

```
## Error: package or namespace load failed for 'CARBayes':  
## .onLoad failed in loadNamespace() for 'units', details:  
## call: udunits_init(path)  
## error: function 'Rcpp_precious_remove' not provided by package 'Rcpp'
```

# Bayesian Spatial Analysis

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# Spatial modeling

Three main types of spatial data:

- Point/geo-referenced
- Areal-referenced
- Point process/pattern

# Point-referenced spatial data

## Features

- Some spatial domain  $\mathcal{D}$  is under study
- Measured spatial locations  $s \in \mathcal{D}$  are **pre-determined**
- Some quantity,  $Y(s)$ , is measured at each location  $s \in \mathcal{D}$

## Examples

- Air quality monitoring
- Coastal tide level monitoring
- Earthquake monitoring
- Bird point counts

# Areal-referenced spatial data

## Features

- Some set of spatial regions  $1, \dots, S$  are pre-determined
- Some quantity,  $Y_s$ , is measured as an aggregate over that region

## Examples

- Disease occurrence per county
- Unemployment rate per state
- Inflation per country

# Point-process spatial data

## Features

- Some spatial domain  $\mathcal{D}$  is under study
- Spatial locations  $s \in \mathcal{S} \subset \mathcal{D}$  are **random**
- $Y(s) = 1$  indicates an occurrence of the event

## Examples

- Locations of Mayan ruins
- Locations of invasive species
- Locations of caught Lingcod

# Point-referenced spatial data

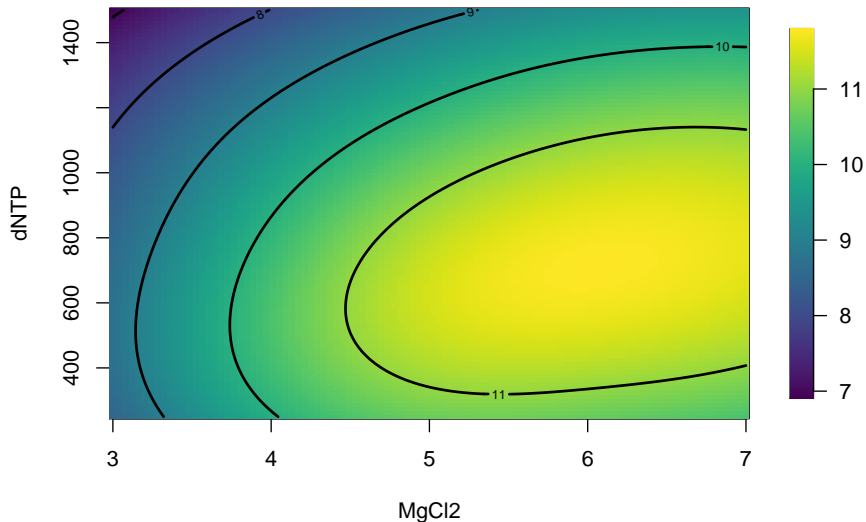
Let  $Y(s)$  for  $s \in \mathcal{D} \subseteq \mathbb{R}^d$  be a spatial process. Let  $E[Y(s)] = 0$  for all  $s \in \mathcal{D}$  because we will model the mean separately.

Assumptions:

- Stationarity
  - Intrinsic stationarity
  - Weak stationarity
  - Strong stationarity
- Isotropy
- Gaussian process

## Example spatial process

log of DNA amplification rate (KCL=29.77, KPO4=32.13)



# Intrinsic stationarity

## Definition

A process  $Y(s)$  is **intrinsically stationary** if  $(E[Y(s+h) - Y(s)] = 0$  and)

$$E[(Y(s+h) - Y(s))^2] = \text{Var}[Y(s+h) - Y(s)] = 2\gamma(h)$$

when  $s, s+h \in \mathcal{D}$ . We call  $2\gamma(h)$  the **variogram** and  $\gamma(h)$  the **semivariogram**.

## Definition

A process  $Y(s)$  is **isotropic** if the semivariogram function depends only on  $\|h\|$ , the length of the separation vector. Otherwise the process is **anisotropic**.



# Weak stationarity

## Definition

A process  $Y(s)$  has **weak stationarity** if ( $E[Y(s)] = \mu$  and)  
 $Cov[Y(s), Y(s+h)] = C(h)$  when  $s, s+h \in \mathcal{D}$ . We call  $C(h)$  the covariance function or covariogram.

Since  $\gamma(h) = C(0) - C(h)$ , a weakly stationary process is also intrinsically stationary.

