Math 342: Homework 1

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Documentation: I used ChatGPT solely for looking up Latex commands. The main Homework 1 MatLab script and all required dependencies are located in the Homework 1 folder found here: https://github.com/Connor-Lemons/Emmons-Math-342. No other resources used.

Problem 1 (13a,b):

Find the third Taylor polynomial $P_3(x)$ for the function $f(x) = (x-1)ln(x)$ about $x_0 = 1$.

First, begin with Taylor's Theorem, which gives that the polynomial expansion of some function is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \tag{1}$$

where n determines the highest derivative included in the polynomial. For the third Taylor polynomial, $n = 3$. To find $P_3(x)$, find the higher order derivatives of $f(x)$ up to the third derivative.

$$f(x) = (x-1)ln(x) \tag{2}$$

$$f^{(1)}(x) = \ln(x) + \frac{x-1}{x} \tag{3}$$

$$f^{(2)}(x) = \frac{2}{x} - \frac{x-1}{x^2} \tag{4}$$

$$f^{(3)}(x) = \frac{2(x-1)}{x^3} - \frac{3}{x^2} \tag{5}$$

Expanding out the summation and factorials in equation (1) for $n = 3$ gives

$$P_3(x) = f(x_0) + f^{(1)}(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2}(x-x_0)^2 + \frac{f^{(3)}(x_0)}{6}(x-x_0)^3$$

and plugging in the derivatives of $f(x)$ evaluated at x_0 and x_0 gives

$$P_3(x) = (x-1)^2 - \frac{(x-1)^3}{2} \tag{6}$$

Use $P_3(0.5)$ to approximate $f(0.5)$.

Plugging in $x = 0.5$ gives an approximation of $f(x)$.

$$f(0.5) \approx P_3(0.5) = \frac{5}{16} \tag{7}$$

Find an upper bound for the error $|f(0.5) - P_3(0.5)|$ using the error formula and compare it to the actual error.

In order to find an upper bound for the error of such a Taylor polynomial, consider the next term of the Taylor polynomial, denoted $R_n(x)$. For $P_3(x)$, this is:

$$R_n(x) = \frac{f^{(4)}(\xi(x))}{24}(x-1)^4 \tag{8}$$

The function $\xi(x)$ represents the truncation error associated with the Taylor polynomial, though its value depends on the value of x at which the original function is approximated by $P_3(x)$. However, it is known that the value of this function will always be between x and x_0. In order to determine the upper bound on the error for this polynomial from this, first expand out the 4th derivative in $R_n(x)$.

$$R_n(x) = \frac{\frac{8}{\xi(x)^3} - \frac{6(\xi(x)-1)}{\xi(x)^4}}{24}(\mathrm{x}-1)^4 \tag{9}$$

The value of $\xi(x)$ that will maximize $R_n(x)$ is the value of x which, when the evaluation of R_n is taken at it, will give the upper bound of the error. For this fraction and for $0.5 \leq \xi(x) \leq x$, the error is maximized when $\xi(x) = 0.5$. Thus, the upper bound of the error of $P_3(x)$ is:

$$\text{Error}_{upper} \leq |R_n(0.5)| = \frac{(0.5-1)^4(0.5+3)}{12(0.5)^4}\Big| = 0.292 \tag{10}$$

Compare this to the actual error, which is given by:

$$Error = |f(0.5) - P_3(0.5)| = 0.0341 \tag{11}$$

The upper bound on the error is greater than the actual error by an order of magnitude.

Find a bound for the error $|f(x) - P_3(x)|$ in using $P_3(x)$ to approximate $f(x)$ on the interval $[0.5,1.5]$.

For the given interval, the upper bound on the error will be maximized when approximating $f(0.5)$, specifically when $\xi(x)$ (which itself is bounded by $0.5 \leq \xi(x) \leq 1$) is allowed to equal 0.5. Thus, the upper bound on the error for approximating $f(x)$ in the interval $[0.5,1.5]$ using $P_3(x)$ is:

$$\text{Error}_{upper} \leq |R_n(0.5)| = \frac{(0.5-1)^4(0.5+3)}{12(0.5)^4}\Big| = 0.292 \tag{12}$$

Problem 2 (19):

Let $f(x) = e^x$ and $x_0 = 0$. Find the nth Taylor polynomial $P_n(x)$ for $f(x)$ about x_0.

Begin with Taylor's Theorem, which gives:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \tag{1}$$

Because the derivative of e^x is itself, and $\mathrm{f(x)} = e^x$ evaluated at $x_0 = 0$ gives $f(0) = 1$, equation (1) simplifies to:

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + \mathrm{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} \tag{2}$$

Find a value of n necessary for $P_n(x)$ to approximate $f(x)$ to within 10^{-6} on the interval $[0,0.5]$.

