Math 342: Project 1

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Documentation: I used ChatGPT solely for looking up Latex commands. The main Project 1 MatLab script and all required dependencies are located in the Project 1 folder found here: https://github.com/Connor-Lemons/Emmons-Math-342. No other resources used.

Project 1:

Part 1: Aitken's delta-squared process

Problem 1: Consider the problem of approximating the root p of $f(x) = x + ln(x)$ using Fixed-Point iteration.

Part A: Show that the root p of $f(x)$ is also a fixed point of the function $g(x) = e^{-x}$.

Begin with the function $f(x) = x + ln(x)$. Finding the root p is equivalent to finding the value of x which satisfies the equation:

$$f(x) = 0 = x + ln(x) \tag{1}$$

Rewriting equation (1) gives:

$$-x = ln(x) \tag{2}$$

$$x = e^{-x} \tag{3}$$

Defining $g(x) = e^{-x}$ gives:

$$g(x) = x = e^{-x} \tag{4}$$

Finding the value of x which satisfies $g(x) = x$ is equivalent to finding the fixed point of $g(x)$. Because this is the same value of x that gives p, the root p of $f(x)$ is the same as the fixed point of $g(x)$.

Part B: Use the Fixed-Point Theorem to show that Fixed-Point Iteration with $g(x) = e^{-x}$ will converge to p for any $p_0 \neq p$ in the interval $\left[\frac{1}{e}, 1\right]$.

Theorem 2.4 states that if $g \in C[a, b]$ (i.e., $g(x) \in [a, b]$ for all $x \in [a, b]$), g' exists on (a, b), and there exists some constant $0 < k < 1$ such that $\left(\forall x \in (a, b)\right)(|g'(x)| \leq k)$, then for any $p_0 \in [a, b]$ the sequence $p_n = g(p_{n-1})$ converges to a unique fixed point $p \in [a, b]$.

Because $g(x)$ is a decreasing monotone function (i.e., $(\forall x \in R)(y \in R \geq x \Rightarrow g(y) \leq g(x))$, if $g(a) \in [a, b]$ and $g(b) \in [a, b]$, then $g \in C[a, b]$.

$$a = \frac{1}{e} = 0.3679 \tag{1}$$

$$b = 1 \tag{2}$$

$$g(a) = 0.6922 \Rightarrow g(a) \in [a, b] \tag{3}$$

$$g(b) = \frac{1}{e} = 0.3679 \Rightarrow g(b) \in [a, b] \tag{4}$$

Thus, both $g(a)$ and $g(b)$ are in the interval $[a,b]$.

Consider $g(x) = e^{-x}$ and its derivative $g'(x) = -e^{-x}$. Both of these functions are defined for all $x \in R$, and are continuous functions for real number inputs. Notably, this means that $g'(x)$ exists on the interval (a,b) because $(a,b) \subseteq R$. Because $g'(x)$ is a monotone increasing function, $g'(x)$ will have the largest value on the interval $[a,b]$ at a. This gives:

$$|g'(a)| = 0.6922 \tag{1}$$

Thus, $k = 0.7$ satisfies the condition

$$0 < 0.7 < 1: \big(\forall x \in (a,b)\big)(|g^{'}(x)| \leq k) \tag{2}$$

because

$$\big(\forall x \in (a,b)\big)(max(|g^{'}(x)|) = g'(a) = 0.6922 < 0.7) \tag{3}$$

Therefore, because $g(x)$ has been shown to satisfy all the conditions of Theorem 2.4 on the interval $[a,b]$, where $a = \frac{1}{e}$ and $b = 1$, $g(x)$ will converge to a unique fixed point p for any $p_0 \neq p \in \left[\frac{1}{e}, 1\right]$.

