Math 342: Homework 4

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Documentation: I used ChatGPT solely for looking up Latex commands. The main Homework 4 MatLab script and all required dependencies are located in the Homework 4 folder found here: https://github.com/Connor-Lemons/Emmons-Math-342. No other resources used.

Problem 1 (5c, 7c):

$f(x) = x\cos(x) - x^2 \sin(x)$

x	$f(x)$	$f'(x)$ approx	Error bound	Actual Error
2.9	-4.827866	5.1014	0.0181	0.012
3.0	-4.240058	6.6548	0.0090	0.0049
3.1	-3.496909	8.2163	0.0049	0.00048
3.2	-2.596792	9.786	0.0099	0.0014

All work can be found in MatLab script attached at the end of the document, or in the GitHub page.

Problem 2 (29):

Consider the function which describes total error $e(h) = \frac{\epsilon}{\mathrm{h}} + \frac{h^2}{6}M$. The minimum of this function will occur when its derivative is equal to zero, which gives:

$$e'(h) = \frac{1}{3}Mh^3 - \epsilon = 0 \tag{1}$$

Rearranging gives:

$$h = \sqrt[3]{\frac{3\text{\\epsilion}}{M}} \tag{2}$$

Thus the error function will be minimized at this value of h.

Problem 3 (15c):

Approximate $\int_{1.1}^{1.5} e^x dx$ using closed Newton-Cotes Formulas up to $n = 4$ and open Newton-Cotes Formulas up to $n = 3$.

	Trapezoid	Simpson	Simpson Three-Eights	Closed $n = 4$	Midpoint	Open $n = 1$	Open $n = 2$	Open $n = 3$
Value	1.497171	1.477536	1.477529	1.477523	1.467719	1.470981	1.477512	1.477515
Error Bound	0.0239	1.59e-5	7.08e-6	3.79e-9	0.0120	0.00797	1.39e-5	9.69e-6
Actual Error	0.0196	1.31e-5	5.81e-6	3.11e-9	0.00980	0.00654	1.14e-5	7.95e-6

The actual error of each of these methods is within the error bound for each of these methods. The most accurate method is the closed Newton-Cotes with $n = 4$. All work can be found in MatLab script attached at the end of the document, or in the GitHub page.

Problem 4 (13b):

Approximate $\int_0^2 \frac{1}{x+4}$ within 10^{-5} using composite Simpson's rule.

For the approximation to be within 10^{-5}, begin with:

$$error = \frac{b-a}{180} h^4 f^{(4)}(\mu) \leq 10^{-5} \tag{1}$$

Substituting a and b for the endpoints gives:

$$\frac{2}{180} h^4 f^{(4)}(\mu) \leq 10^{-5} \tag{2}$$

Note that the fourth derivative of $\frac{1}{x+4}$ is $24(x+4)^{-5}$, which gives:

$$\frac{2}{180} h^4 (24(x+4)^{-5}) \leq 10^{-5} \tag{3}$$

The derivative is maximized at $x = 0$ with a value $\frac{1}{4^5}$, which gives:

$$\frac{2}{180} h^4 \left(\frac{24}{4^5}\right) = \frac{1}{3840} h^4 \leq 10^{-5} \tag{4}$$

Solving for h gives that $h \leq 0.44267$, which gives $n \geq 4.518$ by $h = \frac{b-a}{n}$. Note that this means that $n \geq 6$, which is the smallest even integer that satisfies the given condition.

Implementing Composite Simpson's Rule with $n = 6$ gives the following approximation:

$$\int_0^2 \frac{1}{x+4} \approx 0.405466 \tag{5}$$

MatLab code can be found in the GitHub page or at the end of the document.

