AN IMPROVED DISTRIBUTED CONSENSUS KALMAN FILTER DESIGN APPROACH

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MOTIVATION

Cooperative state estimation in sensor networks...

Cooperative Trading

Cooperative Missile Defence

MOTIVATION

Cooperative state estimation in sensor networks...

Cooperative Trading

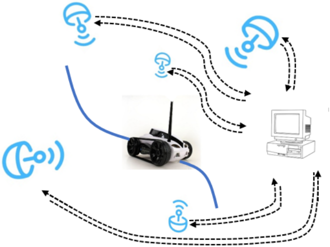
Cooperative Missile Defence

- ◇ local computational loads
- ◇ energy (the amount of data shared)
- ◇ available local information
- ◇ stability
- ◇ overall system performance

N agents observing a process $\mathcal{P}$ with observation model $\mathcal{OM}$

$$\mathcal{P} : x_{k+1} = Ax_k + Bw_k$$

$$\mathcal{OM} : z_k^i = H^i x_k + v_k^i$$



Centralized estimation

Prediction

$$\bar{x}_k = A\hat{x}_{k-1}$$

$$\bar{P}_k = A\hat{P}_{k-1}A^T + BQB^T$$

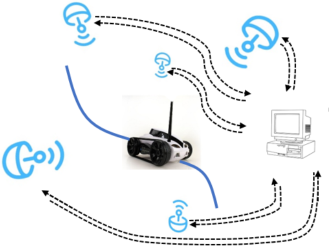
Estimation

$$K_k = P_k\mathbf{H}^T\left(\mathbf{R} + \mathbf{H}\bar{P}_k\mathbf{H}^T\right)^{-1}$$

$$\hat{P}_k = (I - K_k\mathbf{H})\,\bar{P}_k\,(I - K_k\mathbf{H})^T + K_k\mathbf{R}K_k^T$$

$$\hat{x}_k = \bar{x}_k + K_k\left(\mathbf{z}_k - \mathbf{H}\bar{x}_k\right),$$

CENTRALIZED KALMAN FILTER



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$$\hat{x}_k = \bar{x}_k + K_k\left(\mathbf{z}_k - \mathbf{H}\bar{x}_k\right),$$

◇ $\mathbf{z}_k = \left[{z_k^1}^T, \ldots, {z_k^N}^T\right]^T$, $\mathbf{H} = \left[{H^1}^T, \ldots, {H^N}^T\right]^T$, $\mathbf{R} = \mathtt{diag}\{R^i\}_{i=1..N}$

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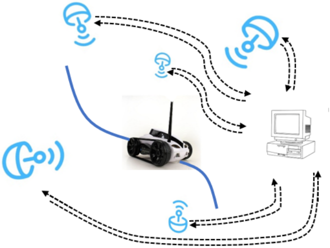


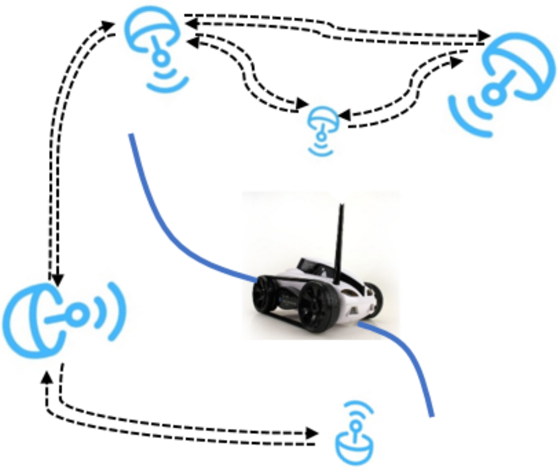
Centralized estimation

$$\begin{cases} \textbf{Prediction} \\ \bar{x}_k = A\hat{x}_{k-1} \\ \bar{P}_k = A\hat{P}_{k-1}A^T + BQB^T \\ \textbf{Estimation} \\ K_k = P_k\mathbf{H}^T\left(\mathbf{R} + \mathbf{H}\bar{P}_k\mathbf{H}^T\right)^{-1} \\ \hat{P}_k = (I - K_k\mathbf{H})\,\bar{P}_k\,(I - K_k\mathbf{H})^T \\ \quad +K_k\mathbf{R}K_k^T \\ \hat{x}_k = \bar{x}_k + K_k\left(\mathbf{z}_k - \mathbf{H}\bar{x}_k\right), \end{cases}$$

