Math 342: Homework 3

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Documentation: I used ChatGPT solely for looking up Latex commands. The main Homework 3 MatLab script and all required dependencies are located in the Homework 3 folder found here: https://github.com/Connor-Lemons/Emmons-Math-342. No other resources used.

Problem 1 (6c, 8c):

Use Lagrange interpolating polynomials to approximate $f(0.18)$ if $f(0.1) = -0.29004986$, $f(0.2) = -0.56079734$, $f(0.3) = -0.81401972$, $f(0.4) = -1.0526302$.

In order to find the first- and second-degree interpolating polynomials, use the first two and the first three points respectively. The full interpolation will produce a third-degree polynomial. Using Lagrange interpolation in MatLab gives the following interpolating polynomials:

$$P_1(x) = -2.707x - 0.0193 \tag{1}$$

$$P_2(x) = 0.8763x^2 - 2.97x - 0.001777 \tag{2}$$

$$P_3(x) = -0.4855x^3 + 1.168x^2 - 3.024x + 0.001136 \tag{3}$$

Evaluating these at $P(0.18)$ gives:

$$f(0.18) \approx P_1(0.18) = -0.50655 \tag{4}$$

$$f(0.18) \approx P_2(0.18) = -0.50805 \tag{5}$$

$$f(0.18) \approx P_3(0.18) = -0.50814 \tag{6}$$

The function was generated by $f(x) = x^2 cos(x) - 3x$. Use the error formula to find the bound on the error and compare it to the actual error for $n = 1{,}2$.

The error for Lagrange Polynomials is given by:

$$E(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\ldots(x - x_n) \tag{7}$$

First begin by finding the higher derivatives of f necessary for the error calculations. For $n = 1$ and $n = 2$, $|f^{(n+1)}|$ is maximized at 0.1 and 0.3 respectively. This gives:

$$f^{(2)}(0.1) = 1.9401 \tag{8}$$

$$f^{(3)}(0.3) = -3.4661 \tag{9}$$

Using MatLab to solve for where $E(x)$ is maximized on the considered interval (by finding where the derivative of $E(x) = 0$) gives the following upper bounds on error for $n = 1$ and $n = 2$.

$$n = 1: |E(0.15)| = 0.0024252 \tag{10}$$

$$n = 2: |E(0.14226)| = 0.00022235 \tag{11}$$

The actual error at these points is given by:

$$error_1 = \frac{|f(0.18) - (-0.50655)|}{f(0.18)} = 0.0029041 \tag{12}$$

$$error_2 = \frac{|f(0.18) - (-0.50805)|}{f(0.18)} = 0.00014487 \tag{13}$$

Problem 2 (20):

Use Lagrange interpolation to approximate the average weight curve for each sample and find the approximate maximum average weight for each sample via the maximum of the interpolating polynomial.

Following the definition in the previous problem, the two interpolating polynomials $P_1(x)$ and $P_2(x)$ (which corresponds to sample 1 and sample 2) are given by:

$$P_1(x) = 4.095 \times 10^{-5}x^6 - 0.003672x^5 + 0.1269x^4 - 2.095x^3 + 16.14x^2 - 42.64x + 6.67 \quad (1)$$

$$P_2(x) = 8.362 \times 10^{-6}x^6 - 0.0007525x^5 + 0.02584x^4 - 0.4138x^3 + 2.913x^2 - 5.678x + 6.67 \quad (2)$$

Finding the x-values on the interval $[0,28]$ where these polynomials have derivatives equal to zero give the possible location of maxima, and plugging in these possible values allow the determination of the maximum average weight for each sample. For these polynomials, this gives:

$$P_1(10.189) = 42.708mg \quad (3)$$

$$P_2(8.769) = 19.416mg \quad (4)$$

Problem 3 (2c):

Use Neville's method to obtain the approximations for Lagrange interpolating polynomials of degrees one, two, and three for the function described in problem 1.