Problem 5 (1b, 3b):

Compute the Simpson's rule approximations $S(a,b)$, $S\left(a, \frac{a+b}{2}\right)$, and $S\left(\frac{a+b}{2}, b\right)$ for $\int_0^1 x^2 e^{-x}$.

$$S(0,1) = 0.162402 \tag{1}$$

$$S(0,0.5) = 0.028861 \tag{2}$$

$$S(0.5,1) = 0.131861 \tag{3}$$

Consider Simpson's rule for some integral:

$$\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} F^{(4)}(\xi); S(a,b) = \frac{h}{3}\big(f(a) + 4f(a+h) + f(b)\big) \tag{4}$$

Applying Composite Simpson's Rule with $n = 4$ and $stepsize = \frac{h}{2}$ gives:

$$\int_a^b f(x)dx = \frac{h}{6}\left(f(a) + 4f\left(a + \frac{h}{2}\right) + 2f(a+h) + 4f\left(a + \frac{3h}{2}\right) + f(b)\right) - \left(\frac{h}{2}\right)^4 \frac{b-a}{180} f^{(4)}(\xi) \tag{5}$$

Letting $S\left(a,\frac{a+b}{2}\right)=\frac{h}{6}\left(f(a)+4f\left(a+\frac{h}{2}\right)+f(a+h)\right)$ and $S\left(\frac{a+b}{2},b\right)=\frac{h}{6}\left(f(a+h)+4f\left(a+\frac{3h}{2}\right)+f(b)\right)$ simplifies equation (5) to:

$$\int_a^b f(x)dx = S\left(a,\frac{a+b}{2}\right)+S\left(\frac{a+b}{2},b\right)-\frac{1}{16}\frac{h^5}{90}f^{(4)}(\xi) \quad (6)$$

Assuming that ξ for Simpson's method and ξ for Composite Simpson's Method are approximately equal (and therefore the value of the fourth derivative of the function evaluated at ξ is also relative equal), then:

$$S\left(a,\frac{a+b}{2}\right)+S\left(\frac{a+b}{2},b\right)-\frac{1}{16}\frac{h^5}{90}f^{(4)}(\xi)\approx S(a,b)-\frac{h^5}{90}F^{(4)}(\xi) \quad (7)$$

Simplifying gives:

$$\frac{h^5}{90}F^{(4)}(\xi)\approx\frac{16}{15}\left(S(a,b)-S\left(a,\frac{a+b}{2}\right)-S\left(\frac{a+b}{2},b\right)\right) \quad (8)$$

Using this estimate in conjunction with equation (6) gives:

$$\left|\int_a^b f(x)dx-S\left(a,\frac{a+b}{2}\right)-S\left(\frac{a+b}{2},b\right)\right|\approx\frac{1}{16}\frac{h^5}{90}f^{(4)}(\xi)\approx\frac{16}{15}\left|S(a,b)-S\left(a,\frac{a+b}{2}\right)-S\left(\frac{a+b}{2},b\right)\right| \quad (9)$$

Let $\epsilon = 10^{-3}$. This means that, for $\int_0^1 x^2e^{-x}$, $\frac{1}{15}|S(0,1)-S(0,0.5)-S(0.5,1)|<\epsilon$. Computing this gives:

$$\frac{1}{15}|S(0,1)-S(0,0.5)-S(0.5,1)|=1.1195e-4<10^-3 \quad (9)$$

Because the inequality holds, $S(0,0.5)+S(0.5,1)=0.1607$ is assumed to be a good approximation for $\int_0^1 x^2e^{-x}$.

Problem 6 (13):

Consider the Trapezoid Rule for some integral:

$$\int_a^b f(x)dx = T(a,b)-\frac{h^3}{12}f^{(2)}(\xi) \quad (1)$$

Additionally, consider the Composite Trapezoid Rule for the same integral:

$$\int_a^b f(x)dx=\frac{h}{4}\left(f(a)+2f(a+h)+f(b)\right)-\left(\frac{h}{2}\right)^2\frac{b-a}{12}f^{(2)}(\bar{\xi}) \quad (2)$$