- $\mathbf{z}_k = \left[{z_k^1}^T, \ldots, {z_k^N}^T\right]^T$, $\mathbf{H} = \left[{H^1}^T, \ldots, {H^N}^T\right]^T$, $\mathbf{R} = \mathtt{diag}\{R^i\}_{i=1..N}$

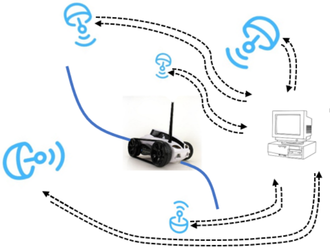
- Optimal Kalman gain $\longrightarrow$ $\boxed{\dfrac{\partial \mathbb{E}\left[(\hat{x} - x)^T(\hat{x} - x)\right]}{\partial K_k} = 0}$

DISTRIBUTED COOPERATIVE ESTIMATION

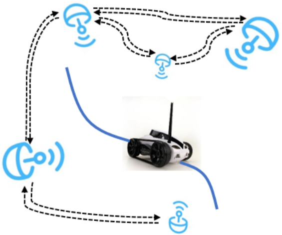


Assumption: each agent has at least one connection.

DISTRIBUTED COOPERATIVE ESTIMATION

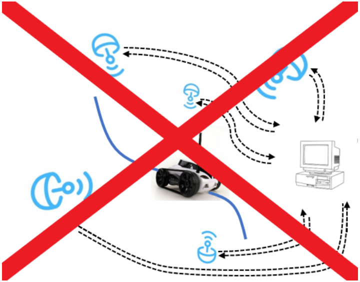


Centralized estimation



Distributed cooperative estimation

DISTRIBUTED COOPERATIVE ESTIMATION



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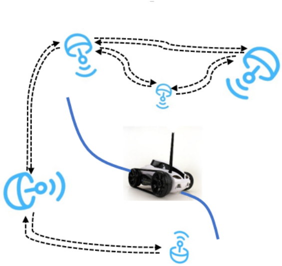

Distributed cooperative estimation

Distributed *Consensus Kalman* estimator (DCKE) [1]

$$\hat{x}_k^i = \bar{x}_k^i + K_k^i \left(z_k^i - H^i \bar{x}_k \right) + \underbrace{C_k^i \sum_{j \in \mathcal{N}_i} \left(\bar{x}_k^j - \bar{x}_k^i \right)}_{\textit{Consensus}},$$

[1] R. Olfati-Saber, J. A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," Proceedings of the IEEE, vol. 95, no. 1, pp. 215–233, 2007.

OPTIMAL KALMAN GAIN

Distributed *Consensus Kalman* estimator (DCKE)

$$\hat{x}_k^i = \bar{x}_k^i + K_k^i \left(z_k^i - H^i \bar{x}_k \right) + C_k^i \sum_{j \in \mathcal{N}_i} \left(\bar{x}_k^j - \bar{x}_k^i \right),$$

K^i and C^i are the i^{th} agent's Kalman and consensus gains, respectively.

Distributed *Consensus Kalman* estimator (DCKE)

$$\hat{x}_k^i = \bar{x}_k^i + K_k^i \left(z_k^i - H^i \bar{x}_k \right) + C_k^i \sum_{j \in \mathcal{N}_i} \left(\bar{x}_k^j - \bar{x}_k^i \right),$$

Optimal Kalman gain -

$$\frac{\partial \mathbb{E}\left[\left(\hat{x} - x \right)^T \left(\hat{x} - x \right) \right]}{\partial K_k} = 0$$

Optimal consensus gain -

$$\frac{\partial \mathbb{E}\left[\left(\hat{x} - x \right)^T \left(\hat{x} - x \right) \right]}{\partial C_k} = 0$$

solved -

R. Deshmukh, C. Kwon, and I. Hwang, "Optimal discrete-time kalman consensus filter," in 2017 American ControlConference (ACC), pp. 5801–5806, IEEE, 2017