Part C: Write a MatLab script to implement Fixed-Point Iteration $p_n = g(p_{n-1})$ with $g(x) = e^{-x}$ and $p_0 = 0.4$. Use $TOL = |p_n - p_{n-1}| \leq 10^{-6}$ as the stopping criterion.

n	p_n	$\lvert p_n - p_{n-1} \rvert$
1	0.67032	0.27032
2	0.51154	0.15878
3	0.59957	0.088024
4	0.54905	0.05052
5	0.5775	0.028451
6	0.5613	0.016199
7	0.57047	0.0091664
8	0.56526	0.0052052
9	0.56821	0.00295
10	0.56654	0.0016737
11	0.56749	0.00094903
12	0.56695	0.00053831
13	0.56725	0.00030528
14	0.56708	0.00017314
15	0.56718	9.8194e-05
16	0.56712	5.5691e-05
17	0.56715	3.1584e-05
18	0.56714	1.7913e-05
19	0.56715	1.0159e-05
20	0.56714	5.7617e-06
21	0.56714	3.2677e-06
22	0.56714	1.8533e-06
23	0.56714	1.0511e-06

24	0.56714	5.9611e-07

Code can be found in Appendix A.

Part D: Show that the solution obtained using Fixed-Point Iteration converges linearly to p by showing that

$$\frac{|p_{n+1}-p|}{|p_n-p|} \approx \text{constant} \tag{0}$$

for the 4 largest values of n.

For the iteration above, use $n = 21,22,23$.

For $n = 21$:

$$\frac{|p_{n+1}-p|}{|p_n-p|} = 0.32537 \tag{1}$$

For $n = 22$:

$$\frac{|p_{n+1}-p|}{|p_n-p|} = 1.3102 \tag{2}$$

For $n = 23$:

$$\frac{|p_{n+1}-p|}{|p_n-p|} = 0 \tag{3}$$

Note that this quotient equals zero for $n = 23$ because the value of p is taken to be the value of p_{24}, which was the first p_n that satisfied the error requirement. It is difficult to determine from these points alone whether $\frac{|p_{n+1}-p|}{|p_n-p|} \approx \text{constant}$ holds. Calculating this quotient for all $n \in [1,23]$ gives a more accurate picture. When accounting for all $n \in [1,23]$, it is clearer that the quotient is approximately constant, and thus this iteration converges linearly.

n	$\frac{\lvert p_{n+1}-p\rvert}{\lvert p_n-p\rvert}$
1	0.53886
2	0.58321
3	0.55804
4	0.57232
5	0.56418
6	0.56886
7	0.5661
8	0.56786
9	0.56652
10	0.56787
11	0.56606
12	0.56894
13	0.56406
14	0.57258

15	0.55768
16	0.58411
17	0.53811
18	0.62109
19	0.48028
20	0.74799
21	0.32537
22	1.3102
23	0

Code can be found in Appendix A.

Problem 2: Consider the problem of approximating the root p of $f(x) = x + ln(x)$ using Steffensen's Method.

Part A: Write a MatLab script to implement Steffensen's Method with $g(x) = e^{-x}$ and $p_0 = 0.4$. Use $TOL = |p_n - p_{n-1}| \leq 10^{-6}$ as the stopping criterion.

n	p_n	$\lvert p_n - p_{n-1} \rvert$
1	0.5703	0.1703
2	0.56714	0.003151
3	0.56714	1.0178e-06
4	0.56714	1.0633e-13

Code can be found in Appendix A.

Part B: Show that the solution obtained using Fixed-Point Iteration converges linearly to p by showing that

$$\frac{|p_{n+1}-p|}{|p_n-p|^2} \approx constant \tag{0}$$

for the 4 largest values of n.

For the iteration above, use $n = 1,2,3$.

