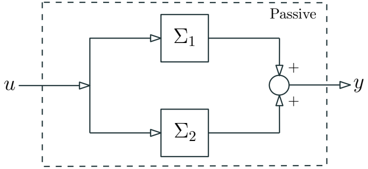
$$u^T y \geq \dot{S} = \frac{\partial S}{\partial x} f(x,u), \quad \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^p$$

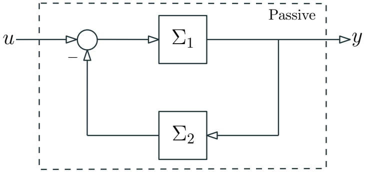
Moreover, it is said to be

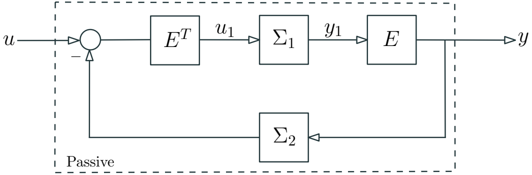
- Input-strictly passive if $\dot{S} \leq u^T y - u^T \phi(u)$ and $u^T \phi(u) > 0, \forall u \neq 0$
- Output-strictly passive if $\dot{S} \leq u^T y - y^T \rho(y)$ and $y^T \rho(y) > 0, \forall y \neq 0$

INTERCONNECTION OF PASSIVE SYSTEMS

- Parallel Interconnection
- Negative Feedback Interconnection
- Symmetric Interconnection







Theorem [Arcak, 2007]

Consider the network system $(\Sigma, \Pi, \mathcal{G})$ comprised of SISO agents and controllers. Suppose that there are vectors $\mathrm{u_i}, \mathrm{y_i}, \zeta_e$ and μ_e such that

i) the systems Σ_i are output strictly-passive with respect to $\mathrm{u_i}$ and $\mathrm{y_i}$;
ii) the systems Π_e are passive with respect to ζ_e and μ_e;
iii) the vectors $\mathrm{u}, \mathrm{y}, \zeta$ and μ satisfy $\mathrm{u} = -\mathcal{E}\mu$ and $\zeta = \mathcal{E}^T \mathrm{y}$.

Then the output vector $y(t)$ converges to y as $t \to \infty$.[1]

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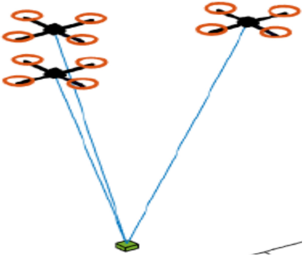
Then the output vector $y(t)$ converges to y as $t \to \infty$.[1]

- requires passivity w.r.t. to specific equilibrium configuration

PASSIVITY W.R.T. FORCED EQUILIBRIUM POINTS

Large-scale Networked Systems



- Not feasible to calculate the equilibrium point for the overall network
- Operate the network at multiple desired equilibrium points (formation of UAVs carrying a suspension load)

Passivity w.r.t. forced equilibra (u, y)

$$\frac{d}{dt} S(x(t)) \leq (u - \mathrm{u})^T (y - \mathrm{y})$$

Incremental Passivity: A close concept however restricted as passivation inequality must be satisfy along <u>any two arbitrary trajectories</u>

Equilibrium Independent Passivity: Requires passivity inequality to hold between <u>any trajectory and forced equilibrium point</u>

STEADY-STATE INPUT-OUTPUT MAPS

Assumption 1

Each agent Σ_i and controller Π_e admit forced steady-state solutions.

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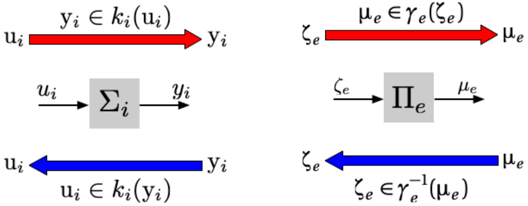
Each agent Σ_i and controller Π_e admit forced steady-state solutions.

Input-Output Maps

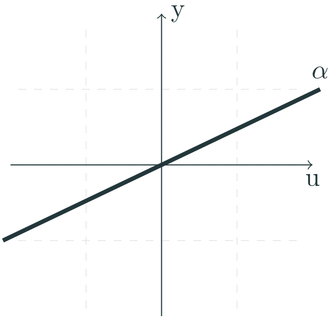
The *steady-state input-output map* $k : \mathcal{U} \to \mathcal{Y}$ associated with Σ is the set consisting of all steady-state input-output pairs (u, y) of the system.



$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\Rightarrow k(\mathrm{u}) = \{\mathrm{y} \mid \underbrace{(-CA^{-1}B + D)}_{\alpha}\,\mathrm{u}\}$$



SISO and stable linear system

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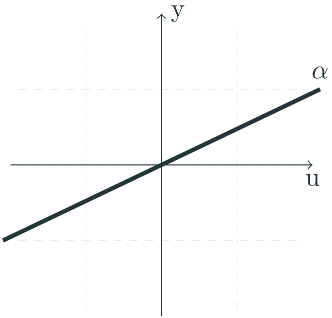
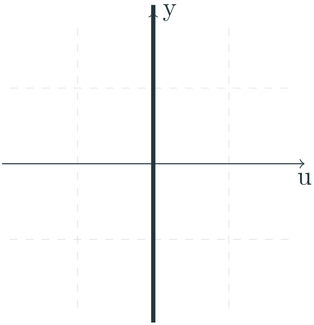


SISO and stable linear system

$$\dot{x} = u$$
$$y = x$$
$$\Rightarrow k = \{(0, \mathrm{y}),\ \mathrm{y} \in \mathbb{R}\}$$



simple integrator

MEIP [Bürger, Z, Allgöwer 2016]

A dynamical SISO system Σ is *maximal equilibrium independent* passive if the following conditions hold:

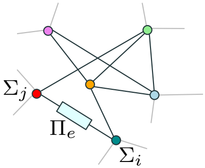
- The system Σ is passive with respect to any steady-state $(\bar{u}, \bar{y}) \in k$.
- The relation k is maximally monotone, that is, $(\bar{u}_1, \bar{y}_1), (\bar{u}_2, \bar{y}_2) \in k$ then either $(\bar{u}_1 \leq \bar{u}_2$ and $\bar{y}_1 \leq \bar{y}_2)$, or $(\bar{u}_1 \geq \bar{u}_2$ and $\bar{y}_1 \geq \bar{y}_2)$, and k is not contained in any larger monotone relation.