Using the same points in problem 1 to give each approximation gives:

$$Q_1 = \begin{bmatrix} 0.1 & -0.29005 & 0 \\ 0.2 & -0.5608 & -0.50665 \end{bmatrix} \quad (1)$$

$$Q_2 = \begin{bmatrix} 0.1 & -0.29005 & 0 & 0 \\ 0.2 & -0.5608 & -0.50665 & 0 \\ 0.3 & -0.81402 & -0.51015 & -0.50805 \end{bmatrix} \quad (2)$$

$$Q_1 = \begin{bmatrix} 0.1 & -0.29005 & 0 & 0 & 0 \\ 0.2 & -0.5608 & -0.50665 & 0 & 0 \\ 0.3 & -0.81402 & -0.51015 & -0.50805 & 0 \\ 0.4 & -1.0526 & -0.52769 & -0.5084 & -0.50814 \end{bmatrix} \quad (3)$$

Note that the first column contains the x-values used, the second column contains the function at those x-values, and the lower right entry is the final approximation. The approximations are the same as those obtained in problem 1.

$$n = 1: f(0.18) \approx -0.50665 \quad (4)$$

$$n = 2: f(0.18) \approx -0.50805 \quad (5)$$

$$n = 3: f(0.18) \approx -0.50814 \quad (6)$$

Problem 4 (10):

Consider an incorrect approximation using Neville's Method of $f(0)$ using $f(-2), f(-1), f(1), f(2)$ where the correct values are $f^*(-2) = f(-2), f^*(-1) = f(-1) - 2, f^*(1) = f(1) + 3, f^*(2) = f(2)$ and compare the error.

To accomplish this, define all of the original functions as $f(-2) = f(-1) = f(1) = f(2) = 0$, which can be represented by the vector:

$$f_{vec_{wrong}} = [0 \quad 0 \quad 0 \quad 0] \tag{1}$$

The correct vector can thus be represented by:

$$f_{vec_{right}} = f_{vec_{wrong}} + [0 \quad -2 \quad 3 \quad 0] = [0 \quad -2 \quad 3 \quad 0] \tag{2}$$

Running both of these through Neville's Method gives the following approximations for $f(0)$:

$$Q_{wrong} = 0 \tag{3}$$

$$Q_{right} = \frac{2}{3} \tag{4}$$

The incorrect values of f produce an approximation that is $\frac{2}{3}$ less than the correct approximation for $x = 0$.

Problem 5 (18):

Find the interpolating polynomial to predict a ¾ mile time using divided differences.

The divided differences formula gives the following coefficient matrix:

$$F = \begin{bmatrix} 25.2 & 0 & 0 & 0 \\ 49.2 & 96.0 & 0 & 0 \\ 96.4 & 94.4 & -2.1333 & 0 \\ 119.4 & 92.0 & -3.2 & -1.0667 \end{bmatrix} \tag{1}$$

Where the coefficients are contained on the main diagonal. Labeling the coefficients starting with one on the top left and incrementing by one moving down the diagonal, the polynomial is given by:

$$P(x) = F_1 + F_2(x - x_1) + \cdots + F_n(x - x_1)(x - x_2) \ldots (x - x_n) \tag{2}$$

For this problem, this gives:

$$P(x) = -1.0667x^3 - 0.26667x^2 + 96.667x + 1.0667 \tag{3}$$

Taking $P(0.75)$ gives:

$$P(0.75) = 1{:}12.967 \tag{4}$$

This gives an absolute error of $\frac{|1{:}13 - 1{:}12.967|}{1{:}13} = 0.00045662$.

Problem 6 (9):

Use Hermite interpolation to interpolate the function.