For $n = 1$:

$$\frac{|p_{n+1}-p|}{|p_n-p|^2} = 0.10244 \tag{1}$$

For $n = 2$:

$$\frac{|p_{n+1}-p|}{|p_n-p|^2} = 0.10262 \tag{2}$$

For $n = 3$:

$$\frac{|p_{n+1}-p|}{|p_n-p|^2} = 0 \tag{3}$$

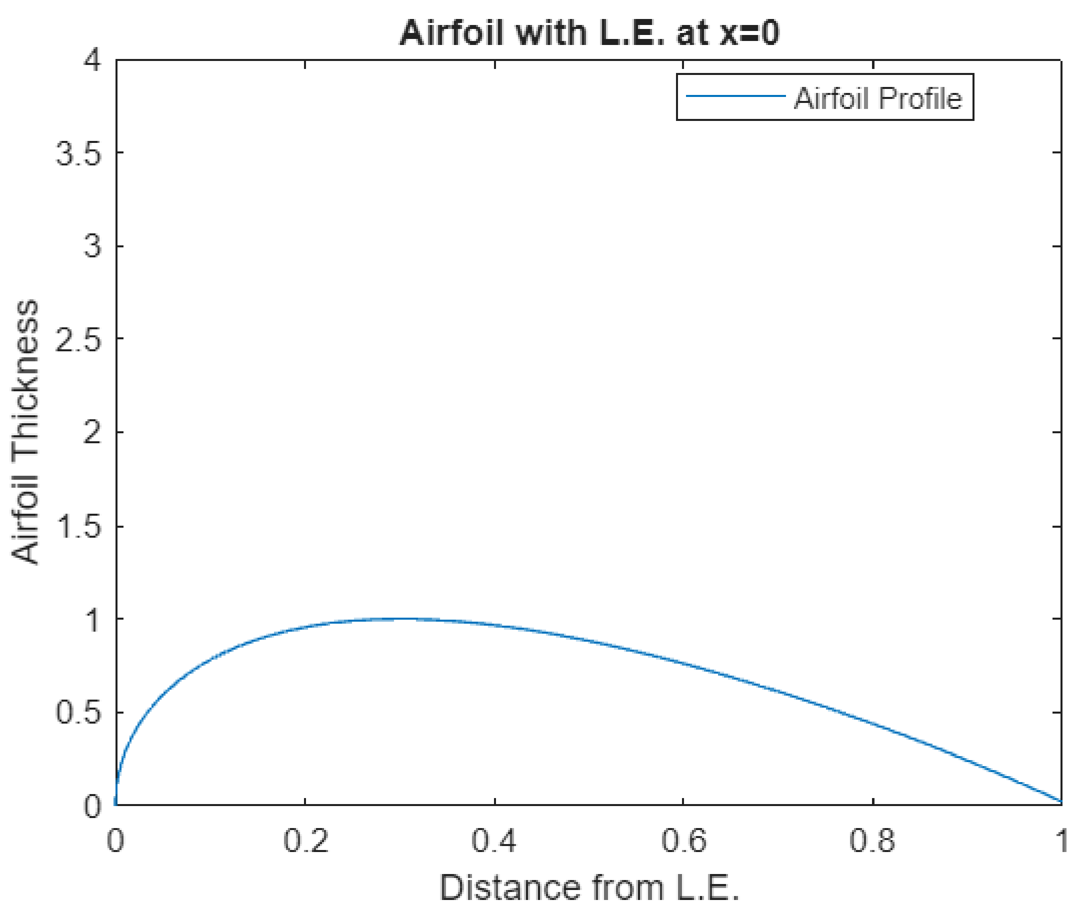
Note that this quotient equals zero for $n = 3$ because the value of p is taken to be the value of p_4, which was the first p_n that satisfied the error requirement. This quotient is relatively constant across all iterations of Steffensen's Method, and thus this iteration converges quadratically.

n	$\frac{\lvert p_{n+1}-p\rvert}{\lvert p_n-p\rvert^2}$
1	0.10244
2	0.10262
3	0

Code can be found in Appendix A.

Part 2: Newton's Method

Problem 1: Plot the airfoil described by $y = 2.969\sqrt{x} - 1.26x - 3.516x^2 + 2.843x^3 - 1.015x^4$ from $x = 0$ to $x = 1$ where y is the thickness of the airfoil and x is the distance from the leading edge of the airfoil ($x = 0$).