Definition

A map F on $\mathbb{R}^n$ is called monotone if

$$(F(x) - F(y))^T (x - y) \geq 0, \; \forall x, y \in \mathbb{R}^n,$$

and it is called maximally monotone if there is no monotone operator that properly contains it.



Theorem [Rockafellar *Convex Analysis*]

The subdifferential for the convex functions on $\mathbb{R}$ are the maximal monotone relations from $\mathbb{R}$ to $\mathbb{R}$.

Main Points

- One can associate a convex function $K_i : \mathbb{R} \to \mathbb{R}$ to the system Σ_i such that

$$\partial K_i(\bar{u}_i) = k_i(\bar{u}_i), \quad \forall \bar{u}_i \in \overline{\mathcal{U}}_i.$$

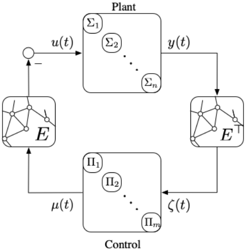
- If k is a continuous single-valued function from $\mathbb{R}$ to $\mathbb{R}$ then $K_i(\bar{u}_i)$ is differentiable and

$$\nabla K_i(\bar{u}_i) = k_i(\bar{u}_i).$$

- Its convex conjugate, defined as:

$$K_i^\star(\bar{y}_i) = \sup_{\bar{u}_i}\{\bar{y}_i\bar{u}_i - K_i(\bar{u}_i)\} = -\inf_{\bar{u}_i}\{K_i(\bar{u}_i) - \bar{y}_i\bar{u}_i\},$$

Note that $\partial K_i^\star(\bar{y}_i) = k_i^{-1}(\bar{y}_i)$ is maximally monotone.

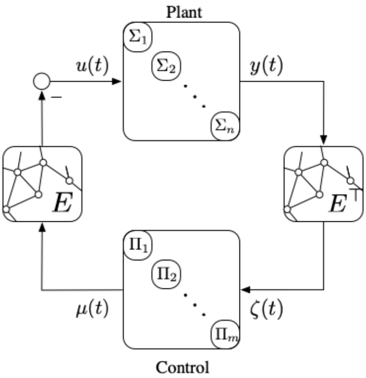


Theorem [Bürger, Z, Allgöwer 2016]

Steady-state values $\bar{u}, \bar{y}, \bar{\zeta}$ and $\bar{\mu}$ are the *(primal-dual)* solutions of the *family* of the following pair of **convex** optimization problems:

Optimal Potential Problem	Optimal Flow Problem
$\min_{\bar{y},\bar{\zeta}} \quad \sum_{i=1}^{\lvert\mathcal{V}\rvert} K_i^\star(\bar{y}_i) + \sum_{e=1}^{\lvert\mathcal{E}\rvert} \Gamma_e(\bar{\zeta}_e)$ $s.t. \quad E^T\bar{y} = \bar{\zeta}.$	$\min_{\bar{u},\bar{\mu}} \quad \sum_{i=1}^{\lvert\mathcal{V}\rvert} K_i(\bar{u}_i) + \sum_{e=1}^{\lvert\mathcal{E}\rvert} \Gamma_e^\star(\bar{\mu}_e)$ $s.t. \quad \bar{u} = -E\bar{\mu}.$

Furthermore, if Σ_is are MEIP and Π_es are output-strictly MEIP, then the signals $u(t), y(t), \zeta(t), \mu(t)$ converge to these solutions.



Node dynamics

$$\Sigma_i : \quad \dot{x}_i(t) = f_i(x_i(t), u_i(t)), \\ y_i(t) = h_i(x_i(t), u_i(t)).$$

Edge functions

$$\Pi_k : \mu_k(t) = \psi_k(\zeta_k(t)),$$

Assumption 1: Feasibility of Consensus

Σ_i is MEIP, and the equilibrium input-output relations of the nodes satisfy $\boldsymbol{k}(\mathbf{0}) \cap \mathcal{N}(E^T) \neq \emptyset$.

In many applications, nonlinear protocols are used to achieve the desired behavior of the network.

Motivation

Generalize the notion of signed networks to the nonlinear case, and analyze its convergence properties.

Passivity

A system $\eta = \pi(t, \xi)$, where ξ, η are the system input vector and system output vector, respectively, is

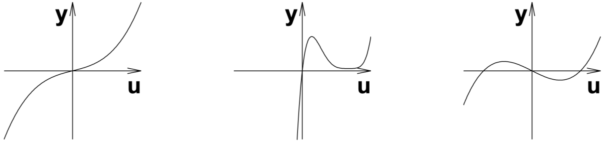
i) passive if $\xi^T \eta \geq 0$;

ii) input strictly passive if there exists $\epsilon > 0$, such that $\xi^T \eta \geq \epsilon \xi^T \xi$;

iii) active if $\xi^T \eta \leq 0$;

iv) input strictly active if there exists $\epsilon > 0$, such that $\xi^T \eta \leq -\epsilon \xi^T \xi$.

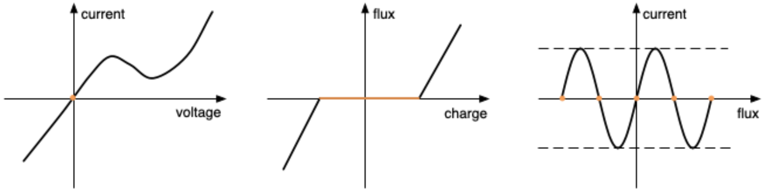
In all above cases, the inequality should hold for all (t, ξ).



Definition

Suppose Π_k is a map from $\mathbb{R}$ to $\mathbb{R}$ with $\psi_k(0) = 0$. Then edge k is termed

i) (strictly) positive if Π_k is (input strictly) passive;
ii) (strictly) negative if Π_k is (input strictly) active.



tunnel diode — (strictly) positive

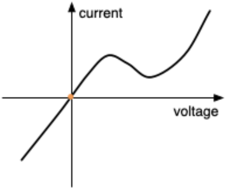
memristor — positive

Josephson junction — neither positive nor negative

Definition: Signed Nonlinear Networks

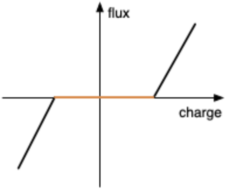
A networked system $(\mathcal{G}, \Sigma, \Pi)$ is a

- positive network if all the edges are positive, and $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$;
- strictly positive network if all the edges are strictly positive, and $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{>})$;
- signed network if not all the edges are positive.