Letting $T\left(a,\frac{a+b}{2}\right)=\frac{h}{4}\left(f(a)+f(a+h)\right)$ and $T\left(\frac{a+b}{2},b\right)=\frac{h}{4}\left(f(a+h)+f(b)\right)$ gives:

$$\int_a^b f(x)dx=T\left(a,\frac{a+b}{2}\right)+T\left(\frac{a+b}{2},b\right)-\frac{1}{4}\frac{h^3}{12}f^{(2)}(\bar{\xi}) \quad (3)$$

Assuming that $\xi\approx\bar{\xi}$ gives:

$$T\left(a,\frac{a+b}{2}\right)+T\left(\frac{a+b}{2},b\right)-\frac{1}{4}\frac{h^3}{12}f^{(2)}(\xi)\approx T(a,b)-\frac{h^3}{12}f^{(2)}(\xi) \quad (4)$$

Rearranging gives:

$$\frac{h^3}{12} \approx \frac{4}{3}\left(T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right) \tag{5}$$

In conjunction with equation (3), this gives:

$$\left|\int_a^b f(x)dx - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right| \approx \frac{1}{4}\frac{h^3}{12} f^{(2)}(\xi) \approx \frac{1}{3}\left|T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right| \tag{6}$$

Problem 7 (2a, 4a):

Approximate $\int_0^{\frac{\pi}{4}} e^{3x} sin(2x)dx$ using Gaussian quadrature. Transforming this integral so that Gaussian quadrature may be used by $x = \frac{1}{2}[(b-a)t + a + b]$ gives:

$$x = \frac{1}{2}\left[\left(\frac{\pi}{4} - 0\right)t + 0 + \frac{\pi}{4}\right] = \frac{\pi}{8}t + \frac{\pi}{8} \tag{1}$$

$$\int_{-1}^{1} e^{3\left(\frac{\pi}{8}t+\frac{\pi}{8}\right)} sin\left(2\left(\frac{\pi}{8}t + \frac{\pi}{8}\right)\right) dt \tag{2}$$

To approximate this integral, find the coefficients and x-values such that:

$$\int_{-1}^{1} f(t)dt \approx c_1 f(t_1) + c_2 f(t_2); f(t) = e^{3\left(\frac{\pi}{8}t+\frac{\pi}{8}\right)} sin\left(2\left(\frac{\pi}{8}t + \frac{\pi}{8}\right)\right) \tag{3}$$

This is done using the Legendre polynomials, and the coefficients and x-values are tabulated for $n \leq 5$. For the $n = 2$ case, table 4.12 gives:

$$\int_0^{\frac{\pi}{4}} e^{3x} sin(2x)dx \approx 1f(0.5773502692)\frac{\pi}{8} + 1f(-0.5773502692)\frac{\pi}{8} = 2.591324 \tag{4}$$

And for the $n = 3$ case:

$$\int_0^{\frac{\pi}{4}} e^{3x} sin(2x)dx \approx \frac{5}{9}f(0.7745966692)\frac{\pi}{8} + \frac{0}{9}f(0)\frac{\pi}{8} + \frac{5}{9}f(-0.7745966692)\frac{\pi}{8} = 2.589258 \tag{5}$$

The error for each case is:

$n = 2$:

$$E = 0.00269608 \tag{6}$$

$n = 3$:

$$E = 0.000629371 \tag{7}$$

Going from $n = 2$ to $n = 3$ produces an error which is one order of magnitude smaller.

The MatLab code can be found in the GitHub page or at the back of this document.

Problem 8 (1a):

$$\int_{2.1}^{2.5}\int_{1.2}^{1.4} xy^2 dydx \approx 0.311573 \tag{1}$$

The MatLab code can be found in the GitHub page or at the back of this document.

