Due in Gradescope by 2359 on Friday, 9 May 2025 (Lesson T40).

DIRECTIONS

Assignment. In groups of size at least two and at most three (you choose the groups) you will complete the set of tasks given below and assemble your work in a single coherent, well-written document (with attachments as needed). Remember that this project is the last graded event for this course for the rest of the semester, so you should use the time you would have spent completing another Progress Assessment, studying for a GR, and preparing for a Final Exam as an estimate for how much time and effort you should plan to invest in this assignment.

Authorized Resources. There are no limitations to the resources you can use for this project but be sure to appropriately cite or give credit to any references that you use. This includes online or print information, software tools, consultations with other humans, and interactions with AI platforms. While your groups will work independently and will each be responsible for turning in their own completed final report, there is no prohibition on informal consultation between groups. I am also available by appointment as a consultant.

It is your responsibility to ensure that **all** members of a group contribute to the effort and have an understanding of the content in your final report. You are all competent students with a variety of individual skills and talents—work together to use them effectively. Do **not** approach this assignment in a "divide and conquer" manner.

Computing. All of the tasks will require coding, unless you want to do some of the computations by hand, which is perhaps possible but would not be pleasant. While MATLAB is well-suited to perform these computations, you are free to use other languages or packages if you prefer. I can definitely give good advice and help with MATLAB, and likely with Python (depending on what packages you choose to use); if you choose to use something else then the help I can provide will likely be less helpful. Include a clearly commented copy of your code as an attachment to your report, and send a working copy to me in a Teams message when you submit your report to Gradescope.

Suspense. Group final reports will be due by 2359 on Friday, May 9 (Lesson T40), prior to the start of Final Examinations, in accord with DF policy. You have a great deal of flexibility in determining the style and format for your report, and creativity is encouraged in how you present your results—but do make sure that the end product reflects both scholarship and professionalism.

Problem:

1. For this problem you will be working with the following "toy" PDE in order to learn about a numerical method we did not cover in class, descriptively called the *method of lines*.

$$\begin{cases} u_t = 2u_{xx} \\ u(0,t) = u(3,t) = 0 \\ u(x,0) = e^{-2\pi^2 t}\sin(\pi x) + e^{-8\pi^2 t}\sin(2\pi x) \end{cases}$$

You should recognize this as a basic heat equation. Perform the following tasks.

A. Solve this IBVP analytically using methods from class to obtain an exact solution.
B. Use finite-difference approximations in the spatial variable to convert the PDE to a coupled system of ODEs in time, and determine an appropriate set of initial conditions for this system of ODEs.
C. Write and employ a fourth-order Runge-Kutta (RK4) method and use it to solve the coupled system of ODEs you found in (B) on the time interval $0 \leq t \leq 2$. Your solution should be a grid of numerical approximations to the actual solution on a grid of spatial and time values determined by your choices of Δx and Δt.
D. Redo (C), but this time use MATLAB's pre-built solver `ode45()` in place of your own RK4 method. Again, your solution should be a grid of numerical approximations to the actual solution on a grid of spatial and time values determined by your choices of Δx and Δt.
E. Compare the solutions you obtained in (C) and (D) against the true solution you found in (A). Give error for each solution in terms of (1) maximum absolute error, and (2) error in the L^2 norm.

2. For this problem, you will need to write code that numerically solves an IBVP involving the wave equation in one spatial variable. You should test your code using problems you have already solved in class this semester. On Lesson M39 I will provide a test problem (specific PDE with BCs and ICs) for you to solve with your method as part of preparing your final report. You don't know exactly what the test problem will look like, so you should take care to ensure that your code can be adapted to fit whatever might come. Things to think about include: Dirichlet vs. Neumann BCs, homogeneous vs. nonhomogeneous BCs and PDE, constant vs. time-dependent BCs, string length, and stability limitations in choosing Δx and Δt.

3. For this problem, you will need to write code that numerically solves an IBVP involving the heat equation in two spatial variables. You should test your code using problems you have already solved in class this semester. On Lesson M39 I will provide a test problem (specific PDE with BCs and ICs) for you to solve with your method as part of preparing your final report. You don't know exactly what the test problem will look like, so you should take care to ensure that your code can be adapted to fit whatever might come. Things to think about include: Dirichlet vs. Neumann BCs, homogeneous vs. nonhomogeneous BCs and PDE, constant vs. time-dependent BCs, and stability limitations in choosing Δx and Δt. The geometry of the test problem will be either rectangular or can be approximated by a subset of a rectangular grid.