Because the Taylor polynomial for e^x consistently underestimates the actual value of the function (i.e., the Taylor polynomial never crosses the function e^x, at least in the RHP), the error will be maximized at $x = 0.5$. Running through a while loop to determine the n value at which the error when compared to $f(0.5)$ is strictly less than 10^{-6} gives that $n = 7$. The resulting Taylor polynomial is:

$$P_7(x) = \frac{x^7}{5040} + \frac{x^6}{720} + \frac{x^5}{120} + \frac{x^4}{24} + \frac{x^3}{6} + \frac{x^2}{2} + x + 1 \tag{3}$$

and the error is:

$$|f(0.5) - P_7(0.5)| = 1.03 \times 10^{-7} \tag{4}$$

Problem 3 (1d, 4c):

Compute the absolute error and relative error in approximations of p by p^* for $p = \sqrt{2}$ and $p^* = 1.414$.

The absolute error is given by:

$$|p - p^*| = |\sqrt{2} - 1.414| = 2.14 \times 10^{-4} \tag{5}$$

The relative error is given by:

$$\frac{|p-p^*|}{|p|} = \frac{|\sqrt{2}-1.414|}{\sqrt{2}} = 1.51 \times 10^{-4} \tag{6}$$

Note that no computer can store the value of $\sqrt{2}$ exactly, and thus there is some potential for additional error due to rounding.

Find the largest interval in which p^* must lie to approximate p with a relative error of at most 10^{-4} for $p = \sqrt{2}$.

For the relative error to be at most 10^{-4}, the following inequality must hold:

$$\frac{|\sqrt{2}-p^*|}{|\sqrt{2}|} \leq 10^{-4} \tag{7}$$

Solving for p^* gives:

$$p^* \geq \sqrt{2} - \sqrt{2} \times 10^{-4}; p^* \leq \sqrt{2} \tag{8a}$$

$$p^* \leq \sqrt{2} + \sqrt{2} \times 10^{-4}; p^* > \sqrt{2} \tag{8b}$$

These equations give the upper and lower bounds of p^* as

$$\sqrt{2} - \sqrt{2} \times 10^{-4} \leq p^* \leq \sqrt{2} + \sqrt{2} \times 10^{-4}$$

which gives the closed interval for which p^* satisfies the conditions as

$$\left[\sqrt{2} - \sqrt{2} \times 10^{-4}, \sqrt{2} + \sqrt{2} \times 10^{-4}\right] \tag{9}$$

Note that the numerical evaluation of this interval is prone to additional error due to the aforementioned limitations of computers.

$$[1.414072, 1.414355] \tag{10}$$

Problem 4 (6b,c):

Use three-digit rounding arithmetic to perform the calculations: $133 - 0.499$ and $(121 - 0.327) - 119$. Compute the absolute error and relative error with the exact value determined to at least five digits.

Each of the two entries is already represented with three significant digits. The calculation without the rounding arithmetic is:

$$133 - 0.499 = 132.501 \tag{1}$$

Applying the three-digit rounding gives:

$$133 - 0.499 = 133 \quad (2)$$

The absolute error is given by

$$|132.501 - 133| = 0.49900 \quad (3)$$

and the relative error is given by

$$\frac{|132.501 - 133|}{|132.501|} = 0.00377 \quad (4)$$

For the calculation $(121 - 0.327) - 119$, once again, all the numbers are already represented by three significant digits. The calculation performed properly gives:

$$(121 - 0.327) - 119 = 1.673 \quad (5)$$

To perform this calculation with three-digit rounding, first compute $121 - 0.327$.

$$121 - 0.327 = 121 \quad (6)$$

Next, compute $121 - 119$.

$$121 - 119 = 2 \quad (7)$$

The absolute error is given by

$$|1.673 - 2| = 0.32700 \quad (8)$$

and the relative error is given by

$$\frac{|1.673 - 2|}{|1.673|} = 0.19546 \quad (9)$$

Problem 5 (16b,c):

Use four-digit rounding arithmetic and the formulas (1.1), (1.2), and (1.3) to find the most accurate approximations to the roots of $\pi x^2 + 13x + 1 = 0$ and $x^2 + x - e = 0$.

		$x = -0.0784$ (1.1)	$x = -4.0596$ (1.1)	$x = -0.0784$ (1.2) and (1.3)	$x = -4.0596$ (1.2) and (1.3)
$\pi x^2 + 13x + 1 = 0$	Absolute	0.00043	0.00038	0.000008	0.02238
	Relative	0.00547	0.00009	0.00011	0.00551
$x^2 + x - e = 0$	Absolute	0.00013	0.00013	0.00013	0.00013
	Relative	0.00011	0.00006	0.00011	0.00006

Please refer to the GitHub page in the documentation for the MatLab script containing the work.

Problem 6 (21):

Suppose two points (x_0, y_0) and (x_1, y_1) are on a straight line with $y_1 \neq y_0$. Show that both $x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0}$ and $x = x_0 - \frac{(x_1 - x_0) y_0}{y_1 - y_0}$ are valid formulas to find the x-intercept of the line.