OPTIMAL KALMAN GAIN

The optimal (MSE) distributed Kalman gain [2]

$$K_k^i = \left(\bar{P}_k^i H^{iT} + C_k^i \sum_{j \in \mathcal{N}_i} \left(\bar{P}_k^{j,i} - \bar{P}_k^i \right) H^{iT} \right) \left(R^i + H^i \bar{P}_k^i H^{iT} \right)^{-1},$$

[2] R. Olfati-Saber, "Kalman-consensus filter: Optimality, sta-bility, and performance," in Proceedings of the 48h IEEEConference on Decision and Control (CDC) held jointly with2009 28th Chinese Control Conference, pp. 7036–7042, IEEE,2009.

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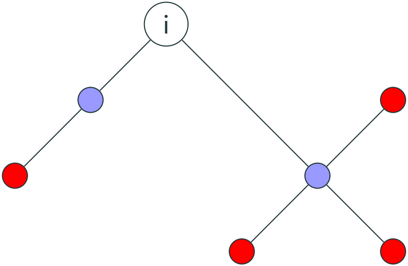

$$\bar{P}_{k+1}^{j,i} = f\left(\bar{P}_k^{j,i}, \bar{P}_k^{j,s} \right), s \in \mathcal{N}_j$$

one-hop neighbors

two-hop neighbors

The corresponding update equations incorporate **two-hop neighbors information exchange!**

Sub-optimal consensus Kalman update equations:

$$\begin{cases} \textbf{Prediction} \\ \bar{x}_k^i = A\hat{x}_{k-1}^i \\ \bar{P}_k^i = A\hat{P}_{k-1}^i A^T + BQB^T \\ \textbf{Estimation} \\ K_k^i = P_k^i H^{iT} \left(R^i + H^i \bar{P}_k^i H^{iT}\right)^{-1} \\ \hat{P}_k^i = \underbrace{\left(I - K_k^i H^i\right)}_{F_k^i} \bar{P}_k^i \left(I - K_k^i H^i\right)^T + K_k^i R^i K_k^{iT} \\ \hat{x}_k^i = \bar{x}_k^i + K_k^i \left(z_k^i - H^i \bar{x}_k\right) + C_k^i \sum_{j \in \mathcal{N}_i} \left(\bar{x}_k^j - \bar{x}_k^i\right). \end{cases}$$

- we define the error dynamics:

$$\begin{aligned}\bar{\eta}_k^i &= A\eta_{k-1}^i \\ \eta_k^i &= \underbrace{(I - K_k^i H^i)}_{F_k^i}\bar{\eta}_k^i + C_k^i \sum_{j \in \mathcal{N}_j} \left(\bar{\eta}_k^j - \bar{\eta}_k^i\right).\end{aligned}$$

- we define the error dynamics:

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\eta_k^i &= \underbrace{\left(I - K_k^i H^i\right)}_{F_k^i} \bar{\eta}_k^i + C_k^i \sum_{j \in \mathcal{N}_j} \left(\bar{\eta}_k^j - \bar{\eta}_k^i\right).
\end{aligned}$$

- we construct a Lyapunov function for the noiseless error dynamics:

$$V_k = \sum_{i=1}^{N} \eta_k^{iT} \hat{P}_k^{i^{-1}} \eta_k^i.$$

- we obtain the Lyapunov step difference function:

$$\delta V_k = \sum_{i=1}^{N} \eta_{k-1}^{iT} \Psi_k^i \eta_{k-1}^i + 2 \sum_{i=1}^{N} \left[\eta_{k-1}^{iT} A^T F_k^{iT} \hat{P}_k^{i^{-1}} C_k^i A \sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right) \right]$$
$$+ \sum_{i=1}^{N} \left[\sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right)^T A^T C_k^{iT} \hat{P}_k^{i^{-1}} C_k^i A \sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right) \right]$$