If the spatial process is **ergodic**, then  $C(h) \rightarrow 0$  as  $\|h\| \rightarrow \infty$  and  $\lim_{\|h\| \rightarrow \infty} \gamma(h) = C(0)$ . Thus

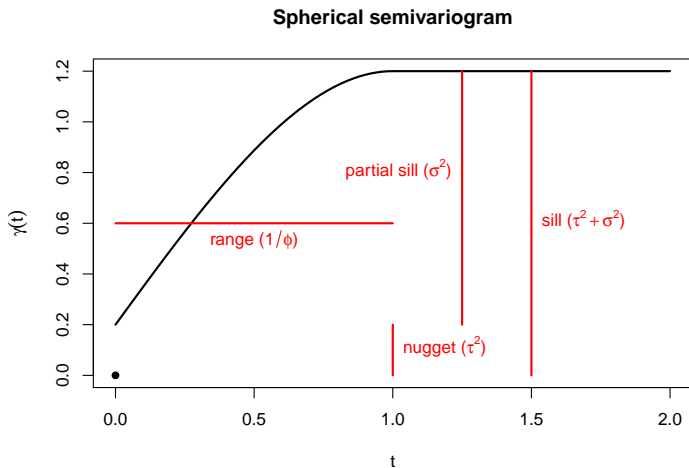
$$C(h) = C(0) - \gamma(h) = \lim_{\|u\| \rightarrow \infty} \gamma(u) - \gamma(h).$$

Thus, if the process is ergodic, an intrinsically stationary process is also weakly stationary.

# Covariance functions for isotropic models

| Model                  | Covariance function, $C(t)$   | Semivariogram, $\gamma(t)$  |
|------------------------|---|---|
| Linear                 | $C(t)$ does not exist   | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 t & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$  |
| Spherical              | $C(t) = \begin{cases} 0 & t > \tau \\ \sigma^2 \left[ 1 - \frac{3}{2}\phi t + \frac{1}{2}(\phi t)^3 \right] & \text{otherwise} \end{cases}$                             | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 & \text{if } t > \tau \\ \tau^2 + \sigma^2 \left[ \frac{3}{2}\phi t - \frac{1}{2}(\phi t)^3 \right] & \text{otherwise} \end{cases}$                                |
| Exponential            | $C(t) = \begin{cases} \sigma^2 \exp(-\phi t) & t > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$   | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 [1 - \exp(-\phi t)] & t > 0 \\ 0 & \text{otherwise} \end{cases}$   |
| Powered exponential    | $C(t) = \begin{cases} \sigma^2 \exp(- \phi t ^p) & t > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$   | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 [1 - \exp(- \phi t ^p)] & t > 0 \\ 0 & \text{otherwise} \end{cases}$   |
| Gaussian               | $C(t) = \begin{cases} \sigma^2 \exp(-\phi^2 t^2) & t > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$   | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 [1 - \exp(-\phi^2 t^2)] & t > 0 \\ 0 & \text{otherwise} \end{cases}$   |
| Rational quadratic     | $C(t) = \begin{cases} \sigma^2 \left( 1 - \frac{t^2}{(1+\phi^2)} \right) & t > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$                                   | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \frac{t^2}{(1+\phi^2)} & t > 0 \\ 0 & \text{otherwise} \end{cases}$  |
| Wave                   | $C(t) = \begin{cases} \sigma^2 \frac{\sin(\phi t)}{\phi t} & t > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$   | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \left[ 1 - \frac{\sin(\phi t)}{\phi t} \right] & t > 0 \\ 0 & \text{otherwise} \end{cases}$  |
| Power law              | $C(t)$ does not exist   | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 t^\lambda & t > 0 \\ 0 & \text{otherwise} \end{cases}$   |
| Matérn                 | $C(t) = \begin{cases} \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} (2\sqrt{\nu}t\phi)^\nu K_\nu(2\sqrt{\nu}t\phi) & t > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$ | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \left[ 1 - \frac{(2\sqrt{\nu}t\phi)^\nu}{2^{\nu-1}\Gamma(\nu)} (2\sqrt{\nu}t\phi)^\nu K_\nu(2\sqrt{\nu}t\phi) \right] & t > 0 \\ 0 & \text{otherwise} \end{cases}$ |
| Matérn ( $\nu = 3/2$ ) | $C(t) = \begin{cases} \sigma^2 (1 + \phi t) \exp(-\phi t) & t > 0 \\ \tau^2 + \sigma^2 & \text{otherwise} \end{cases}$  | $\gamma(t) = \begin{cases} \tau^2 + \sigma^2 [1 - (1 + \phi t) \exp(-\phi t)] & t > 0 \\ 0 & \text{otherwise} \end{cases}$  |

# Spherical semivariogram



# Matérn

Perhaps the most important isotropic process is the Matérn process with covariance

$$C(t) = \begin{cases} \frac{\sigma^2}{2^{\nu-1}\Gamma(\nu)} (2\sqrt{\nu}t\phi)^\nu K_\nu(2\sqrt{\nu}t\phi) & t > 0 \\ \tau^2 + \sigma^2 & t = 0 \end{cases}$$

and variogram

$$\gamma(t) = \begin{cases} \tau^2 + \sigma^2 \left[ 1 - \frac{(2\sqrt{\nu}t\phi)^\nu}{2^{\nu-1}\Gamma(\nu)} K_\nu(2\sqrt{\nu}t\phi) \right] & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

where

- $\nu$  controls the smoothness of the spatial process ( $\lfloor \nu \rfloor$  number of times process realizations are mean square differentiable) while
- $\phi$  is a spatial scale parameter.

