Math 342: Homework 4

Connor Emmons

Documentation: I used ChatGPT solely for looking up Latex commands. The main Homework 4 MatLab script and all required dependencies are located in the Homework 4 folder found here: https://github.com/Connor-Lemons/Emmons-Math-342. No other resources used.

Problem 1 (5c, 7c):

$f(x) = xcos(x) - x^2 sin(x)$

x	$f(x)$	$f'(x)$ approx	Error bound	Actual Error
2.9	-4.827866	5.1014	0.0181	0.012
3.0	-4.240058	6.6548	0.0090	0.0049
3.1	-3.496909	8.2163	0.0049	0.00048
3.2	-2.596792	9.786	0.0099	0.0014

All work can be found in MatLab script attached at the end of the document, or in the GitHub page.

Problem 2 (29):

Consider the function which describes total error $e(h) = \frac{\epsilon}{\mathrm{h}} + \frac{h^2}{6}M$. The minimum of this function will occur when its derivative is equal to zero, which gives:

$$e'(h) = \frac{1}{3}Mh^3 - \epsilon = 0 \tag{1}$$

Rearranging gives:

$$h = \sqrt[3]{\frac{3\text{\epsilon}\text{silon}}{M}} \tag{2}$$

Thus the error function will be minimized at this value of h.

Problem 3 (15c):

Approximate $\int_{1.1}^{1.5} e^x dx$ using closed Newton-Cotes Formulas up to $n = 4$ and open Newton-Cotes Formulas up to $n = 3$.

	Trapezoid	Simpson	Simpson Three-Eights	Closed $n = 4$	Midpoint	Open $n = 1$	Open $n = 2$	Open $n = 3$
Value	1.497171	1.477536	1.477529	1.477523	1.467719	1.470981	1.477512	1.477515
Error Bound	0.0239	1.59e-5	7.08e-6	3.79e-9	0.0120	0.00797	1.39e-5	9.69e-6
Actual Error	0.0196	1.31e-5	5.81e-6	3.11e-9	0.00980	0.00654	1.14e-5	7.95e-6

The actual error of each of these methods is within the error bound for each of these methods. The most accurate method is the closed Newton-Cotes with $n = 4$. All work can be found in MatLab script attached at the end of the document, or in the GitHub page.

Problem 4 (13b):

Approximate $\int_0^2 \frac{1}{x+4}$ within 10^{-5} using composite Simpson's rule.

For the approximation to be within 10^{-5}, begin with:

$$error = \frac{b-a}{180} h^4 f^{(4)}(\mu) \leq 10^{-5} \tag{1}$$

Substituting a and b for the endpoints gives:

$$\frac{2}{180} h^4 f^{(4)}(\mu) \leq 10^{-5} \tag{2}$$

Note that the fourth derivative of $\frac{1}{x+4}$ is $24(x+4)^{-5}$, which gives:

$$\frac{2}{180} h^4 (24(x+4)^{-5}) \leq 10^{-5} \tag{3}$$

The derivative is maximized at $x = 0$ with a value $\frac{1}{4^5}$, which gives:

$$\frac{2}{180} h^4 \left(\frac{24}{4^5}\right) = \frac{1}{3840} h^4 \leq 10^{-5} \tag{4}$$

Solving for h gives that $h \leq 0.44267$, which gives $n \geq 4.518$ by $h = \frac{b-a}{n}$. Note that this means that $n \geq 6$, which is the smallest even integer that satisfies the given condition.

Implementing Composite Simpson's Rule with $n = 6$ gives the following approximation:

$$\int_0^2 \frac{1}{x+4} \approx 0.405466 \tag{5}$$

MatLab code can be found in the GitHub page or at the end of the document.

Problem 5 (1b, 3b):

Compute the Simpson's rule approximations $S(a, b)$, $S\left(a, \frac{a+b}{2}\right)$, and $S\left(\frac{a+b}{2}, b\right)$ for $\int_0^1 x^2 e^{-x}$.

$$S(0,1) = 0.162402 \tag{1}$$

$$S(0,0.5) = 0.028861 \tag{2}$$

$$S(0.5,1) = 0.131861 \tag{3}$$

Consider Simpson's rule for some integral:

$$\int_a^b f(x)dx = S(a,b) - \frac{h^5}{90} F^{(4)}(\xi); S(a,b) = \frac{h}{3}\big(f(a) + 4f(a+h) + f(b)\big) \tag{4}$$

Applying Composite Simpson's Rule with $n = 4$ and $stepsize = \frac{h}{2}$ gives:

$$\int_a^b f(x)dx = \frac{h}{6}\left(f(a) + 4f\left(a + \frac{h}{2}\right) + 2f(a+h) + 4f\left(a + \frac{3h}{2}\right) + f(b)\right) - \left(\frac{h}{2}\right)^4 \frac{b-a}{180} f^{(4)}(\xi) \tag{5}$$

Letting $S\left(a, \frac{a+b}{2}\right) = \frac{h}{6}\left(f(a) + 4f\left(a + \frac{h}{2}\right) + f(a+h)\right)$ and $S\left(\frac{a+b}{2}, b\right) = \frac{h}{6}\left(f(a+h) + 4f\left(a + \frac{3h}{2}\right) + f(b)\right)$ simplifies equation (5) to:

$$\int_a^b f(x)dx = S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\frac{h^5}{90} f^{(4)}(\xi) \tag{6}$$

Assuming that ξ for Simpson's method and ξ for Composite Simpson's Method are approximately equal (and therefore the value of the fourth derivative of the function evaluated at ξ is also relative equal), then:

$$S\left(a, \frac{a+b}{2}\right) + S\left(\frac{a+b}{2}, b\right) - \frac{1}{16}\frac{h^5}{90} f^{(4)}(\xi) \approx S(a,b) - \frac{h^5}{90} F^{(4)}(\xi) \tag{7}$$