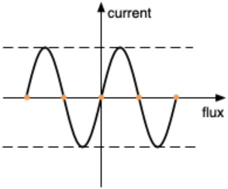
tunnel diode

(strictly) positive



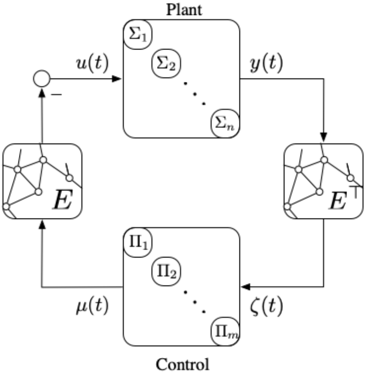
memristor

positive



Josephson junction

neither positive
nor negative



Node dynamics

$$\Sigma_i : \quad \dot{x}_i(t) = f_i(x_i(t), u_i(t)),$$
$$y_i(t) = h_i(x_i(t), u_i(t)).$$

"Signed" Edge functions

$$\Pi_k : \mu_k(t) = \psi_k(\zeta_k(t)),$$

Assumption 1: Feasibility of Consensus

Σ_i is MEIP, and the equilibrium input-output relations of the nodes satisfy $\boldsymbol{k}(\mathbf{0}) \cap \mathcal{N}(E^T) \neq \emptyset$.

Theorem

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. Then $\lim_{t \to \infty} \zeta(t) = \tilde{\zeta}$ exists, and $\tilde{\zeta} \in \boldsymbol{I}$.

Theorem

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. Then $\lim_{t\to\infty} \zeta(t) = \tilde{\zeta}$ exists, and $\tilde{\zeta} \in \boldsymbol{I}$.

PROOF

Consider the Lyapunov function $V(\boldsymbol{x}(t)) = \sum_{i=1}^{|\mathcal{V}|} S_i(x_i(t))$

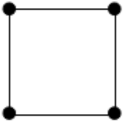
$$\dot{V} = \sum_{i=1}^{|\mathcal{V}|} \dot{S}_i \leq (\boldsymbol{u}(t) - \mathbf{0})^T (\boldsymbol{y}(t) - \tilde{\mathbf{y}}) = -\boldsymbol{\zeta}(t)^T \boldsymbol{\mu}(t) \leq 0.$$

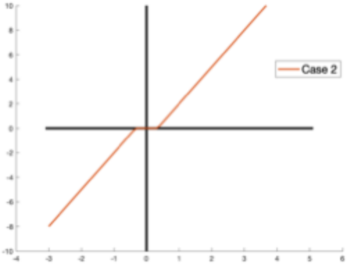
By using LaSalle's invariance principle, $\lim_{t\to\infty} \boldsymbol{\zeta}(t)^T \boldsymbol{\mu}(t) = \mathbf{0}$, meaning $\lim_{t\to\infty} \boldsymbol{\mu}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \boldsymbol{\zeta}(t) \in \boldsymbol{I}$. As a result, $\lim_{t\to\infty} \boldsymbol{u}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \boldsymbol{y}(t) = \tilde{\mathbf{y}} \in \boldsymbol{k}(\mathbf{0})$, therefore $\lim_{t\to\infty} \boldsymbol{\zeta}(t) = \tilde{\zeta} = E^T \tilde{\mathbf{y}}$ exists.

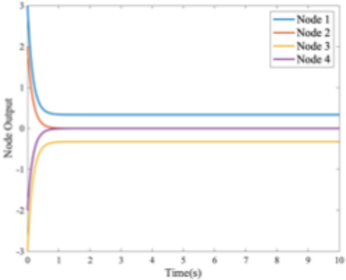
Theorem

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. Then $\lim_{t\to\infty} \zeta(t) = \tilde{\zeta}$ exists, and $\tilde{\zeta} \in \boldsymbol{I}$.

$$\mu_k = \operatorname{sign}(\zeta_k) \max\{3|\zeta_k| - 1, 0\}$$







Corollary: Cluster Bounds

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. Then

$$\lim_{t\to\infty} y_i(t) - y_j(t) \in [z_{\min}, z_{\max}],$$

where

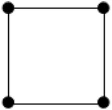
$$z_{\min} = \max_{P_{i,j}} \sum_{k\in P_{i,j}} (-1)^{p_k} I_k^L,\ z_{\max} = \min_{P_{i,j}} \sum_{k\in P_{i,j}} (-1)^{p_k} I_k^R;$$

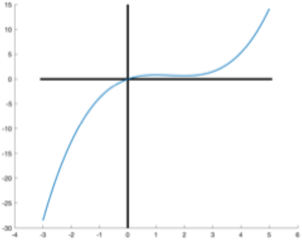
$P_{i,j}$ - directed path from node i to node j
$p_k = 0$ if the original orientation of edge k is consistent with the direction of path, and $p_k = 1$ otherwise.

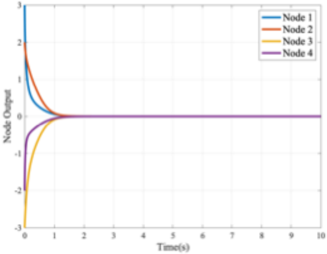
Corollary

Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{>})$ and suppose Assumption 1 holds. Then $\lim_{t\to\infty} \zeta(t) = \mathbf{0}$, and $\lim_{t\to\infty} \boldsymbol{y}(t) = \beta\mathbf{1},\ \beta \in \mathbb{R}$.

$$\mu_k = \frac{1}{3}\zeta_k^3 - \frac{3}{2}\zeta_k^2 + 2\zeta_k$$







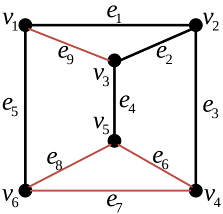
Corollary

Consider a positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\geq})$ and suppose Assumption 1 holds. If there exists a connected subgraph spanning all nodes and strictly positive edges, then

$$\lim_{t \to \infty} \boldsymbol{\zeta}(t) = \mathbf{0} \text{ and } \lim_{t \to \infty} \boldsymbol{y}(t) = \beta \mathbf{1}, \ \beta \in \mathbb{R}.$$

A strictly positive spanning subgraph
is needed to guarantee consensus!