Special cases are the exponential ( $\nu = 1/2$ ) and Gaussian ( $\nu \rightarrow \infty$ ).

# Strong stationarity

## Definition

A process  $Y(s)$  is **strongly (or strictly) stationary** if, for any set of  $n \geq 1$  sites  $\{s_1, \dots, s_n\}$  and any  $h \in \mathbb{R}^d$ ,

$$(Y(s_1), \dots, Y(s_n))^{\top} \stackrel{d}{=} (Y(s_1 + h), \dots, Y(s_n + h))^{\top}$$

where  $\stackrel{d}{=}$  means equal in distribution.

If we assume all variances exist, then strong stationarity implies weak stationarity.

The reverse is not necessarily true.

# Gaussian process

## Definition

$Y(s)$  is a **Gaussian process** if, for any  $n \geq 1$  and any set of sites  $\{s_1, \dots, s_n\}$ ,  $Y = (Y(s_1), \dots, Y(s_n))^T$  has a multivariate normal distribution.

For a Gaussian process, weak stationarity and strong stationarity are equivalent.

# Bayesian estimation of Gaussian process parameters

Suppose we observe data at some locations  $s_1, \dots, s_n$ . Collectively, we have  $y = (y(s_1), \dots, y(s_n))$ . Let's assume the data arise from a Gaussian process and according to a particular covariance function. Collectively refer to the parameters as  $\theta$ , then our objective is

$$p(\theta|y) \propto p(y|\theta)p(\theta).$$

Suppose we assume the Matérn covariance function and a common mean  $\mu$  so that  $\theta = (\mu, \nu, \phi, \tau^2, \sigma^2)$ . Then we have

$$p(\mu, \nu, \phi, \tau^2, \sigma^2|y) \propto N(y; \mu, \Sigma)p(\mu, \nu, \phi, \tau^2, \sigma^2)$$

where  $\Sigma$  is constructed from the parameters  $\nu$ ,  $\phi$ ,  $\tau^2$ , and  $\sigma^2$  and the distances between locations, e.g.  $\|s_1 - s_2\|$ .

Consider point-referenced data at spatial locations  $s_1, \dots, s_n$ , model this data as

$$Y(s) = \mu(s) + w(s) + \epsilon(s)$$

If we constrain ourselves to isotropic models, the Matérn class is suggested as a general tool (Banerjee pg. 37). If  $w = (w(s_1), \dots, w(s_n))^T$  and  $\epsilon = (\epsilon(s_1), \dots, \epsilon(s_n))^T$ , then a general model is

$$\text{Var}[w] = \sigma^2 H(\phi) \quad \text{Var}[\epsilon] = \tau^2 \mathbf{I}$$

where  $H$  is a correlation matrix with  $H_{ij} = \rho(s_i - s_j; \phi)$  and  $\rho$  is a valid isotropic correlation function on  $\mathbb{R}^r$ , i.e. Matérn:

$$\rho(u; \nu, \phi) = \frac{(u/\phi)^\nu K_\nu(u/\phi)}{2^{\nu-1} \Gamma(\nu)}$$

as defined in `geoR:matern`. The overall mean is modeled separately and uses covariates  $x(s)$  via

$$\mu(s) = x(s)^\top \beta.$$



# Bayesian estimation for spatial random effects

Let  $\theta = (\beta, \sigma^2, \tau^2, \phi)$ , then parameter estimates may be obtained from the posterior distribution:

$$p(\theta|y) \propto p(y|\theta)p(\theta)$$

where

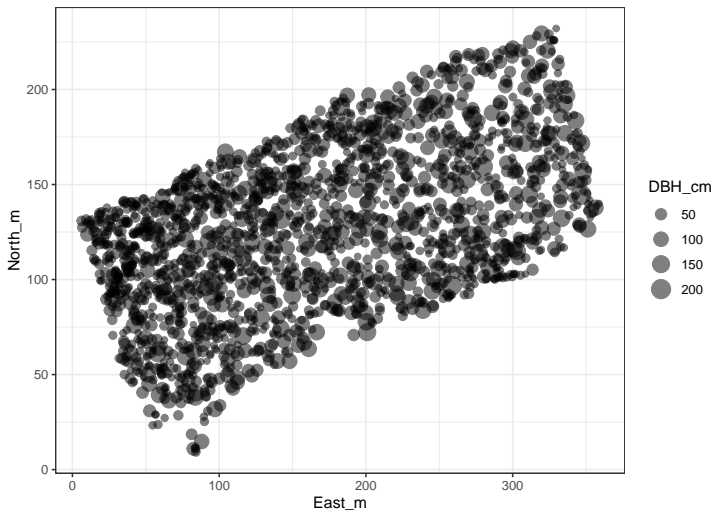
$$Y|\theta \sim N(X\beta, \sigma^2 H(\phi) + \tau^2 I).$$