◇ we obtain the Lyapunov step difference function:

$$\delta V_k = \sum_{i=1}^{N} \eta_{k-1}^{iT} \Psi_k^i \eta_{k-1}^i + 2\gamma_k \sum_{i=1}^{N} \left[\eta_{k-1}^{iT} A^T \underbrace{F_k^{iT} \hat{P}_k^{i^{-1}} C_k^i}_{I} A \sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right) \right]$$

$$+ \gamma_k^2 \sum_{i=1}^{N} \left[\sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right)^T A^T \underbrace{F_k^{i^{-1}} \hat{P}_k^i F_k^{iT^{-1}}}_{Y_k^i} A \sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right) \right]$$

◇ we use the consensus gain structure - $C_k^i = \gamma_k \hat{P}_k^i F_k^{iT^{-1}}$

RECIPE FOR FINDING A CONSENSUS GAIN

- we obtain the Lyapunov step difference function:

$$\delta V_k = \sum_{i=1}^{N} \eta_{k-1}^{iT} \Psi_k^i \eta_{k-1}^i + 2\gamma_k \sum_{i=1}^{N} \left[\eta_{k-1}^{iT} A^T A \sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right) \right]$$
$$+ \gamma_k^2 \sum_{i=1}^{N} \left[\sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right)^T A^T Y_k^i A \sum_{j \in \mathcal{N}_j} \left(\eta_{k-1}^j - \eta_{k-1}^i \right) \right]$$

- we use the consensus gain structure - $C_k^i = \gamma_k \hat{P}_k^i F_k^{iT^{-1}}$

- ultimately, we need to find γ_k.

The error dynamics:

$$\begin{aligned}\bar{\eta}_k^i &= A\eta_{k-1}^i \\ \eta_k^i &= F_k^i \bar{\eta}_k^i + \gamma_k \hat{P}_k^i F_k^{iT^{-1}} \sum_{j\in\mathcal{N}_j} \left(\bar{\eta}_k^j - \bar{\eta}_k^i\right).\end{aligned}$$

The Lyapunov step difference function:

$$\begin{aligned}\delta V_k &= -\eta_{k-1}^T \left[\Psi_k - \gamma_k^2 \left(L\otimes A\right)^T Y_k \left(L\otimes A\right) + 2\gamma_k \left(L\otimes A^T A\right)\right] \eta_{k-1} \\ &= -\eta_{k-1}^T \mathcal{K}_k \eta_{k-1},\end{aligned}$$

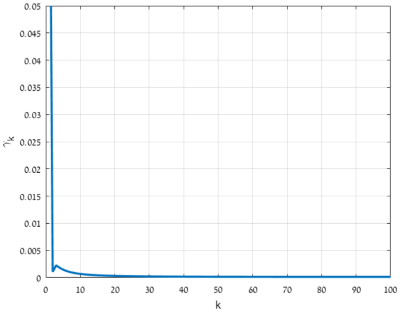
where $\Psi_k = \mathrm{diag}\{\hat{P}_{k-1}^{i^{-1}} - A^T F_k^{iT} \hat{P}_k^{i^{-1}} F_k^i A\}_{i=1}^N$ and $Y_k = \mathrm{diag}\{Y_k^i\}_{i=1}^N$

Olfati- Saber[2]:

$$\gamma_k = \sqrt{\frac{\lambda_{min}\left(\Psi_k\right)}{\lambda_{max}\left(\left(L \otimes A\right) Y_{k+1} \left(L \otimes A\right)\right)}},$$

This will ensure stability but...

$\gamma_k \to 0$ as $k \to \infty$



[2] R. Olfati-Saber, “Kalman-consensus filter: Optimality, sta-bility, and performance,” in Proceedings of the 48h IEEEConference on Decision and Control (CDC) held jointly with2009 28th Chinese Control Conference, pp. 7036–7042, IEEE,2009.

CONSENSUS GAIN FACTOR

We aim to extract the maximal consensus factor!