Problem 2: Consider the problem of finding the thickest point of the airfoil using Newton's Method.

Part A: Derive a function $f(x)$ such that the root p of $f(x)$ corresponds to the location of the thickest point of the airfoil.

Note that the thickest point of the airfoil described by y will simply be the maximum value of y. For some function of x, $g(x) = y = 2.969\sqrt{x} - 1.26x - 3.516x^2 + 2.843x^3 - 1.015x^4$, this will occur when the function's derivative is equal to zero. Thus, the function is maximized (i.e., the airfoil achieves its maximum thickness) at some from the leading edge $0 < p < 1$ such that $f(p) = g'(p) = 0$. Taking this derivative gives

$$f(x) = 1.484x^{-\frac{1}{2}} - 1.26 - 7.032x + 8.529x^2 - 4.06x^3 \quad (1)$$

where the root of $f(x)$ corresponds to the location of the thickest point of the airfoil.

Part B: Write a MatLab script to implement Newton's Method with $f(x)$ and $p_0 = 0.1$. Use $TOL = |p_n - p_{n-1}| \leq 10^{-7}$ as the stopping criterion.

n	p_n	$\lvert p_n - p_{n-1}\rvert$
1	0.1972486	0.09724864
2	0.2762193	0.0789707
3	0.2986261	0.02240674
4	0.2998248	0.001198702
5	0.2998279	3.103562e-06
6	0.2998279	2.069539e-11

Code can be found in Appendix A.

Problem 3: Now consider the problem of finding the thickest point of the airfoil using Steffensen's Method.

Part A: Derive Steffensen's Method from Newton's Method by replacing $f'(x)$ with the forward difference and applying the substitution $h = f(x)$. Explain why $h = f(x)$ is a valid choice.

Begin with Newton's Method:

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad (1)$$

The forward difference approximation for the derivative of the function is:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad (2)$$

This comes from the definition of the derivative, which is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3}$$

Note that this means that the approximation in (2) gets more accurate the closer to zero h gets. Because of this, letting $h = f(x)$ is a valid choice because Newton's method iterates to find the x such that $f(x) = 0$, and subsequent x values will drive $f(x)$ to zero. Applying this gives the function $g(x)$:

$$f'(x) \approx g(x) = \frac{f(x+f(x)) - f(x)}{f(x)} \tag{4}$$

Substituting into equation (1) gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{g(p_{n-1})}; g(p_{n-1}) = \frac{f(p_{n-1} + f(p_{n-1})) - f(p_{n-1})}{f(p_{n-1})} \tag{5}$$

which is Steffensen's Method.

Part B: Write a MatLab script to implement Steffensen's Method with $f(x)$ and $p_0 = 0.1$. Use $TOL = |p_n - p_{n-1}| \leq 10^{-7}$ as the stopping criterion.

n	p_n	$\lvert p_n - p_{n-1}\rvert$
1	0.25318	0.15318
2	0.33455	0.081362
3	0.31674	0.017805
4	0.30374	0.012996
5	0.30004	0.0037032
6	0.29983	0.00021297
7	0.29983	6.4019e-07
8	0.29983	5.7534e-12

Code can be found in Appendix A.

Note that for Steffensen's Method, the equation $f(x) + x$ was in the iteration used to account for the fact that Steffensen's Method finds $p = g(p)$.

Part C: Show that the solution obtained in Part B converges quadratically to p by calculating $g'(p_n) \approx 0$ for your largest value of n.

For $n = 6$:

$$g'(p_6) = 0.08628 \approx 0 \tag{1}$$

The rest of the derivative values are as follows:

n	$g'(p_n)$
1	-1.563
2	1.043
3	0.4613

4	0.1087
5	0.005993
6	1.797e-5
7	0.03098
8	0.08628

Code can be found in Appendix A.