AN EXAMPLE



Edges e_k $(k = 1, \dots, 5)$ form a spanning tree.

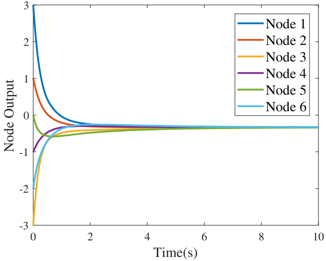
Two possible edge functions:

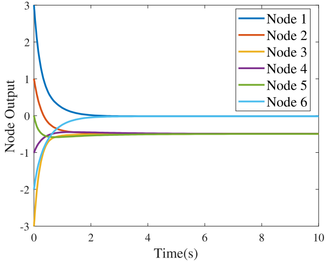
$$\mu_k(t) = \zeta_k(t), \quad (1)$$

$$\mu_k(t) = \text{sign}(\zeta_k(t)) \cdot \max\{\zeta_k(t) - 1, 0\}, \quad (2)$$

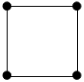
Two scenarios:

- Edges e_k $(k = 1, \dots, 5)$ use (1), and edges e_k $(k = 6, \dots, 9)$ use (2).
- Edges e_k $(k = 2, \dots, 5)$ use (1), and edges e_k $(k = 1, 6, \dots, 9)$ use (2).

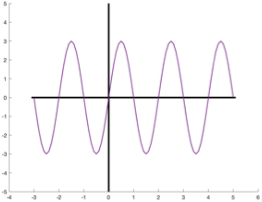


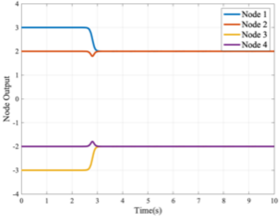


AN EXAMPLE: NON-POSITIVE NETWORKS

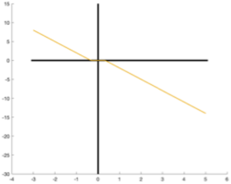


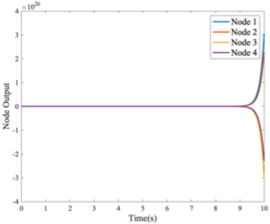
$$\mu_k = 3\sin(\pi\zeta_k)$$

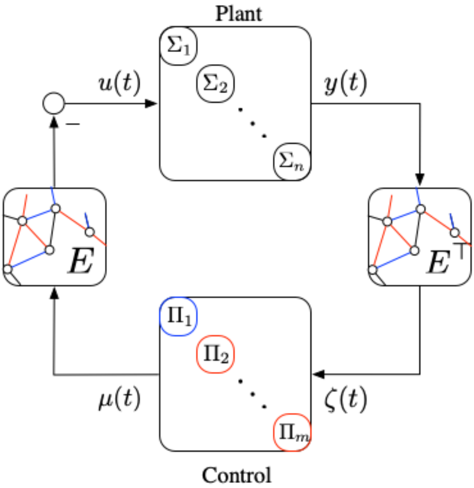




$$\mu_k = -\mathrm{sign}(\zeta_k)\max\{3|\zeta_k| - 1, 0\}$$





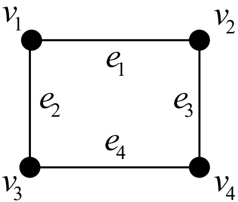


How do we analyze networks with positive and negative edges?

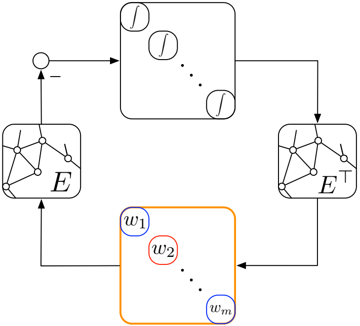
Linear Consensus as an RC-Circuit

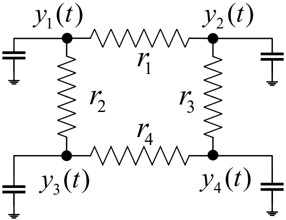
Capacitors ⇔ Node Dynamics (integrators)

Resistors ⇔ Edge Dynamics (linear gain)

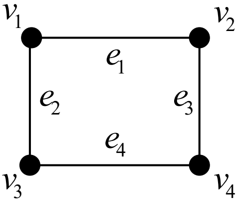


$\mathcal{G}$





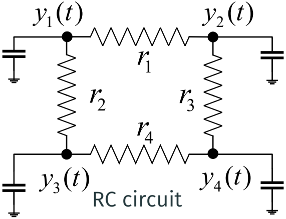
RC circuit



$\mathcal{G}$

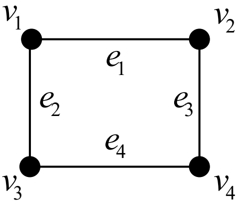
Edge Functions:

$$\mu_k = w_k \zeta_k = w_k(y_i - y_j)$$



Control : $\boldsymbol{u} = -EWE^T\boldsymbol{y}$

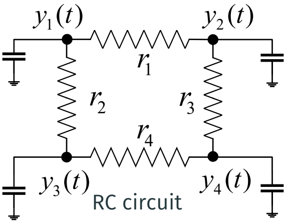
$$w_k = \frac{1}{r_k}$$



$\mathcal{G}$

Edge Functions:

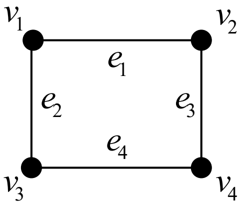
$$\mu_k = w_k \zeta_k = w_k(y_i - y_j)$$



Control : $\boldsymbol{u} = -EWE^T\boldsymbol{y}$

$$w_k = \frac{1}{r_k}$$

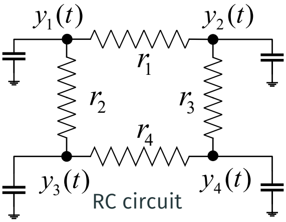
In steady-state, $\boldsymbol{u} = 0$, and the network corresponds to a resistive circuit.