Theorem (DCKE Stability)

The noiseless estimation error for the consensus gain structure - $C_k^i = \gamma_k P_k^i F_k^{T^{-1}}$ *is asymptotically stable with any* $\gamma_k \in [0, \gamma_k^*] \, \forall \, k$*, where* γ_k^* *is the solution to the following SDP:*

$$\max_{\gamma_k} \gamma_k$$
$$\texttt{s.t.} \begin{bmatrix} \Psi_{k-1} + 2\gamma_k (L \otimes A^T A) & \gamma_k (L \otimes A)^T \\ \gamma_k (L \otimes A) & Y_k^{-1} \end{bmatrix} \succeq 0.$$

Proof

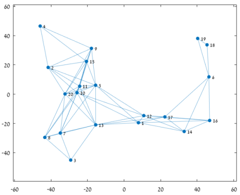
Recall:

$$\mathcal{K}_k = \Psi_k - \gamma_k^2 \left(L \otimes A\right)^T Y_k \left(L \otimes A\right) + 2\gamma_k \left(L \otimes A^T A\right)$$

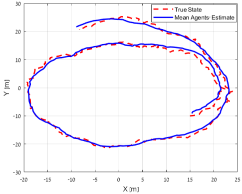
Using the Schur complement, we can now construct the following semi-definite program:

$$\max_{\gamma_k} \gamma_k$$

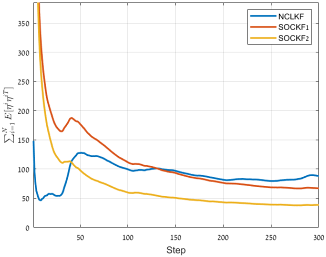
$$\texttt{s.t.} \begin{bmatrix} \Psi_{k-1} + 2\gamma_k(L \otimes A^T A) & \gamma_k(L \otimes A)^T \\ \gamma_k(L \otimes A) & Y_k^{-1} \end{bmatrix} \succeq 0,$$



Communication graph



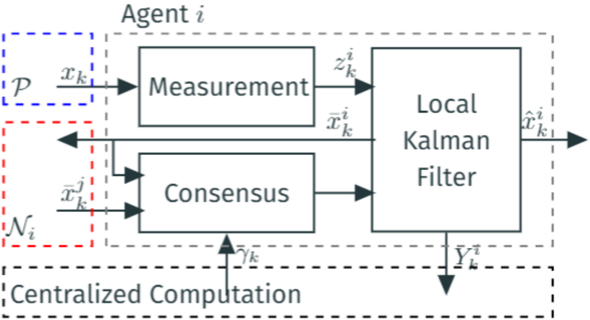
True state Vs. agents' mean estimate



Sum of agents' MSE

- Non-cooperative Kalman filter
- Olfati-Saber
- Zelazo & Priel

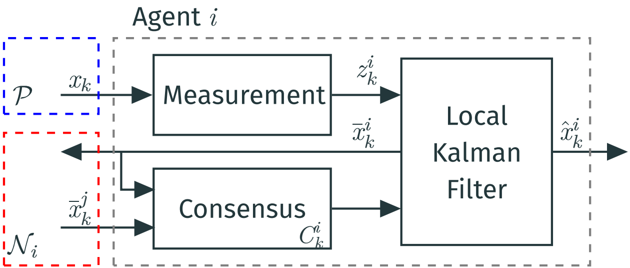
CENTRALIZED CONSENSUS GAIN FACTOR – DISADVANTAGES



- requires the knowledge of global network properties
- a change in network structure or noise properties would require re-calibration
- “heavy” for large scaled systems

Recall the DCKE:

$$\hat{x}_k^i = \bar{x}_k^i + K_k^i \left(z_k^i - H^i \bar{x}_k \right) + C_k^i \sum_{j \in \mathcal{N}_i} \left(\bar{x}_k^j - \bar{x}_k^i \right),$$



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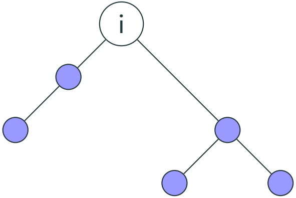
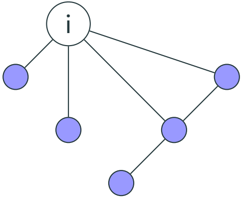
Consider now the decentralized consensus gain,

$$C_k^i = \frac{1}{|\mathcal{N}_{i,k}|} \underbrace{(I - K_k^i H^i)}_{F_k^i},$$

where $\mathcal{N}_{i,k}$ denotes the neighborhood of agent i at time step k.