First, begin with the equation of the line with $y = 0$.

$$-y_1 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \tag{1}$$

Solve for x.

$$x_0 y_0 - x_1 y_0 = (y_1 - y_0)(x - x_0) \tag{2}$$

$$\frac{x_0 y_0 - x_1 y_0}{y_1 - y_0} = \mathrm{x} - x_0 \tag{3}$$

$$\frac{x_0 y_0 - x_1 y_0}{y_1 - y_0} + x_0 = x \tag{4}$$

$$x = \frac{x_0 y_1 - x_1 y_0}{y_1 - y_0} \tag{5}$$

Returning to equation (4), rewrite the fraction on the LHS.

$$\mathrm{x} = x_0 - \frac{x_1 y_0 - x_0 y_0}{y_1 - y_0} \tag{6}$$

$$\mathrm{x} = x_0 - \frac{\mathrm{y}_0(x_1 - x_0)}{y_1 - y_0} \tag{7}$$

Use the data $(x_0, y_0) = (1.31, 3.24)$ and $(x_1, y_1) = (1.93, 4.76)$ and three-digit rounding arithmetic to compute the x-intercept both ways. Which method is better, and why?

Assigning equation (5) the label of Method 1 and equation (7) the label of Method 2, computing the x-intercept gives the following:

	Calculated Value	Absolute Error
Actual	-0.011579	0
Method 1	-0.00658	0.004999
Method 2	-0.01	0.001579

Based on the absolute error, Method 2 is better. This is because Method 2 reduces the number of times an operation is performed, rounded according to three-digit rounding arithmetic, then used in another operation. In other words, by moving one of the operations out of the fraction, Method 2 reduces the error which propagates through to the final value of the calculated fraction and instead introduces it at the very end of the calculation where it has less effect.

Problem 7 (6a, 7a):

Find the rate of convergence of the following sequence as $\mathrm{n} \to \infty$.

$$\lim_{n \to \infty} \sin\left(\frac{1}{n}\right) \tag{1}$$

First consider the first three terms of the Taylor expansion of $sin(x)$.

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{2}$$

Note that substituting $x = \frac{1}{n}$ gives $\frac{1}{n}$ as the first term, and higher powers of $\frac{1}{n}$ in the following terms. For such an expansion, the rate of convergence as $n \to \infty$ is determined by the lowest power of $\frac{1}{n}$. The rate of convergence of $\lim_{n\to\infty} sin\left(\frac{1}{n}\right)$ is $\frac{1}{n}$ or

$$sin\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right) \tag{3}$$

Find the rate of convergence of the following as $h \to 0$.

$$\lim_{h\to 0} \frac{sin(h)}{h} = 1 \tag{4}$$

Using similar logic to 6a, $sin(h)$ converges at a rate of h, or

$$sin(h) = O(h) \tag{5}$$

Of course, the rate of convergence of h as $h \to 0$ is h, or $O(h)$. For $\frac{sin(h)}{h}$, because the top and the bottom converge at the same rate, the rate of convergence of this function is 1, or

$$\frac{sin(h)}{h} = O(1) \tag{6}$$

Problem 8 (8a):

Suppose that $0 < q < p$ and that $\alpha_n = \alpha + O(n^{-p})$. Show that $\alpha_n = \alpha + O(n^{-q})$.

Consider the rate of convergence for $\frac{1}{n^p}$ and $\frac{1}{n^q}$. Because p and q are necessarily positive numbers, and because only large values of n are considered for convergence (i.e., $n > 1$), the rate of convergence of $\frac{1}{n^q}$ is guaranteed to be slower than the rate of convergence of $\frac{1}{n^p}$. Thus, if α_n is known to converge at a rate of $\frac{1}{p}$, then α_n is, by extension of the previous statement, guaranteed to converge at a rate of at least $\frac{1}{n^q}$. Thus, though perhaps not the fastest convergence α_n is known to have, $\alpha_n = \alpha + O(n^{-q})$ holds.

Problem 9 (13, 14b):

Describe the output of the algorithm and compare it to the illustration.

The output of the algorithm shown will return the summation of all elements from 1 to n of some sequence defined: $x_1, x_2, x_3, \dots, x_n$. As long as the subsequent x values are relatively on the same order, the summation will grow in a stable, (relatively) linear fashion. In other words, unless subsequent x values follow unstable exponential growth themselves or some other unstable growth pattern, the summation will grow in a stable linear pattern. For the algorithm in problem 14b, the output will be the product of all the elements of the same sequence with indices $1{:}n$. For this algorithm, even if subsequent x values grow in a stable linear fashion, the recursive product will show unstable exponential growth. As before, it is possible to choose subsequent x values such that they exponentially decay in such a way as to create a product that exhibits stable linear growth, but as long as the elements stay on relatively the same order, the growth will be unstable. Both unstable exponential growth and stable linear growth are illustrated on page 32.