$\mathcal{G}$

Edge Functions:

$\mu_k = w_k \zeta_k = w_k(y_i - y_j)$



Control : $\boldsymbol{u} = -EWE^T\boldsymbol{y}$

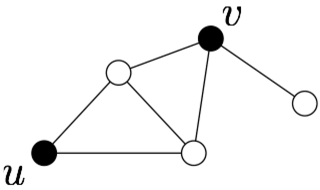
$w_k = \frac{1}{r_k}$

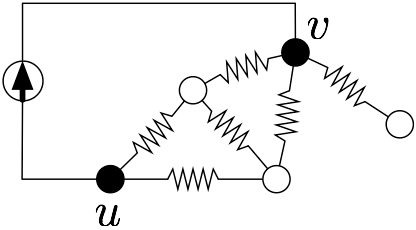
In steady-state, $\boldsymbol{u} = 0$, and the network corresponds to a resistive circuit.

What happens if we add a negative resistor?

EFFECTIVE RESISTANCE

The effective resistance between two nodes u and v is the electrical resistance measured across the nodes when the graph represents a resistive circuit.





Effective Resistance Calculation [Klein and Randić 1993]

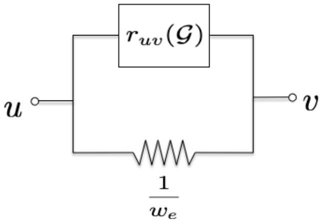
$$r_{uv} = [L^{\dagger}(\mathcal{G})]_{uu} + 2[L^{\dagger}(\mathcal{G})]_{uv} + [L^{\dagger}(\mathcal{G})]_{vv}$$

Theorem [Z, Bürger CDC2014]

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$ be a strictly positive network with edge functions $\mu_k = w_k\zeta_k$ (i.e., $w_k > 0$ for all $k \in \mathcal{E}$) and let $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E}_> \cup e)$ where $e = (u, v)$ is a negative edge with $\mu_k = w_e\zeta_e$ and $w_e < 0$. Then the signed consensus network reaches agreement if and only if

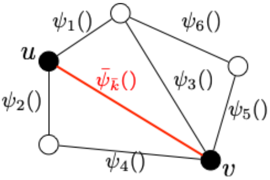
$$|w_e| \leq r_{uv}^{-1},$$

where r_{uv} is the effective resistance in $\mathcal{G}$ between nodes u and v.



The negative edge weights effectively creates an open circuit

Generalizations for non-linear resistors



Equivalent (non-linear) Edge Functions

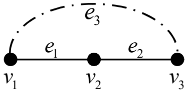
Consider a network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and $\bar{\mathcal{G}} = (\mathcal{V}, \mathcal{E} \cup \bar{k})$, with $k = (u, v)$.
For each given $\zeta_{\bar{k}}$, if there exists a unique $(\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{\mu}})$ and some $\boldsymbol{y} \in \mathbb{R}^{|\mathcal{V}|}$ such that

$$\begin{cases} \bar{E}^T \boldsymbol{y} & = \bar{\boldsymbol{\zeta}} \\ \boldsymbol{\mu} & = \Psi(\boldsymbol{\zeta}) \\ \bar{E}\bar{\boldsymbol{\mu}} & = \mathbf{0} \end{cases},$$

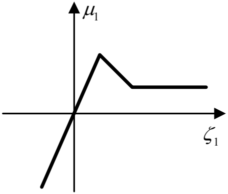
then the flow $\mu_{\bar{k}}$ on the edge $\bar{k}$ can be represented as a function of $\zeta_{\bar{k}}$, which we denote as $\mu_{\bar{k}} = -\bar{\psi}_{\bar{k}}(\zeta_{\bar{k}})$.

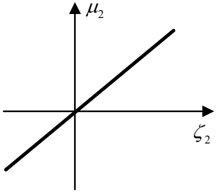
$\bar{\psi}_{\bar{k}}(\cdot) : \mathbb{R} \to \mathbb{R}$ is termed the equivalent edge function between nodes u and v.

DO EQUIVALENT EDGE FUNCTIONS ALWAYS EXIST?



$$\bar{E} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$





$$\mu_1 = \begin{cases} 2\zeta_1, & \zeta_1 \in (-\infty, 1]; \\ 3 - \zeta_1, & \zeta_1 \in (1, 2]; \\ 1, & \zeta_1 \in (2, \infty). \end{cases}$$

$$\mu_2 = \zeta_2.$$

For each ζ_3, there should exist a unique μ_3 such that

$$\begin{cases} \bar{E}^T \boldsymbol{y} = \bar{\boldsymbol{\zeta}} \\ \boldsymbol{\mu} = \Psi(\boldsymbol{\zeta}) \\ \bar{E}\bar{\boldsymbol{\mu}} = \mathbf{0} \end{cases}$$

Suppose $\zeta_3 = 3$,

- if $\zeta_1 = 1$, then $\zeta_2 = 2$, $\mu_1 = \mu_2 = 2$, $\mu_3 = -2$;
- if $\zeta_1 = 2$, then $\zeta_2 = 1$, $\mu_1 = \mu_2 = 1$, $\mu_3 = -1$.

Not unique!

Lemma [L. O. Chua (1969)]

Consider a circuit containing only resistors and independent voltage sources. If the current-voltage function of each resistor is strictly monotonically increasing with the current tending to $\pm\infty$ as the voltage tends to $\pm\infty$, and if the circuit contains no cycles of voltage sources, then for each pair of the voltage source values, the current and voltage drop on each resistor, as well as the current on each source, are unique.

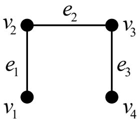
Proposition

Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$. Identify $p, q \in \mathcal{V}$ as two terminals of interest. If for each $k \in \mathcal{E}_>$, its edge function $\mu_k(t) = \psi_k(\zeta_k(t))$ is strictly monotonically increasing, and $\mu_k(t) \to \pm\infty$ as $\zeta_k(t) \to \pm\infty$, then the equivalent edge function of the two-terminal network between nodes p and q exists.

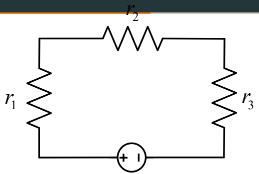
Algorithm 1 Computation of Equivalent Edge Functions

1: Obtain the corresponding resistive circuit, and add a voltage source between p and q;
2: Obtain a finite set $\{\zeta_{pq}\}$ by sampling the interval $[-N, N]$;
3: **for all** ζ_{pq} **do**
4: Set ζ_{pq} as the value of voltage source;
5: Calculate the flow on the voltage source $\bar{\mu}_{pq}$ while satisfying KVL and KCL;
6: **end for**
7: Approximate the equivalent edge function by interpolation based on $\{(\zeta_{pq}, \bar{\mu}_{pq})\}$.