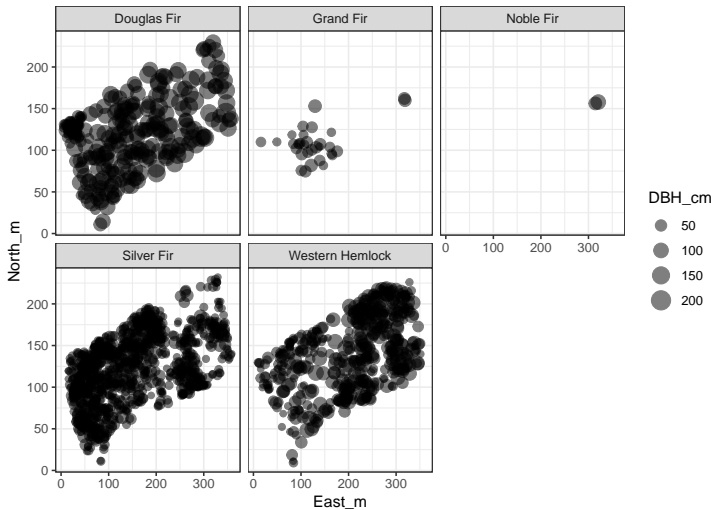
Typically, independent priors are chosen so that

$$p(\theta) = p(\beta)p(\sigma^2)p(\tau^2)p(\phi).$$

As a general rule, non-informative priors can be chosen for  $\beta$ , e.g.  $p(\beta) \propto 1$ . However, improper (or vague proper) priors for the variance parameters can lead to improper (or computationally improper) posteriors.

# Diameter at breast height (DBH) for an experimental forest





# Interpolation of mean DBH (ignoring species)

```
## Error in library("MBA"): there is no package called 'MBA'  
## Error in mba.surf(d[c("East_m", "North_m", "DBH_cm")], no.X = x.res, no.Y = y.res, : could not find  
function "mba.surf"  
## Error in imagePlotInfo(..., breaks = breaks, nlevel = nlevel): object 'surf' not found  
## Error in contour(surf, add = T): object 'surf' not found
```

# Regression

```
## variog: computing omnidirectional variogram
## variofit: covariance model used is exponential
## variofit: weights used: equal
## variofit: minimisation function used: nls
##
## Call:
## lm(formula = DBH_cm ~ Species, data = d)
##
## Residuals:
```

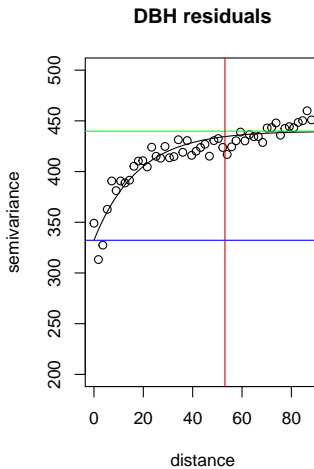
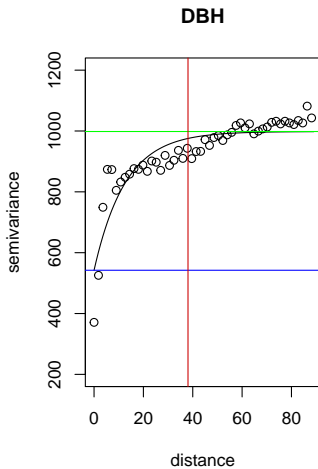
|  | Min     | 1Q     | Median | 3Q     | Max     |
|--|---------|--------|--------|--------|---------|
|  | -78.423 | -9.969 | -3.561 | 10.924 | 118.277 |

```
##
## Coefficients:
```

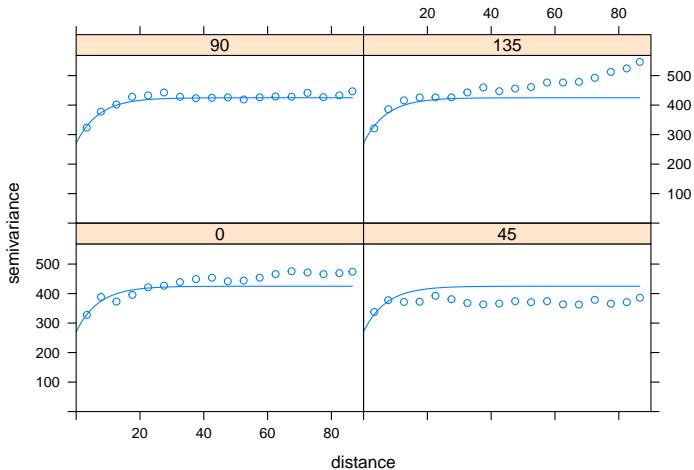
|                        | Estimate | Std. Error | t value | Pr(> t )   |
|------------------------|----------|------------|---------|------------|
| (Intercept)            | 89.423   | 1.303      | 68.629  | <2e-16 *** |
| SpeciesGrand Fir       | -51.598  | 4.133      | -12.483 | <2e-16 *** |
| SpeciesNoble Fir       | -5.873   | 15.744     | -0.373  | 0.709      |
| SpeciesSilver Fir      | -68.347  | 1.461      | -46.784 | <2e-16 *** |
| SpeciesWestern Hemlock | -48.062  | 1.636      | -29.377 | <2e-16 *** |

```
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 22.19 on 1950 degrees of freedom
## Multiple R-squared:  0.5332, Adjusted R-squared:  0.5323
## F-statistic: 556.9 on 4 and 1950 DF,  p-value: < 2.2e-16
```