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where $\mathcal{N}_{i,k}$ denotes the neighborhood of agent i at time step k.
The local noiseless error dynamics are

$$\begin{aligned}
\eta_k^i &= F_k^i A \eta_{k-1}^i + \frac{1}{|\mathcal{N}_{i,k}|} F_k^i A \sum_{j \in \mathcal{N}_{j,k}} \left[\eta_{k-1}^j - \eta_{k-1}^i\right] \\
&= \frac{1}{|\mathcal{N}_{i,k}|} F_k^i A \sum_{j \in \mathcal{N}_{j,k}} \eta_{k-1}^j,
\end{aligned}$$

Proposition

Assume that each sensor in the network measures the process $\mathcal{P}$ using the same observation model

$$z_k^i = Hx_k + v_k^i, i = 1, \dots, N,$$

where v_k^i is the zero-mean Gaussian measurement noise with $\mathbb{E}[v_k^i v_l^i] = R\delta_{kl}$. Then the error dynamics, with the consensus gain $C_k^i = \frac{1}{|\mathcal{N}_{i,k}|}\bar{F}_k$,are asymptotically stable.

Proof

- the augmented error dynamics can be simplified to

$$\eta_k = (I_N \otimes \bar{F}_k A) \underbrace{\left((I_N - \left(\mathcal{D}_k^{-1} L_k\right)) \otimes I_n\right)}_{\textbf{row stochastic}} \eta_{k-1},$$

with $\mathcal{D}_k = \mathtt{diag}\{|\mathcal{N}_{i,k}|\}_{i=1}^{N}$.

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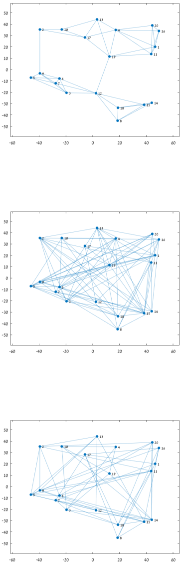
$$\eta_k = (I_N \otimes \bar{F}_k A) \underbrace{\left((I_N - (\mathcal{D}_k^{-1} L_k)) \otimes I_n\right)}_{\textbf{row stochastic}} \eta_{k-1},$$

  with $\mathcal{D}_k = \mathtt{diag}\{|\mathcal{N}_{i,k}|\}_{i=1}^N$.
- This leads to the following inequality

$$\begin{aligned} &\lim_{k\to\infty} \left\| \prod_k \left((I_N - (\mathcal{D}_k^{-1} L_k)) \otimes \bar{F}_k A\right) \right\| \\ &\leq \lim_{k\to\infty} \left\| \prod_k (\bar{F}_k A) \right\| \underbrace{\lim_{k\to\infty} \left\| \prod_k (I_N - (\mathcal{D}_k^{-1} L_k)) \right\|}_{<\infty} = 0. \end{aligned}$$

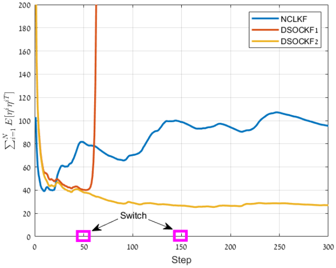
Therefore, the error dynamics are asymptotically stable.



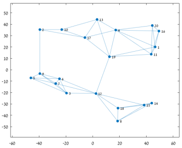
Time varying graph

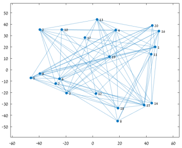
Olfati-Saber and Sandell [3]

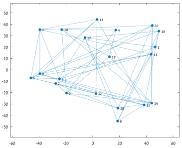
$$C_k^i = \frac{\epsilon}{1 + |\hat{P}_k^i|_F}\hat{P}_k^i,$$



[3] N. F. Sandell and R. Olfati-Saber, "Distributed data associ-ation for multi-target tracking in sensor networks," in 47thIEEE Conference on Decision and Control, pp. 1085–1090,IEEE, 2008.