Problem 1

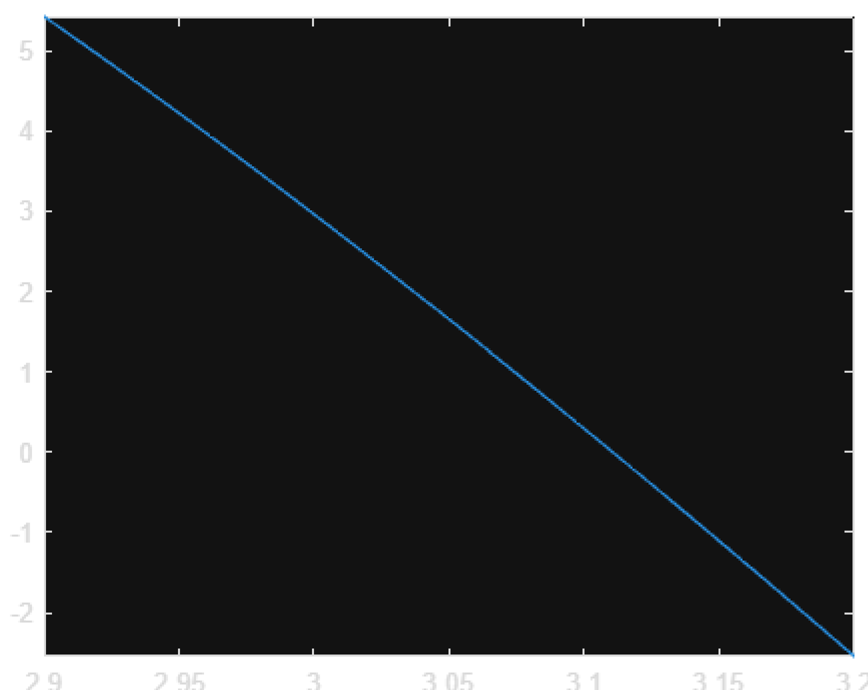
```
clear; clc;

syms x

f(x) = x*cos(x) - x^2*sin(x)
```

f(x) = $x\cos(x) - x^2\sin(x)$

```
f_prime(x) = diff(f,3)
```

f_prime(x) = $x^2\cos(x) - 9\cos(x) + 7\,x\sin(x)$

```
fplot(f_prime)
xlim([2.9, 3.2])
```




```
approx1 = 1/(2*0.1)*(-3*-4.827866 + 4*-4.240058 - -3.496909)
```

approx1 = 5.1014

```
approx2 = 1/(2*0.1)*(-3.496909 - -4.827866)
```

approx2 = 6.6548

```
approx3 = 1/(2*0.1)*(-2.596792 - -4.240058)
```

approx3 = 8.2163

```
approx4 = 1/(2*0.1)*(-4.240058 - 4*-3.496909 + 3*-2.596792)
```

approx4 = 9.7860

```
0.1^2/3*f_prime(2.9)
```

ans = 0.0181

```
0.1^2/6*f_prime(2.9)
```

ans = 0.0090

```
0.1^2/6*f_prime(3)
```

ans = 0.0049

```
0.1^2/3*f_prime(3)
```

ans = 0.0099

```
f_prime(x) = diff(f,1)
```

f_prime(x) = $\cos(x) - x^2\cos(x) - 3\,x\sin(x)$

```
abs(approx1 - f_prime(2.9))
```

ans = 0.0120

```
abs(approx2 - f_prime(3))
```

ans = 0.0049

```
abs(approx3 - f_prime(3.1))
```

ans = $4.7652\text{e-}04$

```
abs(approx4 - f_prime(3.2))
```

ans = 0.0014

Problem 2 (29)

```
clear; clc;

syms h epsilon M

e(h) = epsilon/h + h^2/6*M
```