# Variogram (exponential model)



# Isotropy?



# spBayes

```

p = nlevels(d$Species)
r = spLM(DBH_cm ~ Species,
  data = d,
  coords = as.matrix(d[c('East_m','North_m')]),
  knots = c(6,6,.1), # for spatial prediction
  cov.model = 'exponential',

  starting = list(tau.sq = fit.DBH.resid$nugget,
    sigma.sq = fit.DBH.resid$cov.pars[1],
    phi = fit.DBH.resid$cov.pars[2]),

  tuning = list(tau.sq = 0.015,
    sigma.sq = 0.015,
    phi = 0.015),

  priors = list(beta.Norm = list(rep(0,p), diag(1000,p)),
    phi.Unif = c(3/1,3/0.1),
    sigma.sq.IG = c(2,200),
    tau.sq.IG = c(3,300)),

  n.samples = 10000,
  n.report = 200,
  verbose=TRUE)

```



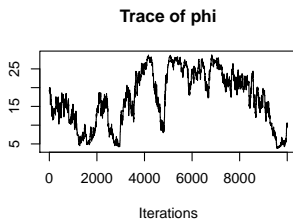
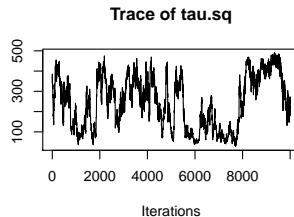
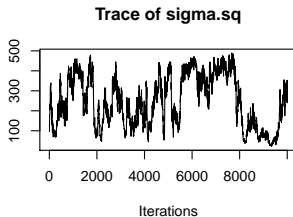
```

## -----
## General model description
## -----
## Model fit with 1955 observations.
##
## Number of covariates 5 (including intercept if specified).
##
## Using the exponential spatial correlation model.
##
## Using modified predictive process with 36 knots.
##
## Number of MCMC samples 10000.
##
## Priors and hyperpriors:
##   beta normal:
##   mu: 0.000 0.000 0.000 0.000 0.000
##   cov:
##   1000.000 0.000 0.000 0.000 0.000
##   0.000 1000.000 0.000 0.000 0.000
##   0.000 0.000 1000.000 0.000 0.000
##   0.000 0.000 0.000 1000.000 0.000
##   0.000 0.000 0.000 0.000 1000.000
##
## sigma.sq IG hyperpriors shape=2.00000 and scale=200.00000
## tau.sq IG hyperpriors shape=3.00000 and scale=300.00000
## phi Unif hyperpriors a=3.00000 and b=30.00000
## -----
## Sampling
## -----
## Sampled: 200 of 10000, 2.00%
## Report interval Metrop. Acceptance rate: 36.50%
## Overall Metrop. Acceptance rate: 36.50%
## -----
## Sampled: 400 of 10000, 4.00%
## Report interval Metrop. Acceptance rate: 36.50%

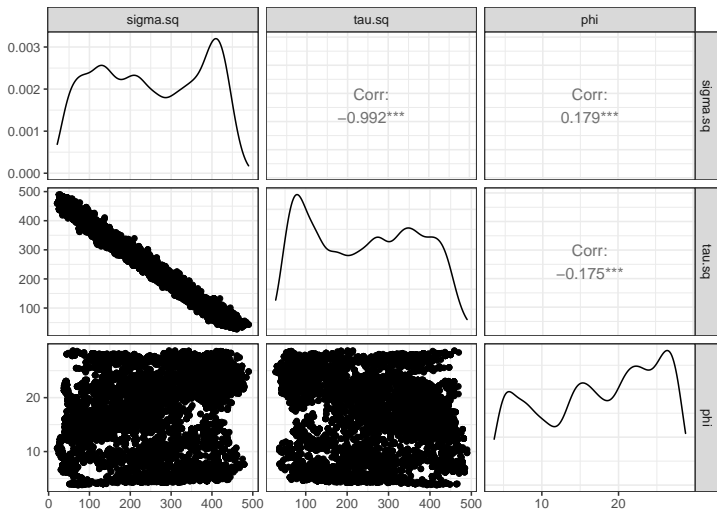
```

