Final Project
Numerical Solutions to PDEs

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Documentation UPDATE THIS
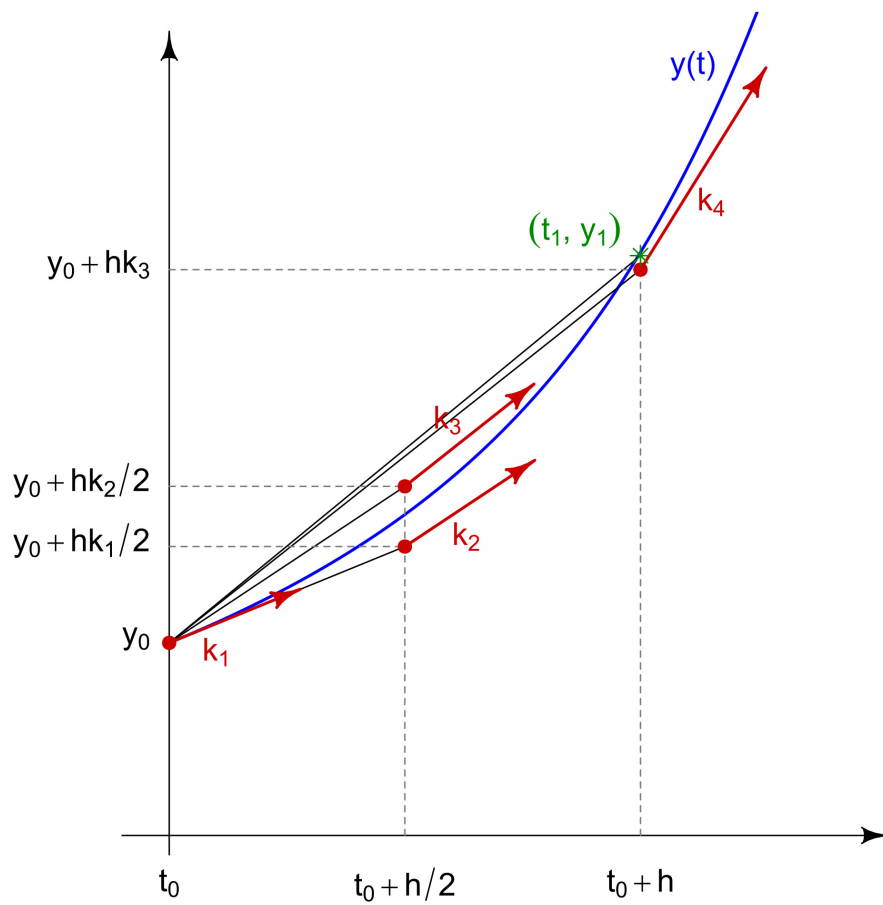
1   Basic Heat Equation

$u_t = 2u_{xx}$
$u(0,t) = u(3,t) = 0$
$u(x,0) = \sin(\pi x) + \sin(2\pi x)$

2   General Wave Equation

$u_{tt} = 0.16u_{xx} + 0.02\sin(x+t)$
$u(0,t) = 0.01\sin(t)$
$u_x(2,t) = 0$
$u(x,0) = \sin\left(\frac{\pi x}{2}\right)$
$u_t(x,0) = \cos\left(\frac{\pi x}{2}\right)$

Using the central difference approximation for second derivatives gives the following:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = 0.16\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right) + 0.02\sin(x+t)$$

$$u_i^{n+1} = \frac{0.16\Delta t^2}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + 0.02\Delta t^2 \sin(x+t) + 2u_i^n - u_i^{n-1}$$

(1)

Note that the i index represents the spatial steps and the n index represents the temporal steps. This gives a formulation for the next time step. Rewriting this gives a formulation for the current time step based on the previous time steps.

$$u_i^n = \frac{0.16\Delta t^2}{\Delta x^2}\left(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}\right) + 0.02\Delta t^2 \sin(x+t) + 2u_i^{n-1} - u_i^{n-2}$$

(2)

This is the form which will be implemented to produce the numerical solution.

3   Two-Dimensional General Heat Equation

$u_t = 0.5\nabla^2 u + e^{-t}u$
$u_x(0,y,t) = 0$
$u_y(x,0,t) = 0$
$u(x,4,t) = e^{-t}$
$u(4,y,t) = 0$
$u(x,y,0) = \begin{cases} 1 & \textbf{for } y \geq x \\ 0 & \textbf{for } y < x \end{cases}$

Using the forward difference approximation for first derivatives and the central difference approximation for second derivatives gives the following:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} = 0.5\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right) + e^{-t}u_{i,j}^n$$

$$u_{i,j}^{n+1} = 0.5\Delta t\left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{\Delta x^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{\Delta y^2}\right) + \left(e^{-t}\Delta t + 1\right)u_{i,j}^n$$

(3)

In this case, i and j represent the x and y spatial steps, respectively. This gives a formulation for the next time step. Assuming the same step size for both spatial dimensions allows the simplificaiton $\Delta x = \Delta y = h$. Along with rewriting the formulation to give the current time step as a function of the previous time steps, this gives:

$$u_{i,j}^n = \frac{0.5\Delta t}{h^2}\left(u_{i+1,j}^{n-1} + u_{i-1,j}^{n-1} + u_{i,j+1}^{n-1} + u_{i,j-1}^{n-1} - 4u_{i,j}^{n-1}\right) + \left(e^{-t}\Delta t + 1\right)u_{i,j}^{n-1}$$

(4)

This is the form which will be implemented to produce the numerical solution.

4 Poisson's Equation

$\nabla^2 u = 1 + 0.2\delta_{(1,3)}(x, y)$
$u(x, 4) = x$
$u(2, y) = 1$
$u(0, y) = 1$ **for** $2 \leq y \leq 4$
$u(1, y) = 0$ **for** $0 \leq y \leq 2$
$u(x, 2) = 0$ **for** $0 \leq x \leq 1$
$u(x, 0) = 1$ **for** $1 \leq x \leq 2$

5 Spacecraft Application