Simplifying gives:

$$\frac{h^5}{90} F^{(4)}(\xi) \approx \frac{16}{15}\left(S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right) \tag{8}$$

Using this estimate in conjunction with equation (6) gives:

$$\left|\int_a^b f(x)dx - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \approx \frac{1}{16}\frac{h^5}{90} f^{(4)}(\xi) \approx \frac{16}{15}\left|S(a,b) - S\left(a, \frac{a+b}{2}\right) - S\left(\frac{a+b}{2}, b\right)\right| \tag{9}$$

Let $\epsilon = 10^{-3}$. This means that, for $\int_0^1 x^2 e^{-x}$, $\frac{1}{15}|S(0,1) - S(0,0.5) - S(0.5,1)| < \epsilon$. Computing this gives:

$$\frac{1}{15}|S(0,1) - S(0,0.5) - S(0.5,1)| = 1.1195e - 4 < 10^-3 \tag{9}$$

Because the inequality holds, $S(0,0.5) + S(0.5,1) = 0.1607$ is assumed to be a good approximation for $\int_0^1 x^2 e^{-x}$.

Problem 6 (13):

Consider the Trapezoid Rule for some integral:

$$\int_a^b f(x)dx = T(a,b) - \frac{h^3}{12} f^{(2)}(\xi) \tag{1}$$

Additionally, consider the Composite Trapezoid Rule for the same integral:

$$\int_a^b f(x)dx = \frac{h}{4}\left(f(a) + 2f(a+h) + f(b)\right) - \left(\frac{h}{2}\right)^2 \frac{b-a}{12} f^{(2)}(\bar{\xi}) \tag{2}$$

Letting $T\left(a, \frac{a+b}{2}\right) = \frac{h}{4}\left(f(a) + f(a+h)\right)$ and $T\left(\frac{a+b}{2}, b\right) = \frac{h}{4}\left(f(a+h) + f(b)\right)$ gives:

$$\int_a^b f(x)dx = T\left(a, \frac{a+b}{2}\right) + T\left(\frac{a+b}{2}, b\right) - \frac{1}{4}\frac{h^3}{12} f^{(2)}(\bar{\xi}) \tag{3}$$

Assuming that $\xi \approx \bar{\xi}$ gives:

$$T\left(a, \frac{a+b}{2}\right) + T\left(\frac{a+b}{2}, b\right) - \frac{1}{4}\frac{h^3}{12} f^{(2)}(\xi) \approx T(a,b) - \frac{h^3}{12} f^{(2)}(\xi) \tag{4}$$

Rearranging gives:

$$\frac{h^3}{12} \approx \frac{4}{3}\left(T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right) \tag{5}$$

In conjunction with equation (3), this gives:

$$\left|\int_a^b f(x)dx - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right| \approx \frac{1}{4}\frac{h^3}{12} f^{(2)}(\xi) \approx \frac{1}{3}\left|T(a,b) - T\left(a, \frac{a+b}{2}\right) - T\left(\frac{a+b}{2}, b\right)\right| \tag{6}$$

Problem 7 (2a, 4a):

Approximate $\int_0^{\frac{\pi}{4}} e^{3x} sin(2x)dx$ using Gaussian quadrature. Transforming this integral so that Gaussian quadrature may be used by $x = \frac{1}{2}[(b-a)t + a + b]$ gives:

$$x = \frac{1}{2}\left[\left(\frac{\pi}{4} - 0\right)t + 0 + \frac{\pi}{4}\right] = \frac{\pi}{8}t + \frac{\pi}{8} \tag{1}$$

$$\int_{-1}^{1} e^{3\left(\frac{\pi}{8}t+\frac{\pi}{8}\right)} sin\left(2\left(\frac{\pi}{8}t + \frac{\pi}{8}\right)\right) dt \tag{2}$$

To approximate this integral, find the coefficients and x-values such that:

$$\int_{-1}^{1} f(t)dt \approx c_1 f(t_1) + c_2 f(t_2); f(t) = e^{3\left(\frac{\pi}{8}t+\frac{\pi}{8}\right)} sin\left(2\left(\frac{\pi}{8}t + \frac{\pi}{8}\right)\right) \tag{3}$$

This is done using the Legendre polynomials, and the coefficients and x-values are tabulated for $n \leq 5$. For the $n = 2$ case, table 4.12 gives:

$$\int_0^{\frac{\pi}{4}} e^{3x} sin(2x)dx \approx 1f(0.5773502692)\frac{\pi}{8} + 1f(\text{-}0.5773502692)\frac{\pi}{8} = 2.591324 \tag{4}$$

And for the $n = 3$ case:

$$\int_0^{\frac{\pi}{4}} e^{3x} sin(2x)dx \approx \frac{5}{9}f(0.7745966692)\frac{\pi}{8} + \frac{0}{9}f(0)\frac{\pi}{8} + \frac{5}{9}f(-0.7745966692)\frac{\pi}{8} = 2.589258 \tag{5}$$

The error for each case is:

$n = 2$:

$$E = 0.00269608 \tag{6}$$

$n = 3$:

$$E = 0.000629371 \tag{7}$$

Going from $n = 2$ to $n = 3$ produces an error which is one order of magnitude smaller.

The MatLab code can be found in the GitHub page or at the back of this document.

Problem 8 (1a):

$$\int_{2.1}^{2.5} \int_{1.2}^{1.4} xy^2 dydx \approx 0.311573 \tag{1}$$

The MatLab code can be found in the GitHub page or at the back of this document.