# Traceplots

```
plot(r$sp.theta.samples, density=FALSE)
```

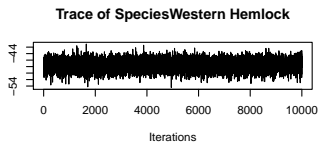
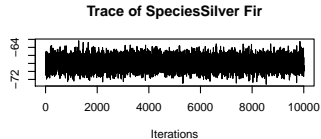
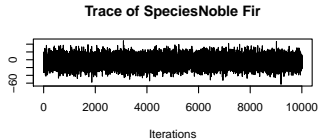
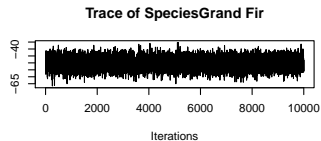
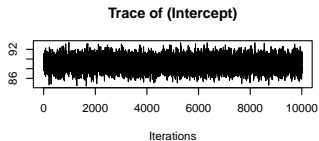


```
r$p.theta.samples[burnin:nreps,] %>%
  as.data.frame %>%
  GGally::ggpairs() +
  theme_bw()
```



# Traceplot 2s

```
plot(r$beta.samples, density=FALSE)
```



# Summary statistics

```
summary(r$p.theta.samples)

##
## Iterations = 1:10000
## Thinning interval = 1
## Number of chains = 1
## Sample size per chain = 10000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##           Mean      SD Naive SE Time-series SE
## sigma.sq 246.59 127.324  1.27324      31.224
## tau.sq   244.94 126.725  1.26725      30.490
## phi      17.24   7.204  0.07204       2.743
##
## 2. Quantiles for each variable:
##
##           2.5%   25%   50%   75% 97.5%
## sigma.sq 43.200 133.07 238.13 366.61 446.7
## tau.sq   51.208 124.40 252.81 355.24 449.0
## phi      4.593  11.46  17.55  23.58  27.9
```

# Summary statistics 2

```
summary(r$p.beta.samples)

##
## Iterations = 1:10000
## Thinning interval = 1
## Number of chains = 1
## Sample size per chain = 10000
##
## 1. Empirical mean and standard deviation for each variable,
##    plus standard error of the mean:
##
##              Mean      SD Naive SE Time-series SE
## (Intercept)    89.011  1.291  0.01291      0.01291
## SpeciesGrand Fir -50.361  4.125  0.04125      0.04125
## SpeciesNoble Fir  -4.406 14.034  0.14034      0.14034
## SpeciesSilver Fir -67.923  1.458  0.01458      0.01458
## SpeciesWestern Hemlock -47.605  1.631  0.01631      0.01631
##
## 2. Quantiles for each variable:
##
##              2.5%    25%    50%    75%   97.5%
## (Intercept)    86.48  88.13  89.006  89.873  91.53
## SpeciesGrand Fir -58.49 -53.11 -50.306 -47.617 -42.21
## SpeciesNoble Fir -32.44 -13.91  -4.382   5.187  22.83
## SpeciesSilver Fir -70.79 -68.90 -67.917 -66.944 -65.09
## SpeciesWestern Hemlock -50.79 -48.69 -47.615 -46.506 -44.44
```

# Spatial surface

If interest resides in  $w$ , draws can be obtained using the following relationship

$$p(w|y) = \int p(w|\sigma^2, \phi, y) p(\sigma^2, \phi|y) d\sigma^2 d\phi$$

which suggests the following strategy:

1. Run the MCMC sampler to obtain draws  $(\sigma^2, \phi)^{(g)} \sim p(\sigma^2, \phi|y)$
2. After burn-in and for  $g = 1, \dots, G$ , sample  $w^{(g)} \sim p(w|(\sigma^2, \phi)^{(g)}, y)$ .

# Prediction

For prediction at points  $s_{01}, \dots, s_{0m}$  and denoting  $Y_0 = (Y(s_{01}), \dots, Y(s_{0m}))^\top$  and design matrix  $X_0$  having rows  $x(s_{0j})^\top$ , we have the following relationship

$$p(y_0|y, X, X_0) = \int p(y_0|y, \theta, X_0)p(\theta|y, X)d\theta \approx \frac{1}{G} \sum_{g=1}^G p(y_0|y, \theta^{(g)}, X_0).$$

It is more common to take draws  $y_0^{(g)} \sim p(y_0|y, \theta^{(g)}, X_0)$  and estimate the predictive distribution using

$$p(y_0|y, X, X_0) \approx \frac{1}{G} \sum_{g=1}^G \delta_{y_0^{(g)}}$$

where  $p(y_0|y, \theta, X_0)$  has a conditional normal distribution.



# Predictions are not conditionally independent

Consider the joint distribution for  $y$  and  $y_0 = y(s_0)$  (a scalar for simplicity), then

$$\begin{pmatrix} y \\ y_0 \end{pmatrix} \sim N \left( \begin{bmatrix} X\beta \\ X_0\beta \end{bmatrix}, \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{bmatrix} \right)$$

where

$$\Omega_{11} = \sigma^2 H(\phi) + \tau^2 \mathbf{I}$$

$$\Omega_{22} = \sigma^2 + \tau^2$$

$$\Omega_{12}^\top = \sigma^2 [\rho(d_{01}; \phi), \dots, \rho(d_{0n}; \phi)]$$

and  $d_{ij} = \|s_i - s_j\|$ .