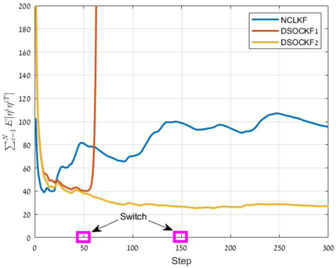




Time varying graph

Olfati-Saber and Sandell

$$C_k^i = \frac{\epsilon}{1 + |\hat{P}_k^i|_F} \hat{P}_k^i,$$



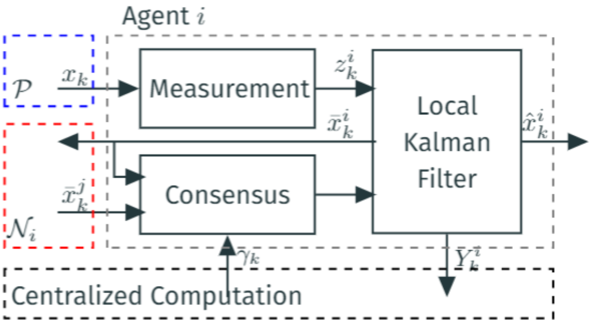
Sum of agent's MSE

Non-cooperative KF

Olfati-Saber

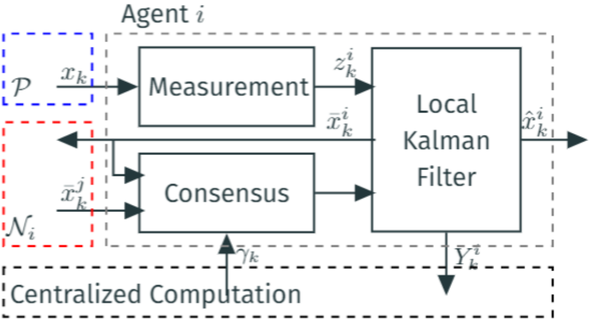
Our proposed gain

◇ widely common sub-optimal *Consensus Kalman* filter scheme



Centralized architecture

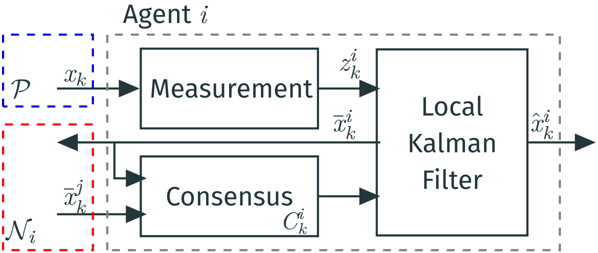
- widely common sub-optimal *Consensus Kalman* filter scheme
- SDP for extracting an upper bound on the consensus factor



Centralized architecture

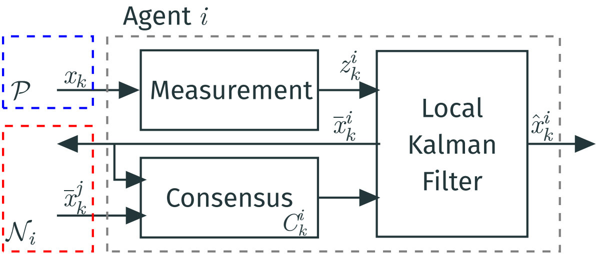
SUMMARY

- decentralized consensus gain which does not require global knowledge of network properties



Decentralized architecture

- decentralized consensus gain which does not require global knowledge of network properties
- performance superiority of both consensus gains over existing solutions in the literature and over the non-cooperative Kalman filter



Decentralized architecture

FUTURE WORK

- event-triggered cooperative estimation

FUTURE WORK

- ◇ event-triggered cooperative estimation
- ◇ CKF with partial non-observability

FUTURE WORK

- ◇ event-triggered cooperative estimation
- ◇ CKF with partial non-observability
- ◇ cooperative estimation with control authorities

FUTURE WORK

- event-triggered cooperative estimation
- CKF with partial non-observability
- cooperative estimation with control authorities
- expand research to account for EKF, Unscented and more...