4. For this problem, you will need to write code that numerically solves an BVP involving either Laplace's equation or Poisson's equation in one spatial variable. You should test your code using problems you have already solved in class this semester. On Lesson M39 I will provide a test problem (specific PDE with BCs) for you to solve with your method as part of preparing your final report. You don't know exactly what the test problem will look like, so you should take care to ensure that your code can be adapted to fit whatever might come. Things to think about include: Dirichlet vs. Neumann BCs, homogeneous vs. nonhomogeneous BCs and PDE, and stability limitations in choosing Δx and Δt. The geometry of the test problem will be either rectangular or can be approximated by a subset of a rectangular grid.

5. This final problem asks you to combine what you learned in Problems 1 and 4.

Interstellar spacecraft such as Voyager 1 and 2 utilize a radioisotope thermoelectric generator, which produces heat through radioactive decay of plutonium and subsequently power through a thermocouple process. Spacecraft need to constantly radiate excess heat to provide the required temperature gradient to produce power through the thermocouples/shed waste heat, and past our solar system the temperatures approach that of absolute zero (on the order of 2.7K).

You are designing a radiator to shed the waste heat in this scenario. The radiator consists of multiple fins, and each fin is heated with the temperature profile given below. The other three edges of the fins are essentially held at 0K to simplify the problem, approximating the 2.7K found in interstellar space. Because the fins are thin, there is essentially no change in temperature through the thickness of the fin material (so that $\frac{\partial^2 T}{\partial z^2} = 0$), allowing a two dimensional model. In addition, the spacecraft has been out there for a long time, so that the heat generation and heat radiating from the fin reached a steady-state heat profile decades ago. A formulation of the corresponding BVP is on the next page.

Tasks:

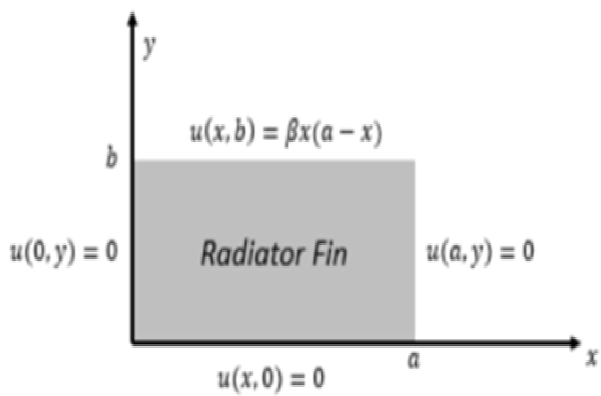
A. Determine the exact analytic temperature profile across the entire surface of the fin.
B. Generate a temperature distribution plot for your solution from (A). Does this appear as you expect? Since weight is a premium on spacecraft, do you suppose there may be a better design for the radiator fin in this idealized case?
C. Use your solver from Problem 4 to find a numerical approximation to the solution. Assess how well this numerical solution approximates the true solution.
D. As in Poblem 1, implement the method of lines to numerically solve this problem. Start by "slicing" along the x-axis and using a finite-difference formula in x to reduce the PDE to a coupled system of second-order ODEs. Then you will need to use reduction of order to transform this to a coupled system of first-order ODEs. Then use an RK method of your choice (please not Euler/RK1) to solve the system. Assess how well this method approximates the true solution.

   Note: You will need initial conditions to implement your RK method. The original problem's boundary condition $u(x, 0) = 0$ should provide half of the required initial conditions for solving this system. You will need to play a clever "guess and check" game to determine suitable values for the remaining initial conditions. The "guess" part is up to you, but for the check you will want the corresponding values of the boundary condition $u(x, b) = \beta x(a - x)$ to closely match with the value of your RK output at $y = b$. If your output for a particular guess of initial conditions doesn't match, then you will need to adjust your guess and try again (and again, and again...) to bring the values closer together. This is often descriptively called the "shooting method" for solving a boundary value problem (which is what you are really facing with this system of ODEs). The shooting method gives us a way to leverage our RK methods in solving boundary value problems in certain cases.

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < a, \quad 0 < y < b$$

$$u(0, y) = 0, \qquad u(a, y) = 0, \qquad 0 < y < b$$

$$u(x, 0) = 0, \qquad u(x, b) = \beta x(a - x), \qquad 0 < x < a$$



Physical properties:

$a = \dfrac{\pi}{8}$ Length of fin (m)

$b = \dfrac{\pi}{16}$ Width of fin (m)

$\beta = 10000$ Thermal constant $\left(\dfrac{K}{m^2}\right)$