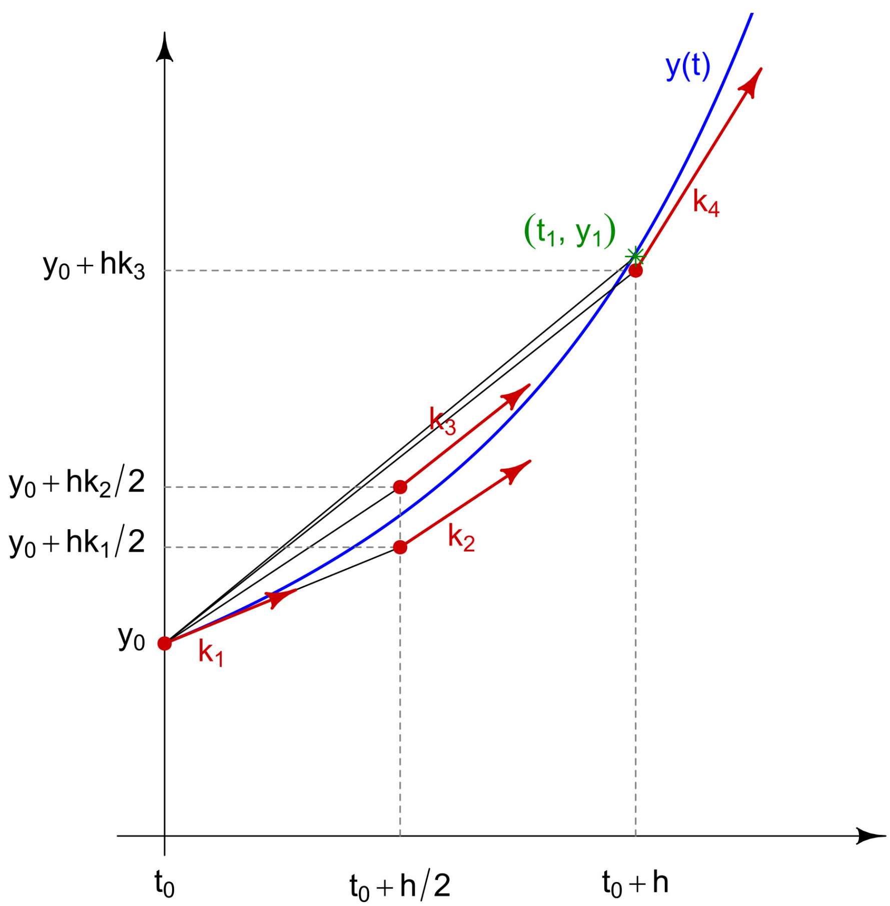
Final Project
Numerical Solutions to PDEs

Connor Emmons and Noah Wells

9 May 2025



Documentation UPDATE THIS

1 Basic Heat Equation

$u_t = 2u_{xx}$
$u(0,t) = u(3,t) = 0$
$u(x,0) = \sin(\pi x) + \sin(2\pi x)$

2 General Wave Equation

$u_{tt} = 0.16u_{xx} + 0.02\sin(x+t)$
$u(0,t) = 0.01\sin(t)$
$u_x(2,t) = 0$
$u(x,0) = \sin\left(\frac{\pi x}{2}\right)$
$u_t(x,0) = \cos\left(\frac{\pi x}{2}\right)$

Using the central difference approximation for second derivatives gives the following:

$$\begin{aligned} \frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} &= 0.16\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right) + 0.02\sin(x+t) \\ u_i^{n+1} &= \frac{0.16\Delta t^2}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) + 0.02\Delta t^2 \sin(x+t) + 2u_i^n - u_i^{n-1} \end{aligned} \tag{1}$$

Note that the i index represents the spatial steps and the n index represents the temporal steps. This gives a formulation for the next time step. Rewriting this gives a formulation for the current time step based on the previous time steps.

$$u_i^n = \frac{0.16\Delta t^2}{\Delta x^2}\left(u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}\right) + 0.02\Delta t^2 \sin(x+t) + 2u_i^{n-1} - u_i^{n-2} \tag{2}$$

This is the form which will be implemented to produce the numerical solution. Note that this equation requires knowledge of the two previous time steps, and thus a different method is required for initialization. In order to do this, begin with the forward difference equation in time and the given initial condition.

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \cos\left(\frac{\pi x}{2}\right) \tag{3}$$

For the wave equation, the other initial condition gives the value, but the wave equation requires two previous time steps. Using this formulation, a "ghost step" can be obtained and used to generate the first iteration of the wave equation. This is given by:

$$u_i^0 = u_i^1 - \Delta t \cos\left(\frac{\pi x}{2}\right) \tag{4}$$

For the boundary condition at the left endpoint, it is readily apparent that the formulation for this must be:

$$u_{end}^n = u_{end-1}^n \tag{5}$$

3 Two-Dimensional General Heat Equation

$u_t = 0.5\nabla^2 u + e^{-t} u$
$u_x(0, y, t) = 0$
$u_y(x, 0, t) = 0$
$u(x, 4, t) = e^{-t}$
$u(4, y, t) = 0$
$u(x, y, 0) = \begin{cases} 1 & \textbf{for } y \geq x \\ 0 & \textbf{for } y < x \end{cases}$

Using the forward difference approximation for first derivatives and the central difference approximation for second derivatives gives the following:

$$\begin{aligned} \frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} &= 0.5 \left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right) + e^{-t} u_{i,j}^{n} \\ u_{i,j}^{n+1} &= 0.5\Delta t \left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2} \right) + \left(e^{-t} \Delta t + 1 \right) u_{i,j}^{n} \end{aligned} \tag{6}$$

In this case, i and j represent the x and y spatial steps, respectively. This gives a formulation for the next time step. Assuming the same step size for both spatial dimensions allows the simplificaiton $\Delta x = \Delta y = h$. Along with rewriting the formulation to give the current time step as a function of the previous time steps, this gives:

$$u_{i,j}^{n} = \frac{0.5\Delta t}{h^2} \left(u_{i+1,j}^{n-1} + u_{i-1,j}^{n-1} + u_{i,j+1}^{n-1} + u_{i,j-1}^{n-1} - 4u_{i,j}^{n-1} \right) + \left(e^{-t} \Delta t + 1 \right) u_{i,j}^{n-1} \tag{7}$$

This is the form which will be implemented to produce the numerical solution.

4 Poisson's Equation

$\nabla^2 u = 1 + 0.2\delta_{(1,3)}(x, y)$
$u(x, 4) = x$
$u(2, y) = 1$
$u(0, y) = 1$ **for** $2 \leq y \leq 4$
$u(1, y) = 0$ **for** $0 \leq y \leq 2$
$u(x, 2) = 0$ **for** $0 \leq x \leq 1$
$u(x, 0) = 1$ **for** $1 \leq x \leq 2$

5 Spacecraft Application