Thus  $y_0|y, \theta, X, X_0$  is normal with

$$\begin{aligned} E[Y(s_0)|y, \theta, X, X_0] &= x_0^\top \beta + \Omega_{12}^\top \Omega_{22}^{-1} (y - X\beta) \\ \text{Var}[Y(s_0)|y, \theta, X, X_0] &= \sigma^2 + \tau^2 - \Omega_{12}^\top \Omega_{22}^{-1} \Omega_{12} \end{aligned}$$

# Generalized linear spatial modeling

Let  $Y(s)$  be the response of interest with

$$E[Y(s)] = g^{-1}(x(s)^\top \beta + w(s))$$

where  $w(s)$  is our spatial random effect.

For example, Poisson regression

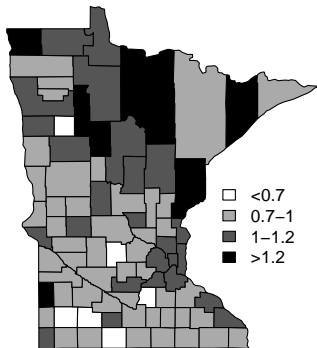
$$Y(s) \sim Po(e^{x(s)^\top \beta + w(s)}).$$

For GLMs (other than linear models),  $w(s)$  cannot be integrated out and therefore a common MCMC strategy is

1. Sample  $\beta | \dots$
2. Sample  $w | \dots$
3. Sample  $\theta | \dots$  (the spatial parameters [no nugget]).

# Choropleth

## MN Lung Cancer SMR



# Modeling areal units

Let  $Y_i$  represent the SMR for lung cancer in MN county  $i$ . Consider the model defined by conditional distributions:

$$Y_i|y_{-i} \sim N \left( \sum_{j \in n_i} y_j / m_i, \tau^2 / m_i \right)$$

where

- $n_i$  indicates the neighbors of  $i$
- $m_i$  indicates the number of neighbors for  $i$

This defines a *Markov Random Field*.

# Brook's Lemma

It is clear that given  $p(y_1, \dots, y_n)$ , the *full conditionals*, i.e.  $p(y_i|y_{-i})$ , are determined.

## Definition

**Brook's Lemma** states that

$$\frac{p(y_1, \dots, y_n)}{p(y'_1, \dots, y'_n)} = \frac{p(y_1|y_2, \dots, y_n)}{p(y'_1|y_2, \dots, y_n)} \cdot \frac{p(y_2|y'_1, y_3, \dots, y_n)}{p(y'_2|y'_1, y_3, \dots, y_n)} \dots \frac{p(y_n|y'_1, \dots, y'_{n-1})}{p(y'_n|y'_1, \dots, y'_{n-1})}$$

for all  $(y'_1, \dots, y'_n)$ .

If

$$p(y'_1, \dots, y'_n) = \int \frac{p(y_1|y_2, \dots, y_n)}{p(y'_1|y_2, \dots, y_n)} \cdot \frac{p(y_2|y'_1, y_3, \dots, y_n)}{p(y'_2|y'_1, y_3, \dots, y_n)} \dots \frac{p(y_n|y'_1, \dots, y'_{n-1})}{p(y'_n|y'_1, \dots, y'_{n-1})} dy_1, \dots, dy_n < \infty$$

then  $p(y_1, \dots, y_n)$  is a proper joint distribution.

# Conditionally autoregressive models

More generally, we can consider

$$Y_i|y_{-i} \sim N\left(\sum_{j \neq i} b_{ij}y_j, \tau_i^2\right)$$

Through Brook's Lemma, we have

$$p(y_1, \dots, y_n) \propto \exp\left(-\frac{1}{2}y^\top D^{-1}[\mathbf{I} - B]y\right)$$

where

- $B$  has elements  $b_{ij}$
- $D$  is diagonal with elements  $\tau_i^2$

In order for  $D^{-1}[\mathbf{I} - B]$  to be symmetric, we need  $\frac{b_{ij}}{\tau_i^2} = \frac{b_{ji}}{\tau_j^2}$  for all  $i, j$ .

# Proximity matrix

## Definition

A **proximity matrix** is a an  $n \times n$  matrix,  $W$ , with elements

- $w_{ii} = 0$  and
- $w_{ij}$  representing the “distance” between unit  $i$  and unit  $j$

Common choices for  $w_{ij}$  are

- 1 if  $i$  is a neighbor of  $j$  and 0 otherwise
  - neighbors defined by those who share an edge
  - neighbors defined by those who share a point
  - neighbors defined by those who are within distance  $\delta$
  - $K$ -nearest neighbors
- “distance”
  - inverse intercentroidal distance
  - inverse minimum distance plus  $c$

# Intrinsically autoregressive model

Recall

$$p(y_1, \dots, y_n) \propto \exp \left( -\frac{1}{2} y^\top D^{-1} [I - B] y \right)$$

if we set  $w_{i+} = \sum_{j=1}^n w_{ij}$ ,  $b_{ij} = w_{ij}/w_{i+}$ , and  $\tau_i^2 = \tau^2/w_{i+}$ , we have

$$p(y_1, \dots, y_n) \propto \exp \left( -\frac{1}{2\tau^2} y^\top [D_w - W] y \right)$$

where

- $W$  is our proximity matrix and
- $D_w$  has diagonal elements  $w_{i+}$

This can be rewritten as

$$p(y_1, \dots, y_n) \propto \exp \left( -\frac{1}{2\tau^2} \sum_{i \neq j} w_{ij} (y_i - y_j)^2 \right)$$

This is called the *intrinsically autoregressive* model.