The resulting coefficient matrix is:

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 75.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.0 & 225.0 & 75.0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3.0 & 225.0 & 77.0 & 0.6667 & 0.2222 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5.0 & 383.0 & 79.0 & 1.0 & 0.06667 & -0.03111 & 0 & 0 & 0 & 0 & 0 \\ 5.0 & 383.0 & 80.0 & 0.5 & -0.25 & -0.06333 & -0.006444 & 0 & 0 & 0 & 0 \\ 8.0 & 623.0 & 80.0 & 0 & -0.1 & 0.03 & 0.01167 & 0.002264 & 0 & 0 & 0 \\ 8.0 & 623.0 & 74.0 & -2.0 & -0.6667 & -0.1133 & -0.02867 & -0.005042 & -0.0009132 & 0 & 0 \\ 13.0 & 993.0 & 74.0 & 0 & 0.25 & 0.1146 & 0.02279 & 0.005146 & 0.0007837 & 0.0001305 & 0 \\ 13.0 & 993.0 & 72.0 & -0.4 & -0.08 & -0.04125 & -0.01948 & -0.004227 & -0.0009373 & -0.0001324 & -2.022 \times 10^{-5} \end{bmatrix} \quad (1)$$

This gives the following polynomial:

$$P(x) = 1.0\text{e-}37\,x\,(-2.022\text{e+}32\,x^8 + 1.041\text{e+}34\,x^7 - 2.188\text{e+}35\,x^6 + 2.43\text{e+}36\,x^5 - 1.538\text{e+}37\,x^4 + 5.508\text{e+}37\,x^3 - 1.01\text{e+}38\,x^2 + 7.162\text{e+}37\,x + 7.5\text{e+}38) \quad (2)$$

Which, at $x = 10$, evaluates to:

$$P(10) = 742.5\ feet \quad (3)$$

Problem 7 (25):

Repeat problem 2 with natural cubic spline interpolation.

The interpolating polynomials for each case are:

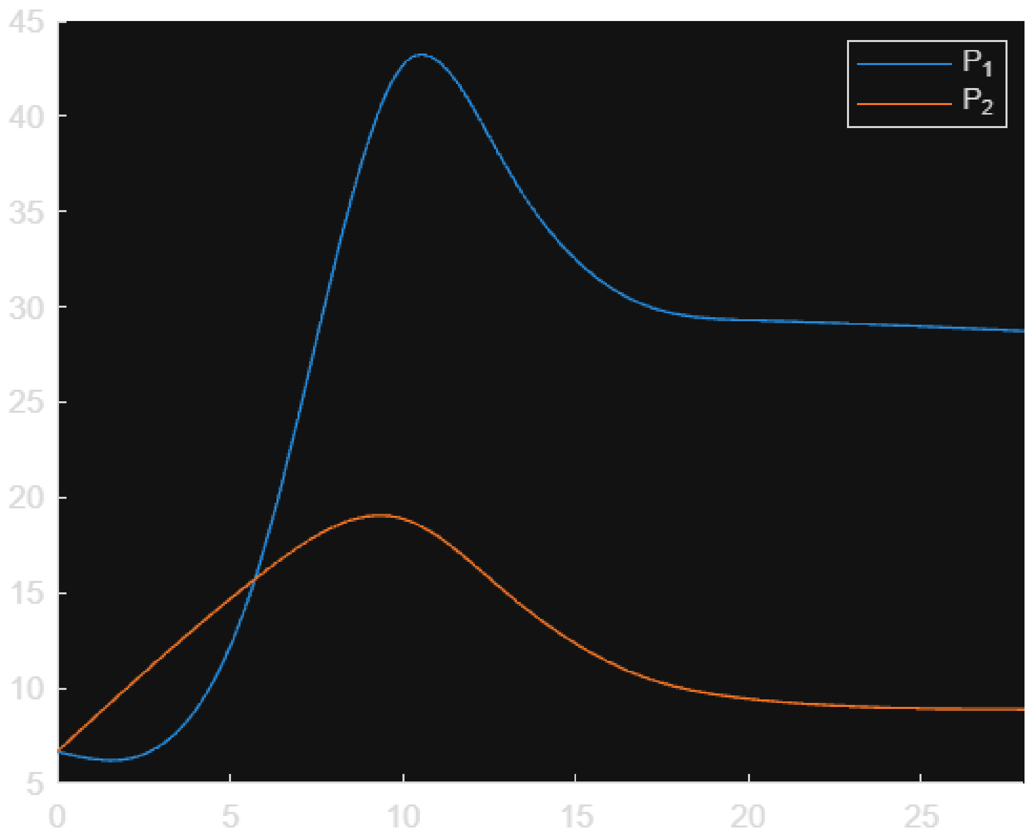
$P_1(x) =$

$$\begin{cases} 0.06176\,x^3 - 0.4469\,x + 6.67 & \text{if } x \in [0.0, 6.0) \\ -0.271\,x^3 + 1.112\,x^2 + 6.224\,x + 17.33 & \text{if } x \in [6.0, 10.0) \\ 0.2811\,x^3 - 2.14\,x^2 + 2.11\,x + 42.67 & \text{if } x \in [10.0, 13.0) \\ -0.01411\,x^3 + 0.3897\,x^2 - 3.141\,x + 37.33 & \text{if } x \in [13.0, 17.0) \\ -0.02491\,x^3 + 0.2204\,x^2 - 0.7002\,x + 30.1 & \text{if } x \in [17.0, 20.0) \\ 0.0001607\,x^3 - 0.003857\,x^2 - 0.05068\,x + 29.31 & \text{if } x \in [20.0, 28.0] \end{cases} \quad (4)$$

$P_2(x) =$

$$\begin{cases} -0.002487\,x^3 + 1.663\,x + 6.67 & \text{if } x \in [0.0, 6.0) \\ -0.03251\,x^3 - 0.04477\,x^2 + 1.394\,x + 16.11 & \text{if } x \in [6.0, 10.0) \\ 0.05916\,x^3 - 0.4349\,x^2 - 0.5244\,x + 18.89 & \text{if } x \in [10.0, 13.0) \\ 0.002264\,x^3 + 0.09756\,x^2 - 1.536\,x + 15.0 & \text{if } x \in [13.0, 17.0) \\ -0.01113\,x^3 + 0.1247\,x^2 - 0.6473\,x + 10.56 & \text{if } x \in [17.0, 20.0) \\ -0.001022\,x^3 + 0.02453\,x^2 - 0.1996\,x + 9.44 & \text{if } x \in [20.0, 28.0] \end{cases} \quad (5)$$

Graphical analysis gives that the maximum value occurs between 10 and 13 for the first case, and between 6 and 10 for the second case. Analyzing the derivatives of the splines on these intervals will provide the location of the maximum. Plugging this value in gives the maximum value of the interpolating polynomial on the interval [0,28].



$$P_1' = 0 \Rightarrow x_{max_1} = 10.553 \tag{6}$$

$$P_2' = 0 \Rightarrow x_{max_2} = 9.3496 \tag{7}$$

$$P_1(x_{max_1}) = 43.23mg \tag{8}$$

$$P_2(x_{max_2}) = 19.06mg \tag{9}$$

Problem 8 (3):

Find the least squares polynomials of degrees 1, 2, 3.

The polynomials are as follows:

$$P_1(x) = 1.22x + 0.6209 \tag{1}$$

$$P_2(x) = -0.01085x^2 + 1.253x + 0.5966 \tag{2}$$

$$P_3(x) = -0.01005x^3 + 0.03533x^2 + 1.185x + 0.629 \quad (3)$$

The error for each of these polynomials can be calculated:

$$E = \sum_{i=1}^{m} [y_i - P(x_i)]^2 \quad (4)$$

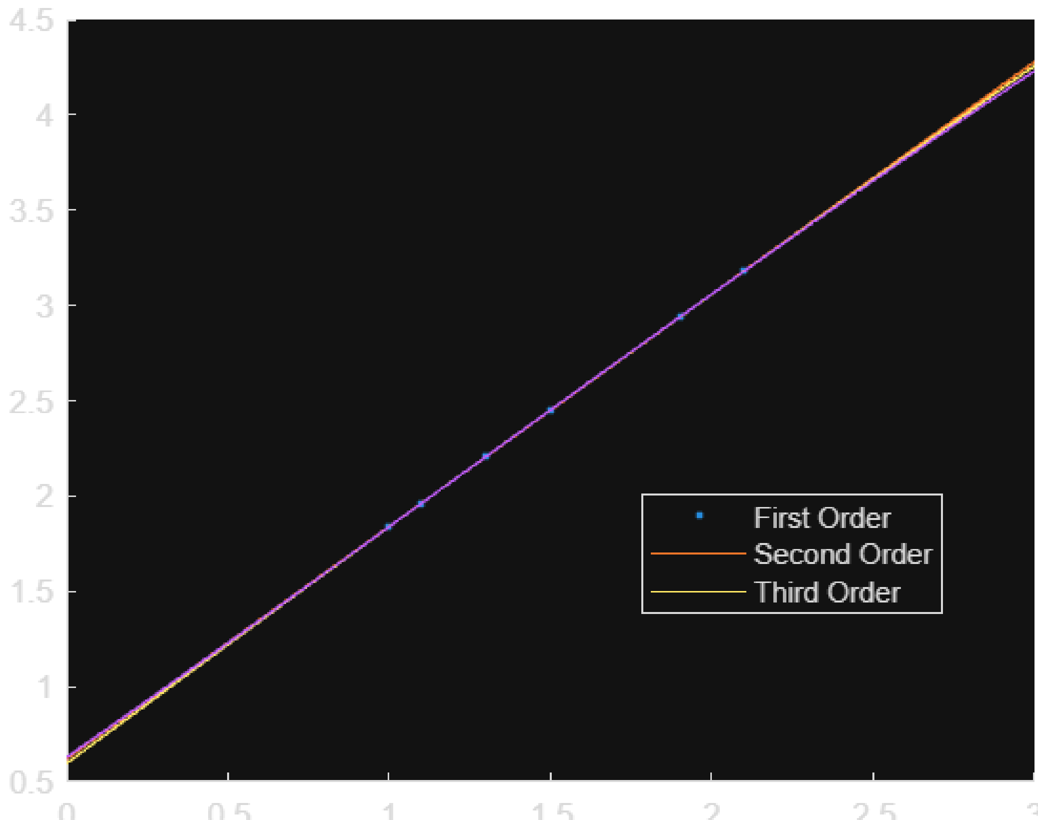
For each of these polynomials, this gives:

$$E_1 = 0.000027194 \quad (5)$$

$$E_2 = 0.000018015 \quad (6)$$

$$E_3 = 0.000017407 \quad (7)$$

The graph of each approximation with the data is as follows:



Problem 9 (1b, 3b, 5b):

Find the least squares polynomial of degree one and two to approximate $f(x) = x^3$.

The polynomials are given:

$$P_1(x) = 3.6x - 1.6 \quad (1)$$

$$P_2(x) = 3x^2 - 2.4x + 0.4 \quad (2)$$

The error is given by:

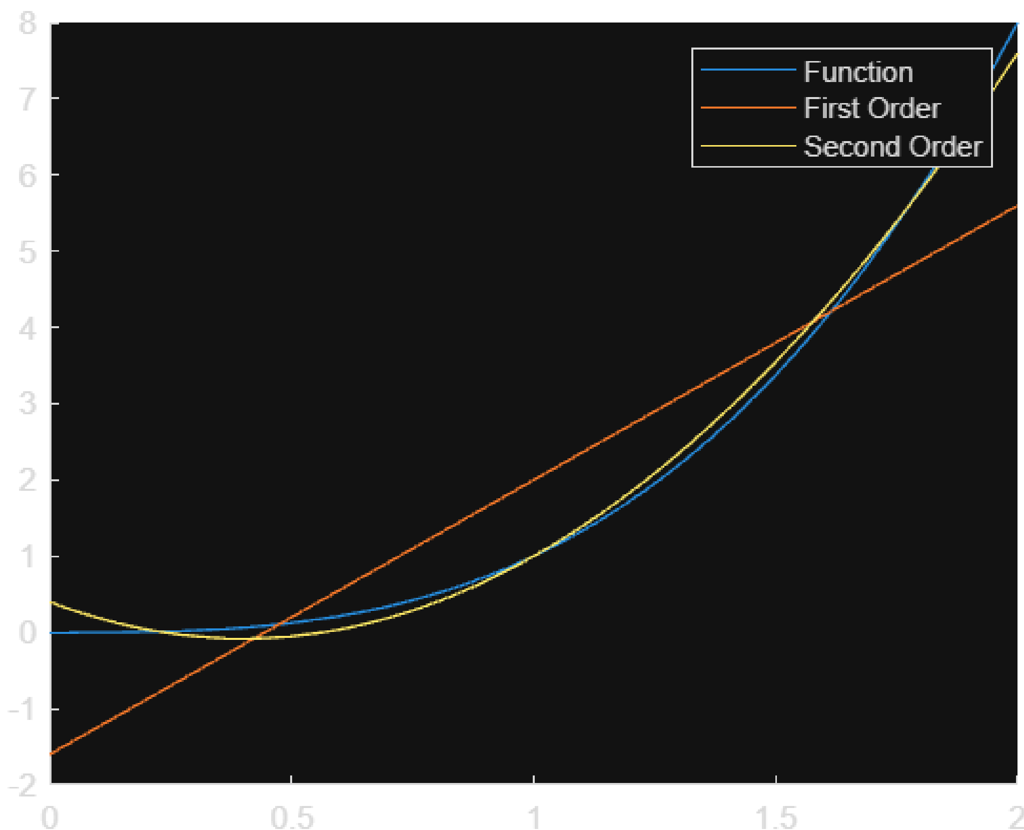
$$E = \int_a^b \left(f(x) - P(x)\right)^2 dx \tag{3}$$

This gives the error for the two approximating polynomials:

$$E_1 = 1.6457 \tag{4}$$

$$E_2 = 0.045714 \tag{5}$$

The graph of the function and the two approximating polynomials is as follows:



Problem 10 (2a):

Use Lagrange, Hermite, and Cubic Spline with the zeros given by the monic Chebyshev polynomials.

The Chebyshev polynomials up to $\tilde{T}_4$ are:

$$\tilde{T}_0 = 1 \tag{1}$$

$$\tilde{T}_1 = x \tag{2}$$

$$\tilde{T}_2 = x^2 - \frac{1}{2} \tag{3}$$

$$\tilde{T}_3 = x^3 - \frac{3}{4}x \tag{4}$$

$$\tilde{T}_4 = x^4 - x^2 + \frac{1}{8} \tag{5}$$

The zeros of these polynomials are:

$$\tilde{T}_0 : \{\} \tag{6}$$

$$\tilde{T}_1 = \{0\} \tag{7}$$

$$\tilde{T}_2 = \left\{\pm\frac{\sqrt{2}}{2}\right\} \tag{8}$$

$$\tilde{T}_3 = \left\{0, \frac{\pm\sqrt{3}}{2}\right\} \tag{9}$$

$$\tilde{T}_4 = \left\{\pm\frac{\sqrt{2\pm\sqrt{2}}}{2}\right\} \tag{10}$$

The union of these sets gives the set containing the zeros of all the Chebyshev polynomials up to $n = 4$. Adding the values ± 1 gives the values over which to interpolate. Performing Lagrange Interpolation, Hermite Interpolation, and Cubic Spline Interpolation on these nodes gives the following interpolating polynomials:

$$\begin{aligned} P_{Lagrange}(x) = {} & 2.838\text{e-}7\,x^{10} + 2.839\text{e-}6\,x^{9} + 2.479\text{e-}5\,x^{8} + 0.0001983\,x^{7} + 0.001389\,x^{6} \\ & + 0.008333\,x^{5} + 0.04167\,x^{4} + 0.1667\,x^{3} + 0.5\,x^{2} + 1.0\,x + 1.0 \end{aligned}$$

$$\begin{aligned} P_{Hermite}(x) = {} & -1.026\text{e-}11\,x^{21} - 4.086\text{e-}11\,x^{20} + 5.851\text{e-}11\,x^{19} + 2.325\text{e-}10\,x^{18} - 1.433\text{e-}10\,x^{17} \\ & - 5.696\text{e-}10\,x^{16} + 1.976\text{e-}10\,x^{15} + 7.964\text{e-}10\,x^{14} - 5.616\text{e-}12\,x^{13} + 1.422\text{e-}9\,x^{12} \\ & + 2.514\text{e-}8\,x^{11} + 2.759\text{e-}7\,x^{10} + 2.756\text{e-}6\,x^{9} + 2.48\text{e-}5\,x^{8} + 0.0001984\,x^{7} \\ & + 0.001389\,x^{6} + 0.008333\,x^{5} + 0.04167\,x^{4} + 0.1667\,x^{3} + 0.5\,x^{2} + 1.0\,x + 1.0 \end{aligned}$$

$P_{Spline}(x) =$

$$\begin{cases} 1.105\,x^3 + 3.314\,x^2 + 3.69\,x + 1.849 & \text{if } x \in [-1.0, -0.9239] \\ -0.2862\,x^3 - 0.5411\,x^2 + 0.1282\,x + 0.7516 & \text{if } x \in [-0.9239, -0.866] \\ 0.092\,x^3 + 0.4416\,x^2 + 0.9793\,x + 0.9972 & \text{if } x \in [-0.866, -0.7071] \\ 0.09272\,x^3 + 0.4432\,x^2 + 0.9804\,x + 0.9975 & \text{if } x \in [-0.7071, -0.3827] \\ 0.1374\,x^3 + 0.4944\,x^2 + 1.0\,x + 1.0 & \text{if } x \in [-0.3827, 0.0] \\ 0.1986\,x^3 + 0.4944\,x^2 + 1.0\,x + 1.0 & \text{if } x \in [0.0, 0.3827] \\ 0.3039\,x^3 + 0.3735\,x^2 + 1.046\,x + 0.9941 & \text{if } x \in [0.3827, 0.7071] \\ 0.2375\,x^3 + 0.5145\,x^2 + 0.9466\,x + 1.018 & \text{if } x \in [0.7071, 0.866] \\ 3.03\,x^3 - 6.74\,x^2 + 7.229\,x - 0.7959 & \text{if } x \in [0.866, 0.9239] \\ -7.257\,x^3 + 21.77\,x^2 - 19.11\,x + 7.316 & \text{if } x \in [0.9239, 1.0] \end{cases}$$

Taking the integral of each of these functions on the interval $[-1,1]$ yields:

$$\int_{-1}^{1} P_{Lagrange}(x)dx = 2.350402387287516 \tag{11}$$

$$\int_{-1}^{1} P_{Hermite}(x)dx = 2.350402387287604 \tag{12}$$

$$\int_{-1}^{1} P_{Spline}(x)dx = 2.350398604412972 \tag{13}$$

When compared to the value of $\int -1^1 e^x dx$, the relative errors are:

$$E_L = 8.6541 \times 10^{-14} \tag{14}$$

$$E_H = 7.6395 \times 10^{-16} \tag{15}$$

$$E_S = 7.6395 \times 10^{-16} \tag{16}$$

The Hermite polynomial and the Cubic Spline polynomial are about equivalent in approximating the true integral.