e(h) =

$$0.1667\, M\, h^2 + \frac{\varepsilon}{h}$$

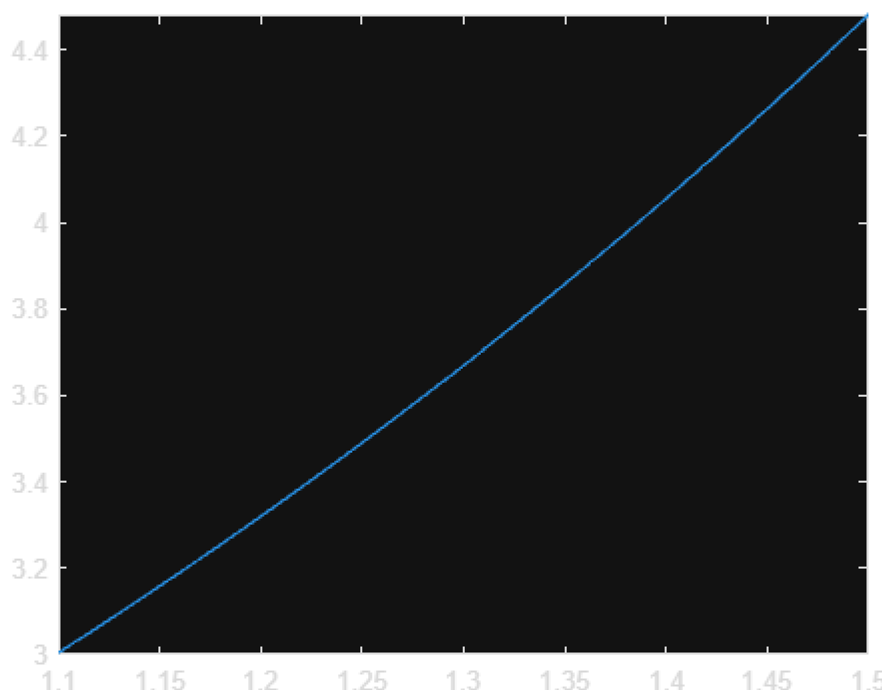
```
diff(e, h)
```

ans(h) =

$$0.3333\, M\, h - \frac{\varepsilon}{h^2}$$

Problem 3 (15c)

```
clear; clc;
format longg

syms x

f(x) = exp(x);
lim = [1.1 1.5];

fplot(f)
xlim(lim)
```




```
approx = zeros(8,1);
error = zeros(8,1);

n = 1;
```

```
h = (lim(2) - lim(1))/n;
points = linspace(lim(1), lim(2), n+1);
approx(n) = h/2*(f(points(1)) + f(points(2)));
error(n) = h^3/12*f(lim(end));

n = 2;
h = (lim(2) - lim(1))/n;
points = linspace(lim(1), lim(2), n+1);
approx(n) = h/3*(f(points(1)) + 4*f(points(2)) + f(points(3)));
error(n) = h^5/90*f(lim(end));

n = 3;
h = (lim(2) - lim(1))/n;
points = linspace(lim(1), lim(2), n+1);
approx(n) = 3*h/8*(f(points(1)) + 3*f(points(2)) + 3*f(points(3)) + f(points(4)));
error(n) = 3*h^5/80*f(lim(end));

n = 4;
h = (lim(2) - lim(1))/n;
points = linspace(lim(1), lim(2), n+1);
approx(n) = 2*h/45*(7*f(points(1)) + 32*f(points(2)) + 12*f(points(3)) +
32*f(points(4)) + 7*f(points(5)));
error(n) = 8*h^7/945*f(lim(end));

n = 0;
h = (lim(2) - lim(1))/(n+2);
points = linspace(lim(1), lim(2), n+3);
points = points(2:end-1);
approx(n + 5) = 2*h*f(points(1));
error(n + 5) = h^3/3*f(lim(end));

n = 1;
h = (lim(2) - lim(1))/(n+2);
points = linspace(lim(1), lim(2), n+3);
points = points(2:end-1);
approx(n + 5) = 3*h/2*(f(points(1)) + f(points(2)));
error(n + 5) = 3*h^3/4*f(lim(end));

n = 2;
h = (lim(2) - lim(1))/(n+2);
points = linspace(lim(1), lim(2), n+3);
points = points(2:end-1);
approx(n + 5) = 4*h/3*(2*f(points(1)) - f(points(2)) + 2*f(points(3)));
error(n + 5) = 14*h^5/45*f(lim(end));

n = 3;
h = (lim(2) - lim(1))/(n+2);
points = linspace(lim(1), lim(2), n+3);
points = points(2:end-1);
```

```
approx(n + 5) = 5*h/24*(11*f(points(1)) + f(points(2)) + f(points(3)) +
11*f(points(4)));
error(n + 5) = 95*h^5/144*f(lim(end));

actual_error = abs(int(f, lim(1), lim(2)) - approx);

double(approx)
```

```
ans = 8×1
    1.4971710188569
   1.47753611765077
   1.47752885891182
   1.47752304950232
    1.4677186670477
   1.47098147226346
   1.47751161487243
   1.47751510112139
```

```
double(error)
```