# Proper CAR models

To make this proper,

$$p(y_1, \dots, y_n) \propto \exp \left( -\frac{1}{2\tau^2} y^\top [D_w - \rho W] y \right)$$

with

- $\rho \in (1/\lambda_{(n)}, 1/\lambda_{(1)})$  where
- $\lambda_{(1)} < \dots < \lambda_{(n)}$  are the ordered eigenvalues of  $D_w^{-1/2} W D_w^{-1/2}$ .

The full conditionals are

$$Y_i | y_{-i} \sim N \left( \rho \sum_{j \neq i} w_{ij} y_j / w_{i+}, \tau^2 / w_{i+} \right)$$

a prior for  $\rho$  that induces a reasonable amount of spatial association should put most of its mass near 1.

# Issues with the proper CAR

The full condition for the proper CAR, i.e.

$$Y_i|y_{-i} \sim N \left( \rho \sum_{j \neq i} w_{ij} y_j / w_{i+}, \tau^2 / w_{i+} \right)$$

indicates some issues with this model:

- $\tau^2$  does not play a role in spatial association.
- $\rho$  is a proportional reaction to the weighted average of its neighbors.
- If  $\rho < 1$ , then the expected value of the current location is less than the weighted average of its neighbors.
- If  $\rho = 0$ , then we have conditional independence. But the variance decreases with the number of neighbors which is perplexing.
- $\rho$  needs to be very close to 1 to obtain a consequential amount of spatial association.

# Dealing with $\rho$ in the proper CAR

- Choose  $\rho$  so the CAR model is proper
- Choose  $\rho = 1$  (improper IAR model) and constrain  $\sum_{i=1}^n Y_i = 0$
- Choose  $\rho = 1$  and estimate a mean (remove mean from the fixed effect)
- Let  $\rho \sim Be(18, 2)$  (Banerjee pg 164) and estimate it.

# Leroux CAR

The Leroux et al. (1999) CAR tries to ameliorate these issues. The joint distribution is

$$Y \sim N(0, \tau^2[\rho(D_w - W) + (1 - \rho)I]^{-1})$$

and the conditional distributions are

$$Y_i|y_{-i} \sim N\left(\frac{\rho \sum_{j \neq i} w_{ij} y_j}{\rho \sum_{j \neq i} w_{ij} y_j + 1 - \rho}, \frac{\tau^2}{\rho \sum_{j \neq i} w_{ij} y_j + 1 - \rho}\right).$$

This distribution is proper so long as  $0 \leq \rho < 1$ . Lee (2011) argued that this CAR should be preferred for a variety of reasons.

# CAR as a model for random effects

Let

- $Y_i$  represent the (continuous) response for observation  $i$
- $X_i$  represent explanatory variables for observation  $i$
- $s[i]$  represent the areal unit for observation  $i$

then a possible model is

$$Y_i = X_i^\top \beta + \omega_{s[i]} + \epsilon_i$$

where

- $\epsilon_i \stackrel{ind}{\sim} N(0, \sigma^2)$  is noise
- $\omega_s$  is the spatial random effect associated with areal unit  $s$ , e.g.

$$p(\omega_1, \dots, \omega_S) \propto \exp \left( -\frac{1}{2\tau^2} \omega^\top [D_w - \rho W] \omega \right)$$

# Housing price model

Let

- $Y_i$  be the logarithm of the median home price in each Intermediate Geography (IG) to the north of the river Clude in the Greater Glasgow and Clyde health board,
- use explanatory variables
  - crime: crime rate (number of crimes per 10,000 people) in each IG (logged),
  - rooms: median number of rooms in a property in each IG,
  - type: predominant property type in each IG with levels: detached, flat, semi, terrace,
  - sales: percentage of properties that sold in each IG in a year, and
  - driveshop: average time taken to drive to a shopping centre in minutes (logged).

# Housing price model

Assume

$$Y_i \stackrel{\text{ind}}{\sim} N(X_i\beta + \omega_{s[i]}, \nu^2)$$

or, alternatively,

$$Y_i = X_i\beta + \omega_{s[i]} + \epsilon_i, \quad \epsilon_i \stackrel{\text{ind}}{\sim} N(0, \nu^2)$$

where

- $\beta$  are the regression parameters and
- $\omega_s$  are assumed to come from an intrinsic CAR model with proximity matrix indicating those regions that share a border