EQUIVALENT EDGE FUNCTION

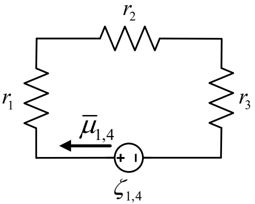


Original underlying graph.

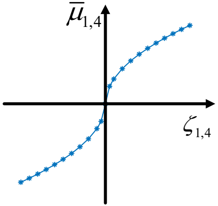


Obtain the corresponding resistive circuit, and add a voltage source.

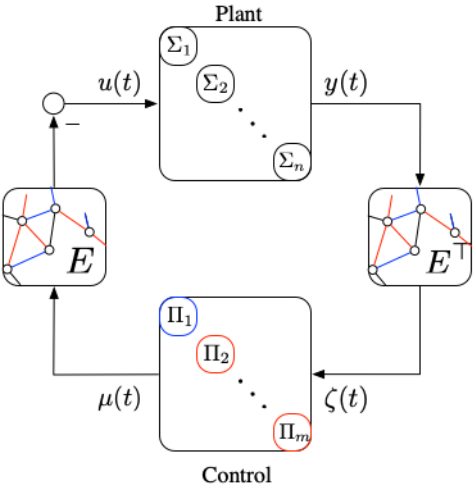
Obtain a finite set $\{\zeta_{1,4}\}$ by sampling the interval $[-N, N]$.



Calculating $\bar{\mu}_{1,4}$ corresponding to each $\zeta_{1,4}$



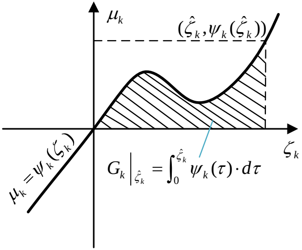
Approximate the equivalent edge function by interpolation based on $\{(\zeta_{1,4}, \bar{\mu}_{1,4})\}$.



How do we analyze networks with positive and negative edges?

The cocontent[1] of a nonlinear resistor (edge k) when its tension ζ_k is specified, is defined as

$$G_k|_{\zeta_k} = \int_0^{\zeta_k} \psi_k(\tau) \cdot d\tau.$$



Cocontent is area under curve. Content ($G_k^*|_{\mu_k}$) is area above curve.

Total Power: $G_k|_{\zeta_k} + G_k^*|_{\mu_k} = \mu_k \zeta_k$

[1] M. Parodi and M. Storace, Linear and Nonlinear Circuits: Basic and Advanced Concepts. Cham, Switzerland: Springer International Publishing, 2018.

COCONTENT FUNCTION

The cocontent[1] of a nonlinear resistor (edge k) when its tension ζ_k is specified, is defined as

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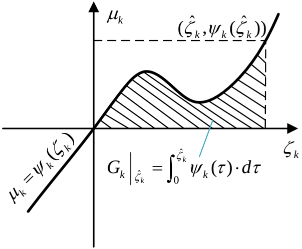
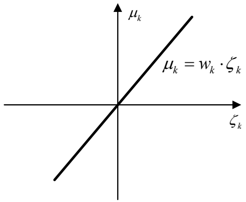


Cocontent of the network denoted as $\mathbf{G} = \sum_{k=1}^{|\mathcal{E}|} G_k|_{\zeta_k}$.

[1] M. Parodi and M. Storace, Linear and Nonlinear Circuits: Basic and Advanced Concepts. Cham, Switzerland: Springer International Publishing, 2018.

COCONTENT FUNCTION

Example 1: A resistor with constant resistance.



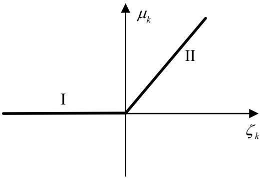
Edge function:

$$\mu_k = w_k \cdot \zeta_k.$$

Cocontent function:

$$\begin{aligned} G_k|_{\zeta_k} &= \int_0^{\zeta_k} w_k \cdot \tau d\tau \\ &= \frac{1}{2} w_k \zeta_k^2. \end{aligned}$$

Example 2: An ideal diode.



Edge function:

$$\mu_k = \begin{cases} 0, & \text{if } \zeta_k < 0. \\ w_k \cdot \zeta_k, & \text{if } \zeta_k \geq 0. \end{cases}$$

Cocontent function:

$$G_k|_{\zeta_k} = \begin{cases} 0, & \text{if } \zeta_k < 0. \\ \frac{1}{2} w_k \zeta_k^2, & \text{if } \zeta_k \geq 0. \end{cases}$$

MINIMUM OF COCONTENT FUNCTION

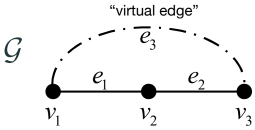
Lemma

Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$, and the augmented graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}})$ obtained by adding the virtual edge $\bar{k}$ ($\bar{\mathcal{E}} = \mathcal{E}_> \cup \{\bar{k}\}$). Suppose for each $k \in \mathcal{E}_>$, $\mu_k = \psi_k(\zeta_k)$ is monotonically increasing. For any fixed $\zeta_{\bar{k}}$, if there exists $\bar{\boldsymbol{\zeta}}^0, \bar{\boldsymbol{\mu}}^0, \boldsymbol{y}^0$, such that

$$\begin{cases} \bar{E}^T \boldsymbol{y}^0 & = \bar{\boldsymbol{\zeta}}^0 \\ \boldsymbol{\mu}^0 & = \Psi(\boldsymbol{\zeta}^0) \\ \bar{E}\bar{\boldsymbol{\mu}}^0 & = \mathbf{0}, \end{cases}$$

then $\sum_{k=1}^{|\mathcal{E}|} G_k|_{\zeta_k}$ reaches its minimum at $(\bar{\boldsymbol{\zeta}}^0, \bar{\boldsymbol{\mu}}^0, \boldsymbol{y}^0)$.