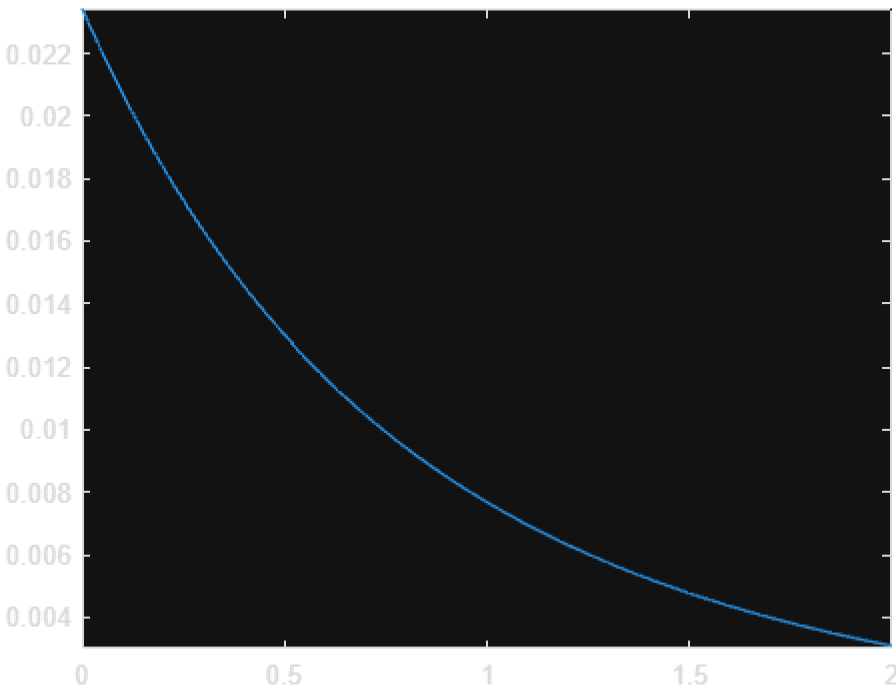
```
ans = 8×1
     0.0239023417084697
   1.59348944723131e-05
   7.08217532102805e-06
   3.79402249340788e-09
     0.0119511708542348
    0.00796744723615656
    1.3943032663274e-05
   9.68841583916637e-06
```

```
double(actual_error)
```

```
ans = 8×1
     0.0196479724652679
   1.30712591333801e-05
   5.81252018924203e-06
   3.11068623168394e-09
    0.00980437934393411
    0.00654157412817039
    1.1431519205162e-05
   7.94527024142646e-06
```

Problem 4 (13b)

```
clear; clc;
format longg

syms x
f(x) = 1/(x+4);
f_prime4(x) = diff(f, 4);

fplot(f_prime4)
xlim([0 2])
```




```
(3.84e-2)^.25
```

```
ans =
        0.442672767880129
```

```
2/(3.84e-2)^.25
```

```
ans =
         4.51801001804922
```

```
double(compositeSimpson(f, 0, 2, 6))
```

```
ans =
        0.405466374584022
```