$\star$ A generalization of *Maxwell's Minimum Heat Theorem* which states that a circuit composed of linear passive resistors and sources, the currents are such that the dissipated power is minimum.



$$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \qquad \bar{E} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Edge Functions

$$\begin{aligned} \mu_1 &= \tfrac{1}{2}\zeta_1 \\ \mu_2 &= \zeta_2 \end{aligned}$$

Network Co-Content

$$\begin{aligned} \mathbf{G} &= \int_0^{\zeta_1} \tfrac{1}{2}\tau \cdot d\tau + \int_0^{\zeta_2} \tau \cdot d\tau \\ &= \tfrac{1}{4}\zeta_1^2 + \tfrac{1}{2}\zeta_2^2 \end{aligned}$$

Example 1

$\zeta_1 = 2$ potential drop between v1-v2

$\zeta_2 = 1$ potential drop between v2-v3

$\zeta_3 = 3$ potential drop between v1-v3

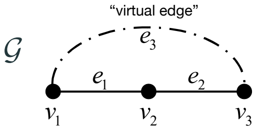
network equations

$$\bar{E}^T y = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\bar{\mu} = \Psi(\zeta) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{E}\bar{\mu} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{G} = 1.5$$



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Example 2

$\zeta_1 = 1$ potential drop between v1-v2

$\zeta_2 = 2$ potential drop between v2-v3

$\zeta_3 = 3$ potential drop between v1-v3

network equations

$$\bar{E}^T y = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\bar{\mu} = \Psi(\zeta) = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

$$\bar{E}\bar{\mu} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 2 \\ \star \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boxed{\mathbf{G} = 2.25 > 1.5}$$

Proposition: Minimum of Cocontent

Consider a strictly positive network system $(\mathcal{G}, \Sigma, \Pi)$ with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_>)$, and all edge functions are monotonically increasing. Identify $p, q \in \mathcal{V}$ as two terminals of interest, and the tension between nodes p and q is specified as ζ_{pq}. If the equivalent edge function between nodes p and q exists, then the minimum cocontent of the network system $(\mathcal{G}, \Sigma, \Pi)$ is the cocontent of the equivalent edge function between nodes p and q, denoted as $\min \mathbf{G}|_{\zeta_{pq}} = G_{pq}|_{\zeta_{pq}}$.

Establishes connection between
equivalent edge functions and cocontent

Nonlinear integrator dynamics:

$$\Sigma_i: \quad \dot{x}_i(t) = \gamma_i(u_i(t)), \quad y_i(t) = x_i(t), \quad i \in \mathcal{V}.$$

where $\gamma_i(\cdot)$ satisfies $u_i \cdot \gamma_i(u_i) \geq 0$, and equality holds if and only if $u_i = 0$.

Theorem

Consider a signed network system $(\mathcal{G}, \Sigma, \Pi)$ of nonlinear integrators with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose there is only one non-strictly positive edge $\hat{k}$ in $\mathcal{E}$, with edge function $\mu_{\hat{k}}(t) = \psi_{\hat{k}}(\zeta_{\hat{k}}(t))$, and $\psi_{\hat{k}}(0) = 0$. Furthermore, $\forall\, k \in \mathcal{E}_{>}$, $\psi_k(\cdot)$ is monotonically increasing. Identify $p, q \in \mathcal{V}$, which are connected by edge $\hat{k}$, as the two terminals of the strictly positive subnetwork system $(\mathcal{G}_{>}, \Sigma, \bar{\Pi})$. If the equivalent edge function $\bar{\mu}_{pq}(t) = \bar{\psi}_{pq}(\zeta_{\hat{k}}(t))$ between p and q in $(\mathcal{G}_{>}, \Sigma, \bar{\Pi})$ exists, and

$$(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) \geq 0$$

holds for any $\zeta_{\hat{k}}(t) \in \mathbb{R}$, then $\lim_{t\to\infty} \boldsymbol{u}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t) = 0$.

Proof

Let $V(t) := G_{\hat{k}}|_{\zeta_{\hat{k}}(t)} + \mathbf{G}_{>}|_{\zeta_{\hat{k}}(t)}$ be the cocontent of the network system $(\mathcal{G}, \Sigma, \Pi)$, where $G_{\hat{k}}|_{\zeta_{\hat{k}}(t)}$ is the cocontent of edge $\hat{k}$, and $\mathbf{G}_{>}|_{\zeta_{\hat{k}}(t)}$ is the cocontent of the subnetwork system $(\mathcal{G}_{>}, \Sigma, \bar{\Pi})$ for a fixed value of $\zeta_{\hat{k}}(t)$. Then

$$V(t) \overset{(a)}{\geq} G_{\hat{k}}|_{\zeta_{\hat{k}}(t)} + G_{pq}|_{\zeta_{\hat{k}}(t)} \overset{(b)}{\geq} 0,$$

where (b) follows from $(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) \geq 0$.

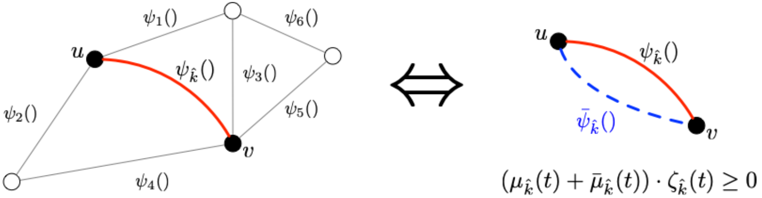
As $\dot{G}_k|_{\zeta(t)}(t) = \mu_k(t)\dot{\zeta}_k(t), \ \forall\, k \in \mathcal{E}$, then

$$\dot{V}(t) = \boldsymbol{\mu}(t)^T \dot{\boldsymbol{\zeta}}(t) = -\boldsymbol{u}(t)^T \dot{\boldsymbol{y}}(t) = -\sum_{i=1}^{|\mathcal{V}|} u_i(t) \cdot \gamma_i(u_i(t)) \leq 0.$$

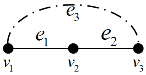
From LaSalle's invariance principle, $\lim_{t\to\infty} \boldsymbol{u}(t) = \mathbf{0}$, and $\lim_{t\to\infty} \mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t) = 0$ follows the definition of equivalent edge function.

Main Point

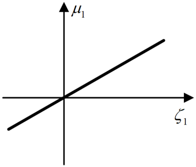
Cocontent function works as a Lyapunov Function

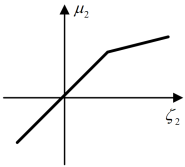


parallel edges must be positive (passive)!

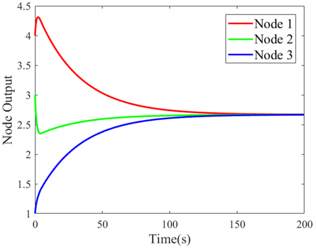


$$\bar{E} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

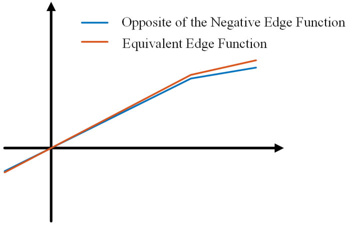


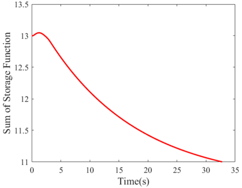


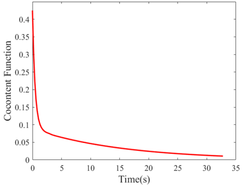
$$\mu_1 = 0.5\zeta_1 \qquad \mu_2 = \begin{cases} \zeta_2, & \text{when} \quad \zeta_2 < 1; \\ 0.2\zeta_2 + 0.8, & \text{when} \quad \zeta_2 \geq 1. \end{cases}$$



Consensus!







Derivative of storage function is indefinite!

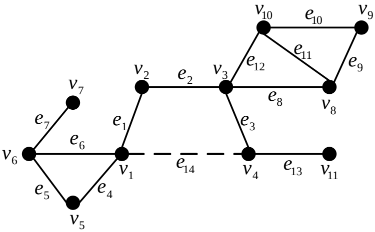
Corollary: Agreement

Consider a signed network system $(\mathcal{G}, \Sigma, \Pi)$ of nonlinear integrators with connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Suppose there is only one non-strictly positive edge $\hat{k}$ in $\mathcal{E}$, with edge function $\mu_{\hat{k}}(t) = \psi_{\hat{k}}(\zeta_{\hat{k}}(t))$, and $\psi_{\hat{k}}(0) = 0$. Furthermore, $\forall\, k \in \mathcal{E}_{>}$, $\psi_k(\cdot)$ is monotonically increasing. Identify $p, q \in \mathcal{V}$, which are connected by edge $\hat{k}$, as the two terminals of the strictly positive subnetwork system $(\mathcal{G}_{>}, \Sigma, \bar{\Pi})$. If the equivalent edge function $\bar{\mu}_{pq}(t) = \bar{\psi}_{pq}(\zeta_{\hat{k}}(t))$ between p and q in $(\mathcal{G}_{>}, \Sigma, \bar{\Pi})$ exists, and

$$(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) \geq 0$$

holds for any $\zeta_{\hat{k}}(t) \in \mathbb{R}$, and $(\mu_{\hat{k}}(t) + \bar{\mu}_{pq}(t)) \cdot \zeta_{\hat{k}}(t) = 0$ if and only if $\zeta_{\hat{k}}(t) = 0$, then $\lim\limits_{t \to \infty} \boldsymbol{\zeta}(t) = \mathbf{0}$, and $\lim\limits_{t \to \infty} \mathbf{y}(t) = \beta \mathbf{1},\ \beta \in \mathbb{R}$.

SIMULATION RESULTS

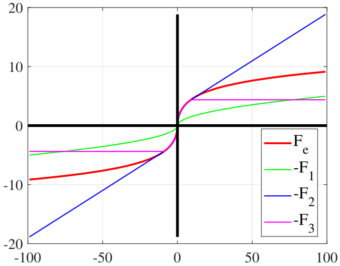


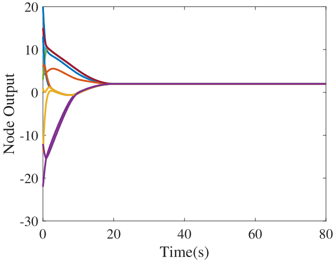
- The original network consists of 11 nodes and 13 edges, and e_{14} is introduced by an attacker.
- The original 13 edges are strictly positive, with edge functions described by

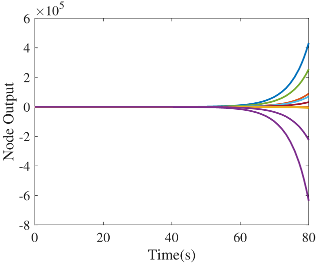
$$\mu_k(t) = w_k \cdot \mathrm{sign}(\zeta_k(t)) \cdot |\zeta_k(t)|^{\alpha_k},$$

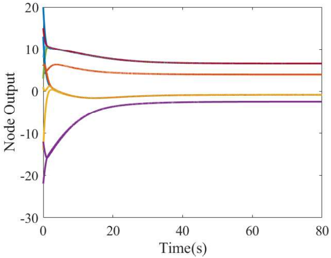
  and $\boldsymbol{w} = (3, 2, 4, 1, 2, 1, 3, 2, 2, 1, 1, 1, 2)^T$,
  $\boldsymbol{\alpha} = (0.4, 0.5, 0.2, 0.8, 0.4, 0.4, 0.5, 0.5, 0.5, 0.6, 0.8, 0.2, 0.5)^T$.
- Only one cycle contains e_{14}, i.e., the cycle consisting of nodes v_1, v_2, v_3 and v_4, and edges e_1, e_2, e_3 and e_{14}.
- We consider three strictly negative candidate functions for e_{14}.

SIMULATION RESULTS









CONCLUSION

- We generalize the definition of signed linear networks to graphs with nonlinear functions on the edges.
- Convergence analysis for positive and strictly positive networks. Spanning subgraphs of strictly positive edges are required for consensus.
- For networks comprised of nonlinear integrator agents, we show a connection to notions from electrical circuit theory and the equivalent circuit model to derive convergence results for networks with non-positive edges. We also propose an algorithm for constructing equivalent edge functions.

Future research directions include:

- Convergence analysis of general MEIP nodes in a signed network of more than one non-strictly positive edges.
- Real-world applications.

References

Signed Linear Networks

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D. Zelazo and M. Bürger, *On the robustness of uncertain consensus networks*, IEEE Control of Network Systems, 4(2):170–178, 2017.

Signed Nonlinear Networks

H. Chen, D. Zelazo, X. Wang, L. Shen, *Convergence analysis of signed nonlinear networks*, to appear in IEEE Control of Network Systems, 2019.

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