Problem 5 (1b, 3b)

```
clear; clc;

syms x
f(x) = x^2*exp(-x);
lim = [0, 1];

n = 2;
h = (lim(2) - lim(1))/n;
points = linspace(lim(1), lim(2), n+1);
approx(1) = h/3*(f(points(1)) + 4*f(points(2)) + f(points(3)));
```

```
n = 2;
h = (lim(2)/2 - lim(1))/n;
points = linspace(lim(1), lim(2)/2, n+1);
approx(2) = h/3*(f(points(1)) + 4*f(points(2)) + f(points(3)));

n = 2;
h = (lim(2) - lim(2)/2)/n;
points = linspace(lim(2)/2, lim(2), n+1);
approx(3) = h/3*(f(points(1)) + 4*f(points(2)) + f(points(3)));

double(approx')
```

```
ans = 3×1
    0.162401683480679
   0.0288610717246675
     0.13186140414724
```

```
(1/15)*(approx(1) - approx(2) - approx(3))
```

ans = 1.1195e-04

```
approx(2) + approx(3)
```

ans = 0.1607

Problem 7 (2a, 4a)

```
clear; clc;

syms x t

a = 0;
b = pi/4;

f(x) = exp(3*x)*sin(2*x);
real = int(f, a, b)
```

real = 2.5886

```
replace(t) = (1/2)*((b-a)*t + a + b);

approx2 = f(replace(0.5773502692))*(b - a)/2 + f(replace(-0.5773502692))*(b - a)/2;
approx3 = 5/9*f(replace(0.7745966692))*(b - a)/2 + 8/9*f(replace(0))*(b - a)/2 +
5/9*f(replace(-0.7745966692))*(b - a)/2;

double(approx2)
```

```
ans =
     2.59132471568316
```

```
double(approx3)
```

```
ans =
        2.58925800303196
```

```
error2 = double(abs(approx2 - real))
```

```
error2 =
     0.00269608317598261
```

```
error3 = double(abs(approx3 - real))
```

```
error3 =
    0.000629370524787804
```

Problem 8 (1a)

```
clear; clc;

syms x y

a = 2.1;
b = 2.5;
c(x) = 1.2*x/x;
d(x) = 1.4*x/x;
f(x, y) = x*y^2;
n = 4;
m = 4;

double(doubleSimpson(a, b, m, n, c, d, f))
```

```
ans =
       0.311573333333333
```

```
function [XI] = compositeSimpson(f, a, b, n)

    syms x
    f(x) = f;

    h = (b - a)/n;

    XI0 = f(a) + f(b);
    XI1 = 0;
    XI2 = 0;

    for i = 1:n-1

        X = a + i*h;

        if (mod(i, 2) == 0)
            XI2 = XI2 + f(X);
        elseif (mod(i, 2) == 1)
            XI1 = XI1 + f(X);
        end

    end

    XI = h*(XI0 + 2*XI2 + 4*XI1)/3;

end
```

```
function [J] = doubleSimpson(a, b, m, n, c, d, f)

    syms x y

    c(x) = c;
    d(x) = d;
    f(x, y) = f;

    h = (b - a)/n;
    J1 = 0;
    J2 = 0;
    J3 = 0;

    for i = 0:n

        x = a + i*h;
        HX = (d(x)-c(x))/m;
        K1 = f(x, c(x)) + f(x, d(x));
        K2 = 0;
        K3 = 0;

        for j = 1:m-1

            y = c(x) + j*HX;
            Q = f(x, y);

            if (mod(j, 2) == 0)
                K2 = K2 + Q;
            elseif (mod(j, 2) == 1)
                K3 = K3 + Q;
            end

        end

        L = (K1 + 2*K2 + 4*K3)*HX/3;

        if (i == 0 || i == n)
            J1 = J1 + L;
        elseif (mod(i, 2) == 0)
            J2 = J2 + L;
        else
            J3 = J3 + L;
        end

    end

    J = h*(J1 + 2*J2 + 4*J3)/3;

end
```